Self-similar cosmological solutions in symmetric teleparallel theory: Friedmann-Lemaître-Robertson-Walker spacetimes

N. Dimakis,[1,*] M. Roumeliotis,[2,†] A. Paliathanasis,[3,4,5,‡] Pantelis S. Apostolopoulos,[5,§] and T. Christodoulakis[2,∥]

[1]*Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu 610064, China*
[2]*Nuclear and Particle Physics section, Physics Department, University of Athens, 15771 Athens, Greece*
[3]*Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa*
[4]*Departamento de Matemáticas, Universidad Católica del Norte, Avenida Angamos 0610, Casilla 1280 Antofagasta, Chile*
[5]*Mathematical Physics and Computational Statistics Research Laboratory, Department of Environment, Ionian University, Zakinthos 29100, Greece*

(Received 19 October 2022; accepted 29 November 2022; published 21 December 2022)

The existence of self-similar solutions is discussed in symmetric teleparallel $f(Q)$ theory for a Friedmann-Lemaître-Robertson-Walker background geometry with zero and nonzero spatial curvature. For the four distinct families of connections that describe the specific cosmology in symmetric teleparallel gravity, the functional form of $f(Q)$ is reconstructed. Finally, to see if the analogy with GR holds, we discuss the relation of the self-similar solutions with the asymptotic behavior of more general $f(Q)$ functions.

DOI: 10.1103/PhysRevD.106.123516

I. INTRODUCTION

With the term of self-similar solutions, we refer to a family of solutions with the characteristic to map to itself after an appropriate scale of the dependent or independent variables. In gravitational physics, self-similar solutions of the Einstein field equations are mainly related with the similarity solutions provided by the existence of a proper homothetic vector field for the physical space [1,2]. With the homothetic symmetry, we refer to the generator of the infinitesimal transformation in the physical space, which preserves the angles between the lines but not the scale. Previous studies on exact solutions on homogeneous and inhomogeneous spacetimes indicate that self-similar solutions correspond to asymptotic limits of more general solutions [3–7]. Recall that self-similar spacetimes cannot describe asymptotically flat or asymptotically spatially compact geometries [8].

Because of the importance of the self-similar solutions, there is a plethora of studies in the literature where the existence of a proper homothetic vector field is investigated in various geometries. The conformal symmetries for the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes are investigated in [9], where it is found that for a power-law scale factor, a proper homothetic vector field exists. Indeed, the exact solutions of an ideal gas for a spatially flat FLRW geometry or the Milne Universe, admit a proper homothetic symmetry. The existence of homothetic vector field for a Bianchi I geometry was studied in [10]. It was found that the Kasner solution as also the Kasner-like solutions are self-similar solutions. An analysis of the homothetic vector field in Bianchi III and Bianchi V can be found in [11], while the four-dimensional stationary axisymmetric vacuum spacetimes with a homothetic vector field were investigated in detail in [12]. On the other hand, anisotropic and homogeneous self-similar exact solutions for Bianchi VIII, Bianchi IX, and Bianchi VI_0 geometries with tilted perfect fluid were studied in [13–15], while Bianchi class B spacetimes with a homothetic vector field was the subject of study in [16,17]. Some inhomogeneous self-similar solutions are presented in [18].

The fundamental invariant of Einstein's general relativity (GR) is the Ricci scalar R defined by the symmetric Levi-Civita. Nevertheless, more general connections from that of the Levi-Civita have been used in gravitational physics. Indeed, in teleparallelism, the torsion scalar T defined by the curvatureless Weitzenböck connection [19] is the geometric object which has been used to define gravity, leading to the teleparallel equivalent of general relativity (TEGR) [20,21]. Furthermore, from the nonmetricity components of a general connection, the scalar Q can be defined, where in the case of a torsion-free and flat connection, we end up with the symmetric teleparallel equivalent of general relativity (STEG) [22]. Scalars R, T, and Q form the so-called geometrical trinity of gravity [23]. The gravitational

[*]nsdimakis@scu.edu.cn; nsdimakis@gmail.com
[†]microum@phys.uoa.gr
[‡]anpaliat@phys.uoa.gr
[§]papostol@ionio.gr
[∥]tchris@phys.uoa.gr

Lagrangians, which are linear in each element of the trinity, lead to the same gravitational theory. The latter equivalence is violated when nonlinear terms of the geometric scalars are introduced in the gravitational Lagrangian.

The modification of the Einstein-Hilbert action integral is the simplest geometric mechanism for the introduction of new degrees of freedom in order to explain the observational phenomena [24,25]. There exist a family of modified theories of gravity known as $f(X)$ theory, where the gravitational Lagrangian is a function f of the geometric scalar X. Usually, X is one element of the geometric trinity, such that when f is a linear function, GR (or TEGR or STEG) is recovered (respectively); of course, more complicated configurations can be constructed as well involving combinations of scalars [26]. The $f(R)$ theory of gravity introduced in [27] is a fourth-order theory of gravity, which can be written in a dynamical equivalent form of the Brans-Dicke scalar field with zero Brans-Dicke parameter. Thus, the $f(R)$ theory is equivalent with a minimally coupled scalar field under a conformal transformation [28]. $f(R)$ theory has been used to describe various epochs of the cosmological history, such as the inflation [29] or the late time acceleration [30]. Similarly, the $f(\mathrm{T})$ teleparallel theory of gravity introduced as a geometric dark energy model [31], while it has been widely applied in various gravitational configurations [32], for more details and applications of $f(\mathrm{T})$ theory, we refer the reader to the recent review [33]. Only recently, the $f(Q)$-symmetric teleparallel theory [34] has drawn the attention of the scientific society [35–54].

In $f(Q)$ theory, we make use of a flat connection pertaining to the existence of affine coordinates in which all its components vanish, that is, turning covariant derivatives into partial (coincident gauge). Consequently, in this theory, it is possible to separate the inertial effects from gravity. Thus, the coincident gauge is always achievable through an appropriate coordinate transformation. Recently, in [55], the effects of the use of different connections for the definition of the scalar Q in the dynamics of FLRW cosmology in $f(Q)$ theory was the subject of study. For the FLRW spacetimes, there exist four distinct families of connections, compatible with the isometries of the FLRW metric [56,57], three for the spatially flat case and one when the spatial curvature is present; for these connections, exact solutions were derived. In a later study, the effects of the different connections were investigated in the case of static spherical symmetric spacetimes [58].

In this work, we are interested in the existence of self-similar solutions in $f(Q)$ theory and on the effects of the different connections on the existence of the homothetic symmetry vector for the case of a FLRW background geometry. Self-similar solutions were investigated before in $f(R)$ theory [59,60] and in the $f(T)$-teleparallel theory of gravity [61].

The structure of the paper is as follows: In Sec. II, we present the basic geometric elements of the $f(Q)$-symmetric teleparallel theory and the different connections for the FLRW spacetime with or without spatial curvature. In Sec. III, we consider the spatially flat case, and we reconstruct closed-form expressions for the $f(Q)$ function, where self-similar solutions exist. The analysis for a nonzero spatial curvature is presented in Sec. IV. Finally, in Sec. V, we summarize our results and draw our conclusions.

