Lorentz-covariant nonlocal collision term for spin-1/2 particles

David Wagner,[1] Nora Weickgenannt,[1] and Dirk H. Rischke[1,2]
[1]*Institute for Theoretical Physics, Goethe University, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany*
[2]*Helmholtz Research Academy Hesse for FAIR, Campus Riedberg, Max-von-Laue-Str. 12, D-60438 Frankfurt am Main, Germany*

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We revisit the derivation of the nonlocal collision term in the Boltzmann equation for spin-1/2 particles, using both the Wigner-function approach by de Groot, van Leeuwen, and van Weert, and the Kadanoff-Baym equation in T-matrix approximation. Contrary to previous calculations, our results maintain full Lorentz covariance of the nonlocal collision term.

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I. INTRODUCTION AND SUMMARY

The observation of polarization phenomena in relativistic heavy-ion collisions [1–8] motivated a plethora of theoretical developments [9–33] in recent years, which aim at describing the dynamics of spin polarization in relativistic fluids. Analogous to the nonrelativistic Barnett effect [34], a rotating relativistic fluid with spin degrees of freedom can be polarized along the vorticity direction through the mutual conversion of orbital and spin angular momentum. In order to understand this process within a theoretical framework and to calculate the resulting spin polarization in a quantitatively reliable manner, considerable efforts have been made to derive a theory of relativistic spin hydrodynamics [35–72].

It is convenient to first derive a kinetic theory from quantum field theory, which in turn serves as the starting point for the derivation of spin hydrodynamics, e.g., by a Chapman-Enskog expansion [40,44] or by the method of moments [66,71]. It was found in previous works [41,73] (see also Refs. [74–82] for related studies) that, in spin kinetic theory, a nonlocal collision term is responsible for the mutual conversion of orbital and spin angular momentum, and therefore polarizes the fluid along the vorticity when equilibrium is approached.

In Refs. [41,73], we derived the Boltzmann equation to order $\mathcal{O}(\hbar)$ for massive spin-1/2 particles in the approach of Ref. [83], termed the "GLW approach" in this paper. For Dirac particles with spin 1/2 and for binary elastic collisions, it assumes the form

$$p\cdot\partial_x f(x,p,\mathfrak{s}) = \mathfrak{C}[f] = \frac{1}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2) \times \mathcal{W}[f(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2) - f(x,p,\bar{\mathfrak{s}})f(x+\Delta'-\Delta,p',\mathfrak{s}')]. \tag{1}$$

Here, $\mathcal{W}$ is the transition probability for the scattering process and $f(x,p,\mathfrak{s})$ is the spin-dependent single-particle distribution function in extended phase space, i.e., ordinary phase space extended by spin degrees of freedom. We also defined $\mathrm{d}\Gamma := \mathrm{d}P\mathrm{d}S(p)$ as the integration measure over on-shell momentum space $\mathrm{d}P$ and spin space $\mathrm{d}S(p)$, see Eqs. (6) and (7). Note that in Eq. (1) there is in principle also a contribution from collisions which exchange only spin but not momentum, cf., e.g., Eq. (24) in Ref. [41]. It can be shown that such a contribution corresponds to corrections to the drift term, as well as an additional Vlasov term, on the left-hand side of the Boltzmann equation. We defer a detailed discussion to a subsequent work [84] and, for the sake of simplicity, omit such a contribution in this paper. We also note that the collision term (1) agrees with the one given in Refs. [41,73] only when replacing $x-\Delta$ by x. The form given in Eq. (1) is actually the more accurate one. The overall shift of all positions by $-\Delta$ was neglected in Refs. [41,73] by an argument assuming an expansion around local thermodynamical equilibrium, for details see Appendix D of Ref. [73].

The nonlocality of the collision term (1) manifests itself in the fact that the distribution functions of the collision partners are taken at space-time points shifted from position x^μ by $\Delta'^\mu-\Delta^\mu$, $\Delta_1^\mu-\Delta^\mu$, and $\Delta_2^\mu-\Delta^\mu$, respectively. These shifts were calculated in Ref. [73] and are of the order of the Compton wavelength of the particles, e.g.,

$$\Delta_\mu := \frac{\hbar}{2m(p\cdot\hat{t}+m)}\epsilon_{\mu\nu\alpha\beta}\hat{t}^\nu p^\alpha \mathfrak{s}^\beta, \tag{2}$$

and similarly for Δ'^μ, Δ_1^μ, and Δ_2^μ, replacing $(p^\alpha, \mathfrak{s}^\beta)$ by $(p'^\alpha, \mathfrak{s}'^\beta)$, $(p_1^\alpha, \mathfrak{s}_1^\beta)$, and $(p_2^\alpha, \mathfrak{s}_2^\beta)$, respectively. In Eq. (2), $\hat{t}^\nu := (1, \mathbf{0})$ is a timelike unit vector defining the frame where p^α is measured. The nonlocality of the collision term allows for the conversion of orbital into spin angular momentum. We remark that the expression (2) is identical with the so-called Berry curvature, cf., e.g., Ref. [85].

However, a serious shortcoming of the previous work [41,73] is that the Berry curvature violates Lorentz covariance, since it explicitly depends on the frame vector $\hat{t}^\mu$, which does not transform under Lorentz transformations. The technical reason for the occurrence of the Berry curvature is that at some point of the calculation one has to take momentum derivatives of the Dirac spinors appearing in the Wigner functions. At the same time, momentum derivatives of the scattering matrix elements are neglected, arguing that these derivatives should be small if the interaction is sufficiently localized. However, in doing so one neglects momentum derivatives of the Dirac spinors appearing in the matrix elements, which otherwise would lead to similar Berry-curvature terms, which, if kept, restore Lorentz covariance.

In this paper, we improve on the previously made approximation and derive a nonlocal collision term which manifestly respects Lorentz covariance. The form of this term is similar to Eq. (1), but the space-time shift (2) is replaced by a (much more complicated, but) Lorentz-covariant expression. For a current-current interaction as in the time-honored Nambu-Jona-Lasinio (NJL) model [86,87], i.e., an interaction which couples the fermion current $\bar{\psi}\Gamma^{(c)}\psi$ with itself with coupling strength G_c, where $\Gamma^{(c)}$ is a Dirac matrix and the index c characterizes the particular interaction channel (for details see Sec. III), we obtain

$$\Delta_1^\mu := \frac{\hbar}{m}\frac{4G_cG_d}{\hbar^2}\frac{m^4}{\mathcal{W}}\mathrm{Im}\{\mathrm{Tr}[h\Gamma^{(d)}h_2\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}h_1\gamma^\mu\Gamma^{(c)}h'] - \mathrm{Tr}[h\Gamma^{(d)}h_1\gamma^\mu\Gamma^{(c)}h'\Gamma^{(d)}h_2\Gamma^{(c)}]\}, \tag{3a}$$

$$\Delta_2^\mu := \frac{\hbar}{m}\frac{4G_cG_d}{\hbar^2}\frac{m^4}{\mathcal{W}}\mathrm{Im}\{\mathrm{Tr}[h\Gamma^{(d)}h_2\gamma^\mu\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}h_1\Gamma^{(c)}h'] - \mathrm{Tr}[h\Gamma^{(d)}h_1\Gamma^{(c)}h'\Gamma^{(d)}h_2\gamma^\mu\Gamma^{(c)}]\}, \tag{3b}$$

$$\Delta'^\mu := \frac{\hbar}{m}\frac{4G_cG_d}{\hbar^2}\frac{m^4}{\mathcal{W}}\mathrm{Im}\{\mathrm{Tr}[h\Gamma^{(d)}h_2\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}h_1\Gamma^{(c)}h'\gamma^\mu] - \mathrm{Tr}[h\Gamma^{(d)}h_1\Gamma^{(c)}h'\gamma^\mu\Gamma^{(d)}h_2\Gamma^{(c)}]\}, \tag{3c}$$

$$\Delta^\mu := \frac{\hbar}{m}\frac{4G_cG_d}{\hbar^2}\frac{m^4}{\mathcal{W}}\mathrm{Im}\{\mathrm{Tr}[h\gamma^\mu\Gamma^{(d)}h_2\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}h_1\Gamma^{(c)}h'] - \mathrm{Tr}[h\gamma^\mu\Gamma^{(d)}h_1\Gamma^{(c)}h'\Gamma^{(d)}h_2\Gamma^{(c)}]\}, \tag{3d}$$

where

$$\mathcal{W} := m^4\frac{G_cG_d}{\hbar^2}16\mathrm{Re}\{\mathrm{Tr}[h\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}]\mathrm{Tr}[\Gamma^{(d)}h_1\Gamma^{(c)}h'] - \mathrm{Tr}[h\Gamma^{(d)}h_1\Gamma^{(c)}h'\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}]\}. \tag{4}$$

Here,

$$h \equiv h(p,\mathfrak{s}) := \frac{1}{4m}(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m), \tag{5}$$

and similarly for h', h_1, h_2, and $\bar{h}$, with $(p,\mathfrak{s})$ replaced by $(p',\mathfrak{s}')$, $(p_1,\mathfrak{s}_1)$, $(p_2,\mathfrak{s}_2)$, and $(p,\bar{\mathfrak{s}})$, respectively. Equations (3) are the main result of this work. Note that they pertain to the special case of a pointlike interaction, but can be readily generalized to the case where the interaction is mediated over a nonvanishing distance, see, e.g., Eqs. (100), which are also valid for more general interactions.

The paper is organized as follows. In Sec. II we recall some facts about the Wigner function. In Sec. III we define the Lagrangian underlying our investigations. In Sec. IV we carefully repeat the calculation of the collision term via the GLW approach [83] performed in Refs. [41,73], but now paying attention to maintaining full Lorentz covariance throughout the calculation. For the purpose of making our calculations more concise, various definitions and conventions differ from those of Ref. [83] and used previously in Refs. [41,73]. This also facilitates comparison with other works, which adhere to more commonly used notations. Our result is the nonlocal Boltzmann equation (1) with the space-time shifts (2) replaced by the expressions (3). In Sec. V we then confirm our results by repeating the calculation in the Kadanoff-Baym (KB) approach (see, e.g., Refs. [88,89]), showing that the results coincide for an interaction of NJL-type. The KB approach was previously used by some of us (N.W., D.H.R.) in Ref. [81], where the nonlocal collision term was derived in T-matrix approximation. However, the use of the matrix-valued spin distribution functions in that work prevented a direct comparison with that of the GLW approach of Refs. [41,73]. In this paper, we complete the derivation of the nonlocal collision term in the KB approach, using the scalar distribution function $f(x,p,\mathfrak{s})$ in extended phase space that was used in the GLW approach. We note that a similar study in the nonrelativistic limit has been performed in a recent paper [90]. Finally, in Sec. VI, we conclude our work with an outlook for future studies.

We define the Lorentz-invariant measure in momentum space as

$$\mathrm{d}P := \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3 p^0}. \tag{6}$$

The Lorentz-invariant measure in spin space for particles with spin 1/2 is defined as [41,73]

$$\mathrm{d}S(p) \coloneqq \sqrt{\frac{p^2}{3\pi^2}}\mathrm{d}^4\mathfrak{s}\delta(\mathfrak{s}\cdot\mathfrak{s}+3)\delta(p\cdot\mathfrak{s}). \tag{7}$$

This measure implies the following relations for integration over the spin 4-vector $\mathfrak{s}^\mu$:

$$\int \mathrm{d}S(p) = 2, \qquad \int \mathrm{d}S(p)\mathfrak{s}^\mu = 0,$$
$$\int \mathrm{d}S(p)\mathfrak{s}^\mu\mathfrak{s}^\nu = -2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \tag{8}$$

The measure (7) is chosen in such a way that integrating over the whole spin space gives the number of spin degrees of freedom, see the first relation in Eq. (8).

We adopt the following conventions: the metric tensor is $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ and the four-dimensional unit matrix in Dirac space is denoted as $\mathbb{1}$, while the Dirac matrices are denoted as γ^μ. The four-dimensional Levi-Civitá symbol is $\epsilon^{0123} = -\epsilon_{0123} = 1$, and summation over repeated indices is implied if not stated explicitly. The scalar product of two four-vectors a^μ and b^μ is $a\cdot b \coloneqq a^\mu b_\mu$. Furthermore, we define $\not{a} \coloneqq \gamma^\mu a_\mu$. (Anti-) symmetrization in Lorentz indices is denoted as $a_{[\mu}b_{\nu]} \coloneqq a_\mu b_\nu - a_\nu b_\mu$ and $a_{(\mu}b_{\nu)} \coloneqq a_\mu b_\nu + a_\nu b_\mu$. We choose natural Heaviside-Lorentz units, $c = \varepsilon_0 = \mu_0 = k_B = 1$, but the reduced Planck constant $\hbar$ is kept explicitly in order to perform the power counting. Lorentz indices are denoted by Greek indices, except for α, β, γ and δ, which are used for Dirac indices (if necessary with appropriate sub-indices, e.g., $\alpha', \alpha_1, \alpha_2, \ldots$). Spin indices are denoted by the letters $r, s, \ldots$.

II. WIGNER FUNCTION

In this section, we collect some well-known facts about the Wigner function, which will be used in the calculation of the collision term in the GLW as well as the KB approach. We start with a discussion of the two-particle correlation function in the closed-time path formalism and then focus on the definition of the Wigner function, its Clifford decomposition, as well as its equation of motion. We then establish a relation between the Wigner function and the single-particle distribution function in extended phase space.

A. Two-particle correlation function in closed-time path formalism

On the closed-time path (see, e.g., Ref. [88]), the two-particle correlation function assumes the following matrix form,

$$G(x_1,x_2) = \begin{pmatrix} G^{++}(x_1,x_2) & G^{+-}(x_1,x_2) \\ G^{-+}(x_1,x_2) & G^{--}(x_1,x_2) \end{pmatrix} \equiv \begin{pmatrix} G^{F}(x_1,x_2) & G^{<}(x_1,x_2) \\ G^{>}(x_1,x_2) & G^{\bar{F}}(x_1,x_2) \end{pmatrix}. \tag{9}$$

where $G^{ij}(x_1,x_2)$ (with $i,j=+,-$) means that the first time argument $t_1 = x_1^0$ lives on the time branch i and the second time argument $t_2 = x_2^0$ lives on the time branch j. The definitions of the various Green's functions are

$$G^F_{\alpha\beta}(x_1,x_2) \coloneqq \langle T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle, \tag{10a}$$

$$G^{\bar{F}}_{\alpha\beta}(x_1,x_2) \coloneqq \langle T_A\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle, \tag{10b}$$

$$G^{<}_{\alpha\beta}(x_1,x_2) \coloneqq \langle \bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle, \tag{10c}$$

$$G^{>}_{\alpha\beta}(x_1,x_2) \coloneqq \langle \psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle, \tag{10d}$$

where T and T_A denote the time-ordering and anti-time-ordering operators, respectively, and angular brackets denote averages computed with respect to some initial state. Note that we define $G^<$ with opposite sign as in Ref. [81], but with the same sign as in Ref. [80].

Not all components of the correlation function (9) are independent. In fact, one may express G^F and $G^{\bar{F}}$ by $G^<$ and $G^>$ using the definition of the time-ordering and anti-time-ordering operators, respectively,

$$G^F_{\alpha\beta}(x_1,x_2) = \theta(t_1-t_2)G^>_{\alpha\beta}(x_1,x_2) - \theta(t_2-t_1)G^<_{\alpha\beta}(x_1,x_2), \tag{11a}$$

$$G^{\bar{F}}_{\alpha\beta}(x_1,x_2) = -\theta(t_1-t_2)G^<_{\alpha\beta}(x_1,x_2) + \theta(t_2-t_1)G^>_{\alpha\beta}(x_1,x_2). \tag{11b}$$

B. Wigner function

The Wigner function $G^<(x,p)$ in the KB approach is defined as the Fourier transform of the two-point function (10c) with respect to the difference $y \coloneqq x_1 - x_2$ of the two space-time points x_1 and x_2,

$$G^<_{\alpha\beta}(x,p) \coloneqq \int \mathrm{d}^4y e^{ip\cdot y/\hbar}G^<_{\alpha\beta}(x_1,x_2) \equiv \int \mathrm{d}^4y e^{ip\cdot y/\hbar}\left\langle \bar{\psi}_\beta\left(x-\frac{y}{2}\right)\psi_\alpha\left(x+\frac{y}{2}\right)\right\rangle, \tag{12}$$

where $x \coloneqq (x_1+x_2)/2$ is the arithmetic mean (or center) of the two space-time points x_1 and x_2. Similarly,

$$G^{>}_{\alpha\beta}(x,p) \coloneqq \int \mathrm{d}^4y e^{ip\cdot y/\hbar} G^{>}_{\alpha\beta}(x_1,x_2) \equiv \int \mathrm{d}^4y e^{ip\cdot y/\hbar} \left\langle \psi_\alpha\left(x+\frac{y}{2}\right)\bar{\psi}_\beta\left(x-\frac{y}{2}\right)\right\rangle. \quad (13)$$

On the other hand, in the GLW approach [83], the Wigner function is defined as

$$W_{\alpha\beta}(x,p) \coloneqq \int \mathrm{d}^4y e^{-\frac{i}{\hbar}p\cdot y}\left\langle :\bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right):\right\rangle, \quad (14)$$

i.e., similar as in Eq. (12) when substituting $y \to -y$ (in order to facilitate the comparison with previous work, we do not perform this substitution explicitly), but with an additional normal-ordering operation on the field operators. This bears no further consequences, since we will neglect antiparticles anyway.

Note that compared to Refs. [39,41,73,83] the factor $(2\pi\hbar)^{-4}$ in the integration measure in Eq. (14) is absent, because we choose to absorb such factor into the four-dimensional momentum-space measure $\mathrm{d}^4p/(2\pi\hbar)^4$. A further consequence is that the single-particle distribution function does not have a prefactor $(2\pi\hbar)^{-3}$ as, e.g., in Eq. (85) of Ref. [41]. This facilitates the expression for Pauli-blocking factors, which simply read $1-f$, instead of $1-(2\pi\hbar)^3 f$ in the notation of Ref. [83].

C. Clifford decomposition

We can expand $G^{<}(x,p)$ (or $W(x,p)$) in terms of the 16 independent generators of the Clifford algebra, Γ_a, $a = 1,\ldots,16$, with

$$\Gamma_a \in \{\mathbb{1}, \gamma^\mu, -i\gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}, \quad (15)$$

where $\sigma^{\mu\nu} \coloneqq \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, such that

$$G^{<}(x,p) \equiv W(x,p) = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right). \quad (16)$$

Similarly,

$$G^{>}(x,p) = \frac{1}{4}\left(\bar{\mathcal{F}} + i\gamma^5\bar{\mathcal{P}} + \gamma^\mu\bar{\mathcal{V}}_\mu + \gamma^5\gamma^\mu\bar{\mathcal{A}}_\mu + \frac{1}{2}\sigma^{\mu\nu}\bar{\mathcal{S}}_{\mu\nu}\right). \quad (17)$$

The real-valued coefficient functions $\mathcal{F}$, $\bar{\mathcal{F}}$, $\mathcal{P}$, $\bar{\mathcal{P}}$, $\mathcal{V}_\mu$, $\bar{\mathcal{V}}_\mu$, $\mathcal{A}_\mu$, $\bar{\mathcal{A}}_\mu$, $\mathcal{S}_{\mu\nu}$, and $\bar{\mathcal{S}}_{\mu\nu}$ are the scalar, pseudoscalar, vector, axial-vector, and tensor components of $G^{\lessgtr}(x,p)$ (or $W(x,p)$), respectively, which can be obtained by taking the trace of $G^{\lessgtr}(x,p)$ (or $W(x,p)$), multiplied with the appropriate generator Γ_a of the Clifford algebra.

