Analogy of the Lorentz-violating fermion-gravity and fermion photon couplings

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By adopting a methodology proposed by R. J. Adler *et al.*, we study the interesting analogy between the fermion-gravity and the fermion-electromagnetic interactions in the presence of the minimal Lorentz-violating (LV) fermion coefficients. The single-fermion matrix elements of gravitational interaction (SMEGI) are obtained with a prescribed Lense-Thirring (LT) metric assuming test particle assumption. Quite distinct from the extensively studied linear gravitational potential, the LT metric is essentially curved and thus reveals the anomalous LV matter-gravity couplings as a manifestation of the so-called gravito-magnetic effects, which go beyond the conventional equivalence principle predictions. By collecting all the spin-dependent operators from the SMEGI with some reasonable assumptions, we get a LV nonrelativistic Hamiltonian, from which we derive the anomalous spin precession and gravitational acceleration due to LV. By combining these results with certain spin gravity experiments, we get some rough bounds on several LV coefficients, such as $|3\vec{\tilde{H}} - 2\vec{b}| \leq 1.46 \times 10^{-5}$ GeV, with some *ad hoc* assumptions.

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I. INTRODUCTION

Classical electrodynamics and its quantum version, QED, are ideal paradigms for modern physics. As a quantum theory of matter-electromagnetic coupling, QED has reached an unprecedented precision for the match between theory and observations [1]. In fact, the success of QED nourishes many branches of physics, such as the Yang-Mills theory. Exactly paralleling the historical precedent of QED, we expect to gain some insight by studying matter-gravity couplings in the *semiclassical regime* in weak gravity, given that gravity still resists successful quantization after decades of endeavor. As a supporting fact, the Einstein field equation and the geodesic equation resemble the Maxwell equation and the Lorentz force law [2] for a slowly moving particle in the weak field limit, though this analogy breaks down when gravity is sufficiently strong. The conceptual reasons are rooted in the peculiar differences between gravity and electromagnetism: 1. Gravity is extremely weak and universal. 2. Gravity is highly nonlinear.

Another motivation for the study lies in the fact that Lorentz violation (LV) may be a testable signal of some unified theory at Planck scale [3]. Many different scenarios leading to LV have been proposed, such as noncommutative field theory [4], loop gravity [5], very special relativity [6], etc. To systematically study the possible LV effects, an effective field theory framework incorporating all standard model fields and tiny tensorial coefficients controlling LV has been developed, the Standard-Model Extension, or briefly SME [7,8]. This framework facilitates the test of the common foundation of the strong nuclear force, electroweak theory, and gravity, namely the Lorentz symmetry. Only in the presence of gravity, the Lorentz symmetry is a local symmetry instead of a global one. In the SME, the close resemblance between gravity and electromagnetics has been utilized to map a solution of the Maxwell equation with a restricted class of the $(k_F)^{\kappa\lambda\mu\nu}$ term to the solution of the Einstein equation with the $\bar{s}_{\mu\nu}$ term [9], though the nonlinear acceleration $\vec{a}_{\rm NL}$ spoils the exact formal analogy of weak gravity to electrodynamics even restricted to terms with linear velocity and in the stationary limit. Combined with the precision measurements of Gravity Probe B [10,11], new bounds on $\bar{s}_{\mu\nu}$ have been extracted from the anomalous spin precession caused by the LV gravito-electromagnetic (GEM) fields [12]. With the observation of the structural similarity for the couplings between the gravito-magnetic field and the LV $\tilde{b}$-type coefficient to intrinsic spin [13], the bounds on $\bar{s}^{0k} - \frac{\alpha}{m}(\bar{a}^{\rm S}_{\rm eff})^k$ have been obtained from various comagnetometer experiments [14,15], by reinterpreting $\tilde{b}$ as the gravito-magnetic field caused by the off-diagonal metric perturbation due to LV.

In comparison, in this paper, we try to explore the resemblance of the LV fermion-gravity couplings with the Lense-Thirring metric to the LV fermion-photon

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couplings with Lorentz invariant (LI) electromagnetic field in the framework of SME. In other words, we focus on the quantum matter effects induced by LV in this analogy. For simplicity, we consider only the fermion LV coefficients in the minimal SME and keep the gravity sector intact. Partially because we are more interested in the LV fermion sector and partially because the LV coefficient $\bar{s}_{\mu\nu}$ in minimal gravity sector can be switched into $\bar{c}_{\mu\nu}$ by a proper field redefinition [16], we do not consider the LV fermion-gravity couplings arising from pure gravity. No doubt the backreaction of LV matter fields on spacetime geometry necessarily generates a LV metric perturbation, and this fact has already been thoroughly explored for the $(\bar{a}_{\rm eff})_\mu, \bar{c}_{\mu\nu}$ coefficients in [13,16]. However, by adopting the test particle assumption and ignoring the backreaction in our simple setting, there is no need to worry about the extra modes from diffeomorphism breaking unless the pure gravity sector were also affected by LV. As for the extra modes due to spontaneous local Lorentz symmetry breaking, which may play the role of photon or graviton, such as in the bumblebee or cardinal models ([8,17–19]), or mediate new forces [20], they suffer severe experimental constraints [21] and lie outside of the scope of our present discussion, we disregard them for simplicity.

It is interesting to note that a systematic and thorough treatment of all possible LV matter-gravity couplings, both in formalism and in conceptual issues, have been developed recently [22], where no room is left for spontaneous local LV with diffeomorphism invariance. However, this superficial conflict is because we omit the backreaction of the LV matter field to spacetime geometry in the test particle assumption. Since spontaneous symmetry breaking is assumed, the no-go constraints [8] can also be avoided. In comparison, the signals beyond-Riemann geometry have been explored with an effective field theory incorporating all linear fermion-gravity operators up to dimension 5 [23], based on the assumption of local LI but explicit diffeomorphism breaking. In contrast to Ref. [23], where the typical gravitational acceleration is uniform as the exploration mainly focuses on laboratory experiments on the Earth, our study assumes the Lense-Thirring metric [24], which is essentially curved and has nonzero source angular momentum. This setting is particularly suitable for a tentative study of LV gravitomagnetic effects.

As the fermion in the analogy is nonrelativistic (NR) for practical purposes, it seems necessary to perform the Foldy-Wouthuysen (FW) transformation ([13,25,26]) first; however, a different method first proposed in Ref. [27] is adopted, where the one-fermion matrix elements for a NR fermion scattering off external fields are studied. The NR feature relies on the assumption that the field quantum carries negligible energy and fermion quantization is truncated on positive energy states only. The rest of the paper is arranged as follows. In Sec. II, we briefly review the basic background of gravito-electromagnetism, an analogy of weak gravity in general relativity (GR) to electromagnetism. In Sec. III, we derive the energy-momentum tensor (EMT) for a LV fermion in flat spacetime as a warm-up exercise for the discussion of LV matter-gravity couplings, since gravity couples exactly to the EMT of matter fields, just as photon couples to the electromagnetic current. In Sec. IV, we briefly review the formalism describing a LV fermion coupled with gravity in the weak field approximation. In Sec. V, we outline the main methodology for obtaining the one-fermion matrix elements of a LV fermion coupled with external fields. To make transparent the analogy, we demonstrate the fermion-photon couplings together with the fermion-gravity couplings in the static limit. Possible experimental constraints on LV spin-gravity couplings are discussed in Sec. VI, and we summarize our main results in Sec. VII.

II. THE GRAVITO-ELECTROMAGNETISM

The electromagnetic (EM) analogy for weak gravity can be found in many textbooks on GR [28] or review papers [2]. The inhomogeneous Maxwell equations and Lorentz force law for a charged particle moving in the EM fields are

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0}, \qquad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}_e \tag{1}$$

$$\frac{d(\gamma\vec{v})}{dt} = \frac{e}{m}\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]. \tag{2}$$

For sufficiently weak gravity and slow-moving source, we can expand the metric around Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3}$$

When the source is stationary $\dot{T}_{\mu\nu} = 0$, and in the harmonic gauge $\Gamma^\rho \equiv \Gamma^\rho_{\mu\nu} g^{\mu\nu} = 0$, the Einstein field equation $G_{\mu\nu} = \kappa T_{\mu\nu}$ ($\kappa \equiv \frac{8\pi G}{c^4}$) can be cast into the form similar to (1),

$$\nabla \cdot \vec{E}_g = -\frac{\kappa c^4}{2}\rho_m, \quad \nabla \times \vec{B}_g - \frac{1}{c}\partial_t \vec{E}_g = -2\kappa c^3 \vec{j}_m, \tag{4}$$

where $\vec{E}_g \equiv -\nabla\phi_g - \frac{1}{c}\partial_t \vec{A}_g$ is the so-called gravito-electric field, or just the local gravitational acceleration when $\dot{\vec{A}}_g = 0$, $B^i_g \equiv c^2 \epsilon_{ijk}\partial_j h_{0k}$ is the gravito-magnetic field, and $\rho_m, \vec{j}_m = \rho_m \vec{v}$ are the matter mass density and mass current, respectively. It is easy to check that the homogeneous equations similar to $\partial_\mu \tilde{F}^{\mu\nu} = 0$ ($\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\rho\sigma\mu\nu}F_{\rho\sigma}$) in electrodynamics are also satisfied; see Appendix A. In fact, up to the first Post-Newtonian order, PNO(1) [e.g., $\mathcal{O}(c^{-2})$ for h_{ij}], these GEM equations can be further generalized to the case when matter source does have time dependence, as long as the gravitating system is moving slowly; see [2,29,30]. In that case, one can even derive a formal equation $[\nabla^2 - \frac{4}{c^2}\partial_t^2]\vec{B}_g = 0$ for the fields outside the source current, which may indicate that gravitational waves propagate with the same speed as light *in vacuo*. The extra numerical factor 2^2, which can also be seen in $\frac{2\kappa c^3}{\kappa c^4/2} = \frac{4}{c}$ in parallel to the ratio of $\mu_0\epsilon_0$ in Eq. (1), is because gravity is a spin-2 instead of a spin-1 field. The minus sign $-\frac{\kappa c^4}{2}\rho_m$ in Eq. (4) compared with $\frac{\rho_e}{\epsilon_0}$ in Eq. (1), the Gauss law, reflects the fact that the "charges" of the same sign in gravity attract rather than repel each other. The geodesic equation $\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\rho\sigma}u^\rho u^\sigma = 0$ can also be put into the form [2]

$$\frac{d\vec{v}}{dt} = \frac{m}{m}\left[\vec{E}_g\left(1+\frac{\vec{v}^2}{c^2}\right) + \frac{\vec{v}}{c}\times\vec{B}_g\right], \tag{5}$$

analogous to the Lorentz force law, Eq. (2). In this analogy, gravitational mass can be regarded as the charge responsible for gravito-electromagnetic (GEM) field, and the weak equivalence principle (WEP) ensures that the "charge-to-mass" ratio is unity. Substituting $h_{0j} = \epsilon_{ijk}x^j\omega^k/c$ for an observer stuck to a rotating noninertial frame in Minkowski spacetime into (5), the corresponding force $2m\vec{\omega}\times\vec{v}$ is exactly the Coriolis force, confirming that the noninertial force and gravity may have a common origin, which is partially encoded in the Mach principle. However, we have to keep caution that the formal analogy cannot be extended too far, though it proves quite fruitful, such as the prediction of gravito-magnetic precession of a spinning gyroscope in analogy with the magnetic dipole precession in magnetic fields, confirmed in Gravity Probe B project [10,11], and also in deriving solutions of the LV-modified Einstein equation from the known ones in LV electrodynamics [9]. The reason is that gravity is quite different from the EM field: 1. The Maxwell equation is linear and the EM field is abelian, while the Einstein equation is notoriously difficult to solve for its nonlinearity. 2. The EM acceleration can be quite different for different particles with different charge-to-mass ratios, while gravity is universal for all kinds of matter (attractive except for the cosmological constant [31]) due to the equivalence principle. This distinguishes gravity from all the other three forces in nature, i.e., gravity can be geometrized and pointlike particles propagate freely along geodesics of curved spacetime. It is not a force at all in GR. Technically, 1. Maxwell equation and Lorentz force law are gauge invariant, and thus, we can choose any gauge we like. This is not true in the case of gravito-electromagnetism, where only a restricted class of gauge transformations $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ with $\partial^2\xi^\nu = 0$ (satisfying the harmonic gauge) are allowed, otherwise the Maxwell-like equations (4) cannot hold. 2. The Eqs. (4) and (5) are essentially not gauge invariant due to the two-layer structure of gravity: The metric $g_{\mu\nu}$ can be viewed as the potential of the connection $\Gamma^\alpha_{\beta\gamma}$, just as the definitions of $\vec{E}_g, \vec{B}_g$ express (in this sense, Eqs. (4), (5) are gauge invariant); while the connection $\Gamma^\alpha_{\beta\gamma}$ is again the potential of the Riemann tensor $R^\lambda{}_{\rho\mu\nu}$, and the latter is the intrinsically "gauge invariant" field strength. In other words, by working in the observer's local inertial frame or the Riemann normal coordinates, we can always gauge away the force $m\frac{d\vec{v}}{dt}$ (derived from $m\frac{du^\mu}{d\tau}$). In this respect, a set of essentially gauge invariant Maxwell-like equations must be based on equations with covariant tensor forms, such as the Einstein equation and the geodesic deviation equation [32]. The bonus of this choice is that we can go beyond linear approximations, and the corresponding equations are more robust for further applications. A detailed discussion of the essentially gauge invariant gravitational analogy of Maxwell electrodynamics in the context of LV will be very interesting; however, this is beyond the scope of our present investigation.

III. THE FERMION ENERGY-MOMENTUM TENSOR IN FLAT SPACETIME

The EM analogy in weak gravity is very useful because electrodynamics is easier and more intuitive to deal with, and we are more familiar with it, so we expect the similarity also arises between fermion-gravity (FG) and fermion-electromagnetic (FE) couplings. The usual minimal FE coupling is in the form of $A_\mu j_e{}^\mu$, where $j_e{}^\mu = -e\bar{\psi}\gamma^\mu\psi$ is the conserved current. The conservation is ensured by the gauge invariance of the FE coupling under gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$. Similarly, in the weak field limit, we expect the minimal FG coupling takes a similar form $-\frac{1}{2}h_{\mu\nu}\Theta^{\mu\nu}$, where $\Theta^{\mu\nu}$ is the symmetric energy-momentum tensor (EMT). In fact, from the gravitational definition of EMT [33],

$$\Theta^{\mu\nu}(x) \equiv \frac{2\delta I_{\mathrm{M}}}{\sqrt{-g(x)}\delta g_{\mu\nu}(x)}, \tag{6}$$

for a gauge transformation $\delta g_{\mu\nu}(x) = 2\nabla_{(\mu}\xi_{\nu)}$, the matter action I_{M} in (6) transforms as

$$\begin{aligned}\delta I_{\mathrm{M}} &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{\mu\nu}\Theta^{\mu\nu} \\ &= -\int d^4x\xi_\nu\sqrt{-g}\left\{\frac{\partial_\mu[\sqrt{-g}\Theta^{\mu\nu}]}{\sqrt{-g}} + \Gamma^\nu_{\mu\rho}\Theta^{\mu\rho}\right\} \\ &\quad + \int d^4x\partial_\mu[\sqrt{-g}\xi_\nu\Theta^{\mu\nu}],\end{aligned} \tag{7}$$

where the terms in the large brace above are exactly $\nabla_\mu\Theta^{\mu\nu}$. Ignoring the surface term $\sqrt{-g}\xi_\nu\Theta^{\mu\nu}$, gauge invariance again ensures the covariant conservation of EMT, $\nabla_\mu\Theta^{\mu\nu} = 0$. Unlike the case of EM matter couplings, there is no simple conservation law of EMT $\partial_\mu\Theta^{\mu\nu} = 0$ for the case of gravity, though the linear gauge transformation $\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$ may lead to the ordinary current conservation for the coupling $-\frac{1}{2}h_{\mu\nu}\Theta^{\mu\nu}$. This is quite similar to the non-Abelian Yang-Mills theory, where no simple conservation law exists for a current constructed purely from matter field, $J_a{}^{\nu} = -i\frac{\partial \mathcal{L}_M}{\partial D_\nu \psi} t_a \psi$ [34]. $J_a{}^{\nu}$ is only covariantly conserved, $D_\nu J_a{}^{\nu} = 0$. To construct an ordinary conserved EMT $\partial_\mu \tau^{\mu\nu} = 0$, just like the ordinary conserved current $\mathcal{J}^\mu_a \equiv J^\mu_a - C^c_{ab} F^{\mu\nu}_c A^b_\nu$ $(\partial_\mu \mathcal{J}^\mu_a = 0)$ contains contribution from non-Abelian gauge field itself, the ordinary conserved EMT $\tau^{\mu\nu}$ must also contain a contribution from the gravitational field itself, i.e., terms proportional to the summation of powers of metric tensors and their derivatives, such as the Landau-Lifschitz pseudotensor $t^{\mu\nu}_{\rm LL}$; then $\tau^{\mu\nu} = (-g)[T^{\mu\nu} + t^{\mu\nu}_{\rm LL}]$ [35]. In other words, the gravitational field itself carries energy and momentum and thus contributes to the source of gravity. This has already been dramatically verified by the direct observation of gravitational waves [36]. In fact, the stress-energy tensor for the GEM field in the stationary case can be shown to be exactly proportional to the pseudotensor $t^{\mu\nu}_{\rm LL}$ [2].

In flat spacetime, the canonical formalism gives another way to obtain EMT as the zero-gravity limit of the matter "source current" for gravity, provided the Belinfante-Rosenfeld symmetrization procedure (BR-procedure) is performed. However, in the presence of LV, the usual Belinfante-Rosenfeld symmetrization may not be attainable [8]. As an example, consider the following SME Lagrangian [8],

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \delta\mathcal{L}_{\rm LV},\\
\mathcal{L}_0 &= \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{D}_\mu \psi - m_\psi \bar{\psi}\psi - \frac{1}{2}{\rm Tr}[F^{\mu\nu}F_{\mu\nu}],\\
\delta\mathcal{L}_{\rm LV} &= \frac{i}{2}\bar{\psi}\delta\Gamma^\mu \overleftrightarrow{D}_\mu \psi - \bar{\psi}\delta M \psi,\\
\bar{\chi}\Gamma^\mu \overleftrightarrow{D}_\mu \psi &\equiv \bar{\chi}\Gamma^\mu D_\mu \psi - \bar{\chi}\bar{D}_\mu \Gamma^\mu \psi,
\end{aligned} \tag{8}$$

where $\bar{\chi}\bar{D}_\mu\Gamma^\mu\psi \equiv [(\partial_\mu - ieA_\mu)\bar{\chi}]\Gamma^\mu\psi$, $\delta\Gamma^\mu \equiv \Gamma^\mu - \gamma^\mu \equiv -[c^{\nu\mu}\gamma_\nu + d^{\nu\mu}\gamma_5\gamma_\nu]$ and $\delta M \equiv a^\mu\gamma_\mu + b^\mu\gamma_5\gamma_\mu + \frac{1}{2}H^{\mu\nu}\sigma_{\mu\nu}$. Note for simplicity, the $e^\mu, f^\mu, g^{\lambda\mu\nu}$ coefficients are dropped. Except the $c^{\nu\mu}$ and a^μ, all the other LV coefficients are responsible for the LV spin interactions [37]. We include a^μ term, as in the presence of gravity, the a^μ coefficient cannot be totally removed by field redefinition even for fermions with a single flavor, unlike in the case of flat spacetime [38]. Also note there is a sign difference for the c, d coefficients in Γ^μ, as the signature for Minkowski metric is $\eta_{\mu\nu} = {\rm diag}(-1,+1,+1,+1)$, the one conventionally adopted in the gravity community, rather than the one in QFT [7]. Only in this section, we use Greek indices to denote variables in Minkowski spacetime, while in the following sections, we use Latin indices $a, b, c\ldots$ from the beginning for tangent space variables and the Latin indices $i, j, k, \ldots$ in the middle for purely spatial indices, while the Greek indices $\mu, \nu, \rho, \ldots$ are for manifold variables. Similarly, the convention for the totally antisymmetric tensor is fixed by $\epsilon_{0123} = 1$.

