Constructing generic effective field theory for all masses and spins

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We solve the long-standing problem of operator basis construction for fields with all masses and spins. Based on the on-shell method, we propose a novel method to systematically construct a complete set of lowest dimensional amplitude bases at any given dimension through semistandard Young tableaus of Lorentz subgroup $SU(2)_r$ and global symmetry $U(N)$ (N is the number of external legs), which can be directly mapped into physical operator bases. We first construct a complete set of independent monomial bases whose dimension is not the lowest and a redundant set of bases that always contains a complete set of amplitude bases with the lowest dimension. Then we decompose the bases of the redundant set into the complete monomial bases from low to high dimension and eliminate the linear correlation bases. Finally, the bases with the lowest dimension can be picked up. We also propose a matrix projection method to construct the massive amplitude bases involving identical particles. The operator bases of a generic massive effective field theory can be efficiently constructed by the computer programs. A complete set of four-vector operators at dimensions up to six is presented.

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I. INTRODUCTION

Effective field theory (EFT) of massive fields is widely applied in particle physics, such as lower energy QCD [1–5], Higgs EFT (HEFT) [6,7], dark matter EFT [8–12], and low energy EFT [13]. Compared with massless EFTs, massive EFTs have many advantages in new physics (NP) study. For example, HEFT can fully describe the IR effects of the NP models in which electroweak symmetry is nonlinearly realized, but standard model EFT (SMEFT) cannot [6,7]. Massive EFT is more convenient for studying IR effects of NP theory at the electroweak symmetry breaking (EWSB) phase. For example, there is no field normalization issue, and generally, low point (three- and four-point) operator bases are enough for most low-energy phenomenology studies.

Constructing a generic massive EFT is still a long-standing problem. A complete set of EFT bases is essential to fully categorize and parametrize the infrared (IR) effects of any ultraviolet (UV) theory. Nevertheless, constructing independent EFT bases is challenging in traditional field theory because of operator redundancy from the equation of motion (EOM) and integration by part (IBP).

On-shell scattering amplitude is efficient in dealing with some problems of EFT, such as calculating the running of EFT operators [14–20], deriving EFT selecting rules [21–23], and constructing scalar EFT with nontrivial soft-limit [24–27]. Especially it is very efficient in constructing EFT bases of massless fields (called amplitude bases) [20,28–31]. A complete set of the amplitude bases without the EOM and IBP redundancy can be systematically constructed by the semistandard Young tableaus (SSYTs) of the global symmetry of massless spinors [32] (more applications can be found in [33,34]).

However, this method is not applicable in constructing amplitude bases for massive fields. In massive EFT, besides the issues of EOM and IBP redundancy, the dimension of massive amplitude bases should be minimized so that they can describe the leading IR effects of a UV theory (some primary explorations can be found in [35–38].). Therefore, we propose a novel method based on our previous work [39] to systematically construct a complete set of amplitude bases with the lowest dimension. We first construct a complete set of amplitude bases through the SSYTs of Lorentz subgroup $SU(2)_r$ ($SO(3,1) \simeq SU(2)_l \times SU(2)_r$) and global symmetry $U(N)$ (N is the number of external particles), and then systematically construct an over redundant set of amplitude bases that always contains a complete set of amplitude bases with the lowest dimension based on polarization tensor classification. Then we decompose the bases of this redundant set from low to high dimension into the simplified amplitude bases and eliminate the linear correlation bases. Finally, the complete set of bases with the lowest dimension can be picked up. We also prove that the leading order decomposition without including the terms containing mass factors is enough to determine the independence of bases. So the decomposition can be very efficient, and a complete set of operator bases at any dimension can be easily constructed.

Within the framework of this theory, we propose a matrix projection method to get the amplitude bases involving identical particles. Instead of constructing the gauge structure and kinematic part separately [33], we act with the matrix representation of the Young operator on a complete set of amplitude bases, and the bases satisfying Bose/Fermi statistics can be projected out. Based on our theory, we write a *Mathematica* code that can construct a complete set of massive amplitude bases at any dimension and explicitly list all bases of four massive vectors at dimension-four and six.

The paper is organized as follows. Section II, briefly introduces the spinors and LG symmetry of on-shell scattering amplitudes. Section III demonstrates in detail how to construct systematically massive amplitude bases and gives some examples. Section IV explains how to construct the simplified amplitude bases and prove their independence and completeness. Section V illustrates how to decompose a polynomial of spinor products into a complete set of simplified amplitude bases. Section VI discusses how to systematically construct the redundant set of amplitude bases that contains a complete set of amplitude bases with minimal dimension and gives an example to explain how to obtain the minimal dimension bases. Section VII briefly discusses how to construct amplitude bases involving identical particles. We conclude in Sec. VIII. The appendices explain how to deal with identical particles and show some proofs. We also list all the independent operator bases of four vectors at dimension-four and six.

II. MASSLESS AND MASSIVE SPINORS

In this section, we briefly discuss the massive and massless spinors and the basic property of on-shell scattering amplitudes. For a particle-i, its momentum can be written as a product of two spinors [40,41],

$$(p_i)_{\dot{\alpha}\alpha} \equiv (p_i)_\mu (\sigma^\mu)_{\dot{\alpha}\alpha} = |i^I]_{\dot{\alpha}} \langle i_I|_\alpha,$$
$$(p_i)^{\alpha\dot{\alpha}} \equiv (p_i)_\mu (\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} = |i^I]^\alpha \langle i_I|^{\dot{\alpha}}, \tag{1}$$

where $\sigma^\mu(\bar{\sigma}^\mu) \equiv \{1, (-)\sigma^i\}$, σ^i is Pauli matrices, and the right-handed and left-handed spinors $|i^I]_{\dot{\alpha}}$ and $|i^I\rangle_\alpha$ are in the fundamental representation of Lorentz subgroup $SU(2)_r$ and $SU(2)_l$ respectively [the Lorentz group $SO(3,1)$ is isomorphic to $SU(2)_l \otimes SU(2)_r$)]. I is the index of little group (LG) $SU(2)_i$ for massive particle-i [$I = 1$, 2 is summed over in Eq. (1)]. For the massless spinor, its index I is trivial and can be neglected. The massive right-handed and left-handed spinors $|i^I]_{\dot{\alpha}}$ and $|i^I\rangle_\alpha$ are both in the fundamental representation of massive LG $SU(2)_i$. The massless right-handed and left-handed spinors $|i]_{\dot{\alpha}}$ and $|i\rangle_\alpha$ take $+$ and $-$ unit charge of massless LG $U(1)_i$. Two spinors with same chirality can contract their Lorentz indices with the two-index Levi-Civita tensor $\epsilon^{\dot{\alpha}\dot{\beta}}$ ($\epsilon^{\alpha\beta}$) to form a Lorentz singlet, and the spinor products of different chiral spinors usually are denoted by square spinor bracket $[ij]$ and angle spinor bracket $\langle ij\rangle$,

$$[ij]^{IJ} \equiv \epsilon^{\dot{\alpha}\dot{\beta}} |i^I]_{\dot{\beta}} |j^J]_{\dot{\alpha}}, \qquad \langle ij\rangle^{IJ} \equiv \epsilon^{\alpha\beta} |i^I\rangle_\beta |j^J\rangle_\alpha. \tag{2}$$

The left- and right-handed massive spinors can be related to each other through EOM,

$$p_i|i^I] = m_i|i^I\rangle, \qquad p_i|i^I\rangle = -m_i|i^I]. \tag{3}$$

The on-shell scattering amplitudes are the functions of the angle and square brackets (also include the gauge structures determined by gauge symmetry of external legs). For an external massive particle-i with spin s_i, its scattering amplitude should be in the $2s_i$ indices symmetric representation of $SU(2)_i$ (i.e., $(2s_i + 1)$-dimension representation). For example, the amplitude of a massive vector particle-i should transform under LG $SU(2)_i$ as [41]

$$\mathcal{M}^{\{I_1,I_2\}}(w^i_{II'}|i^{I'}], w^i_{II'}|i^{I'}\rangle, \ldots)$$
$$= w^i_{I_1I'_1} w^i_{I_2I'_2} \mathcal{M}^{\{I'_1,I'_2\}}(|i^{I'}], |i^{I'}\rangle, \ldots), \tag{4}$$

where w^i is LG $SU(2)_i$ element and the superscript bracket $\{I^i_1, \ldots, I^i_{2s_i}\}$ means that these $2s_i$ indices of $SU(2)_i$ should be totally symmetric. For the external massless particle-i with helicity h_i, its amplitudes should take $2h_i$ charge of massless LG $U(1)_i$ [40].

III. CONSTRUCTING AMPLITUDE BASIS

According to the LG transformation of the scattering amplitude, its general structure can be factorized into two parts: the massive little group tensor structure (MLGTS) and the massive little group neutral structure (MLGNS), which are charged and neutral under LGs of massive particles, respectively. This indicates that to get the complete amplitude bases, we can separately construct the complete sets of MLGTS bases and the corresponding MLGNS bases and then contract them. We find that MLGTS and MLGNS bases can be constructed completely through the Lorentz subgroup $SU(2)_r$ and a $U(N)$ global symmetry, respectively (N is the number of external particles) without the EOM and IBP redundancy. In this section, we will discuss how to construct these two parts systematically.

A. Massive LG tensor structures

As said before, the scattering amplitudes of m massive and n massless particles can be factorized into MLGTS part $\mathcal{A}$ and MLGNS part G,

$$\mathcal{M}^I_{m,n} = \sum_{\{\dot\alpha\}} \mathcal{A}^{\{I\}}_{\{\dot\alpha\}}(\{\epsilon_i\}) G^{\{\dot\alpha\}}(|j],|j\rangle,p_i), \tag{5}$$

where $\epsilon_i \equiv |i]^{\{I_1}_{\dot\alpha_1},\ldots,|i]^{I_{2s_i}\}}_{\dot\alpha_{2s_i}}$ is the polarization tensor of massive particle-i with spin s_i, $\{I\}$ generally denotes the LG indices and $\{\dot\alpha\}$ collectively denote Lorentz indices. The LG indices of ϵ_i are required to be total symmetric, so its quantum number under $SU(2)_i \otimes SU(2)_r$ is $(2s_i+1, 2s_i+1)$. Since the massive left and right handed spinors are related by the EOMs in Eq. (3), the MLGTS $\mathcal{A}$ can always be made the holomorphic function of right-handed massive spinors $|i^I]$ without losing generality. Since the $\mathcal{A}$ is in the same massive LG representation as polarization ϵ_i, $\mathcal{A}$ should be the linear function of ϵ_i. Moreover, since all the spinors $|i^I]$ in $\mathcal{A}$ are totally symmetric, two $|i^I]$s cannot be contracted together by antisymmetric Levi-Civita tensor. So $\mathcal{A}$ is free of the EOM redundancy. It is also free of the IBP redundancy because it does not contain any momentum. The MLGNS $G(|j],|j\rangle,p_i)$ is only charged under massless LG and neutral under massive LG so it is the function of massless spinors ($|j]$ and $|j\rangle$) and massive momentums (p_i).

Since $\mathcal{A}$ is linear in each polarization tensor, all the MLGTS bases must belong to the outer product of all the polarization tensors' $SU(2)_r$ representations, $\mathcal{A} \subset \otimes_{i=1}^m (\mathbf{2s_i+1})$. A complete set of $\mathcal{A}$ bases can be constructed by finding all the possible $SU(2)_r$ irreducible representations decomposed from this $\otimes_{i=1}^m (\mathbf{2s_i+1})$ representation. Next, we will discuss in detail how to systematically construct the complete MLGTS bases.

As said before, the quantum number of massive right-handed spinor $|i^I]_{\dot\alpha}$ under $SU(2)_i \otimes SU(2)_r$ is (2,2), represented by Young diagram (YD) as

$$|i^I]_{\dot\alpha} = \boxed{i}_{\,i} \otimes \boxed{i}_{\,r}, \tag{6}$$

where we use i filled in the box to label YD of spinor $|i^I]_{\dot\alpha}$ and the subscript i or r to label $SU(2)_i$ or $SU(2)_r$ YD. Thus any YD of massive LGs and $SU(2)_r$ corresponds to a holomorphic function of right-handed spinors and can be written down according to the group indices permutation symmetry in the YD. For example, two massive right-handed spinors product can be read from the following YD of the group $SU(2)_i \otimes SU(2)_j \otimes SU(2)_r$,

$$\boxed{i}_{\,i} \otimes \boxed{j}_{\,j} \otimes \begin{array}{|c|}\hline i\\ \hline j\\ \hline\end{array}_{\,r} = |i^I]_{\dot\alpha}|j^J]_{\dot\beta} - |i^I]_{\dot\beta}|j^J]_{\dot\alpha} = [i^I j^J]. \tag{7}$$

The massive polarization tensor ϵ_i in the $(2s_i+1, 2s_i+1)$ representation of $SU(2)_i \otimes SU(2)_r$ can be read from the direct product of two YDs with one row and $2s_i$ columns,

$$\underbrace{\boxed{i}\cdots\boxed{i}}_{(2s_i)}{}_{i} \otimes \underbrace{\boxed{i}\cdots\boxed{i}}_{(2s_i)}{}_{r} = |i^{\{I_1}]_{\dot\alpha_1}\cdots|i^{I_{2s_i}\}}]_{\dot\alpha_{2s_i}} = \epsilon_i\,. \tag{8}$$

Since the LG indices of these m polarization tensors in $\mathcal{A}$ should be bare, only the different contraction patterns of their $SU(2)_r$ indices can give the different structures of $\mathcal{A}$. So MLGTS can be classified by Lorentz $SU(2)_r$ irreducible representations of these polarization tensors. To construct the complete MLGTS $\mathcal{A}$, we just need to find all the irreducible $SU(2)_r$ representations decomposed from the outer product of these m massive polarizations' YDs in Eq. (8), based on Littlewood-Richardson rule. Then, we can read out the expressions of $\mathcal{A}$ bases from the YDs according to $SU(2)_r$ indices permutation symmetry.