II. SYMMETRIC TELEPARALLEL THEORY

The metric $g_{\mu\nu}$ and the connection $\Gamma^{\kappa}{}_{\mu\nu}$ are the basic dynamical objects in metric-affine gravitational theories. We define the following fundamental tensors: the curvature $R^{\kappa}{}_{\lambda\mu\nu}$, the torsion $\mathrm{T}^{\lambda}_{\mu\nu}$, and the nonmetricity $Q_{\lambda\mu\nu}$,

$$R^{\kappa}{}_{\lambda\mu\nu} = \frac{\partial\Gamma^{\kappa}{}_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial\Gamma^{\kappa}{}_{\lambda\mu}}{\partial x^{\nu}} + \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\kappa}{}_{\mu\sigma} \tag{1}$$

$$\mathrm{T}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \tag{2}$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\sigma}{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma}. \tag{3}$$

In the latter expressions, the symbol ∇_{μ} means covariant derivative with respect to the affine connection $\Gamma^{\kappa}{}_{\mu\nu}$. For a symmetric connection, as the one we consider in this work, the torsion tensor vanishes, $\mathrm{T}^{\lambda}_{\mu\nu} = 0$. Moreover, in symmetric teleparallelism, the curvature tensor is also zero, that is $R^{\kappa}{}_{\lambda\mu\nu} = 0$, while for the nonmetricity part, we have $Q_{\lambda\mu\nu} \neq 0$.

The fundamental nonmetricity scalar Q of the symmetric teleparallel theory is defined as

$$Q = Q_{\lambda\mu\nu} P^{\lambda\mu\nu}, \tag{4}$$

where $P^{\lambda}_{\mu\nu}$ is the nonmetricity conjugate tensor expressed as

$$P^{\lambda}_{\mu\nu} = -\frac{1}{4} Q^{\lambda}{}_{\mu\nu} + \frac{1}{2} Q_{(\mu}{}^{\lambda}{}_{\nu)} + \frac{1}{4}(Q^{\lambda} - \bar{Q}^{\lambda}) g_{\mu\nu} - \frac{1}{4}\delta^{\lambda}{}_{(\mu} Q_{\nu)}, \tag{5}$$

in which the contractions $Q_{\lambda} = Q_{\lambda\mu}{}^{\mu}$, $\bar{Q}_{\lambda} = Q^{\mu}{}_{\lambda\mu}$ are introduced.

A. $f(Q)$ theory

In $f(Q)$ theory, the gravitational Lagrangian density is defined by a generally nonlinear function $f(Q)$ namely [62],

$$S = \frac{1}{2}\int d^4x\sqrt{-g} f(Q) + \int d^4x\sqrt{-g}\mathcal{L}_M + \lambda_{\kappa}{}^{\lambda\mu\nu} R^{\kappa}{}_{\lambda\mu\nu} + \tau_{\lambda}{}^{\mu\nu}\mathrm{T}^{\lambda}{}_{\mu\nu}, \tag{6}$$

where, $g = \det(g_{\mu\nu})$, $\mathcal{L}_M$ is the matter fields' Lagrangian density, and $\lambda_\kappa{}^{\lambda\mu\nu}$, $\tau_\lambda{}^{\mu\nu}$ are Lagrange multipliers, whose variation impose the conditions $R^\kappa{}_{\lambda\mu\nu} = 0$ and $\mathrm{T}^\lambda{}_{\mu\nu} = 0$.

The gravitational field equations of the $f(Q)$ theory, for the metric $g_{\mu\nu}$, are

$$\frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f'(Q)P^\lambda{}_{\mu\nu}) - \frac{1}{2}f(Q)g_{\mu\nu} + f'(Q)(P_{\mu\rho\sigma}Q_\nu{}^{\rho\sigma} - 2Q_{\rho\sigma\mu}P^{\rho\sigma}{}_\nu) = T_{\mu\nu}, \quad (7)$$

where now a prime denotes total derivative with respect to the variable Q, that is, $f'(Q) = \frac{df}{dQ}$ and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial g^{\mu\nu}}$ is the energy-momentum tensor, which describes the matter components of the gravitational fluid.

An equivalent way to write the field equations (7) is with the use of the Einstein-tensor $G_{\mu\nu}$ such that [62],

$$f'(Q)G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f'(Q)Q - f(Q)) + 2f''(Q)(\nabla_\lambda Q)P^\lambda{}_{\mu\nu} = T_{\mu\nu}, \quad (8)$$

where $G_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}$, with $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ being the Riemannian Ricci tensor and scalar, respectively, which are constructed from the Levi-Civita connection. A direct comparison with general relativity can be perceived as the effect of an effective energy momentum tensor,

$$\mathcal{T}_{\mu\nu} = -\frac{1}{f'(Q)}\left[\frac{1}{2}g_{\mu\nu}(f'(Q)Q - f(Q)) + 2f''(Q)(\nabla_\lambda Q)P^\lambda{}_{\mu\nu}\right]. \quad (9)$$

With the help of (9), the resulting field equations can be written in the simple form,

$$G_{\mu\nu} = \mathcal{T}_{\mu\nu} + \frac{1}{f'(Q)}T_{\mu\nu}. \quad (10)$$

In addition, the variation of the action integral (6) with respect to the connection gives the field equations,

$$\nabla_\mu\nabla_\nu(\sqrt{-g}f'(Q)P^{\mu\nu}{}_\sigma) = 0. \quad (11)$$

From the definition of the effective energy momentum tensor, $\mathcal{T}_{\mu\nu}$, we observe that, if $f(Q) \propto Q$, the limit of general relativity is recovered since $\mathcal{T}_{\mu\nu} = 0$. Moreover, when $Q = \mathrm{const}$, Eq. (11) leads to solutions of general relativity with a cosmological constant Λ.

The basic objects in the theory are the metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$, for which, Eqs. (7) and (11) have to be solved, respectively. Note also that these equations incorporate the effect of the variation of the Lagrange multipliers. Therefore, for whatever connection is obtained through the field equations, there can always be found a coordinate transformation $x \mapsto \tilde{x}$, under whose effect, the transformed connection becomes zero,

$$\bar{\Gamma}^\lambda{}_{\mu\nu}(\tilde{x}) = \frac{\partial\tilde{x}^\lambda}{\partial x^\rho}\frac{\partial x^\eta}{\partial\tilde{x}^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}\Gamma^\rho{}_{\eta\sigma}(x) - \frac{\partial x^\rho}{\partial\tilde{x}^\nu}\frac{\partial x^\sigma}{\partial\tilde{x}^\mu}\frac{\partial^2\tilde{x}^\lambda}{\partial x^\rho\partial x^\sigma} = 0. \quad (12)$$

This stems from the two basic properties of $\mathrm{T}^\lambda{}_{\mu\nu}$, being both symmetric and flat, $R^\kappa{}_{\lambda\mu\nu} = 0$ [63,64].

Since trivializing the connection is a matter of a coordinate transformation, a possible strategy is to enforce *a priori* $\Gamma^\lambda{}_{\mu\nu} = 0$ into the field equations; this is referred in the literature as the adoption of the coincident gauge. However, special care is needed when also making some particular ansatz for the metric. Assuming a specific type of $g_{\mu\nu}$ already consists a partial gauge fixing. So, it may happen that $\Gamma^\lambda{}_{\mu\nu} = 0$, together with the ansatz for $g_{\mu\nu}$, overrestrict the system. An example of this can be seen in the simple case of a spatially flat FLRW space-time. If you write the metric in spherical coordinates, then the condition $\Gamma^\lambda{}_{\mu\nu} = 0$ in the equations is no longer a gauge choice, but rather a restriction [62]. The problem is resolved if instead you consider the FLRW metric in Cartesian coordinates, which are compatible with having $\Gamma^\lambda{}_{\mu\nu} = 0$.

In this work, we do not assume blindly the coincident gauge. We rather explore the different possibilities that exist and which are compatible with the system of equations for a given form of the metric.