From the Lagrangian (8), we get the canonical momenta from the definition $\Pi_l \equiv \partial\mathcal{L}/\partial\Psi^l$,

$$\begin{aligned}
&\Pi^\mu_\psi \equiv \frac{i}{2}\bar{\psi}[\gamma^\mu + \delta\Gamma^\mu], \qquad \Pi^\mu_{\bar{\psi}} \equiv -\frac{i}{2}[\gamma^\mu + \delta\Gamma^\mu]\psi,\\
&-\Pi^\mu_{A^a{}_\rho} \equiv F^{a\mu\rho} = \partial^\mu A^{a\rho} - \partial^\rho A^{a\mu} + f^a{}_{bc}A^{b\mu}A^{a\rho}.
\end{aligned} \tag{9}$$

The canonical EMT denoted as $T^{\mu\nu}$ is obtained below:

$$\begin{aligned}
T^{\mu\nu} &\equiv \Pi^\mu_\psi \partial^\nu\psi + \partial^\nu\bar{\psi}\Pi^\mu_{\bar{\psi}} + \Pi^\mu_{A^a{}_\rho}\partial^\nu A^a{}_\rho - \eta^{\mu\nu}\mathcal{L}\\
&= T_0{}^{\mu\nu} + \delta T^{\mu\nu},\\
T_0{}^{\mu\nu} &\equiv \frac{i}{2}[\bar{\psi}\gamma^\mu\partial^\nu\psi - (\partial^\nu\bar{\psi})\gamma^\mu\psi] - F^{a\mu\rho}\partial^\nu A^a{}_\rho - \eta^{\mu\nu}\mathcal{L}_0,\\
\delta T^{\mu\nu} &\equiv \frac{i}{2}[\bar{\psi}\delta\Gamma^\mu\partial^\nu\psi - (\partial^\nu\bar{\psi})\delta\Gamma^\mu\psi] - \eta^{\mu\nu}\delta\mathcal{L}_{\rm LV}.
\end{aligned} \tag{10}$$

In the absence of gravity, the violation of Lorentz invariance does not conflict with the spacetime translation invariance, which is assumed to hold since we do not want to lose the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$, provided the fields and their derivatives vanish sufficiently quickly at spatial infinity. However, the BR procedure does not work, as it crucially relies on the fact that the total angular momentum tensor density is conserved, $\partial_\mu \mathcal{J}^\mu{}_{\alpha\beta} = 0$, where

$$\mathcal{J}^\mu{}_{\alpha\beta} \equiv \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Psi(x)]}[S_{\alpha\beta}]\Psi(x) + x_\alpha T^\mu{}_\beta - x_\beta T^\mu{}_\alpha$$

includes the intrinsic spin contribution $\mathcal{S}^\mu{}_{\alpha\beta} \equiv \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Psi(x)]}[S_{\alpha\beta}]\Psi(x)$ due to the nontrivial field representation of the Poincaré group. Since Lorentz invariance is broken, the total angular momentum needs not to be conserved, $\partial_\mu\mathcal{J}^\mu{}_{\alpha\beta} \neq 0$. To see why LV blocks the construction of a symmetric EMT, first, we note that the antisymmetric part of the canonical EMT is

$$T^{[\alpha\beta]} \equiv \frac{1}{2}(T^{\alpha\beta} - T^{\beta\alpha}) = \frac{1}{2}\partial_\mu[\mathcal{J}^{\mu\alpha\beta} - \mathcal{S}^{\mu\alpha\beta}], \tag{11}$$

where we have used $\partial_\alpha T^{\alpha\beta} = 0$. Now suppose one adds $T^{\alpha\beta}$ with a total derivative $\partial_\rho\mathcal{A}^{\rho\alpha\beta}$, provided that $\mathcal{A}^{\rho\alpha\beta}$ vanishes sufficiently quickly at spatial infinity. Then the improved EMT is

$$\begin{aligned}
\Theta^{\alpha\beta} &= T^{\alpha\beta} + \partial_\rho\mathcal{A}^{\rho\alpha\beta}\\
&= T^{(\alpha\beta)} + \partial_\rho\left\{\frac{1}{2}[\mathcal{J}^{\rho\alpha\beta} - \mathcal{S}^{\rho\alpha\beta}] + \mathcal{A}^{\rho\alpha\beta}\right\},
\end{aligned} \tag{12}$$

where the unaltered conservations law requires $\mathcal{A}^{\rho\alpha\beta} = -\mathcal{A}^{\alpha\rho\beta}$. Still adopting the Belinfante-Rosenfeld formalism [39,40] and letting

$$\mathcal{A}^{\rho\alpha\beta} \equiv \frac{1}{2}[\mathcal{S}^{\rho\alpha\beta} - \mathcal{S}^{\alpha\rho\beta} - \mathcal{S}^{\beta\rho\alpha}],$$

we can confirm that $\mathcal{A}^{\rho\alpha\beta} = -\mathcal{A}^{\alpha\rho\beta}$ from its definition. The BR procedure indeed guarantees the current conservation $\partial_\alpha \Theta^{\alpha\beta} = 0$, only in general $\Theta^{\alpha\beta} \neq \Theta^{\beta\alpha}$ due to the presence of Lorentz violation; i.e., $\partial_\rho \mathcal{J}^{\rho\alpha\beta} \neq 0$. This feature of EMT has been clarified with an explicit example in [7] and has been discussed in depth including gravity in the Rieman-Cartan geometry [8].

As Lorentz violation forbids a conserved angular momentum current, we do not expect a natural symmetric EMT. Moreover, the BR procedure cannot even necessarily give rise to a gauge invariant EMT. This has been observed in the LV modified electromagnetism with the k_{AF} term [7] already. As another example, we show after the BR procedure, the EMT with only the c coefficient is

$$\begin{aligned}\Theta^{\mu\nu} &\equiv \Theta^{\mu\nu}_\psi + \Theta^{\mu\nu}_A \\ &= \frac{i}{4}\bar{\psi}[(\gamma^\nu \overleftrightarrow{D}^\mu + \gamma^\mu \overleftrightarrow{D}^\nu)]\psi - F^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\mathcal{L}_0 \\ &\quad + \frac{i}{4}\overline{\psi}[c_\rho{}^\nu(\gamma^\rho \overleftrightarrow{D}^\mu - \gamma^\mu \overleftrightarrow{D}^\rho) - c_\rho{}^\mu(\gamma^\rho \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\rho) \\ &\quad + (\gamma^\mu c^\nu{}_\rho - \gamma^\nu c^\mu{}_\rho)\overleftrightarrow{D}^\rho]\psi + e\overline{\psi}[(c_\rho{}^\mu A^\nu + c_\rho{}^\nu A^\mu)\gamma^\rho \\ &\quad - (\gamma^\nu c_\rho{}^\mu + \gamma^\mu c_\rho{}^\nu)A^\rho]\psi - \eta^{\mu\nu}\delta\mathcal{L}_{\mathrm{LV}}, \end{aligned} \tag{13}$$

where we have ignored the second order LV corrections. Clearly, the terms proportional to c coefficients block the symmetrization, $\Theta^{[\mu\nu]} \neq 0$, and the terms in the third square bracket even block the gauge invariance. Without LV, the terms in the second line are manifestly symmetric and gauge invariant and coincide with the gauge invariant EMT, Eqs. (4)–(5) for quark and gluon in [41], up to a sign difference. Interestingly, the BR procedure does give a gauge invariant EMT for pure LV gauge field with Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}[F^{a\mu\nu}F^a{}_{\mu\nu} + (k_F)^{\mu\nu\rho\sigma}F^a{}_{\mu\nu}F^a{}_{\rho\sigma}], \tag{14}$$

where $F^a{}_{\mu\nu} \equiv \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + f^a_{bc}A^b{}_\mu A^c{}_\nu$ is the field strength for the gauge field A^a. The BR procedure improved EMT is

$$\Theta^{\mu\nu}_A = -F^{a\mu}{}_\kappa F^{a\nu\kappa} - (k_F)^\mu{}_\kappa{}^{\alpha\beta}F^a{}_{\alpha\beta}F^{a\nu\kappa} - \eta^{\mu\nu}\mathcal{L}_A, \tag{15}$$

which is apparently gauge invariant, but still not symmetric. In view of these examples, we see that to have a gauge invariant improvement of the canonical EMT, seems other improvement procedures rather than the BR procedure are required; the latter is not even attainable. Not only because symmetrization is blocked by the presence of LV, which is equivalent to the presence of background tensor fields causing the asymmetry, but also because symmetrization is only indicated by the metric framework of gravitational theory. For a generic gravitational theory allowing other degrees of freedom (d.o.f.), such as torsion or nonmetricity [8,42], the generalized Einstein equation does not require a symmetric EMT as the source of gravity, though an effective symmetric EMT is always attainable if we separate the Einstein tensor into the Riemannian part and incorporate the non-Riemannian part into the effective EMT [42]. However, the cost is that it plagues a proper interpretation of the gravitation and matter d.o.f. We will postpone a further investigation of LV EMT in the future and turn to the discussion of FG couplings in the next section.

IV. PRELIMINARY FOR FERMION-GRAVITY INTERACTIONS

To consider the fermion-gravity couplings, the flat space LV fermion Lagrangian (8) has to be replaced by the curved space version [8,16]

$$\mathcal{L}_\psi = e\left[\frac{i}{2}e^\mu{}_a\bar{\psi}\Gamma^a\overleftrightarrow{\nabla}_\mu\psi - \bar{\psi}M\psi\right], \tag{16}$$

$$\begin{aligned}\bar{\psi}\Gamma^a\overleftrightarrow{\nabla}_\mu\psi &\equiv \bar{\psi}\Gamma^a\left[\vec{D}_\mu + \frac{i}{4}\omega_\mu{}^{bc}\sigma_{bc}\right]\psi - \bar{\psi}[\overleftarrow{D}_\mu - \frac{i}{4}\omega_\mu{}^{bc}\sigma_{bc}]\Gamma^a\psi, \\ \Gamma^a &\equiv \gamma^a - [c_{\rho\nu}\gamma^b + d_{\rho\nu}\gamma_5\gamma^b]e^{\nu a}e^\rho{}_b, \end{aligned} \tag{17}$$

$$M \equiv m + a_\mu e^\mu{}_a\gamma^a + b_\mu e^\mu{}_a\gamma_5\gamma^a + \frac{1}{2}H_{\mu\nu}e^\mu{}_a e^\nu{}_b\sigma^{ab}. \tag{18}$$

Note we use $\vec{D}_\mu\psi = (\partial_\mu + igA_\mu)\psi$ to represent pure gauge coupling and $\vec{\nabla}_\mu\psi = [\vec{D}_\mu + \frac{i}{4}\omega_\mu{}^{bc}\sigma_{bc}]\psi$ to represent the covariant derivatives including both the minimal gauge field (the gauge field means photon in this context) coupling and the spin connection coupling. Also note that we use g instead of e to represent gauge coupling to avoid confusion with the determinant of vierbein, as it will be easier to distinguish the determinant of metric from the coupling constant g in this context. The gravity sector is assumed to be intact to largely simplify the original construction with torsion in Riemann-Carton spacetime [8]. We mention that torsion and nonmetricity can also be tightly constrained in the context of SME [43–45], though they draw great attention to the gravity community even in the LI context [42,46,47]. Considering the weak gravity limit up to the lowest order of metric perturbation

$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, the vierbein and spin connection can thus be written as

$$e_\mu{}^a \simeq \delta_\mu{}^a + \frac{1}{2}h_\mu{}^a + \chi_\mu{}^a, \qquad e^\mu{}_a = \delta^\mu_a - \frac{1}{2}h^\mu{}_a + \chi^\mu{}_a,$$
$$\omega_\mu{}^{ab} = \frac{1}{2}[e^{\nu a}(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) - e_\mu{}^c \partial_\alpha e_{\beta c} e^{\alpha a} e^{\beta b}] - (a \leftrightarrow b)$$
$$\simeq \frac{1}{2}(h_{\mu,}{}^{ab} - h_{\mu,}{}^{ba}) + \chi^{ab}{}_{,\mu} + \chi_{\mu,}{}^{ab} - \chi_{\mu,}{}^{ba}. \tag{19}$$

The $\chi_{ab} = -\chi_{ba}$ contains the 6 local Lorentz degrees of freedom in the vierbein and can be totally removed by fermion field redefinition $\psi(x) \to \exp[-\frac{i}{4}\chi_{ab}(x)\sigma^{ab}]\psi(x)$ [16,18]. This redefinition may still leave imprints on the fluctuations of LV coefficients [16]; however, due to stringent experimental constraints [21] and our sole interest in the effects caused by the vacuum expectation values (VEVs) of the LV coefficients, we can safely ignore χ in the following. For example,

$$e^\mu{}_a\Gamma^a = \gamma^\mu - \frac{1}{2}h_\mu{}^a\gamma^a - \left[c_b{}^\mu - c_{b\nu}h^{\mu\nu} - \frac{1}{2}c_\rho{}^\mu h^\rho{}_b\right]\gamma^b$$
$$- \left[d_b{}^\mu - d_{b\nu}h^{\mu\nu} - \frac{1}{2}d_\rho{}^\mu h^\rho{}_b\right]\gamma_5\gamma^b. \tag{20}$$

Note that different from the notation in [16], we use mixed Latin and Greek indices to keep track of their origin, though all the indices can be put into the Greek ones since we take $h_{\mu\nu}$ as the metric deviation from the vacuum Minkowski background. In other words, in the following discussions of linearized weak gravity, there is no need to distinguish Latin and Greek indices, as all the upper and lower indices, whether Latin or Greek, are raised or lowered by the corresponding Minkowski metric. As the LV coefficients are linear on the level of Lagrangian, we can treat them one by one. First, note we can separate the Lagrangian (16) into LI and LV parts, $\mathcal{L}_\psi = (1 + \frac{1}{2}h)[\mathcal{L}_{\rm LI} + \mathcal{L}_{\rm LV}]$, where the determinant of the vierbein $e = \sqrt{-g} = 1 + \frac{1}{2}h$. The LI Lagrangian can be written as

$$\mathcal{L}_{\rm LI} = \frac{i}{2}e^\mu{}_a\bar\psi\gamma^a\overset{\leftrightarrow}{\nabla}_\mu\psi - \bar\psi m\psi$$
$$\simeq \frac{i}{2}\bar\psi\left[\gamma^a\overset{\leftrightarrow}{D}_a - \frac{1}{2}h^\mu{}_a\gamma^a\overset{\leftrightarrow}{D}_\mu\right]\psi - \bar\psi m\psi, \tag{21}$$

where the "$\simeq$" means preserving only terms up to linear order of $h_{\mu\nu}$, and we have utilized the identity $h_{ab,c}\{\gamma^a, \sigma^{bc}\} = 0$. The LV counterpart is

$$\mathcal{L}_{\rm LV} = \frac{i}{2}e^\mu{}_a\bar\psi\delta\Gamma^a\overset{\leftrightarrow}{\nabla}_\mu\psi - \bar\psi\delta M\psi$$
$$\simeq \frac{i}{2}\bar\psi\left[\delta\Gamma^a_\circ\left(\delta^\mu_a - \frac{1}{2}h^\mu_a\right)\overset{\leftrightarrow}{D}_\mu + \delta\Gamma^a_h\overset{\leftrightarrow}{D}_a\right]\psi$$
$$- \bar\psi(\delta M_\circ + \delta M_h)\psi + \frac{1}{4}\epsilon^{bcmn}h_{am,n}\bar\psi$$
$$\times [c_b{}^a\gamma_5\gamma_c + d_b{}^a\gamma_c]\psi, \tag{22}$$

where, for simplicity, we have defined

$$\delta\Gamma^a_h \equiv \frac{1}{2}[h^{\nu a}(c_{b\nu} + d_{b\nu}\gamma_5) + h^\rho{}_b(c_\rho{}^a + d_\rho{}^a\gamma_5)]\gamma^b,$$
$$\delta\Gamma^a_\circ \equiv -[c_b{}^a\gamma^b + d_b{}^a\gamma_5\gamma^b],$$
$$\delta M_h \equiv -\frac{1}{2}h^\mu{}_a\left[(a_\mu + b_\mu\gamma_5)\gamma^a + \frac{1}{2}H_{\mu b}\sigma^{ab}\right] - \frac{1}{4}h^\nu{}_b H_{a\nu}\sigma^{ab},$$
$$\delta M_\circ \equiv \left(a_a\gamma^a + b_a\gamma_5\gamma^a + \frac{1}{2}H_{ab}\sigma^{ab}\right).$$

To the linear order of metric perturbation, the Euler-Lagrangian equation to $\mathcal{L}_{\rm LI} + \mathcal{L}_{\rm LV}$ is

$$\left\{i\left[(\Gamma^a_\circ + \delta\Gamma^a_h)\vec{D}_a - \frac{h^\mu{}_a}{2}\Gamma^a_\circ\vec{D}_\mu\right] - (M_\circ + \delta M_h)\right\}\psi$$
$$+ \frac{i}{2}\left[\partial_a\delta\Gamma^a_h - \frac{1}{2}\partial_a h^a{}_c\Gamma^c_\circ - \frac{i}{2}\epsilon^{bcd}_e h_{ab,c}(c_d{}^a\gamma_5 + d_d{}^a)\gamma^e\right]\psi$$
$$= 0, \tag{23}$$

where $\Gamma^a_\circ = \gamma^a + \delta\Gamma^a_\circ$ and $M_\circ = m + \delta M_\circ$. Note we haven't considered the so-called geometric term

$$\mathcal{L}_{\rm geo} = (e-1)\mathcal{L}_\psi \simeq \frac{h}{2}\mathcal{L}_{\rm flat}, \tag{24}$$

where $\mathcal{L}_{\rm flat} \equiv \frac{i}{2}\bar\psi\Gamma^a_\circ\overset{\leftrightarrow}{D}_a\psi - \bar\psi M_\circ\psi$, since the geometric term comes from the artifact of linearization, which amounts to nothing but multiplying Eq. (23) with a rescaling factor $e = 1 + \frac{h}{2}$. If pick up back these terms, the equation is exactly the one obtained from the linearization of the full Dirac equation with respect to the Lagrangian (16).