D. Equation of motion

The equation of motion for the Wigner function $G^{<}(x,p)$ can be derived from the Dyson-Schwinger equation for the two-particle correlation function. In the quasi-particle approximation one obtains [81]

$$(\not{K} - m)G^{<}(x,p) = I_{\mathrm{coll}}, \quad (18)$$

where m is the mass of the particles and

$$K^\mu \coloneqq p^\mu + \frac{i\hbar}{2}\partial^\mu_x. \quad (19)$$

The collision term I_{coll} is given by

$$I_{\mathrm{coll}} \coloneqq \frac{i\hbar}{2}[\Sigma^{<}(x,p)G^{>}(x,p) - \Sigma^{>}(x,p)G^{<}(x,p)] + \frac{\hbar^2}{4}[\{\Sigma^{<}(x,p),G^{>}(x,p)\}_{\mathrm{PB}} - \{\Sigma^{>}(x,p),G^{<}(x,p)\}_{\mathrm{PB}}]. \quad (20)$$

Note the change of sign in the collision term as compared to Ref. [81], which is due to the opposite sign in the definition of $G^{<}$. In Eq. (20), $\Sigma^{\gtrless}(x,p)$ are the Wigner transforms of the self-energies $\Sigma^{\gtrless}(x_1,x_2)$ on the closed-time path and we introduced the Poisson bracket

$$\{A,B\}_{\mathrm{PB}} \coloneqq (\partial_x A)\cdot(\partial_p B) - (\partial_p A)\cdot(\partial_x B). \quad (21)$$

By multiplying Eq. (18) with the generators of the Clifford algebra and taking the trace, we can derive a system of equations of motion for the Clifford components of the Wigner function. The real parts of these equations read

$$p^\mu\mathcal{V}_\mu - m\mathcal{F} = \mathrm{ReTr}(I_{\mathrm{coll}}), \quad (22a)$$

$$m\mathcal{P} + \frac{\hbar}{2}\partial^\mu_x\mathcal{A}_\mu = \mathrm{ReTr}(i\gamma^5 I_{\mathrm{coll}}), \quad (22b)$$

$$p_\mu\mathcal{F} - m\mathcal{V}_\mu + \frac{\hbar}{2}\partial^\nu_x\mathcal{S}_{\mu\nu} = \mathrm{ReTr}(\gamma_\mu I_{\mathrm{coll}}), \quad (22c)$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}p^\nu\mathcal{S}^{\rho\sigma} + m\mathcal{A}_\mu - \frac{\hbar}{2}\partial_{x,\mu}\mathcal{P} = \mathrm{ReTr}(\gamma^5\gamma_\mu I_{\mathrm{coll}}), \quad (22d)$$

$$\epsilon_{\mu\nu\rho\sigma}p^\rho\mathcal{A}^\sigma + m\mathcal{S}_{\mu\nu} - \frac{\hbar}{2}\partial_{x[\mu}\mathcal{V}_{\nu]} = -\mathrm{ReTr}(\sigma_{\mu\nu}I_{\mathrm{coll}}), \quad (22e)$$

while the imaginary parts are

$$\frac{\hbar}{2}\partial_x^\mu \mathcal{V}_\mu = \mathrm{Im}\mathrm{Tr}(I_{\mathrm{coll}}), \tag{22f}$$

$$p^\mu \mathcal{A}_\mu = \mathrm{Im}\mathrm{Tr}(-i\gamma^5 I_{\mathrm{coll}}), \tag{22g}$$

$$p^\nu \mathcal{S}_{\nu\mu} + \frac{\hbar}{2}\partial_{x,\mu}\mathcal{F} = \mathrm{Im}\mathrm{Tr}(\gamma_\mu I_{\mathrm{coll}}), \tag{22h}$$

$$p_\mu \mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\rho\sigma}\partial_x^\nu \mathcal{S}^{\rho\sigma} = \mathrm{Im}\mathrm{Tr}(\gamma^5\gamma_\mu I_{\mathrm{coll}}), \tag{22i}$$

$$p_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\rho\sigma}\partial_x^\rho \mathcal{A}^\sigma = -\mathrm{Im}\mathrm{Tr}(\sigma_{\mu\nu} I_{\mathrm{coll}}). \tag{22j}$$

Acting with the operator $\not{K} + m$ onto Eq. (18) and combining the resulting equation with its Hermitian conjugate, multiplied from the left- and the right-hand side with γ^0, we can derive a mass-shell constraint and a Boltzmann equation for the Wigner function $G^<(x,p)$,

$$\left(p^2 - m^2 - \frac{\hbar^2}{4}\partial_x^2\right) G^<(x,p) = \frac{1}{2}\{(\not{K} + m) I_{\mathrm{coll}} + \gamma^0[(\not{K} + m) I_{\mathrm{coll}}]^\dagger \gamma^0\}, \tag{23a}$$

$$\hbar p\cdot\partial_x G^<(x,p) = -\frac{i}{2}\{(\not{K} + m) I_{\mathrm{coll}} - \gamma^0[(\not{K} + m) I_{\mathrm{coll}}]^\dagger \gamma^0\}, \tag{23b}$$

where we have used $\gamma^0 (G^<)^\dagger \gamma^0 \equiv G^<$, which follows from Eq. (16). Taking the trace with the appropriate basis elements of the Clifford algebra, we obtain for the components of the Wigner function

$$\left(p^2 - m^2 - \frac{\hbar^2}{4}\partial_x^2\right)\mathrm{Tr}(\Gamma_a G^<) = \mathrm{Re}\mathrm{Tr}[\Gamma_a(\not{K} + m) I_{\mathrm{coll}}], \tag{24a}$$

$$\hbar p\cdot\partial_x \mathrm{Tr}(\Gamma_a G^<) = \mathrm{Im}\mathrm{Tr}[\Gamma_a(\not{K} + m) I_{\mathrm{coll}}], \tag{24b}$$

where we have used $\gamma^0\Gamma_a^\dagger\gamma^0 = \Gamma_a$. It was shown in Ref. [81] that off-shell contributions in Eq. (24b) are of higher order, $\mathcal{O}(G_c^4)$, in the coupling constant. They are therefore neglected in the following. In this approximation, we will also show by an explicit computation that, at least to order $\mathcal{O}(\hbar)$, all propagators are on the mass shell.

Similarly, in the GLW approach the equations of motion for the Wigner function (14) read [73,83]

$$\left(p^2 - m^2 - \frac{\hbar^2}{4}\partial_x^2\right) W(x,p) = \hbar\delta M(x,p), \tag{25a}$$

$$p\cdot\partial_x W(x,p) = \mathcal{C}(x,p). \tag{25b}$$

Here we defined

$$\delta M_{\alpha\beta}(x,p) := \frac{1}{2}\int \mathrm{d}^4 y e^{-\frac{i}{\hbar}y\cdot p}\langle : [\bar\rho(x_1)(i\hbar\overleftarrow{\not\partial}_x + m)]_\beta \psi_\alpha(x_2) + \bar\psi_\beta(x_1)[(-i\hbar\not\partial_x + m)\rho(x_2)]_\alpha : \rangle, \tag{26a}$$

$$\mathcal{C}_{\alpha\beta}(x,p) := \frac{i}{2}\int \mathrm{d}^4 y e^{-\frac{i}{\hbar}y\cdot p}\langle : [\bar\rho(x_1)(-i\hbar\overleftarrow{\not\partial}_x + m)]_\beta \psi_\alpha(x_2) - \bar\psi_\beta(x_1)[(i\hbar\not\partial_x + m)\rho(x_2)]_\alpha : \rangle, \tag{26b}$$

where

$$\rho(x) := -\frac{1}{\hbar}\frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial\bar\psi(x)}, \tag{27}$$

with the interaction Lagrangian $\mathcal{L}_{\mathrm{int}}$ of the theory under consideration, and where $\bar\psi(x) := \psi^\dagger(x)\gamma^0$, $\bar\rho(x) := \rho^\dagger(x)\gamma^0$ are the Dirac adjoints of the fermion field $\psi(x)$ and the source term $\rho(x)$. Note the formal similarity between Eqs. (23) and (25).

Similar to Eqs. (23a) and (23b), Eq. (25a) is a mass-shell equation for the Wigner function, while Eq. (25b) represents a Boltzmann-type equation. Again, off-shell terms in Eq. (25b) can be shown to cancel [73], such that we have

$$p\cdot\partial_x W_{\text{on-shell}}(x,p) = \mathcal{C}_{\text{on-shell}}(x,p), \tag{28}$$

where the Wigner function and collision terms are decomposed as

$$W(x,p) = 4\pi m\hbar\delta(p^2 - m^2) W_{\text{on-shell}}(x,p) + W_{\text{off-shell}}(x,p), \tag{29a}$$

$$\mathcal{C}(x,p) = 4\pi m\hbar\delta(p^2 - m^2)\mathcal{C}_{\text{on-shell}}(x,p) + \mathcal{C}_{\text{off-shell}}(x,p). \tag{29b}$$

Note that the prefactor is chosen such that the usual momentum-space measure is recovered for the on-shell terms, as is discussed in Appendix A. This also implies that the $\hbar$ in the prefactor of Eqs. (29a), (29b) does not participate in the $\hbar$-power counting.

E. Single-particle distribution function in extended phase space

Our goal is to derive a Boltzmann equation for the single-particle distribution function $f(x,p,\mathfrak{s})$ in extended phase space from the Boltzmann-type equations (23b) and (25b), respectively. To this end, we need to establish a relation between $f(x,p,\mathfrak{s})$ and $G^<(x,p)$ or $W(x,p)$, respectively. Since we work in the semiclassical expansion, we can do this order by order in $\hbar$. We work up to order $\mathcal{O}(\hbar^2)$ in the equation of motion (18) for the Wigner function. Since the collision term I_{coll} is already of order $\mathcal{O}(\hbar)$, cf. Eq. (20), for the computation of the latter we only need to determine $f(x,p,\mathfrak{s})$ and $G^<(x,p)$ or $W(x,p)$ up to first order in $\hbar$.

First of all, following Refs. [41,73] we introduce a scalar single-particle distribution function in extended phase space,

$$\mathfrak{f}(x,p,\mathfrak{s}) \coloneqq \frac{1}{2}[\mathcal{F}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)], \tag{30}$$

and analogously

$$\bar{\mathfrak{f}}(x,p,\mathfrak{s}) \coloneqq \frac{1}{2}[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\bar{\mathcal{A}}(x,p)]. \tag{31}$$

Note that $\mathfrak{f}(x,p,\mathfrak{s})$ and $\bar{\mathfrak{f}}(x,p,\mathfrak{s})$ are distribution-valued and thus not identical, but proportional, to $f(x,p,\mathfrak{s})$ and $\bar{f}(x,p,\mathfrak{s})$, respectively.

We now expand all quantities in powers of $\hbar$, e.g.,

$$\mathfrak{f}(x,p,\mathfrak{s}) = \mathfrak{f}^{(0)}(x,p) + \hbar\mathfrak{f}^{(1)}(x,p,\mathfrak{s}) + \hbar^2\mathfrak{f}^{(2)}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^3)\ldots, \tag{32}$$

and similarly for all other quantities. Here we have assumed that spin effects enter $\mathfrak{f}$ first at order $\mathcal{O}(\hbar)$, cf. the discussion in Refs. [41,73].

1. Zeroth order in $\hbar$

Since the collision term I_{coll} is already of first order in $\hbar$, cf. Eq. (20), at zeroth order in $\hbar$ we obtain from Eqs. (22a)–(22d)

$$p^\mu\mathcal{V}_\mu^{(0)} - m\mathcal{F}^{(0)} = 0, \tag{33a}$$

$$\mathcal{P}^{(0)} = 0, \tag{33b}$$

$$p_\mu\mathcal{F}^{(0)} - m\mathcal{V}_\mu^{(0)} = 0, \tag{33c}$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}p^\nu\mathcal{S}^{(0)\rho\sigma} + m\mathcal{A}_\mu^{(0)} = 0. \tag{33d}$$

In order to proceed, we make the additional assumption that $\mathcal{A}_\mu = \hbar\mathcal{A}_\mu^{(1)} + \mathcal{O}(\hbar^2)$, i.e., that $\mathcal{A}_\mu^{(0)} = 0$, see Refs. [41,73]. This can be justified by assuming that polarization effects are at most generated dynamically within the system, but are not induced already from the outset by, e.g., external fields. In this case, Eqs. (33b)–(33d) imply that

$$\mathcal{P}^{(0)} = 0, \quad \mathcal{V}_\mu^{(0)} = \frac{p_\mu}{m}\mathcal{F}^{(0)}, \quad \mathcal{A}_\mu^{(0)} = 0, \quad \mathcal{S}_{\mu\nu}^{(0)} = 0, \tag{34}$$

and the only independent Lorentz component of the Wigner function at order $\mathcal{O}(\hbar^0)$ is $\mathcal{F}^{(0)}$. Analogous relations hold for the Clifford components $\bar{\mathcal{P}}^{(0)}$, $\bar{\mathcal{V}}_\mu^{(0)}$, $\bar{\mathcal{A}}_\mu^{(0)}$, and $\bar{\mathcal{S}}_{\mu\nu}^{(0)}$ of the Wigner function $G^>$.

Furthermore, we conclude by combining Eqs. (33a) and (33c) that

$$p^2\mathcal{F}^{(0)} = mp^\mu\mathcal{V}_\mu^{(0)} = m^2\mathcal{F}^{(0)}, \tag{35}$$

i.e., $\mathcal{F}^{(0)}$ is on-shell, and thus also $\mathcal{V}_\mu^{(0)}$ is on-shell. Similar arguments apply to $\bar{\mathcal{F}}^{(0)}$ and $\bar{\mathcal{V}}_\mu^{(0)}$.

Inserting Eq. (34) into Eq. (16), we immediately derive

$$G^{<(0)}(x,p) \equiv W^{(0)}(x,p) = \frac{\not{p}+m}{4m}\mathcal{F}^{(0)}(x,p) \equiv \frac{1}{2}\Lambda^+(p)\mathcal{F}^{(0)}(x,p), \tag{36}$$

and, similarly,

$$G^{>(0)}(x,p) = \frac{1}{2}\Lambda^+(p)\bar{\mathcal{F}}^{(0)}(x,p), \tag{37}$$

where we used the projector

$$\Lambda^+(p) \coloneqq \frac{\not{p}+m}{2m} \tag{38}$$

onto positive-energy states. On the other hand, because $\mathcal{A}_\mu^{(0)} = 0$, to order $\mathcal{O}(\hbar^0)$, Eq. (30) reads

$$\mathfrak{f}^{(0)}(x,p) = \frac{1}{2}\mathcal{F}^{(0)}(x,p), \tag{39}$$

and similarly Eq. (31) reads

$$\bar{\mathfrak{f}}^{(0)}(x,p) = \frac{1}{2}\bar{\mathcal{F}}^{(0)}(x,p). \tag{40}$$

Since $\mathcal{F}^{(0)}$ and $\bar{\mathcal{F}}^{(0)}$ are on-shell, we can factor out a mass-shell delta function from $\mathfrak{f}^{(0)}$ and $\bar{\mathfrak{f}}^{(0)}$,

$$\mathfrak{f}^{(0)}(x,p) \coloneqq 4\pi m\hbar\delta(p^2-m^2)f^{(0)}(x,p),$$
$$\bar{\mathfrak{f}}^{(0)}(x,p) \coloneqq 4\pi m\hbar\delta(p^2-m^2)\bar{f}^{(0)}(x,p), \tag{41}$$

where the prefactor $4\pi m\hbar$ is introduced to make $f^{(0)}$ and $\bar{f}^{(0)}$ dimensionless and to ensure that $f^{(0)}$ converges to the Fermi distribution function in the thermodynamical limit (see Appendix A), while $\bar{f}^{(0)}$ becomes a Pauli-blocking factor $1 - f^{(0)}$. Note that the factor $2\pi\hbar$ in the prefactor in Eq. (41) does not contribute to the power counting, since it does not appear with either a gradient or a spin-related quantity. It merely serves to cancel a $(2\pi\hbar)^{-1}$ from the four-dimensional momentum-space measure, cf. Appendix A. We thus obtain the final expressions for $G^{\lessgtr(0)}$ and $W^{(0)}$ in terms of $f^{(0)}$ and $\bar{f}^{(0)}$,

$$G^{<(0)}(x,p) \equiv W^{(0)}(x,p) = 4\pi m\hbar\delta(p^2-m^2)\Lambda^+(p)f^{(0)}(x,p), \tag{42a}$$

$$G^{>(0)}(x,p) = 4\pi m\hbar\delta(p^2-m^2)\Lambda^+(p)\bar{f}^{(0)}(x,p). \tag{42b}$$

Following the discussion in Sec. V of Ref. [81], we expect $\bar{f}^{(0)} \equiv 1 - f^{(0)}$ (note the sign difference in our definition of $G^<$ with respect to that reference), i.e., it is sufficient to know $f^{(0)}$ to reconstruct both $G^{<(0)}$ and $G^{>(0)}$.

2. First order in $\hbar$

To first order in $\hbar$, we employ Eqs. (22a)–(22e), as well as Eq. (22g), which read

$$p^\mu \mathcal{V}^{(1)}_\mu - m\mathcal{F}^{(1)} = \mathrm{ReTr}(I^{(1)}_{\mathrm{coll}}), \tag{43a}$$

$$\frac{1}{2}\partial^\mu_x \mathcal{A}^{(0)}_\mu + m\mathcal{P}^{(1)} = \mathrm{ReTr}(i\gamma_5 I^{(1)}_{\mathrm{coll}}), \tag{43b}$$

$$\frac{1}{2}\partial^\nu_x \mathcal{S}^{(0)}_{\nu\mu} - p_\mu \mathcal{F}^{(1)} + m\mathcal{V}^{(1)}_\mu = -\mathrm{ReTr}(\gamma_\mu I^{(1)}_{\mathrm{coll}}), \tag{43c}$$

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} p^\nu \mathcal{S}^{(1)\rho\sigma} + m\mathcal{A}^{(1)}_\mu = \mathrm{ReTr}(\gamma^5\gamma_\mu I^{(1)}_{\mathrm{coll}}), \tag{43d}$$

$$\frac{1}{2}\partial_{x[\mu}\mathcal{V}^{(0)}_{\nu]} - \epsilon_{\mu\nu\rho\sigma} p^\rho \mathcal{A}^{(1)\sigma} - m\mathcal{S}^{(1)}_{\mu\nu} = \mathrm{ReTr}(\sigma_{\mu\nu} I^{(1)}_{\mathrm{coll}}), \tag{43e}$$

$$p^\mu \mathcal{A}^{(1)}_\mu = \mathrm{ImTr}(-i\gamma^5 I^{(1)}_{\mathrm{coll}}). \tag{43f}$$

Because $G^{>(1)} \equiv -G^{<(1)}$ [81], we have $\bar{\mathcal{F}}^{(1)} \equiv -\mathcal{F}^{(1)}$, $\bar{\mathcal{P}}^{(1)} \equiv -\mathcal{P}^{(1)}$, $\bar{\mathcal{V}}^{(1)}_\mu \equiv -\mathcal{V}^{(1)}_\mu$, $\bar{\mathcal{A}}^{(1)}_\mu \equiv -\mathcal{A}^{(1)}_\mu$, and $\bar{\mathcal{S}}^{(1)}_{\mu\nu} \equiv -\mathcal{S}^{(1)}_{\mu\nu}$, respectively. We also have $\bar{f}^{(1)} = -f^{(1)}$.

With Eq. (34) we conclude from Eq. (43b) that

$$\mathcal{P}^{(1)} = \frac{1}{m}\mathrm{ReTr}(i\gamma_5 I^{(1)}_{\mathrm{coll}}) = \mathcal{O}(G_c^2). \tag{44}$$

Since we are ultimately interested in computing the collision term, we may make further approximations. Namely, when using the Clifford decomposition for $G^{<(1)}$ in the collision term, a term such as in Eq. (44) gives an overall contribution of order $\mathcal{O}(G_c^4)$, which can be neglected. We can therefore safely assume that $\mathcal{P}^{(1)} \simeq 0$.

Equation (43c) together with Eq. (34) yields

$$\mathcal{V}^{(1)}_\mu = \frac{p_\mu}{m}\mathcal{F}^{(1)} - \frac{1}{m}\mathrm{ReTr}(\gamma_\mu I^{(1)}_{\mathrm{coll}}) \simeq \frac{p_\mu}{m}\mathcal{F}^{(1)}. \tag{45}$$

In the last step, we have again neglected the contribution of order $\mathcal{O}(G_c^2)$ from the collision term, since this gives rise to an overall contribution of order $\mathcal{O}(G_c^4)$ when inserting the Clifford decomposition for $G^{<(1)}$ into the collision term.

Finally, Eq. (43e) together with Eq. (34) gives

$$\begin{aligned}\mathcal{S}^{(1)}_{\mu\nu} &= -\frac{1}{m}\epsilon_{\mu\nu\rho\sigma} p^\rho \mathcal{A}^{(1)\sigma} + \frac{1}{2m}\partial_{x,[\mu}\mathcal{V}^{(0)}_{\nu]} - \frac{1}{m}\mathrm{ReTr}(\sigma_{\mu\nu} I^{(1)}_{\mathrm{coll}}) \\ &\simeq -\frac{1}{m}\epsilon_{\mu\nu\rho\sigma} p^\rho \mathcal{A}^{(1)\sigma} - \frac{1}{2m^2} p_{[\mu}\partial_{x,\nu]}\mathcal{F}^{(0)}.\end{aligned} \tag{46}$$

By the same arguments as above, the contribution from the collision term can be neglected.

Furthermore, combining Eqs. (43a) and (43c) and using Eq. (34), we conclude that

$$p^2\mathcal{F}^{(1)} = m^2\mathcal{F}^{(1)} + \mathrm{ReTr}[(\not{p} + m) I^{(1)}_{\mathrm{coll}}] \simeq m^2\mathcal{F}^{(1)}, \tag{47}$$

i.e., $\mathcal{F}^{(1)}$ is on-shell up to collisional contributions, which can be neglected when inserting the Clifford decomposition for $G^{<(1)}$ into the collision term.

We can also derive a mass-shell condition for $\mathcal{A}^{(1)}_\mu$. To this end, multiply Eq. (43e) by $p_\lambda \epsilon^{\mu\nu\lambda\tau}$ and insert Eqs. (43d) and (43f). Using Eq. (34), this results in

$$\begin{aligned}p^2\mathcal{A}^{(1)\tau} &= m^2\mathcal{A}^{(1)\tau} - \mathrm{ReTr}[\gamma_5\gamma^\tau(\not{p} + m) I^{(1)}_{\mathrm{coll}}] \\ &\simeq m^2\mathcal{A}^{(1)\tau},\end{aligned} \tag{48}$$

i.e., $\mathcal{A}^{(1)}_\mu$ is on-shell up to collisional contributions, which can be neglected when inserting the Clifford decomposition for $G^{<(1)}$ into the collision term. On account of Eqs. (30) and (31), also $\mathfrak{f}^{(1)}$ and $\bar{\mathfrak{f}}^{(1)}$ are on-shell up to collisional contributions.