For the matter energy-momentum tensor, the constraint $T^\mu{}_{\nu;\mu} = 0$ is still valid, which is the conservation law of mass. With ";" we denote the covariant derivative with respect to the Christoffel symbols. Thus, the $\mathcal{T}^\mu{}_{\nu;\mu} = 0$ relation resulting of the Eq. (11) for the connection, can be considered as a conservation law for the theory [65].

B. FLRW background geometry

The FLRW line element in spherical coordinates reads

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (13)$$

where k is the spatial curvature, $k = 0$ describes a spatially flat universe, $k = 1$ describes a closed universe and $k = -1$ denotes an open universe. Moreover, function $a(t)$ is the scale factor describes the radius of the universe and $N(t)$ is the lapse function. Without loss of generality, we assume that $N(t) = 1$. The FLRW spacetimes admit a six-dimensional symmetry group with generators,

$$\zeta_1 = \sin\phi\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi, \qquad \zeta_2 = -\cos\phi\partial_\theta + \frac{\sin\phi}{\tan\theta}\partial_\phi, \qquad \zeta_3 = -\partial_\phi \quad (14)$$

and

$$\xi_1 = \sqrt{1-kr^2}\sin\theta\cos\phi\partial_r + \frac{\sqrt{1-kr^2}}{r}\cos\theta\cos\phi\partial_\theta - \frac{\sqrt{1-kr^2}}{r}\frac{\sin\phi}{\sin\theta}\partial_\phi$$
$$\xi_2 = \sqrt{1-kr^2}\sin\theta\sin\phi\partial_r + \frac{\sqrt{1-kr^2}}{r}\cos\theta\sin\phi\partial_\theta + \frac{\sqrt{1-kr^2}}{r}\frac{\cos\phi}{\sin\theta}\partial_\phi$$
$$\xi_3 = \sqrt{1-kr^2}\cos\theta\partial_r - \frac{\sqrt{1-kr^2}}{r}\sin\theta\partial_\phi. \tag{15}$$

In two independent recent studies, Hohmann [56] and D'Ambrosio *et al.* [57] derived the general form of all compatible connections for the line element (13) for the symmetric teleparallel theory by enforcing on a generic connection the six Killing symmetries and the requirement $R^\kappa{}_{\lambda\mu\nu} = 0$. In what follows, we briefly summarize the results.

For the spatially flat spacetime, i.e., $k = 0$, there are three compatible connections. The common nonzero components of all three are

$$\Gamma^r{}_{\theta\theta} = -r, \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta,$$
$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r},$$
$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta. \tag{16}$$

However, they do differ in the way a free function of time enters in some of their other components. The first connection, named hereafter Γ_1, has only one additional nonzero component,

$$\Gamma^t{}_{tt} = \gamma(t), \tag{17}$$

where $\gamma(t)$ is a function of the time variable t.

The second connection, Γ_2, possesses the—additional to (16)—nonzero components,

$$\Gamma^t{}_{tt} = \frac{\dot\gamma(t)}{\gamma(t)} + \gamma(t),$$
$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \gamma(t), \tag{18}$$

where the dot denotes differentiation with respect to t.

Finally, the third connection, Γ_3, is characterized by the extra nonzero components,

$$\Gamma^t{}_{tt} = -\frac{\dot\gamma(t)}{\gamma(t)}, \qquad \Gamma^t{}_{rr} = \gamma(t), \qquad \Gamma^t{}_{\theta\theta} = \gamma(t)r^2,$$
$$\Gamma^t{}_{\phi\phi} = \gamma(t)r^2\sin^2\theta. \tag{19}$$

In the case where $k \neq 0$, there exist only one compatible connection with nonzero coefficients,

$$\Gamma^t{}_{tt} = -\frac{k+\dot\gamma(t)}{\gamma(t)}, \qquad \Gamma^t{}_{rr} = \frac{\gamma(t)}{1-kr^2} \qquad \Gamma^t{}_{\theta\theta} = \gamma(t)r^2, \qquad \Gamma^t{}_{\phi\phi} = \gamma(t)r^2\sin^2(\theta)$$
$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = -\frac{k}{\gamma(t)}, \qquad \Gamma^r{}_{rr} = \frac{kr}{1-kr^2},$$
$$\Gamma^r{}_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2(\theta)(1-kr^2), \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r},$$
$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta. \tag{20}$$

We will refer to this last connection as $\Gamma^{(k)}$. We observe that when $k = 0$, the latter connection reduces to the third connection of the flat case, that is $\Gamma^{(0)} = \Gamma_3$.

III. SPATIALLY FLAT CASE $k = 0$

Enforcing the homothetic restriction in the metric tensor (13) for the spatially flat case, i.e., $L_\xi g_{\mu\nu} = 2g_{\mu\nu}$, where L stands for the Lie derivative, we conclude that

$$\xi = t\partial_t + (1-\lambda)r\partial_r \tag{21}$$

is a homothetic vector field, and the corresponding line element reads

$$ds^2 = -dt^2 + t^{2\lambda}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{22}$$

When $\lambda = 0$, the flat space is recovered; thus, we exclude this value from our considerations.

We proceed to investigate the $f(Q)$ theories that admit such a line element as a solution for the different connections we discussed in the previous section. Note that, as previously stated, there exists a coordinate system, where each of the connections becomes zero. However, this coordinate system is different for each of the admitted connections and—in most situations—leads to a transformed metric that loses its obvious homogeneity and isotropy. Of the various connections, only the Γ_1 can be made zero by going to Cartesian coordinates and by adopting a simple time reparametrization, thus, maintaining the line element in its commonly encountered form. Here, we work in spherical coordinates, with the line element given by (22) and consider the corresponding nonzero connections compatible with this expression.

A. The first connection Γ_1

For the first connection, the nonmetricity scalar is derived

$$Q = -6\left(\frac{\dot{a}}{a}\right)^2; \tag{23}$$

thus, for the line element (22), it follows

$$Q = -\frac{6c_1^2}{t^2}. \tag{24}$$

Note that, with the definition (4) we used, the resulting Q in the case of Γ_1 results to be negative. This is purely a matter of convention; in other works in the literature, the opposite definition is sometimes adopted.

The equations of motion for the connection (11), are identically satisfied. Thus, the connection Γ_1 plays no role in the dynamics. This corresponds to the usual case studied in the literature; that of the spatially flat FLRW geometry, with the metric taken in Cartesian coordinates and the connection in the coincident gauge, $\Gamma^{\kappa}{}_{\lambda\mu} = 0$.

It is easily seen that the gravitational field equations are independent of the function $\gamma(t)$, and they are

$$\frac{3\lambda^2 f'(Q)}{t^2} - \frac{1}{2}Qf'(Q) + \frac{1}{2}f(Q) = \rho, \tag{25}$$

$$\lambda(2-3\lambda)t^{-2}f'(Q) - 2\lambda t^{-1}\dot{Q}f''(Q) + \frac{1}{2}(Qf'(Q) - f(Q)) = p, \tag{26}$$

where we have considered the tensor $T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho(t), p(t), p(t), p(t))$ to describe the energy momentum tensor.