In comparison with the EM coupling $-j^\mu A_\mu$, we can also collect all the terms proportional to $h_{\mu\nu}$ in the Lagrangian, which is

$$\mathcal{L}_{\rm hI} = -\frac{1}{2}h^\mu{}_a\left\{\frac{i}{2}\bar\psi\Gamma^a_\circ\overset{\leftrightarrow}{D}_\mu\psi - \bar\psi[(a_\mu + b_\mu\gamma_5)\gamma^a + H_{\mu b}\sigma^{ab}]\psi\right\}$$
$$+ \frac{i}{4}\bar\psi[h^{\nu a}(c_{b\nu} + d_{b\nu}\gamma_5) + h^\rho{}_b(c_\rho{}^a + d_\rho{}^a\gamma_5)]\gamma^b\overset{\leftrightarrow}{D}_a\psi$$
$$+ \frac{1}{4}\epsilon^{mnb}{}_c h_{am,n}\bar\psi[c_b{}^a\gamma_5 + d_b{}^a]\gamma^c\psi + \frac{h}{2}\mathcal{L}_{\rm flat}$$
$$\doteq -\frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \tag{25}$$

where all the Greek and Lain indices are raised or lowered by the corresponding Minkowski metric and thus lose the distinctive features they have before the linearization. Also note "$\doteq$" means equal up to a total derivative since we have dropped a total derivative term proportional to $\epsilon^{\mu bcd}$, and the energy-momentum tensor $T^{\mu\nu}$ is given explicitly,

$$T^{\mu\nu} = \frac{i}{2}\bar{\psi}\Gamma^{\nu}_{\circ}\overset{\leftrightarrow}{D^{\mu}}\psi - \overline{\psi}[(a^{\mu} + b^{\mu}\gamma_5)\gamma^{\nu} + H^{\mu}{}_{b}\sigma^{\nu b}]\psi - \frac{i}{2}\bar{\psi}[(c_{b}{}^{\mu} + d_{b}{}^{\mu}\gamma_5)\gamma^{b}\overset{\leftrightarrow}{D^{\nu}} + (c^{\mu}{}_{b} + d^{\mu}{}_{b}\gamma_5)\gamma^{\nu}\overset{\leftrightarrow}{D^{b}}]\psi - \eta^{\mu\nu}\mathcal{L}_{\mathrm{flat}} + i\epsilon^{\mu bcd}\partial_{d}[\overline{\psi}(c_{b}{}^{\nu}\gamma_5 + d_{b}{}^{\nu})\gamma_{c}\psi]. \tag{26}$$

Aside from the last term coming from spin connection, the EMT obtained in this way is gauge invariant and symmetric, as the apparently asymmetric part $T^{[\mu\nu]}$ does not contribute due to the coupling with $h_{\mu\nu} = h_{\nu\mu}$. The LV coefficients in the above are just VEVs and thus are assumed to be spacetime independent at least in the post-Newtonian approximation of the SME [18,48].

Now the form $\mathcal{L}_{\mathrm{hI}} \doteq -\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$ is similar to the EM coupling $-A_{\mu}j^{\mu}$ and can be regarded as a linear approximation of $\delta I = \frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}T^{\mu\nu}$ up to the determinant $\sqrt{-g}$. It is interesting to note that the geometric contribution in (24) cannot be ignored as stated in [27]; otherwise, the resultant EMT will differ by a term proportional to $\eta^{\mu\nu}$ compared to the EMT obtained with the canonical formalism. However, this term doesn't contribute if the matter fields are on the mass shell since we only consider metric couplings up to linear order.

As mentioned already, we can study the NR fermion-gravity interaction from the well-known FW transformation method [25,49,50], which requires a relativistic Hamiltonian with conventional time evolution as the starting point. As our main concern, the other way is to calculate the interaction energy between a pair of one-fermion states, $\int d^3\vec{x}\langle p',\beta| - \mathcal{L}_{\mathrm{int}}|p,\alpha\rangle = \frac{1}{2}\int d^3\vec{x}h_{\mu\nu}\langle p',\beta|T^{\mu\nu}|p,\alpha\rangle$, where $\langle p',\beta|T^{\mu\nu}|p,\alpha\rangle$ is the gravitational form factor extensively studied in hadron spin structures [51]. However, even for the latter approach, to find out the proper eigenspinors for proper Fourier expansion of $\psi(x)$, we still face the same necessity of field redefinition. In fact, even for a covariant Dirac equation without unconventional time derivatives impeding the proper identification of the time evolution operator [16], field redefinition is still an essential step to get a hermitian Hamiltonian [50] and has been well developed in the context of SME [7,16,37] to study perturbative LV effects, such as the effects due to LV fermion-gravity couplings. We will discuss the field redefinition later.

V. NONRELATIVISTIC FERMION-GRAVITY COUPLING AND THE ANALOGY

The method to get NR interaction energies is adopted from Ref. [27], where the basic idea is from the lessons we learn in QED. In QED, the electrostatic force is mediated by the photon exchange between two charged particles, and the full relativistic interaction is described by the vector-current interaction $-j^{\mu}A_{\mu}$ with $j^{\mu} = \bar{\psi}\Gamma^{\mu}\psi$, where $\Gamma^{\mu} = \gamma^{\mu}$ in LI QED. Likewise, the gravitational interaction is mediated by the graviton exchange between two energy carriers (without NR approximation, massless particles such as photons are also allowed), and the full relativistic interaction is described by the tensor-current interaction $-\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$, where $T^{\mu\nu}$ is given by Eq. (6) in general, and only the symmetrized part of $T^{\mu\nu}$ essentially contributes. For the Lagrangian (16), $T^{\mu\nu}$ is explicitly given by (26) in the linear approximation. Following the same logic, we try to get the leading order NR one-fermion interaction matrix elements from the fully relativistic interaction Lagrangian (25).

In standard QFT, the spinor ψ can be expanded as

$$\psi(x) = \sum_{\sigma=1,2}\int d\tilde{k}[\hat{b}_{\sigma}(\vec{k})u_{\sigma}(\vec{k})e^{ik\cdot x} + \hat{d}^{\dagger}_{\sigma}(\vec{k})v_{\sigma}(\vec{k})e^{-ik\cdot x}], \tag{27}$$

where $d\tilde{k} \equiv \frac{d^3k}{(2\pi)^3}\frac{m}{k^0}$, $k\cdot x \equiv \vec{k}\cdot\vec{x} - k^0x^0$, and u_{σ}, v_{σ} are the eigenspinors describing electron and positron, respectively. In the LI situation, the explicit forms of u_{σ}, v_{σ} can be found in any textbooks of QFT, say [52,53]. However, in the presence of generic LV couplings, the physical free-particle states cannot be directly described by ψ due to unconventional time evolution imposed by the LV derivative couplings, such as the c, d terms [54]. To eliminate the extra time derivatives, we have to invoke the spinor redefinition $\psi = \hat{U}\chi$ to cast the kinematic term into the conventional structure $\frac{1}{2}i\bar{\chi}\gamma^0\overset{\leftrightarrow}{\partial_0}\chi$, which leads to a Hermitian Hamiltonian with the usual scalar product $\langle\Psi|\Phi\rangle \equiv \int d^3x\Psi^{\dagger}(x)\Phi(x)$ in flat space [16]; otherwise, we have to redefine the scalar product via the prescription adopted in Ref. [55].

For the flat space Lagrangian (8), $\hat{U} = (\gamma^0\Gamma^0_{\circ})^{-\frac{1}{2}}$ is a nonsingular spacetime-independent matrix [54,56]. For a generic fermion Lagrangian, the redefinition matrix is given by Eq. (30) in Ref. [16] up to the leading order of perturbation parameters of $h_{\mu\nu}$ and LV coefficients. Thus, in general, $\hat{U}$ can be quite complicated and spacetime dependent. The explicit form of $\hat{U}$ corresponding to Lagrangian (16) is given in Appendix C, and can be shown to satisfy $\hat{U}^{\dagger}\gamma^0\Gamma^0\hat{U} = \hat{I}$ [57]. As what we concerned about is $\mathcal{L}_{\mathrm{hI}}$ in Eq. (25), it suffices to use the flat-space redefinition matrix $\hat{U}_0 \equiv 1 + \frac{1}{2}(d_{b0}\gamma_5 - c_{b0})\gamma^0\gamma^b$ for a linear approximation. Detailed calculations lead to additional h couplings from the flat-space Lagrangian $\mathcal{L}_{\mathrm{flat}}$, as the spinor redefinition matrix $\hat{U} = 1 + \delta\hat{U}_0 + \delta\hat{U}^h$; however, these terms do not contribute to $\langle p',\beta|\int d^3x\mathcal{L}_{\mathrm{flat}}|p,\alpha\rangle$, provided the external fermions are on mass shell.

For nonderivative LV couplings such as a, b, H coefficients, the LI eigenspinor may serve as a first-order approximation in Eq. (27), while for the c, d coefficients with extra time derivatives, the quantization expansion in terms of $\hat{b}_{\sigma}, \hat{d}_{\sigma}$ has to be done with a redefined spinor χ

directly. Of course, the eigenspinor can always be written as $S_\alpha = S_{\alpha 0} + S_{\alpha 1}$ (S_α refers to u_α or v_α), where $S_{\alpha 0}, S_{\alpha 1}$ denote the LI and LV contributions, respectively. For the one-fermion matrix elements $\langle p', \beta|\hat{O}|p, \alpha\rangle$ at leading order approximation, the key ingredient in the Fourier expansion can be written as

$$\bar{S}_\beta \hat{O} S_\alpha = \bar{S}_{\beta 0}(\hat{O}_0 + \hat{O}_1) S_{\alpha 0} + (\bar{S}_{\beta 1}\hat{O}_0 S_{\alpha 0} + \bar{S}_{\beta 0}\hat{O}_0 S_{\alpha 1}), \quad (28)$$

where $\bar{S}_\alpha \equiv S_\alpha{}^\dagger \gamma^0$ is the Dirac adjoint of the eigenspinor S_α, $\hat{O}$ denotes any operator we are interested in, such as $e\vec{A}\cdot\vec{\Gamma}$, and $\hat{O}_0, \hat{O}_1$ denote the LI and LV separations of $\hat{O}$. Since in the NR limit, the contribution from the spinor $v_\sigma(\vec{k})$ with negative energy can be totally ignored, and the scattered fermion is assumed to be always on the mass shell, Eq. (27) becomes (for c, d coefficients, ψ has to be replaced by $\chi = \hat{U}^{-1}\psi$)

$$\psi(x) = \int d\tilde{k} \sum_{\sigma=1,2} \hat{b}_\sigma(\vec{k}) u_\sigma(\vec{k}) e^{ik\cdot x}, \quad (29)$$

where $k^0 = k^0[\vec{k}, m, X]$ is the LV modified dispersion relation, and X represents a set of generic LV coefficients with indices suppressed. The LI eigenspinor is

$$u_{\sigma 0}(\vec{k}) = \sqrt{\frac{\omega_0 + m}{2m}} \begin{pmatrix} \xi^\sigma \\ U_0(k)\xi^\sigma \end{pmatrix} \overset{\mathrm{NR}}{\simeq} \begin{pmatrix} \xi^\sigma \\ \frac{\vec{\sigma}\cdot\vec{k}}{2m}\xi^\sigma \end{pmatrix}, \quad (30)$$

where $U_0(k) \equiv \frac{\vec{\sigma}\cdot\vec{k}}{\omega_0 + m}$ and $\omega_0 = \sqrt{\vec{k}^2 + m^2}$. For calculational simplicity, we simply ignore the $\mathcal{O}(\frac{\vec{k}^2}{m^2})$ corrections of the normalization factor $\sqrt{\frac{\omega_0+m}{2m}}$ and set it equal to 1 in the last step. This can largely simplify our calculations but may induce $\mathcal{O}(1)$ numerical differences for those terms of $\mathcal{O}(\frac{\vec{p}^2}{m^2})$ in comparison with the corresponding terms in [27]. For a, b type coefficients, the eigenspinor can be directly found in the appendix in [7]. For completeness, we collect them together with the eigenspinors for c, d, H coefficients in Appendix D. The key idea is that since LV is supposed to be tiny by observational constraints [21], we only need to keep linear order LV corrections and hence, can treat various LV coefficients one by one as if the other LV coefficients are absent. Thus, in calculating LV contributions of FG or FE interaction energies from matrix elements, we can classify them into three categories:

- (i) Explicit LV vertices, such as $\mathcal{O}_{\mathrm{LV}} = \frac{h_{ba}}{2}\bar{\psi}[(a^b + b^b\gamma_5)\gamma^a + H^b{}_c\sigma^{ac}]\psi$, where Eq. (27) with LI eigenspinors is sufficient;
- (ii) Eigenspinor induced LV to the superficially LI vertices, such as $\mathcal{O}_{\mathrm{LI}} = \frac{ih_{ba}}{4}\bar{\psi}\gamma^a \overset{\leftrightarrow}{D}{}^b\psi$, where ψ and $\bar{\psi}$ receive LV corrections and thus induce LV corrections to interaction energy. In this case, the eigenspinor correction appears through the LV corrected matrix connecting the upper and lower two bispinors ξ^α and $U_X(k)\xi^\alpha$, i.e., $U_0(k) \to U_X(k)$, where X again represents certain LV coefficient with Lorentz indices suppressed;
- (iii) LV correction to dispersion relations, which in the NR limit, may also induce LV corrections; for example, $\frac{1}{E(p,X)+m} \simeq \frac{1}{2m}[1 - \frac{X}{4m}]$, where X represents some indices suppressed LV coefficient with dimension 1 and $E(p, X) = [\vec{p}^2 + m^2 + Xm]^{\frac{1}{2}}$.

Equipping with these tools and following the spirit of [27], we calculate the interaction energy

$$\hat{E}_{\mathrm{int}} = -\int d^3\vec{x}\mathcal{L}_{\mathrm{int}}$$

in the following subsections. As an objective in analogy, we calculate the fermion-photon interaction first with the interaction Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathrm{int}} = \mathcal{L}_{\mathrm{AI}} &= -g\bar{\psi}\Gamma^a A_a\psi \\ &= -gA_a[\bar{\psi}\gamma^a\psi - \bar{\psi}(c_b{}^a\gamma^b + d_b{}^a\gamma_5\gamma^b)\psi], \end{aligned} \quad (31)$$

while for fermion-gravity interaction, $\mathcal{L}_{\mathrm{int}}$ is replaced by $\mathcal{L}_{\mathrm{hI}}$ in Eq. (25).

A. Nonrelativistic fermion-photon interaction

The interaction energy between two one-electron states $|p', \beta\rangle$ and $|p, \alpha\rangle$ is

$$\begin{aligned} E_{\mathrm{AI}} &\equiv E_{\mathrm{AI}}^{\mathrm{LI}} + E_{\mathrm{AI}}^{\mathrm{LV}} \\ &= g\langle p', \beta| \int d^3\vec{x}[\bar{\psi}\vec{\Gamma}\psi\cdot\vec{A} - \bar{\psi}\Gamma^0\psi A^0]|p, \alpha\rangle \\ &= g\sum_{s_1, s_2} \int d^3\vec{x} \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} \langle 0|b_\beta(p')b_{s_1}^\dagger(k_1) \\ &\quad \times [\bar{u}_{s_1}(k_1)\Gamma\cdot A u_{s_2}(k_2)]b_{s_2}(k_2)b_\alpha^\dagger(p)|0\rangle \\ &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}[u_\beta^\dagger(p')\gamma^0\Gamma\cdot A u_\alpha(p)], \end{aligned} \quad (32)$$

where $q \equiv p' - p$ and $\Gamma\cdot A \equiv \vec{\Gamma}\cdot\vec{A}(x) - \Gamma^0 A^0(x)$, and we have used $\{b_\alpha(p), b_\sigma^\dagger(k)\} = (2\pi)^3\delta_{\alpha\sigma}\delta^3(\vec{p} - \vec{k})$. Note we assume that the field redefinition has already been done implicitly, so for the c, d coefficients, $\gamma^0\Gamma\cdot A$ has to be replaced by $(\Gamma\hat{U})^\dagger\gamma^0\cdot A\hat{U}$. In the following, we will always deal with a, b, H terms first and treat c, d terms later, and we omit the subscript "0" for denoting LI spinor

$u_{\sigma 0}$ unless necessary. The LI part of the interaction energy is

$$\begin{aligned} E_{\mathrm{AI}}^{\mathrm{LI}} &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}(\bar{u}_\beta(p')[\vec{A}(x)\cdot\vec{\gamma}-A^0(x)\gamma^0]u_\alpha(p)) \\ &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\left(\left[\frac{\vec{l}\cdot\vec{A}+i\vec{q}\times\vec{A}\cdot\vec{\sigma}}{E+m}\right]\right. \\ &\quad \left. -A^0\left[1+\frac{\vec{p'}\cdot\vec{p}+i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{(E+m)^2}\right]\right)\xi_\alpha(p) \\ &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\left[\left(\frac{\vec{A}\cdot\vec{p}}{m}-A^0\left(1+\frac{\vec{p}^2}{4m^2}\right)\right)\right. \\ &\quad \left. +\frac{\vec{\sigma}\cdot\vec{B}}{2m}+\frac{(\vec{E}\times\vec{p})\cdot\vec{\sigma}}{4m^2}\right]\xi_\alpha, \end{aligned} \tag{33}$$

which is exactly the same as the Eqs. (7.7) and (7.9) in [27] if the signature difference is concerned. Note we have defined $l\equiv p'+p$ and assumed the fermion is always on the mass shell such that the energy transfer is zero, $q^0=0$ (elastic scattering), just as in [27]. However, in the presence of a generic LV coefficient X, the dispersion relation is modified. Thus, $p'^0=p^0$ does not imply $\vec{q}\cdot(2\vec{p}+\vec{q})=0$, but rather

$$\frac{\vec{q}\cdot\vec{p}}{4m^2}\simeq-\frac{\vec{q}^2}{8m^2}+\frac{\delta\omega_{p'}-\delta\omega_p}{4m}, \tag{34}$$

where $\delta\omega_p=\delta\omega(p,m,X)\equiv p^0(p,m,X)-\sqrt{\vec{p}^2+m^2}$, and we divided $\vec{q}\cdot\vec{p}$ by $4m^2$ to fit the factor appearing in the third line of (33). The extra term in (34) means that there is an extra LV contribution due to the modified dispersion relation, even in the calculation of the superficially LI $E_{\mathrm{AI}}^{\mathrm{LI}}$. To facilitate the analogy, we also assume that the four-potential of the photon field is static, $\dot{A}^\mu=0$. For simplicity, we choose the Coulomb gauge $\nabla\cdot\vec{A}=0$, which is equivalent to the Lorenz gauge in static limit. The absence of $\vec{q}\cdot\vec{A}$ is simply due to this gauge choice. The third term in (33) is exactly the standard Dirac's prediction, the magnetic moment interaction $\frac{\vec{\sigma}\cdot\vec{B}}{2m}$, and the last term $(\vec{E}\times\vec{p})\cdot\vec{\sigma}$ contributes to the fine-structure corrections of the hydrogen atom. Apart from the spin-orbit (SO) coupling, there are the Darwin term (though it vanishes in the static limit) and the relativistic corrections of the kinematics that also contribute. The first term $g\vec{A}\cdot\vec{p}/m$ is simply the cross term in the gauge invariant kinetic energy $\frac{(\vec{p}+g\vec{A})^2}{2m}$ in the Coulomb gauge, and $-gA^0$ is the static Coulomb energy with the correction factor $1+\frac{\vec{p}^2}{4m^2}$ for a charged particle in its comoving frame. The vanishing of $\vec{q}^2A^0$ term is because this term is proportional to $\nabla^2A^0(\vec{x})=-\rho_e\delta(\vec{x}-\vec{x}_s)$ by Coulomb's law, where $\vec{x}_s$ denotes the position of source particle for the external EM field, and the fermion is assumed to be far away from the source particle. Note that we have made a replacement $-i\vec{q}A^0\to\vec{E}$ and $\vec{q}\times\vec{A}\to\vec{B}$ in the fourth and fifth lines of Eq. (33) and dropped out total derivative terms from partial integration.

For LV eigenspinor contribution to EM interaction energy, we give an explicit formula for a generic LV coefficient X,

$$\begin{aligned} E_{\mathrm{AI}}^{\mathrm{X-spinor}} &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger([\vec{\sigma}\cdot\vec{A}\delta U_X(p)+\delta U_X^\dagger(p')\vec{\sigma}\cdot\vec{A}] \\ &\quad -A^0[U_0^\dagger(p')\delta U_X(p)+\delta U_X^\dagger(p')U_0(p)])\xi_\alpha(p), \end{aligned} \tag{35}$$

where $U_0(p)\equiv\frac{\vec{\sigma}\cdot\vec{p}}{\omega_0+m}$ and U_X are the LI and LV matrices connecting the upper and lower bispinors. For example, for a given Dirac spinor $u(p)=(\xi(p),\eta(p))^{\mathrm{T}}$, $\eta=U_X\xi$ and $\delta U_X(p)\equiv U_X(p)-U_0(p)$. For details, see Appendix D.