Next, we will demonstrate how to use YD to construct MLGTS $\mathcal{A}$ bases. Take the 4-point vertices of massive fermion-fermion-vector-scalar $\psi\psi' Vh$ as an example. As said before, the massive polarization tensor ϵ_i is in the $(\mathbf{2s_i+1})$ representation of $SU(2)_r$, so the $SU(2)_r$ YDs of these particles' polarizations are

$$\psi \sim \boxed{1} \quad \psi' \sim \boxed{2} \quad Z \sim \boxed{3}\boxed{3} \quad h \sim \bullet, \tag{9}$$

where the number in the box is used to label the $SU(2)_r$ indices of different polarization tensors, and the bullet • represents the $SU(2)_r$ singlet. Then we can reduce the outer product of these $4SU(2)_r$ tensors to the irreducible representations by Littlewood-Richardson rule and get four representations,

$$\begin{aligned} &\boxed{1}\otimes\boxed{2}\otimes\boxed{3}\boxed{3}\otimes\bullet \\ &= \begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 3\\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline 1&3&3\\ \hline 2\\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\ \hline\end{array} \\ &\equiv \mathcal{A}^I_{[2,2]} \oplus \mathcal{A}^I_{[(3,1)^1]} \oplus \mathcal{A}^I_{[(3,1)^2]} \oplus \mathcal{A}^I_{[4]}\,, \end{aligned} \tag{10}$$

where the subscript $[r_1, r_2, \ldots, r_n]$ denotes the shape of the YD, having n rows and r_j boxes at the jth row, and superscripts in $[(3,1)^{1,2}]$ denote two different SSYTs in the same shape. Then the complete MLGTS bases can be read out from the above YDs filled with numbers. For example, according to the $SU(2)_r$ index permutation symmetry of the YD, the first base $\mathcal{A}^I_{[2,2]}$ is given by

$$\begin{aligned}\mathcal{A}^I_{[2,2]} &\equiv \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array} \\ &= (|1^I]_{\dot\alpha}|2^J]_{\dot\beta}|3^{K_1}]_{\dot\gamma_1}|3^{K_2}]_{\dot\gamma_2} + \text{perms in } SU(2)_r \text{ indices}) \\ &= [1^I 3^{\{K_1}][2^J 3^{K_2\}}].\end{aligned} \tag{11}$$

We find that the amplitude bases $\mathcal{A}\cdot G$ with MLGTS bases in different $SU(2)_r$ representations must be independent. Because if the MLGTSs of any two amplitude bases are in different $SU(2)_r$ YD representations, they must be two linearly independent tensors of massive LGs $\otimes_{i=1}^m SU(2)_i$. So, no matter the $SU(2)_r$ representations of their G bases, they must be two independent $SU(2)_r$ singlet states.

We can also use group theory to systematically construct the complete independent G bases by embedding the spinors of the $N\equiv m+n$ external legs into a global symmetry $U(N)$ representation. Similar to the above discussion, G can be classified by $U(N)$ representation. Each G corresponds to a basis of a particular $U(N)$ representation, so all the G are independent. Also, this method can systematically get rid of the redundancy from the EOM and IBP. In the next subsection, we will discuss them in detail.

B. Massive LG neutral structures

The amplitude bases of massless particles can automatically get rid of the EOM redundancy because any basis containing this redundancy should vanish ($\not{p}_j|j]=0$). So the only issue for constructing massless amplitude bases is to remove the IBP redundancy, which is systematically solved by YD method. However, the situation is different when constructing on-shell massive EFT. Since the EOM of massive spinors is not as trivial as the massless spinor, $p_i|i^I]=m_i|i^I\rangle$. A redundant on-shell basis can not only be expressed as the combination of the independent bases with the same dimension through the IBP (i.e., momentum conservation) but also the combination of lower-dimensional bases multiplied by the mass factors through the EOM. So both redundancies from the EOM and IBP should be removed in constructing on-shell massive EFTs. Since the MLGTS basis is the linear function of massive polarization tensor, having these two kinds of redundancy is impossible. Thus only MLGNS basis can suffer from the EOM and IBP redundancy. This subsection will explain how to get rid of them systematically.

We first discuss how to remove the EOM redundancy in massive amplitude bases. Since MLGNS G is neutral under massive LGs $\otimes_{i=1}^m SU(2)_i$ and charged under massless LG $\otimes_{j=m+1}^N U(1)_j$, it must be the polynomial function of massless spinors $|j]$ or $|j\rangle$ with $j=m+1,\ldots,m+n$, and massive momentum $p_{i,\dot\alpha\alpha}\equiv|i^I]_{\dot\alpha}\langle i_I|_\alpha$ with $i=1,\ldots,m$. As discussed above, the massless amplitude basis is automatically free of the EOM redundancy, so it is possible to get rid of it if we first construct the complete set of massless limit bases of MLGNT G and then construct the G bases from them. Since $G(|j],|j\rangle,p_i)$ is massive LG singlet, one $G(|j],|j\rangle,p_i)$ will smoothly go to a definite massless limit if all massive momentums go to massless limit,

$$p_{i,\dot\alpha\alpha}\to|i]_{\dot\alpha}\langle i|_\alpha : G(|j],|j\rangle,p_i)\to g(|j],|j\rangle,|i]\langle i|), \tag{12}$$

where $|i]_{\dot\alpha}\langle i|_\alpha$ is the massless limit of massive momentum $p_{i,\alpha\dot\alpha}$ and $g\equiv G(|j],|j\rangle,|i]\langle i|)$ is the limit of G when $p_{i,\dot\alpha\alpha}\to|i]_{\dot\alpha}\langle i|_\alpha$. We know that the difference between two different G, which are related to each other through EOM, must be proportional to terms with mass factors, so their massless limits must be the same. Conversely, the massless spinor polynomial bases $\{g\}$ can be one-to-one mapped to the independent MLGNS bases $\{G\}$ without the EOM redundancy just through replacing massless limit spinors $|i]$ and $|i\rangle$ with the corresponding massive spinors $|i^I]$ and $|i_{I'}\rangle$ and choosing one pattern of LG index contraction between $|i^I]$ s and $|i_{I'}\rangle$ s (equivalent to momentum replacement $|i]_{\dot\alpha}\langle i|_\alpha\to p_{i,\dot\alpha\alpha}$). Notice that different LG indexes contractions in g produce different G bases, but only one is independent, and the others are EOM redundant because they have the same massless limit. Based on these discussions, we find that to construct the complete MLGNS G bases without the EOM redundancy; we should first construct the complete set of its massless limit basis g and then map g to G by restoring the original massive spinors from their massless limits. In the rest of this subsection, we will discuss how to construct the complete massless basis g without the IBP redundancy.

For an MLGTS $\mathcal{A}$ basis, its partner G bases should be in the same $SU(2)_r$ representation and also should be $SU(2)_l$ singlet to form Lorentz singlets with the $\mathcal{A}$ basis, which means that the total number of left-handed spinors should be even $\sum_{k=1}^N n_k=\text{even}\equiv L$, where n_k is the number of massive spinors $|k^I\rangle$ or massless spinors $|k\rangle$ in the G bases. To be massive LG neutral, the number of massive spinors $|i^I]$ and $|i_I\rangle$ should be equal. And massless LG symmetry requires that the difference between the number of massless spinors $|j]$s and $|j\rangle$s in G should equal the twice massless particle-j's helicity,

$$\begin{aligned}\tilde n_i - n_i &= 0, &&\text{with}\quad i=1,\ldots,m\\ \tilde n_j - n_j &= 2h_j, &&\text{with}\quad j=m+1,\ldots,N,\end{aligned} \tag{13}$$

where h_j is the helicity of particle-j and $\tilde n_i$ is the number of right-handed spinors $|i^I]$ (same for $\tilde n_j$). The massless limit g basis should also satisfy these constraints.

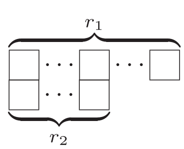
For the g basis, its LGs are just trivial Abelian group $\otimes_{k=1}^{N} U(1)_k$ so we can embed them into a global symmetry $U(N) \supset \otimes_{k=1}^{N} U(1)_k$ through embedding the massless spinor $\tilde{\lambda}^k_{\dot{\alpha}} \equiv |k]$ $(\lambda_{k\alpha} \equiv |k\rangle)$ into the (anti-) fundamental representation of $U(N)$ symmetry with $k = 1, \ldots, N$. So a g basis can be a basis of $U(N)$ representation, which corresponds to an SSYT of the $U(N)$ representation. Conversely, it can also be written down through this SSYT based on the permutation symmetry of the $U(N)$ indices. For example, the product of a right-/left-handed spinor pair can be obtained from the $2/(N-2)$ rows and 1 column $U(N)$ SSYT

$$\begin{array}{|c|}\hline i\\ \hline j\\ \hline\end{array} = (\tilde{\lambda}^i_{\dot{\alpha}}\tilde{\lambda}^j_{\dot{\beta}} - \tilde{\lambda}^j_{\dot{\alpha}}\tilde{\lambda}^i_{\dot{\beta}}) = [ij]$$

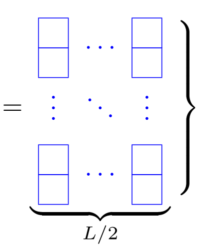
$$N-2\left\{\begin{array}{|c|}\hline k_1\\ \hline k_2\\ \hline \cdot\\ \hline \cdot\\ \hline\end{array}\right. = \frac{\left(\epsilon^{ijk_1\ldots k_{N-2}} + \text{anti-sym in } k_1..k_{N-2}\right)}{(N-2)!}\lambda_{i\alpha}\lambda_{j\beta} = \langle ij\rangle \epsilon^{ijk_1\ldots k_{N-2}}\,. \qquad (14)$$

Notice that the columns in the SSYT associated with the $U(N)$ indices of λ are in blue to distinguish with $\tilde{\lambda}$ indices.

Since g can have the IBP redundancy, the spinor polynomials g from some $U(N)$ representations may not be independent. What kind of SSYT is free of IBP? Next, we will briefly discuss how to obtain these independent SSYTs. If some bases $g(\tilde{\lambda})$ are holomorphic functions of right-handed spinors and furnish a $U(N)$ representation, these bases are independent, and their $U(N)$ YD is in the same shape as its $SU(2)_r$ YD. For example, if these $g(\tilde{\lambda})$ bases have $(R = r_1 + r_2)$ right-handed spinors and are in the $(\mathbf{r_1} - \mathbf{r_2} + \mathbf{1})$ symmetric representation of $SU(2)_r$, their $U(N)$ YD is in the shape of $[r_1, r_2]$,

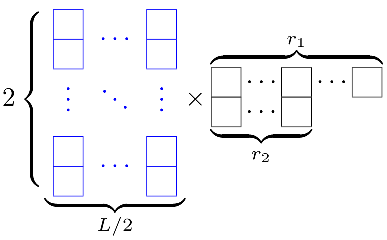
$g(\tilde{\lambda}) =$  . (15)

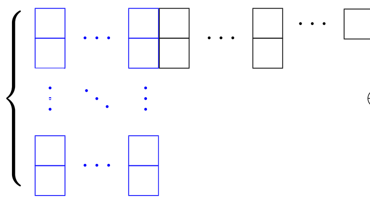
While, if some bases $g(\lambda)$ in a $U(N)$ representation are a holomorphic function of L left-handed spinor λs and also Lorentz singlets, their $U(N)$ YD has $N-2$ rows and $L/2$ columns,

$g(\lambda) =$  $N-2$. (16)

These bases $g(\lambda)$ are also independent. This is because these holomorphic bases $g(\tilde{\lambda})$ $(g(\lambda))$ do not contain momentum factors.

For the nonholomorphic case, if some bases $g(\lambda, \tilde{\lambda})$ are in the reducible $U(N)$ representation which is the out product of the representations of $g(\tilde{\lambda})$ and $g(\lambda)$ in Eqs. (15) and (16), their $U(N)$ representation can be decomposed into $U(N)$ irreducible representations via Littlewood-Richardson rules

$g(\tilde{\lambda}, \lambda) = N-2$ 

$= N-2$  $\oplus \cdots$ (17)

where $\oplus \cdots$ represents the other irreducible representations. Only the first irreducible YD does not contain an overall factor of total momentum $P = \sum_{k=1}^{N} p_k$, so the $g(|k], |k\rangle)$ bases from the first YD, which is obtained by just gluing the blue and white YD simply without shifting around white YD, are independent [32]. Then the complete MLGNS $G(|j], |j\rangle, p_i)$ bases without the EOM and IBP redundancy can be obtained from their massless limit $g(|j], |j\rangle, |i]\langle i|)$ bases by restoring massive spinors and their LG index contractions (equivalent to restoring the massive momentums from their massless limits),

$$G(|j], |j\rangle, p_i) = g(|j], |j\rangle, |i]\langle i|)|_{|i]\langle i| \to p_i}. \qquad (18)$$

We have demonstrated how to construct the complete bases of $\mathcal{A}$ and G through the YD method separately. Thus the complete massive amplitude bases can be obtained by contracting the complete set of $\{\mathcal{A}\}$ bases with the corresponding complete $\{G\}$ bases. Moreover, we prove that the amplitude bases constructed in this way are independent because of the independence of $\{\mathcal{A}\}$ and $\{g\}$ bases [39].

C. Some examples

In this section, we explicitly construct all the 3-point amplitude bases of the massive gauge boson through the YD method discussed above. The general procedure is to first construct the complete MLGTS bases via $SU(2)_r$ YDs, then construct their corresponding MLGNS bases with the same $SU(2)_r$ quantum number via $U(N)$ SSYT, and finally contract them to get the independent amplitude bases.

