Quark and lepton flavor structure in magnetized orbifold models at residual modular symmetric points

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We study quark and lepton mass matrices derived from magnetized $T^2/\mathbb{Z}_2$ orbifold models. Quark and lepton masses have a large hierarchy. In addition, mixing angles are large in the lepton sector, while those are small in the quark sector. We find that this structure can be realized in certain flavor models, which are identified by the zero points of the zero-mode wave functions of fermions and Higgs modes. We classify such realistic flavor models. Fixed points $\tau = i$, $e^{2\pi i/3}$ and $i\infty$ of the modulus τ play a role in realizing a large mass hierarchy through our scenario, where residual S, ST, and T symmetries remain and the lightest Higgs modes can correspond to eigenstates of residual symmetries at the leading order. As a result, we find that there are 24 flavor models in total which can be realistic in a vicinity of S-symmetric vacuum but no flavor models for ST- and T-symmetric vacua.

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I. INTRODUCTION

The standard model (SM) established by the discovery of the Higgs particle is a successful theory describing almost all observations from current experiments. However it still contains several unsolved issues. The origin of the quark and lepton flavors is one of such issues. Quark and lepton masses are required to be hierarchical, and neutrino masses must be extremely light compared with other fermions by observational results. In addition, quark mixings are required to be small but lepton mixings are large. Also we have CP-violating phases. To describe these observables in the quark sector, the SM needs ten real parameters: six quark masses, three mixing angles, and one CP-violating phase. For the lepton sector, it needs twelve real parameters: six lepton masses, three mixing angles, and three Dirac and Majorana CP-violating phases. Understanding the origin of these parameters is one of the most fundamental and challenging issues in the current particle physics.

Superstring theory is a promising candidate for the unified theory. This theory predicts ten-dimensional (10D) space-time. The extra six-dimensional (6D) space must be compactified to be unobserved. The low-energy effective theory of superstring theory leads 10D super-Yang-Mills theory and its compactification can lead to solutions to the issues in particle physics. Hence, we can expect that the quark and lepton flavor structures are originated from this 6D space. For example, torus and orbifold compactifications with magnetic flux background give the four-dimensional chiral theory where wave functions have a generation structure determined by the size of the magnetic flux and Yukawa couplings depend on moduli [1–4]. In this sense, superstring theory on torus compactification and its orbifoldings with magnetic fluxes is attractive. Indeed, several numerical studies have shown that realistic flavor structures can be realized [5–10].

Moreover, torus compactification has another important aspect. The geometrical symmetry of torus is the modular symmetry $\Gamma \equiv SL(2,\mathbb{Z})$ as well as $\bar{\Gamma} \equiv SL(2,\mathbb{Z})/\mathbb{Z}_2$, which has recently drawn attention from the bottom-up approach. It is well known that the finite modular subgroups Γ_N for $N = 2$, 3, 4, and 5 are isomorphic to S_3, A_4, S_4, and A_5, respectively [11]. Inspired by these aspects, flavor models with Γ_N have been studied intensively in the bottom-up approach. (See e.g., Refs. [12–61].) In those models, matter fields are assumed to transform nontrivially under modular symmetry. In addition, Yukawa couplings are also modular forms and depend on the modulus τ. Especially, the modular fixed points, $\tau = i$, $e^{2\pi i/3}$ and $i\infty$, are important. Yukawa couplings as well as mass matrices have $S(\mathbb{Z}_2)$, $ST(\mathbb{Z}_3)$, and $T(\mathbb{Z}_N)$-symmetries at $\tau = i$, $e^{2\pi i/3}$, and $i\infty$, respectively, and these residual symmetries make the structure of Yukawa couplings specific patterns. In Refs. [33,36,43,55–57], realistic results were obtained at the vicinity of the modualr fixed points. (See also Ref. [62].)

Here, we study the quark and lepton mass matrices derived from magnetized $T^2/\mathbb{Z}_2$ orbifold models. Zero modes in magnetized $T^2/\mathbb{Z}_2$ orbifold models transform nontrivially under the modular symmetry [63–70].

In general, compactifications of superstring theory lead to more than one candidate for Higgs modes, which have the same quantum numbers under $SU(3) \times SU(2) \times U(1)$ SM gauge group and can couple with quarks and leptons. It is true for torus and orbifold compactification with magnetic flux. Note that again the generation number of wave functions is determined by the size of magnetic flux. Then, the number of Higgs pairs is also determined by the magnetic fluxes in the quark and lepton sectors and is larger than one, in general. They can couple to both quarks and leptons and give multipair Yukawa couplings among them. Thus mass matrices of quarks and leptons are given by the liner combination of these Yukawa couplings and vacuum expectation values (VEVs) of multipair Higgs fields. We expect that those Higgs modes may have mass terms, i.e., the so-called μ terms, and the lightest linear combination develops its VEV. Thus, the Higgs VEV direction is determined by the lightest direction. The mass terms of Higgs fields are forbidden at the perturbative level in superstring theory. They can be induced by nonperturbative effects such as D-brane instanton effects. However, such analyses are not straightforward in explicit models, and the lightest direction is not clear. In our analysis, as in Refs. [5–8], we use the direction of Higgs VEVs as parameters and find the model conditions to lead to hierarchical masses of quarks and charged lepton, small mixing of quarks and large mixing of leptons. In addition, we assume the vicinity at either of three modular fixed points, $\tau = i$, $e^{2\pi i/3}$, and $i\infty$, and Higgs VEVs are aligned in the eigenbasis of S, ST, and T transformations, respectively. As we will show, such vacuum can be led by the leading order Higgs μ term due to D-brane instanton effects.

Also we need to solve the smallness of neutrino masses. It can be realized by a seesaw mechanism. Majorana mass terms of a right-handed neutrino can be induced by D-brane instanton effects [71–75]. In particular, in Ref. [76], possible forms of right-handed neutrino mass terms induced by D-brane instanton effects on magnetized $T^2/\mathbb{Z}_2$ orbifold were studied. Assuming same D-brane instanton effects, we obtain light neutrino masses through the seesaw mechanism.