We can now insert Eqs. (44)–(46) into the Clifford decomposition (16) of the Wigner function $G^{<(1)}(x,p) \equiv W^{(1)}(x,p)$ and obtain up to collisional contributions

$$G^{<(1)}(x,p) \equiv W^{(1)}(x,p) \simeq G^{<(1)}_{\mathrm{qc}}(x,p) + G^{<(1)}_{\nabla}(x,p), \tag{49}$$

with

$$G^{<(1)}_{\mathrm{qc}}(x,p) = \frac{1}{2}\Lambda^+(p)[\mathcal{F}^{(1)}(x,p) + \gamma_5\gamma\cdot\mathcal{A}^{(1)}(x,p)], \tag{50a}$$

$$G^{<(1)}_{\nabla}(x,p) = \frac{1}{8m^2}\sigma^{\mu\nu} p_\nu \partial^x_\mu \mathcal{F}^{(0)}(x,p), \tag{50b}$$

where in the terminology of Ref. [81] the subscript "qc" denotes the so-called quasiclassical contribution, while the subscript "∇" denotes the gradient contribution. The gradient contribution $G^{<(1)}_\nabla$ can be immediately expressed in terms of $f^{(0)}(x,p)$ using Eqs. (39) and (41),

$$G^{<(1)}_{\nabla}(x,p) = \frac{\pi\hbar}{m}\delta(p^2 - m^2)\sigma^{\mu\nu} p_\nu \partial^x_\mu f^{(0)}(x,p), \tag{51}$$

and similarly

$$G^{>(1)}_{\nabla}(x,p) = \frac{\pi\hbar}{m}\delta(p^2 - m^2)\sigma^{\mu\nu} p_\nu \partial^x_\mu \bar{f}^{(0)}(x,p). \tag{52}$$

Both the quasiclassical and the gradient parts are "quasiparticle" contributions in the sense that they are on the mass shell. Since $\partial^x_\mu \bar{f}^{(0)} \equiv -\partial^x_\mu f^{(0)}$, we confirm that $G^{>(1)}_\nabla \equiv -G^{<(1)}_\nabla$.

In order to express the quasiclassical contribution $G_{\rm qc}^{<(1)}$ in terms of $f^{(1)}$, we need to invert Eq. (30). This can be done with the relations [41,73]

$$\mathcal{F}(x,p) = \int \mathrm{d}S(p)\mathfrak{f}(x,p,\mathfrak{s}),$$
$$\mathcal{A}_\mu(x,p) = \int \mathrm{d}S(p)\mathfrak{s}_\mu\mathfrak{f}(x,p,\mathfrak{s}). \tag{53}$$

Using the definition (30) of $\mathfrak{f}(x,p,\mathfrak{s})$ and the identities (8), one proves that the first relation is strictly valid to all orders in $\hbar$, while the second relation is strictly valid up to order $\mathcal{O}(\hbar^0)$ and valid up to order $\mathcal{O}(\hbar)$ if collisional contributions are disregarded, such that $p\cdot\mathcal{A}^{(1)} \simeq 0$, cf. Eq. (43f).

Inserting the first-order expressions $\mathcal{F}^{(1)}$ and $\mathcal{A}^{(1)}_\mu$ into Eq. (50a), and factoring out a mass-shell delta function from $\mathfrak{f}^{(1)}(x,p,\mathfrak{s})$,

$$\mathfrak{f}^{(1)}(x,p,\mathfrak{s}) = 4\pi m\hbar\delta(p^2-m^2)f^{(1)}(x,p,\mathfrak{s}), \tag{54}$$

we then obtain

$$G_{\rm qc}^{<(1)}(x,p) = 4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}S(p)h(p,\mathfrak{s})f^{(1)}(x,p,\mathfrak{s}), \tag{55}$$

where we have used Eq. (5) and the relation

$$(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\Lambda^+(p) = \Lambda^+(p)(\mathbb{1}+\gamma_5\not{\mathfrak{s}}), \tag{56}$$

which holds since $p\cdot\mathfrak{s}=0$. Similarly,

$$G_{\rm qc}^{>(1)}(x,p) = 4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}S(p)h(p,\mathfrak{s})\bar{f}^{(1)}(x,p,\mathfrak{s}), \tag{57}$$

Since $\bar{f}^{(1)} \equiv -f^{(1)}$, we confirm that also $G_{\rm qc}^{>(1)} = -G_{\rm qc}^{<(1)}$.

To summarize the results of this subsection, we have expressed the Wigner functions $G^{\lessgtr}(x,p)$ and $W(x,p)$ up to first order in $\hbar$ in terms of $f^{(0)}$, $\bar{f}^{(0)}$, $f^{(1)}$, and $\bar{f}^{(1)}$, cf. Eqs. (42) and (49) with Eqs. (51), (52), (55), and (57). For further use, we note that, because of Eq. (8), up to first order in $\hbar$ we may write

$$G^<(x,p) \equiv W(x,p) = 4\pi m\hbar\delta(p^2-m^2) \times \int \mathrm{d}S(p)h(p,\mathfrak{s})f(x,p,\mathfrak{s}) + G^<_\nabla(x,p). \tag{58}$$

III. INTERACTION LAGRANGIAN

For the interaction $\mathcal{L}_{\rm int}$ between fermions we consider one-boson exchange. Assuming the interaction range to be much smaller than all other scales in the problem, we can integrate out the boson fields and reduce the interaction to a four-fermion vertex, similar to the NJL model [86,87]. Thus, the interaction Lagrangian reads

$$\mathcal{L}_{\rm int} = \sum_c G_c \sum_{a,b}[\bar\psi(x)\Gamma_a^{(c)}\psi(x)]g^{ab}_{(c)}[\bar\psi(x)\Gamma_b^{(c)}\psi(x)]. \tag{59}$$

The sum over c runs over all possible interaction channels, e.g., scalar ($c=S$), pseudoscalar ($c=P$), vector ($c=V$), axial-vector ($c=A$), and tensor ($c=T$) channel. The matrices $\Gamma_a^{(c)}, \Gamma_b^{(c)}$ represent the corresponding elements of the Clifford algebra: $\Gamma_a^{(S)} = \mathbb{1}$, $\Gamma_a^{(P)} = -i\gamma_5$, $\Gamma_a^{(V)} = \gamma^\mu$, $\Gamma_a^{(A)} = \gamma_5\gamma^\mu$, and $\Gamma_a^{(T)} = \sigma^{\mu\nu}$. In the scalar and pseudoscalar channels, the sum over a and b only contains one element and $g^{ab}_{(S,P)} \equiv 1$. In the vector and axial-vector channels, a and b are Lorentz indices, which are summed over with $g^{ab}_{(V,A)} \equiv g^{\mu\nu}$. In the tensor channel, a and b represent pairs of (unequal) Lorentz indices, say $a=(\mu\nu)$, $b=(\rho\sigma)$ and $g^{ab}_{(T)} \equiv g^{\mu\rho}g^{\nu\sigma}$. Finally, G_c denotes the four-fermion coupling in channel (c).

In order to simplify the notation, in the remainder of this work we will omit the indices a, b and the metric $g^{ab}_{(c)}$, and just indicate the particular interaction channel c at the element $\Gamma^{(c)}$ of the Clifford algebra, i.e., an appropriate summation over (a,b) is implied,

$$\sum_{a,b}[\bar\psi(x)\Gamma_a^{(c)}\psi(x)]g^{ab}_{(c)}[\bar\psi(x)\Gamma_b^{(c)}\psi(x)] \equiv [\bar\psi(x)\Gamma^{(c)}\psi(x)][\bar\psi(x)\Gamma^{(c)}\psi(x)]. \tag{60}$$

IV. THE NONLOCAL COLLISION TERM IN THE GLW APPROACH

In this section, we rederive the collision term in the GLW approach using slightly different and more commonly used conventions than in Refs. [41,73], in order to facilitate comparison with results from the KB approach. We first repeat parts of the derivation of the collision term as presented in Ref. [73] and then focus separately on the local and the nonlocal parts of the collision term. Finally, we summarize our results.

A. The collision term revisited

In the following we restrict ourselves to the particle sector, i.e., $p^0 > 0$ for all on-shell momenta p^μ; the antiparticle sector can be derived analogously. The positive-frequency "in" fields are given by

$$\psi_{\rm in}(x) = \frac{1}{2}\sum_s \int {\rm d}P u_s(p)\hat{a}_s(p)e^{-ip\cdot x/\hbar}, \quad (61)$$

where p^μ is on shell, ${\rm d}P$ was defined in Eq. (6), and $\hat{a}_s(p)$ annihilates a particle with momentum p and spin s. In our convention (which differs from that of Ref. [83]) the nonzero anticommutation relations for the creation and annihilation operators are

$$\{\hat{a}_s(p), \hat{a}_r^\dagger(p')\} = (2\pi\hbar)^3 2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rs}, \quad (62)$$

while the orthogonality and completeness relation of the basis spinors read

$$\bar{u}_{r,\alpha}(p)u_s^\alpha(p) = 2m\delta_{rs},$$
$$\sum_r u_r^\alpha(p)\bar{u}_{r,\beta}(p) = (\not{p}+m)^\alpha_\beta. \quad (63)$$

Note that, from now on, we will not distinguish between upper and lower Dirac indices. When evaluating the expressions, it is implied that the Dirac indices are on their natural position, e.g., upper (lower) indices for spinors (adjoint spinors), and repeated indices are simply summed over (without additional sign change). Following Ref. [83], we may cast the collision term (26b) in the following form:

$$\mathcal{C}_{\alpha\beta}(x,p) = \frac{1}{2}\frac{1}{(2\pi\hbar)^{12}(2m)^4}\sum_{r^2,s^2}\int {\rm d}^4x^2 \int {\rm d}^4p^2 \int {\rm d}^4u^2$$
$$\times {}_{\rm in}\left\langle p^2-\frac{u^2}{2};r^2\middle|\Phi_{\alpha\beta}(p)\middle|p^2+\frac{u^2}{2};s^2\right\rangle_{\rm in}$$
$$\times \prod_{j=1}^2 e^{\frac{i}{\hbar}u_j\cdot x_j}\bar{u}_{s_j,\alpha_j}\left(p_j+\frac{u_j}{2}\right)$$
$$\times W^{\alpha_j\beta_j}(x+x_j,p_j)u_{r_j,\beta_j}\left(p_j-\frac{u_j}{2}\right). \quad (64)$$

Note that the different prefactor as compared to Eq. (11) of Ref. [73] is due to the different definition of the momentum-space measure. Here, for the sake of brevity we used the notation of Ref. [83] for sums, integrals, and two-particle states,

$$\sum_{r^2,s^2} := \sum_{r_1=1}^2\sum_{r_2=1}^2\sum_{s_1=1}^2\sum_{s_2=1}^2,$$
$$\int {\rm d}^4x^2 := \int {\rm d}^4x_1\int {\rm d}^4x_2,$$
$$\left|p^2\pm\frac{u^2}{2};r^2\right\rangle := \left|p_1\pm\frac{u_1}{2},p_2\pm\frac{u_2}{2};r_1,r_2\right\rangle. \quad (65)$$

The operator Φ in Eq. (64) reads

$$\Phi_{\alpha\beta}(p) := \frac{i}{2}\frac{1}{(2\pi\hbar)^4}\int {\rm d}^4y e^{-\frac{i}{\hbar}p\cdot y}:\left\{\left[P_\mu,\bar{\rho}\left(\frac{y}{2}\right)\gamma^\mu\right]_\beta\psi_\alpha\left(-\frac{y}{2}\right)\right.$$
$$+m\bar{\rho}_\beta\left(\frac{y}{2}\right)\psi_\alpha\left(-\frac{y}{2}\right)-\bar{\psi}_\beta\left(\frac{y}{2}\right)\left[\gamma^\mu\rho\left(-\frac{y}{2}\right),P_\mu\right]_\alpha$$
$$\left.-m\bar{\psi}_\beta\left(\frac{y}{2}\right)\rho_\alpha\left(-\frac{y}{2}\right)\right\}:, \quad (66)$$

where P_μ denotes the momentum operator. The calculation needed to compute the matrix element in Eq. (64) has been discussed in Refs. [73,83], but we will nonetheless show the main steps. After the insertion of a complete set of "out" states and performing the y integration, one arrives at

$${}_{\rm in}\left\langle p^2-\frac{u^2}{2};r^2\middle|\Phi_{\alpha\beta}(p)\middle|p^2+\frac{u^2}{2};s^2\right\rangle_{\rm in}$$
$$= \frac{i}{4}\sum_{r'}\int {\rm d}P'\delta^{(4)}(p+p'-p_1-p_2)$$
$$\times\left[{}_{\rm out}\left\langle p';r'\middle|\psi_\alpha(0)\middle|p^2+\frac{u^2}{2};s^2\right\rangle_{\rm in\,in}\left\langle p^2-\frac{u^2}{2};r^2\middle|:\bar{\rho}_{\alpha'}(0):\middle|p';r'\right\rangle_{\rm out}\left(\not{p}-\frac{\not{k}_1+\not{k}_2}{2}+m\right)_{\alpha'\beta}\right.$$
$$\left.-\left(\not{p}+\frac{\not{k}_1+\not{k}_2}{2}+m\right)_{\alpha\alpha'}{}_{\rm out}\left\langle p';r'\middle|:\rho_{\alpha'}(0):\middle|p^2+\frac{u^2}{2};s^2\right\rangle_{\rm in\,in}\left\langle p^2-\frac{u^2}{2};r^2\middle|\bar{\psi}_\beta(0)\middle|p';r'\right\rangle_{\rm out}\right]. \quad (67)$$

Next we evaluate the matrix elements of the fields $\psi,\bar{\psi}$. These fields can be written in terms of the "in" fields as

$$\psi(0) = \psi_{\rm in}(0) + \int {\rm d}^4x S_R(-x)\rho(x), \quad (68)$$

where S_R is the retarded fermion propagator, whose Fourier transform is given by $\tilde{S}_R(p) = -(1/\hbar)(\not{p} + m)\tilde{G}(p)$, with the scalar propagator $\tilde{G}(p) \coloneqq -\hbar^2/(p^2 - m^2 + i\epsilon p^0)$. Using the orthogonality relation of the momentum eigenstates,

$$ {}_{\text{in}}\langle p^2; r^2 | p'^2; r'^2 \rangle_{\text{in}} = [2(2\pi\hbar)^3]^2 p_1^0 p_2^0 [\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_1')\delta_{r_1 r_1'}\delta^{(3)}(\mathbf{p}_2 - \mathbf{p}_2')\delta_{r_2 r_2'} - \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2')\delta_{r_1 r_2'}\delta^{(3)}(\mathbf{p}_2 - \mathbf{p}_1')\delta_{r_2 r_1'}], \quad (69) $$

in conjunction with the fact that for one-particle states "in" and "out" states are identical, we find

$$ \begin{aligned} {}_{\text{out}}\left\langle p'; r' \middle| \psi(0) \middle| p^2 + \frac{u^2}{2}; s^2 \right\rangle_{\text{in}} &= 2(2\pi\hbar)^3 p'^0 \left[u_{s_1}\left(p_1 + \frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{p}' - \mathbf{p}_2 - \frac{\mathbf{u}_2}{2}\right)\delta_{r' s_2} - (1 \leftrightarrow 2) \right] \\ &\quad + \tilde{S}_R\left(p_1 + \frac{u_1}{2} + p_2 + \frac{u_2}{2} - p'\right) {}_{\text{out}}\left\langle p'; r' \middle| :\rho(0): \middle| p^2 + \frac{u^2}{2}; s^2 \right\rangle_{\text{in}}. \end{aligned} \quad (70) $$

Employing this relation and using the projector $\Lambda^+(p)$ defined in Eq. (38), we can rewrite Eq. (67) as

$$ \begin{aligned} &{}_{\text{in}}\left\langle p^2 - \frac{u^2}{2}; r^2 \middle| \Phi_{\alpha\beta}(p) \middle| p^2 + \frac{u^2}{2}; s^2 \right\rangle_{\text{in}} \\ &= im\Bigg(\Bigg\{ u_{s_1,\alpha}\left(p_1 + \frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{p} - \mathbf{p}_1 + \frac{\mathbf{u}_2}{2}\right)\delta\left[p^0 + \sqrt{\left(\mathbf{p}_2 + \frac{\mathbf{u}_2}{2}\right)^2 + m^2} - p_1^0 - p_2^0\right] \\ &\quad \times {}_{\text{in}}\left\langle p^2 - \frac{u^2}{2}; r^2 \middle| :\bar{\rho}_{\alpha'}(0): \middle| p_2 + \frac{u_2}{2}; s_2 \right\rangle_{\text{out}} \Lambda^+_{\alpha'\beta}\left(p - \frac{u_1 + u_2}{2}\right) + (1 \leftrightarrow 2) \Bigg\} \\ &\quad - \Bigg\{ \bar{u}_{r_1,\beta}\left(p_1 - \frac{u_1}{2}\right)\delta^{(3)}\left(\mathbf{p} - \mathbf{p}_1 - \frac{\mathbf{u}_2}{2}\right)\delta\left[p^0 + \sqrt{\left(\mathbf{p}_2 - \frac{\mathbf{u}_2}{2}\right)^2 + m^2} - p_1^0 - p_2^0\right] \\ &\quad \times \Lambda^+_{\alpha\alpha'}\left(p + \frac{u_1 + u_2}{2}\right) {}_{\text{out}}\left\langle p_2 - \frac{u_2}{2}; r_2 \middle| :\rho_{\alpha'}(0): \middle| p^2 + \frac{u^2}{2}; s^2 \right\rangle_{\text{in}} + (1 \leftrightarrow 2) \Bigg\} \\ &\quad - \frac{m}{\hbar}\sum_{r'} \int \mathrm{d}P' \delta^{(4)}(p + p' - p_1 - p_2)\left[\tilde{G}\left(p + \frac{u_1 + u_2}{2}\right) - \tilde{G}^*\left(p - \frac{u_1 + u_2}{2}\right) \right] \Lambda^+_{\alpha\alpha'}\left(p + \frac{u_1 + u_2}{2}\right) \\ &\quad \times {}_{\text{out}}\left\langle p'; r' \middle| :\rho_{\alpha'}(0): \middle| p^2 + \frac{u^2}{2}; s^2 \right\rangle_{\text{in}} {}_{\text{in}}\left\langle p^2 - \frac{u^2}{2}; r^2 \middle| :\bar{\rho}_{\beta'}(0): \middle| p'; r' \right\rangle_{\text{out}} \Lambda^+_{\beta'\beta}\left(p - \frac{u_1 + u_2}{2}\right) \Bigg). \end{aligned} \quad (71) $$

Here we used that $|p_1, p_2; s_1, s_2\rangle = -|p_2, p_1; s_2, s_1\rangle$. Next, we have to make use of the relation between the source terms and scattering-matrix elements, which is given by [83]

$$ \begin{aligned} (2\pi\hbar)^4 \delta^{(4)}(p + p' - p_1 - p_2) {}_{\text{in}}\langle p, p'; r, r' | \hat{t} | p^2; r^2 \rangle_{\text{in}} &\coloneqq -i {}_{\text{in}}\langle p, p'; r, r' | (\hat{S} - \mathbb{1}) | p^2; r^2 \rangle_{\text{in}} \\ &= -(2\pi\hbar)^4 \delta^{(4)}(p + p' - p_1 - p_2) \bar{u}_r(p) {}_{\text{out}}\langle p'; r' | \rho(0) | p^2; r^2 \rangle_{\text{in}}, \end{aligned} \quad (72) $$

where $\hat{S}$ is the scattering matrix. We now split the transfer matrix into real and imaginary parts and make use of the optical theorem [83],

$$ \frac{i}{2} {}_{\text{in}}\langle p, p'; r, r' | (\hat{t} - \hat{t}^\dagger) | p^2; r^2 \rangle_{\text{in}} = -\frac{(2\pi\hbar)^4}{16} \delta^{(4)}(p + p' - p_1 - p_2) \sum_{s^2} \int \mathrm{d}Q_1 \mathrm{d}Q_2 {}_{\text{in}}\langle p, p'; r, r' | \hat{t} | q^2; s^2 \rangle_{\text{in}} {}_{\text{in}}\langle q^2; s^2 | \hat{t}^\dagger | p^2; r^2 \rangle_{\text{in}}. \quad (73) $$

In the remainder of this paper, we will neglect the real parts of the transfer matrix and defer a more detailed discussion of the latter to a subsequent work [84].