For the 3-pt massive gauge boson bases $W^+W^-Z\partial^n$, since the polarization tensor of each gauge boson is $SU(2)_r$ triplet, their MLGTS $\mathcal{A}$ bases can be obtained through decomposing the outer product of these three $SU(2)_r$ triplet representations. We can get seven independent $\mathcal{A}$ bases in the following $SU(2)_r$ YD representations,

$$\begin{aligned}&\young(11)\otimes\young(22)\otimes\young(33)\\ =&\young(112,233)\oplus\young(1122,33)\oplus\young(1123,23)\oplus\young(1133,22)\\ \oplus&\young(11223,3)\oplus\young(11233,2)\oplus\young(112233)\\ \equiv&\mathcal{A}^I_{[3,3]}\oplus\mathcal{A}^I_{[(4,2)^1]}\oplus\mathcal{A}^I_{[(4,2)^2]}\oplus\mathcal{A}^I_{[(4,2)^3]}\\ &\oplus\mathcal{A}^I_{[(5,1)^1]}\oplus\mathcal{A}^I_{[(5,1)^2]}\oplus\mathcal{A}^I_{[6]}\,.\end{aligned} \tag{19}$$

Then we can read out the tensor structures based on the permutation symmetry of $SU(2)_r$ indices from above YD (the number $i=1,\,2,\,3$ in the YD represent the Lorentz indices of spinor $|i^I]_{\dot\alpha}$). Take the first YD as an example. According to the permutation symmetry of their $SU(2)_r$ indices, we can get the expression of the first MLGTS basis,

$$\begin{aligned}\mathcal{A}^{\{I_1,I_2\},\{J_1,J_2\},\{K_1,K_2\}}_{[3,3]}&\equiv\young(112,233)\\ &=4[1^{I_1}2^{J_1}][1^{I_2}3^{K_1}][2^{J_2}3^{K_2}]\,.\end{aligned} \tag{20}$$

Its corresponding partners G should also be Lorentz scalar. For three particles dynamics, Mandelstam variable s_{ij} is trivial and just a constant function of mass, e.g.,

$$s_{12}=(p_1+p_2)^2=m_3^2. \tag{21}$$

Since G must be the function of Mandelstam variables s_{ij}, G is just a constant. Meanwhile, we also cannot construct a valid SSYT as in Eq. (17), which means it is not a dynamical polynomial. So all the amplitude bases with MLGTS $\mathcal{A}^I_{[3,3]}$ are just $\mathcal{A}^I_{[3,3]}$ itself.

Next, we will consider the nontrivial case, such as the basis $\mathcal{A}_{[(4,2)^1]}$. Its expression is

$$\begin{aligned}\mathcal{A}^{\{I_1,I_2\},\{J_1,J_2\},\{K_1,K_2\}}_{[(4,2)^1]}&\equiv\young(1122,33)\\ &=2[1^{I_1}3^{K_1}][1^{I_2}3^{K_2}]|2^{J_1}]_{\{\dot\alpha}|2^{J_2}]_{\dot\alpha'\}}\\ &+2[2^{J_1}3^{K_1}][2^{J_2}3^{K_2}]|1^{I_1}]_{\{\dot\alpha}|1^{I_2}]_{\dot\alpha'\}}\\ &+8[1^{I_1}3^{K_1}][2^{J_1}3^{K_2}]|1^{I_2}]_{\{\dot\alpha}|2^{J_2}]_{\dot\alpha'\}}\end{aligned} \tag{22}$$

To guarantee its partner G bases to be in the same $SU(2)_r$ quantum number and also LG neutral, they at least contain two right-handed massive spinors to contract with the bare Lorentz indices of $\mathcal{A}^I_{[(4,2)^1]}$, and the number of the left-handed massive spinors should be same as right-handed spinors [see Eq. (13)]. Therefore, the SSYT of G satisfying this constraint can only be the $U(3)$ SSYT which contains one column of blue boxes and two columns of white boxes $\young(123)$. Then we can write down this G basis following its permutation symmetry,

$$\begin{aligned}G^{[3]}&=\young(123)\\ &=\Big(\langle i_1i_2\rangle\epsilon^{1i_1i_2}|2]^{\{\dot\alpha}|3]^{\dot\alpha'\}}+\langle i_1i_2\rangle\epsilon^{2i_1i_2}|3]^{\{\dot\alpha}|1]^{\dot\alpha'\}}\\ &\quad+\langle i_1i_2\rangle\epsilon^{3i_1i_2}|1]^{\{\dot\alpha}|2]^{\dot\alpha'\}}\Big)\big|_{|i]\langle i|\to p_i}\\ &=\langle 2_I3_J\rangle|2^I]^{\{\dot\alpha}|3^J]^{\dot\alpha'\}}+\langle 3_I1_J\rangle|3^I]^{\{\dot\alpha}|1^J]^{\dot\alpha'\}}\\ &\quad\langle 1_I2_J\rangle|1^I]^{\{\dot\alpha}|2^J]^{\dot\alpha'\}}.\end{aligned} \tag{23}$$

Notice that after reading out g from $U(3)$ SSYT, G can be obtained by restoring the LG indices of the massive spinors and arbitrarily choosing one kind of LG contraction pattern. In this case, the massive LG indices only have one contraction pattern. For the g polynomials with larger mass dimensions, we cannot construct a valid YD, so higher dimensional g cannot be independent. This can be seen directly through its dynamics: the extra mass dimension of G^{higher} bases must come from Mandelstam variables s_{ij} compared with $G^{[3]}$, which is just mass constant, so G^{higher} is descendent from $G^{[3]}$. In this case, there is only one independent basis totally which is $\mathcal{A}^I_{[4,2]}\cdot G^{[3]}$.

For the same reason, the independent partners $G^{[n]}$ of the reset $\mathcal{A}$ bases are also unique, so there is a total of seven bases for 3-pt W^+-W^--Z. Following the same procedures, we list all the G bases for all the $\mathcal{A}$ bases,

$$\begin{aligned}\mathcal{A}^I_{[3,3]}&:\;G^{\bullet}=\bullet\\ \mathcal{A}^I_{[(4,2)^{1,2,3}]}&:\;G^{[3]}=\young(123)\\ \mathcal{A}^I_{[(5,1)^{1,2}]}&:\;G^{[6]}=\young(112233)\\ \mathcal{A}^I_{[6]}&:\;G^{[9]}=\young(111222333)\,.\end{aligned} \tag{24}$$

IV. SIMPLIFICATION OF EFT AMPLITUDE BASIS

However, the SSYT's horizontal permutations make the $\mathcal{A}$ and G bases very long polynomials. In order to efficiently decompose any polynomial of spinor products into a complete set of $\{\mathcal{A}\cdot G\}$ bases, we should first simplify $\{\mathcal{A}\cdot G\}$ amplitude bases to make each basis a monomial of spinor products. We find that a set of spinor monomials can be read off from the $SU(2)_r$ YDs of a complete set of $\mathcal{A}$ bases without considering the horizontal permutation symmetry (HPS). Moreover, these monomials are independent of each other because of the Fock condition.[1] Then we can get the simplified MLGTS bases, called $\mathcal{B}$ bases, which are equivalent to $\mathcal{A}$ basis. Following the same logic, we can also get a complete set of monomials from the $U(N)$ SSYTs of G bases, which is also equivalent to G bases and is called H bases. After contracting the $\mathcal{B}$ bases with the corresponding H bases, the complete set of simplified amplitude basis $\{\mathcal{B}\cdot H\}$ can be obtained (the $SU(2)_r$ Lorentz index contraction convention between $\mathcal{B}$ and H can be fixed in the following discussion).

We find that different from $\{\mathcal{A}\cdot G\}$ basis construction, $\{\mathcal{B}\cdot H\}$ can be easily constructed from the enlarged SSYTs. For the SSYT of a g basis without HPS (see Eq. (17)), the blue sub-SSYT and white sub-SSYT correspond to two monomials, two holomorphic functions of left-handed and right-handed spinors, and their product gives the H basis. Since the shape of the white sub-SSYT of g is the same as its $SU(2)_r$ YD, we can equivalently treat the white sub-SSTY of $U(N)$ as a $SU(2)_r$ YD. So one of $SU(2)_r$ indices contraction patterns between H and $\mathcal{B}$ bases can be obtained by gluing the white sub-SSYT of H with the $SU(2)_r$ YD of $\mathcal{B}$ via counterclockwise rotating the YD of $\mathcal{B}$ by 180°. After that, we find that a $\{\mathcal{B}\cdot H\}$ basis corresponds to an enlarged Young tableau (YT). In order to distinguish the H part from the $\mathcal{B}$ part in this enlarged YT, we require the boxes representing spinor $|i^I]$ in $\mathcal{B}$ to be labeled by number with prime superscript i'. But the numbers filled in the boxes associated with H do not take prime superscripts, ranging from 1 to N. Since the sub-YTs associated with $\mathcal{B}$ bases are rotated by 180° to glue with H SSYTs, in order to make the enlarged YT be "SSYT," we define the size order of the numbers filled in the enlarged YT as $1<\cdots<N<m'<\cdots<2'<1'$. Since the i'-boxes in the enlarged SSYT are only associated with right-handed spinors, the enlarged SSYTs of $\{\mathcal{B}\cdot H\}$ bases should not have blue boxes filled in i'. Conversely, we can easily find that the enlarged SSYTs without HPS satisfying this condition one-to-one correspond to $\mathcal{B}\cdot H$ bases.

As we said before, to get a complete set of amplitude bases with the lowest dimension (a complete set of bases with the lowest dimension means that the EOM cannot further reduce the dimension of the bases in it), we should first find the complete but redundant bases, which contain all the lowest dimension bases. Then decompose them into $\{\mathcal{B}\cdot H\}$ bases to pick up the independent and lowest dimension monomials as amplitude bases. For the convenience of this decomposition, we renumber the external legs: the m massive legs are labeled by number $\{1,n+2,\ldots,N\}$, and the n massless legs are labeled by number $\{2,\ldots,n+1\}$ according to their spins in descending order. As before, we define the size order of the numbers filled in the enlarged YT as

$$1<\cdots<N<N'<\cdots<(n+2)'<1', \tag{25}$$

and thus, each enlarged SSYT without blue boxes filled in number-i' and HPS still one-to-one corresponds to a $\mathcal{B}.H$ basis. Moreover, we will see that any polynomial of spinor products can be systematically decomposed into the $\mathcal{B}.H$ bases constructed from this kind of enlarged SSYTs.

A. General property of $\{\mathcal{B}\cdot H\}$ bases and an example

The complete set of $\{\mathcal{B}\cdot H\}$ bases can be constructed from the enlarged SSYTs satisfying the following conditions:

(i) Fill the YDs $[(L+R)/2,(L+R)/2,(L/2)^{N-4}]$ with $L/2$ number-i and $2s_i$ number-i' for massive particle-i with spin-s_i ($i=\{1,n+2,\ldots,N\}$); and $(L/2+2h_j)$ number-j for massless particle-j with helicity-h_j ($j=\{2,\ldots,n+1\}$).

(ii) The number-i' can only appear in the white boxes, corresponding to the right-handed spinors in polarization tensors.

Notice that the power of $(N/2)^{N-4}$ in the above YD shape $[,,(N/2)^{N-4}]$ represents the number of columns with $N/2$ boxes. For a specific $\mathcal{B}.H$ basis with dim-D, its SSYT shape should satisfy the following conditions:

$$R=D-N+\sum s_i+\sum h_j,$$
$$L=D-N-\sum s_i-\sum h_j. \tag{26}$$

Here the dimension D of a $\mathcal{B}.H$ basis is defined to be the dimension of its corresponding operator, $D\equiv[\mathcal{B}.H]+N$, where $[\mathcal{B}.H]$ is dimension of $\mathcal{B}.H$ amplitude and N is external leg number. $\sum s_i$ and $\sum h_j$ are the sums of all massive particle spin and massless particle helicity, respectively.

We take interactions of four massive fields, fermion-fermion-scalar-scalar ($\psi_1\psi_2\phi_3\phi_4$), as an example to explain how to systematically construct a complete set of their $\{\mathcal{B}.H\}$ bases at a given dimension D through enlarged SSYTs.

The polarization tensors of ψ_1 and ψ_2 are just spinor $|1^I]$ and $|2^J]$ so their enlarged SSYTs should contain two white boxes filled with numbers $1'$ and $2'$ representing their polarization tensors. Besides the two polarization tensors, $\{\mathcal{B}.H\}$ basis can also contain any number of massive momentums p_i. It can contain at least zero momentum factor, corresponding to $D=5$ bases. According to the conditions in Eq. (26), we get $R=2$ and $L=0$ for $D=5$. So, based on the above two properties of $\{\mathcal{B}.H\}$ SSYT, its SSYT is in the shape of [1, 1] and is only filled with the number $\{2',1'\}$. Fill in this YD with the numbers $\{2',1'\}$ in

[1]For a general $\mathcal{A}_{\{\dot\alpha\}}$ basis, assuming it has n bare indices, its terms from horizontal permutation are not Semi-standard. Using the Fock condition, which is equivalent to the Schouten identity for spinor calculation, we can convert $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{A}_{\{\dot\alpha_1,\ldots,\dot\alpha_n\}}=\mathcal{B}_{\dot\alpha_1,\ldots,\dot\alpha_n}+\epsilon_{\dot\alpha_i\dot\alpha_j}\mathcal{B}_{\ldots,\dot\alpha_{i-1},\dot\alpha_{i+1}\ldots\dot\alpha_{j-1},\dot\alpha_{j+1}\ldots}+\cdots$ Since all polynomial $\mathcal{A}$ could be decomposed into $\{\mathcal{B}\}$, $\{\mathcal{A}\}\subset\{\mathcal{B}\}$; and they have the same number of bases (SSYT), $rank\{\mathcal{A}\}=rank\{\mathcal{B}\}$. Then we can say that these two sets are equivalent, $\{\mathcal{A}\}=\{\mathcal{B}\}$.

the semistandard pattern according to the number size defined in Eq. (25), and we only get one SSYT,

$$\{\mathcal{B}\cdot H\}^{D=5} = \left\{ \begin{array}{|c|}\hline 2' \\ \hline 1' \\ \hline \end{array} \right\} = \{[1^I 2^J]\}. \qquad (27)$$

Then for higher dimension bases, it may contain two momenta and its dimension $D=7$. Similarly, we can find that $R=4$ and $L=2$. So the YD shape is [3, 3], and the number-i and i' only appear once. We should fill the number $\{1,2,3,4,2',1'\}$ once in this YD in the semistandard pattern and get five bases at $D=7$:

$$\{\mathcal{B}\cdot H\}^{D=7} = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 2' & 1' \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 2' & 1' \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 2' \\ \hline 3 & 4 & 1' \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 2' & 1' \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 2' \\ \hline 2 & 4 & 1' \\ \hline \end{array} \right\}$$
$$= \{[1^I 322^J], [1^I 422^J], s_{24}[1^I 2^J], [1^I 432^J], s_{34}[1^I 2^J]\}, \qquad (28)$$

where $s_{ij} \equiv p_i \cdot p_j$ is the Mandelstam variable, and the bracket $[1^I 322^J]$ is defined as $[1^I 322^J] \equiv [1^I|\sigma^\mu \bar{\sigma}^\nu|2^J](p_3)_\mu (p_2)_\nu$. Now we get the complete set of $\{\mathcal{B}.H\}$ bases at $D=5$ and 7. Following the same procedure, the complete set of bases at higher D can be systematically obtained.