Assume now a perfect fluid with constant equation of state parameter, that is, $p = w\rho$; thus, the equation of motion for the matter fluid, $T^{\mu}{}_{\nu;\mu} = 0$, provides

$$t\dot{\rho} + 3\lambda(1+w)\rho = 0, \tag{27}$$

with exact solution $\rho(t) = \rho_0 t^{-3\lambda(1+w)}$, where ρ_0 is an integration constant. Under the above substitutions, as well as eliminating explicit time dependence through (24), Eq. (25) is transcribed to the following form:

$$\rho_0(6\lambda^2)^{\frac{-3\lambda(w+1)}{2}}(-Q)^{\frac{3}{2}\lambda(w+1)} - Qf'(Q) + \frac{1}{2}f(Q) = 0. \tag{28}$$

The latter equation is a first-order ordinary differential equation in the dependent variable $f(Q)$, while Q is the independent variable. The exact solution of equation (28) is expressed as follows:

$$f(Q) = f_0\sqrt{-Q} + f_1(-Q)^{\frac{3}{2}\lambda(w+1)}, \tag{29}$$

where f_0 is an arbitrary constant, while f_1 is defined as

$$f_1 = \frac{2\lambda^{-3\lambda(1+w)}\rho_0}{6^{\frac{3}{2}\lambda(w+1)}(1-3\lambda(1+w))}. \tag{30}$$

The minus signs are placed in Eq. (29) in order to have a real valued action for real constants of integration; remember that in our conventions we obtain $Q < 0$ in this section, see Eq. (23).

We observe that, self-similarity in the case of Γ_1, implies a power-law type of $f(Q)$ theory where Q is raised to the power $\frac{3}{2}\lambda(w+1)$. The constant λ is the one appearing in the line element (22), while w is the equation of state parameter. If we consider the value which linearizes the relevant term in (29), that is $\frac{3}{2}\lambda(w+1) = 1$, then we recover the solution of general relativity with $a(t) = t^{\lambda} = t^{\frac{2}{3(1+w)}}$.

The extra $\sqrt{-Q}$ term that we see in (29) contributes just as a surface term at the level of the action; see Eq. (23), the $\sqrt{-Q} = \frac{d}{dt}(\sqrt{6}\ln a)$ is a total derivative of a function involving the scale factor. In fact, if we consider the same configuration in the absence of matter, $\rho = p = 0$, the only solution which is obtained is $f(Q) \sim \sqrt{-Q}$, which is trivial in the sense that, for Γ_1, the $f(Q) \sim \sqrt{-Q}$ theory admits all metrics as solutions.

It is quite simple to repeat the above calculations for different equations of state. If we consider a generic barotropic equation $p = p(\rho)$, then the continuity condition $T^{\mu}{}_{\nu;\mu} = 0$, results in

$$2Q\frac{d\rho}{dQ} - 3\lambda(p(\rho(Q)) + \rho(Q)) = 0, \tag{31}$$

where we have used (24) to make the change of variables $t \to Q$. The latter can be integrated to give

$$\int \frac{d\rho}{p(\rho) + \rho} = \frac{3\lambda}{2}\ln Q. \tag{32}$$

However, whether it will be possible to invert (32), in order to obtain the $\rho(Q)$, which is to be used in (25) for the derivation of a differential equation in the $f(Q)$, depends on the particular equation of state, $p(\rho)$, under consideration. In any case, the (25) is going to produce a first order ordinary differential equation, which is of the form,

$$\frac{f(Q)}{2} - Qf'(Q) = \rho(Q), \tag{33}$$

solved by

$$f(Q) = \sqrt{-Q} \int \frac{\rho(Q)}{(-Q)^{\frac{3}{2}}} dQ. \tag{34}$$

For the case of the linear equation of state $p = w\rho$, relation (32) leads to $\rho \sim (-Q)^{\frac{3}{2}\lambda(w+1)}$, and subsequently, through (34) to the theory we obtained in (29).

We can now easily generalize solution (29) by considering an arbitrary number of perfect fluids with different equations of state, $p_i = w_i \rho_i$. If we assume the sufficient (but not necessary) condition that each fluid separately satisfies a continuity equation, so that $\rho_i = C_i (-Q)^{\frac{3}{2}\lambda(w_i+1)}$, then, due to $\rho(Q)$ in (34) being $\rho(Q) = \sum_{i=1}^n \rho_i$, we obtain a theory of the form,

$$f(Q) = f_0\sqrt{-Q} + 2\sum_{i=1}^{n} \frac{C_i(-Q)^{\frac{3}{2}\lambda(w_i+1)}}{1-3\lambda(1+w)}, \qquad n \in \mathbb{Z}_+, \tag{35}$$

where the C_i correspond to constants of integration. We have thus obtained a series, comprised of terms involving the nonmetricity scalar Q, raised in powers associated with the equation of state parameter w_i of each fluid.

B. The second connection Γ_2

For the second connection, the nonmetricity scalar is obtained as

$$Q = -\frac{6\dot{a}^2}{a^2} + 9\gamma\frac{\dot{a}}{a} + 3\dot{\gamma} \tag{36}$$

where it is clear that function $\gamma(t)$ is involved. The field equations in the case of vacuum are

$$\frac{3\dot{a}^2 f'(Q)}{a^2} + \frac{1}{2}(f(Q) - Qf'(Q)) + \frac{3\gamma\dot{Q}f''(Q)}{2} = 0, \tag{37a}$$

$$-2\frac{d}{dt}\left(\frac{f'(Q)\dot{a}}{a}\right) - \frac{3\dot{a}^2}{a^2}f'(Q) - \frac{1}{2}(f(Q) - Qf'(Q)) + \frac{3\gamma\dot{Q}f''(Q)}{2} = 0, \tag{37b}$$

while the one for the connection yields

$$\dot{Q}^2 f'''(Q) + \left[\ddot{Q} + \dot{Q}\left(\frac{3\dot{a}}{a}\right)\right] f''(Q) = 0. \tag{38}$$

Note that, unlike what we saw in the previous section for connection Γ_1, here the vacuum case is not trivial.

For the line element (22), the scalar (36) becomes

$$Q = -\frac{6\lambda^2}{t^2} + \frac{9\lambda\gamma}{t} + 3\dot{\gamma}. \tag{39}$$

The equations of motion for the connection (11) reads

$$3\lambda\dot{Q}f''(Q) + t\dot{Q}^2 f'''(Q) + t\ddot{Q}(t)f''(Q) = 0, \tag{40}$$

and the gravitational field equations reduce to

$$\left(\frac{6\lambda^2}{t^2} - Q\right) f'(Q) + 3\gamma\dot{Q}f''(Q) + f(Q) = 0 \tag{41}$$

$$(-6\lambda^2 + 4\lambda + t^2 Q) f'(Q) + t(3t\gamma(t) - 4\lambda)\dot{Q}f''(Q) - t^2 f(Q) = 0. \tag{42}$$

By solving (41) and (42) algebraically with respect to γ and $f''(Q)$, we find

$$\gamma = \frac{2\lambda((6\lambda^2 - t^2 Q)f'(Q) + t^2 f(Q))}{3t((2\lambda(3\lambda-1) - t^2 Q)f'(Q) + t^2 f(Q))} \tag{43a}$$

$$f''(Q) = \frac{(2\lambda(1-3\lambda) + t^2 Q)f'(Q) - t^2 f(Q)}{2\lambda t\dot{Q}}. \tag{43b}$$

Taking the time derivative of (43), solving it for $f'''(Q)$ and substituting it together with (43) in (40), we arrive at the integrability condition,

$$(4\lambda + t^2 Q)f(Q) - Q(t^2 Q - 6(\lambda-1)\lambda)f'(Q) = 0. \tag{44}$$

This relation provides a relation for the first derivative of $f(Q)$ as long as $t^2 Q - 6(\lambda-1)\lambda \neq 0$. But let us first examine the special case, where $t^2 Q - 6(\lambda-1)\lambda = 0$.