We now employ the optical theorem and the following expression of the transfer-matrix elements

$$\begin{aligned}\langle p,p';r,r'|\hat{t}|p^2;r^2\rangle &= \frac{1}{\hbar}\langle p,p';r,r'|:\mathcal{L}_{\mathrm{int}}(0):|p^2;r^2\rangle \\ &= \bar{u}_{r,\alpha}(p)\bar{u}_{r',\alpha'}(p')u_{r_1,\alpha_1}(p_1)u_{r_2,\alpha_2}(p_2)M^{\alpha\alpha'\alpha_1\alpha_2}(p,p',p_1,p_2),\end{aligned} \tag{74a}$$

$$\begin{aligned}\langle p^2;r^2|\hat{t}^\dagger|p,p';r,r'\rangle &= \frac{1}{\hbar}\langle p^2;r^2|:\mathcal{L}^\dagger_{\mathrm{int}}(0):|p,p';r,r'\rangle \\ &= \bar{u}_{r_1,\alpha_1}(p_1)\bar{u}_{r_2,\alpha_2}(p_2)u_{r,\alpha}(p)u_{r',\alpha'}(p')\bar{M}^{\alpha_1\alpha_2\alpha\alpha'}(p_1,p_2,p,p'),\end{aligned} \tag{74b}$$

where M is the tree-level vertex function of the theory in momentum space, i.e., the Fourier transform of the fourth functional derivative of the classical action with respect to the fields, and $\bar{M}^{\alpha_1\alpha_2\alpha\alpha'} \coloneqq \gamma^0_{\alpha\beta}\gamma^0_{\alpha'\beta'}\gamma^0_{\alpha_1\beta_1}\gamma^0_{\alpha_2\beta_2}M^{*\beta\beta'\beta_1\beta_2}$. With these we are able to rewrite the source terms in the second and fourth lines of Eq. (71) as

$$\begin{aligned}&\Lambda^+_{\alpha\alpha'}\left(p+\frac{u_1+u_2}{2}\right)_{\mathrm{out}}\left\langle p_2-\frac{u_2}{2};r_2\right|:\rho_{\alpha'}(0):\left|p^2+\frac{u^2}{2};s^2\right\rangle_{\mathrm{in}} \\ &\quad= -\frac{i}{4}m^2(2\pi\hbar)^4\int \mathrm{d}Q_1\mathrm{d}Q_2\delta^{(4)}\left(p+p_2+\frac{u_1}{2}-q_1-q_2\right)M^{\alpha_1\alpha_2\beta_1\beta_2}\bar{M}^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\alpha\alpha_1}\left(p+\frac{u_1+u_2}{2}\right) \\ &\qquad\times\Lambda^+_{\beta_1\gamma_1}(q_1)\Lambda^+_{\beta_2\gamma_2}(q_2)\bar{u}_{r_2,\alpha_2}\left(p_2-\frac{u_2}{2}\right)u_{s_1,\delta_1}\left(p_1+\frac{u_1}{2}\right)u_{s_2,\delta_2}\left(p_2+\frac{u_2}{2}\right)\end{aligned} \tag{75a}$$

and

$$\begin{aligned}&{}_{\mathrm{out}}\left\langle p^2-\frac{u^2}{2};s^2\right|:\bar{\rho}_{\alpha'}(0):\left|p_2+\frac{u_2}{2};r_2\right\rangle_{\mathrm{in}}\Lambda^+_{\alpha'\beta}\left(p-\frac{u_1+u_2}{2}\right) \\ &\quad= \frac{i}{4}m^2(2\pi\hbar)^4\int \mathrm{d}Q_1\mathrm{d}Q_2\delta^{(4)}\left(p+p_2-\frac{u_1}{2}-q_1-q_2\right)\bar{M}^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\beta_1\beta}\left(p-\frac{u_1+u_2}{2}\right) \\ &\qquad\times\Lambda^+_{\delta_1\alpha_1}(q_1)\Lambda^+_{\delta_2\alpha_2}(q_2)u_{r_2,\beta_2}\left(p_2+\frac{u_2}{2}\right)\bar{u}_{s_1,\gamma_1}\left(p_1-\frac{u_1}{2}\right)\bar{u}_{s_2,\gamma_2}\left(p_2-\frac{u_2}{2}\right),\end{aligned} \tag{75b}$$

respectively. On the other hand, the source terms in the last line can be written as

$$\begin{aligned}&\sum_{r'}\Lambda^+_{\alpha\alpha'}\left(p+\frac{u_1+u_2}{2}\right)_{\mathrm{out}}\left\langle p';r'\right|:\rho_{\alpha'}(0):\left|p^2+\frac{u^2}{2};s^2\right\rangle_{\mathrm{in}\,\mathrm{in}}\left\langle p^2-\frac{u^2}{2};r^2\right|:\bar{\rho}_{\beta'}(0):\left|p';r'\right\rangle_{\mathrm{out}}\Lambda^+_{\beta'\beta}\left(p-\frac{u_1+u_2}{2}\right) \\ &\quad= 2mM^{\alpha_1\alpha_2\beta_1\beta_2}\bar{M}^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\alpha\alpha_1}\left(p+\frac{u_1+u_2}{2}\right)\Lambda^+_{\delta_1\beta}\left(p-\frac{u_1+u_2}{2}\right)\Lambda^+_{\delta_2\alpha_2}(p') \\ &\qquad\times u_{s_1,\beta_1}\left(p_1+\frac{u_1}{2}\right)u_{s_2,\beta_2}\left(p_2+\frac{u_2}{2}\right)\bar{u}_{r_1,\gamma_1}\left(p_1-\frac{u_1}{2}\right)\bar{u}_{r_2,\gamma_2}\left(p_2-\frac{u_2}{2}\right).\end{aligned} \tag{75c}$$

Furthermore, in the Boltzmann equation only on-shell terms contribute, i.e., it has to hold that $p^2 = m^2$ [73,81]. For this reason we may use the relation

$$\tilde{G}\left(p+\frac{u_1+u_2}{2}\right)-\tilde{G}^*\left(p-\frac{u_1+u_2}{2}\right) = 2\pi i\hbar^2\delta(p^2-m^2) \tag{76}$$

in Eq. (71), since the neglected terms are all off-shell contributions.

Inserting Eqs. (75) and (76) into Eq. (71) and the result into Eq. (64), we arrive with the definition of the on-shell quantities (29) at the following result for the collision term,

$$\begin{aligned}
\mathcal{C}_{\text{on-shell},\alpha\beta}(x,p) = (2\pi\hbar)^4\frac{m^4}{2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P' \int \mathrm{d}^4u^2 M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\alpha\alpha'}\left(p+\frac{u_1+u_2}{2}\right)\Lambda^+_{\beta'\beta}\left(p-\frac{u_1+u_2}{2}\right)\\
\times\left(\delta^{(4)}(p+p'-p_1-p_2)\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1}\Lambda^+_{\delta_2\alpha_2}(p')\Lambda^+_{\beta_1\gamma_1'}\left(p_1+\frac{u_1}{2}\right)\Lambda^+_{\beta_2\gamma_2'}\left(p_2+\frac{u_2}{2}\right)\right.\\
\times\Lambda^+_{\delta_1'\gamma_1}\left(p_1-\frac{u_1}{2}\right)\Lambda^+_{\delta_2'\gamma_2}\left(p_2-\frac{u_2}{2}\right)\prod_{j=1}^{2}\{W^{\gamma_j'\delta_j'}_{\text{on-shell}}(x,p_j)\delta^{(4)}(u_j)-i\hbar[\partial^\rho_{u_j}\delta^{(4)}(u_j)]\partial^x_\rho W^{\gamma_j'\delta_j'}_{\text{on-shell}}(x,p_j)\}\\
-\frac{1}{2}\delta^{\alpha'}_{\gamma_1'}\delta^{\beta'}_{\beta_1}\delta^{(4)}\left(p+p'-p_1-p_2-\frac{u_1}{2}\right)\Lambda^+_{\beta_2\gamma_2'}\left(p'+\frac{u_2}{2}\right)\Lambda^+_{\delta_1'\gamma_1}\left(p+\frac{u_2-u_1}{2}\right)\Lambda^+_{\delta_2'\gamma_2}\left(p'-\frac{u_2}{2}\right)\\
\times\Lambda^+_{\delta_1\alpha_1}(p_1)\Lambda^+_{\delta_2\alpha_2}(p_2)\left\{W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,p+\frac{u_2}{2}\right)\delta^{(4)}(u_1)-i\hbar[\partial^\mu_{u_1}\delta^{(4)}(u_1)]\partial^x_\mu W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,p+\frac{u_2}{2}\right)\right\}\\
\times\{W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,p')\delta^{(4)}(u_2)-i\hbar[\partial^\mu_{u_2}\delta^{(4)}(u_2)]\partial^x_\mu W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,p')\}\\
-\frac{1}{2}\delta^{\alpha'}_{\gamma_1'}\delta^{\beta'}_{\delta_1'}\delta^{(4)}\left(p+p'-p_1-p_2+\frac{u_1}{2}\right)\Lambda^+_{\beta_2\gamma_2'}\left(p'-\frac{u_2}{2}\right)\Lambda^+_{\beta_1\gamma_1'}\left(p+\frac{u_1-u_2}{2}\right)\Lambda^+_{\delta_2'\gamma_2}\left(p'+\frac{u_2}{2}\right)\\
\times\Lambda^+_{\delta_1\alpha_1}(p_1)\Lambda^+_{\delta_2\alpha_2}(p_2)\left\{W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,p-\frac{u_2}{2}\right)\delta^{(4)}(u_1)-i\hbar[\partial^\mu_{u_1}\delta^{(4)}(u_1)]\partial^x_\mu W^{\gamma_1'\delta_1'}_{\text{on-shell}}\left(x,p-\frac{u_2}{2}\right)\right\}\\
\left.\times\{W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,p')\delta^{(4)}(u_2)-i\hbar[\partial^\mu_{u_2}\delta^{(4)}(u_2)]\partial^x_\mu W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,p')\}\right). \qquad (77)
\end{aligned}$$

Here we used the completeness relation of the basis spinors multiple times, assumed that $\bar{M}=M$, and took M to be independent of momentum. Furthermore, we expanded the Wigner function to first order around $x_j=0$ and considered only the on-shell part of the collision term, cf. Eq. (28). The first terms in curly brackets in Eq. (77) provide the local contributions, while the respective second terms, which are proportional to space-time derivatives of the Wigner functions, constitute the nonlocal parts of the collision term.

B. Local collisions

From Eq. (77) we can read off the local collision term as

$$\begin{aligned}
&\mathcal{C}^{\text{local}}_{\text{on-shell},\alpha\beta}(x,p)\\
&=\frac{m^4}{2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\Lambda^+_{\alpha\alpha'}(p)\Lambda^+_{\beta'\beta}(p)\\
&\times\left\{\delta^{\alpha'}_{\alpha_1}\delta^{\beta'}_{\delta_1}\Lambda^+_{\delta_2\alpha_2}(p')\Lambda^+_{\beta_1\gamma_1'}(p_1)\Lambda^+_{\beta_2\gamma_2'}(p_2)\Lambda^+_{\delta_1'\gamma_1}(p_1)\Lambda^+_{\delta_2'\gamma_2}(p_2)\prod_{j=1}^{2}W^{\gamma_j'\delta_j'}_{\text{on-shell}}(x,p_j)\right.\\
&\left.-\frac{1}{2}\Lambda^+_{\beta_2\gamma_2'}(p')\Lambda^+_{\delta_2'\gamma_2}(p')\Lambda^+_{\delta_1\alpha_1}(p_1)\Lambda^+_{\delta_2\alpha_2}(p_2)[\delta^{\alpha'}_{\gamma_1'}\delta^{\beta'}_{\beta_1}\Lambda^+_{\delta_1'\gamma_1}(p)+\delta^{\alpha'}_{\gamma_1}\delta^{\beta'}_{\delta_1'}\Lambda^+_{\beta_1\gamma_1'}(p)]W^{\gamma_1'\delta_1'}_{\text{on-shell}}(x,p)W^{\gamma_2'\delta_2'}_{\text{on-shell}}(x,p')\right\}. \qquad (78)
\end{aligned}$$

Next we show how to translate this expression into extended phase space. To this end, we first note that all Wigner functions in Eq. (78) are sandwiched between energy projectors. Because of the relation

$$\Lambda^+(p)\sigma^{\mu\nu}p_\nu\Lambda^+(p)=0 \qquad (79)$$

this has the consequence that the gradient part of the Wigner function vanishes, cf. Eqs. (50b) and (58), such that

$$\Lambda^+(p)W_{\text{on-shell}}(x,p)\Lambda^+(p)=\int \mathrm{d}S(p)h(p,\mathfrak{s})f(x,p,\mathfrak{s}). \qquad (80)$$

One may wonder whether this introduces a discrepancy to the KB approach (where no such cancellation occurs). However, in the GLW approach another gradient contribution is generated at order $\mathcal{O}(\hbar)$ by an integration by parts, so that in the end both approaches yield the same result.

Inserting Eq. (80) into Eq. (78), the local collision term becomes

$$
\begin{aligned}
&\mathcal{C}^{\text{local}}_{\text{on-shell},\alpha\beta}(x,p)\\
&\quad=\frac{m^4}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\
&\quad\times\{\Lambda^+_{\alpha\alpha_1}(p)\Lambda^+_{\delta_1\beta}(p)h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)\\
&\quad-f(x,p,\bar{\mathfrak{s}})f(x,p',\mathfrak{s}')h_{\delta_1\alpha_1}(p_1,\mathfrak{s}_1)h_{\delta_2\alpha_2}(p_2,\mathfrak{s}_2)[h_{\alpha\gamma_1}(p,\bar{\mathfrak{s}})\Lambda^+_{\beta_1\beta}(p)+\Lambda^+_{\alpha\gamma_1}(p)h_{\beta_1\beta}(p,\bar{\mathfrak{s}})]h_{\beta_2\gamma_2}(p',\mathfrak{s}')\},
\end{aligned}
\tag{81}
$$

where we used that

$$
\int \mathrm{d}S(p)h(p,\mathfrak{s})=\Lambda^+(p). \tag{82}
$$

Lastly, we employ that $\mathfrak{C}_{\text{on-shell}}:=\frac{1}{2}(1+\gamma_5\not{s})_{\beta\alpha}\mathcal{C}^{\alpha\beta}_{\text{on-shell}}$, which then gives

$$
\begin{aligned}
\mathfrak{C}^{\text{local}}_{\text{on-shell}}(x,p,\mathfrak{s})=\frac{m^4}{4}\int &\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\
&\times h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')[h_{\delta_1\beta}(p,\mathfrak{s})h_{\beta\alpha_1}(p,\bar{\mathfrak{s}})+h_{\delta_1\beta}(p,\bar{\mathfrak{s}})h_{\beta\alpha_1}(p,\mathfrak{s})]\\
&\times[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)-f(x,p,\bar{\mathfrak{s}})f(x,p',\mathfrak{s}')],
\end{aligned}
\tag{83}
$$

where we also made use of the relation

$$
\begin{aligned}
&\int \mathrm{d}\bar{S}(p)[h_{\delta_1\beta}(p,\mathfrak{s})h_{\beta\alpha_1}(p,\bar{\mathfrak{s}})+h_{\delta_1\beta}(p,\bar{\mathfrak{s}})h_{\beta\alpha_1}(p,\mathfrak{s})]\\
&\quad=2h_{\delta_1\alpha_1}(p,\mathfrak{s}).
\end{aligned}
\tag{84}
$$

Note that the dependence on $\bar{\mathfrak{s}}$ in Eq. (83) can be eliminated employing a so-called "weak equivalence principle" [41,73]. This then gives a clearer interpretation of the last term in Eq. (83) as a loss term corresponding to particles with momentum p and spin $\mathfrak{s}$. The new collision term then has the form

$$
\begin{aligned}
&\tilde{\mathfrak{C}}^{\text{local}}_{\text{on-shell}}(x,p,\mathfrak{s})\\
&\quad=\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\tilde{\mathcal{W}}[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)\\
&\quad-f(x,p',\mathfrak{s}')f(x,p,\mathfrak{s})],
\end{aligned}
\tag{85}
$$

with

$$
\begin{aligned}
\tilde{W}:=&\frac{m^4}{2}(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\
&\times M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)\\
&\times h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\delta_1\alpha_1}(p,\mathfrak{s}).
\end{aligned}
\tag{86}
$$

This agrees with the local collision term derived in Refs. [41,73] up to the part corresponding to collisions without momentum exchange.

If we employ an NJL-type interaction according to Eq. (59), we obtain

$$
M^{\alpha_1\alpha_2\beta_1\beta_2}=\frac{2G_c}{\hbar}(\Gamma^{(c)\alpha_1\beta_1}\Gamma^{(c)\alpha_2\beta_2}-\Gamma^{(c)\alpha_1\beta_2}\Gamma^{(c)\alpha_2\beta_1}), \tag{87}
$$

from which it follows that

$$
\begin{aligned}
&M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)\\
&\quad\times h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\delta_1\beta}(p,\mathfrak{s})h_{\beta\alpha_1}(p,\bar{\mathfrak{s}})\\
&\equiv\frac{8G_cG_d}{\hbar^2}\{\mathrm{Tr}[h_2\Gamma^{(d)}h'\Gamma^{(c)}]\mathrm{Tr}[h_1\Gamma^{(d)}h\bar{h}\Gamma^{(c)}]\\
&\quad-\mathrm{Tr}[h_2\Gamma^{(d)}h\bar{h}\Gamma^{(c)}h_1\Gamma^{(d)}h'\Gamma^{(c)}]\},
\end{aligned}
\tag{88}
$$

where we abbreviated $h_1:=h(p_1,\mathfrak{s}_1)$ and likewise for $h_2,h',\bar{h}$, and h. Here, the symbol "$\equiv$" means that the expressions are equal under the respective integrals where they appear. This allowed us to use the symmetry under exchanging the integrations over $(p_1,\mathfrak{s}_1)$ and $(p_2,\mathfrak{s}_2)$, which is reflected in an additional factor of two in Eq. (88). Taking the complex conjugate of Eq. (88) and using that $h^\dagger=\gamma^0h\gamma^0$ as well as $\gamma^0\Gamma^{(c)\dagger}\gamma^0\equiv\Gamma^{(c)}$, we find

$$
\begin{aligned}
&[M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)\\
&\quad\times h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\delta_1\beta}(p,\mathfrak{s})h_{\beta\alpha_1}(p,\bar{\mathfrak{s}})]^*\\
&\equiv\frac{8G_cG_d}{\hbar^2}\{\mathrm{Tr}[h_2\Gamma^{(d)}h'\Gamma^{(c)}]\mathrm{Tr}[h_1\Gamma^{(d)}\bar{h}h\Gamma^{(c)}]\\
&\quad-\mathrm{Tr}[h_2\Gamma^{(d)}\bar{h}h\Gamma^{(c)}h_1\Gamma^{(d)}h'\Gamma^{(c)}]\}.
\end{aligned}
\tag{89}
$$

This allows us to bring Eq. (83) into the following form,

$$\begin{aligned}\mathfrak{C}^{\text{local}}_{\text{on-shell}}(x,p,\mathfrak{s}) = \frac{4G_cG_d}{\hbar^2}m^4\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ \times \mathrm{Re}\{\mathrm{Tr}[h_2\Gamma^{(d)}h'\Gamma^{(c)}]\mathrm{Tr}[h_1\Gamma^{(d)}h\bar{h}\Gamma^{(c)}] - \mathrm{Tr}[h_2\Gamma^{(d)}h\bar{h}\Gamma^{(c)}h_1\Gamma^{(d)}h'\Gamma^{(c)}]\}\\ \times [f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) - f(x,p',\bar{\mathfrak{s}})f(x,p,\mathfrak{s}')].\end{aligned} \tag{90}$$

C. Nonlocal collisions

In order to obtain the nonlocal contribution to the collision term, we have to integrate by parts in the variables u_1, u_2 in Eq. (77). Fortunately, the derivatives acting on the projectors are straightforwardly evaluated, e.g.,

$$\partial_u^\mu\Lambda^+\left(p+\frac{u}{2}\right)\Big|_{u=0} = \frac{1}{4m}\gamma^\mu. \tag{91}$$

We split the effect of the u_1, u_2 derivatives into four contributions (enumerated by capital Roman numbers): First, the derivatives act on the projectors $\Lambda^+(p\pm u_1/2\pm u_2/2)$ in front of everything, giving

$$\begin{aligned}\mathcal{C}^{\text{nonlocal}}_{\text{on-shell,I},\alpha\beta}(x,p) = \frac{i\hbar}{4m}\frac{m^4}{2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\ \times [\gamma^\mu_{\alpha\alpha'}\Lambda^+_{\beta'\beta}(p) - \Lambda^+_{\alpha\alpha'}(p)\gamma^\mu_{\beta'\beta}]\{\delta_{\alpha'\alpha_1}\delta_{\beta'\delta_1}\Lambda^+_{\delta_2\alpha_2}(p')\partial^x_\mu[W^{\beta_1\gamma_1}_{\text{on-shell}}(x,p_1)W^{\beta_2\gamma_2}_{\text{on-shell}}(x,p_2)]\\ -\frac{1}{2}\Lambda^+_{\delta_1\alpha_1}(p_1)\Lambda^+_{\delta_2\alpha_2}(p_2)[\delta_{\alpha'\gamma_1'}\delta_{\beta_1\beta'}\delta_{\delta_1'\gamma_1} + \delta_{\alpha'\gamma_1}\delta_{\delta_1'\beta'}\delta_{\beta_1\gamma_1'}]\partial^x_\mu[W^{\gamma_1'\delta_1'}_{\text{on-shell}}(x,p)W^{\beta_2\gamma_2}_{\text{on-shell}}(x,p')]\},\end{aligned} \tag{92}$$

where we already simplified some contractions of energy projectors and Wigner functions. Translating this expression into extended phase space, we find

$$\begin{aligned}\mathfrak{C}^{\text{nonlocal}}_{\text{on-shell,I}}(x,p,\mathfrak{s}) = \frac{i\hbar}{4m}\frac{m^4}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\ \times [h(p,\mathfrak{s}),\gamma^\mu]_{\beta'\alpha'}\{\delta_{\alpha'\alpha_1}\delta_{\beta'\delta_1}h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)\partial^x_\mu[f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)]\\ - h_{\delta_1\alpha_1}(p_1,\mathfrak{s}_1)h_{\delta_2\alpha_2}(p_2,\mathfrak{s}_2)h_{\beta_2\gamma_2}(p',\mathfrak{s}')[h_{\alpha'\gamma_1}(p,\bar{\mathfrak{s}})\delta_{\beta'\beta_1} + h_{\beta_1\beta'}(p,\bar{\mathfrak{s}})\delta_{\alpha'\gamma_1}]\partial^x_\mu[f(x,p,\bar{\mathfrak{s}})f(x,p',\mathfrak{s}')]\}.\end{aligned} \tag{93}$$

Since the parts of $f(x,p,\mathfrak{s})$ that are proportional to $\mathfrak{s}^\mu$ are at least of order $\mathcal{O}(\hbar)$, we may perform the $\mathrm{d}\bar{S}(p)$ integral trivially to obtain

$$\begin{aligned}\mathfrak{C}^{\text{nonlocal}}_{\text{on-shell,I}}(x,p,\mathfrak{s}) = \frac{i\hbar}{4m}\frac{m^4}{2}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\ \times h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)[h(p,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1}\partial^x_\mu[f(x,p_1)f(x,p_2) - \frac{1}{2}f(x,p)f(x,p')].\end{aligned} \tag{94}$$

Here we used that $\{[h(p,\mathfrak{s}),\gamma^\mu],\Lambda^+(p)\} = [h(p,\mathfrak{s}),\gamma^\mu]$ since $h(p,\mathfrak{s})(\not{p}-m) = 0$.