V. DECOMPOSITION

In this section, we will systematically discuss how to decompose any polynomial into the $\{\mathcal{B}\cdot H\}$ basis. Since the massive polarization tensors of $\{\mathcal{B}\cdot H\}$ basis are the holomorphic functions of right-handed spinors, any left-handed spinors in the massive polarization tensors of a polynomial should be replaced by right-handed spinors through EOM ($|i^I\rangle_\alpha = p_{i\alpha\dot{\alpha}}|i^I]^{\dot{\alpha}}/m_i$) for the decomposition. Since $\{\mathcal{B}\cdot H\}$ bases are constructed from enlarged SSYTs without HPS, the momentum $p_{1\alpha\dot{\alpha}}$ and Lorentz scalar $\langle 2_I 3_J\rangle[3^K 2^L]$ cannot appear in semistandard H bases.[2]

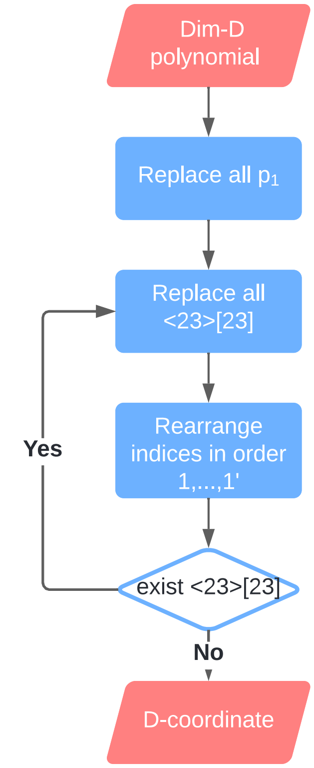


FIG. 1. Decomposition procedures of any polynomial into $\{\mathcal{B}\cdot H\}$ bases.

So to decompose a polynomial into $\{\mathcal{B}\cdot H\}$ bases, all the $p_{1\alpha\dot{\alpha}}$ and $\langle 2_I 3_J\rangle[3^K 2^L]$ in it should be eliminated by some identities (see below). After decomposition, each generated spinor monomial can be mapped to an enlarged SSYT, which means this polynomial is correctly decomposed into $\{\mathcal{B}\cdot H\}$ bases. Otherwise, do not stop using the Schouten identity until the polynomial is converted into the correct one that can be mapped into the enlarged SSYTs (this can always be realized).

In the following, we summarize how to systematically do the decomposition, which is also shown in Fig. 1. Note that all the polarization's spinors $|i^I\rangle_\alpha$ in the input polynomial are replaced with $p_{i\alpha\dot{\alpha}}|i^I]^{\dot{\alpha}}/m_i$.

(i) *Step-1:* using momentum conservation, replace all the momentum $p_{1\alpha\dot{\alpha}} = |1^I\rangle_\alpha [1_I|_{\dot{\alpha}}$ by $-\sum_{k=2}^N p_k$ in this polynomial and then simplify it by EOMs,

$$(p_i)_{\alpha\dot{\alpha}}(p_i)^{\dot{\alpha}}_{\beta} = m_i^2 \epsilon_{\alpha\beta}, \qquad (p_i)_{\alpha\dot{\alpha}}(p_i)^{\alpha}_{\dot{\beta}} = m_i^2 \epsilon_{\dot{\alpha}\dot{\beta}},$$
$$\langle jj\rangle = [jj] = 0. \qquad (29)$$

To guarantee that the polarization tensor is always $|i^I]^{2s_i}$, we do not apply EOM $p_i|i^I] = m_i|i^I\rangle$ to the polarization's spinor. If some terms get an overall factor m_i^2 from EOMs, these terms should be discarded (we will see that these terms do not affect finding a set of lowest dimension bases).

[2] The Schouten Identity guarantees that the interior of the blue or white part is semistandard. Now the only nonsemistandard part is the place where blue and white boxes meet. Since particle-1 is massive, there are a total of $L/2$ numbers-1 that need to be filled in the $U(N)$ conjugating Young tableau (left-handed spinors are in the antifundamental representation of $U(N)$), and they are just filled in all the boxes in the first row. So after turning this conjugating Young Tableau into the blue boxes by using $U(N)$ epsilon tensor to raise $\lambda U(N)$ index, there will be no blue boxes with number-1 in SSYT. This indicates that there is no momentum $|1^I]\langle 1_I|$ in $\mathcal{B}\cdot H$. In the absence of p_1, the only possible nonsemistandard Young tableau where blue and white meet is $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}$, which corresponds to polynomials with a factor $\langle 23\rangle[23]$ in massless limit.

(ii) *Step-2:* If one monomial contains $\langle 2^I 3^J\rangle[3_J 2_I]$, directly replace it with the identity

$$p_2\cdot p_3 = -\frac{m_2^2+m_3^2-m_1^2}{2}-\frac{(\sum_{k=4}^N p_k)^2}{2} - (p_2+p_3)\cdot\sum_{k=4}^N p_k. \quad (30)$$

Suppose one monomial contains $\langle 2_I 3_J\rangle$ and $[2^K 3^L]$ simultaneously, and their LG indices are not bare (these spinors come from momentums p_2 and p_3). In that case, we use Schouten identity to adjust the LG indices of $\langle 2_I 3_J\rangle$ and the other two spinors, $|2\rangle_K$ and $|3\rangle_L$, in this monomial to generate factor $p_2.p_3$. However, in practice, we can directly exchange the LG indices of $\langle 2_I 3_J\rangle$ with the spinor $|2\rangle_K$ and $|3\rangle_L$ (if particle-2 or 3 is massless, ignore this step). This is because each time we use Schouten identity to adjust the LG index, an additional term with mass factor will be generated, which we should discard. The following is the proof,

$$\begin{aligned}&\langle 2_I 3_J\rangle\langle 2_K x\rangle\langle 3_L y\rangle[2^K 3^L]\\ &= \langle 2_K 3_J\rangle\langle 2_I x\rangle\langle 3_L y\rangle[2^K 3^L] + \mathcal{O}(m_2^2)\\ &= \langle 2_K 3_L\rangle\langle 2_I x\rangle\langle 3_J y\rangle[2^K 3^L] + \mathcal{O}(m_2^2) + \mathcal{O}(m_3^2).\end{aligned} \quad (31)$$

Then we can get $p_2\cdot p_3$ factor and replace it by Eq. (30).

(iii) *Step-3:* Since $\{\mathcal{B}\cdot H\}$ basis is constructed based on the SSYT, its spinor contractions are arranged in the order of Eq. (25) (called semistandard). Generally, after step-2, the generated polynomial is not semistandard. So to be decomposed into $\{\mathcal{B}\cdot H\}$ bases, the spinor contraction pattern in this polynomial should be adjusted by Schouten identity to become semistandard. There is only one kind of spinor contraction pattern that is not semistandard. That is $[ij][kl]$ ($\langle ij\rangle\langle kl\rangle$) if $i<k<l<j$. We can easily convert it into the semistandard pattern as

$$[ij][kl] = [il][kj] - [ik][lj]. \quad (32)$$

The two terms on the right side of this equation can be mapped into the sub-SSYT, which indicates that any nonsemistandard polynomial can be converted into the combination of semistandard monomials by Schouten identity.

(iv) *Step-4:* Repeat step-2 and -3 until there are no $p_2.p_3$ factors in the generated polynomial, and each term is the semistandard monomial, which is $\{\mathcal{B}\cdot H\}$ base.

Following the above four steps, we can systematically decompose any monomial into $\{\mathcal{B}\cdot H\}$ bases and get its coordinate in $\{\mathcal{B}\cdot H\}$ base space.

VI. OVERREDUNDANT $\{\mathcal{C}\cdot F\}$ BASIS

Now we know how to systematically decompose any polynomial into $\{\mathcal{B}\cdot H\}$ bases and determine their independence. The only problem is systematically constructing a complete but redundant set of amplitude bases that always contains all lowest dimensional amplitude bases. After constructing such a basis set, we can decompose its bases into $\{\mathcal{B}\cdot H\}$ bases in the ascending order of dimension and eliminate the linear correlation bases according to their coordinates. Finally, we get a complete set of lowest dimensional amplitude bases from the redundant basis set. This section will discuss how to construct this kind of basis set systematically.

A. Lowest dimension basis $\{\mathcal{C}\cdot F\}$

In this subsection, we briefly discuss why the simplified $\{\mathcal{B}\cdot H\}$ bases or $\{\mathcal{A}\cdot G\}$ bases cannot be directly mapped into the physical operator bases. The complete set of physical operator bases always refers to the basis set with the lowest dimension (EOM or other identities cannot further reduce the dimension of the bases). Since the polarization tensors of $\{\mathcal{B}\cdot H\}$ bases are always the holomorphic function of massive right-handed spinors, they cannot be mapped into the operator bases whose polarization tensors contain left-handed spinors. If we replace the left-handed spinors in polarizations through EOM $|i^I\rangle = p_i|i^I]/m_i$, the operator basis is the linear combination of the $\{\mathcal{B}\cdot H\}$ bases with higher dimensions. We can take four massive particle vertex fermion-fermion-scalar-scalar ($\psi_1\psi_2\phi_3\phi_4$) as an example. Obviously, it has two dim-5 operator bases $\{\bar\psi_{1L}\psi_{2R}\phi_3\phi_4, \bar\psi_{1R}\psi_{2L}\phi_3\phi_4\}$, corresponding to the lowest dimensional amplitude bases $\{[1^I 2^J], \langle 1^I 2^J\rangle\}$ respectively. As said above, $\langle 1^I 2^J\rangle$ base does not exist in $\{\mathcal{B}\cdot H\}$ set, which can be expressed as the combination of $\{\mathcal{B}\cdot H\}$ bases,

$$\langle 1^I 2^J\rangle = \frac{m_2[1^I 2^J]}{m_1} + \frac{[1^I 322^J]}{m_1 m_2} + \frac{[1^I 422^J]}{m_1 m_2}. \quad (33)$$

Clearly, the last two $\{\mathcal{B}\cdot H\}$ bases on the right side of this equation have higher dimensions than the $\langle 1^I 2^J\rangle$ basis.

In order to find the lowest dimensional operator bases, we need to know how the amplitude bases are mapped into operator bases and the correlation between the dimension of operator bases and amplitude bases. Generally, the maps between these two kinds of bases follow the rules,

$$\begin{aligned}&\phi_i\leftrightarrow \mathbf{1},\quad \psi_{iL}\leftrightarrow |i^I\rangle,\quad \psi_{iR}\leftrightarrow |i^I],\quad F^+_{i\dot\alpha\dot\beta}\leftrightarrow |i^{\{I_1}]|i^{I_2\}}],\\ &A_{i\mu}\leftrightarrow \frac{|i^{\{I_1}]\langle i^{I_2\}}|}{m},\quad F^-_{i\alpha\beta}\leftrightarrow |i^{\{I_1}\rangle|i^{I_2\}}\rangle,\quad \partial_i\leftrightarrow p_i,\end{aligned} \quad (34)$$

where ϕ_i is scalar, $\psi_{iL,R}$ is left-handed or right-handed massive fermion, $A_{i\mu}$ is massive vector, $F^\pm_{i\dot\alpha\dot\beta}\equiv 1/2(F_{i\mu\nu}\pm i\epsilon_{\mu\nu\rho\sigma}F_i^{\rho\sigma})$, and $F_{i\mu\nu}$ is field strength of $A_{i\mu}$. We find that when the operator contains bare vector fields $A_{i\mu}$ (not refer to A_μ in $F^\pm$), the operator dimension d is not equal to the dimension of the corresponding amplitude basis plus the number of external legs N, but equal to its amplitude base dimension plus the number of external legs and minus the number of bare vector fields,

$$d = [\mathcal{M}] + N - n_A, \tag{35}$$

where $[\mathcal{M}]$ is the dimension of amplitude base $\mathcal{M}$, N is the number of external legs, and n_A is the number of $A_{i\mu}$ vector fields.