1. *Special case*

If we require $t^2 Q - 6(\lambda-1)\lambda = 0$, then, as long as $f(Q) \neq 0$, Eq. (44) implies $4\lambda + t^2 Q = 0$. These two conditions for Q combined lead to the algebraic condition $(1-3\lambda)\lambda = 0$, that is, $\lambda = 0$ or $\lambda = \frac{1}{3}$. The case $\lambda = 0$ corresponds to the flat space and leads to $Q = 0$, so we concentrate our attention at the value $\lambda = \frac{1}{3}$. For this λ, we have

$$Q = -\frac{4}{3t^2}. \tag{45}$$

By calculating $\dot{Q}$ and $\ddot{Q}$ and by inverting (45) to substitute $t \to Q$ inside (40), we arrive to a differential equation for $f(Q)$,

$$Qf'''(Q) + f''(Q) = 0. \tag{46}$$

Its solution is $f(Q) = f_1 Q + f_2 Q \ln(-Q) + f_3$, with the f_1, f_2, and f_3 being constants of integration. In addition, since we know Q from (45), we can use it in (39) and integrate the latter to obtain the function which enters the connection Γ_2. Thus, we derive

$$\gamma = \frac{C}{t} - \frac{2}{9t}\ln t, \tag{47}$$

where C is a constant of integration. The above relations (45), (47), with the derived expression for $f(Q)$ satisfy the field equations under the following conditions for the constants of integration:

$$f_3 = 0 \quad \text{and} \quad C = \frac{f_1 + f_2(3 + \ln(\frac{4}{3}))}{9 f_2}. \tag{48}$$

Thus, the theory we obtain in this case is

$$f(Q) = f_1 Q + f_2 Q \ln(-Q), \tag{49}$$

which introduces a logarithmic modification to GR, $f(Q) \sim Q$. Notice that the minus sign enters the logarithm due to the Q of (45) being negative definite.

2. General case

Leaving aside the above special case, if we now consider $t^2 Q - 6(\lambda - 1)\lambda \neq 0$, then Eq. (44) gives

$$f'(Q) = \frac{(4\lambda + t^2 Q) f(Q)}{Q(t^2 Q - 6(\lambda - 1)\lambda)}. \tag{50}$$

We can take the time derivative of the above expression and solve with respect to $f''(Q)$, then we substitute this relation together with (50) inside (43) and its time derivative. By substituting the latter two in (39), we arrive at a differential equation for Q,

$$(1 - 3\lambda)Q + t\dot{Q} = 0, \tag{51}$$

which yields

$$Q = q_0 t^{3(\lambda - 1)}, \tag{52}$$

where q_0 is a constant. With the use of the above expression in (39), we can integrate the latter to obtain the corresponding connection that results in

$$\gamma = \frac{q_0 t^{3\lambda} + 12\lambda^2 t}{6(3\lambda - 1)t^2} + C t^{-3\lambda}, \tag{53}$$

with C denoting once more a constant of integration. By calculating the derivatives of (52) and by also inverting (52) to obtain a mapping $t \to Q$, we obtain from the connection equation (40) the following condition on $f(Q)$:

$$3(\lambda - 1) Q f'''(Q) + 2(3\lambda - 2) f''(Q) = 0, \tag{54}$$

which is solved by

$$f(Q) = f_1 Q + f_2 Q^{\frac{2}{3(1-\lambda)}} + f_3, \tag{55}$$

with f_1, f_2, and f_3 denoting the integration constants. However, we need to substitute the above acquired expressions in the rest of the field equations (41) and (42). When we do so, we observe that the following conditions must be set:

$$C = f_3 = 0 \quad \text{and} \quad f_2 = -6 f_1 (\lambda - 1)\lambda q_0^{\frac{2}{3(\lambda - 1)}}, \tag{56}$$

which results in a theory characterized by

$$f(Q) = f_1 \left(Q - 6(\lambda - 1)\lambda q_0^{\frac{2}{3(\lambda-1)}} Q^{\frac{2}{3(1-\lambda)}} \right). \tag{57}$$

Thus, in the generic case $\lambda \neq \frac{1}{3}$, we get a theory which involves a power-law modification to general relativity. The case $\lambda = 1$ corresponds to a GR solution and yields $Q = \text{const}$.

C. The third connection Γ_3

The nonmetricity scalar is calculated

$$Q = -\frac{6\dot{a}^2}{a^2} + \frac{3\gamma}{a^2}\frac{\dot{a}}{a} + \frac{3\dot{\gamma}}{a^2}, \tag{58}$$

while the equations of motion for the metric in the case of vacuum are

$$\begin{aligned} &\frac{3\dot{a}^2 f'(Q)}{a^2} + \frac{1}{2}(f(Q) - Q f'(Q)) - \frac{3\gamma \dot{Q} f''(Q)}{2a^2} = 0, \\ &-2\frac{d}{dt}\left(\frac{f'(Q)\dot{a}}{a}\right) - \frac{3\dot{a}^2}{a^2} f'(Q) - \frac{1}{2}(f(Q) - Q f'(Q)) \\ &\quad + \frac{\gamma \dot{Q} f''(Q)}{2a^2} = 0 \end{aligned} \tag{59}$$

and for the connection, we have

$$\dot{Q}^2 f'''(Q) + \left[\ddot{Q} + \dot{Q}\left(\frac{\dot{a}}{a} + \frac{2\dot{\gamma}}{\gamma}\right)\right] f''(Q) = 0. \tag{60}$$

For the self-similar line element (22), we calculate

$$Q(t) = t^{-2(\lambda+1)}(-6\lambda^2 t^{2\lambda} + 3\lambda t \gamma + 3t^2 \dot{\gamma}), \tag{61}$$

or equivalently,

$$\dot{\gamma} = \frac{1}{3}t^{2\lambda}Q + \frac{\lambda(2\lambda t^{2\lambda} - t\gamma)}{t^2}. \quad (62)$$

The field equations are written in the equivalent form,

$$\left(\frac{6\lambda^2}{t^2} - Q\right)f'(Q) - 3t^{-2\lambda}\gamma Q' f''(Q) + f(Q) = 0, \quad (63)$$

$$2\left(\frac{\gamma}{4} - \lambda t^{2\lambda-1}\right)\dot{Q}f''(Q) - \frac{1}{2}t^{2\lambda}(f(Q) - Qf'(Q)) - \lambda(3\lambda-2)t^{2\lambda-2}f'(Q) = 0, \quad (64)$$

and

$$\gamma(\lambda\dot{Q}f''(Q) + t\ddot{Q}f''(Q) + t\dot{Q}^2 f'''(Q)) + 2t\dot{Q}\dot{\gamma}f''(Q) = 0. \quad (65)$$

We notice that if we divide the equation of motion for the connection (65) with $t\gamma\dot{Q}f''(Q)$, we can infer an integral of motion,

$$t^{\lambda}\gamma^2\dot{Q}f''(Q) = m, \quad (66)$$

that is

$$f''(Q) = \frac{mt^{-\lambda}}{\gamma^2\dot{Q}}, \quad (67)$$

where we assume $m \neq 0$ to avoid the linear $f(Q) \sim Q$ case. We replace (67) into (63), (64), and we find the following relations:

$$f'(Q) = \frac{t^2(\gamma f(Q) - 3mt^{-3\lambda})}{\gamma(t^2Q - 6\lambda^2)}, \quad (68)$$

$$f(Q) = \frac{mt^{-3\lambda-1}(t^2Q(2\lambda t^{2\lambda} + t\gamma) - 6\lambda(2\lambda^2t^{2\lambda} + (\lambda-1)t\gamma))}{2\lambda\gamma^2}. \quad (69)$$

If we take the first derivative with respect to the time (68) and divide it by $\dot{Q}$, the result must be the same as in (69). We thus end up with the equation,

$$(6\lambda^2 - t^2Q)(6(4\lambda^3t^{4\lambda} + (\lambda-1)t^2\gamma^2) + t^{2(\lambda+1)}Q(4\lambda t^{2\lambda} + t\gamma)) = 0. \quad (70)$$

The above relation allows for an algebraic derivation of $\gamma(t)$ with respect to t and Q. The subsequent derivation of $Q(t)$ by integrating (61) is straightforward. However, the resulting $Q(t)$ relation cannot in general be inverted to obtain the mapping $t \to Q$, which would allow us to write a differential equation for $f(Q)$ from the field equations and thus obtain the general family of theories allowing for self-similar solutions.