Our purpose of this paper is to realize realistic quark and lepton mass matrices in magnetized orbifold models. One of the key points is how to derive a large hierarchy of fermion masses. Such a hierarchy can be realized in mass matrices with almost rank one. In Ref. [9], the texture structure of quark mass matrices was studied and it was shown that (approximate) rank one quark mass matrices can be derived through the texture structure. Here, we show another method to find (approximate) rank one mass matrices by using properties of wave functions in compact space. Such a method will be applied to both the quark sector and charged lepton sector, although only the quark sector was studied in Ref. [9]. Another key point is how to drive the large mixing in the lepton sector and the small mixing in the quark sector. As will be shown, the above Higgs VEV directions leading to approximate rank one mass matrices have a problem when we apply it to the neutrino sector, too. Such a difficulty can be avoided in some models, which satisfy certain consistency conditions. We will study such conditions and show explicit models.

The paper is organized as follows. In Sec. II, we review the zero-mode wave functions and flavor models on the torus and orbifold with magnetic fluxes. In Sec. III, we study the conditions to realize quark and lepton flavor structure and classify the flavor models consistent with them. In Sec. IV, we also classify the flavor models consistent with the modular symmetric vacuum. In Sec. V, we give the numerical studies for the quark and lepton mass matrices in our models. Section VI concludes this study. In Appendixes A and B, we review the Majorana mass terms of right-handed neutrinos and Higgs μ terms induced by D-brane instanton effects, respectively. In Appendix C, we summarize the Yukawa couplings and Majorana mass terms on the model studied numerically in Sec. V.

II. MAGNETIZED ORBIFOLD MODEL

A. Torus and orbifold compactifications

First of all, we briefly review the zero-mode wave functions on the magnetized T^2. As in Ref. [9], to make our analysis simple, we assume $U(1)$ background magnetic flux on T^2:

$$F = dA = \frac{\pi i M}{\mathrm{Im}\tau} dz \wedge d\bar{z}, \qquad A = \frac{\pi M}{\mathrm{Im}\tau} \mathrm{Im}(\bar{z} dz), \quad (1)$$

where z denotes the complex coordinate on T^2, τ denotes the complex structure modulus, M is a value of flux, and A is the vector potential one-form. Then the torus identification $z \sim z + m + n\tau$, $m, n \in \mathbb{Z}$, makes the value of flux M be the integer, which is called the Dirac quantization condition.

The two-dimensional spinor with $U(1)$ unit charge $q = 1$, $\psi = (\psi_+, \psi_-)^T$, must satisfy the following boundary conditions:

$$\begin{aligned} \psi(z+1) &= e^{2\pi i \alpha_1} e^{\pi i M \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \psi(z), \\ \psi(z+\tau) &= e^{2\pi i \alpha_2} e^{\pi i M \frac{\mathrm{Im} \bar{\tau} z}{\mathrm{Im}\tau}} \psi(z), \end{aligned} \quad (2)$$

where α_i, $i = 1, 2$ denote Scherk-Schwarz (SS) phases, which cannot be removed by the gauge transformation. Imposing these conditions on the massless Dirac equation, $i\not{D}\psi = 0$, it is found that when $M > 0$ ($M < 0$), ψ_+ (ψ_-) has the $|M|$ number of degenerate solutions but ψ_- (ψ_+) has no solution. Then the jth zero modes of ψ_+ for $M > 0$ are expressed as

$$\psi_+^{(j+\alpha_1,\alpha_2),|M|}(z,\tau) = \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{2\pi i\frac{(j+\alpha_1)\alpha_2}{|M|}} e^{\pi i|M|z\frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \vartheta\begin{bmatrix}\frac{j+\alpha_1}{|M|}\\ -\alpha_2\end{bmatrix}(|M|z,|M|\tau),$$
$$j=0,1,\ldots,|M|-1, \tag{3}$$

where $\mathcal{A}$ denotes the area of T^2 and ϑ denotes the Jacobi theta function defined by

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(\nu,\tau)=\sum_{\ell\in\mathbb{Z}} e^{\pi i(a+\ell)^2\tau} e^{2\pi i(a+\ell)(\nu+b)}. \tag{4}$$

Similarly, ones of ψ_- for $M<0$ are given. Hereafter we assume $M>0$ and denote $\psi_+^{(j+\alpha_1,\alpha_2),|M|}$ as $\psi_{T^2}^{(j+\alpha_1,\alpha_2),M}$ although one can study the $M<0$ case in the same way. It can be shown that the zero modes satisfy the normalization,

$$\int d^2z\psi^{(i+\alpha_1,\alpha_2),M}(z,\tau)(\psi^{(j+\alpha_1,\alpha_2),M}(z,\tau))^* = (2\mathrm{Im}\tau)^{-1/2}\delta_{i,j}. \tag{5}$$

Second, we review the zero-mode wave functions on the $T^2/\mathbb{Z}_2$ twisted orbifold. The $T^2/\mathbb{Z}_2$ twisted orbifold is obtained by the $\mathbb{Z}_2$ twist identification, $z\sim -z$, in addition to torus identification. Then the zero modes on $T^2/\mathbb{Z}_2$ are given by

$$\begin{aligned}\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(z) &= \mathcal{N}^{(j+\alpha_1)}\left(\psi_{T^2}^{(j+\alpha_1,\alpha_2),M}(z)+(-1)^m\psi_{T^2}^{(j+\alpha_1,\alpha_2),M}(-z)\right)\\ &= \mathcal{N}^{(j+\alpha_1)}\left(\psi_{T^2}^{(j+\alpha_1,\alpha_2),M}(z)+(-1)^{m-2\alpha_2}\psi_{T^2}^{(M-(j+\alpha_1),\alpha_2),M}(z)\right)\\ &\equiv O_m^{jk,\alpha_1,\alpha_2,M}\psi_{T^2}^{(k+\alpha_1,\alpha_2),M}(z),\end{aligned} \tag{6}$$

where $m\in\{0,1\}$ denotes parities under $\mathbb{Z}_2$ twist, $\mathcal{N}^{(j+\alpha_1)}$ is a normalization factor defined by

$$\mathcal{N}^{(j+\alpha_1)} = \begin{cases}1/2 & (j+\alpha_1=0, M/2),\\ 1/\sqrt{2} & (\text{otherwise}),\end{cases} \tag{7}$$

and

$$O_m^{jk,\alpha_1,\alpha_2,M} = \mathcal{N}^{(j+\alpha_1)}(\delta_{j,k}+(-1)^{m-2\alpha_2}\delta_{M-j-2\alpha_1,k}). \tag{8}$$