First, we can calculate the contribution of a, b, H coefficients to the fermion-photon interaction separately. As they do not superficially alter the conserved currents, there is no modification of the fermion-photon vertex due to these coefficients. In other words, for a, b, H, it is sufficient to take into account eigenspinor contributions $E_{\mathrm{AI}}^{\mathrm{X-spinor}}$ and corrections due to modified dispersion relations. For the a coefficient, its effects can be simply shown by replacing $\vec{p}\to\vec{p}+\vec{a}$ in (33) and omitting $\vec{a}^2$ terms, which are of higher order. This manifestly shows that for EM interaction, a term only shifts the four-momentum and causes no observable physical effects and thus can be removed by proper field redefinitions [7,16]. However, it does have effects on gravitational interaction [16] and will be explicitly shown in the next subsection.

The LV correction for the b coefficient is

$$\begin{aligned} E_{\mathrm{AI}}^{b} &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\left[\frac{i(\vec{q}\times\vec{A})\cdot\vec{b}-b^0A^0\vec{l}\cdot\vec{\sigma}/2}{2m^2}\right. \\ &\quad \left.+\frac{b^0\vec{\sigma}\cdot\vec{A}}{m}+\frac{\vec{A}\cdot(\vec{l}\,\vec{b}-\vec{b}\,\vec{l})\cdot\vec{\sigma}}{2m^2}\right]\xi_\alpha \\ &= g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\left[\frac{b^0\vec{\sigma}\cdot\vec{A}}{m}+\frac{\vec{B}\cdot\vec{b}-b^0A^0\vec{p}\cdot\vec{\sigma}}{2m^2}\right. \\ &\quad \left.+\frac{2\vec{A}\cdot[(\vec{b}\times\vec{p})\times\vec{\sigma}]+i\vec{\sigma}\cdot\vec{\nabla}(\vec{b}\cdot\vec{A})}{2m^2}\right]\xi_\alpha. \end{aligned} \tag{36}$$

Note that $-\frac{b^0A^0\vec{q}\cdot\vec{\sigma}}{4m^2}$ in (36) is canceled by the correction due to LV dispersion relation; see Eq. (34). Interestingly, the $\vec{B}\cdot\vec{b}/2m^2$ term seems to indicate that the $\vec{b}$ vector behaves like a "cosmic magnetic dipole moment" $\delta\vec{\mu}=-\frac{g\vec{b}}{2m^2}$ in comparison to the conventional magnetic dipole moment (MDM) $\vec{\mu}=-\frac{g\vec{\sigma}}{2m}$. Contrary to the dynamical $\vec{\mu}$, which can be manipulated by spin polarization, $\delta\vec{\mu}$ is supposed to be a constant background, whose projection on a specific direction, say $\vec{B}$, varies due to the relative motion of the charged particle with respect to the cosmic background, and may cause a sidereal variation in terrestrial experiments. A similar $\vec{\Omega}\cdot\vec{b}$ coupling also arises when a LV fermion is coupled to the gravitational field due to a large rotating mass. For a fermion coupled with some kind of cosmic anisotropic vector [58], or the axial vector part of a torsion tensor by the identification $\frac{\epsilon^{\mu\alpha\beta\gamma}}{8}T_{\alpha\beta\gamma}\to b^{\mu}_{\rm eff}$ [16], we may expect similar forms of interactions. In fact, the nonminimal $\frac{1}{2}b_F^{ijk}F_{jk}\overline{\psi}\gamma_5\gamma_i\psi$ term [59] may also produce a term looking like $b_F\vec{\sigma}\cdot\vec{B}$ with similar structure if $b_F^{ijk}=b_F\epsilon^{ijk}$, and thus, the terms within quite different scenarios may be constrained by similar phenomenological observations, such as the comagnetometer experiments [14,15].

The LV correction for the H coefficient is

$$\begin{aligned}E^H_{\rm AI}&=g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\left[\frac{\vec{A}\times\vec{H}\cdot\vec{\sigma}}{m}-\frac{(\vec{\sigma}\cdot\vec{A})(\vec{l}\cdot\tilde{\vec{H}})}{2m^2}\right.\\&\quad\left.+\frac{iA^0\vec{q}\cdot\vec{H}}{4m^2}-\frac{A^0\vec{l}\times\vec{H}\cdot\vec{\sigma}}{4m^2}\right]\xi_\alpha\\&=g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\left[\frac{\vec{A}\times\vec{H}\cdot\vec{\sigma}}{m}-\frac{(\vec{\sigma}\cdot\vec{A})(\vec{p}+\vec{q}/2)\cdot\tilde{\vec{H}}}{m^2}\right.\\&\quad\left.-\frac{\vec{E}\cdot\vec{H}}{4m^2}-\frac{A^0(\vec{p}+\vec{q})\times\vec{H}\cdot\vec{\sigma}}{2m^2}\right]\xi_\alpha,\end{aligned}\tag{37}$$

where we have decomposed $H_{\mu\nu}$ into an "electric" part $\vec{H}^i\equiv H_{0i}$ and a "magnetic" part $\tilde{\vec{H}}^i\equiv\frac{1}{2}\epsilon_{ijk}H_{jk}$. This decomposition is meaningful, as seen from various couplings such as $-\vec{E}\cdot\vec{H}/4m^2$. Just like the "cosmic MDM" induced by the $\vec{b}$ vector, $-\vec{E}\cdot\vec{H}/4m^2$ behaves like a "cosmic electric dipole moment" $\frac{g\vec{H}}{4m^2}$ for a charged fermion. Also just like $-b^0A^0\vec{\sigma}\cdot\vec{p}/2m^2$ in (36) coupling spin $\vec{\sigma}/2$ with momentum $\vec{p}$ and hence, behaving as a spin-orbit-like (SOL) operator (here, we adopt the literal instead of the conventional meaning of the nomenclature; i.e., a "spin-orbit" like operator simply means an operator coupling spin and momentum), $A^0\vec{H}\times\vec{p}\cdot\vec{\sigma}/2m^2=-A^0\vec{\sigma}\times\vec{p}\cdot\vec{H}/2m^2$ can also be viewed as a tiny LV SOL correction to the LI SO coupling term, $g\frac{(\vec{E}\times\vec{p})\cdot\vec{\sigma}}{4m^2}$. However, the external $\vec{E}$ in the LI operator is controllable, while the "cosmic" $\vec{H}$ term is not, though it may receive a sidereal variation for any terrestrial experiment. Moreover, it depends on the local electric potential, which is like the term $\phi_g\frac{\vec{p}\times\vec{H}\cdot\vec{\sigma}}{m}$ in (51). These distinctive features mean the LV SOL couplings can be testable and distinguished from any LI background in the ultrahigh precision fine structure observations.

For the c, d coefficients, as they not only lead to eigenspinor corrections, but also bring corrections to conserved current, and hence, impose the need of spinor redefinition to cure the otherwise non-Hermitian Hamiltonian if the spinor ψ is improperly used, we treat them separately.

After redefinition, the fermion-photon interaction is

$$\begin{aligned}gA_a\bar{\psi}[\delta_b{}^a-c_b{}^a-d_b{}^a\gamma_5]\gamma^b\psi&=g\chi^\dagger[(\vec{\alpha}\cdot\vec{A}-A^0)\\&\quad+A^j(\tilde{d}_{ij}\gamma_5\alpha^i+2d_{(0j)}\gamma_5)-A^j(\tilde{c}_{ij}\alpha^i+2c_{(0j)})]\chi,\end{aligned}\tag{38}$$

where we defined $c_{(0j)}\equiv\frac{1}{2}(c_{0j}+c_{j0})$, $d_{(0j)}\equiv\frac{1}{2}(d_{0j}+d_{j0})$, $\tilde{c}_{ij}\equiv c_{00}\delta_{ij}+c_{ij}$, $\tilde{d}_{ij}\equiv d_{00}\delta_{ij}+d_{ij}$, and again we keep terms only up to the linear order of LV coefficients. The LV c, d corrections to the conserved current are the terms in the second line in (38), where the terms in the first line in the large bracket correspond to LI current. It is interesting to note that the consistency of the field redefinition for c, d terms lies in the fact that there is no LV A^0 coupling operator in the second line in (38), as the goal of the field redefinition is just to remove the unconventional kinematic couplings caused by the c, d terms, and the A^0 coupling will in no doubt be removed due to *the minimal coupling* schemes. Inserting the quantization expansion of χ in terms of annihilation and creation operators as Eq. (27), we get

$$\begin{aligned}E^{c_1}_{\rm AI}&=-g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}u^\dagger_\beta(p')[\alpha^i\tilde{c}_{ij}A^j+2c_{(0j)}A^j]u_\alpha(p)\\&=-g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\left[2\vec{c}\cdot\vec{A}\left[1+\frac{\vec{p'}\cdot\vec{p}+i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{4m^2}\right]\right.\\&\quad\left.+\frac{c_{00}}{2m}[\vec{l}\cdot\vec{A}+\vec{B}\cdot\vec{\sigma}]+\frac{c_{ij}A^j}{2m}(l^i+i\epsilon_{kil}q^k\sigma^l)\right]\xi_\alpha,\end{aligned}\tag{39}$$

$$\begin{aligned}E^{d_1}_{\rm AI}&=g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}u^\dagger_\beta(p')\{\tilde{d}_{ij}\Sigma^iA^j+2\vec{d}\cdot\vec{A}\gamma_5\}u_\alpha(p)\\&=g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\left\{\vec{d}\cdot\vec{A}\frac{\vec{\sigma}\cdot\vec{l}}{m}+\tilde{d}_{ij}A^j\cdot\left[\sigma^i+\frac{(2p^i+q^i)\vec{\sigma}\cdot\vec{p}+p^i\vec{\sigma}\cdot\vec{q}-\vec{p'}\cdot\vec{p}\sigma^i+i(\vec{p}\times\vec{q})^i}{4m^2}\right]\right\}\xi_\alpha,\end{aligned}\tag{40}$$

where we defined $\vec{c}^j \equiv c_{(0j)}$ and $\vec{d}^j \equiv d_{(0j)}$ for notational simplicity. Note both in (39) and (40), $u_\sigma(k) = u_{\sigma 0}(\vec{k})$ in (30), as we only need to keep terms of the linear order of LV coefficients. By the comparison of (39) with (33), we see c_{00} acts like a scale factor to the corresponding LI terms such as $\frac{\vec{\sigma}\cdot\vec{B}}{m}$, $\frac{\vec{p}\cdot\vec{A}}{m}$, while c_{ij} plays the role of a shear factor, and $\vec{c}^i = c_{(0i)}$ mixes the coupling of $\vec{A}$ into those originally coupled with A^0 if LV were absent. In short, $c_{\mu\nu}$ acts like a perturbation of metric tensor: It not only scales isotropically but also shears slightly the original LI EM interactions, as if the original terms being viewed in slightly sheared coordinates. However, we should avoid confusion with the so-called "coordinate transformations," which have no physical effects [7]. In comparison, the c-coefficient induced effects are in principle testable, such as constraints of the sidereal variation by measuring the transition frequency in atomic clocks [60]. For d coefficient, due to the γ_5 factor, it mediates the SOL couplings with the EM field, except the $\tilde{d}_{ij}A^j\sigma^i$ and $\frac{i\tilde{d}_{ij}A^j(\vec{p}\times\vec{q})^i}{4m^2}$ terms. For example, the $\frac{\tilde{d}_{ij}A^jp^i\vec{\sigma}\cdot\vec{p}}{2m^2}$ term looks much like an anomalous magnetic moment (AMM) coupling term $\frac{\mu'(\vec{B}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{2m^2}$ [61], which comes from the FW transformation of the Pauli term $-\frac{\mu'}{2}\overline{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$, and μ' is the AMM coupling constant put by hand.

Next, we consider the LV eigenspinor corrections to the superficially LI term, the term in the first line in the large square bracket in (38). The eigenspinor for c, d coefficients in the quantization of χ has to be obtained from the free LV modified Dirac equation,

$$i\dot{\chi} = -i[(\delta_{ij} - \tilde{c}_{ij} + \tilde{d}_{ij}\gamma_5)\alpha^i - 2(c_{(0j)} - d_{(0j)}\gamma_5)]\nabla_j\chi \\ + m[\gamma^0(1 - c_{00}) - d_{j0}\gamma_5\gamma^j]\chi. \tag{41}$$

Assuming the eigenspinor takes the form $\chi = e^{ip\cdot x}\binom{\xi}{\eta}$, where $\eta = U_X\xi$, we obtain the U_X with $X = c$, d; see (D4), (D5) in the Appendix D.

We still treat c, d terms separately in the spirit of keeping only the linear order of LV coefficients. For the c coefficient, the LV eigenspinor contrition to EM interaction can be obtained by substituting $\delta U_c(k) \equiv U_c(k) - U_0(k)$ in (35),

$$E^{c_2}_{\rm AI} = g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg\{\Bigg[A^0\left(-\frac{c_{(0j)}q^j}{2m} + \frac{c_{ij}p^ip^j}{2m^2}\right) \\ -c_{ij}\frac{p^jA^i}{m}\Bigg] + \Bigg[A^0\frac{2ic_{ij}q^{[j}p^{k]}\epsilon_{ikl}\sigma^l - c_{ij}q^iq^j}{4m^2} \\ -c_{ij}\frac{q^jA^i + i\epsilon_{ikl}q^jA^k\sigma^l}{2m}\Bigg]\Bigg\}\xi_\alpha(p). \tag{42}$$

Again, we have added the correction $A^0(\delta\omega'_p - \delta\omega_p)/4m$ into (42) by substituting (D14); see Eq. (34). The total LV FE interaction energy due to c coefficient is $E^{c_1}_{\rm AI} + E^{c_2}_{\rm AI}$.

For the d coefficients, the LV eigenspinor correction is

$$E^{d_2}_{\rm AI} = g\int d^3\vec{x}e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg(A^0\Bigg[\frac{d_{j0}\left(q^{(i}p^{j)} + \frac{q^jq^i}{2}\right) + d_{0j}q^iq^j}{4m^2} \\ + \frac{\tilde{d}_{ij}q^j}{4m} - \frac{d_{0j}p^ip^j}{2m^2}\Bigg]\sigma^i - \frac{i\epsilon_{jkl}d_{ji}A^l(2q^{(k}p^{i)} + q^kq^i)}{2m^2} \\ + d_{ji}\frac{2A^{[j}\sigma^{k]}[p^kp^i + q^{(k}p^{i)} + q^kq^i/2]}{m^2} \\ + d_{0j}\frac{l^j\vec{\sigma}\cdot\vec{A}}{2m}\Bigg)\xi_\alpha(p). \tag{43}$$

Unlike (39) and (40), $E^{c_2}_{\rm AI}$ and $E^{d_2}_{\rm AI}$ do contain contributions from the LV interaction with scalar potential A^0. These terms would be absent if corrections from LV eigenspinors were not taken into account; see the second lines in (38). The total LV fermion-photon interaction energy due to the d coefficient is $E^{d_1}_{\rm AI} + E^{d_2}_{\rm AI}$. We separately write them out to make the nature of where they originate (from LV corrected current or LV eigenspinor) more clear. Also note, Eq. (43) contains similar SOL terms, such as $d_{ij}\frac{(A^i\vec{\sigma} - \sigma^i\vec{A})\cdot\vec{p}p^j}{m^2}$, as the SO coupling term $\frac{iA^0c_{ij}q^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m^2}$ in (42), only now the role of scalar potential A^0 being replaced by vector potential $\vec{A}$. The $\frac{iA^0c_{ij}q^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m^2}$ term is called SO coupling as it is equivalent to $c_{ij}\frac{E^j(\vec{\sigma}\times\vec{p})^i - p^j(\vec{\sigma}\times\vec{E})^i}{4m^2}$ after the integration by part and the identification $E^i = -\partial_iA^0$ for a static EM field, and as usual, the total derivatives have been ignored. The reason for the structure differences can date back to the additional γ_5 factor in LV kinematic d term compared with the corresponding c term in $\mathcal{L} \supset \frac{i}{2}\bar{\psi}\delta\Gamma^\mu\overleftrightarrow{\partial}_\mu\psi$.

At last, we stress that all the above FE couplings can be expressed in terms of gauge invariant $\vec{E}$, $\vec{B}$ fields by the already mentioned replacement $-i\vec{q}A^0 \to \vec{E}$ and $\vec{q}\times\vec{A} \to \vec{B}$. The reasons we favor the gauge four-potential A^μ instead of the gauge invariant field strength are the following: 1. A^μ directly comes from the current interaction $-j^\mu A_\mu$, and using A^μ makes the calculation more convenient. 2. Expressing all the interaction terms in terms of A^μ and A^μ_g facilitates the comparison of the FE and the FG interactions. Moreover, since the FG couplings do not come from a strict $U(1)$ gauge interaction as the FE couplings, there is no gravitational analogy of the $U(1)$ gauge invariance to guarantee that all the FG couplings can be expressed purely in terms of the $\vec{E}_g$, $\vec{B}_g$ fields. Though the two forces do bare some similarities as mentioned in Sec. II, the GEM is allowed only for a restricted class of gauge transformation $h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$, which only reduces to an approximate $U(1)$ gauge transformation under some special assumptions, such as $\vec{\xi} \sim \mathcal{O}(c^{-4}), \xi^0 \sim \mathcal{O}(c^{-3})$ and $\partial_0 \xi = 0$.

B. Nonrelativistic fermion-gravity interaction

Now we calculate the fermion-gravity interaction. Since we assume the scattered fermions are on the mass shell, which means the equation of motion is satisfied, the term proportional to h does not contribute. The interaction energy $-\frac{1}{2}\int d^3x h_{\mu\nu} T^{\mu\nu}$ in (25) is proportional to

$$\frac{1}{2} h^b{}_a \left(\frac{i}{2} \bar{\psi} \Gamma^a \overleftrightarrow{D}_b \psi - \bar{\psi}[(a_b + b_b \gamma_5)\gamma^a + H_{bc}\sigma^{ac}]\psi \right) - \frac{i}{4}\bar{\psi}[h^{\nu a}(c_{b\nu} + d_{b\nu}\gamma_5) + h^\rho{}_b (c_\rho{}^a + d_\rho{}^a \gamma_5)]\gamma^b \overleftrightarrow{D}_a \psi - \frac{1}{4}\epsilon^{bcmn} h_{am,n} \bar{\psi}[c_b{}^a \gamma_5 + d_b{}^a]\gamma_c \psi. \quad (44)$$

Note we ignore all photon couplings by replacing $D_\mu \to \partial_\mu$, not only for calculational simplicity, but also to facilitate the discussions of the test of equivalence principle (EP), where photon interaction not only complicates but may even spoil the precision test of WEP [62]. For comparison convenience, we adopt the conventional definition of the GEM potentials,

$$h_{00} = -2\phi_g, \qquad h_{ij} = -\delta_{ij} 2\phi_g, \qquad h_{0j} = h_{j0} = A^j_g, \qquad \vec{g} \equiv \nabla\phi_g, \qquad \vec{\Omega} \equiv \nabla \times \vec{A}_g, \quad (45)$$

where $\vec{A}_g$ differs slightly from the definition of A^j_g in the Appendix A. Note that the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\vec{A}_g = \frac{2G_N \vec{r} \times \vec{J}}{r^2}$ is exactly the Lense-Thirring metric [24], and $\vec{J}$ is the total angular momentum of the spinning body. We will call an operator gravito-electric coupling if it couples with gravito-electric fields ϕ_g or $\vec{g}$ or call it gravito-magnetic coupling if it couples with gravito-magnetic fields $\vec{A}_g$ or $\vec{\Omega}$.