For a scattering process involving massive external particle-i with spin-s_i, we can classify all of its possible amplitude bases according to the particle-i's polarization tensor configuration (PTC),

$$\epsilon_i^{l_i} \equiv (|i^{\{I\}}\rangle)^{l_i}(|i^{I\}}])^{2s_i - l_i}, \tag{36}$$

where $l_i \in [0, 2s_i]$ is the number of left-handed spinors in the polarization tensor, all the LG indices should be totally symmetric. So different sets of l_i values represent different PTCs. We use the symbol $\{\mathcal{C}\cdot F\}_d^{\{\ldots,l_i,l_{i+1},\ldots\}}$ to denote the complete set of dim-d bases with PTC $\{\ldots,\epsilon_i^{l_i},\epsilon_{i+1}^{l_{i+1}},\ldots\}$. In this kind of set, the basis can also be generally factorized into two parts, similar to $\{\mathcal{B}\cdot H\}$ bases,

$$\begin{aligned}&(\mathcal{C}\cdot F)^{\{\ldots,l_i,l_{i+1},\ldots\}}\\ &\quad= \mathcal{C}^{\{\ldots,l_i,\ldots\}}(|i^{\{I\}}]^{2s_i-l_i})\cdot F^{\{\ldots,l_i,\ldots\}}(|i^{I\}}\rangle^{l_i}, p_i, |j], |j\rangle),\end{aligned} \tag{37}$$

where the $\mathcal{C}^{\{\ldots,l_i,\ldots\}}$ part is a holomorphic function of massive right-handed spinors and is required to be a linear function of $(2s_i - l_i)$ right-handed spinors $|i^I]$, while the $F^{\{\ldots,l_i,\ldots\}}$ part takes both massive and massless LG charges, and the massive LG indices of its (l_i) left-handed spinors $|i^I\rangle$ are bare. Similar to $\{\mathcal{B}\cdot H\}$ bases, we can first construct $\mathcal{C}^{\{\ldots,l_i,\ldots\}}(|i^{\{I\}}]^{2s_i-l_i})$ bases and $F^{\{\ldots,l_i,\ldots\}}(|i^{I\}}]^{l_i}, p_i, |j], |j\rangle)$ bases separately, and then the complete set of $\{\mathcal{C}\cdot F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ bases can be obtained by contracting these two sets of bases. Same as the construction of $\{\mathcal{B}\}$ and $\{H\}$ bases, $\mathcal{C}^{\{\ldots,l_i,\ldots\}}(|i^{\{I\}}]^{2s_i-l_i})$ bases can be constructed by $SU(2)_r$ YDs, and $F^{\{\ldots,l_i,\ldots\}}(|i^I\rangle^{l_i}, p_i, |j], |j\rangle)$ bases can be obtained by first constructing their massless limit bases $\{f^{\{\ldots,l_i,\ldots\}}\}$ via $U(N)$ SSYTs and then restoring the massive spinors' LG indices in $\{f^{\{\ldots,l_i,\ldots\}}\}$ bases to obtain $F^{\{\ldots,l_i,\ldots\}}(|i^I\rangle^{l_i}, p_i, |j], |j\rangle)$ bases. The massless limit of an $F^{\{\ldots,l_i,\ldots\}}(|i^I\rangle^{l_i}, p_i, |j], |j\rangle)$ basis is defined to be that the LG indices of all its massive spinors are stripped,

$$\begin{aligned}&f^{\{\ldots,l_i,\ldots\}}(|i\rangle^{l_i}, |i\rangle[i|, |j], |j\rangle)\\ &\quad\equiv F^{\{\ldots,l_i,\ldots\}}(|i^I\rangle^{l_i}, p_i, |j], |j\rangle)|_{|i^I\rangle\to|i\rangle, |i^I]\to|i]}.\end{aligned} \tag{38}$$

Notice that $\{f^{\{\ldots,l_i,\ldots\}}\}$ basis is not rigorously the massless limit of $\{F^{\{\ldots,l_i,\ldots\}}\}$ basis, but rather like a particular LG component of $\{F^{\{\ldots,l_i,\ldots\}}\}$ basis. So independent $f^{\{\ldots,l_i,\ldots\}}$ bases one-to-one correspond to independent $F^{\{\ldots,l_i,\ldots\}}$ bases. If some $F^{\{\ldots,l_i,\ldots\}}$ bases correspond to the same $f^{\{\ldots,l_i,\ldots\}}$ basis, these bases must be related by EOM so only one of them is independent. Since the proof of $F^{\{\ldots,l_i,\ldots\}}$ bases independence is the same as $\{H\}$ bases, we will not discuss it in detail (similar proof can be found in [39]).

So in this way, a complete set of $\{\mathcal{C}\cdot F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ with one kind of PTC $\{\ldots,l_i,l_{i+1},\ldots\}$ can be constructed.[3] The redundant but complete set of $\{\mathcal{C}\cdot F\}$ bases is the one that contains all the complete basis sets, each with a different PTC,

$$\{\mathcal{C}.F\} = \sum_{\{\ldots,l_i,l_{i+1},\ldots\}}^{\ldots,0\le l_i\le 2s_i,\ldots} \{\{\mathcal{C}.F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}\}. \tag{39}$$

Since $\{\mathcal{C}\cdot F\}$ contains the complete bases for each PTC, all the lowest dimensional bases must be contained in it (each lowest dimension basis with a kind of PTCs must belong to the $\{\mathcal{C}\cdot F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ with the same PTC). Note that there is redundancy between different sets $\{\mathcal{C}\cdot F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$, while the bases within each set are independent.

Same as $\mathcal{B}\cdot H$ bases, the massless limit bases $\{\mathcal{C}\cdot f\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ with $L'/2$ $(R'/2)$ left-handed (right-handed) spinor products can also be constructed through the enlarged SSYTs with only vertical permutation (without HPS) satisfying the following conditions:

(i) Fill YD $[(L'+R')/2, (L'+R')/2, (L'/2)^{N-4}]$ with $(L'/2 - l_i)$ number-i, $(2s_i - l_i)$ number-i' for massive particle-i $(i = \{1, n+2, \ldots, N\})$, and $(L'/2 + 2h_j)$ number-j for massless particle-j $(j = \{2, \ldots, n+1\})$.

(ii) Number-i' can only be filled in white boxes corresponding to the right-handed spinors in polarization tensor.

For a $\{\mathcal{C}\cdot F\}$ basis with physical dim-d, its SSYT should satisfy the following conditions:

$$\begin{aligned}R' &= d - N + n_A + \sum(s_i - l_i) + \sum h_j,\\ L' &= d - N + n_A - \sum(s_i - l_i) - \sum h_j.\end{aligned} \tag{40}$$

Finally the $\{\mathcal{C}\cdot F\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ bases can be obtained from $\{\mathcal{C}\cdot f\}^{\{\ldots,l_i,l_{i+1},\ldots\}}$ enlarged SSYTs as follows. We assign

[3] Any polynomial with the polarization in Eq. (36) can be expressed as the inner product of tensors $\mathcal{C}$ and F. As long as we construct complete set $\{\mathcal{C}^{\{\ldots,l_i,\ldots\}}\}$ and $\{F^{\{\ldots,l_i,\ldots\}}\}$, then $\{\mathcal{C}.F\}$ must be complete (may not be independent).

the totally symmetric $SU(2)_i$ LG indices $\{I_1, \ldots, I_{2s_i}\}$ to the first l_i left-handed spinors $|i\rangle$ in f and $(2s_i - l_i)$ right-handed spinors $|i']$ in $\mathcal{C}$ (spinor $|i']$ represents the massive spinor $|i^I]$) and treat them as massive polarization tensor $\epsilon_i^{l_i}$. Then pair all the remaining $|i\rangle$ s and $|i]$ s into massive momentums in any way, $|i\rangle|i] \to p_{i\alpha\dot{\alpha}} = |i^I\rangle_\alpha[i_I|_{\dot{\alpha}}$. The procedures to systematically construct $\{\mathcal{C}\cdot F\}$ bases is the same as $\{\mathcal{B}\cdot H\}$ bases. An example to explain $\{\mathcal{C}\cdot F\}$ base construction is shown in Sec. VI C.

B. Decompose $\{\mathcal{C}\cdot F\}$ into $\{\mathcal{B}\cdot H\}$

After obtaining the complete $\{\mathcal{C}\cdot F\}$ bases, we can do the decomposition $\{\mathcal{C}\cdot F\} \to \{\mathcal{B}\cdot H\}$ following the procedure in Fig. 1, and then eliminate the linear correlation bases. As said before, when doing the decomposition, all the left-handed spinors $|i^I\rangle$ in $\epsilon_i^{l_i}$ should be replaced by $p_i|i^I]/m_i$ to convert $\epsilon_i^{l_i}$ to holomorphic tenser ϵ_i^0, which results in the dimension of $\{\mathcal{C}\cdot F\}$ amplitude bases being increased. The generated amplitude bases, denoted as $\{\mathcal{C}'\cdot F'\}$ (called holomorphic bases), are equivalent to $\{\mathcal{C}\cdot F\}$, and the $\mathcal{C}'$ and F' part is defined the same as $\mathcal{B}$ and H. After converting $\{\epsilon_i^{l_i}\}$ to $\{\epsilon_i^0\}$, we can define the operator dimension of $\{\mathcal{C}'\cdot F'\}_d^{\{\ldots,l_i,\ldots\}}$ as the fake dimension D' of $\{\mathcal{C}\cdot F\}_d^{\{\ldots,l_i,\ldots\}}$ bases,

$$D' = d + \sum l_i + n_A, \tag{41}$$

where d is the operator basis dimension defined in Eq. (35) and $\sum l_i$ is the total number of left-handed spinors in the polarization tensors (fake dimension D' of a $\{\mathcal{B}\cdot H\}$ base equals to its operator dimension D).

We can easily find that a $\{\mathcal{C}\cdot F\}_d^{\{\ldots,l_i,\ldots\}}$ basis with fake dimension $D' = D_{\mathcal{C}\cdot F}$ can be decomposed into the $\{\mathcal{B}\cdot H\}$ bases with the highest fake dimension equal to $D_{\mathcal{C}\cdot F}$,

$$\begin{aligned}\{\mathcal{C}\cdot F\}_d^{\{l\}} &\to \{\mathcal{C}'\cdot F'\}_d^{\{l\}} \\ &\to \{\mathcal{B}\cdot H\}^{D=D_{\mathcal{C}\cdot F}} + m^2\{\mathcal{B}\cdot H\}^{D_{\mathcal{C}\cdot F}-2} + \cdots.\end{aligned} \tag{42}$$

So decompose $\{\mathcal{C}\cdot F\}_d$ into $\{\mathcal{B}\cdot H\}$ bases in the ascending order of d, and remove all the linear correlation bases according to their coordinates in $\{\mathcal{B}\cdot H\}$ space. Finally, we can obtain a complete set of amplitude bases with the lowest dimension, denoted as $\{\mathcal{O}^{\text{phy}}\}$.

The above full decomposition of $\{\mathcal{C}\cdot F\}_d$ bases is very inefficient. We find that their coordinates in the $\{\mathcal{B}\cdot H\}^{D=D_{\mathcal{C}\cdot F}}$ basis space is enough to pick up the independent $\{\mathcal{O}^{\text{phy}}\}$ bases (proof is shown in Appendix A). Therefore, we can discard all terms with mass factors in Eq. (42) during the decomposition,

$$\{\mathcal{C}\cdot F\}_d^{\{l\}} \to \{\mathcal{C}'\cdot F'\}_d^{\{l\}} \to \{\mathcal{B}\cdot H\}^{D=D_{\mathcal{C}\cdot F}}. \tag{43}$$

That is to say, any mass factors that appear in the decomposition procedures discussed in Sec. V should be discarded.

C. Example

In this subsection, we explain how to systematically construct $\{\mathcal{C}\cdot F\}$ bases and do the decomposition. We still take four-point interactions $\psi_1 - \psi_2 - \phi_3 - \phi_4$ as the example.

According to above discussions, since $\psi_{1,2}$ and $\phi_{1,2}$ are fermions and scalars, the range of their polarization parameters l_i is [see Eq. (36)]

$$l_{1,2} \in [0,1], \qquad l_{3,4} = 0. \tag{44}$$

So there is a total of four different PTCs. To obtain all the $d \leq 7$ physical operators, we need to construct all the related $\{\mathcal{C}\cdot f\}_d^{\{l\}}$ bases in ascending order of $d \leq 7$. According to Eq. (40), since L' and R' are even integers, we can determine the allowed PTCs for different dimensions $d \leq 7$, and all are listed as follows,

$$\begin{aligned}&D' = 5\colon\ \{\mathcal{C}\cdot F\}_5^{0000}.\\ &D' = 7\colon\ \{\mathcal{C}\cdot F\}_5^{1100}, \{\mathcal{C}\cdot F\}_6^{0100}, \{\mathcal{C}\cdot F\}_6^{1000}, \{\mathcal{C}\cdot F\}_7^{0000}.\\ &D' = 9\colon\ \{\mathcal{C}\cdot F\}_7^{1100}, \cdots\end{aligned} \tag{45}$$

where the sets in the same line have the same fake dimension-D'. Then, following the procedures in Fig. 1 and Eq. (43), decompose the sets in the three lines into $\{\mathcal{B}\cdot H\}^{D=5}$, $\{\mathcal{B}\cdot H\}^{D=7}$, and $\{\mathcal{B}\cdot H\}^{D=9}$ respectively (discard all the mass factors during the decompositions). Here we explicitly show how to construct the above bases at second line ($D' = 7$) and decompose them. Following Eq. (40), we can find that the SSYTs of all these $\{\mathcal{C}\cdot f\}_d^{\{l\}}$ bases are

$$\begin{aligned}
\{\mathcal{C}\cdot f\}_5^{1100} &= \left\{ \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \right\},\\
\{\mathcal{C}\cdot f\}_6^{0100} &= \left\{ \begin{array}{|c|c|}\hline 1 & 3\\\hline 4 & 1'\\\hline\end{array}\,, \begin{array}{|c|c|}\hline 1 & 4\\\hline 3 & 1'\\\hline\end{array} \right\},\\
\{\mathcal{C}\cdot f\}_6^{1000} &= \left\{ \begin{array}{|c|c|}\hline 2 & 3\\\hline 4 & 2'\\\hline\end{array}\,, \begin{array}{|c|c|}\hline 2 & 4\\\hline 3 & 2'\\\hline\end{array} \right\},\\
\{\mathcal{C}\cdot f\}_7^{0000} &= \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 4 & 2' & 1'\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1 & 2 & 4\\\hline 3 & 2' & 1'\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1 & 2 & 2'\\\hline 3 & 4 & 1'\\\hline\end{array}\,,\right.\\
&\qquad \left. \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 2 & 2' & 1'\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1 & 3 & 2'\\\hline 2 & 4 & 1'\\\hline\end{array} \right\}.
\end{aligned} \tag{46}$$

Read out all the massive amplitude bases $\{\mathcal{C}\cdot F\}_d^{\{l\}}$ according to the discussions in Sec. VI A and convert them into holomorphic bases $\{\mathcal{C}'\cdot F'\}_d^{\{l\}}$ by EOMs. And then decompose them into $\{\mathcal{B}\cdot H\}^{D=7}$ in the ascending order of d following the four steps in Sec. V (see Fig. 1),

$$\begin{aligned}
&\langle 1^I 2^J\rangle \to \langle 12\rangle[11^I][22^J] = \langle 23\rangle[31^I][22^J] + \langle 24\rangle[41^I][22^J],\\
&\langle 2^J 3\rangle[31^I] \to \langle 23\rangle[31^I][22^J] = \langle 23\rangle[31^I][22^J],\\
&\langle 42^J\rangle[41^I] \to \langle 42\rangle[41^I][22^J] = -\langle 24\rangle[41^I][22^J],\\
&\cdots, \qquad (47)
\end{aligned}$$

where the bases in the left-side of the arrow are $\{\mathcal{C}\cdot F\}_d^{\{l\}}$ bases, and the right-side are holomorphic bases $\{\mathcal{C}'\cdot F'\}_d^{\{l\}}$. Here since the decomposition is at the leading order in mass factor, the LG indices of all the spinors can be neglected except the ones of polarization tensor. We can see that these three bases are linearly correlated, so we should remove the third redundant base with higher d. After removing all the linear correlation terms in Eq. (45) and recovering the massive momentums from the massless limit, we get the bases with the lowest dimension, which is equivalent to $\{\mathcal{B}\cdot H\}$ bases,

$$\begin{aligned}
&\{\mathcal{B}\cdot H\}^5 \to \{[1^I 2^J]\},\\
&\{\mathcal{B}\cdot H\}^7 \to \{\langle 1^I 2^J\rangle, [1^I 32^J\rangle, \langle 1^I 32^J], s_{24}[1^I 2^J], s_{34}[1^I 2^J]\},\\
&\{\mathcal{B}\cdot H\}^9 \to \{s_{34}\langle 1^I 2^J\rangle, \langle 1^I 342^J\rangle, \cdots\}. \qquad (48)
\end{aligned}$$

With these lowest dimensional bases at dimension $d = 5, 6, 7$ on the right-hand side ("$\cdots$" refer to higher dim-d operators), we can directly map them into operator bases following the rules in Eq. (34).