Note that SS phases on $T^2/\mathbb{Z}_2$ are restricted to

$$(\alpha_1,\alpha_2)=(0,0),(1/2,0),(0,1/2),(1/2,1/2), \tag{9}$$

due to the $\mathbb{Z}_2$ twist identification. The number of zero modes is summarized in Table I.

TABLE I. The number of zero modes on $T^2/\mathbb{Z}_2$. $\mathbb{Z}_2$ parities 0 and 1 denote even and odd modes, respectively.

($\mathbb{Z}_2$ parity, α_1, α_2)	$M=$ even	$M=$ odd
(0, 0, 0)	$\frac{M}{2}+1$	$\frac{M+1}{2}$
(1, 0, 0)	$\frac{M}{2}-1$	$\frac{M-1}{2}$
(0, 1/2, 0)	$\frac{M}{2}$	$\frac{M+1}{2}$
(1, 1/2, 0)	$\frac{M}{2}$	$\frac{M-1}{2}$
(0, 0, 1/2)	$\frac{M}{2}$	$\frac{M+1}{2}$
(1, 0, 1/2)	$\frac{M}{2}$	$\frac{M-1}{2}$
(0, 1/2, 1/2)	$\frac{M}{2}$	$\frac{M-1}{2}$
(1, 1/2, 1/2)	$\frac{M}{2}$	$\frac{M+1}{2}$

B. Zero points of zero modes on $T^2/\mathbb{Z}_2$

Here we study the zero points of zero-mode wave functions on $T^2/\mathbb{Z}_2$. Zero points of zero-mode wave functions are the coordinates on $T^2/\mathbb{Z}_2$ where all zero-mode wave functions vanish, $\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(z)=0$ for all of j. We focus on zero points at the fixed points which are invariant points under $\mathbb{Z}_2$ twist up to lattice translations of torus. As we will see soon, all zero-mode wave functions become zero or nonzero at each fixed point. The fixed points on $T^2/\mathbb{Z}_2$ are obtained as follows:

$$P_F=\left\{0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\right\}. \tag{10}$$

First we consider the boundary conditions at the fixed points in order to find the zero points. From Eq. (3), we can check that the zero modes on T^2 satisfy the following conditions:

$$\psi_{T^2}^{(j+\alpha_1,\alpha_2),M}\left(z+\frac{n_1+n_2\tau}{2}\right) = e^{\pi i(j+\alpha_1)n_1} e^{\frac{\pi iM}{4}n_1n_2} e^{\frac{\pi iM}{2}\frac{\mathrm{Im}(n_1+n_2\bar{\tau})z}{\mathrm{Im}\tau}}\cdot\psi_{T^2}^{(j+\alpha_1+\frac{M}{2}n_2,\alpha_2+\frac{M}{2}n_1),M}(z), \tag{11}$$

where $n_1,n_2\in\mathbb{Z}$. They lead to

$$(-1)^{m-2\alpha_2}\psi_{T^2}^{(M-(j+\alpha_1),\alpha_2),M}\left(z+\frac{n_1+n_2\tau}{2}\right)$$
$$= e^{\pi i(j+\alpha_1)n_1}e^{\frac{\pi iM}{4}n_1n_2}e^{\frac{\pi iM}{2}\frac{\mathrm{Im}(n_1+n_2\bar{\tau})z}{\mathrm{Im}\tau}}(-1)^{(m-Mn_1n_2-2(\alpha_1n_1+\alpha_2n_2))-2(\alpha_2+\frac{M}{2}n_1)}\psi_{T^2}^{(M-(j+\alpha_1+\frac{M}{2}n_2),\alpha_2+\frac{M}{2}n_1),M}(z). \quad (12)$$

Thus, the zero modes on $T^2/\mathbb{Z}_2$ also satisfy the following conditions:

$$\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}\left(z+\frac{n_1+n_2\tau}{2}\right) = e^{\pi i(j+\alpha_1)n_1}e^{\frac{\pi iM}{4}n_1n_2}e^{\frac{\pi iM}{2}\frac{\mathrm{Im}(n_1+n_2\bar{\tau})z}{\mathrm{Im}\tau}}\psi_{T^2/\mathbb{Z}_2^{m'}}^{(j+\alpha_1+\frac{M}{2}n_2,\alpha_2+\frac{M}{2}n_1),M}(z), \quad (13)$$

where

$$m' = m - Mn_1n_2 - 2(\alpha_1n_1+\alpha_2n_2) \pmod 2. \quad (14)$$

Note that $m, m' = 0$ and 1 denote $\mathbb{Z}_2$ even and odd modes, respectively. At $z=0$, we obtain

$$\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}\left(\frac{n_1+n_2\tau}{2}\right) = e^{\pi i(j+\alpha_1)n_1}e^{\frac{\pi iM}{4}n_1n_2}\psi_{T^2/\mathbb{Z}_2^{m'}}^{(j+\alpha_1+\frac{M}{2}n_2,\alpha_2+\frac{M}{2}n_1),M}(0). \quad (15)$$

This means that the zero-mode wave function values on $T^2/\mathbb{Z}_2$ at the fixed points are given in terms of ones at $z=0$; therefore we can find whether zero modes at the fixed points vanish or not by the values of zero modes at $z=0$.