As a second nonlocal contribution, after integration by parts the u_1, u_2 derivatives in Eq. (77) act on the remaining projectors. Performing the same steps that led to Eq. (94), we find

$$\begin{aligned}\mathfrak{C}^{\text{nonlocal}}_{\text{on-shell,II}}(x,p,\mathfrak{s}) = -\frac{i\hbar}{4m}\frac{m^4}{2}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\\ \times\{f(x,p_2)[\partial^x_\mu f(x,p_1)]h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p_1,\mathfrak{s}_1),\gamma^\mu]_{\beta_1\gamma_1}\\ + f(x,p_1)[\partial^x_\mu f(x,p_2)]h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p_2,\mathfrak{s}_2),\gamma^\mu]_{\beta_2\gamma_2}\\ - f(x,p)[\partial^x_\mu f(x,p')]h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p',\mathfrak{s}'),\gamma^\mu]_{\delta_2\alpha_2}\\ -\frac{1}{2}[f(x,p')\partial^x_\mu f(x,p) - f(x,p)\partial^x_\mu f(x,p')]h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')[h(p,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1}\}.\end{aligned} \tag{95}$$

Third, a u_2 derivative acts on the Wigner functions $W^{\gamma_1'\delta_1'}_{\text{on-shell}}(x, p \pm u_2/2)$ in the loss term, yielding

$$\mathfrak{C}^{\text{nonlocal}}_{\text{on-shell,III}}(x,p,\mathfrak{s}) = -\frac{i\hbar}{4m}\frac{m^4}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \times [\partial_p^\mu f(x,p)][\partial_\mu^x f(x,p')]h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')[h(p,\mathfrak{s}),h(p,\bar{\mathfrak{s}})]_{\delta_1\alpha_1}. \tag{96}$$

Note that, after performing the $\mathrm{d}\bar{S}(p)$ integration trivially, this term vanishes as a consequence of our assumptions that polarization effects enter at order $\mathcal{O}(\hbar)$ and that $\bar{M} = M$.

Lastly, there are u_1 derivatives acting on the momentum-conserving delta function in the loss term, which can be rewritten as derivatives with respect to p' and then act both on projectors and the Wigner function, giving

$$\mathfrak{C}^{\text{nonlocal}}_{\text{on-shell,IV}}(x,p,\mathfrak{s}) = -\frac{i\hbar}{4m}\frac{m^4}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2} \times h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)[\partial_\mu^x f(x,p)][h_{\delta_2\alpha_2}(p',\mathfrak{s}')\partial_{p'}^\mu f(x,p') + \{h(p',\mathfrak{s}'),\gamma^\mu\}_{\delta_2\alpha_2}f(x,p')] \times [h(p,\mathfrak{s}),h(p,\bar{\mathfrak{s}})]_{\delta_1\alpha_1}. \tag{97}$$

Like Eq. (96), this contribution vanishes due to our assumptions.

D. Summary

Collecting both the local and the nonvanishing nonlocal contributions, we find

$$p\cdot\partial_x f(x,p,\mathfrak{s}) = \frac{1}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\mathcal{W} \times [f(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2) - f(x,p,\bar{\mathfrak{s}})f(x+\Delta'-\Delta,p',\mathfrak{s}')], \tag{98}$$

where we defined the local transition rate

$$\mathcal{W} \coloneqq m^4 M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')\{h(p,\mathfrak{s}),h(p,\bar{\mathfrak{s}})\}_{\delta_1\alpha_1}, \tag{99}$$

and the nonlocal shifts read

$$\Delta_1^\mu \coloneqq -\frac{i\hbar}{4m}\frac{m^4}{\mathcal{W}}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p_1,\mathfrak{s}_1),\gamma^\mu]_{\beta_1\gamma_1}, \tag{100a}$$

$$\Delta_2^\mu \coloneqq -\frac{i\hbar}{4m}\frac{m^4}{\mathcal{W}}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\delta_2\alpha_2}(p',\mathfrak{s}')h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p_2,\mathfrak{s}_2),\gamma^\mu]_{\beta_2\gamma_2}, \tag{100b}$$

$$\Delta'^\mu \coloneqq -\frac{i\hbar}{4m}\frac{m^4}{\mathcal{W}}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_1\alpha_1}(p,\mathfrak{s})[h(p',\mathfrak{s}'),\gamma^\mu]_{\delta_2\alpha_2}, \tag{100c}$$

$$\Delta^\mu \coloneqq -\frac{i\hbar}{4m}\frac{m^4}{\mathcal{W}}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}h_{\beta_1\gamma_1}(p_1,\mathfrak{s}_1)h_{\beta_2\gamma_2}(p_2,\mathfrak{s}_2)h_{\delta_2\alpha_2}(p',\mathfrak{s}')[h(p,\mathfrak{s}),\gamma^\mu]_{\delta_1\alpha_1}. \tag{100d}$$

At this point, two remarks are in order. First, the factor m^4 in the local transition rate (99) does not necessarily imply that $\mathcal{W}$ vanishes in the massless limit, since it cancels with appropriate inverse factors in the energy projectors $\Lambda^+(p)$. Indeed, considering the case where the distribution functions do not depend on spin, it is apparent that the spin-integrated transition rate becomes in the massless limit

$$\int \mathrm{d}S_1(p_1)\mathrm{d}S_2(p_2)\mathrm{d}S'(p')\mathrm{d}\bar{S}(p)\mathcal{W} = \frac{1}{16}M^{\alpha_1\alpha_2\beta_1\beta_2}M^{\gamma_1\gamma_2\delta_1\delta_2}\not{p}_{1,\beta_1\gamma_1}\not{p}_{2,\beta_2\gamma_2}\not{p}'_{\delta_2\alpha_2}[(\mathbb{1}+\gamma_5\not{s})\not{p}]_{\delta_1\alpha_1}, \tag{101}$$

which is (assuming that the vertices M do not diverge) manifestly finite. Nevertheless, in order to properly assess the collision term in the massless limit, the calculation in the GLW approach should be repeated taking into account the different equations of motion for $W(x,p)$ in this case, which is beyond the scope of this work.

Second, it is reassuring to see how Eq. (98) agrees with the known expression for binary elastic scattering in the limit of local collisions and distribution functions independent of spin [83]. In this case, the integration over $\mathrm{d}\bar{S}(p)$ in the first line of Eq. (98) will produce a factor of 2, similar to the other spin-space integrals. Finally, there is a factor of $1/2$ on the right-hand side of Eq. (98), which can be interpreted as a symmetry factor due to the indistinguishable nature of the particles.

Upon inserting the explicit NJL-type interaction (87), we arrive at the main result of this work, i.e., Eqs. (3) with Eq. (4). Note that, in order to arrive at the precise form of those equations, we switched the index pairs $\beta_1 \leftrightarrow \beta_2$, $\gamma_1 \leftrightarrow \gamma_2$, and used the symmetries of the vertices, i.e., $M^{\alpha_1\alpha_2\beta_1\beta_2} = -M^{\alpha_1\alpha_2\beta_2\beta_1} = -M^{\alpha_2\alpha_1\beta_1\beta_2}$.

V. THE NONLOCAL COLLISION TERM IN THE KB APPROACH

In this section, we derive the collision term within the KB approach. We first discuss the mass-shell constraint and the Boltzmann-type equation in the semiclassical expansion, i.e., order by order in $\hbar$. We then compute the various collision terms in T-matrix approximation. The advantage of the KB approach as compared to the GLW approach is that full quantum statistics is retained.

A. Equations of motion in semiclassical expansion

We first derive a mass-shell constraint and a Boltzmann-type equation for the single-particle distribution function $\mathfrak{f}(x,p,\mathfrak{s})$ in extended phase space. Taking Eqs. (24a), (24b) for $\Gamma_a = \mathbb{1}$ and $\Gamma_a = \gamma_5\gamma^\mu$, multiplying the latter with $\mathfrak{s}^\mu$, and adding them, we obtain with Eq. (30)

$$\left(p^2 - \frac{\hbar^2}{4}\partial_x^2 - m^2\right)\mathfrak{f}(x,p,\mathfrak{s}) = \frac{1}{2}\mathrm{ReTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{K}+m)I_{\mathrm{coll}}], \tag{102a}$$

$$\hbar p\cdot\partial_x \mathfrak{f}(x,p,\mathfrak{s}) = \frac{1}{2}\mathrm{ImTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{K}+m)I_{\mathrm{coll}}]. \tag{102b}$$

We now expand Eqs. (102) up to second order in $\hbar$, i.e., we need $\mathfrak{f}(x,p,\mathfrak{s})$ up to second order in $\hbar$, cf. Eq. (32). Furthermore, the collision term (20) is already of order $\mathcal{O}(\hbar)$, i.e.,

$$I_{\mathrm{coll}} = \hbar I_{\mathrm{coll}}^{(1)} + \hbar^2 I_{\mathrm{coll}}^{(2)} + \mathcal{O}(\hbar^3). \tag{103}$$

The Wigner functions $G^{\lessgtr}$ and the self-energies $\Sigma^{\gtrless}$ entering the collision term are therefore only required up to order $\mathcal{O}(\hbar)$,

$$G^{\lessgtr}(x,p) = G^{\lessgtr(0)}(x,p) + \hbar G^{\lessgtr(1)}(x,p), \tag{104a}$$

$$\Sigma^{\gtrless}(x,p) = \Sigma^{\gtrless(0)}(x,p) + \hbar\Sigma^{\gtrless(1)}(x,p). \tag{104b}$$

Inserting this into Eq. (20), we obtain

$$I_{\mathrm{coll}}^{(1)} = \frac{i}{2}(\Sigma^{<(0)}G^{>(0)} - \Sigma^{>(0)}G^{<(0)}), \tag{105a}$$

$$I_{\mathrm{coll}}^{(2)} = \Delta I_{\mathrm{coll}}^{(1)} + I_{\mathrm{coll,PB}}^{(0)}, \tag{105b}$$

where

$$\Delta I_{\mathrm{coll}}^{(1)} := \frac{i}{2}(\Sigma^{<(1)}G^{>(0)} - \Sigma^{>(1)}G^{<(0)} + \Sigma^{<(0)}G^{>(1)} - \Sigma^{>(0)}G^{<(1)}), \tag{106a}$$

$$I_{\mathrm{coll,PB}}^{(0)} := \frac{1}{4}(\{\Sigma^{<(0)}, G^{>(0)}\}_{\mathrm{PB}} - \{\Sigma^{>(0)}, G^{<(0)}\}_{\mathrm{PB}}). \tag{106b}$$

1. Zeroth order in $\hbar$

At $\mathcal{O}(\hbar^0)$, the collision term (20) vanishes, $I_{\mathrm{coll}}^{(0)} = 0$, since it is at least of order $\mathcal{O}(\hbar)$. Equation (102b) is trivially fulfilled, while Eq. (102a) becomes

$$\mathcal{O}(\hbar): (p^2 - m^2)\mathfrak{f}^{(0)}(x,p) = 0. \tag{107}$$

This confirms that $\mathfrak{f}^{(0)}(x,p) \sim \delta(p^2-m^2)$, i.e., it is on shell, cf. Eq. (42).

2. First order in $\hbar$

At $\mathcal{O}(\hbar)$, Eqs. (102a) and (102b) become

$$(p^2-m^2)\mathfrak{f}^{(1)}(x,p,\mathfrak{s}) = -\frac{1}{4}\mathrm{ImTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m) \times (\Sigma^{<(0)}G^{>(0)} - \Sigma^{>(0)}G^{<(0)})], \tag{108a}$$

$$p\cdot\partial_x\mathfrak{f}^{(0)}(x,p) = \frac{1}{4}\mathrm{ReTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m) \times (\Sigma^{<(0)}G^{>(0)} - \Sigma^{>(0)}G^{<(0)})]. \tag{108b}$$

3. Second order in $\hbar$

At $\mathcal{O}(\hbar^2)$, we also need to take into account the $\mathcal{O}(\hbar)$ contribution to the operator K^μ, cf. Eq. (19), when computing the right-hand sides of Eqs. (102a) and (102b). We thus obtain

$$(p^2-m^2)\mathfrak{f}^{(2)}(x,p,\mathfrak{s})-\frac{1}{4}\partial_x^2\mathfrak{f}^{(0)}(x,p)=-\frac{1}{8}\mathrm{ReTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\not{\partial}_x(\Sigma^{<(0)}G^{>(0)}-\Sigma^{>(0)}G^{<(0)})]$$
$$-\frac{1}{4}\mathrm{ImTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m)(\Sigma^{<(1)}G^{>(0)}-\Sigma^{>(1)}G^{<(0)}+\Sigma^{<(0)}G^{>(1)}-\Sigma^{>(0)}G^{<(1)})]$$
$$+\frac{1}{8}\mathrm{ReTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m)(\{\Sigma^{<(0)},G^{>(0)}\}_{\mathrm{PB}}-\{\Sigma^{>(0)},G^{<(0)}\}_{\mathrm{PB}})], \tag{109a}$$

$$p\cdot\partial_x\mathfrak{f}^{(1)}(x,p,\mathfrak{s})=-\frac{1}{8}\mathrm{ImTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\not{\partial}_x(\Sigma^{<(0)}G^{>(0)}-\Sigma^{>(0)}G^{<(0)})]$$
$$+\frac{1}{4}\mathrm{ReTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m)(\Sigma^{<(1)}G^{>(0)}-\Sigma^{>(1)}G^{<(0)}+\Sigma^{<(0)}G^{>(1)}-\Sigma^{>(0)}G^{<(1)})]$$
$$+\frac{1}{8}\mathrm{ImTr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m)(\{\Sigma^{<(0)},G^{>(0)}\}_{\mathrm{PB}}-\{\Sigma^{>(0)},G^{<(0)}\}_{\mathrm{PB}})]. \tag{109b}$$

B. Collision terms

1. Self-energies in T-matrix approximation

For binary elastic scattering, the self-energies $\Sigma^{\gtrless}(x,p)$ will be taken in T-matrix approximation, where they are given by the Feynman diagrams shown in Fig. 1,

$$\Sigma^{\gtrless}(x,p)=4\frac{G_cG_d}{\hbar^2}\int\frac{\mathrm{d}^4p_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4p_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4p'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)$$
$$\times\{\mathrm{Tr}[\Gamma^{(d)}G^{\gtrless}(x,p_1)\Gamma^{(c)}G^{\lessgtr}(x,p')]\Gamma^{(d)}G^{\gtrless}(x,p_2)\Gamma^{(c)}-\Gamma^{(d)}G^{\gtrless}(x,p_1)\Gamma^{(c)}G^{\lessgtr}(x,p')\Gamma^{(d)}G^{\gtrless}(x,p_2)\Gamma^{(c)}\}. \tag{110}$$

The first term in the second line of Eq. (110) corresponds to the "direct diagrams" of Figs. 1(a) and 1(b), while the last term corresponds to the "exchange diagrams" of Figs. 1(c) and 1(d). The coupling constants G_c and G_d are associated with the one-boson-exchange interactions in channel (c) or channel (d), respectively. Each vertex carries a factor $\hbar^{-1}$, giving rise to the factor $\hbar^{-2}$ [which was missed in Eq. (131) of Ref. [81]].

2. First order in $\hbar$

To first order in $\hbar$, we need to compute the real and imaginary part of

$$\mathcal{I}_0 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})(\not{p}+m)(\Sigma^{<(0)}G^{>(0)}-\Sigma^{>(0)}G^{<(0)})], \tag{111}$$

cf. Eqs. (108a), (108b). The self-energies $\Sigma^{\lessgtr(0)}$ are given by Eq. (110), with all Wigner functions taken at zeroth order in $\hbar$, i.e., by Eq. (42). Inserting these expressions into $\Sigma^{\lessgtr(0)}$ and the result into the trace (111), we obtain with the cyclicity property of the trace, with Eq. (56), as well as using the idempotency of $\Lambda^+(p)$ the result

$$\mathcal{I}_0=8m^4\frac{G_cG_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)$$
$$\times\int\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)$$
$$\times\mathcal{T}_0[f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)}-\bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}f^{(0)}], \tag{112}$$

where we introduced the abbreviations

$$f_1^{(0)}\coloneqq f^{(0)}(x,p_1),\qquad f_2^{(0)}\coloneqq f^{(0)}(x,p_2),$$
$$f'^{(0)}\coloneqq f^{(0)}(x,p'),\qquad f^{(0)}\coloneqq f^{(0)}(x,p), \tag{113}$$

and similarly for $\bar{f}_1^{(0)}$, $\bar{f}_2^{(0)}$, $\bar{f}'^{(0)}$, and $\bar{f}^{(0)}$, respectively. We also defined

$$\mathcal{T}_0\coloneqq\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]$$
$$\times\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]$$
$$-\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{\mathfrak{s}})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)$$
$$\times\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]. \tag{114}$$

In App. B 1 we prove that, because of the symmetry of the integrand in Eq. (112) under the exchange $p_1^\mu\leftrightarrow p_2^\mu$, only the real part of $\mathcal{T}_0$ contributes, therefore in the following we will set $\mathrm{Im}\mathcal{T}_0=0$ under the integral.

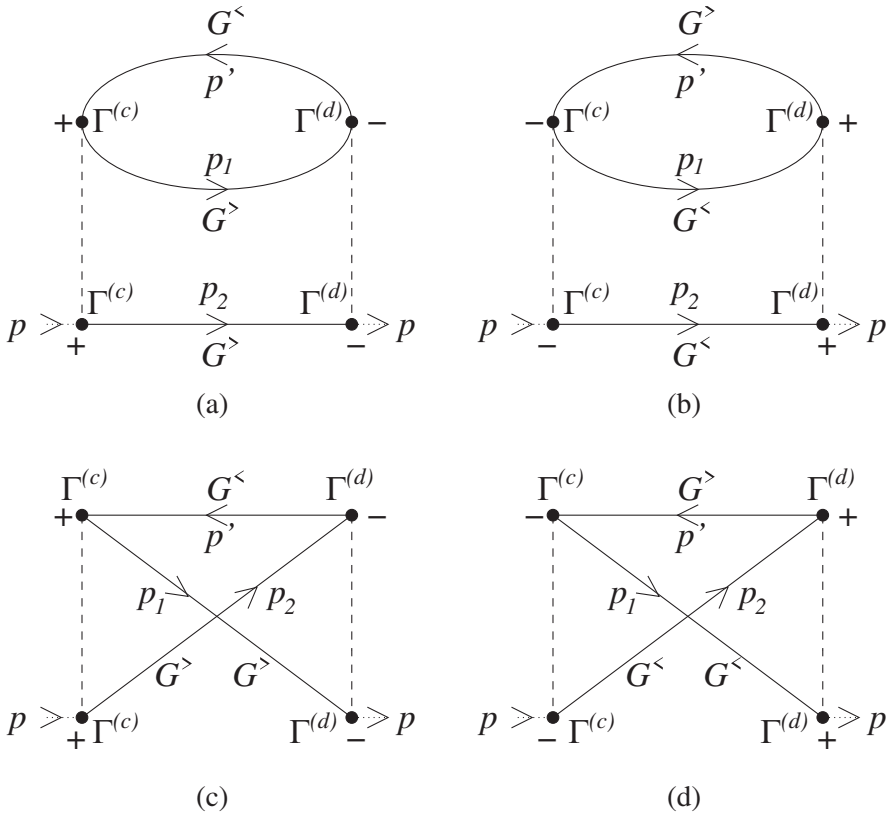

FIG. 1. Feynman diagrams for (a),(c) $\Sigma^{>}(x,p)$ and (b),(d) $\Sigma^{<}(x,p)$. Solid lines represent fermion propagators, dashed lines represent the one-boson-exchange interaction of coupling strength G_c or G_d, respectively. Vertices denote elements $\Gamma^{(c)}$, $\Gamma^{(d)}$ of the Clifford algebra corresponding to the interaction channels (c) or (d), respectively.

Inserting Eq. (112) into Eqs. (108a) and (108b), we obtain with Eq. (41) for the mass-shell constraint and the Boltzmann equation at order $\mathcal{O}(\hbar)$,

$$(p^2 - m^2)\mathfrak{f}^{(1)}(x,p,\mathfrak{s}) = 0, \tag{115a}$$

$$\begin{aligned} p\cdot\partial_x f^{(0)}(x,p) = {} & 2m^4\frac{G_c G_d}{\hbar^2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4 \\ & \times\delta^{(4)}(p+p'-p_1-p_2) \\ & \times \mathrm{Re}\mathcal{T}_0[f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}f^{(0)}]. \end{aligned} \tag{115b}$$

The right-hand side of Eq. (115) vanishes because $\mathrm{Im}\mathcal{T}_0 = 0$. This has the consequence that $\mathfrak{f}^{(1)}$ is on shell, which is consistent with Eq. (54), see also Ref. [79].