The LI fermion-gravity interaction is

$$E^{\rm LI}_{\rm GI} = {}_0\langle p', \beta| \int d^3x \left\{ \frac{i}{4} h^b{}_a \bar{\psi}\gamma^a \overleftrightarrow{\partial}_b \psi \right\} |p, \alpha\rangle_0$$
$$= \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi^\dagger_\beta \left\{ p^0 \left[\phi_g \left(1 + \frac{\vec{p}^2 + \vec{q}\cdot\vec{p} + i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{4m^2} \right) - \frac{\vec{A}_g \cdot \vec{l} + i\vec{q}\times\vec{A}_g\cdot\vec{\sigma}}{4m} \right] + \left(\vec{p} + \frac{\vec{q}}{2} \right) \cdot \left[\frac{\phi_g}{2m} (\vec{l} - i\vec{q}\times\vec{\sigma}) - \frac{\vec{A}_g}{2} \left(1 + \frac{\vec{p}^2 + \vec{q}\cdot\vec{p} + i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{4m^2} \right) \right] \right\} \xi_\alpha$$
$$= \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi^\dagger_\beta \left\{ m\phi_g \left(1 + \frac{7\vec{p}^2}{4m^2} \right) + \frac{3\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m} - \overrightarrow{A_g}\cdot\vec{p} - \frac{\vec{\Omega}\cdot\vec{\sigma}}{4} \right\} \xi_\alpha, \quad (46)$$

where $|p, \alpha\rangle_0$ denotes the pure LI eigen-vector, and we preserve only terms up to PNO(1). Specific in detail, we keep

$$h_{00} \sim \mathcal{O}(v^4), \qquad h_{0i} \sim \mathcal{O}(v^3), \qquad h_{ij} \sim \mathcal{O}(v^2), \quad (47)$$

where v is the velocity of the fermion. Though the metric component $h_{00} = -2\phi_g$ only contains the $\mathcal{O}(v^2)$ term ϕ_g, it is further suppressed by keeping the NR factors, such as $\phi_g \frac{\vec{p}^2}{m^2} \sim \mathcal{O}(v^4)$. Also note only a quarter of the spin-orbit FG interaction $\frac{3\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m}$ comes from the pure temporal metric h_{00} contribution, and the other half comes from the spatial metric h_{ij} contribution. In comparison with the Eq. (20) in [63], the spin-orbit operator from h_{00} exactly coincides with the corresponding term due to a NR fermion coupled with the noninertial force, thus confirming the WEP [63], since the only nonzero metric perturbation for a linear acceleration is $h_{00} = \vec{a}\cdot\vec{r}$.

It will be interesting to compare the result (46) with those obtained by the FW transformation, such as Eq. (2.44) in [64] for static spherically symmetric metric. For this purpose, we also keep

$$\partial_i h_{00} \left| \frac{p^i}{m} \right| \sim \frac{m\lambda_c}{\bar{r}} \mathcal{O}(v^3), \qquad h_{0i} p^k \sim m\mathcal{O}(v^4), \qquad \partial_j h_{0i} \sim \frac{m\lambda_c}{\bar{r}} \mathcal{O}(v^3), \quad (48)$$

where $\lambda_c = \frac{\hbar}{mc}$ is the Compton wavelength of the fermion we are concerned with, say, a neutron, and $\bar{r}$ is the characteristic length scale of the gravitational source, such as the Earth's radius. In general, $\lambda_c/\bar{r} \ll \frac{v}{c}$ (we temporarily restore c for clearness), for example, on the Earth, $\lambda_c/\bar{r} \sim 10^{-22}$ and $v/c \sim 10^{-6}$ for thermal neutron ($v \sim 10^3$ m/s); thus, on numerical grounds, we can totally ignore terms involving $\partial_j h_{0i}$ and $\partial_i h_{00} |\frac{\vec{p}}{m}|$. However, not only for parallel comparison but also in preparation for exotic situations such as neutron stars, where both $h_{00} \sim 0.1$, $|\vec{\Omega}| \sim 10^{-23}$ GeV are much larger than the corresponding values on Earth, we keep these terms in the following. In deriving Eq. (46), we also assume $\vec{\Omega}$ is constant such that $\vec{A}_g = \frac{1}{2}\vec{\Omega} \times \vec{r}$ and utilize the equations $\nabla^2 \phi_g = 4\pi G_N \rho_m$ and $\nabla^2 \vec{A}_g = 16\pi G_N \vec{j}_m$ to eliminate the $\vec{p} \cdot \vec{q} \phi_g$ and $\vec{p} \cdot \vec{q} \vec{A}_g$ terms, since the neutral fermion is assumed to be outside the matter source of gravity, where ρ_m and $\vec{j}_m$ vanish. This also explains why there is no $i\vec{g} \cdot \vec{p}$ term compared with the NR Hamiltonian obtained by the FW approach.

Now we also consider the a, b, H terms first as these terms do not involve derivative couplings. However, unlike the EM current $\bar{\psi}\Gamma^a \psi$, the a, b, H terms do contribute to the LV energy-momentum tensor $T^{\mu\nu}$, which receives any kind of contribution from matter source. For simplicity, we discuss the LV eigenspinor corrections first. As in Eq. (35), we write down a general formula for the LV eigenspinor correction,

$$E_{\rm GI}^{\rm X-spinor} = \frac{1}{4}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \{[2l^0\phi_g - \vec{l}\cdot\vec{A}_g] \cdot [\delta U_X^\dagger(\vec{p'}) U_0(\vec{p}) + U_0^\dagger(\vec{p'}) \delta U_X(\vec{p})] + [2\phi_g \vec{l} - \vec{A}_g l^0] \cdot [\delta U_X^\dagger(\vec{p'})\vec{\sigma} + \vec{\sigma}\delta U_X(\vec{p})]\} \xi_\alpha, \tag{49}$$

where the 2×2 matrices U_0 and δU_X are defined under Eq. (35). Note the structure similarity between (35) and (49), where ϕ_g and A_g replace the role of ϕ and $\vec{A}$, respectively, while the remaining terms, $\vec{A}_g \cdot \vec{l}$ and $\phi_g \vec{l}$, reflect the tensor nature of gravitational coupling. Both the similarity and differences between FG and FE couplings may stem from this peculiar structure.

Substituting $\delta U_X(\vec{p})$ with $X = a, b, H, c, d$ separately into (49) with δU_X given in Appendix D, the LV eigenspinor corrections to FG interaction due to a, b, H coefficients are

$$\begin{aligned} E_{\rm GI}^{ab-1} &= \frac{1}{4}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ [2l^0\phi_g - \vec{l}\cdot\vec{A}_g] \left[\frac{(\vec{a} + b^0\vec{\sigma})\cdot\vec{l}}{4m^2} + \frac{i\vec{q}\times\vec{a}\cdot\vec{\sigma}}{4m^2} \right] + [2\phi_g\vec{l} - \vec{A}_g l^0] \cdot \left[\frac{\vec{a} + b^0\vec{\sigma}}{m} - \frac{i\vec{q}\times\vec{b}}{2m^2} - \frac{(\vec{l}\times\vec{b})\times\vec{\sigma}}{2m^2} \right] \right\} \xi_\alpha \\ &\simeq \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ \frac{\vec{g}\times\vec{a}\cdot\vec{\sigma}}{4m} + \phi_g \left[\frac{3\vec{a}\cdot(\vec{p}+\vec{q}/2)}{2m} + \frac{3b^0\vec{l}\cdot\vec{\sigma}}{4m} + \frac{(\vec{b}\cdot\vec{l})(\vec{\sigma}\cdot\vec{l})}{4m^2} - \frac{(\vec{p}^2 + \vec{p}\times\vec{q} + \vec{q}^2/4)}{m^2} (\vec{b}\cdot\vec{\sigma}) \right] \right. \\ &\quad \left. - \left[\vec{A}_g \cdot \left(\vec{a} + \frac{b^0\vec{\sigma}}{2} \right) + \frac{\vec{\Omega}\cdot\vec{b}}{4m} + \frac{\vec{A}_g\cdot(\vec{l}\vec{b} - \vec{b}\vec{l})\cdot\vec{\sigma}}{4m} \right] \right\} \xi_\alpha, \end{aligned} \tag{50}$$

$$\begin{aligned} E_{\rm GI}^{H-1} &= \frac{1}{4}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ [2l^0\phi_g - \vec{l}\cdot\vec{A}_g] \left[\frac{\vec{l}\times\vec{H}\cdot\vec{\sigma}}{4m^2} - \frac{i\vec{q}\cdot\vec{H}}{4m^2} \right] + [2\phi_g\vec{l} - \vec{A}_g l^0] \cdot \left[\frac{(\vec{H}\times\vec{\sigma})}{m} - \frac{\vec{l}\cdot\vec{\tilde{H}}}{2m^2}\vec{\sigma} \right] \right\} \xi_\alpha \\ &\simeq \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ \phi_g \left[\frac{3\vec{l}\times\vec{H}\cdot\vec{\sigma}}{4m} - \vec{l}\cdot\vec{\sigma}\frac{\vec{l}\cdot\vec{\tilde{H}}}{4m^2} \right] - \frac{\vec{g}\cdot\vec{H}}{4m} - \vec{A}_g \cdot \frac{(\vec{H}\times\vec{\sigma})}{2} + \vec{A}_g\cdot\vec{\sigma}\frac{\vec{l}\cdot\vec{\tilde{H}}}{4m} \right\} \xi_\alpha, \end{aligned} \tag{51}$$

where we have ignored terms of order $\frac{\vec{g}}{m^2}$ and $\frac{\vec{A}_g}{m^2}$ in the last two approximations. Note $\frac{\vec{\Omega}\cdot\vec{b}}{4m}$ appears in (50) just as $\frac{g\vec{B}\cdot\vec{b}}{2m^2}$ appears in (36). The lesser suppression by the inverse power of m is because in the gravitational case, m plays the role of coupling constant g. As mentioned before, the corresponding terms can be found from (36) and (37) by replacing $\phi, \vec{A}$ with $\phi_g, \vec{A}_g$, though the associated numerical factors are different. Due to the tensor nature of gravity, there are additional terms such as $-\phi_g \frac{(\vec{l}\cdot\vec{\sigma})(\vec{l}\cdot\vec{\tilde{H}})}{4m^2}$, $\phi_g \frac{(\vec{b}\cdot\vec{l})(\vec{\sigma}\cdot\vec{l})}{4m^2}$ and $\frac{-\vec{p}^2\phi_g\vec{b}\cdot\vec{\sigma}}{4m^2}$ in comparison with the FE couplings for the corresponding LV coefficients.

For the apparent LV interaction vertices due to a, b, H, their contribution to the interaction energy is

$$
\begin{aligned}
E_{\rm GI}^{\rm ab-2} &= -\frac{1}{2}\langle p',\beta| \int d^3x\{h_{ba}\overline{\psi}[(a^b+b^b\gamma_5)\gamma^a]\psi\}|p,\alpha\rangle \\
&= \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\Bigg\{\left(\phi_g a^0 - \frac{\vec{a}\cdot\vec{A}_g}{2}\right)\left(1+\frac{\vec{p}'\cdot\vec{p}+i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{4m^2}\right)+\left[\frac{\phi_g\vec{a}}{2m}-\frac{\vec{A}_g a^0}{4m}\right]\cdot[\vec{l}-i\vec{q}\times\vec{\sigma}]+\left(\frac{\vec{A}_g\cdot\vec{b}}{2}-\phi_g b^0\right)\frac{\vec{\sigma}\cdot\vec{l}}{2m} \\
&\quad+\left(\frac{b^0\vec{A}_g}{2}-\phi_g\vec{b}\right)\cdot\left[\left(1-\frac{\vec{p}\cdot\vec{p}'}{4m^2}\right)\vec{\sigma}+\frac{i\vec{p}\times\vec{q}+(\vec{p}\,\vec{p}'+\vec{p}'\,\vec{p})\cdot\vec{\sigma}}{4m^2}\right]\Bigg\}\xi_\alpha \\
&\simeq \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\Bigg\{\phi_g\left[a^0\left(1+\frac{\vec{p}'\cdot\vec{p}}{4m^2}\right)+\frac{(\vec{a}-b^0\vec{\sigma})\cdot\vec{l}}{2m}\right]-\frac{\vec{a}\cdot\vec{A}_g}{2}+\frac{\vec{g}\times\vec{a}\cdot\vec{\sigma}}{2m}-a^0\left[\frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4m}\right]+\frac{\vec{A}_g}{2}\cdot\left(b^0\vec{\sigma}+\frac{\vec{b}\,\vec{\sigma}\cdot\vec{l}}{2m}\right) \\
&\quad-\phi_g\vec{b}\cdot\left[\left(1-\frac{\vec{p}\cdot\vec{p}'}{4m^2}\right)\vec{\sigma}+\frac{(\vec{p}\,\vec{p}'+\vec{p}'\,\vec{p})\cdot\vec{\sigma}}{4m^2}\right]\Bigg\}\xi_\alpha. \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
E_{\rm GI}^{\rm H-2} &= -\frac{1}{2}\langle p',\beta| \int d^3x\{h_{ba}\overline{\psi}H^{bc}\sigma_c^a\psi\}|p,\alpha\rangle \\
&= \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\Bigg\{\left(\frac{\vec{A}_g\times\vec{H}}{2}+2\phi_g\vec{\tilde{H}}\right)\cdot\left[\left(1+\frac{\vec{p}\cdot\vec{p}'}{4m^2}\right)\vec{\sigma}-\frac{i\vec{p}\times\vec{q}+(\vec{p}\,\vec{p}'+\vec{p}'\,\vec{p})\cdot\vec{\sigma}}{4m^2}\right] \\
&\quad+\frac{\vec{\tilde{H}}}{4m}\cdot(\vec{l}\,\vec{\sigma}-\vec{\sigma}\,\vec{l})\cdot\vec{A}_g-\frac{i\vec{q}\times\vec{A}_g\cdot\vec{\tilde{H}}}{4m}\Bigg\}\xi_\alpha \\
&\simeq \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi_\beta^\dagger\Bigg\{\frac{\vec{A}_g\times\vec{H}}{2}\cdot\vec{\sigma}+\frac{\vec{\tilde{H}}}{4m}\cdot[(\vec{l}\,\vec{\sigma}-\vec{\sigma}\,\vec{l})\cdot\vec{A}_g]+2\phi_g\vec{\tilde{H}}\cdot\left[\left(1+\frac{\vec{p}\cdot\vec{p}'}{4m^2}\right)\vec{\sigma}+\frac{(\vec{p}\,\vec{p}'+\vec{p}'\,\vec{p})\cdot\vec{\sigma}}{4m^2}\right]-\frac{\vec{\Omega}\cdot\vec{\tilde{H}}}{4m}\Bigg\}\xi_\alpha. \qquad (53)
\end{aligned}
$$

By comparison with Eq. (37), there should not have any scalar potential coupling to the "magnetic" part of the H coefficient; however, due to the tensor nature, the nonzero spatial metric h_{ij} induces gravito-electric couplings to $\vec{\tilde{H}}$, such as the terms proportional to $\phi_g\vec{\tilde{H}}\cdot\vec{\sigma}$ and $(\phi_g\vec{\tilde{H}}\cdot\vec{l})(\vec{l}\cdot\vec{\sigma})$.

A striking difference from the fermion-photon interaction is the presence of a-coupling terms in (50) and (52). Comparing (52) with (46), we see the a^μ coefficient couples to ϕ_g and $\vec{A}_g$ in exactly the same way as the four-momentum p^μ. This is not surprising as in the momentum space,

$$
\begin{aligned}
&-\frac{1}{2}h_{ba}\overline{\psi}a^b\gamma^a\psi+\frac{i}{4}h^b{}_a\bar{\psi}\gamma^a\overleftrightarrow{\partial}_b\psi \\
&\Rightarrow -\frac{1}{2}h_{ba}\bar{u}_\beta(p')\left[a^b+\frac{p'^b+p^b}{2}\right]\gamma^a u_\alpha(p),
\end{aligned}
$$

and is also the same reason that a^μ can be shifted away by a phase redefinition of the fermion field; thus, it does not have any observable consequence for a single fermion coupled with a photon field in flat space. However, the above reasoning does not apply to a fermion coupled with gravity [38]. This can be verified by inspecting Eq. (46), where the simple replacement $p^\mu \to (p+a)^\mu$ cannot lead to the a-coupling terms in (50).

Note that we also need to consider the implicit correction to the fermion-gravity interaction energy induced by LV dispersion relation $p^0=\omega_0(\vec{p},m)+\delta\omega(\vec{p},m,X)$. This correction comes from the substitution of $\vec{p}\cdot\vec{q}$ in the superficially LI term $\frac{i}{4}\int d^3xh^b{}_a\bar{\psi}\gamma^a\overleftrightarrow{\partial}_b\psi$, just as what we did with Eq. (34). However, in the gravitational case, an additional contribution comes from p^0 term in (46) and thus is proportional to $\delta\omega_p$. Inspection of $\delta\omega_p-\delta\omega_{p'}$ for various LV coefficients in the Appendix D, we see these terms are at least of $\mathcal{O}(v)$, so in making a substitution of $\vec{q}\cdot\vec{p}=-\frac{\vec{q}^2}{2}+m[\delta\omega_p-\delta\omega_{p'}]$ and $p^0=\omega_0+\delta\omega_p$, the following correction

$$
\begin{aligned}
&\phi_g\left[\left(1+\frac{\vec{p}^2+i\vec{q}\times\vec{p}\cdot\vec{\sigma}}{4m^2}\right)\delta\omega_p+\frac{5}{4}(\delta\omega_p-\delta\omega_{p'})\right] \\
&\quad-\left[\frac{\vec{A}_g\cdot\vec{l}+i\vec{q}\times\vec{A}_g\cdot\vec{\sigma}}{4m}\right]\delta\omega_p \qquad (54)
\end{aligned}
$$

has to be added for each type of LV coefficient. For completeness, the dispersion relation for a coefficient is $k^0 = \sqrt{(\vec{k}+\vec{a})^2+m^2} - a^0$ for a positive energy fermion, and hence, $\delta\omega_p \simeq \vec{k}\cdot\vec{a}/\omega_0 - a^0$. For a, b, H coefficients, the corrections due to LV dispersion relations are listed below:

$$E_{\rm GI}^{\rm ab-3} = \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ \phi_g \left(\frac{(4\vec{p}-5\vec{q})\cdot(\vec{a}+b^0\vec{\sigma})}{4m} \right) + (a^0+\vec{b}\cdot\vec{\sigma}) \left(\frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4m} - \left[\phi_g + \frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m^2} \right] \right) + \phi_g \left[\frac{\vec{p}^2}{4m^2}\cdot(\vec{b}\cdot\vec{\sigma}-a^0) + \frac{5(\vec{p}'\cdot\vec{b})(\vec{p}'\cdot\vec{\sigma}) - 9(\vec{p}\cdot\vec{b})(\vec{p}\cdot\vec{\sigma})}{8m^2} \right] \right\} \xi_\alpha, \tag{55}$$

$$E_{\rm GI}^{\rm H-3} = \int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ \phi_g \left[\frac{\vec{H}\times(4\vec{p}-5\vec{q})\cdot\vec{\sigma}}{4m} + \left(1+\frac{\vec{p}^2}{4m^2}\right)\vec{\tilde{H}}\cdot\vec{\sigma} + \frac{5\vec{\tilde{H}}\cdot\vec{p}'\,\vec{\sigma}\cdot\vec{p}' - 9\vec{\tilde{H}}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p}}{8m^2} \right] - \left[\frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4m} - \frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m^2} \right] \vec{\tilde{H}}\cdot\vec{\sigma} \right\} \xi_\alpha. \tag{56}$$

The total NR fermion-gravity interaction energy from a, b, H contributions is the summation of Eqs. (50)–(53) and (55) and (56). Though it is easy to see that several terms in the above equations can be combined or even canceled, such as the terms proportional to $\phi_g \vec{b}\cdot\vec{\sigma}$ and $a^0 \frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4m}$ in (52) and (55), or the terms proportional to $\vec{g}\times\vec{a}\cdot\vec{\sigma}$ in (50) and (52), we keep them separately for the clarity of their origin.