It follows from the zero-mode formula in Eq. (6) that $\mathbb{Z}_2$ even and odd modes satisfy

$$\psi_{T^2/\mathbb{Z}_2^0}^{(j+\alpha_1,\alpha_2),M}(0) \neq 0, \qquad \psi_{T^2/\mathbb{Z}_2^1}^{(j+\alpha_1,\alpha_2),M}(0) = 0 \quad (16)$$

at $z=0$ for all j, α_1, α_2, and M. This means that if the $\mathbb{Z}_2$ parity of the zero mode on the right-hand side in Eq. (15) is odd, the zero mode on the left-hand side vanishes. Thus the zero modes with (m,α_1,α_2) at the fixed point $z=\frac{n_1+n_2\tau}{2}$ become zero if

$$m' = m - Mn_1n_2 - 2(\alpha_1n_1+\alpha_2n_2) = 1 \pmod 2 \quad (17)$$

are satisfied. In Table II we have summarized the zero points of each zero mode at the fixed points.

Also we find the zero points of the derivative of zero-modes wave functions. The boundary condition in Eq. (13) leads to the following relation:

$$\frac{\partial}{\partial z}\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}\left(z+\frac{n_1+n_2\tau}{2}\right) = e^{\pi i(j+\alpha_1)n_1}e^{\frac{\pi iM}{4}n_1n_2}e^{\frac{\pi iM}{2}\frac{\mathrm{Im}(n_1+n_2\bar{\tau})z}{\mathrm{Im}\tau}}\left(\frac{\partial}{\partial z}+\frac{\pi M}{4\mathrm{Im}\tau}(n_1+n_2\bar{\tau})\right)\psi_{T^2/\mathbb{Z}_2^{m'}}^{(j+\alpha_1+\frac{M}{2}n_2,\alpha_2+\frac{M}{2}n_1),M}(z). \quad (18)$$

At $z=0$, we obtain

$$\frac{\partial}{\partial z}\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}\left(\frac{n_1+n_2\tau}{2}\right) = e^{\pi i(j+\alpha_1)n_1}e^{\frac{\pi iM}{4}n_1n_2}\left(\frac{\partial}{\partial z}+\frac{\pi M}{4\mathrm{Im}\tau}(n_1+n_2\bar{\tau})\right)\psi_{T^2/\mathbb{Z}_2^{m'}}^{(j+\alpha_1+\frac{M}{2}n_2,\alpha_2+\frac{M}{2}n_1),M}(0). \quad (19)$$

Since the derivatives of $\mathbb{Z}_2$ even and odd modes are $\mathbb{Z}_2$ odd and even modes respectively, they satisfy

$$\frac{\partial}{\partial z}\psi_{T^2/\mathbb{Z}_2^0}^{(j+\alpha_1,\alpha_2),M}(0) = 0, \qquad \frac{\partial}{\partial z}\psi_{T^2/\mathbb{Z}_2^1}^{(j+\alpha_1,\alpha_2),M}(0) \neq 0 \quad (20)$$

at $z=0$ for all j, α_1, α_2, and M. This and Eq. (16) mean that either the first or second terms on the right-hand side in Eq. (19) vanishes but the remaining term does not vanish. Thus only the derivatives of $\mathbb{Z}_2$ even modes vanish at $z=0$ and do not vanish at $z=\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}$, and other derivatives do not vanish at all fixed points. In Table III we have summarized the results.

C. Yukawa couplings

Here we study Yukawa couplings which are obtained by the overlap integrals of the wave functions of the left-handed fermion, right-handed fermion, and Higgs fields. First we study Yukawa couplings on T^2 instead of ones on $T^2/\mathbb{Z}_2$. Yukawa couplings on T^2 are given by the overlap integral of zero modes on T^2:

TABLE II. Zero points of each zero mode at the fixed points. $\mathbb{Z}_2$ parities 0 and 1 denote even and odd modes, respectively.

($\mathbb{Z}_2$ parity, α_1, α_2)	$M=$ even	$M=$ odd
(0,0,0)	None	$\frac{1+\tau}{2}$
(1,0,0)	$0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$	$0, \frac{1}{2}, \frac{\tau}{2}$
(0,1/2,0)	$\frac{1}{2}, \frac{1+\tau}{2}$	$\frac{1}{2}$
(1,1/2,0)	$0, \frac{\tau}{2}$	$0, \frac{\tau}{2}, \frac{1+\tau}{2}$
(0,0,1/2)	$\frac{\tau}{2}, \frac{1+\tau}{2}$	$\frac{\tau}{2}$
(1,0,1/2)	$0, \frac{1}{2}$	$0, \frac{1}{2}, \frac{1+\tau}{2}$
(0,1/2,1/2)	$\frac{1}{2}, \frac{\tau}{2}$	$\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$
(1,1/2,1/2)	$0, \frac{1+\tau}{2}$	0

TABLE III. Zero points of derivatives of each zero mode at the fixed points.