In order to facilitate comparison with the result from the GLW approach, we extend the integration on the right-hand side of Eq. (115b) to extended phase space, $\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P' \to \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)$, using the relations (8). Since the integrand does not depend on any of the spin variables $\mathfrak{s}_1^\mu$, $\mathfrak{s}_2^\mu$, $\mathfrak{s}'^\mu$, and $\bar{\mathfrak{s}}^\mu$, we may also extend the definition of $\mathcal{T}_0$ under the integral,

$$\begin{aligned} \mathcal{T}_0 \equiv \mathcal{T} \coloneqq 32\{ & \mathrm{Tr}[h\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}]\mathrm{Tr}[\Gamma^{(d)}h_1\Gamma^{(c)}h'] \\ & - \mathrm{Tr}[h\Gamma^{(d)}h_1\Gamma^{(c)}h'\Gamma^{(d)}h_2\Gamma^{(c)}\bar{h}]\}, \end{aligned} \tag{116}$$

where we used Eq. (5). With this, we obtain from Eq. (115b)

$$\begin{aligned} p\cdot\partial_x f^{(0)}(x,p) = {} & \frac{1}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4 \\ & \times\delta^{(4)}(p+p'-p_1-p_2) \\ & \times\mathcal{W}[f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}f^{(0)}], \end{aligned} \tag{117}$$

where

$$\mathcal{W} \equiv \frac{m^4}{2}\frac{G_c G_d}{\hbar^2}\mathrm{Re}\mathcal{T}, \tag{118}$$

cf. Eq. (4).

3. Second order in $\hbar$

At order $\mathcal{O}(\hbar^2)$, we need to compute three different traces, cf. Eqs. (109a) and (109b). The first one is

$$\mathcal{I}_1 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\not{\partial}_x(\Sigma^{<(0)}G^{>(0)}-\Sigma^{>(0)}G^{<(0)})]. \tag{119}$$

The last term in parentheses under the trace is the same which already occurred in Eq. (111). Accounting for the additional γ_μ matrix in $\not{\partial}_x$ and the fact that the partial derivative in this term acts on all distribution functions appearing in the self-energies $\Sigma^{\lessgtr(0)}$ and in the Wigner functions $G^{\gtrless(0)}$, we readily obtain

$$\begin{aligned}\mathcal{I}_1 = {} & 4m^3\frac{G_cG_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ & \times\mathcal{T}^{(a)}_\mu[(\partial^\mu_x f^{(0)}_1)f^{(0)}_2\bar{f}'^{(0)}\bar{f}^{(0)}-(\partial^\mu_x\bar{f}^{(0)}_1)\bar{f}^{(0)}_2f'^{(0)}f^{(0)}+f^{(0)}_1(\partial^\mu_x f^{(0)}_2)\bar{f}'^{(0)}\bar{f}^{(0)}-\bar{f}^{(0)}_1(\partial^\mu_x\bar{f}^{(0)}_2)f'^{(0)}f^{(0)}\\ & +f^{(0)}_1f^{(0)}_2(\partial^\mu_x\bar{f}'^{(0)})\bar{f}^{(0)}-\bar{f}^{(0)}_1\bar{f}^{(0)}_2(\partial^\mu_x f'^{(0)})f^{(0)}+f^{(0)}_1f^{(0)}_2\bar{f}'^{(0)}(\partial^\mu_x\bar{f}^{(0)})-\bar{f}^{(0)}_1\bar{f}^{(0)}_2f'^{(0)}(\partial^\mu_x f^{(0)})],\end{aligned} \tag{120}$$

where

$$\begin{aligned}\mathcal{T}^{(a)}_\mu \coloneqq {} & \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\gamma_\mu\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]\\ & -\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\gamma_\mu\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}].\end{aligned} \tag{121}$$

The second trace we need to compute is

$$\mathcal{I}_2 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})(\not{p}+m)(\Sigma^{<(1)}G^{>(0)}-\Sigma^{>(1)}G^{<(0)}+\Sigma^{<(0)}G^{>(1)}-\Sigma^{>(0)}G^{<(1)})]. \tag{122}$$

Here,

$$\begin{aligned}\Sigma^{\gtrless(1)}(x,p) = {} & 4\frac{G_cG_d}{\hbar^2}\int\frac{\mathrm{d}^4p_1}{(2\pi\hbar)^4}\frac{\mathrm{d}^4p_2}{(2\pi\hbar)^4}\frac{\mathrm{d}^4p'}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ & \times\{\mathrm{Tr}[\Gamma^{(d)}G^{\gtrless(1)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(0)}(x,p')]\Gamma^{(d)}G^{\gtrless(0)}(x,p_2)\Gamma^{(c)}\\ & -\Gamma^{(d)}G^{\gtrless(1)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(0)}(x,p')\Gamma^{(d)}G^{\gtrless(0)}(x,p_2)\Gamma^{(c)}\\ & +\mathrm{Tr}[\Gamma^{(d)}G^{\gtrless(0)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(1)}(x,p')]\Gamma^{(d)}G^{\gtrless(0)}(x,p_2)\Gamma^{(c)}\\ & -\Gamma^{(d)}G^{\gtrless(0)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(1)}(x,p')\Gamma^{(d)}G^{\gtrless(0)}(x,p_2)\Gamma^{(c)}\\ & +\mathrm{Tr}[\Gamma^{(d)}G^{\gtrless(0)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(0)}(x,p')]\Gamma^{(d)}G^{\gtrless(1)}(x,p_2)\Gamma^{(c)}\\ & -\Gamma^{(d)}G^{\gtrless(0)}(x,p_1)\Gamma^{(c)}G^{\lessgtr(0)}(x,p')\Gamma^{(d)}G^{\gtrless(1)}(x,p_2)\Gamma^{(c)}\}.\end{aligned} \tag{123}$$

According to Eq. (49), each of the Wigner functions $G^{\gtrless(1)}$ contains two terms, a quasiclassical contribution and a gradient contribution. Since $G^{\gtrless(1)}$ appears linearly in all terms in the trace (122), the latter also splits into two parts,

$$\mathcal{I}_2 \equiv \mathcal{I}_{2\mathrm{qc}}+\mathcal{I}_{2\nabla}. \tag{124}$$

The first part, $\mathcal{I}_{2\mathrm{qc}}$, contains the quasiclassical parts $G^{\gtrless(1)}_{\mathrm{qc}}$, and the second part, $\mathcal{I}_{2\nabla}$, contains the gradient parts $G^{\gtrless(1)}_{\nabla}$.

Let us first focus on $\mathcal{I}_{2\mathrm{qc}}$. Inserting $G^{\lessgtr(1)}_{\mathrm{qc}}$ from Eqs. (55) and (57) as well as $G^{\lessgtr(0)}$ from Eq. (42), we obtain

$$\begin{aligned}\mathcal{I}_{2\mathrm{qc}} = {} & \frac{m^4}{2}\frac{G_cG_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)\int\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ & \times\{\mathcal{T}_1[f^{(1)}_1f^{(0)}_2\bar{f}'^{(0)}\bar{f}^{(0)}-\bar{f}^{(1)}_1\bar{f}^{(0)}_2f'^{(0)}f^{(0)}]+\mathcal{T}_2[f^{(0)}_1f^{(1)}_2\bar{f}'^{(0)}\bar{f}^{(0)}-\bar{f}^{(0)}_1\bar{f}^{(1)}_2f'^{(0)}f^{(0)}]\\ & +\mathcal{T}'[f^{(0)}_1f^{(0)}_2\bar{f}'^{(1)}\bar{f}^{(0)}-\bar{f}^{(0)}_1\bar{f}^{(0)}_2f'^{(1)}f^{(0)}]+\bar{\mathcal{T}}[f^{(0)}_1f^{(0)}_2\bar{f}'^{(0)}\bar{f}^{(1)}-\bar{f}^{(0)}_1\bar{f}^{(0)}_2f'^{(0)}f^{(1)}]\},\end{aligned} \tag{125}$$

where we defined

$$\mathcal{T}_1 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}(\mathbb{1}+\gamma_5\not{s}1)\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}(\mathbb{1}+\gamma_5\not{s}_1)\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}], \tag{126a}$$

$$\mathcal{T}_2 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}(\mathbb{1}+\gamma_5\not{s}_2)\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}(\mathbb{1}+\gamma_5\not{s}_2)\Lambda^+(p_2)\Gamma^{(c)}], \tag{126b}$$

$$\mathcal{T}' \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}(\mathbb{1}+\gamma_5\not{s}')\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}(\mathbb{1}+\gamma_5\not{s}')\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}], \tag{126c}$$

$$\bar{\mathcal{T}} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}(\mathbb{1}+\gamma_5\not{\bar{s}})]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}(\mathbb{1}+\gamma_5\not{\bar{s}})], \tag{126d}$$

and introduced the abbreviations

$$f_1^{(1)} \coloneqq f^{(1)}(x,p_1,\mathfrak{s}_1), \qquad f_2^{(1)} \coloneqq f^{(1)}(x,p_2,\mathfrak{s}_2), \qquad f'^{(1)} \coloneqq f^{(1)}(x,p',\mathfrak{s}'), \qquad f^{(1)} \coloneqq f^{(1)}(x,p,\bar{\mathfrak{s}}), \tag{127}$$

and similarly for $\bar{f}_1^{(1)}$, $\bar{f}_2^{(1)}$, $\bar{f}'^{(1)}$, and $\bar{f}^{(1)}$, respectively. Note that, in Eq. (125), we extended the phase-space integration from $\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'$ to $\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)/16$. Because of Eq. (8), this merely inserts a factor of 1 in all terms which do not depend on the respective spin vector. Because of this, we may also extend the definition of the quantities (126) so that, under the extended phase-space integration, all become identical,

$$\mathcal{T}_1 \equiv \mathcal{T}_2 \equiv \mathcal{T}' \equiv \bar{\mathcal{T}} \equiv \mathcal{T}, \tag{128}$$

where we used Eq. (116). Consequently,

$$\begin{aligned}\mathcal{I}_{2\mathrm{qc}} = \frac{m^4}{2}\frac{G_c G_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ \times\mathcal{T}[f_1^{(1)}f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(1)}\bar{f}_2^{(0)}f'^{(0)}f^{(0)} + f_1^{(0)}f_2^{(1)}\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(1)}f'^{(0)}f^{(0)}\\ + f_1^{(0)}f_2^{(0)}\bar{f}'^{(1)}\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(1)}f^{(0)} + f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(1)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}f^{(1)}].\end{aligned} \tag{129}$$

We now consider the gradient part $\mathcal{I}_{2\nabla}$. Inserting $G_\nabla^{\lessgtr(1)}$ from Eqs. (51) and (52) as well as $G^{\lessgtr(0)}$ from Eq. (42), we obtain

$$\begin{aligned}\mathcal{I}_{2\nabla} = 2m^2\frac{G_c G_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ \times\{\mathcal{T}_{1,\mu\nu}p_1^\nu[(\partial_x^\mu f_1^{(0)})f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - (\partial_x^\mu\bar{f}_1^{(0)})\bar{f}_2^{(0)}f'^{(0)}f^{(0)}] + \mathcal{T}_{2,\mu\nu}p_2^\nu[f_1^{(0)}(\partial_x^\mu f_2^{(0)})\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}(\partial_x^\mu\bar{f}_2^{(0)})f'^{(0)}f^{(0)}]\\ + \mathcal{T}'_{\mu\nu}p'^\nu[f_1^{(0)}f_2^{(0)}(\partial_x^\mu\bar{f}'^{(0)})\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_x^\mu f'^{(0)})f^{(0)}] + \mathcal{T}_{\mu\nu}p^\nu[f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}(\partial_x^\mu\bar{f}^{(0)}) - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}(\partial_x^\mu f^{(0)})]\},\end{aligned} \tag{130}$$

where we defined

$$\mathcal{T}_{1,\mu\nu} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\sigma_{\mu\nu}\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\sigma_{\mu\nu}\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}], \tag{131a}$$

$$\mathcal{T}_{2,\mu\nu} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\sigma_{\mu\nu}\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\sigma_{\mu\nu}\Gamma^{(c)}], \tag{131b}$$

$$\mathcal{T}'_{\mu\nu} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\sigma_{\mu\nu}] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\sigma_{\mu\nu}\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}], \tag{131c}$$

$$\mathcal{T}_{\mu\nu} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\sigma_{\mu\nu}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\sigma_{\mu\nu}]. \tag{131d}$$

The third trace we need to compute is

$$\mathcal{I}_3 \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})(\not{p}+m)(\{\Sigma^{<(0)},G^{>(0)}\}_{\mathrm{PB}}-\{\Sigma^{>(0)},G^{<(0)}\}_{\mathrm{PB}})]. \tag{132}$$

For the Poisson-bracket terms, we need

$$\partial_x^\mu G^{<(0)} = 4\pi m\hbar\delta(p^2-m^2)\Lambda^+(p)\partial_x^\mu f^{(0)}, \qquad \partial_p^\mu G^{<(0)} = 4\pi m\hbar\delta(p^2-m^2)\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p)\partial_p^\mu\right]f^{(0)}, \tag{133a}$$

$$\partial_x^\mu G^{>(0)} = 4\pi m\hbar\delta(p^2-m^2)\Lambda^+(p)\partial_x^\mu \bar{f}^{(0)}, \qquad \partial_p^\mu G^{>(0)} = 4\pi m\hbar\delta(p^2-m^2)\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p)\partial_p^\mu\right]\bar{f}^{(0)}, \tag{133b}$$

where we neglected off-shell terms $\sim\delta'(p^2-m^2)$ in the equalities on the right-hand side. We also need

$$\begin{aligned}\partial_x^\mu\Sigma^{<(0)}(x,p) = {}& 4m^3\frac{G_cG_d}{\hbar^2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\{\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}-\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\}\\ &\times[(\partial_x^\mu f_1^{(0)})f_2^{(0)}\bar{f}'^{(0)}+f_1^{(0)}(\partial_x^\mu f_2^{(0)})\bar{f}'^{(0)}+f_1^{(0)}f_2^{(0)}(\partial_x^\mu\bar{f}'^{(0)})],\end{aligned} \tag{134a}$$

$$\begin{aligned}\partial_x^\mu\Sigma^{>(0)}(x,p) = {}& 4m^3\frac{G_cG_d}{\hbar^2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\{\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}-\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\}\\ &\times[(\partial_x^\mu \bar{f}_1^{(0)})\bar{f}_2^{(0)}f'^{(0)}+\bar{f}_1^{(0)}(\partial_x^\mu\bar{f}_2^{(0)})f'^{(0)}+\bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_x^\mu f'^{(0)})],\end{aligned} \tag{134b}$$

$$\begin{aligned}\partial_p^\mu\Sigma^{<(0)}(x,p) = {}& -4m^3\frac{G_cG_d}{\hbar^2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\left(\mathrm{Tr}\left\{\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p')\partial_{p'}^\mu\right]\right\}\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\right.\\ &\left.-\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p')\partial_{p'}^\mu\right]\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\right)f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)},\end{aligned} \tag{134c}$$

$$\begin{aligned}\partial_p^\mu\Sigma^{>(0)}(x,p) = {}& -4m^3\frac{G_cG_d}{\hbar^2}\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\left(\mathrm{Tr}\left\{\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p')\partial_{p'}^\mu\right]\right\}\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\right.\\ &\left.-\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\left[\frac{\gamma^\mu}{2m}+\Lambda^+(p')\partial_{p'}^\mu\right]\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\right)\bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}.\end{aligned} \tag{134d}$$

In Eqs. (134c) and (134d), we used the fact that, because of the energy-momentum conserving delta function, $\partial_p^\mu\equiv\partial_{p'}^\mu$, and then integrated by parts. In the course of the latter, we have neglected off-shell terms $\sim\delta'(p'^2-m^2)$. Inserting Eqs. (133a)–(134d) into Eq. (132), we arrive at

$$\begin{aligned}\mathcal{I}_3 = {}& 4m^3\frac{G_cG_d}{\hbar^2}4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P'(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\{2m\mathcal{T}_0[(\partial_x^\mu f_1^{(0)})f_2^{(0)}\bar{f}'^{(0)}(\partial_\mu^p\bar{f}^{(0)})-(\partial_x^\mu\bar{f}_1^{(0)})\bar{f}_2^{(0)}f'^{(0)}(\partial_\mu^p f^{(0)})+f_1^{(0)}(\partial_x^\mu f_2^{(0)})\bar{f}'^{(0)}(\partial_\mu^p\bar{f}^{(0)})-\bar{f}_1^{(0)}(\partial_x^\mu\bar{f}_2^{(0)})f'^{(0)}(\partial_\mu^p f^{(0)})\\ &+f_1^{(0)}f_2^{(0)}(\partial_x^\mu\bar{f}'^{(0)})(\partial_\mu^p\bar{f}^{(0)})-\bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_x^\mu f'^{(0)})(\partial_\mu^p f^{(0)})+f_1^{(0)}f_2^{(0)}(\partial_{p'}^\mu\bar{f}'^{(0)})(\partial_\mu^x\bar{f}^{(0)})-\bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_{p'}^\mu f'^{(0)})(\partial_\mu^x f^{(0)})]\\ &+\mathcal{T}_\mu^{(b)}[(\partial_x^\mu f_1^{(0)})f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)}-(\partial_x^\mu\bar{f}_1^{(0)})\bar{f}_2^{(0)}f'^{(0)}f^{(0)}+f_1^{(0)}(\partial_x^\mu f_2^{(0)})\bar{f}'^{(0)}\bar{f}^{(0)}-\bar{f}_1^{(0)}(\partial_x^\mu\bar{f}_2^{(0)})f'^{(0)}f^{(0)}\\ &+f_1^{(0)}f_2^{(0)}(\partial_x^\mu\bar{f}'^{(0)})\bar{f}^{(0)}-\bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_x^\mu f'^{(0)})f^{(0)}]+\mathcal{T}_\mu^{(c)}[f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}(\partial_x^\mu\bar{f}^{(0)})-\bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}(\partial_x^\mu f^{(0)})]\},\end{aligned} \tag{135}$$

where we have used Eq. (114) and defined

$$\mathcal{T}_\mu^{(b)} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\gamma_\mu]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] - \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\gamma_\mu], \tag{136a}$$

$$\mathcal{T}_\mu^{(c)} \coloneqq \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu] - \mathrm{Tr}[(\mathbb{1}+\gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]. \tag{136b}$$

In Appendix B 2 we prove that, under an integral of the same type as in Eq. (135), $\mathcal{T}_\mu^{(b)} \equiv \mathcal{T}_\mu^{(a)*}$, while in Appendix B 3 we prove that under the same type of integral the imaginary part of $\mathcal{T}_\mu^{(c)}$ vanishes.