In order to proceed and obtain an $f(Q)$ theory, given in terms of elementary functions, at least as a partial solution, we follow a different procedure. Specifically, we make use of the existence of the homothetic vector field and use it to set an additional restriction on the connection; in a sense, demand self-similarity for the connection as well, not just for the metric. Due to the fact that the connection is not a tensor, we choose to set a "homothetic" condition over the nonmetricity tensor and demand $L_\xi Q_{\lambda\mu\nu} = 2\sigma Q_{\lambda\mu\nu}$, where σ is a constant. Interestingly enough, this equation is solved if the homothetic factor is the same as that of the metric, i.e., $\sigma = 1$, while γ satisfies the differential equation,

$$(2\lambda - 1)\gamma - t\dot{\gamma} = 0, \quad (71)$$

that is, $\gamma = \gamma_0 t^{2\lambda-1}$. However, for the compatable connections that we consider in this work condition, $L_\xi Q_{\lambda\mu\nu} = 2Q_{\lambda\mu\nu}$ is equivalent with the condition $L_\xi \Gamma^{\lambda}{}_{\mu\nu} = 0$. Thus, we demand the autoparallels, that is, the equations of motion for a test particle, to be invariant under the action of the homothetic vector field as in the case of the Levi-Civita connection.

With this $\gamma(t)$, the nonmetricity scalar reads

$$Q(t) = -\frac{3(2\lambda^2 - 3\lambda\gamma_0 + \gamma_0)}{t^2}. \quad (72)$$

The field equations are now written as

$$(6\lambda^2 - t^2Q)f'(Q) - 3\gamma_0 t\dot{Q}f'' + t^2f(Q) = 0, \quad (73)$$

$$(-6\lambda^2 + 4\lambda + t^2Q)f'(Q) + t(\gamma_0 - \lambda)\dot{Q}(t)f'' - t^2f(Q) = 0, \quad (74)$$

and

$$((5\lambda - 2)\dot{Q}f''(Q) + tQ'^2f'''(Q) + t\ddot{Q}f''(Q)) = 0. \quad (75)$$

By using (72) to convert Eq. (75) into a differential equation for $f(Q)$, we obtain

$$5(\lambda - 1)f''(Q) - 2Qf'''(Q) = 0,$$

from where it follows

$$f(Q) = f_1 Q^{\frac{5\lambda-1}{2}} + f_2 Q + f_3, \quad (76)$$

with $\lambda \neq \frac{1}{5}$ and $\lambda \neq \frac{3}{5}$.

If we replace (76) in the field equations, we find that the flat spacetime is recovered, that is, $\lambda = 0$, when $f_3 = f_1 = 0$, which corresponds to GR since $f(Q) \sim Q$. On the other hand, for $f_2 = f_3 = 0$, i.e., a power-law $f(Q)$, there exists an analytic solution with the constraint $\gamma_0 = \frac{2\lambda(2-5\lambda)}{5\lambda-3}$. Finally, for $\lambda = \frac{1}{5}$ or $\lambda = \frac{3}{5}$, there is not any valid solution for a function γ obtained under the condition (71).

IV. NONZERO SPATIAL CURVATURE $k \neq 0$

Imposing the homothetic constraint, $L_\xi g_{\mu\nu} = 2g_{\mu\nu}$, for the case of nonzero spatial curvature, we end with the line element,

$$ds^2 = -dt^2 + (a_0 t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{77}$$

where the homothetic vector field is $\xi = t\partial_t$.

For a nonzero spatial curvature k, the nonmetricity scalar becomes

$$Q = -\frac{6\dot{a}^2}{a^2} + \frac{3\gamma}{a^2}\left(\frac{\dot{a}}{a}\right) + \frac{3\dot{\gamma}}{a^2} + k\left[\frac{6}{a^2} + \frac{3}{\gamma}\left(\frac{\dot{\gamma}}{\gamma} - \frac{3\dot{a}}{a}\right)\right]. \tag{78}$$

The gravitational field equations in vacuum are

$$\frac{3\dot{a}^2 f'(Q)}{a^2} + \frac{1}{2}(f(Q) - Qf'(Q)) - \frac{3\gamma\dot{Q}f''(Q)}{2a^2} + 3k\left(\frac{f'(Q)}{a^2} - \frac{\dot{Q}f''(Q)}{2\gamma}\right) = 0, \tag{79a}$$

$$-2\frac{d}{dt}\left(\frac{f'(Q)\dot{a}}{a}\right) - \frac{3\dot{a}^2}{a^2}f'(Q) - \frac{1}{2}(f(Q) - Qf'(Q)) + \frac{\gamma\dot{Q}f''(Q)}{2a^2} - k\left(\frac{f'(Q)}{a^2} + \frac{3\dot{Q}f''(Q)}{2\gamma}\right) = 0, \tag{79b}$$

and the equation of motion for the connection assumes the form,

$$\dot{Q}^2 f'''(Q)\left(1 + \frac{ka^2}{\gamma^2}\right) + \left[\ddot{Q}\left(1 + \frac{ka^2}{\gamma^2}\right) + \dot{Q}\left(\left(1 + \frac{3ka^2}{N^2\gamma^2}\right)\frac{\dot{a}}{a} + \frac{2\dot{\gamma}}{\gamma}\right)\right]f''(Q) = 0. \tag{80}$$

Once more, the problem is too complex to proceed without setting some restriction on the function $\gamma(t)$. We try the same ansatz as in the $k = 0$ case of Γ_3 in the previous section, $L_\xi Q_{\lambda\mu\nu} = 2Q_{\lambda\mu\nu}$, which can be seen to be equivalent to $L_\xi \Gamma^\lambda{}_{\mu\nu} = 0$. One of the nonzero independent terms of $L_\xi \Gamma^\lambda{}_{\mu\nu} = 0$ is $\gamma(t) - t\dot{\gamma}(t) = 0$. The solution is $\gamma(t) = \gamma_0 t$, with $\gamma_0 = \text{const}$, which satisfies the entire homothetic equation. Thus, by replacing it in (78), we find

$$Q(t) = -\frac{6(a_0^2 - \gamma_0)(\gamma_0 + k)}{a_0^2\gamma_0 t^2}. \tag{81}$$

The field equations are

$$\gamma_0(6(a_0^2 + k) - a_0^2 t^2 Q)f'(Q) - 3t(a_0^2 k + \gamma_0^2)\dot{Q}f''(Q) + a_0^2\gamma_0 t^2 f(Q) = 0, \tag{82}$$

$$t(a_0^2(4\gamma_0 + 3k) - \gamma_0^2)\dot{Q}f''(Q) + \gamma_0(2(a_0^2 + k) - a_0^2 t^2 Q)f'(Q) + a_0^2\gamma_0 t^2 f(Q) = 0, \tag{83}$$

and

$$3(a_0^2 k + \gamma_0^2)(tf'''(Q)\dot{Q}^2 + t\ddot{Q}f''(Q) + 3\dot{Q}f''(Q)) = 0. \tag{84}$$