($\mathbb{Z}_2$ parity; SS phases)	$M=$ even	$M=$ odd
(0, 0, 0)	0	0
(1, 0, 0)	None	None
(0, 1/2, 0)	0	0
(1, 1/2, 0)	None	None
(0, 0, 1/2)	0	0
(1, 0, 1/2)	None	None
(0, 1/2, 1/2)	0	0
(1, 1/2, 1/2)	None	None

$$Y^{ijk}_{T^2} = g(2\mathrm{Im}\tau)^{1/2} \int_{T^2} d^2z \psi^{(i+\alpha_{1L},\alpha_{2L}),M_L}_{T^2}(z) \cdot \psi^{(j+\alpha_{1R},\alpha_{2R}),M_R}_{T^2}(z) \cdot (\psi^{(k+\alpha_{1H},\alpha_{2H}),M_H}_{T^2}(z))^*, \tag{21}$$

where $(M_f, \alpha_{1f}, \alpha_{2f})$ with $f \in \{L, R, H\}$ are (flux, SS phases) of the left-handed fermion (L), right-handed fermion (R), and Higgs fields (H), and g denotes the 3-point coupling in higher dimensional theory. Using the normalization in Eq. (5), we find

$$Y^{ijk}_{T^2} = g\mathcal{A}^{-1/2}\left|\frac{M_L M_R}{M_H}\right|^{1/4} e^{2\pi i(\frac{(i+\alpha_{1L})\alpha_{2L}}{M_L} + \frac{(j+\alpha_{1R})\alpha_{2R}}{M_R} - \frac{(k+\alpha_{1H})\alpha_{2H}}{M_H})} \sum_{m=0}^{M_H-1} \vartheta\begin{bmatrix} \frac{M_R(i+\alpha_{1L}) - M_L(j+\alpha_{1R}) + M_L M_R m}{M_L M_R M_H} \\ 0 \end{bmatrix}$$
$$\times (M_L\alpha_{2R} - M_R\alpha_{2L}, M_L M_R M_H \tau) \cdot \delta_{i+j-k, M_H\ell - M_L m}, \tag{22}$$

where $M_L + M_R = M_H$, $(\alpha_{1L},\alpha_{2L}) + (\alpha_{1R},\alpha_{2R}) = (\alpha_{1H},\alpha_{2H})$, and $\ell \in \mathbb{Z}$.

Similarly Yukawa couplings on $T^2/\mathbb{Z}_2$ are given by the overlap integral of zero modes on $T^2/\mathbb{Z}_2$:

$$Y^{ijk}_{T^2/\mathbb{Z}_2} = g(2\mathrm{Im}\tau)^{1/2} \int d^2z \psi^{(i+\alpha_{1L},\alpha_{2L}),M_L}_{T^2/\mathbb{Z}_2^{m_L}}(z) \cdot \psi^{(j+\alpha_{1R},\alpha_{2R}),M_R}_{T^2/\mathbb{Z}_2^{m_R}}(z) \cdot (\psi^{(k+\alpha_{1H},\alpha_{2H}),M_H}_{T^2/\mathbb{Z}_2^{m_H}}(z))^*, \tag{23}$$

where m_f with $f \in \{L, R, H\}$ is $\mathbb{Z}_2$ parity of left-handed fermion (L), right-handed fermion (R), and Higgs fields (H) and we have $m_f = 0$ for $\mathbb{Z}_2$ even and $m_f = 1$ for $\mathbb{Z}_2$ odd. From Eq. (8), this can be rewritten by Yukawa couplings on T^2 as

$$Y^{ijk}_{T^2/\mathbb{Z}_2} = O^{ii',\alpha_{1L},\alpha_{2L},M_L}_{m_L} O^{jj',\alpha_{1R},\alpha_{2R},M_R}_{m_R} (O^{kk',\alpha_{1H},\alpha_{2H},M_H}_{m_H})^* Y^{i'j'k'}_{T^2}. \tag{24}$$

Hereafter we denote Yukawa couplings on $T^2/\mathbb{Z}_2$ as Y^{ijk} instead of $Y^{ijk}_{T^2/\mathbb{Z}_2}$.

D. Quark and lepton flavor models

Here we study quark and lepton flavor models on the magnetized orbifold model. We start with higher dimensional theory with larger gauge group, e.g., $SU(3)\times SU(2) \times U(1)_Y \times U(1)^n$. We introduce magnetic flux background along $U(1)$ directions so as to obtain three generations of quarks and leptons. $U(1)$ gauge bosons may become massive except $U(1)_Y$. See for details of model building Refs. [6,77]. We consider all possible zero-mode assignments into left-handed quark doublets $Q = (u_L, d_L)^T$, right-handed up-sector (down-sector) quark singlets u_R (d_R), left-handed lepton doublets $L = (\nu_L, e_L)^T$, right-handed neutrino (charged lepton) singlets ν_R (e_R), and up and down type Higgs fields $H_{u,d}$. Here and in what follows we denote (flux, $\mathbb{Z}_2$ parity, SS phases) of zero modes assigned into each field of $f \in \{Q = (u_L, d_L)^T, u_R, d_R | L = (\nu_L, e_L)^T, \nu_R, e_R | H_u, H_d\}$ by B_f. In addition, we denote the jth zero-mode wave function of each field as ψ_{f^j}. Then mass matrices for up-sector quarks, down-sector quarks, and charged leptons, M_u, M_d, and M_e, are given by

$$M_u = Y^{ijk}_u \langle H^k_u \rangle, \quad M_d = Y^{ijk}_d \langle H^k_d \rangle, \quad M_e = Y^{ijk}_e \langle H^k_d \rangle, \tag{25}$$

where

$$Y_u^{ijk} = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{u_L^i}\cdot\psi_{u_R^j}\cdot(\psi_{H_u^k})^* \\ = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{Q^i}\cdot\psi_{u_R^j}\cdot(\psi_{H_u^k})^*, \quad (26)$$

$$Y_d^{ijk} = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{d_L^i}\cdot\psi_{d_R^j}\cdot(\psi_{H_d^k})^* \\ = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{Q^i}\cdot\psi_{d_R^j}\cdot(\psi_{H_d^k})^*, \quad (27)$$

$$Y_e^{ijk} = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{e_L^i}\cdot\psi_{e_R^j}\cdot(\psi_{H_d^k})^* \\ = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{L^i}\cdot\psi_{e_R^j}\cdot(\psi_{H_d^k})^*, \quad (28)$$

and $\langle H_{u,d}^k\rangle$ denote Higgs VEVs. On the other hand the light neutrino mass, M_ν, can be induced through the seesaw mechanism as follows:

$$M_\nu = M_D M_{RR}^{-1} M_D^T, \quad (29)$$

where M_{RR} is Majorana mass matrix of right-handed neutrinos, $M_D^{ij} = Y_\nu^{ijk}\langle H_u^k\rangle$ is Dirac mass, and

$$Y_\nu^{ijk} = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\psi_{\nu_L^i}\cdot\psi_{\nu_R^j}\cdot(\psi_{H_u^k})^*. \quad (30)$$