Inserting Eqs. (120), (124) with (129) and (130), and (135) into Eqs. (109a) and (109b), and using the identities of Appendix B, we obtain

$$\begin{aligned}
&(p^2-m^2)\mathfrak{f}^{(2)}(x,p,\mathfrak{s}) - \pi m\hbar\delta(p^2-m^2)\partial_x^2 f^{(0)}(x,p)\\
&= -\frac{m^3}{16}\frac{G_c G_d}{\hbar^2} 4\pi m\hbar\delta(p^2-m^2)\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\
&\quad\times\{[2m\mathrm{Im}\mathcal{T}f_1^{(1)} + \mathrm{Re}X_{1,\mu}(\partial_x^\mu f_1^{(0)})]f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - [2m\mathrm{Im}\mathcal{T}\bar{f}_1^{(1)} + \mathrm{Re}X_{1,\mu}(\partial_x^\mu \bar{f}_1^{(0)})]\bar{f}_2^{(0)}f'^{(0)}f^{(0)}\\
&\quad + f_1^{(0)}[2m\mathrm{Im}\mathcal{T}f_2^{(1)} + \mathrm{Re}X_{2,\mu}(\partial_x^\mu f_2^{(0)})]\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}[2m\mathrm{Im}\mathcal{T}\bar{f}_2^{(1)} + \mathrm{Re}X_{2,\mu}(\partial_x^\mu \bar{f}_2^{(0)})]f'^{(0)}f^{(0)}\\
&\quad + f_1^{(0)}f_2^{(0)}[2m\mathrm{Im}\mathcal{T}\bar{f}'^{(1)} + \mathrm{Re}X'_\mu(\partial_x^\mu \bar{f}'^{(0)})]\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}[2m\mathrm{Im}\mathcal{T}f'^{(1)} + \mathrm{Re}X'_\mu(\partial_x^\mu f'^{(0)})]f^{(0)}\\
&\quad + f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}[2m\mathrm{Im}\mathcal{T}\bar{f}^{(1)} + \mathrm{Re}X_\mu(\partial_x^\mu \bar{f}^{(0)})] - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}[2m\mathrm{Im}\mathcal{T}f^{(1)} + \mathrm{Re}X_\mu(\partial_x^\mu f^{(0)})]\\
&\quad - m\mathrm{Re}\mathcal{T}_0[(\partial_x^\mu f_1^{(0)})f_2^{(0)}\bar{f}'^{(0)}(\partial_\mu^p \bar{f}^{(0)}) - (\partial_x^\mu \bar{f}_1^{(0)})\bar{f}_2^{(0)}f'^{(0)}(\partial_\mu^p f^{(0)}) + f_1^{(0)}(\partial_x^\mu f_2^{(0)})\bar{f}'^{(0)}(\partial_\mu^p \bar{f}^{(0)}) - \bar{f}_1^{(0)}(\partial_x^\mu \bar{f}_2^{(0)})f'^{(0)}(\partial_\mu^p f^{(0)})\\
&\quad + f_1^{(0)}f_2^{(0)}(\partial_x^\mu \bar{f}'^{(0)})(\partial_\mu^p \bar{f}^{(0)}) - \bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_x^\mu f'^{(0)})(\partial_\mu^p f^{(0)}) + f_1^{(0)}f_2^{(0)}(\partial_{p'}^\mu \bar{f}'^{(0)})(\partial_\mu^x \bar{f}^{(0)}) - \bar{f}_1^{(0)}\bar{f}_2^{(0)}(\partial_{p'}^\mu f'^{(0)})(\partial_\mu^x f^{(0)})]\}, \qquad (137a)
\end{aligned}$$

$$\begin{aligned}
p\cdot\partial_x f^{(1)}(x,p,\mathfrak{s}) &= \frac{m^3}{16}\frac{G_c G_d}{\hbar^2}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\\
&\quad\times\{[2m\mathrm{Re}\mathcal{T}f_1^{(1)} - \mathrm{Im}X_{1,\mu}(\partial_x^\mu f_1^{(0)})]f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - [2m\mathrm{Re}\mathcal{T}\bar{f}_1^{(1)} - \mathrm{Im}X_{1,\mu}(\partial_x^\mu \bar{f}_1^{(0)})]\bar{f}_2^{(0)}f'^{(0)}f^{(0)}\\
&\quad + f_1^{(0)}[2m\mathrm{Re}\mathcal{T}f_2^{(1)} - \mathrm{Im}X_{2,\mu}(\partial_x^\mu f_2^{(0)})]\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}[2m\mathrm{Re}\mathcal{T}\bar{f}_2^{(1)} - \mathrm{Im}X_{2,\mu}(\partial_x^\mu \bar{f}_2^{(0)})]f'^{(0)}f^{(0)}\\
&\quad + f_1^{(0)}f_2^{(0)}[2m\mathrm{Re}\mathcal{T}\bar{f}'^{(1)} - \mathrm{Im}X'_\mu(\partial_x^\mu \bar{f}'^{(0)})]\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}[2m\mathrm{Re}\mathcal{T}f'^{(1)} - \mathrm{Im}X'_\mu(\partial_x^\mu f'^{(0)})]f^{(0)}\\
&\quad + f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}[2m\mathrm{Re}\mathcal{T}\bar{f}^{(1)} - \mathrm{Im}X_\mu(\partial_x^\mu \bar{f}^{(0)})] - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}[2m\mathrm{Re}\mathcal{T}f^{(1)} - \mathrm{Im}X_\mu(\partial_x^\mu f^{(0)})]\}, \qquad (137b)
\end{aligned}$$

where we have used Eq. (54), employed $\mathrm{Im}\mathcal{T}_0 \equiv 0$ under the $\mathrm{d}P_1\mathrm{d}P_2$ integral, and defined

$$X_{1,\mu} \coloneqq \frac{1}{2}[\mathcal{T}_\mu^{(a)} - \mathcal{T}_\mu^{(a)*}] - \frac{i}{2m}\mathcal{T}_{1,\mu\nu}p_1^\nu = i\left[\mathrm{Im}\mathcal{T}_\mu^{(a)} - \frac{1}{2m}\mathcal{T}_{1,\mu\nu}p_1^\nu\right], \tag{138a}$$

$$X_{2,\mu} \coloneqq \frac{1}{2}[\mathcal{T}_\mu^{(a)} - \mathcal{T}_\mu^{(a)*}] - \frac{i}{2m}\mathcal{T}_{2,\mu\nu}p_2^\nu = i\left[\mathrm{Im}\mathcal{T}_\mu^{(a)} - \frac{1}{2m}\mathcal{T}_{2,\mu\nu}p_2^\nu\right], \tag{138b}$$

$$X'_\mu \coloneqq \frac{1}{2}[\mathcal{T}_\mu^{(a)} - \mathcal{T}_\mu^{(a)*}] - \frac{i}{2m}\mathcal{T}'_{\mu\nu}p'^\nu = i\left[\mathrm{Im}\mathcal{T}_\mu^{(a)} - \frac{1}{2m}\mathcal{T}'_{\mu\nu}p'^\nu\right], \tag{138c}$$

$$X_\mu \coloneqq \frac{1}{2}[\mathcal{T}_\mu^{(a)} - \mathcal{T}_\mu^{(c)}] - \frac{i}{2m}\mathcal{T}_{\mu\nu}p^\nu \equiv \mathrm{Re}\mathcal{T}_\mu^{(a)} - \frac{1}{2}\mathcal{T}_\mu^{(c)} - \frac{p_\mu}{2m}\mathcal{T}_0. \tag{138d}$$

In Eqs. (138a)–(138d), we used the fact that $\mathcal{T}_\mu^{(b)} \equiv \mathcal{T}_\mu^{(a)*}$ under the $\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2$ integral. For the second equality in Eq. (138d) we employed Eq. (B19). Note that X_μ is purely real under the $\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2$ integral.

In the following, we focus on the Boltzmann equation (137b) and define

$$\bar{\Delta}_{1,\mu} := -\frac{\mathrm{Im}X_{1,\mu}}{2m\mathrm{Re}\mathcal{T}} \equiv \frac{1}{2m\mathrm{Re}\mathcal{T}}\left[\frac{1}{2m}\mathrm{Re}\mathcal{T}_{1,\mu\nu}p_1^\nu - \mathrm{Im}\mathcal{T}_\mu^{(a)}\right] \equiv \frac{1}{\hbar}(\Delta_{1,\mu}-\Delta_\mu), \tag{139a}$$

$$\bar{\Delta}_{2,\mu} := -\frac{\mathrm{Im}X_{2,\mu}}{2m\mathrm{Re}\mathcal{T}} \equiv \frac{1}{2m\mathrm{Re}\mathcal{T}}\left[\frac{1}{2m}\mathrm{Re}\mathcal{T}_{2,\mu\nu}p_2^\nu - \mathrm{Im}\mathcal{T}_\mu^{(a)}\right] \equiv \frac{1}{\hbar}(\Delta_{2,\mu}-\Delta_\mu), \tag{139b}$$

$$\bar{\Delta}'_{\mu} := -\frac{\mathrm{Im}X'_{\mu}}{2m\mathrm{Re}\mathcal{T}} \equiv \frac{1}{2m\mathrm{Re}\mathcal{T}}\left[\frac{1}{2m}\mathrm{Re}\mathcal{T}'_{\mu\nu}p'^{\nu} - \mathrm{Im}\mathcal{T}_\mu^{(a)}\right] \equiv \frac{1}{\hbar}(\Delta'_{\mu}-\Delta_\mu), \tag{139c}$$

$$\bar{\Delta}_{\mu} := -\frac{\mathrm{Im}X_{\mu}}{2m\mathrm{Re}\mathcal{T}} \equiv 0. \tag{139d}$$

All identities right after the definitions hold under the integral. The final identities, which relate the barred Δ's on the left-hand sides to the space-time shifts (3) on the right-hand sides are proven in Appendix B 5, using identities from Appendix B 4.

Then, with Eq. (118) the Boltzmann equation (137b) reads

$$\begin{aligned} p\cdot\partial_x f^{(1)}(x,p,\mathfrak{s}) = \frac{1}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\mathcal{W} \\ \times\{[f_1^{(1)}+\bar{\Delta}_{1,\mu}(\partial_x^\mu f_1^{(0)})]f_2^{(0)}\bar{f}'^{(0)}\bar{f}^{(0)} - [\bar{f}_1^{(1)}+\bar{\Delta}_{1,\mu}(\partial_x^\mu \bar{f}_1^{(0)})]\bar{f}_2^{(0)}f'^{(0)}f^{(0)} \\ + f_1^{(0)}[f_2^{(1)}+\bar{\Delta}_{2,\mu}(\partial_x^\mu f_2^{(0)})]\bar{f}'^{(0)}\bar{f}^{(0)} - \bar{f}_1^{(0)}[\bar{f}_2^{(1)}+\bar{\Delta}_{2,\mu}(\partial_x^\mu \bar{f}_2^{(0)})]f'^{(0)}f^{(0)} \\ + f_1^{(0)}f_2^{(0)}[\bar{f}'^{(1)}+\bar{\Delta}'_{\mu}(\partial_x^\mu \bar{f}'^{(0)})]\bar{f}^{(0)} - \bar{f}_1^{(0)}\bar{f}_2^{(0)}[f'^{(1)}+\bar{\Delta}'_{\mu}(\partial_x^\mu f'^{(0)})]f^{(0)} \\ + f_1^{(0)}f_2^{(0)}\bar{f}'^{(0)}[\bar{f}^{(1)}+\bar{\Delta}_{\mu}(\partial_x^\mu \bar{f}^{(0)})] - \bar{f}_1^{(0)}\bar{f}_2^{(0)}f'^{(0)}[f^{(1)}+\bar{\Delta}_{\mu}(\partial_x^\mu f^{(0)})]\}. \end{aligned} \tag{140}$$

C. Summary

Expanding $f(x+\hbar\bar{\Delta},p,\mathfrak{s})$ in powers of $\hbar$ around $f^{(0)}(x,p)$,

$$f(x+\hbar\bar{\Delta},p,\mathfrak{s}) = f^{(0)}(x,p) + \hbar f^{(1)}(x,p,\mathfrak{s}) + \hbar\bar{\Delta}_\mu\partial_x^\mu f^{(0)}(x,p) + \mathcal{O}(\hbar^2), \tag{141}$$

and similarly for f_1, f_2, f', $\bar{f}$, $\bar{f}_1$, $\bar{f}_2$, and $\bar{f}'$, we now combine the results (117), (139), and (140) to write the complete Boltzmann equation for $f(x,p,\mathfrak{s})$ up to first order in $\hbar$ as,

$$\begin{aligned} p\cdot\partial_x f(x,p,\mathfrak{s}) = \frac{1}{4}\int \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(p)(2\pi\hbar)^4\delta^{(4)}(p+p'-p_1-p_2)\mathcal{W} \\ \times[f(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2)\bar{f}(x+\Delta'-\Delta,p',\mathfrak{s}')\bar{f}(x,p,\bar{\mathfrak{s}}) \\ -\bar{f}(x+\Delta_1-\Delta,p_1,\mathfrak{s}_1)\bar{f}(x+\Delta_2-\Delta,p_2,\mathfrak{s}_2)f(x+\Delta'-\Delta,p',\mathfrak{s}')f(x,p,\bar{\mathfrak{s}})] + \mathcal{O}(\hbar^2). \end{aligned} \tag{142}$$

This result agrees with that of the GLW approach, Eq. (1), in the limit of Boltzmann statistics, where $\bar{f},\bar{f}',\bar{f}_1,\bar{f}_2\to 1$, and generalizes it to the case of quantum statistics.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have revisited the nonlocal collision term derived in Refs. [41,73]. In those works, the nonlocality of the collision term manifested itself by certain space-time shifts [given by Eq. (2) for a particle with momentum p^μ and spin vector $\mathfrak{s}^\mu$] of the collision partners. However, the explicit dependence of these space-time shifts on the frame vector $\hat{t}^\mu=(1,\mathbf{0})$ violates Lorentz covariance. In this work, we restored Lorentz covariance by carefully recomputing the collision term in the GLW approach [83], and confirmed the result by a calculation within the KB [89] approach. This results in the more complicated, but manifestly Lorentz-covariant expressions (3) for the space-time shifts.

In future work, one should repeat the derivation of spin hydrodynamics along the lines of Ref. [66,71] with the

Lorentz-covariant space-time shifts (3). For such a calculation one cannot use the space-time shifts (3) directly, since the trace $\mathcal{T}$ appears in the denominator, which depends on the spin variables, cf. Eq. (116). However, in order to perform the integrations over spin space, the latter should appear in the numerator, where one can apply the relations (8). One therefore needs to resort to the form of the nonlocal collision term as given, e.g., in Eq. (137b). Here, both $\mathcal{T}$ and the first-order distribution functions are linear in the spin variables and thus appear under the integral in a form where the relations (8) are applicable. In Appendix C we give the trace terms required for such a calculation for scalar boson exchange as a simple example.

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APPENDIX A: PREFACTOR IN EQS. (29) AND (41)

In this Appendix, we convince ourselves of the correctness of the prefactor in Eqs. (29) and (41). To this end, consider the expression for the canonical energy-momentum tensor in terms of the vector component of the Wigner function,

$$T^{\mu\nu} = \int \frac{\mathrm{d}^4 p}{(2\pi\hbar)^4} p^\nu \mathcal{V}^\mu. \tag{A1}$$

At order $\mathcal{O}(\hbar^0)$, $\mathcal{V}^{(0)\mu} = p^\mu \mathcal{F}^{(0)}/m$, cf. Eq. (34), and we obtain with Eqs. (39) and (41),

$$\begin{aligned} T^{\mu\nu} &= 2\int \frac{\mathrm{d}^4 p}{(2\pi\hbar)^4} p^\nu \frac{p^\mu}{m} 4\pi m\hbar\delta(p^2 - m^2) f^{(0)}(x,p) \\ &= 2\int \mathrm{d}P p^\mu p^\nu f^{(0)}(x,p). \end{aligned} \tag{A2}$$

This is the standard expression for the energy-momentum tensor for a non-interacting gas in kinetic theory. In equilibrium, $f^{(0)}$ is the Fermi-Dirac distribution function, $f^{(0)}_{\mathrm{eq}} = [e^{(u\cdot p-\mu)/T} + 1]^{-1}$. The overall factor 2 counts the two spin degrees of freedom. Thus, Eqs. (29a) and (41) are correct.

APPENDIX B: PROPERTIES OF TRACE TERMS

In this appendix, we prove a set of identities for the traces $\mathcal{T}_0$, $\mathcal{T}^{(a)}_\mu$, $\mathcal{T}^{(b)}_\mu$, $\mathcal{T}^{(c)}_\mu$, $\mathcal{T}_{1,\mu\nu}p_1^\nu$, $\mathcal{T}_{2,\mu\nu}p_2^\nu$, $\mathcal{T}'_{\mu\nu}p'^\nu$, and $\mathcal{T}_{\mu\nu}p^\nu$, respectively. These identities make use of the symmetry properties of the $\mathrm{d}P_1\mathrm{d}P_2$ integrals under which these various traces appear.

1. $\mathcal{T}_0$ is real

We prove that, under an integral of the form

$$\int \mathrm{d}P_1 \mathrm{d}P_2 \mathcal{T}_0 f(p_1,p_2), \tag{B1}$$

where $f(p_1,p_2) = f(p_2,p_1)$, the imaginary part of $\mathcal{T}_0$ can be set to zero. To this end, we compute

$$\begin{aligned} \mathcal{T}_0^* &= \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]^* \mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]^* \\ &\quad - \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]^* \\ &= \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger\Gamma^{(d)\dagger}\Lambda^+(p)^\dagger(\mathbb{1} + \gamma_5 \not{s})^\dagger]\mathrm{Tr}[\Lambda^+(p')^\dagger\Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger\Gamma^{(d)\dagger}] \\ &\quad - \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger\Gamma^{(d)\dagger}\Lambda^+(p')^\dagger\Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger\Gamma^{(d)\dagger}\Lambda^+(p)^\dagger(\mathbb{1} + \gamma_5 \not{s})^\dagger]. \end{aligned} \tag{B2}$$

Using $\Lambda(p)^\dagger = \gamma^0\Lambda^+(p)\gamma^0$ [and similarly for $\Lambda^+(p_1)$, $\Lambda^+(p_2)$, and $\Lambda^+(p')$], $(\mathbb{1} + \gamma_5 \not{s})^\dagger = \gamma^0(\mathbb{1} + \gamma_5 \not{s})\gamma^0$, as well as $\Gamma^{(c)\dagger} = \gamma^0\Gamma^{(c)}\gamma^0$ we obtain

$$\begin{aligned} \mathcal{T}_0^* &= \mathrm{Tr}[\Gamma^{(c)}\Lambda^+(p_2)\Gamma^{(d)}\Lambda^+(p)(\mathbb{1} + \gamma_5 \not{s})]\mathrm{Tr}[\Lambda^+(p')\Gamma^{(c)}\Lambda^+(p_1)\Gamma^{(d)}] \\ &\quad - \mathrm{Tr}[\Gamma^{(c)}\Lambda^+(p_2)\Gamma^{(d)}\Lambda^+(p')\Gamma^{(c)}\Lambda^+(p_1)\Gamma^{(d)}\Lambda^+(p)(\mathbb{1} + \gamma_5 \not{s})]. \end{aligned} \tag{B3}$$

We now exchange the summation indices $c \leftrightarrow d$ in both terms, which is possible, since the prefactor $\sim G_c G_d$ of the collision integral is also symmetric under this exchange. Under an integral of the form (B1), we are also allowed to exchange the integration variables $p_1^\mu \leftrightarrow p_2^\mu$ in the second term, because $f(p_1,p_2)$ is symmetric under this exchange. Finally, using Eq. (56) and the cyclic property of the trace, we arrive at ac

$$\mathcal{T}_0^* \equiv \mathcal{T}_0, \tag{B4}$$

which proves that the imaginary part of $\mathcal{T}_0$ vanishes under the integral (B1).

2. $\mathcal{T}_\mu^{(b)}$ is complex conjugate of $\mathcal{T}_\mu^{(a)}$

We prove that, under an integral of the form

$$\int \mathrm{d}P_1 \mathrm{d}P_2 \mathcal{T}_\mu^{(b)} f(p_1, p_2), \tag{B5}$$

where $f(p_1, p_2) = f(p_2, p_1)$, we may set $\mathcal{T}_\mu^{(b)} \equiv \mathcal{T}_\mu^{(a)*}$. From Eq. (121) we compute

$$\begin{aligned}
\mathcal{T}_\mu^{(a)*} &= \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\gamma_\mu \Gamma^{(d)} \Lambda^+(p_2)\Gamma^{(c)}]^* \mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]^* \\
&\quad - \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\gamma_\mu \Gamma^{(d)} \Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]^* \\
&= \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger \Gamma^{(d)\dagger}\gamma_\mu^\dagger \Lambda^+(p)^\dagger (\mathbb{1} + \gamma_5 \not{s})^\dagger] \mathrm{Tr}[\Lambda^+(p')^\dagger \Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger \Gamma^{(d)\dagger}] \\
&\quad - \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger \Gamma^{(d)\dagger}\Lambda^+(p')^\dagger \Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger \Gamma^{(d)\dagger}\gamma_\mu^\dagger \Lambda^+(p)^\dagger (\mathbb{1} + \gamma_5 \not{s})^\dagger].
\end{aligned} \tag{B6}$$

Performing similar steps as in Appendix B 1, because $f(p_1, p_2)$ in the integral (B5) is symmetric under $p_1^\mu \leftrightarrow p_2^\mu$, we then show that

$$\begin{aligned}
\mathcal{T}_\mu^{(a)*} &= \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\gamma_\mu] \mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] \\
&\quad - \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}\gamma_\mu] \equiv \mathcal{T}_\mu^{(b)},
\end{aligned} \tag{B7}$$

cf. Eq. (136).

3. $\mathcal{T}_\mu^{(c)}$ is real

We show that, under an integral

$$\int \mathrm{d}P_1 \mathrm{d}P_2 \mathcal{T}_\mu^{(c)} f(p_1, p_2), \tag{B8}$$

where $f(p_1, p_2) = f(p_2, p_1)$, the imaginary part of $\mathcal{T}_\mu^{(c)}$ vanishes. To this end, we compute

$$\begin{aligned}
\mathcal{T}_\mu^{(c)*} &= \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]^* \mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu]^* \\
&\quad - \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu \Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]^* \\
&= \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger \Gamma^{(d)\dagger}\Lambda^+(p)^\dagger (\mathbb{1} + \gamma_5 \not{s})^\dagger] \mathrm{Tr}[\gamma_\mu^\dagger \Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger \Gamma^{(d)\dagger}] \\
&\quad - \mathrm{Tr}[\Gamma^{(c)\dagger}\Lambda^+(p_2)^\dagger \Gamma^{(d)\dagger}\gamma_\mu^\dagger \Gamma^{(c)\dagger}\Lambda^+(p_1)^\dagger \Gamma^{(d)\dagger}\Lambda^+(p)^\dagger (\mathbb{1} + \gamma_5 \not{s})^\dagger].
\end{aligned} \tag{B9}$$

Employing similar steps as in Appendix B 1, we obtain

$$\begin{aligned}
\mathcal{T}_\mu^{(c)*} &= \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}] \mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu] \\
&\quad - \mathrm{Tr}[(\mathbb{1} + \gamma_5 \not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu \Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}] \equiv \mathcal{T}_\mu^{(c)}.
\end{aligned} \tag{B10}$$

4. Identities for $\mathcal{T}_{1,\mu\nu}p_1^\nu$, $\mathcal{T}_{2,\mu\nu}p_2^\nu$, $\mathcal{T}'_{\mu\nu}p'^\nu$, and $\mathcal{T}_{\mu\nu}p^\nu$

We first show that, under an integral of the form

$$\int \mathrm{d}P_1 \mathrm{d}P_2 \mathcal{T}'_{\mu\nu} p'^\nu f(p_1, p_2), \tag{B11}$$

where $f(p_1, p_2) = f(p_2, p_1)$, $\mathcal{T}'_{\mu\nu}p'^\nu$ assumes the form

$$\begin{aligned}\mathcal{T}'_{\mu\nu}p'^\nu \equiv 2m\mathrm{Im}\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\gamma_\mu] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\gamma_\mu\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\}.\end{aligned} \tag{B12}$$

To this end, note that $\sigma_{\mu\nu}p'^\nu = \frac{i}{2}[\gamma_\mu, \not{p}'] = im[\gamma_\mu, \Lambda^+(p')]$. Therefore, Eq. (131c) (multiplied by p'^ν) can be written as

$$\begin{aligned}\mathcal{T}'_{\mu\nu}p'^\nu = im\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu\Lambda^+(p')] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\gamma_\mu\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\gamma_\mu] \\ &+ \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\gamma_\mu\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\}.\end{aligned} \tag{B13}$$

Using similar steps as in Appendix B 1, we exploit the symmetry under the integral (B11) to show that the terms in the first two lines in Eq. (B13) are the complex conjugates of the terms in the last two lines. This proves Eq. (B12).