VII. IDENTICAL PARTICLES

If the scattering process involves n identical bosons (fermions), the scattering amplitude should be in the totally (ant-) symmetric representation of the permutation group S_n (correspond to the S_n YD $[n]$ ($[1^n]$)). Next, we will discuss systematically constructing the complete set of amplitude bases involving identical particles.

The Young operator $\mathcal{Y}_{[R_n]}$ of the S_n representation $[R_n] = [n]$ or $[1^n]$ is the permutation operation that make the wave function of the n identical particles totally symmetric or anti-symmetric. So the eigenstate of $\mathcal{Y}_{[R_n]}$ must be in the $[R_n]$ representation. Thus, the basic idea to construct a complete set of amplitude bases in the representation $[R_n]$ is that: first, use the Young operator $\mathcal{Y}_{[R_n]}$ to act on the space of the complete bases $\{\mathcal{M}_d\}$ at dim-d, then get the representation matrix $M_{[R_n]}$ of $\mathcal{Y}_{[R_n]}$, and finally the eigenvectors with nonzero eigenvalues (actually is 1) correspond to the amplitude bases in the $[R_n]$ representation. In contrast, the eigenvectors with zero eigenvalues correspond to the bases that vanish under identical particle permutations.

Generally, an amplitude base consists of two parts: the gauge structure (T) and Lorentz structure ($\mathcal{D}$),

$$\mathcal{M} = T\times\mathcal{D}. \qquad (49)$$

So the complete set of amplitude bases can be constructed by combining the complete sets of gauge structure and Lorentz structure bases, $\{T\}$ and $\{\mathcal{D}\}$,

$$\{\mathcal{M}\} = \{T\}\times\{\mathcal{D}\}. \qquad (50)$$

Notice that any Young operator of the permutation group S_n can be expressed as a function of permutation elements (12) and $(1\ldots n)$, so we only need to get their representation matrices in $\{T\}$ and $\{\mathcal{D}\}$ space, denoted as $M^T_{(12),(1,\ldots,n)}$ and $M^{\mathcal{D}}_{(12),(1,\ldots,n)}$. So representation matrix $M_{[R_n]}$ is determined by the outer product of matrices $M^T_{(12),(1,\ldots,n)}$ and $M^{\mathcal{D}}_{(12),(1,\ldots,n)}$. For example, the matrix of totally symmetric representation [3] of S_3 group can be expressed as

$$\begin{aligned}
&M_{\boxed{1|2|3}} = (\mathbb{1} + x + y + xy + yx + xyx)/6,\\
&x = M^T_{(12)}\otimes M^{\mathcal{D}}_{(12)}, \quad y = M^T_{(123)}\otimes M^{\mathcal{D}}_{(123)},
\end{aligned} \qquad (51)$$

where $M^a_{(12),(123)}$ is the representation matrix of (12) or (123) in $a\in\{T,\mathcal{D}\}$ space. In Appendix B, we present the systematical method to calculate their representation matrices.

Similarly, suppose that the amplitude bases have different identical bosons (fermions). In that case, we only need to multiply the matrix $M_{[R_n]}$ of each permutation group together to get a total matrix, and the eigenvectors with nonzero eigenvalues are the bases allowed by identical particle statistics.

Generally, the amplitude bases $\{\mathcal{M}_{id}\}$ satisfying identical particle statistics are polynomials of monomial bases $\{\mathcal{O}^{\rm phy}\}$. When we map these amplitudes into operator bases through the relations in Eq. (34), the operator bases are also very long polynomials. Practically, when a monomial amplitude basis, which is not in $[R_n]$ representation, is mapped into an operator, this operator automatically satisfies identical particle statistics (the Feynman rule of any operator automatically enforces identical particle permutation symmetries in this operator). So we do not need to map all the terms of the $\mathcal{M}_{id}$ basis into operators, but one of its independent monomial terms into an operator basis. Obviously, this amplitude monomial's operator must be equivalent to the operator fully mapped from the amplitude basis $\mathcal{M}_{id}$. So, in this way, we can get a complete set of simplest operator bases equivalent to $\{\mathcal{M}_{id}\}$, and they can be used more conveniently in calculations. In Appendix C, we list a complete set of four-vector operator bases at dimension-4 and 6.

VIII. CONCLUSION

EFT of massive fields is widely applied in various fields of physics. However, how to systematically construct the complete set of EFT bases of massive fields is still a long-standing problem. We propose a novel theory based on on-shell scattering amplitude to construct the complete set of lowest dimensional amplitude bases at any given dimension for massive fields with any spins. These bases can be directly mapped into physical operator bases without any redundancy.

The massive amplitude bases with the lowest dimension can be constructed through three steps. First, we systematically construct a complete set of massive amplitude bases $\{\mathcal{B}\cdot H\}$ by the enlarged SSYTs without horizontal permutation symmetries, which are constructed by gluing the SSYTs of Lorentz subgroup $SU(2)_r$ and global symmetry $U(N)$. These bases are just monomials of spinor products but not the lowest dimensional amplitude bases. Second, since massive amplitude bases can be classified by the configurations of massive polarization tensors, we can systematically construct a complete but redundant basis set $\{\mathcal{C}\cdot F\}$ that consists of all the complete sets of massive amplitude bases with different polarization configurations. Since the bases with the lowest dimension must have a kind of polarization configuration, this redundant set always contains a complete set of lowest dimensional amplitude bases. Finally, since $\{\mathcal{B}\cdot H\}$ bases are complete and independent monomials, we can systematically decompose the $\{\mathcal{C}\cdot F\}$ bases into $\{\mathcal{B}\cdot H\}$ bases from low to high dimension and eliminate the linear correlation bases through their coordinates in $\{\mathcal{B}\cdot H\}$ space. After these procedures, we can always obtain a complete set of the lowest dimensional amplitude bases. We also give an example to explain how to get this kind of base systematically.

The amplitude bases involving identical particles can also be systematically constructed. First, we find the representation matrices of the Young operators, associated with the permutation symmetry representations required by spin-statistics, in the amplitude basis space and then multiply these matrices together to get a total matrix. Finally, the eigenvectors with nonzero eigenvalues of this matrix correspond to the bases satisfying spin statics.

Based on this theory, we write the *Mathematica* codes that can automatically construct a complete set of lowest dimensional amplitude bases at a given dimension. We show the complete set of all four-vectors operator bases at dimension-4 and 6 in Appendix C (also the complete bases involving identical particles). A complete set of other kinds of massive operator bases will be presented in the later work [42].

Within this theory, constructing massive EFT is not a problem. Our work provides an efficient tool to study the low energy effects of UV theories at the EWSB phase. The wave function normalization of massive particles at the EWSB phase does not need to be cared about, and the complete sets of three-point and four-point massive EFT bases are enough for phenomenology study generally. So when doing the massive operator matching, we do not need to deal with the high point EFT bases, which can simplify the calculations very much. While in massless EFT, such as SMEFT, many higher point bases involving Higgs doublets always contribute to wave functions of particles, three-point, and four-point interactions at the EWSB phase, which makes EFT calculations complicated. We can also use them to study dark matter interactions with experimental detections and analyze the dark matter signals from different UV models. Massive EFT could have some advantages in various scenarios of physics, and a lot of its exciting applications deserve to be explored in the future.

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Note added.—While our paper was being finalized, Ref. [43] appeared, which presents a similar topic. This work uses a graphic method to construct the massive amplitude bases, equivalent to the Young tableaux method used here. Nevertheless, some of the assumptions in Ref. [43] are only numerically checked without rigorous proof. On the contrary, our work has a solid mathematical foundation.

APPENDIX A: PROOF OF LEADING ORDER DECOMPOSITION

In this section, we prove that the independence of physical amplitude bases $\{\mathcal{O}^{\rm phy}\}$ with fake dimension $D' = D_{\mathcal{O}^{\rm phy}}$ is determined by their coordinates in the space of $\{\mathcal{B}.H\}^{D=D_{\mathcal{O}^{\rm phy}}}$ bases with same fake dimension.

For the first case, we suppose that two physical bases $\mathcal{O}^{\text{phy}}_{1,2}$ with the same PTC and the same fake dimension $D' = D_{\mathcal{O}^{\text{phy}}}$ are independent (for simplicity only l_i in their PTC is assumed to be nonzero $\vec{l} = \{\ldots, 0, l_i, 0\ldots\}$) and their coordinates in the $\{\mathcal{B}\cdot H\}^{D=D_{\mathcal{O}^{\text{phy}}}}$ space are assumed to be the same. So, after converting the $\mathcal{O}^{\text{phy}}_{1,2}$ into holomorphic bases $\mathcal{O}^{\text{phy}}_{1,2} \to \mathcal{O}'^{\text{phy}}_{1,2} = ([i^J i^I])^{l_i}\mathcal{O}^{\text{phy}\{J^{l_i}\}}_{1,2}$ (J^{l_i} represents the LG indices of l_i $|i^J\rangle$ s in PTC), we can get the decomposition of their difference,

$$([i^J i^I])^{l_i}\left(\mathcal{O}^{\text{phy}\{J^{l_i}\}}_1 - \mathcal{O}^{\text{phy}\{J^{l_i}\}}_2\right) \to \{\mathcal{B}\cdot H\}^{D=D_{\mathcal{O}^{\text{phy}}}-2} + \{\mathcal{B}\cdot H\}^{D=D_{\mathcal{O}^{\text{phy}}}-4} + \cdots. \quad \text{(A1)}$$

Since the PTCs at both sides are the same, the decomposition is only determined by the MLGNSs of the holomorphic bases $\mathcal{O}'^{\text{phy}}_{1,2}$. If their MLGNSs go to massless limit (massive momentums go to massless limit) and the spinors in PTCs keep intact, above decomposition should become null,

$$([i^J i^I])^{l_i}\left(\mathcal{O}^{\text{phy}\{J^{l_i}\}}_1 - \mathcal{O}^{\text{phy}\{J^{l_i}\}}_2\right)|_{p_i\to|i\rangle[i|} = ([ii^I])^{l_i}\left((\mathcal{C}.f)^{\vec{l}}_{\mathcal{O}^{\text{phy}}_1} - (\mathcal{C}.f)^{\vec{l}}_{\mathcal{O}^{\text{phy}}_2}\right) = 0, \quad \text{(A2)}$$

where $(\mathcal{C}.f)^{\vec{l}}_{\mathcal{O}^{\text{phy}}_{1,2}}$ are the massless limits of $\mathcal{O}^{\text{phy}}_{1,2}$ defined in Sec. VI A. This is because the coordinates in $\{\mathcal{B}\cdot H\}^{D\leq D_{\mathcal{O}^{\text{phy}}}-2}$ space always proportional to mass factor due to fake dimension mismatch at both sides of Eq. (A1) and the mass factor is only generated from spinor EOMs in $\mathcal{O}'^{\text{phy}}_{1,2}$ MLGNSs. Since the identity in Eq. (A2) is independent of the spinors in PTCs, the factor $([ii^I])^{l_i}$ can be treated as independent variables. So the only solution to above equation is $(\mathcal{C}.f)^{\vec{l}}_{\mathcal{O}^{\text{phy}}_1} = (\mathcal{C}.f)^{\vec{l}}_{\mathcal{O}^{\text{phy}}_2}$, which means the two bases $\mathcal{O}^{\text{phy}}_{1,2}$ are the same (see the discussions in Sec. VI A). It conflicts with our assumption, so we prove that the independence of lowest dimensional bases $\{\mathcal{O}^{\text{phy}}_{1,2}\}$ with the same PTC is determined by leading decomposition. Above proof can be easily generalized to the case for any number bases.