In Appendix A, we briefly review Majorana mass terms of right-handed neutrinos induced by the D-brane instanton effects on the magnetized $T^2/\mathbb{Z}_2$ model. To obtain non-vanishing Yukawa couplings for quarks and leptons (flux, $\mathbb{Z}_2$ parity, SS phases) of each field must satisfy

$$B_Q + B_{u_R} = B_L + B_{\nu_R} = B_{H_u}, \quad (31)$$

$$B_Q + B_{d_R} = B_L + B_{e_R} = B_{H_d}. \quad (32)$$

Because of these conditions, the number of Higgs modes is larger than one, in general. Furthermore, to cancel the chiral anomaly the number of up- and down-type Higgs fields must be the same in four-dimensional supersymmetric models. Under these conditions, we have obtained 6,460 flavor models in total on the magnetized orbifold. However realistic quark and lepton flavor structure cannot be realized in most of models. In the following section, we will see the difficulties to realize realistic quark and lepton flavor structure, and find the conditions to avoid them. We will also classify the flavor models satisfying such conditions.

III. CONDITIONS TO REALIZE QUARK AND LEPTON FLAVORS

As we have seen in the previous section, we have obtained 6,460 candidate models for quark and lepton flavors on the magnetized orbifold models. However it is not easy to realize realistic flavors due to mass hierarchies and the differences of mixing angles between quarks and leptons. Here we study the conditions to realize realistic quark and lepton flavor observables.

A. Conditions for flavors

First we show four conditions to realize realistic quark and lepton flavor structure. Here, we assume a linear combination $H_{u,d}^k$ corresponds to the lightest mode, and it develops its VEV.

1. Condition for up quark masses

Since an up-sector quark has large mass hierarchy, its mass matrix can be regarded approximately as a rank one matrix,

$$M_u = Y_u^{ijk}\langle H_u^k\rangle = (U_L^u)^\dagger\begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t\end{pmatrix}U_R^u \\ \propto (U_L^u)^\dagger\begin{pmatrix} \mathcal{O}(10^{-6}) & & \\ & \mathcal{O}(10^{-3}) & \\ & & 1\end{pmatrix}U_R^u \\ \sim (U_L^u)^\dagger\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1\end{pmatrix}U_R^u, \quad (33)$$

where U_L^u and U_R^u are unitary matrices to diagonalize M_u. That is

$$M_u^{ij} = Y_u^{ijk}\langle H_u^k\rangle \sim M_{\mathrm{rank}-1} \quad (34)$$

are required, where $M_{\mathrm{rank}-1}$ denotes a rank one matrix. To realize such a mass matrix, the following direction h_u^k must exist:

$$\exists\, h_u^k \mathrm{s.t.} Y_u^{ijk}h_u^k = M_{\mathrm{rank}-1} \quad (\mathrm{condition\,I}). \quad (35)$$

Then it is possible to realize the mass hierarchy in the up-sector quark by taking $\langle H_u^k\rangle = h_u^k + \varepsilon_u^k$ such that $\varepsilon_u/h_u \sim \mathcal{O}(\frac{m_c}{m_t}) \sim \mathcal{O}(10^{-3})$.

2. Condition for down quark and charged lepton masses

The down-sector quarks and charged leptons also have large hierarchies, and their mass matrices can be regarded approximately as rank one matrices,

$$M_d = Y_d^{ijk}\langle H_d^k\rangle = (U_L^d)^\dagger \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix} U_R^d \propto (U_L^d)^\dagger \begin{pmatrix} \mathcal{O}(10^{-4}) & & \\ & \mathcal{O}(10^{-2}) & \\ & & 1 \end{pmatrix} U_R^d \sim (U_L^d)^\dagger \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} U_R^d, \tag{36}$$

$$M_e = Y_e^{ijk}\langle H_d^k\rangle = (U_L^e)^\dagger \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix} U_R^e \propto (U_L^e)^\dagger \begin{pmatrix} \mathcal{O}(10^{-4}) & & \\ & \mathcal{O}(10^{-2}) & \\ & & 1 \end{pmatrix} U_R^e \sim (U_L^e)^\dagger \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} U_R^e, \tag{37}$$

where U_L^d and U_R^d are unitary matrices to diagonalize M_d, and U_L^e and U_R^e are ones for M_e. That is

$$M_d^{ij} = Y_d^{ijk}\langle H_d^k\rangle \sim M_{\text{Rank}-1}, \quad M_e^{ij} = Y_e^{ijk}\langle H_d^k\rangle \sim M_{\text{rank}-1} \tag{38}$$

are required. To realize such mass matrices, the following direction h_d^k must exist:

$$\exists\, h_d^k \text{s.t.} Y_d^{ijk} h_d^k = M_{\text{rank}-1}, \quad Y_e^{ijk} h_d^k = M_{\text{rank}-1} \quad (\text{condition II}). \tag{39}$$

Then it is possible to realize the down-sector quark and charged lepton mass hierarchies by taking $\langle H_d^k\rangle = h_d^k + \varepsilon_d^k$ such that $\varepsilon_d/h_d \sim \mathcal{O}(\frac{m_s}{m_b}) \sim \mathcal{O}(\frac{m_\mu}{m_\tau}) \sim \mathcal{O}(10^{-2})$.