Hence, with the use of (81) in (84) and given $Q \neq 0$, it follows

$$(a_0^2 k + \gamma_0^2)f'''(Q) = 0. \tag{85}$$

For $(a_0^2 k + \gamma_0^2) \neq 0$, we find $f(Q) = f_1 + f_2 Q + f_3 Q^2$, and by replacing it in the field equations, for arbitrary k, we end up with the constraints $f_1 = f_2 = 0$ and $\gamma_0 = -3a_0^2$. There is also the special case $\gamma_0 = a_0^2$, which however leads to $Q = 0$. For a constant Q, the equation of the connection is identically zero and sets no restriction in the functional form of $f(Q)$. Through the rest of the equations of motion, we see that, the $\gamma_0 = a_0^2$, $Q = 0$ case allows for two possibilities: either $k = -a_0^2 = -1$ (Milne universe) or a_0 arbitrary and $k = \pm 1$; the first requires $f(0) = 0$, while the second leads to the constraint $f(0) = f'(0) = 0$.

On the other hand, for $k = -1$ and $\gamma_0 = \pm a_0$, from (81) and (82), it follows that $Q = -\frac{6(a_0 \mp 1)^2}{a_0^2 t^2}$. From Eqs. (82) and (83), we further obtain

$$f(Q) = q_0 Q^{\frac{a_0 \mp 1}{2a_0}}, \tag{86}$$

where q_0 is a constant and it is required that $a_0 \neq \pm 1$. The $a_0 = \mp 1$ case leads to $f(Q) \sim Q$, which leads to the Milne universe solution with $k = -1$.

We briefly summarize the results we obtained in this and in the previous sections, for the various types of spatial geometry and the corresponding connections, in Table I.

V. CONCLUSIONS

In this study we investigate the existence of self-similar solutions in $f(Q)$-symmetric teleparallel gravitational theory for four distinct families of connections related with a homogeneous and isotropic background geometry described by the FLRW line element. Three of the distinct connections describe spatially flat FLRW spacetimes, while

TABLE I. The functional form of the $f(Q)$ theory resulting in self-similar gravitation field for the various types of spatial geometry and connections. We include in the table only the cases consisting of modifications of GR. Notice, that for the last three cases (Γ_3 and the two $k \neq 0$), the $f(Q)$ function is derived under the extra condition of self-similarity over the nonmetricity tensor.

Spatial curvature	$k=0$			$k=+1$	$k=-1$
Connection	Γ_1	Γ_2	Γ_3	$\Gamma^{(+1)}$	$\Gamma^{(-1)}$
$f(Q)$	$\sim\lvert Q\rvert^{\frac{3}{2}\lambda(w+1)}$	$f_1 Q + f_2 Q \ln\lvert Q\rvert$, $\lambda=\frac{1}{3}$ or $f_1 Q + f_2 Q^{\frac{2}{3(1-\lambda)}}$,	$\sim Q^{\frac{5\lambda-1}{2}}$	$\sim Q^2$	$\sim Q^2$, $\forall\ a_0 \in \mathbb{R}^*$ or $\sim Q^{\frac{a_0 \mp 1}{2a_0}}$
Matter content	$p = w\rho$	…	…	…	…

the fourth connection corresponds to a nonzero spatial FLRW geometry. For each family of connection, we present the gravitational field equations and the equation of motion for the connection. We assume that the background geometry admits an homothetic vector field such that to be a self-similar spacetime. Hence, for the exact functional form of the dynamical variables of the metric, we solve the field equations in order to constraint the connection and the $f(Q)$ function.

For the first connection, we observe that it is necessary to introduce an external matter source such that the self-similar solution to exist. The resulting $f(Q)$ function is of power-law, and it can be extended into a series by adding different fluids in the matter sector. For the second and third families of connections, it is not necessary to introduce an external matter source, and for power-law $f(Q)$ and logarithmic modifications to GR, self-similar solutions exist. Finally, for the fourth connection, which corresponds to a universe with a nonzero spatial curvature, self-similar solutions exist for $f(Q) = Q^2$, when the curvature is positive, or for a general power-law $f(Q)$ theory when the space is hyperbolic. For the latter case, we also demonstrated the consistency of the equations by obtaining the Milne universe, when $f(Q) \sim Q$.

As we know, the general relativistic limit in the case of symmetric teleparallel gravity lies in the linear theory $f(Q) \sim Q$. If we purely concentrate on the form of the action, most of the reconstructed $f(Q)$ theories, which are seen in Table I, can approach this limit by constraining appropriately the remaining free parameters. It is particularly interesting the case of connection Γ_2, where this limit can be directly achieved by simply requiring $f_2 \to 0$. In the rest of the cases, a constant which enters the space-time metric (either λ or a_0) needs to be constrained, but the limit is still achievable.

The use of the different connections in this work reveals additional applicable dynamics. We see how the same gravitational field $g_{\mu\nu}$ is produced by different $f(Q)$ theories depending on the type of connection you introduce. However, the physical interpretation of the different connections is not obvious, especially if you rely on observed quantities whose value depends purely on the metric, e.g., the Hubble function. Possible observable effects owed to the nonmetricity have been explored in [66] by considering matter which couples to the connection (e.g., fermions). However, even in this setting, it has been shown that the result depends on the level where you introduce the non-Riemannian modification [67]. If you start by modifying the Dirac equation, then the nonmetricity introduces an extra coupling among the fermions. On the other hand, if you insert the modification at the level of the action, the effect of the nonmetricity disappears from the resulting field equation. In the first case, for a fixed metric $g_{\mu\nu}$, the differences owed to the distinct admissible $\Gamma^{\kappa}{}_{\lambda\mu}$, characterized by the $\gamma(t)$, could, in principle, be quantified.

The importance of self-similar spacetimes is well established in general relativity [68]. They form simple solutions, which however play a significant role as asymptotic limits of more complicated gravitational configurations; for example, consider the role of the Kasner solution in the application of the Belinski-Khalatnikov-Lifshitz (BKL) conjecture [69]. At this point, it is difficult to evaluate if self-similarity will prove as important and as general in $f(Q)$ theory. But, given the fact that the field equations of the theory can be formulated as that of an effective fluid in general relativity, it is expected that, at least under specific configurations, self-similarity will continue playing a role in limiting cases. From the analysis performed in previous works [38,55] in FLRW spacetimes and for the case of the first connection we study here, it appears that self-similar cosmological solutions do indeed describe the asymptotic limits of more general solutions. A similar conclusion with that of general relativity. For the rest of the connections and for the function form of the nonmetricity scalar, we do the hypothesis that a similar conclusion is valid. However, that should be investigated further. Lastly, partial results from ongoing work of ours show that the general connections, compatible to the symmetries of a chosen background geometry, can be retrieved by the mere knowledge of the corresponding flat, self-similar connections.

In a future work, we plan to investigate if the latter conclusion is valid in the case of anisotropic geometries by investigating self-similar solutions in Bianchi spacetimes.

ACKNOWLEDGMENTS

N. D. acknowledges the support of the Fundamental Research Funds for the Central Universities, Sichuan University Full-time Postdoctoral Research and Development Fund No. 2021SCU12117. A. P. was supported in part by the National Research Foundation of South Africa (Grants No. 131604). A.P. acknowledges financial support from the Universidad Catolica del Norte Resolución VRIDT N°098/2022.