For the second case, we assume that the two physical bases $\mathcal{O}^{\text{phy}}_{1',2'}$ with different PTCs and the same fake dimension $D = D'_{\mathcal{O}^{\text{phy}}}$ have the same coordinates in the $\{\mathcal{B}\cdot H\}^{D=D'_{\mathcal{O}^{\text{phy}}}}$ space. For simplicity we assume that, only l_i and l_j in their PTCs are nonzero respectively, $\vec{l}_1 = \{0,\ldots,0,l_i,0,\ldots,0\}$ and $\vec{l}_2 = \{0,\ldots,0,l_j,0,\ldots,0\}$. Following the same logic, decomposing their difference into $\{\mathcal{B}\cdot H\}$ space, we can also get the similar massless limit identity,

$$\left(([i^{I'} i^I])^{l_i}\mathcal{O}^{\text{phy}\{(I')^{l_i}\}}_{1'} - ([j^{J'} j^J])^{l_j}\mathcal{O}^{\text{phy}\{(J')^{l_j}\}}_{2'}\right)|_{p_i\to|i\rangle[i|} = ([ii^I])^{l_i}(\mathcal{C}.f)^{\vec{l}_1}_{\mathcal{O}^{\text{phy}}_1} - ([jj^J])^{l_j}(\mathcal{C}.f)^{\vec{l}_2}_{\mathcal{O}^{\text{phy}}_2} = 0, \quad \text{(A3)}$$

where $(\mathcal{C}.f)^{\vec{l}_1}_{\mathcal{O}^{\text{phy}}_1}$ ($(\mathcal{C}.f)^{\vec{l}_2}_{\mathcal{O}^{\text{phy}}_2}$) is the massless limit of $\mathcal{O}^{\text{phy}}_1$ ($\mathcal{O}^{\text{phy}}_2$) defined in Sec. VI A. As discussed above, since $([ii^I])^{l_i}$ and $([jj^J])^{l_j}$ are two independent LG tensors, their massless limit bases $(\mathcal{C}.f)^{\vec{l}_1}_{\mathcal{O}^{\text{phy}}_1}$ and $(\mathcal{C}.f)^{\vec{l}_2}_{\mathcal{O}^{\text{phy}}_2}$ should contain the tensor factor $([jj^J])^{l_j}$ and $([ii^I])^{l_i}$ respectively to guarantee that this identity can be satisfied. If so, it means that the massive basis $\mathcal{O}^{\text{phy}}_{1'} = (\mathcal{C}.F)^{\vec{l}_1}_{\mathcal{O}^{\text{phy}}_1}$ ($\mathcal{O}^{\text{phy}}_{2'} = (\mathcal{C}.F)^{\vec{l}_2}_{\mathcal{O}^{\text{phy}}_2}$) should contain factor $([j^{J'} j^J])^{l_j}$ ($([i^{I'} i^I])^{l_i}$). However this factor is proportional to mass, so it means that $\mathcal{O}^{\text{phy}}_{1',2'}$ are not the lowest dimension bases, conflicting with $\mathcal{O}^{\text{phy}}_{1',2'}$ definition. So the independence of two lowest dimensional bases with the same fake dimension and different PTCs is only determined by the coordinates in $\{\mathcal{B}\cdot H\}^{D=D'_{\mathcal{O}^{\text{phy}}}}$ space.

The above two proofs can be easily generalized to the case for any number of bases. So in summary the $\{\mathcal{O}^{\text{phy}}\}$ bases can be picked up just through the leading decomposition of $\{\mathcal{C}\cdot F\}_d$ in Eq. (42),

$$\{\mathcal{C}\cdot F\}^{\{l\}}_d \to \{\mathcal{C}'\cdot F'\}^{\{l\}}_d \to \{\mathcal{B}\cdot H\}^{D=D_{\mathcal{C}\cdot F}}. \quad \text{(A4)}$$

Since the coefficients of this leading decomposition are independent of mass factors, any mass factors that appear in the decomposition procedures discussed in Sec. V should be discarded.

APPENDIX B: REPRESENTATION MATRIX FOR S_n

1. Gauge part

In this section we explain how to calculate the Young operator matrix $M^T_{[R_n]}$ of S_n representation $[R_n]$ in gauge structure $\{T\}$ space. For concreteness, we take the $SU(3)_c$ color group as an example.

We suppose that the amplitude bases contain QCD quarks, antiquarks, and gluons. According to their $SU(3)_c$ quantum numbers, the YDs of their $SU(3)_c$ representation are in the forms,

$$\psi^a \sim \boxed{a} \qquad \epsilon^{abc}\bar{\psi}_a \sim \begin{array}{|c|}\hline b\\\hline c\\\hline\end{array} \qquad \epsilon^{abc}\lambda^{id}_a g^i \sim \begin{array}{|c|c|}\hline b & d\\\hline c\\\cline{1-1}\end{array}. \quad \text{(B1)}$$

where λ^i refer to the eight Gell-Mann matrices. The complete set of $SU(3)_c$ $\{T\}$ bases consists of all the Standard Young tableaus (SYTs) in the shape of $[x,x,x]$

($SU(3)_c$ singlet) satisfying above $SU(3)_c$ index permutation symmetries of all $\bar{\psi}_a$s and g^is. In the following, we will discuss how to systematically construct the $\{T\}$ bases and then discuss how to get the matrices $M^T_{[R_n]}$.

We can first find a set of $SU(3)_c$ gauge structure bases $\{T'\}$ consists of all the SYTs with the shape of $[x,x,x]$, where the length x of each row is determined by the $SU(3)_c$ quantum numbers of external fields. The set of $\{T'\}$ bases contain the $\{T\}$ base we need. Then we can use the projection matrix $\mathcal{P}$ (see below) to project out the structures whose indices satisfy the permutation symmetry of all $\bar{\psi}_a$s' and g^is' $SU(3)_c$ indices. So these projected out bases are the $\{T\}$ bases. Next, we will list the procedures to get matrices $M^T_{[R_n]}$.

(i) Get the representation matrix $M^{T'}_{(12)/(1\ldots n)}$ of S_n element (12) or $(1\ldots n)$ in $\{T'\}$ space: Since there are totally $n_Y \equiv \frac{2(3x)!}{x!(x+1)!(x+2)!}$ SYTs with shape $[x,x,x]$, the number of $\{T'\}$ bases is equal to n_Y. The $(n_Y\times n_Y)$ representation matrix $M^{T'}_{(12)}$ of (12) can be obtained by first using permutation element (12) to act on the color indices of $\{T'\}$ base associated with particle-1 and -2 and then decomposing them into the n_Y SYTs. $M^{T'}_{(1\ldots n)}$ can also be obtain in the same way.

(ii) Get the projection matrix $\mathcal{P}$: $\mathcal{P}$ is the product of adjoint (anti-fundamental) Young operators of each external particle gs' ($\bar{\psi}$s) YDs.

$$\mathcal{P} \equiv \overbrace{\mathcal{Y}_{\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, \\ \cline{1-1}\end{array}} \cdots \mathcal{Y}_{\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, \\ \cline{1-1}\end{array}}}^{adjoint} \overbrace{\mathcal{Y}_{\tiny\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}} \cdots \mathcal{Y}_{\tiny\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}}}^{anti-fund.}. \qquad \text{(B2)}$$

The amplitude gauge structure bases $\{T\}$ can be projected out from the enlarged $\{T'\}$ bases by acting $\mathcal{P}$ on the n_Y SYTs and decomposing them back to the n_Y SYTs. Finally we can get the $(n_Y \times n_Y)$ matrix representation P of $\mathcal{P}$ in $\{T'\}$ space. The eigenvectors with nonzero eigenvalues of P are the $\{T\}$ bases consistent with $SU(3)_c$ quantum number of external legs.

The method used in the above decompositions is using just the Fock condition. Using this method, we can turn any Young Tableau into a combination of SYTs. Finally, we get the $SU(3)_c$ representation matrix for (12) and $(1\cdots n)$ in the space of $\{T\}$ bases (P is to project out $\{T\}$ bases from $\{T'\}$),

$$M^T_{(12)/(1\cdots n)} = M^{T'}_{(12)/(1\cdots n)} \cdot P. \qquad \text{(B3)}$$

Since $M^{T'}$ and P are commutative, we can commutate all P in Eq. (51) to the most right position, which can simplify the calculation. Finally, the totally symmetric representation of S_3 can be expressed as

$$M_{\tiny\boxed{1}\boxed{2}\boxed{3}} = (\mathbb{1} + x + y + xy + yx + xyx)(P\otimes \mathbb{1}^{\mathcal{D}})/6,$$
$$x = M^{T'}_{(12)} \otimes M^{\mathcal{D}}_{(12)}, \quad y = M^{T'}_{(123)} \otimes M^{\mathcal{D}}_{(123)}. \qquad \text{(B4)}$$

2. Lorentz part

To get $M^{\mathcal{D}}_{(12)}$ in bases $\{\mathcal{O}^{\rm phy}\}$ space, we need to act the permutation element (12) on $\{\mathcal{O}^{\rm phy}\}$ and then decompose the generated bases back into the combinations of $\{\mathcal{O}^{\rm phy}\}$ bases,

$$(12)(\mathcal{O}^{\rm phy})^D_d = \sum m^{(d-d')}(\mathcal{O}^{\rm phy})^{D'}_{d'}, \qquad \text{(B5)}$$

where the superscript D and D' are fake dimensions. And due to the following reasons,

(i) No matter whether using Schouten identity or momentum conservation, it will only lower its fake dim-D;

(ii) $\{\mathcal{O}^{\rm phy}_d\}$ is the complete set of bases with the lowest dimension, so $(12)(\mathcal{O}^{\rm phy}_d)$ could only be decomposed into some bases with dimension lower than d,

each term on the right-hand side of Eq. (B5) should satisfy the following constraints,

$$d' \le d, \qquad D' \le D. \qquad \text{(B6)}$$

It means that the representation matrix $M^{\mathcal{D}}_{(12)}$ is a partitioned upper triangular matrix in the space of the bases arranged in the ascending order of both dim-D and dim-d, as shown in Fig. 2. This property tells us that the eigenvectors with nonzero eigenvalues of the diagonal sub-matrix $M^{\mathcal{D}}_{D(12)}$ in the base space with a certain fake dimension D one-to-one correspond to the eigenvectors with nonzero eigenvalues of the full matrix $M^{\mathcal{D}}_{(12)}$. So these diagonal sub-matrices $M^{\mathcal{D}}_{D(12)}$ in $M^{\mathcal{D}}_{(12)}$ are enough to project out the bases involving identical particles ($M^{\mathcal{D}}_{D(12)}$ correspond to the diagonal gray boxes in Fig. 2). Using the following steps, we can get the submatrix $M^{\mathcal{D}}_{D(12)}$ for a certain dim-D (the box within blue circle in Fig. 2).

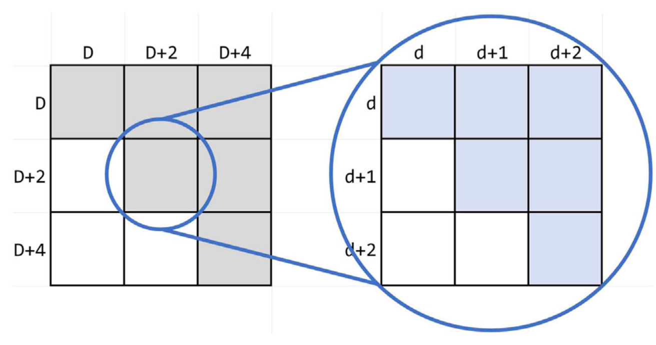

FIG. 2. Partitioned upper triangular matrix $M^{\mathcal{D}}_{(12)/(1\cdots n)}$.

(i) Decompose $(12)\{\mathcal{O}^{\rm phy}\}^D$ to $\{\mathcal{B}\cdot H\}^D$, and get the matrix $M_{(12)}^{BH\leftarrow\mathcal{O}}$
(ii) Decompose $\{\mathcal{O}^{\rm phy}\}^D$ to $\{\mathcal{B}\cdot H\}^D$, and get the matrix $M_e^{BH\leftarrow\mathcal{O}}$
(iii) Finally $M_{D(12)}^{\mathcal{D}} = [M_e^{BH\leftarrow\mathcal{O}}]^{-1}\times M_{(12)}^{BH\leftarrow\mathcal{O}}$

Since $M_{D(12)}^{\mathcal{D}}$ is still partitioned upper triangular matrix, now we only need to calculate the eigenvectors of its diagonal blocks (the diagonal blue boxes), which have certain dim-D and dim-d. When we map an independent monomial of an amplitude basis to an EFT operator, this operator will automatically satisfy identical particle statistics enforced by Feynman rules. So we do not need to map the full eigenvectors with nonzero eigenvalues (correspond long polynomials of spinor products) into operators but just need to find one independent component of each eigenvector and map it into an EFT operator, which is the simplest operator basis satisfying spin statistics.

APPENDIX C: CODE OUTPUT: 4-PT GAUGE BOSON EFT OPERATORS

In this section, we list all the four-point amplitude bases and the corresponding operator bases for massive gauge boson Z and $W^\pm$ at dim-4 and 6. Notice that for simplicity, we do not symmetrize the expressions of the amplitude bases involving identical particles, and the full expressions can be obtained by just symmetrizing these amplitude bases in the following tables. In the first column of the following Tables, the bold bracket is defined as $[\mathbf{i}\,\mathbf{j}]\equiv[i^I j^J]$ (we follow the conventions in [41]).