3. Condition for quark mixing

The absolute values of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements are observed as

$$|V_{\text{CKM}}| \equiv |(U_L^u)^\dagger U_L^d| = \begin{pmatrix} 0.974 & 0.227 & 0.00361 \\ 0.226 & 0.973 & 0.0405 \\ 0.00854 & 0.0398 & 0.999 \end{pmatrix}, \tag{40}$$

which is approximately a unit matrix. This implies that the following relation

$$U_L^u \sim U_L^d \tag{41}$$

is required. To realize this relation, here we introduce the unitary matrices $u_{L,R}^{u,d}$ which diagonalize rank one matrices $Y_{u,d}^{ijk} h_{u,d}^k$:

$$[(u_L^{u,d})^\dagger]^{ii'} Y_{u,d}^{i'j'k} h_{u,d}^k [u_R^{u,d}]^{j'j} \propto \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}^{ij}, \tag{42}$$

and impose the following condition:

$$u_L^u = u_L^d \quad (\text{condition III}). \tag{43}$$

Under this condition, in a basis where u_L^u, u_L^d, u_R^u, and u_R^d are unit matrices, the quark mass matrices should have the following forms:

$$M_u = Y_u^{ijk}\langle H_u^k\rangle = Y_u^{ijk}(h_u^k + \varepsilon_u^k) \propto \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \mathcal{O}\left(\frac{m_c}{m_t}\right), \tag{44}$$

$$M_d = Y_d^{ijk}\langle H_d^k\rangle = Y_d^{ijk}(h_d^k + \varepsilon_d^k) \propto \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \mathcal{O}\left(\frac{m_s}{m_b}\right), \tag{45}$$

because of the mass hierarchies. From the above, the unitary matrices $U_{L,R}^{u,d}$ which diagonalize $M_{u,d}$ can be estimated to be

$$U_{L,R}^u \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathcal{O}(\frac{m_c}{m_t}) \\ 0 & \mathcal{O}(\frac{m_c}{m_t}) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathcal{O}(\frac{m_c}{m_t}) \\ 0 & 1 & 0 \\ \mathcal{O}(\frac{m_c}{m_t}) & 0 & 1 \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} * & * & \mathcal{O}(\frac{m_c}{m_t}) \\ * & * & \mathcal{O}(\frac{m_c}{m_t}) \\ \mathcal{O}(\frac{m_c}{m_t}) & \mathcal{O}(\frac{m_c}{m_t}) & 1 \end{pmatrix}, \tag{46}$$

$$U^d_{L,R} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathcal{O}(\frac{m_s}{m_b}) \\ 0 & \mathcal{O}(\frac{m_s}{m_b}) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathcal{O}(\frac{m_s}{m_b}) \\ 0 & 1 & 0 \\ \mathcal{O}(\frac{m_s}{m_b}) & 0 & 1 \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} * & * & \mathcal{O}(\frac{m_s}{m_b}) \\ * & * & \mathcal{O}(\frac{m_s}{m_b}) \\ \mathcal{O}(\frac{m_s}{m_b}) & \mathcal{O}(\frac{m_s}{m_b}) & 1 \end{pmatrix}, \quad (47)$$

where $*$ denotes unestimated values. Then the CKM matrix can be estimated to be

$$V_{\mathrm{CKM}} \sim \begin{pmatrix} * & * & \mathcal{O}(\frac{m_s}{m_b}) \\ * & * & \mathcal{O}(\frac{m_s}{m_b}) \\ \mathcal{O}(\frac{m_s}{m_b}) & \mathcal{O}(\frac{m_s}{m_b}) & 1 \end{pmatrix} \sim \begin{pmatrix} * & * & \mathcal{O}(10^{-2}) \\ * & * & \mathcal{O}(10^{-2}) \\ \mathcal{O}(10^{-2}) & \mathcal{O}(10^{-2}) & 1 \end{pmatrix}, \quad (48)$$

and this is consistent with the observations in Eq. (40). Thus it is possible to realize realistic quark mixing under the condition III in Eq. (43).

4. Condition for lepton mixing

If we take the direction of up-type Higgs VEVs, $\langle H^k_u \rangle = h^k_u + \varepsilon^k_u$, satisfying Eq. (38), the light neutrino mass matrix becomes

$$\begin{aligned} M^{ij}_\nu &= Y^{ii'm}_\nu \langle H^m_u \rangle (M^{i'j'}_{RR})^{-1} (Y^{j'jn}_\nu)^T \langle H^n_u \rangle \\ &= Y^{ii'm}_\nu h^m_u (M^{i'j'}_{RR})^{-1} (Y^{j'jn}_\nu)^T h^n_u + \mathcal{O}(\varepsilon_u) + \mathcal{O}(\varepsilon^2_u). \end{aligned} \quad (49)$$

If the first term is nonvanishing and ε_u is small enough, the first term is dominant. In this case, the direction of h_u is determined to satisfy Eq. (34) and there are no parameters to be used for realizing lepton mixing. Thus it is difficult to realize realistic lepton mixing unless M_{RR} and $Y^{ijk}_\nu h^k_u$ have ideal structures. To avoid this difficulty, here we impose the following condition:

$$Y^{ijk}_u h^k_u = M_{\mathrm{rank}-1} \Rightarrow Y^{ijk}_\nu h^k_u = 0 \quad (\text{condition IV}). \quad (50)$$

In such case the mass matrix of light neutrino is given by

$$M^{ij}_\nu = Y^{ii'm}_\nu \varepsilon^m_u (M^{i'j'}_{RR})^{-1} (Y^{j'jn}_\nu)^T \varepsilon^n_u, \quad (51)$$

and we have the possibility to realize realistic lepton mixing by taking appropriate directions of ε_u.

In Table IV, we summarize all conditions.

TABLE IV. The conditions I, II, III, and IV.