[1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions to Einstein's Field Equations*, 2nd ed. (Cambridge University Press, New York, 2003).
[2] M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971).
[3] B. J. Carr and A. A. Coley, Classical Quantum Gravity **16**, R31 (1999).
[4] J. Wainwright and J. F. R. Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge, England, 1997).
[5] G. Haager and M. Mars, Classical Quantum Gravity **15**, 1567 (1998).
[6] L. Hsu and J. Wainwright, Classical Quantum Gravity **3**, 1105 (1986).
[7] T. Harada, H. Maeda, and B. J. Carr, Phys. Rev. D **77**, 024022 (2008).
[8] D. M. Eardley, J. Isenberg, J. E. Marsden, and V. Moncrief, Commun. Math. Phys. **106**, 137 (1986).
[9] A. A. Coley and B. O. J Tupper, Gen. Relativ. Gravit. **22**, 241 (1990).
[10] M. Tsamparlis and P. S. Apostolopoulos, J. Math. Phys. (N.Y.) **41**, 7573 (2000).
[11] A. Mitsopulos, M. Tsamparlis, and A. Paliathanasis, Mod. Phys. Lett. A **34**, 1950326 (2019).
[12] A. M. Sherif, P. K. S. Dunsby, R. Goswami, and S. Majarah, Int. J. Geom. Method Mod. Phys. **18**, 2150121 (2021).
[13] P. S. Apostolopoulos and M. Tsamparlis, Gen. Relativ. Gravit. **35**, 2051 (2003).
[14] P. S. Apostolopoulos, Classical Quantum Gravity **20**, 3371 (2003).
[15] P. S. Apostolopoulos, Gen. Relativ. Gravit. **36**, 1939 (2004).
[16] P. S. Apostolopoulos, Gen. Relativ. Gravit. **37**, 937 (2005).
[17] P. S. Apostolopoulos, Classical Quantum Gravity **22**, 323 (2005).
[18] A. Yahil, Astrophys. J. **265**, 1047 (1983).
[19] R. Weitzenböck, *Invarianten Theorie* (Nordhoff, Groningen, 1923).
[20] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[21] J. W. Maluf, J. Math. Phys. (N.Y.) **35**, 335 (1994).
[22] M. Hohmann, Phys. Rev. D **104**, 124077 (2021).
[23] J. Beltrán-Jiménez, L. Heisenberg, and T. S. Koivisto, Universe **5**, 173 (2019).
[24] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rep. **692**, 1 (2017).
[25] L. Heisenberg, Phys. Rep. **796**, 1 (2019).
[26] T. Harko, N. Myrzakulov, R. Myrzakulov, and S. Shahidi, Phys. Dark Universe **34**, 100886 (2021).
[27] H. A. Buchdahl, Month. Not. R. Astron. Soc. **150**, 1 (1970).
[28] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).
[29] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[30] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini, Phys. Rev. D **77**, 046009 (2008).
[31] R. Ferraro and F. Fiorini, Phys. Rev. D **75**, 084031 (2007).
[32] M. Hohmann, L. Järv, M. Krššák, and C. Pfeifer, Phys. Rev. D **100**, 084002 (2019).
[33] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hohmann, J. L. Said, J. Mifsud, and E. Di Valentino, arXiv:2106.13793.
[34] J. B. Jimenez, L. Heisenberg, and T. Koivisto, Phys. Rev. D **98**, 044048 (2018).
[35] L. Atayde and N. Frusciante, Phys. Rev. D **104**, 064052 (2021).
[36] F. D'Ambrosio, L. Heisenberg, and S. Kuhn, Classical Quantum Gravity **39**, 025013 (2022).
[37] W. Wang, H. Chen, and T. Katsuragawa, Phys. Rev. D **105**, 024060 (2022).
[38] W. Khyllep, A. Paliathanasis, and J. Dutta, Phys. Rev. D **103**, 103521 (2021).
[39] R.-H. Lin and X.-H. Zhai, Phys. Rev. D **103**, 124001 (2021).
[40] A. De, S. Mandal, J. T. Beh, T.-H. Loo, and P. K. Sahoo, Eur. Phys. J. C **82**, 72 (2022).
[41] R. Solanki, A. De, and P. K. Sahoo, Phys. Dark Universe **36**, 100996 (2022).
[42] F. D' Ambrosio, S. D. B. Fell, L. Heisenberg, and S. Kuhn, Phys. Rev. D **105**, 024042 (2022).
[43] F. K. Anagnostopoulos, S. Basilakos, and E. N. Saridakis, Phys. Lett. B **822**, 136634 (2021).
[44] L. Atayde and N. Frusciante, Phys. Rev. D **104**, 064052 (2021).
[45] N. Dimakis, A. Paliathanasis, and T. Christodoulakis, Classical Quantum Gravity **38**, 225003 (2021).
[46] S. Capozziello and M. Shokri, Phys. Dark Universe **37**, 101113 (2022).
[47] S. Bahamonde, J. G. Valcarcel, L. Järv, and J. Lember, J. Cosmol. Astropart. Phys. 08 (2022) 082.
[48] M. Koussour, S. K. J. Pacif, M. Bennai, and P. K. Sahoo, arXiv:2208.04723.
[49] A. Lymperis, J. Cosmol. Astropart. Phys. 11 (2022) 018.
[50] A. De, D. Saha, G. Subramaniam, and A. K. Sanyal, arXiv:2209.12120.
[51] G. N. Gadbail, S. Mandal, and P. K. Sahoo, Phys. Lett. B **835**, 137509 (2022).
[52] S. Capozziello and R. D'Agostino, Phys. Lett. B **832**, 137229 (2022).
[53] M. Calzá and L. Sebastiani, arXiv:2208.13033.

[54] R. D'Agostino and R. C. Nunes, arXiv:2210.11935.
[55] N. Dimakis, A. Paliathanasis, M. Roumeliotis, and T. Christodoulakis, Phys. Rev. D **106**, 043509 (2022).
[56] M. Hohmann, Phys. Rev. D **104**, 124077 (2021).
[57] F. D' Ambrosio, L. Heisenberg, and S. Kuhn, Classical Quantum Gravity **39**, 025013 (2022).
[58] S. Bahamonde and L. Järv, Eur. Phys. J. C **82**, 963 (2022).
[59] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005); **90**, 029902(E) (2014).
[60] P. S. Apostolopoulos, Gen. Relativ. Gravit. **49**, 59 (2017).
[61] A. Paliathanasis, J. D. Barrow, and P. G. L. Leach, Phys. Rev. D **94**, 023525 (2016).
[62] D. Zhao, Eur. Phys. J. C **82**, 303 (2022).
[63] L. P. Eisenhart, Non-Riemannian Geometry, *American Mathematical Society* (Colloquium Publications, New York, 1927), Vol. VIII.
[64] J. B. Jiménez and T. S. Koivisto, Int. J. Geom. Method Mod. Phys. **19**, 2250108 (2022).
[65] J. B. Jiménez, L. Heisenberg, and T. S. Koivisto, J. Cosmol. Astropart. Phys. 08 (2018) 039.
[66] A. Delhom-Latorre, G. J. Olmo, and M. Ronco, Phys. Lett. B **780**, 294 (2018).
[67] A. Delhom, Eur. Phys. J. C **80**, 728 (2020).
[68] D. M. Eardley, Commun. Math. Phys. **37**, 287 (1974).
[69] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).