Inspecting Eqs. (50)–(53) and (55) and (56) reveals that there are abundant interaction structures for the LV spin-gravity coupling, especially for the b, H coefficients. For example, the $-\frac{(\vec{\tilde{H}}+\vec{b})\cdot\vec{\Omega}}{4m}$ term is in analogy with the LV magnetic field coupling term $\frac{g\vec{b}\cdot\vec{B}}{2m^2}$ in (36), only with gravitomagnetic field $\vec{\Omega}$ replacing the magnetic field $\vec{B}$. Similarly, the $\frac{(\vec{b}-\vec{\tilde{H}})\cdot\vec{\sigma}(\vec{\Omega}\cdot\vec{\sigma})}{4m}$ and the spin-orbit coupling terms such as those proportional to $\frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m^2}$ alter the geodetic and frame-dragging precession frequencies of microscopic particles. Since there is no reason for the LV coefficients to be universal for particles with different flavors, the WEP must be violated due to the nonuniversal LV gravitational couplings. These effects are in principle testable, such as in the high precision Gravity Probe B-like experiment [10,11].

Aside from the B-type LV couplings, the E-type LV couplings also show some similarity between the fermion-photon and fermion-gravity couplings, such as $-\frac{\vec{E}\cdot\vec{H}}{4m^2}$ and $-\frac{\vec{g}\cdot\vec{H}}{4m}$, or $\frac{ig\vec{E}\times\vec{H}}{2m^2}\cdot\vec{\sigma}$ and $\frac{-2i\vec{g}\times\vec{H}}{m}\cdot\vec{\sigma}$. The similarities for the LV couplings between the b, H coefficients can be traced back to the operator level by the identity $\overline{\psi}\gamma_5\vec{\gamma}\psi = -\bar{\psi}\gamma_0\vec{\Sigma}\psi$ (where $\Sigma^i = \frac{i\epsilon_{ijk}}{4}[\gamma^j,\gamma^k]$), while γ^0 is effectively equal to $\hat{1}$ for positive energy particles. For example, this fact can be validated by the similar form of couplings between $\vec{b}\cdot\vec{\sigma}$ and $\vec{\tilde{H}}\cdot\vec{\sigma}$ with $[\vec{\Omega}-\vec{g}\times\vec{p}/m]\cdot\vec{\sigma}$ in (55) and (56).

Next, we discuss the fermion-gravity interaction energies due to the c, d coefficients. The contributions due to eigenspinor corrections for c, d coefficients are

$$E_{\rm GI}^{c-1} = \frac{1}{4}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ [A_g^k l^0 - 2\phi_g l^k] \left[c_{ij}\frac{\delta_{ik}l^j + i\epsilon_{ikl}q^j\sigma^l}{2m} \right] + [\vec{l}\cdot\vec{A}_g - 2l^0\phi_g] \left[\frac{c_{(ij)}p'^i p^j + ic_{ij}q^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m^2} \right] \right\} \xi_\alpha, \tag{57}$$

$$E_{\rm GI}^{d-1} = \frac{1}{4}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \xi_\beta^\dagger \left\{ [2l^0\phi_g - \vec{l}\cdot\vec{A}_g] \frac{d_{0j}p'^{(i}p^{j)}\sigma^i}{2m^2} + [2\phi_g l^m - A_g^m l^0] \left[\frac{d_{0j}l^j}{2m}\sigma^m + \frac{id_{ji}\epsilon_{jkm}(p^k p^i - p'^k p'^i)}{2m^2} + \frac{d_{mi}\sigma^k(p^k p^i + p'^k p'^i) - d_{ji}\sigma^j(p^m p^i + p'^m p'^i)}{2m^2} \right] \right\} \xi_\alpha. \tag{58}$$

Comparing the terms in (57) and (58) taking the form of $(A_g^k X_k - 2\phi_g Y)l^0$, where X_k, Y are LV operators such as $\frac{d_{0j}l^j}{2m}\sigma^k$, $-\frac{c_{ij}q^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m}$, with the terms in (42) and (43), we see they also look quite similar, as mentioned in the general discussion of Eq. (49).

The apparently LV vertex contributions due to c, d coefficients are

$$E_{\rm GI}^{\rm cd-V} = -\frac{i}{4}\langle p',\beta| \int d^3x \{ h^b{}_a[\bar{\psi}(c_e{}^a + d_e{}^a\gamma_5)\gamma^e \overleftrightarrow{\partial}_b\psi] + h^{\nu a}[\bar{\psi}(c_{b\nu} + d_{b\nu}\gamma_5)\gamma^b \overleftrightarrow{\partial}_a\psi] + h^\rho{}_b[\bar{\psi}(c_\rho{}^a + d_\rho{}^a\gamma_5)\gamma^b \overleftrightarrow{\partial}_a\psi] \} |p,\alpha\rangle. \tag{59}$$

Note the terms in the first line are in fact equal to the terms in the second line of (59). The terms in the third line of Eq. (44), $-\frac{1}{4}\epsilon^{bcmn}\langle p', \beta| \int d^3x h_{am,n}\bar{\psi}[c_b{}^a\gamma_5 + d_b{}^a]\gamma_c\psi|p, \alpha\rangle$ come from the spin-connection interaction and thus only contain GEM field strength $\partial_\rho h_{\mu\nu}$ and are naively expected to be much smaller than the terms coupled directly with the metric perturbation $h_{\mu\nu}$.

The contributions due to LV dispersion relation corrections for the c, d coefficients are

$$E^{\mathrm{c}-2}_{\mathrm{GI}} = \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg\{c_{00}\left[\frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4} - \frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m}\right] + \phi_g\cdot\left[c_{(0j)}\frac{5q^j-4p^j}{2} + c_{(ij)}\frac{5p'^i p'^j - 9p^i p^j}{4m} - c_{00}\left(m+\frac{3\vec{p}^2}{4m}\right)\right]\Bigg\}\xi_\alpha, \tag{60}$$

$$E^{\mathrm{d}-2}_{\mathrm{GI}} = \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg\{\phi_g\left[\frac{9(\vec{p}\cdot\vec{d})(\vec{p}\cdot\vec{\sigma}) - 5(\vec{p}'\cdot\vec{d})(\vec{p}'\cdot\vec{\sigma})}{2m} + \frac{\tilde{d}_{ji}(4p^i-5q^i)\sigma^j}{4} + \frac{5d_{j0}p'^j\vec{\sigma}\cdot\vec{p}' - 9d_{j0}p^j\vec{\sigma}\cdot\vec{p}}{8m} + d_{j0}\sigma^j\left(m+\frac{\vec{p}^2}{4m}\right)\right] + \frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{4m}d_{j0}\sigma^j - \frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4}d_{j0}\sigma^j\Bigg\}\xi_\alpha. \tag{61}$$

In comparison with Eq. (39), there is also a corresponding term $c_{00}(\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma})/4$, which rescales the gravito-magnetic moment just as the corresponding term rescales the magnetic moment. This and the other similar LV corrections spoil the theorem of zero anomalous gravito-magnetic moment due to the EP [65], which is not unexpected in the presence of LV.

For compactness, we combine these c, d couplings together and disregard the quadratic terms of q^i, as $q^i q^j h_{0\nu} \sim \partial_i\partial_j h_{0\nu}$ and $|\partial_i\partial_j h_{0\nu}| \ll |p^i\partial_j h_{0\nu}|$ in general. We also keep only terms up to $\mathcal{O}(m^{-1})$, and the results are

$$E^{\mathrm{c}}_{\mathrm{GI}} = \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg\{\Bigg[c_{00}\left(\frac{\vec{g}\times\vec{p}\cdot\vec{\sigma}}{2m}+2m\phi_g\right) + c_{(0i)}\phi_g\frac{7q^i}{2} - l^i\phi_g c_{0i} + \frac{1}{2}\epsilon_{ijk}c_{0i}g^k\sigma^j - c_{ij}\epsilon_{ikl}\left(\frac{3g^{(j}p^{k)}}{2}+\frac{p^j g^k}{2}\right)\frac{\sigma^l}{m} - c_{(ij)}\phi_g\frac{2p^i q^j}{m} - \epsilon_{ijk}c_{ij}g^k\frac{\vec{\sigma}\cdot\vec{p}}{4m}\Bigg] + \left[mA^i_g\left(\frac{c_{0i}}{2}+c_{(0i)}\right) - c_{00}\left(\vec{A}_g\cdot\vec{p}+\frac{\vec{\Omega}\cdot\vec{\sigma}}{4}\right) + c_{ij}\frac{A^i_g l^j + i\epsilon_{ikl}q^j A^k_g\sigma^l}{4} + \frac{l^i c_{ij}}{4}A^j_g + [l^i - \epsilon_{ikl}\sigma_l\partial_k]A^j_g\frac{c_{ij}}{2} - \epsilon_{ijk}\frac{c_{il}}{4}\partial_k A^l_g\sigma^j\right]\Bigg\}\xi_\alpha, \tag{62}$$

$$E^{\mathrm{d}}_{\mathrm{GI}} = \int d^3x e^{-i\vec{q}\cdot\vec{x}}\xi^\dagger_\beta\Bigg\{\phi_g\left[d_{ij}\left(4p^j+\frac{q^j}{4}\right)\sigma^i - d_{00}\frac{\vec{\sigma}\cdot(8\vec{p}+11\vec{q})}{4} - \frac{7d_{j0}q^{(i}p^{j)}\sigma^i}{4m}\right] + \frac{1}{2}\epsilon_{ijk}d_{ij}g^k + (d_{i0}+2d_{(0i)})\frac{(\vec{g}\times\vec{p})^i}{2m} + \frac{id_{j0}}{4m}[\vec{\sigma}\times(\vec{g}\times\vec{p})]^j + md_{00}\frac{\vec{A}_g\cdot\vec{\sigma}}{2} + \frac{d_{i0}}{4}(\sigma^i\vec{A}_g - A^i_g\vec{\sigma})\cdot\vec{l} - d_{0i}A^i_g\frac{\vec{\sigma}\cdot\vec{l}}{2} - \frac{id_{j0}}{4}(\vec{\Omega}\times\vec{\sigma})^j - mA^j_g d_{ij}\sigma^i\Bigg\}\xi_\alpha. \tag{63}$$

Inspection of (60)–(63) shows that several LV spin-orbit coupling terms, such as $d_{j0}\sigma^j\vec{g}\times\vec{p}\cdot\vec{\sigma}/4m$, $c_{00}\vec{g}\times\vec{p}\cdot\vec{\sigma}/2m$ and $c_{ij}(\vec{g}\times\vec{\sigma})^i p^j/2m$, are of the similar kind of structure as we found in FE interactions, like $c_{ij}\frac{(\vec{E}\times\vec{\sigma})^i p^j - (\vec{p}\times\vec{\sigma})^i E^j}{4m^2}$. Other more complicated structures of spin-orbit couplings, such as $\frac{c_{0i}}{2}(\vec{g}\times\vec{\sigma})^i$, $-\frac{id_{j0}}{4}(\vec{\Omega}\times\vec{\sigma})^j$, $d_{(0i)}A^i_g\vec{\sigma}\cdot\vec{l}/2$, $\frac{id_{j0}}{4m}[\vec{\sigma}\times(\vec{g}\times\vec{p})]^j$, etc. can also be found in Eqs. (62) and (63). Also we notice that there are only two spin-independent fermion-gravity couplings for the d coefficient, $(d_{i0}+2d_{(0i)})\frac{(\vec{g}\times\vec{p})^i}{2m}$ and $\frac{1}{2}\epsilon_{ijk}d_{ij}g^k$. This is not surprising as in the Lagrangian level, d term is of the $\gamma_5\gamma^a$ structure and is an essentially spin-dependent term from the relativistic point of view.

In summary, due to similar Dirac structures in Lorentz violating fermion-gravity (FG) and fermion-electromagnetic (FE) couplings, there are analog operators for the LV fermion coupled with these two external fields. At a simple glance, we collect several sample operators of FG and FE interactions in Table I. Operators such as $a^0\frac{\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma}}{4m}$, $\frac{(\vec{A}_g\times\vec{H})\cdot\vec{\sigma}}{2}$ exactly cancel and thus in fact do not appear. The mismatch between FG and FE interactions may partly be due to the

TABLE I. Examples of the analogous operators between the LV fermion-gravity and fermion-photon couplings.

FG	$-\frac{(\vec{\tilde{H}}+\vec{b})\cdot\vec{\Omega}}{4m}$	$\frac{(\vec{A}_g\cdot\vec{l})(\vec{b}\cdot\vec{\sigma})}{2m}$	$\phi_g\frac{\vec{p}\times\vec{H}\cdot\vec{\sigma}}{2m}$	$\frac{(\vec{A}_g\cdot\vec{\sigma})(\vec{p}\cdot\vec{\tilde{H}})}{m}$
FE	$\frac{g\vec{b}\cdot\vec{B}}{2m^2}$	$\frac{g(\vec{A}\cdot\vec{l})(\vec{b}\cdot\vec{\sigma})}{2m^2}$	$-\frac{gA^0\vec{p}\times\vec{H}\cdot\vec{\sigma}}{2m^2}$	$-\frac{g(\vec{A}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\tilde{H}})}{m^2}$
FG	$-\frac{c_{00}(2\vec{A}_g\cdot\vec{l}+\vec{\Omega}\cdot\vec{\sigma})}{4}$	$\frac{3c_{ij}g^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m}$	$-\frac{\vec{g}\cdot\vec{H}}{4m}$	$-\frac{(\vec{d}\cdot\vec{A}_g)(\vec{\sigma}\cdot\vec{l})}{2}$
FE	$\frac{gc_{00}(\vec{A}\cdot\vec{l}+\vec{B}\cdot\vec{\sigma})}{2m}$	$\frac{gc_{ij}E^{[j}p^{k]}\epsilon_{ikl}\sigma^l}{2m^2}$	$-\frac{g\vec{E}\cdot\vec{H}}{4m^2}$	$\frac{g(\vec{d}\cdot\vec{A})(\vec{\sigma}\cdot\vec{l})}{m}$

tensor structure of gravity and partly due to the fact that the LV corrections from fermion dispersion relations $p^0 = \omega^0 + \delta\omega_p$ can contribute directly in the case of gravity, in contrast to the case of photon coupling, where only $\delta\omega_p - \delta\omega_{p'}$ enters in the $q \cdot p$ substitution. Anyway, we think even the sample operators in Table I can convince the readers that the LV spin coupling structures are very abundant, which means that the gravitational phenomenologies arising from the LV spin-gravity couplings [23] waiting for us to explore are very rich.

VI. PHENOMENOLOGY IN TEST OF EP

The LV spin-gravity couplings have already been thoroughly explored in the uniform limit $\phi_g = \vec{g} \cdot \vec{z}$ [23], which is a very good approximation for most experiments on the Earth. However, the linear potential is essentially flat and is incapable of capturing the curvature effects of space, as only g_{00} matters in this case. In comparison, the Lense-Thirring metric is an intrinsically curved one and may be able to test LV spin-gravity couplings where the other metric components take effect, such as the frame-dragging (FD) effect of a single fermion due to the rotation of a massive object like a neutron star. For the pure gravity sector, we also note that the spin precession effects in the post-Newtonian approximation up to $\mathcal{O}(3)$ have already been systematically studied [48], and the anomalous precession rates due to LV have also been utilized to constrain the $s_{\mu\nu}$ coefficients [12]. However, these are for macroscopic spinning gyroscopes, not for the intrinsic spin of microscopic fermions.

The Lorentz invariant NR fermion-gravity Hamiltonian has been fully studied in the literature [50,63,64], and it is interesting to note that the LI operators in $E^{\rm LI}_{\rm GI}$, Eq. (46), coincide with those in the NR fermion Hamiltonian obtained in [64] except the higher order term $\phi_g \vec{p}^2/2m$, which differs by an $\mathcal{O}(1)$ numerical factor. This can be attributed to the two following differences between our calculations and those in [64]: 1. The simplification of the normalization factor $\sqrt{\frac{\omega_0+m}{2m}} = 1$ in front of the spinor u_σ and its conjugate $u^\dagger_\sigma$, see Eq. (30), and this can induce a $\frac{\vec{p}^2}{4m^2}$ difference. 2. The operators we studied are sandwiched by the bispinors $\xi^\dagger_\beta$ and ξ_α, while for comparison, the NR Hamiltonian in [64] needs to be sandwiched by $u^\dagger_\beta$ and u_α, which will induce another $\frac{\vec{p}^2}{4m^2}$ difference. Taken together, they give the correct $\frac{3\vec{p}^2}{2m^2}$ factor in front of $m\phi_g$. The vanishing of terms $\nabla^2\phi_g, \frac{i\vec{g}\cdot\vec{p}}{m}$ is due to our on-shell and source free assumptions. This is not surprising, as the LI terms in the one-fermion matrix element $-\frac{1}{2}\langle p', \beta| \int d^3x h_{\mu\nu} T^{\mu\nu} |p, \alpha\rangle$ under the assumption of zero energy transfer, $q^0 = 0$, are just the potential energy at tree-level approximation. For the corresponding LV terms, we may also expect them to be the corresponding LV operators in the NR Hamiltonian obtained by FW transformation [13,25,26], except that each pair of operators obtained from different approaches may differ by an $\mathcal{O}(1)$ numerical factor. As most LV coefficients in the minimal SME have been tightly constrained to be vanishingly small [21], what we really cared about is essentially the order of magnitude; the $\mathcal{O}(1)$ numerical factors may be irrelevant for practical purposes. Thus, we can collect all spin-dependent operators up to $\mathcal{O}(m^{-1})$ [except the a^0 term being of $\mathcal{O}(m^{-2})$]

$$\begin{aligned}\delta\hat{H}_{g\sigma} = {} & \left[\frac{3\vec{g}\times\vec{a}}{4m} - \frac{a^0\vec{g}\times\vec{p}}{4m^2}\right]\cdot\vec{\sigma} + \phi_g\left[\frac{3b^0\vec{p}}{2m} - 2\vec{b}\right]\cdot\vec{\sigma} \\ & + \frac{\vec{\sigma}\cdot\vec{p}}{m}\vec{A}_g\cdot\vec{b} + \phi_g\left[\frac{\vec{p}\times\vec{H}}{2m} + 3\vec{\tilde{H}}\right] \\ & \cdot\vec{\sigma} + \frac{\vec{A}_g\cdot(\vec{\sigma}\,\vec{p} - \vec{p}\,\vec{\sigma})\cdot\vec{\tilde{H}}}{m} + c_{00}\left[\frac{\vec{g}\times\vec{p}}{2m} - \frac{\vec{\Omega}}{4}\right] \\ & \cdot\vec{\sigma} + d_{00}\left[\frac{m\vec{A}_g}{2} - 2\phi_g\vec{p}\right]\cdot\vec{\sigma} \qquad (64)\end{aligned}$$

together, and for simplicity, we also ignore the terms coupled with $c_{\mu\nu}, d_{\mu\nu}$ coefficients, except the c_{00} and d_{00}. Note that the meaning of "spin dependence" here should not be confused with the spin dependence attached to the LV coefficients in free fermion theory, where a, c are spin independent. We boldly assume that the NR Hamiltonian is