For the vertex $D^{2n}ZZW^+W^-$ (D_μ is the QED covariant derivative), we have 2 physical dim-4 operators and 29 dim-6 operators, and we define

$$\acute{F}_{\mu\nu}\equiv F_{\mu\nu}+i\tilde{F}_{\mu\nu}\quad \grave{F}_{\mu\nu}\equiv F_{\mu\nu}-i\tilde{F}_{\mu\nu}. \tag{C1}$$

Amplitude	Operator $D^{2n}ZZW^+W^-$
$\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle[\mathbf{42}][\mathbf{31}]$	$Z^\mu Z^\nu W^+_\mu W^-_\nu$
$\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle[\mathbf{43}][\mathbf{21}]$	$Z^\mu Z_\mu W^{+\nu} W^-_\nu$
$\langle\mathbf{34}\rangle[\mathbf{42}][\mathbf{31}][\mathbf{21}]$	$\acute{Z}^{\mu\nu}Z_{\mu\nu}W^{+\rho}W^-_\rho - i\epsilon^{\mu\nu\rho\sigma}Z_{\mu\gamma}Z^\gamma_\nu W^+_\rho W^-_\sigma$
$\langle\mathbf{24}\rangle[\mathbf{42}][\mathbf{31}]^2$	$\acute{Z}^{\mu\nu}W^+_{\mu\nu}Z^\rho W^-_\rho$
$\langle\mathbf{24}\rangle[\mathbf{43}][\mathbf{31}][\mathbf{21}]$	$\acute{Z}^{\mu\nu}(W^+_{\mu\nu}W^-_\rho Z^\rho + 2Z_\mu W^+_{\rho\nu}W^{-\rho} - 2W^-_\mu W^+_{\rho\nu}Z^\rho)$
$\langle\mathbf{23}\rangle[\mathbf{42}][\mathbf{41}][\mathbf{31}]$	$\acute{Z}^{\mu\nu}(W^-_{\mu\nu}Z^\rho W^+_\rho + 2W^+_\mu W^-_{\rho\nu}Z^\rho - 2Z_\mu W^-_{\rho\nu}W^{+\rho})$
$\langle\mathbf{23}\rangle[\mathbf{43}][\mathbf{41}][\mathbf{21}]$	$\acute{Z}^{\mu\nu}(W^-_{\mu\nu}W^+_\rho Z^\rho + 2Z_\mu W^-_{\rho\nu}W^{+\rho} - 2W^+_\mu W^-_{\rho\nu}Z^\rho)$
$\langle\mathbf{12}\rangle[\mathbf{43}][\mathbf{42}][\mathbf{31}]$	$\acute{W}^{+\mu\nu}W^-_{\mu\nu}Z^\rho Z_\rho$
$\langle\mathbf{24}\rangle\langle\mathbf{34}\rangle[\mathbf{31}][\mathbf{21}]$	$\acute{Z}^{\mu\nu}(W^-_{\mu\nu}Z^\rho W^+_\rho - 2W^+_\mu W^-_{\rho\nu}Z^\rho - 2Z_\mu W^-_{\rho\nu}W^{+\rho})$
$\langle\mathbf{23}\rangle\langle\mathbf{34}\rangle[\mathbf{41}][\mathbf{21}]$	$\acute{Z}^{\mu\nu}(W^+_{\mu\nu}Z^\rho W^-_\rho - 2W^-_\mu W^+_{\rho\nu}Z^\rho - 2Z_\mu W^+_{\rho\nu}W^{-\rho})$
$\langle\mathbf{23}\rangle\langle\mathbf{24}\rangle[\mathbf{41}][\mathbf{31}]$	$\acute{Z}^{\mu\nu}(Z_{\mu\nu}W^{+\rho}W^-_\rho - 2W^-_\mu Z_{\rho\nu}W^{+\rho} - 2W^+_\mu Z_{\rho\nu}W^{-\rho})$
$\langle\mathbf{14}\rangle\langle\mathbf{24}\rangle[\mathbf{32}][\mathbf{31}]$	$\acute{W}^{+\mu\nu}(W^-_{\mu\nu}Z^\rho Z_\rho - 4Z_\mu W^-_{\rho\nu}Z^\rho)$
$\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle[\mathbf{42}][\mathbf{41}]$	$\acute{W}^{-\mu\nu}(W^+_{\mu\nu}Z^\rho Z_\rho - 4Z_\mu W^+_{\rho\nu}Z^\rho)$
$\langle\mathbf{12}\rangle\langle\mathbf{24}\rangle[\mathbf{43}][\mathbf{31}]$	$\acute{W}^{+\mu\nu}(Z_{\mu\nu}W^{-\rho}Z_\rho - 2Z_\mu Z_{\rho\nu}W^{-\rho} - 2W^-_\mu Z_{\rho\nu}Z^\rho)$
$\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{43}][\mathbf{41}]$	$\acute{W}^{-\mu\nu}(Z_{\mu\nu}W^{+\rho}Z_\rho - 2Z_\mu Z_{\rho\nu}W^{+\rho} - 2W^+_\mu Z_{\rho\nu}Z^\rho)$
$\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle\langle\mathbf{34}\rangle[\mathbf{21}]$	$\grave{W}^{+\mu\nu}W^-_{\mu\nu}Z^\rho Z_\rho$
$\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle^2[\mathbf{31}]$	$\grave{Z}^{\mu\nu}W^-_{\mu\nu}W^{+\rho}Z_\rho$
$\langle\mathbf{12}\rangle\langle\mathbf{24}\rangle\langle\mathbf{34}\rangle[\mathbf{31}]$	$\grave{Z}^{\mu\nu}(W^-_{\mu\nu}W^{+\rho}Z_\rho + 2Z_\mu W^-_{\rho\nu}W^{+\rho} - 2W^+_\mu W^-_{\rho\nu}Z^\rho)$
$\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle\langle\mathbf{24}\rangle[\mathbf{41}]$	$\grave{W}^{+\mu\nu}(Z_{\mu\nu}W^{-\rho}Z_\rho + 2Z_\mu Z_{\rho\nu}W^{-\rho} - 2W^+_\mu Z_{\rho\nu}Z^\rho)$
$\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle\langle\mathbf{34}\rangle[\mathbf{41}]$	$\grave{Z}^{\mu\nu}(W^+_{\mu\nu}W^{-\rho}Z_\rho + 2Z_\mu W^+_{\rho\nu}W^{-\rho} - 2W^-_\mu W^+_{\rho\nu}Z^\rho)$
$\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle[\mathbf{43}]$	$\grave{Z}^{\mu\nu}(Z_{\mu\nu}W^{-\rho}W^+_\rho + 2W^+_\mu Z_{\rho\nu}W^{-\rho} - 2W^-_\mu Z_{\rho\nu}W^{+\rho})$
$\langle\mathbf{3}2\mathbf{3}]\langle\mathbf{24}\rangle[\mathbf{41}][\mathbf{21}]$	$i\acute{Z}^{\mu\nu}W^-_\mu(D^\rho Z_\nu)W^+_\rho$
$\langle\mathbf{4}2\mathbf{4}]\langle\mathbf{23}\rangle[\mathbf{31}][\mathbf{21}]$	$i\acute{Z}^{\mu\nu}W^+_\mu(D^\rho Z_\nu)W^-_\rho$
$\langle\mathbf{1}3\mathbf{4}]\langle\mathbf{24}\rangle[\mathbf{32}][\mathbf{31}]$	$iD_\rho(\acute{W}^{+\mu\rho}g^{\nu\sigma} + \acute{W}^{+\rho\sigma}g^{\mu\nu} + \acute{W}^{+\rho\nu}g^{\mu\sigma} + \acute{W}^{+\sigma\mu}g^{\nu\rho} + \acute{W}^{+\nu\mu}g^{\rho\sigma})W^-_\mu Z_\nu Z_\sigma$
$\langle\mathbf{2}3\mathbf{4}]\langle\mathbf{13}\rangle[\mathbf{42}][\mathbf{31}]$	$i\acute{W}^{-\mu\nu}Z_\nu(D_\mu W^+_\rho)Z^\rho$
$\langle\mathbf{4}2\mathbf{3}]\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle[\mathbf{21}]$	$i(\grave{W}^{-\mu\rho}g^{\nu\sigma} + \grave{W}^{-\rho\sigma}g^{\mu\nu} + \grave{W}^{-\rho\nu}g^{\mu\sigma} + \grave{W}^{-\sigma\mu}g^{\nu\rho} + \grave{W}^{-\nu\mu}g^{\rho\sigma} + \grave{W}^{-\sigma\nu}g^{\mu\rho})(D_\sigma Z_\nu)Z_\rho W^+_\mu$
$\langle\mathbf{4}2\mathbf{4}]\langle\mathbf{13}\rangle\langle\mathbf{23}\rangle[\mathbf{21}]$	$i\grave{W}^{+\mu\nu}(D_\rho Z_\mu)Z_\nu W^{-\rho}$

(Table continued)

(Continued)

Amplitude	Operator $D^{2n}ZZW^+W^-$
$\langle 234]\langle 13\rangle\langle 24\rangle[31]$	$i\grave{Z}^{\mu\nu}W^-_\mu(D_\nu W^+_\rho)Z^\rho$
$\langle 434]\langle 12\rangle\langle 23\rangle[31]$	$i\grave{Z}^{\mu\nu}Z_\mu(D_\rho W^+_\nu)W^{-\rho}$
$\langle 232]\langle 423]\langle 13\rangle[41]$	$(i\epsilon^{\mu\nu\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\nu}g^{\rho\sigma})(D_\xi W^+_\mu)W^-_\rho(D_\nu Z_\xi)Z^\sigma$
$\langle 424]\langle 242]\langle 13\rangle[31]$	$Z^\rho W^+_\rho(D_\mu W^{-\nu})(D_\nu Z^\mu)$

And for vertex $D^{2n}W^+W^+W^-W^-$, we have 2 physical dim-4 operators, and 18 dim-6 operators,

Amplitude	Operator $D^{2n}W^+W^+W^-W^-$
$\langle 13\rangle\langle 24\rangle[42][31]$	$W^{+\mu}W^{+\nu}W^-_\mu W^-_\nu$
$\langle 13\rangle\langle 24\rangle[43][21]$	$W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu$
$\langle 34\rangle[42][31][21]$	$\acute{W}^{+\mu\nu}W^+_{\mu\nu}W^{-\rho}W^-_\rho$
$\langle 24\rangle[42][31]^2$	$\acute{W}^{+\mu\nu}W^-_{\mu\nu}W^{+\rho}W^-_\rho$
$\langle 24\rangle[43][31][21]$	$\acute{W}^{+\mu\nu}(W^-_{\mu\nu}W^{-\rho}W^+_\rho+2W^+_\mu W^-_{\rho\nu}W^{-\rho}-2W^-_\mu W^-_{\rho\nu}W^{+\rho}$
$\langle 12\rangle[43][42][31]$	$\acute{W}^{-\mu\nu}W^-_{\mu\nu}W^{+\rho}W^+_\rho$
$\langle 24\rangle\langle 34\rangle[31][21]$	$\acute{W}^{+\mu\nu}(W^-_{\mu\nu}W^{+\rho}W^-_\rho-2W^-_\mu W^-_{\rho\nu}W^{+\rho}-2W^+_\mu W^-_{\rho\nu}W^{-\rho})$
$\langle 23\rangle\langle 24\rangle[41][31]$	$\acute{W}^{+\mu\nu}(W^+_{\mu\nu}W^{-\rho}W^-_\rho-4W^-_\mu W^+_{\rho\nu}W^{-\rho})$
$\langle 14\rangle\langle 24\rangle[32][31]$	$\acute{W}^{-\mu\nu}(W^-_{\mu\nu}W^{+\rho}W^+_\rho-4W^+_\mu W^-_{\rho\nu}W^{+\rho})$
$\langle 12\rangle\langle 24\rangle[43][31]$	$\acute{W}^{-\mu\nu}(W^+_{\mu\nu}W^{-\rho}W^+_\rho-2W^+_\mu W^+_{\rho\nu}W^{-\rho}-2W^-_\mu W^+_{\rho\nu}W^{+\rho})$
$\langle 13\rangle\langle 24\rangle\langle 34\rangle[21]$	$\grave{W}^{-\mu\nu}W^-_{\mu\nu}W^{+\rho}W^+_\rho$
$\langle 13\rangle\langle 24\rangle^2[31]$	$\grave{W}^{-\mu\nu}W^+_{\mu\nu}W^{-\rho}W^+_\rho$
$\langle 12\rangle\langle 24\rangle\langle 34\rangle[31]$	$\grave{W}^{+\mu\nu}(W^-_{\mu\nu}W^{-\rho}W^+_\rho+2W^+_\mu W^-_{\rho\nu}W^{-\rho}-2W^+_\mu W^-_{\rho\nu}W^{+\rho})$
$\langle 12\rangle\langle 13\rangle\langle 24\rangle[43]$	$\grave{W}^{+\mu\nu}W^+_{\mu\nu}W^{-\rho}W^-_\rho$
$\langle 323]\langle 24\rangle[41][21]$	$i\acute{W}^{+\mu\nu}W^-_\mu(D^\rho W^+_\nu)W^-_\rho$
$\langle 134]\langle 24\rangle[32][31]$	$iD_\rho(\acute{W}^{-\mu\rho}g^{\nu\sigma}+\acute{W}^{-\rho\sigma}g^{\mu\nu}+\acute{W}^{-\rho\nu}g^{\mu\sigma}+\acute{W}^{-\sigma\mu}g^{\nu\rho}+\acute{W}^{-\nu\mu}g^{\rho\sigma})W^-_\mu W^+_\nu W^+_\sigma$
$\langle 423]\langle 13\rangle\langle 24\rangle[21]$	$i(\grave{W}^{-\mu\rho}g^{\nu\sigma}+\grave{W}^{-\rho\sigma}g^{\mu\nu}+\grave{W}^{-\rho\nu}g^{\mu\sigma}+\grave{W}^{-\sigma\mu}g^{\nu\rho}+\grave{W}^{-\nu\mu}g^{\rho\sigma}+\grave{W}^{-\sigma\nu}g^{\mu\rho})(D_\sigma W^+_\nu)W^+_\rho W^-_\mu$
$\langle 234]\langle 13\rangle\langle 24\rangle[31]$	$i\grave{W}^{+\mu\nu}W^-_\mu(D_\nu W^-_\rho)W^{+\rho}$
$\langle 232]\langle 423]\langle 13\rangle[41]$	$(i\epsilon^{\mu\nu\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\nu}g^{\rho\sigma})(D_\xi W^-_\mu)W^-_\rho(D_\nu W^{+\xi})W^+_\sigma$
$\langle 424]\langle 242]\langle 13\rangle[31]$	$W^{+\rho}W^-_\rho(D_\mu W^{-\nu})(D_\nu W^{+\mu})$

Finally, for vertex $D^{2n}ZZZZ$, we only have 1 physical dim-4 operator, and 4 dim-6 operators,

Amplitude	Operator $D^{2n}ZZZZ$
$\langle 13\rangle\langle 24\rangle[42][31]$	$Z^\mu Z^\nu Z_\mu Z_\nu$
$\langle 34\rangle[42][31][21]$	$\acute{Z}^{\mu\nu}Z_{\mu\nu}Z^\rho Z_\rho$
$\langle 24\rangle\langle 34\rangle[31][21]$	$\acute{Z}^{\mu\nu}(Z_{\mu\nu}Z^\rho Z_\rho-4Z_\mu Z_{\rho\nu}Z^\rho)$
$\langle 13\rangle\langle 24\rangle\langle 34\rangle[21]$	$\grave{Z}^{\mu\nu}Z_{\mu\nu}Z^\rho Z_\rho$
$\langle 232]\langle 423]\langle 13\rangle[41]$	$(D_\mu Z^\nu)(D_\nu Z^\mu)Z_\rho Z^\rho$

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