Next, we show that, under an integral of the form

$$\int \mathrm{d}P_1 \mathrm{d}P_2 [\mathcal{T}_{1,\mu\nu}p_1^\nu f(p_1, p_2) + \mathcal{T}_{2,\mu\nu}p_2^\nu g(p_1, p_2)], \tag{B14}$$

with real-valued functions $f(p_1, p_2)$, $g(p_1, p_2)$ which fulfill $f(p_1, p_2) = g(p_2, p_1)$, we may use the following expressions for $\mathcal{T}_{1,\mu\nu}p_1^\nu$ and $\mathcal{T}_{2,\mu\nu}p_2^\nu$,

$$\begin{aligned}\mathcal{T}_{1,\mu\nu}p_1^\nu \equiv 2m\mathrm{Im}\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\},\end{aligned} \tag{B15a}$$

$$\begin{aligned}\mathcal{T}_{2,\mu\nu}p_2^\nu \equiv 2m\mathrm{Im}\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\}.\end{aligned} \tag{B15b}$$

The proof follows similar steps as that of Eq. (B12). We first write $\sigma_{\mu\nu}p_{1,2}^\nu = \frac{i}{2}[\gamma_\mu, \not{p}_{1,2}] = im[\gamma_\mu, \Lambda^+(p_{1,2})]$ and insert this into Eq. (131a) (multiplied by p_1^ν) and Eq. (131b) (multiplied by p_2^ν), respectively,

$$\begin{aligned}\mathcal{T}_{1,\mu\nu}p_1^\nu = im\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\gamma_\mu\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\gamma_\mu\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')] \\ &+ \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\},\end{aligned} \tag{B16a}$$

$$\begin{aligned}\mathcal{T}_{2,\mu\nu}p_2^\nu = im\{&\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\gamma_\mu\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\gamma_\mu\Lambda^+(p_2)\Gamma^{(c)}] \\ &- \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')] \\ &+ \mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\}.\end{aligned} \tag{B16b}$$

Multiplying $\mathcal{T}_{1,\mu\nu}p_1^\nu$ with $f(p_1,p_2)$ and $\mathcal{T}_{2,\mu\nu}p_2^\nu$ with $g(p_1,p_2)$, and taking the sum, we then prove that, under the integral (B14) and using $f(p_1,p_2)=g(p_2,p_1)$, the complex conjugate of terms resulting from the last two lines in Eqs. (B16a) and (B16b) is

$$\begin{aligned}
&\{\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')]\\
&\quad-\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\gamma_\mu\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\}^* f(p_1,p_2)\\
&\quad+\{\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]\\
&\quad-\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\gamma_\mu\Gamma^{(c)}]\}^* g(p_1,p_2)\\
&=\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\gamma_\mu\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]f(p_1,p_2)\\
&\quad-\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\gamma_\mu\Lambda^+(p_2)\Gamma^{(c)}]g(p_1,p_2)\\
&\quad+\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\gamma_\mu\Lambda^+(p_2)\Gamma^{(c)}]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]g(p_1,p_2)\\
&\quad-\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\gamma_\mu\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}]f(p_1,p_2).
\end{aligned}\tag{B17}$$

This shows that, under the integral (B14) and in the combination $\mathcal{T}_{1,\mu\nu}p_1^\nu f(p_1,p_2)+\mathcal{T}_{2,\mu\nu}p_2^\nu g(p_1,p_2)$, the terms in the first and second lines of Eqs. (B16a), (B16b) are (up to a minus sign) the complex conjugates of the terms in the third and fourth lines. This completes the proof of Eqs. (B15a), (B15b).

Finally, we compute

$$\begin{aligned}
\frac{i}{2m}\mathcal{T}_{\mu\nu}p^\nu=&\frac{1}{2m}\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}i\sigma_{\mu\nu}p^\nu]\mathrm{Tr}[\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')]\\
&-\frac{1}{2m}\mathrm{Tr}[(\mathbb{1}+\gamma_5\not{s})\Lambda^+(p)\Gamma^{(d)}\Lambda^+(p_1)\Gamma^{(c)}\Lambda^+(p')\Gamma^{(d)}\Lambda^+(p_2)\Gamma^{(c)}i\sigma_{\mu\nu}p^\nu].
\end{aligned}\tag{B18}$$

Using $i\sigma_{\mu\nu}p^\nu=-\gamma_\mu\not{p}+p_\mu$, the cyclicity of the trace, the relation $\not{p}(\mathbb{1}+\gamma_5\not{s})=(\mathbb{1}+\gamma_5\not{s})\not{p}$, which holds since $p\cdot s=0$, and the identity $\not{p}\Lambda^+(p)=m\Lambda^+(p)$, which is valid for on-shell particles, we arrive with Eqs. (114) and (136) at

$$\frac{i}{2m}\mathcal{T}_{\mu\nu}p^\nu=\frac{p_\mu}{2m}\mathcal{T}_0-\frac{1}{2}\mathcal{T}_\mu^{(b)}\equiv\frac{p_\mu}{2m}\mathcal{T}_0-\frac{1}{2}\mathcal{T}_\mu^{(a)*}.\tag{B19}$$

The last identity uses the result of Appendix B 2, which is possible since the term on the left-hand side appears under an integral of the type (B5), cf., e.g., Eq. (B137b).

5. Relation between traces and space-time shifts

Consider the traces in Eqs. (114), (121), (131), and (136). Under the $\mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'$ integral, we can replace the energy projectors by the quantities defined in Eq. (5) as, e.g., $\Lambda^+(p_1)\equiv 2h(p_1,s_1)$, since the additional terms vanish because of Eq. (8). Employing this for $\Lambda^+(p_1),\Lambda^+(p_2)$, and $\Lambda^+(p')$, we find with Eqs. (B4), (B7), (B10), (B12), (B15a), (B15b), and (B19),

$$\mathrm{Im}\mathcal{T}_\mu^{(a)}\equiv-\mathrm{Im}\mathcal{T}_\mu^{(b)}\equiv 4m\frac{\hbar^2}{G_cG_d}\frac{\mathcal{W}}{m^4}\frac{\Delta_\mu}{\hbar},\tag{B20a}$$

$$\mathrm{Im}\mathcal{T}_\mu^{(c)}\equiv 0,\tag{B20b}$$

$$\frac{1}{2m}\mathrm{Re}\mathcal{T}_{1,\mu\nu}p_1^\nu\equiv 4m\frac{\hbar^2}{G_cG_d}\frac{\mathcal{W}}{m^4}\frac{\Delta_{1,\mu}}{\hbar},\tag{B20c}$$

$$\frac{1}{2m}\mathrm{Re}\mathcal{T}_{2,\mu\nu}p_2^\nu\equiv 4m\frac{\hbar^2}{G_cG_d}\frac{\mathcal{W}}{m^4}\frac{\Delta_{2,\mu}}{\hbar},\tag{B20d}$$

$$\frac{1}{2m}\mathrm{Re}\mathcal{T}'_{\mu\nu}p'^\nu\equiv 4m\frac{\hbar^2}{G_cG_d}\frac{\mathcal{W}}{m^4}\frac{\Delta'_\mu}{\hbar},\tag{B20e}$$

$$\frac{1}{2m}\mathrm{Re}\mathcal{T}_{\mu\nu}p^\nu\equiv 2m\frac{\hbar^2}{G_cG_d}\frac{\mathcal{W}}{m^4}\frac{\Delta_\mu}{\hbar},\tag{B20f}$$

with the space-time shifts (3) and $\mathcal{W}$ from Eq. (4). Using Eq. (118), we immediately prove Eqs. (139).

APPENDIX C: TRACES FOR SCALAR BOSON EXCHANGE

In this appendix, we evaluate the various traces occurring in the collision term for scalar boson exchange, $\Gamma^{(c)}\equiv\mathbb{1}$. Using the Mandelstam variables

$$\begin{aligned}
s&:=(p+p')^2=(p_1+p_2)^2,\\
t&:=(p-p_1)^2=(p'-p_2)^2,\\
u&:=(p-p_2)^2=(p'-p_1)^2,
\end{aligned}\tag{C1}$$

with $s+t+u=4m^2$, Eq. (114) becomes

$$\mathcal{T}_0 = \frac{(s+t)(2s+t)}{8m^4} - \frac{s-t}{2m^2} + \frac{i}{m^3}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\mu p^\nu p_1^\alpha p_2^\beta. \quad \text{(C2)}$$

We denote the momentum p^μ in the C.M. frame as $p_\star^\mu := (E_\star, \mathbf{p}_\star)$, with $E_\star := \sqrt{p_\star^2 + m^2}$, where $p_\star := |\mathbf{p}_\star|$. It then follows that $p_\star'^\mu \equiv (E_\star, -\mathbf{p}_\star)$. Furthermore, we define $p_{1\star}^\mu := (E_\star, \mathbf{p}_{1\star})$, such that $p_{2,\star}^\mu \equiv (E_\star, -\mathbf{p}_{1\star})$. Here we have used that, in the c.m. frame, $|\mathbf{p}_\star| = |\mathbf{p}_{1\star}| \equiv p_\star$, such that the on-shell energies of all particles are equal. Introducing the scattering angle $\Theta = \measuredangle(\mathbf{p}_\star, \mathbf{p}_{1\star})$, we have $s = 4(m^2 + p_\star^2)$, $t = -2p_\star^2(1-\cos\Theta)$, $u = -2p_\star^2(1+\cos\Theta)$, and therefore

$$\mathrm{Re}\mathcal{T}_0 = 2\left[1 + \frac{p_\star^2}{m^2}(1+2\cos\Theta) + \frac{p_\star^4}{m^4}(1+\cos\Theta)(3+\cos\Theta)\right]. \quad \text{(C3)}$$

The low-energy limit is therefore $\lim_{p_\star\to 0}\mathrm{Re}\mathcal{T}_0 = 2$, and the limit of small scattering angles is $\lim_{\Theta\to 0}\mathrm{Re}\mathcal{T}_0 = 2(1 + 3\frac{p_\star^2}{m^2} + \frac{5p_\star^4}{4m^4})$.

Similarly, we evaluate the traces (126). This is simplified by taking the difference of these traces and $\mathcal{T}_0$. Using energy-momentum conservation multiple times to eliminate the dependence on p'^μ, we find

$$\begin{aligned}\mathcal{T}_1 - \mathcal{T}_0 = {}& \mathfrak{s}\cdot\mathfrak{s}_1\left[\frac{s-t}{2m^2} + \frac{t(s+t)}{8m^4}\right] \\ & - \frac{\mathfrak{s}\cdot p_1}{m}\left(\frac{u}{4m^2}\frac{\mathfrak{s}_1\cdot p}{m} + \frac{\mathfrak{s}_1\cdot p_2}{m}\right) \\ & - \frac{\mathfrak{s}\cdot p_2 \mathfrak{s}_1\cdot p}{m^2} + \frac{i}{m^3}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}_1^\mu p^\nu p_1^\alpha p_2^\beta, \end{aligned} \quad \text{(C4a)}$$

$$\begin{aligned}\mathcal{T}_2 - \mathcal{T}_0 = {}& \mathfrak{s}\cdot\mathfrak{s}_2\left[\frac{s-t}{2m^2} - \frac{(2s+t)(s+t)}{8m^4}\right] \\ & + \frac{\mathfrak{s}\cdot p_2}{m}\left(\frac{2s+t}{4m^2}\frac{\mathfrak{s}_2\cdot p}{m} - \frac{\mathfrak{s}_2\cdot p_1}{m}\right) \\ & - \frac{\mathfrak{s}\cdot p_1 \mathfrak{s}_2\cdot p}{m^2} + \frac{i}{m^3}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}_2^\mu p^\nu p_1^\alpha p_2^\beta, \end{aligned} \quad \text{(C4b)}$$

$$\begin{aligned}\mathcal{T}' - \mathcal{T}_0 = {}& \mathfrak{s}\cdot\mathfrak{s}'\left[\frac{s-t}{2m^2} + \frac{t(s+t)}{8m^4}\right] \\ & - \frac{\mathfrak{s}\cdot p_1}{m}\left(\frac{\mathfrak{s}'\cdot p}{m} + \frac{u}{4m^2}\frac{\mathfrak{s}'\cdot p_2}{m}\right) \\ & - \frac{\mathfrak{s}\cdot p_2}{m}\left(\frac{\mathfrak{s}'\cdot p}{m} + \frac{t}{4m^2}\frac{\mathfrak{s}'\cdot p_1}{m}\right) \\ & + \frac{i}{m^3}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}'^\mu p^\nu p_1^\alpha p_2^\beta, \end{aligned} \quad \text{(C4c)}$$

$$\begin{aligned}\bar{\mathcal{T}} - \mathcal{T}_0 = {}& \mathfrak{s}\cdot\bar{\mathfrak{s}}\left[\frac{s-t}{2m^2} - \frac{(2s+t)(s+t)}{8m^4}\right] + \frac{\mathfrak{s}\cdot p_1 \bar{\mathfrak{s}}\cdot p_2}{m^2} \\ & - \frac{\mathfrak{s}\cdot p_2 \bar{\mathfrak{s}}\cdot p_1}{m^2} + \frac{i}{m^3}\epsilon_{\mu\nu\alpha\beta}\bar{\mathfrak{s}}^\mu p^\nu p_1^\alpha p_2^\beta. \end{aligned} \quad \text{(C4d)}$$

We note that the last two terms in the first line of Eq. (C4d) are antisymmetric under the exchange of $p_1 \leftrightarrow p_2$ and thus vanish under a $\mathrm{d}P_1\mathrm{d}P_2$ integral where the remainder of the integrand is symmetric under this exchange.

In the low-energy limit, i.e., where the c.m. momentum $\mathbf{p}_\star \to 0$, one can show that all four-products between spin vectors and momenta vanish. This is most easily seen in the c.m. frame and using the orthogonality of the spin four-vector with the four-momentum. Also, in this limit all momenta just have a time component, e.g., $p^\mu \to (m, \mathbf{0})$, and similarly for the other momenta. Then, all imaginary parts in Eqs. (C4) vanish. This then yields

$$\lim_{p_\star\to 0}(\mathcal{T}_1 - \mathcal{T}_0) = 2\mathfrak{s}\cdot\mathfrak{s}_1, \quad \text{(C5a)}$$

$$\lim_{p_\star\to 0}(\mathcal{T}_2 - \mathcal{T}_0) = -2\mathfrak{s}\cdot\mathfrak{s}_2, \quad \text{(C5b)}$$

$$\lim_{p_\star\to 0}(\mathcal{T}' - \mathcal{T}_0) = 2\mathfrak{s}\cdot\mathfrak{s}', \quad \text{(C5c)}$$

$$\lim_{p_\star\to 0}(\bar{\mathcal{T}} - \mathcal{T}_0) = -2\mathfrak{s}\cdot\bar{\mathfrak{s}}. \quad \text{(C5d)}$$

Similarly, we compute from Eqs. (131)

$$\begin{aligned}\mathcal{T}_{1,\mu}p_1^\nu = {}& \epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot p_1}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{2m^2} + \left(1 + \frac{t}{4m^2}\right)\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{m} \right. \\ & \left. + \frac{t-u}{4m^2}\frac{\mathfrak{s}^\nu p^\alpha p_1^\beta}{m}\right] \\ & + i\left[\left(1 - \frac{3s}{4m^2}\right)p_\mu + \frac{s-t}{4m^2}p_{1,\mu} \right. \\ & \left. + \left(1 - \frac{3t}{4m^2}\right)p_{2,\mu}\right], \end{aligned} \quad \text{(C6a)}$$

$$\begin{aligned}\mathcal{T}_{2,\mu}p_2^\nu = {}& -\epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot p_2}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{2m^2} + \left(1 + \frac{u}{4m^2}\right)\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{m} \right. \\ & \left. + \frac{t-u-4(s+t)}{4m^2}\frac{\mathfrak{s}^\nu p^\alpha p_2^\beta}{m}\right] \\ & - i\left[\left(1 - \frac{3s}{4m^2}\right)p_\mu + \left(1 - \frac{3u}{4m^2}\right)p_{1,\mu} \right. \\ & \left. + \frac{s-u}{4m^2}p_{2,\mu}\right], \end{aligned} \quad \text{(C6b)}$$

$$\mathcal{T}'_{\mu\nu}p'^{\nu} = -\epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot(p_1-p_2)}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{2m^2} + \frac{t-u}{4m^2}\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{m} + \left(1+\frac{t}{4m^2}\right)\frac{\mathfrak{s}^\nu p^\alpha p_1^\beta}{m} + \left(1+\frac{u}{4m^2}\right)\frac{\mathfrak{s}^\nu p^\alpha p_2^\beta}{m}\right] - i\left(\frac{u-t}{4m^2}p_\mu + \frac{s-t}{4m^2}p_{1,\mu} + \frac{u-s}{4m^2}p_{2,\mu}\right), \quad \text{(C6c)}$$

and from Eq. (121)

$$\mathcal{T}^{(a)}_\mu = \left(\frac{s+3t}{4m^2}\frac{p_\mu}{m} - \frac{s+t}{4m^2}\frac{p_{1,\mu}}{m} + \frac{s}{2m^2}\frac{p_{2,\mu}}{m}\right) - i\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\nu\left(\frac{p_1^\alpha p_2^\beta}{m^2} - \frac{u}{4m^2}\frac{p^\alpha p_1^\beta}{m^2} + \frac{s+t-u}{4m^2}\frac{p^\alpha p_2^\beta}{m^2}\right), \quad \text{(C7)}$$

as well as from Eq. (136b)

$$\mathcal{T}^{(c)}_\mu = -\left[\frac{u+t}{4m^2}\frac{p_\mu}{m} - \frac{3(s+t)}{4m^2}\frac{p_{1,\mu}}{m} + \frac{s+u}{4m^2}\frac{p_{2,\mu}}{m}\right] + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\nu\left(\frac{p_1^\alpha p_2^\beta}{m^2} - \frac{p^\alpha p_1^\beta}{m^2} + \frac{p^\alpha p_2^\beta}{m^2}\right). \quad \text{(C8)}$$

Note that some terms in these expression can be further simplified using the symmetry under the $\mathrm{d}P_1\mathrm{d}P_2$ integral.

From these results we finally compute the space-time shifts using Eqs. (139),

$$\Delta_{1,\mu} = \frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\frac{1}{2m}\mathrm{Re}\mathcal{T}_{1,\mu\nu}p_1^\nu = \frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot p_1}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{4m^3} + \left(1+\frac{t}{4m^2}\right)\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{2m^2} + \frac{t-u}{4m^2}\frac{\mathfrak{s}^\nu p^\alpha p_1^\beta}{2m^2}\right], \quad \text{(C9a)}$$

$$\Delta_{2,\mu} = \frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\frac{1}{2m}\mathrm{Re}\mathcal{T}_{2,\mu\nu}p_2^\nu = -\frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot p_2}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{4m^3} + \left(1+\frac{u}{4m^2}\right)\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{2m^2} + \frac{t-u-4(s+t)}{4m^2}\frac{\mathfrak{s}^\nu p^\alpha p_2^\beta}{2m^2}\right], \quad \text{(C9b)}$$

$$\Delta'_\mu = \frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\mathrm{Re}\mathcal{T}'_{\mu\nu}p'^{\nu} = -\frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\epsilon_{\mu\nu\alpha\beta}\left[\frac{\mathfrak{s}\cdot(p_1-p_2)}{m}\frac{p^\nu p_1^\alpha p_2^\beta}{4m^3} + \frac{t-u}{4m^2}\frac{\mathfrak{s}^\nu p_1^\alpha p_2^\beta}{2m^2} + \left(1+\frac{t}{4m^2}\right)\frac{\mathfrak{s}^\nu p^\alpha p_1^\beta}{2m^2} + \left(1+\frac{u}{4m^2}\right)\frac{\mathfrak{s}^\nu p^\alpha p_2^\beta}{2m^2}\right], \quad \text{(C9c)}$$

$$\Delta_\mu = \frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\mathrm{Im}\mathcal{T}^{(a)}_\mu = -\frac{\hbar}{2m\mathrm{Re}\mathcal{T}}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\nu\left(\frac{p_1^\alpha p_2^\beta}{m^2} - \frac{u}{4m^2}\frac{p^\alpha p_1^\beta}{m^2} + \frac{s+t-u}{4m^2}\frac{p^\alpha p_2^\beta}{m^2}\right). \quad \text{(C9d)}$$

In the low-energy limit, the spacetime shifts vanish.

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