$$\begin{aligned}\hat{H}_{\rm NR} = {} & \frac{\vec{p}^2}{2m} + m\phi_g + \frac{3}{2m}\left[\phi_g\vec{p}^2 - i\vec{g}\cdot\vec{p} + \vec{g}\times\vec{p}\cdot\frac{\sigma}{2}\right] \\ & - \frac{\vec{\Omega}\cdot\vec{\sigma}}{4} + \delta\hat{H}_{g\sigma}, \qquad (65)\end{aligned}$$

where the first line involves LI contributions. Note we have ignored all the spin-independent LV operators, as they do not directly affect spin dynamics. The spin time evolution is governed by the Heisenberg equation

$$\frac{d\vec{S}}{dt} = \frac{1}{i\hbar}[\vec{S}, \hat{H}_{\rm NR}] = (\vec{\omega}_{\rm LI} + \delta\vec{\omega}_{\rm LV}) \times \vec{S}, \qquad (66)$$

where $\vec{\omega}_{\rm LI} \equiv \vec{\omega}_{\rm geo} + \vec{\omega}_{\rm FD}$, and $\vec{\omega}_{\rm geo} = \frac{3}{2m}\vec{g}\times\vec{p}$, $\vec{\omega}_{\rm FD} = -\frac{\vec{\Omega}}{2}$ describe the geodetic precession and FD precession, respectively. Interestingly, we obtain $\vec{\omega}_{\rm LI}$ from the Heisenberg equation with the semiclassical fermion-gravity couplings, which only relies on the minimal fermion-gravity assumption within the tetrad formalism. Since the geodetic and FD precession angular vectors, $\vec{\omega}_{\rm geo} = \frac{3}{2m}\vec{g}\times\vec{p}$ and $\vec{\omega}_{\rm FD} = -\frac{\vec{\Omega}}{2}$, exactly coincide with those predicted in GR [33] for a probe gyroscope carrying macroscopic angular momentum in the weak gravitational field of a massive rotating object, while the spin geodetic and FD precession frequencies for a quantum object, such as a fermion carrying spin-$\frac{1}{2}$, are not necessarily the same as the corresponding macroscopic terms in a general gravitational theory other than GR, this coincidence can be viewed as a piece of evidence that the WEP is valid even in the quantum regime [49,50,66], though this evidence is far from a conclusive one. Given that WEP has been tested to high precision [66–68], we may reasonably believe that the GR predicted spin precession of microscopic particles can also be tested to the same precision in the future as that of the macroscopic gyroscope in the famous Gravity Probe B (GPB) project [10], in addition to technical difficulties caused by the extremely weak fermion-gravity couplings. The GPB gives a geodetic drift rate of $R_{\mathrm{NS,o}} = 6601.8 \pm 18.3$ mas/yr and a frame-dragging drift rate of $R_{\mathrm{WE,o}} = 37.2 \pm 7.2$ mas/yr, while the corresponding drift rates predicted by GR are of $R_{\mathrm{geo}} = 6606.1$ mas/yr and $R_{\mathrm{FD}} = 39.2$ mas/yr, respectively, so the measured drift rate deviations are $|\Delta R_{\mathrm{NS}}| < 22.6$ mas/yr and $|\Delta R_{\mathrm{WE}}| < 9.2$ mas/yr [12]. The LV-induced anomalous precession is

$$\delta\vec{\omega}_{\mathrm{LV}} = \frac{\vec{g}}{2m} \times \left[3\vec{a} - \frac{a^0\vec{p}}{m}\right] + \phi_g\left[\frac{3b^0\vec{p}}{m} + (6\vec{H} - 4\vec{b})\right] + \frac{2\vec{A}_g\cdot\vec{b}}{m}\vec{p} + \phi_g\frac{\vec{p}\times\vec{H}}{m} + \frac{2}{m}[(\vec{\tilde{H}}\cdot\vec{p})\vec{A}_g - (\vec{A}_g\cdot\vec{p})\vec{\tilde{H}}]c_{00}\left[\frac{\vec{g}\times\vec{p}}{m} - \frac{\vec{\Omega}}{2}\right] + d_{00}[m\vec{A}_g - 4\phi_g\vec{p}]. \tag{67}$$

If we attribute all the drift rate deviations to the LV-induced anomalous precession and assume that the same precision can be achieved for fermion spin precession measurements, we may obtain some very rough bounds on

$$|3\vec{\tilde{H}} - 2\vec{b}| \leq \frac{3\Delta R_{\mathrm{NS}}}{4R_{\mathrm{geo}}}\frac{v}{r} \simeq 5.432 \times 10^{-22}\ \mathrm{GeV}, \tag{68}$$

$$|a^0| \leq 3m\frac{\Delta R_{\mathrm{NS}}}{R_{\mathrm{geo}}} \simeq 9.65 \times 10^{-3}\ \mathrm{GeV}, \tag{69}$$

$$|c_{00}| \leq \mathrm{Min}\left\{\frac{3}{2}\frac{\Delta R_{\mathrm{NS}}}{R_{\mathrm{geo}}}, \frac{\Delta R_{\mathrm{WE}}}{R_{\mathrm{FD}}}\right\} = 5.14 \times 10^{-3}, \tag{70}$$

where we set $r = 7018.0$ km as the GPB polar orbit parameter (orbit altitude 642 km) and assume each type of LV coefficient as the only nonzero one in our estimations. The bounds are weak as they are obtained from the deviation of the essentially weak GR effects. Also note we intentionally choose the above LV coefficients in our naive estimates, because the other LV operators such as $\vec{g}\times\vec{a}$, $b^0\vec{p}$, $\vec{p}\times\vec{H}$ may be even weaker as they may average out in a cycle, not to mention that the data acquisition period is almost 1 year, from August 2004 to August 2005. In other words, if we had transformed to the Sun-centered frame, our estimates could be even weaker. The LAGEOS, LAGEOS 2, and LARES laser-ranged satellites can test the LT nodal shift to the accuracy 0.2% [69], and this in principle may put at least 2 orders of magnitude tighter bounds to the LV coefficients, though it is more unlikely as the test is not even for a gyroscope in an orbit. Another point is that our estimates are based on the assumption that fermion precession can be tested to the same accuracy as for the macroscopic gyroscope. This means our bounds above are best to be viewed as expectations.

If we consider the acceleration

$$\vec{a} \equiv \frac{d\vec{p}}{mdt} = \frac{1}{im}[\vec{p}, \hat{H}_{\mathrm{NR}}] \simeq -\nabla\phi_g\left(1 + \frac{3\vec{p}^2}{2m^2}\right) - \nabla\phi_g\left[\frac{(3\vec{\tilde{H}} - 2\vec{b})}{m} - \frac{2d_{00}}{m}\vec{p}\right]\cdot\vec{\sigma}, \tag{71}$$

where we have ignored all the LV corrections with higher order than m^{-1} and the LV corrections coupled with gravito-magnetic vector potential $\vec{A}_g$ or derivatives of $\vec{g}$, since we expect these terms to be much tinier compared with the remaining ones, and we note that the anomalous acceleration is purely due to the LV spin-gravity couplings, we can then get bounds

$$|3\vec{\tilde{H}} - 2\vec{b}| \leq 1.8 \times 10^{-7} m_{87} \simeq 1.46 \times 10^{-5}\ \mathrm{GeV}, \tag{72}$$

$$|d_{00}| \leq 9 \times 10^{-8}\sqrt{\frac{m_{87}}{3k_B T}} \simeq 4.51 \times 10^{-6}, \tag{73}$$

from the test of WEP with neutral atoms with the precision of $\eta = (0.2 \pm 1.6) \times 10^{-7}$ [68,70]. We choose the temperature as $T = 1.4$ μK [68], mass m_{87} as the 87 atomic mass unit, as the particle involved are ^{87}Sr, ^{88}Sr, and ^{87}Rb, roughly of the same mass range, and $|\eta| \simeq 1.8 \times 10^{-7}$, the most conservative one. Since the time scales for two experiments are much smaller than a day, there is no need to take into account of the sidereal variations for a rough estimate, and these weak bounds are more reliable.

VII. SUMMARY

In this paper, we calculate the one-fermion matrix elements of fermion-electromagnetic (FE) and fermion-gravity (FG) interactions for on-shell fermions. Due to the partial structure similarities between FE and FG interactions, many LV fermion-gravity operators bear resemblance to LV fermion-photon operators. We have shown the resemblance with several sampling operators in Table I. This resemblance can be viewed as a natural manifestation of the well-known gravito-electromagnetism generalized to the LV fermion couplings.

By collecting the spin-dependent LV operators in the matrix elements as leading order LV perturbations and combined with the nonrelativistic LI gravitational interaction in the Lense-Thirring (LT) metric, we obtain a hybrid Hamiltonian, from which we obtain a spin precession equation (66) and a linear acceleration equation (71). By identifying the anomalous spin precession rate as the correction to the geodetic precession and LT frame-dragging precession predicted in GR, we can get some weak bounds on gravitationally coupled LV fermion coefficients, Eqs. (68)–(70). Though these constraints rely on an unrealistic assumption of the measurement capability, which says the fermion-gravity coupling can be measured to the same precision as in the Gravity Probe B project, these bounds are interacting since they reveal another aspect of the WEP test [49,50], namely, the spin precession of a microscopic fermion may be different from that of a macroscopic gyroscope if the LV spin-gravity couplings are allowed. From the WEP test with atoms of nonzero spin, we can also get some relatively stronger and more reliable bounds (72) and (73) on the LV fermion-gravity couplings. These bounds do not require one to take account the sidereal effect induced by the motion of the Earth, as the relevant time scale is much shorter than a sidereal day; however, the analysis of sidereal effects may necessarily render the bound more stringent. Moreover, future high-accuracy experiments with polarized neutral atoms may be able to give tighter bounds on these LV spin-gravity couplings [71,72].

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APPENDIX A: THE GRAVITO-ELECTROMAGNETIC EQUATIONS

The gravito-electromagnetism can be viewed as an analogy to electrodynamics when gravity is sufficiently weak for slowly moving gravitational sources. For weak gravity, we can linearize the Einstein equation $G_{\mu\nu} = \kappa T_{\mu\nu}$ by regarding the metric as a small deviation from Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{A1}$$

The field equation can be further simplified in the harmonic gauge $\partial_\mu \bar{h}^\mu{}_\nu = 0$ with trace reversed rank-2 tensor $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}}{2} h$ (where $h = \eta^{\mu\nu} h_{\mu\nu}$),

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}, \qquad \kappa \equiv \frac{8\pi G}{c^4}. \tag{A2}$$

Then a class of retarded solutions can be found as (A2) is simply a wave equation. For $T^{00} \sim \rho_m c^2$, $T^{0i} \sim \rho_m c u^i$, $T^{ij} \sim \rho_m u^i u^j$, where u^i is the spatial component of the four-velocity, we get up to $\mathcal{O}(c^{-4})$, $\bar{h}_{00} \equiv -\frac{4\phi_g}{c^2}$, $\bar{h}_{0i} = \frac{4A_g^i}{sc^{3-n}}$ and $\bar{h}_{ij} \sim \mathcal{O}(c^{-4})$, where

$$\phi_g(x) = -G \int d^3y \frac{\rho_m\left(t - \frac{|\vec{x}-\vec{y}|}{c}; \vec{y}\right)}{|\vec{x} - \vec{y}|}, \tag{A3}$$

$$A_g^i(x) = -\frac{sG}{c^n} \int d^3y \frac{\rho_m\left(t - \frac{|\vec{x}-\vec{y}|}{c}; \vec{y}\right) u^i}{|\vec{x} - \vec{y}|}. \tag{A4}$$

Then, we define $\vec{E}_g \equiv -\nabla\phi_g - \frac{r}{c}\partial_t \vec{A}_g$ and $\vec{B}_g \equiv \nabla \times \vec{A}_g$. The n, s, r are a set of constants to be determined. It is easy to verify that the homogeneous equations

$$\nabla \cdot \vec{B}_g = 0, \qquad \nabla \times \vec{E}_g = -\frac{r}{c}\partial_t \vec{B}_g \tag{A5}$$

are satisfied automatically. Since the harmonic gauge $\partial_\mu \bar{h}^\mu{}_\nu = 0$ reads

$$0 = \partial_j \bar{h}_{j0} - \partial_0 \bar{h}_{00} = \frac{4}{c^2}\left[\frac{\partial_j A_g^j}{sc^{1-n}} + \frac{1}{c}\partial_t \phi_g\right]$$
$$0 = \partial_0 \bar{h}^0{}_i + \partial_j \bar{h}^j{}_i = -\partial_0 \bar{h}_{0i} = -\frac{4\partial_t A_g^i}{sc^{4-n}}, \tag{A6}$$

the vector potential must be time independent, $\dot{A}_g^i = 0$, and substituting $\nabla \cdot \vec{A}_g = -\frac{s}{c^n}\partial_t \phi_g$ into the inhomogeneous equations gives

$$\nabla \cdot \vec{E}_g = -\nabla^2 \phi_g - \frac{r}{c}\partial_t(\nabla \cdot \vec{A}_g) = -\left[\nabla^2 - \frac{rs}{c^{n+1}}\partial_t^2\right]\phi_g$$
$$\overset{rs=1}{\underset{n=1}{\equiv}} -\Box\phi_g = -4\pi G \rho_m, \tag{A7}$$

$$\nabla \times \vec{B}_g = \nabla(\nabla \cdot \vec{A}_g) - \nabla^2 \vec{A}_g = -\frac{s}{c^n}\partial_t(\nabla\phi_g) - \nabla^2 \vec{A}_g$$
$$\overset{rs=1}{\underset{n=1}{\equiv}} \frac{s\partial_t \vec{E}_g}{c} - \Box \vec{A}_g = s\left[\frac{\partial_t \vec{E}_g}{c} - \frac{4\pi G}{c}\rho_m \vec{u}\right], \tag{A8}$$

where $\Box \equiv [\nabla^2 - \frac{1}{c^2}\partial_t^2]$ is the flat space d' Alembert operator, and in order to make use of $\Box_x \int d^3y \frac{f(t - \frac{|\vec{x}-\vec{y}|}{c}; \vec{y})}{|\vec{x}-\vec{y}|} = -4\pi f(t, \vec{x})$, we have to set $rs = n = 1$.

The geodesic equation $\frac{du^\alpha}{d\tau}+\Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma=0$ can be written as

$$\left.\begin{array}{l}\frac{du^\alpha}{dt}=-\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\frac{dt}{d\tau}\\ \frac{du^\alpha}{dt}=\frac{d}{dt}\left[\frac{dx^\alpha}{dt}\frac{dt}{d\tau}\right]=\frac{d^2x^\alpha}{dt^2}\frac{dt}{d\tau}+\frac{dx^\alpha}{dt}\frac{d^2t}{d\tau^2}\frac{d\tau}{dt}\end{array}\right\}\Rightarrow$$

$$\frac{d^2x^\alpha}{dt^2}=\left[\frac{1}{c}\frac{dx^\alpha}{dt}\Gamma^0{}_{\beta\gamma}-\Gamma^\alpha{}_{\beta\gamma}\right]\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}. \tag{A9}$$

Note in the weak gravitational field limit,

$$\begin{aligned}
&\Gamma^0_{00}\simeq-\frac{1}{2}h_{00,0}=\frac{\partial_t\phi_g}{c^3},\qquad \Gamma^0_{0j}\simeq-\frac{1}{2}h_{00,j}=\frac{\partial_j\phi_g}{c^2},\\
&\Gamma^0_{jk}\simeq\frac{1}{2}(h_{jk,0}-h_{0j,k}-h_{0k,j})=-\delta_{jk}\frac{\partial_t\phi_g}{c^3}-\frac{2}{sc^2}(A^g{}_{j,k}+A^g{}_{k,j}),\\
&\Gamma^i_{00}\simeq\left[h_{i0,0}-\frac{1}{2}h_{00,i}\right]=\frac{4}{sc^3}\partial_tA_g{}^i+\frac{\partial_i\phi_g}{c^2},\\
&\Gamma^i_{0j}\simeq\frac{1}{2}[h_{i0,j}+h_{ij,0}-h_{0j,i}]=\frac{2}{sc^2}[A^g{}_{i,j}-A^g{}_{j,i}]-\delta_{ij}\frac{\partial_t\phi_g}{c^3},\\
&\Gamma^i_{jk}\simeq\frac{1}{2}(h_{ij,k}+h_{ik,j}-h_{jk,i})=\frac{1}{c^2}[\delta_{jk}\partial_i-\delta_{ik}\partial_j-\delta_{ij}\partial_k]\phi_g,
\end{aligned} \tag{A10}$$

where $h_{ij}=-\frac{2\phi_g}{c^2}\delta_{ij}$, $h_{0j}=\frac{4A_g^j}{sc^2}$, $h_{00}=-\frac{2\phi_g}{c^2}$. Substituting the above equations into the geodesic equation (A9), we get

$$\begin{aligned}
a^i\equiv\frac{d^2x^i}{dt^2}&=\left[\frac{v^i}{c}\Gamma^0{}_{00}-\Gamma^i{}_{00}\right]c^2+2\left[\frac{v^i}{c}\Gamma^0{}_{0j}-\Gamma^i{}_{0j}\right]cv^j\\
&+\left[\frac{v^i}{c}\Gamma^0{}_{jk}-\Gamma^i{}_{jk}\right]v^jv^k=\left[\frac{3v^i}{c^2}\partial_t+\frac{4v^i}{c^2}(\vec{v}\cdot\nabla)\right]\phi_g\\
&-\left[\left(\frac{4}{sc}\partial_tA_g{}^i+\partial_i\phi_g\right)+\frac{4}{sc}v^j(A^g{}_{i,j}-A^g{}_{j,i})\right]\\
&-\frac{\vec{v}^2}{c^2}\partial_i\phi_g+\mathcal{O}(c^{-3}).
\end{aligned} \tag{A11}$$

In comparison, if we want to have an analogy to the Lorentz force law, we have to set $\frac{4}{s}=r$, which is in contradiction with the condition $rs=1$ in the Eq. (A7). To compromise, we have to resort to stationary assumption, where $\phi_g,\vec{A}_g$ are time independent, and then $\Box\to\nabla^2$. A convention is $r=1$, $s=4$, and then the

$$a^i\equiv\left(1+\frac{\vec{v}^2}{c^2}\right)E_g^i+(\vec{v}\times\vec{B}_g)^i. \tag{A12}$$

APPENDIX B: THE LINEAR LV LAGRANGIAN DENSITY

The original Dirac equation obtained from (16) is

$$\begin{aligned}
&\left[ie^\mu{}_a\left(\Gamma^a\vec{\nabla}_\mu+\frac{i}{8}\omega_\mu^{bc}[\sigma_{bc},\Gamma^a]\right)-M\right]\psi\\
&\quad+\frac{i}{2}e^\mu{}_a\{\partial_\mu\Gamma^a+\omega_\mu{}^a{}_c\Gamma^c\}\psi=0.
\end{aligned} \tag{B1}$$

Now consider the linearized LV fermion-gravity Lagrangian in metric perturbation $h_{\mu\nu}$. The LV fermion-gravity Lagrangian is

$$\mathcal{L}_{\mathrm{LV}}=\frac{i}{2}e^\mu{}_a\bar\psi\delta\Gamma^a\overleftrightarrow{\nabla}_\mu\psi-\bar\psi\delta M\psi=\mathcal{L}_{c,d}+\mathcal{L}_{a,b,H}, \tag{B2}$$

where $\delta\Gamma^a\equiv\Gamma^a-\gamma^a$ and $\delta M\equiv M-m$. For the c, d coefficients, the corresponding Lagrangian is

$$\begin{aligned}
\mathcal{L}_{c,d}&=-\frac{i}{2}e^\mu{}_a\bar\psi(c_{\rho\nu}+d_{\rho\nu}\gamma_5)\gamma^be^{\nu a}e^\rho{}_b\overleftrightarrow{\nabla}_\mu\psi\\
&=-\frac{i}{2}e^\mu{}_a\bar\psi(c_{\rho\nu}+d_{\rho\nu}\gamma_5)\gamma^be^{\nu a}e^\rho{}_b\overleftrightarrow{D}_\mu\psi+\frac{1}{8}e^\mu{}_a\omega_\mu{}^{cd}\\
&\quad\cdot\bar\psi\{(c_{\rho\nu}+d_{\rho\nu}\gamma_5)\gamma^b,\sigma_{cd}\}e^{\nu a}e^\rho{}_b\psi\\
&\simeq-\frac{i}{2}\bar\psi[c_b{}^a+d_b{}^a\gamma_5]\gamma^b\left[\overleftrightarrow{D}_a-\frac{1}{2}h^\mu{}_a\overleftrightarrow{D}_\mu\right]\psi\\
&\quad+\frac{i}{4}\bar\psi[h^{\nu a}(c_{b\nu}+d_{b\nu}\gamma_5)\gamma^b+h^\rho{}_b(c_\rho{}^a+d_\rho{}^a\gamma_5)\gamma^b]\overleftrightarrow{D}_a\psi\\
&\quad+\frac{1}{4}\epsilon^{bcmn}h_{am,n}\bar\psi[c_b{}^a\gamma_5+d_b{}^a]\gamma_c\psi,
\end{aligned} \tag{B3}$$

while for the a, b, H coefficients, the contributions to the Lagrangian are

$$\begin{aligned}
\mathcal{L}_{a,b,H}&=-\bar\psi\delta M\psi\simeq-\bar\psi\left[(a_a+b_a\gamma_5)\gamma^a+\frac{1}{2}H_{bc}\sigma^{bc}\right]\psi\\
&\quad+\frac{h^\mu{}_a}{2}\bar\psi\left[(a_\mu+b_\mu\gamma_5)\gamma^a+\frac{1}{2}(H_{\mu b}\sigma^{ab}+H_{b\mu}\sigma^{ba})\right]\psi.
\end{aligned} \tag{B4}$$

APPENDIX C: FIELD REDEFINITION PROCEDURE

The field redefinition matrix for the linearized Lagrangian $\mathcal{L}_\psi = (1+\frac{1}{2}h)[\mathcal{L}_{\rm LI}+\mathcal{L}_{\rm LV}]$ is $\hat{U} \equiv 1-\frac{1}{2}\gamma^0 C^0 \equiv 1+\delta\hat{U}_0+\delta\hat{U}^h$, where $\delta\hat{U}_0 \equiv \frac{1}{2}(d_{b0}\gamma_5 - c_{b0})\gamma^0\gamma^b$ is the redefinition matrix in flat space and $\delta\hat{U}^h = \delta\hat{U}^h_I + \delta\hat{U}^h_V$ is the additional contribution due to gravity. The LI and LV pieces of $\delta\hat{U}^h$ are

$$\delta\hat{U}^h_I = -\frac{1}{4}(h + h_{0\mu}\gamma^0\gamma^\mu), \tag{C1}$$

$$\delta\hat{U}^h_V = -\frac{1}{4}[h^{\nu 0}(c_{b\nu} - d_{b\nu}\gamma_5) + h^\rho{}_b(d_{\rho 0}\gamma_5 - c_{\rho 0})]\gamma^0\gamma^b - \frac{1}{4}(h\gamma^0\delta\Gamma^0_\circ - h^0{}_\mu\gamma^0\delta\Gamma^\mu_\circ), \tag{C2}$$

respectively. The spinor redefinition is $\psi = \hat{U}\chi$, and the associated fermion bilinear $\bar{\psi}\hat{\mathcal{O}}\psi$ after spinor redefinition is $\bar{\chi}\gamma^0\hat{U}^\dagger\gamma^0\hat{\mathcal{O}}\hat{U}\chi$. However, up to linear order approximation in $h_{\mu\nu}$, there is an effective distinction between:

(i) Any operator constructed from the Lagrangian $\mathcal{L}_\psi$ linear in $h_{\mu\nu}$. There is no need to take into account $\delta\hat{U}^h$, as otherwise, the resultant operator is of order $\mathcal{O}(h^2)$. In other words, we only need to take $\hat{U}_0 \equiv 1+\delta\hat{U}_0$ as the redefinition matrix.

(ii) Any "flat space" operator such as $\frac{i}{2}\bar{\psi}\Gamma^a_\circ \overleftrightarrow{D}_a\psi$ or $\bar{\psi}M_\circ\psi$. The redefinition matrix can be taken either as $1+\delta\hat{U}^h_I$ or $1+\delta\hat{U}^h_I+\delta\hat{U}^h_V$, depending on whether the original operator contains LV coefficients or not.

The good news is that we can prove that up to linear order of $h_{\mu\nu}$ and LV coefficients, there is no need to consider the redefinition induced "h interaction" arising from the operator $\mathcal{L}_{\rm flat} = \frac{i}{2}\bar{\psi}\Gamma^a_\circ\overleftrightarrow{D}_a\psi - \bar{\psi}M_\circ\psi$ between a pair of one-fermion states $\langle p',\beta|\int d^3x\mathcal{L}_{\rm flat}|p,\alpha\rangle$, once the Dirac equation is utilized; i.e., the external fermions are on mass shell.

APPENDIX D: VARIOUS EIGENSPINORS

The eigenspinor in the presence of LV coefficients will be given separately by assuming only one-type LV coefficient is nonzero. For a more general treatment including nonminimal LV coefficients, the interested reader can resort to Ref. [37,73].

Firstly, the eigenspinor for a and b coefficients can be found in [7], and for completeness, we compile it here. As a term acts like a shift in four-momentum, we will state the corresponding eigenspinor here together with the b term:

$$u_\alpha(k) = \begin{pmatrix} \xi^\alpha \\ U_{ab}(k)\xi^\alpha \end{pmatrix}, \tag{D1}$$

where $U_{ab}(k) \equiv \frac{[(k^0+a^0)+m-\vec{b}\cdot\vec{\sigma}][(\vec{k}+\vec{a})\cdot\vec{\sigma}+b^0]}{[(k^0+a^0)+m]^2-\vec{b}^2}$, and the two-component spinors ξ^α satisfy the eigenvalue equation given by (A4)–(A5) in [7]. It is clear from the above consideration that a^μ serves as a pure shift in four-momentum and thus is usually ignored due to field redefinition. However, we keep a term here as we see gravity concerns the a term. For calculational convenience, we also note

$$U_b(k) = (k^0+m+\vec{b}\cdot\vec{\sigma})^{-1}(\vec{k}\cdot\vec{\sigma}+b^0) \simeq \frac{b^0+\vec{k}\cdot\vec{\sigma}}{\omega_0+m} - \frac{i\vec{b}\times\vec{k}\cdot\vec{\sigma}+\vec{b}\cdot\vec{k}+\delta\omega\vec{k}\cdot\vec{\sigma}}{(\omega_0+m)^2}, \tag{D2}$$

where $\omega_0 = \sqrt{\vec{k}^2+m^2}$ and $\delta\omega \equiv k^0-\omega_0$. As for b, d, g, H type LV coefficients, the fourfold degeneracy between all four eigenspinors is completely broken and the explicit form of k^0 (and hence, $\delta\omega$) is very complicated even at linear order of LV coefficients. Depending on the nature of LV coefficients, a simple form of $\delta\omega$ may be obtained. For example, if $b^2>0$, in an observer frame where $b^0=0$, $\delta\omega = (-1)^\alpha[m^2\vec{b}^2+(\vec{b}\cdot\vec{k})^2]^{\frac{1}{2}}/\omega_0$, where $\alpha=1,2$ denotes the two spin d.o.f. of ξ^α.

Then we turn to H coefficients. The corresponding operator in the Lagrangian is $-\frac{1}{2}H_{ab}\sigma^{ab}$, which implies that $H_{ab}=-H_{ba}$. The antisymmetric property indicates that we can define two vectors, $\vec{H}^i \equiv H_{0i}$ and $\vec{\tilde{H}}^i \equiv \frac{1}{2}\epsilon_{ijk}H_{jk}$. Then the eigenspinor $u^\alpha(k)$ can still be written in the form of (D1), only by replacing $U_{ab}(k)$ with

$$U_H(k) = \frac{(k^0+m-\vec{\sigma}\cdot\vec{\tilde{H}})\vec{\sigma}\cdot(\vec{k}-i\vec{H})}{(k^0+m)^2-\vec{\tilde{H}}^2} \simeq \frac{\vec{\sigma}\cdot(\vec{k}-i\vec{H})}{\omega_0+m} - \frac{\vec{k}\cdot\vec{\tilde{H}}+i\vec{\tilde{H}}\times\vec{k}\cdot\vec{\sigma}+\delta\omega\vec{\sigma}\cdot\vec{k}}{(\omega_0+m)^2}, \tag{D3}$$

where we also keep only linear order corrections due to the LV H coefficients. Note ω_0 and $\delta\omega$ are defined as above, but now $\delta\omega$ only receives LV corrections from H coefficient.

Naively, we can also obtain

$$U_d(k) = [k^0+m-\vec{\mathfrak{d}}\cdot\vec{\sigma}]^{-1}[\mathfrak{d}_0+\vec{k}\cdot\vec{\sigma}],$$

where we defined $\mathfrak{d}_\mu \equiv d_{\mu\nu}k^\nu$ for simplicity. However, as mentioned in the main text that a proper treatment of c, d terms involves field redefinition, which gives the correct $U_c(k), U_d(k)$ by the procedure in getting $U_b(k)$. The $U_c(k), U_d(k)$ up to linear order of c, d coefficients are shown below,

$$\begin{aligned} U_d(k) &= [k^0 + m(1 + d_{j0}\sigma^j) - \tilde{d}_{ij}\sigma^i p^j]^{-1}(\vec{\sigma} + 2\vec{d}) \cdot \vec{p} \\ &\simeq \frac{\vec{k}\cdot(\vec{\sigma}+2\vec{d})}{\omega_0+m} - \frac{(md_{j0} - \tilde{d}_{ji}k^i)k^j}{(\omega_0+m)^2} \\ &\quad - \frac{i\epsilon_{jkl}(md_{j0} - \tilde{d}_{ji}k^i)k^k\sigma^l + \delta\omega\vec{\sigma}\cdot\vec{k}}{(\omega_0+m)^2}, \end{aligned} \tag{D4}$$

$$\begin{aligned} U_c(k) &= \frac{\vec{\sigma}\cdot\vec{p} - \tilde{c}_{ij}\sigma^i p^j}{k^0 + m(1 - c_{00}) + 2\vec{c}\cdot\vec{p}} \\ &\simeq \frac{\vec{k}\cdot\vec{\sigma} - \tilde{c}_{ij}\sigma^i k^j}{\omega_0+m} + \frac{mc_{00} - 2\vec{c}\cdot\vec{k} - \delta\omega}{(\omega_0+m)^2}\vec{k}\cdot\vec{\sigma}, \end{aligned} \tag{D5}$$

where it is easy to separate the formally LV contributions from the LI one, $\frac{\vec{k}\cdot\vec{\sigma}}{\omega_0+m}$. To obtain the correction $\delta\omega$ for b, H, c, d coefficients, we'd better find out their explicit dispersion relations, which can be found in [56,74]. They all share the similar form

$$[(k^0)^2 - \omega_0^2 + Y^2]^2 = 4Z^2, \tag{D6}$$

where

$$Y^2 = \begin{cases} b^2 \\ (\vec{\tilde{H}}^2 - \vec{H}^2) \\ d_{ab}k^b d^a{}_c k^c \end{cases}, \tag{D7}$$

$$Z^2 = \begin{cases} (k\cdot b)^2 - k^2 b^2 \\ H^{*\mu\nu}k_\mu H^*{}_{\zeta\nu}k^\zeta - \left(\frac{1}{4}H^{*\mu\nu}H_{\mu\nu}\right)^2 \\ (k^a d_{ab}k^b)^2 - k^2 d_{ab}k^b d^a{}_c k^c \end{cases}, \tag{D8}$$

where $H^{*\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}H_{\alpha\beta}$, and the three rows of Y^2 and Z^2 correspond to b, H, d terms, respectively, while for c term, the dispersion relation is simply $(c_{\mu\nu} + \eta_{\mu\nu})k^\nu(c^{\mu\rho} + \eta^{\mu\rho})k_\rho + m^2 = 0$, which is spin independent and thus leads to much greater calculational simplicity for $\delta\omega$. From the exact dispersion relation (D6), we can readily obtain

$$\delta\omega = \omega_0\left[\sqrt{1 \pm \frac{2Z}{\omega_0^2} - \frac{Y^2}{2\omega_0^2}} - 1\right] \simeq \pm\frac{Z}{\omega_0} \simeq \pm\frac{Z}{m}, \tag{D9}$$

where we have ignored Y^2, as $Y^2, Z^2 \sim \mathcal{O}(X^2)$ are at least of second order in a generic LV coefficient X. The double signs associated to $\delta\omega$ for b, d, H coefficients reflects the fact that the corresponding terms are spin dependent; i.e., these LV coefficients break the spin degeneracy. Thus, the degeneracy for dispersion relations between relevant eigenspinors is completely removed. However, since in calculating matrix elements, $\delta\omega$ in effect acts on the two-component spinor ξ^α, see Eqs. (D1)–(D3), the sign choice ambiguity can be removed by some kinds of "eigenequations". These "eigenequations" can be obtained by taking the "square root" of the exact quartic dispersion relation $\det[\Gamma\cdot k + M] = 0$.

Take the b term as an example. Left multiplying the k-space positive Dirac equation $(\gamma\cdot k + m + b\cdot\gamma_5\gamma)u(k) = 0$ by $(m - \gamma\cdot k - b\cdot\gamma_5\gamma)$, we get

$$\begin{aligned} &-(\gamma\cdot k - m + b\cdot\gamma_5\gamma)(\gamma\cdot k - m + b\cdot\gamma_5\gamma)u(k) \\ &= (k^2 + m^2 - b^2 + 2ib_\mu k_\nu\gamma_5\sigma^{\mu\nu})u(k) \\ &= \begin{pmatrix} K^2 + 2k_{b\sigma} & 2i\vec{b}\times\vec{k}\cdot\vec{\sigma} \\ 2i\vec{b}\times\vec{k}\cdot\vec{\sigma} & K^2 + 2k_{b\sigma} \end{pmatrix}\begin{pmatrix} \xi^\alpha \\ U_d(k)\xi^\alpha \end{pmatrix} = 0, \end{aligned} \tag{D10}$$

where $K^2 \equiv k^2 + m^2 - b^2$ and $k_{b\sigma} \equiv (b^0\vec{k} - k^0\vec{b})\cdot\vec{\sigma}$. Keeping only b terms to linear order, the upper equation $[k^2 + m^2 + 2(b^0\vec{k} - k^0\vec{b})\cdot\vec{\sigma} + 2i\vec{b}\times\vec{k}\cdot\vec{\sigma}U_0(k)]\xi^\alpha = 0$ for bispinor ξ_α can be rearranged as

$$(k^0 + \omega_0)\delta\omega\xi^\alpha = [4b^{[0}k^{j]}\sigma^j + 2i\vec{b}\times\vec{k}\cdot\vec{\sigma}U_0(k)]\xi^\alpha, \tag{D11}$$

which leads to the eigenequation for LV correction $\delta\omega$

$$\delta\omega\xi^\alpha = \left[-\vec{b}\cdot\vec{\sigma} + \frac{b^0\vec{k}\cdot\vec{\sigma}}{\omega_0} + \frac{\vec{k}^2\vec{b}\cdot\vec{\sigma} - (\vec{k}\cdot\vec{b})(\vec{k}\cdot\vec{\sigma})}{\omega_0(\omega_0+m)}\right]\xi^\alpha. \tag{D12}$$

Similarly for H, d, c terms, we have

$$\delta\omega\xi^\alpha = \left[\vec{\tilde{H}}\cdot\vec{\sigma} + \frac{\vec{H}\times\vec{k}\cdot\vec{\sigma}}{\omega_0} - \frac{\vec{\tilde{H}}\cdot\vec{k}\,\vec{\sigma}\cdot\vec{k}}{\omega_0(\omega_0+m)}\right]\xi^\alpha, \tag{D13}$$

$$\begin{aligned} \delta\omega\xi^\alpha = \Bigg[&(md_{j0} + \tilde{d}_{ji}k^i)\sigma^j + 2\frac{\vec{d}\cdot\vec{k}\,\vec{\sigma}\cdot\vec{k}}{\omega_0} \\ &- \frac{(md_{j0} + i\epsilon_{ikl}d_{ij}k^k\sigma^l)k^j\vec{\sigma}\cdot\vec{k}}{\omega_0(\omega_0+m)}\Bigg]\xi^\alpha, \end{aligned} \tag{D14}$$

$$\delta\omega = -2c_{(0j)}k^j - c_{(ij)}\frac{k^i k^j}{\omega_0} - c_{00}\omega_0, \tag{D15}$$

where due to the spin independence, there is no need to act on bispinors for the c coefficient, compared with other LV coefficients, and we choose the positive sign corresponding to the electron's dispersion relation instead of the positron's. Substituting these $\delta\omega$ back into (D2), (D3), (D4), (D5), we can get

$$\begin{aligned} U_b(k) &= \frac{\vec{k}\cdot\vec{\sigma} + b^0}{\omega_0+m} - \frac{2i\vec{b}\times\vec{k}\cdot\vec{\sigma}}{(\omega_0+m)^2} + \mathcal{O}(\omega_0^{-3}) \\ &\overset{\mathrm{NR}}{\simeq} \frac{\vec{k}\cdot\vec{\sigma} + b^0}{2m} + \frac{i\vec{k}\times\vec{b}\cdot\vec{\sigma}}{2m^2}, \end{aligned} \tag{D16}$$

$$U_H(k) = \frac{\vec{\sigma}\cdot(\vec{k}-i\vec{H})}{\omega_0+m} - \frac{2\vec{k}\cdot\tilde{\vec{H}}}{(\omega_0+m)^2} + \mathcal{O}(\omega_0^{-3}) \overset{\text{NR}}{\simeq} \frac{\vec{\sigma}\cdot(\vec{k}-i\vec{H})}{2m} - \frac{\vec{k}\cdot\tilde{\vec{H}}}{2m^2}, \tag{D17}$$

$$U_d(k) = \frac{\vec{k}\cdot(\vec{\sigma}+2\vec{d})}{\omega_0+m} - 2\frac{(md_{j0}-i\epsilon_{ikl}\tilde{d}_{ij}k^k\sigma^l)k^j}{(\omega_0+m)^2} + \mathcal{O}(m^{-3}) \overset{\text{NR}}{\simeq} \frac{\vec{k}\cdot\vec{\sigma}+d_{0j}k^j}{2m} + \frac{i\epsilon_{jkl}d_{ji}k^k k^i\sigma^l}{2m^2}, \tag{D18}$$

$$U_c(k) = \frac{\vec{\sigma}\cdot\vec{k}-c_{ij}\sigma^i k^j}{\omega_0+m} + \mathcal{O}(\omega_0^{-3}) \overset{\text{NR}}{\simeq} \frac{\vec{\sigma}\cdot\vec{k}-c_{ij}\sigma^i k^j}{2m}, \tag{D19}$$

where we have ignored the terms suppressed by orders higher than ω_0^2 (or m^{-2}). The "NR" at the last steps means we adopt the nonrelativistic approximation. Substituting these $U(k)$ in (D16)–(D19) into

$$u_\alpha(k)_X = \begin{pmatrix} \xi^\alpha \\ \eta^\alpha \end{pmatrix}_X = \begin{pmatrix} \xi^\alpha \\ U_X(k)\xi^\alpha \end{pmatrix}, \tag{D20}$$

where X in the subscript refers to b, H, d, c, we obtain the LV corrected positive frequency eigenspinors up to linear order of LV coefficients.

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