	Conditions
I	$\exists\, h^k_u$ s.t. $Y^{ijk}_u h^k_u =$ rank one matrix
II	$\exists\, h^k_d$ s.t. $Y^{ijk}_d h^k_d =$ rank one matrix, $Y^{ijk}_e h^k_d =$ rank one matirx
III	$u^u_L = u^d_L$
IV	$Y^{ijk}_u h^k_u =$ rank one matrix $\Rightarrow Y^{ijk}_\nu h^k_u = 0$

B. Zero-point analysis

In the previous subsection, we saw four conditions, I, II, III, and IV, to realize quark and lepton masses and their mixing angles. Then Higgs VEV directions $h^k_{u,d}$ leading to rank one or vanishing mass matrices have been required. In this subsection, we show such directions can be realized in several cases by checking the zero points of zero modes. The procedure is as follows. First we start from Yukawa couplings between left-handed fermion zero modes ψ^i_L, right-handed fermion zero modes ψ^j_R, and Higgs field zero modes ψ^k_H. We consider zero points of zero modes ψ^i_L, ψ^j_R, and ψ^k_H. As we will see soon, zero-point patterns on Yukawa couplings have the information which liner combinations of Yukawa matrices lead to rank one or vanishing mass matrix. Second, we will construct unitary matrices for Higgs field zero modes which correspond to this liner combination. Finally, we classify the structure of mass matrices in each pattern of zero points.

Yukawa couplings between ψ^i_L, ψ^j_R, and ψ_H are given by

$$Y^{ijk} = g y^{ijk} = g(2\mathrm{Im}\tau)^{1/2} \int d^2 z \psi^i_L(z) \cdot \psi^j_R(z) \cdot (\psi^k_H(z))^*. \quad (52)$$

This leads to the product expansion,

$$\psi^i_L(z) \cdot \psi^j_R(z) = y^{ijk} \psi^k_H(z). \quad (53)$$

Here we denote sets of the zero points at the fixed points of ψ^i_L, ψ^j_R, and ψ^k_H as P_{ψ_L}, P_{ψ_R}, and P_{ψ_H}, and ones of the derivatives of ψ^i_L, ψ^j_R, and ψ^k_H as P'_{ψ_L}, P'_{ψ_R}, and P'_{ψ_H}.

Next, we choose one point p on $T^2/\mathbb{Z}_2$ (not necessary to be fixed points) and consider a unitary transformation for ψ^i_L such as

$$\psi^i_L \to \hat{\psi}^i_L = U^{ij}_{\psi_L}(p) \psi^j_L, \quad (54)$$

$$U^{ij}_{\psi_L}(p) = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} e^{-i\alpha_0} & 0 & 0 \\ 0 & e^{-i\alpha_1} & 0 \\ 0 & 0 & e^{-i\alpha_2} \end{pmatrix}, \tag{55}$$

where

$$\begin{cases} \alpha^i = \mathrm{Ang}(\psi^i_L(p)) & \text{for } p \notin P_{\psi_L}, \\ \alpha^i = \mathrm{Ang}\left(\frac{\partial}{\partial z}\psi^i_L(p)\right) & \text{for } p \in P_{\psi_L}, \end{cases} \tag{56}$$

$$\begin{cases} \theta_1 = \tan^{-1}\frac{|\psi^1_L(p)|}{|\psi^2_L(p)|}, \quad \theta_2 = \tan^{-1}\frac{|\psi^0_L(p)|}{\sin\theta_1|\psi^1_L(p)|+\cos\theta_1|\psi^2_L(p)|} & \text{for } p \notin P_{\psi_L}, \\ \theta_1 = \tan^{-1}\frac{|\frac{\partial}{\partial z}\psi^1_L(p)|}{|\frac{\partial}{\partial z}\psi^2_L(p)|}, \quad \theta_2 = \tan^{-1}\frac{|\frac{\partial}{\partial z}\psi^0_L(p)|}{\sin\theta_1|\frac{\partial}{\partial z}\psi^1_L(p)|+\cos\theta_1|\frac{\partial}{\partial z}\psi^2_L(p)|} & \text{for } p \in P_{\psi_L}. \end{cases} \tag{57}$$

For $p \notin P_{\psi_L}$ redefined zero modes $\hat{\psi}^i_L(z)$ $(i \neq 2)$ become zero at $z = p$ while for $p \in P_{\psi_L}$ the derivative of redefined zero modes $\frac{\partial}{\partial z}\hat{\psi}^i_L(z)$ $(i \neq 2)$ become zero at $z = p$. In a similar way we can obtain redefined zero modes, $\hat{\psi}_R$ and $\hat{\psi}_H$, for ψ_R and ψ_H by unitary transformations $U_{\psi_R}(p)$ and $U_{\psi_H}(p)$ such that only $\hat{\psi}^2_R$ and $\hat{\psi}^{(g_H-1)}_H$ are nonvanishing. Then we consider the structures of redefined Yukawa couplings,

$$\hat{Y}^{ijk} = g(2\mathrm{Im}\tau)^{1/2}\int d^2z\hat{\psi}^i_L(z)\cdot\hat{\psi}^j_R(z)\cdot(\hat{\psi}^k_H(z))^*. \tag{58}$$

From Table II, we can find that conditions for nonvanishing Yukawa couplings in Eqs. (31) and (32) mean that when one point p is in P_L, it is also in either P_R or P_H; when p is in P_R, it is also in either P_L or P_H; when p is in P_H, it is also in either P_L or P_R. That is, there are four possible patterns of p:

(1) $p \notin P_{\psi_L}$, $\quad p \notin P_{\psi_R}$, $\quad p \notin P_{\psi_H}$,

(2) $p \in P_{\psi_L}$, $\quad p \in P_{\psi_R}$, $\quad p \notin P_{\psi_H}$,

(3) $p \in P_{\psi_L}$, $\quad p \notin P_{\psi_R}$, $\quad p \in P_{\psi_H}(p \notin P'_{\psi_L}, p \notin P'_{\psi_H})$,

(4) $p \notin P_{\psi_L}$, $\quad p \in P_{\psi_R}$, $\quad p \in P_{\psi_H}(p \notin P'_{\psi_R}, p \notin P'_{\psi_H})$.

Note that the derivatives of zero modes which vanish at $z = p \in P_F$ do not vanish at $z = p$ as can be read from Tables II and III. In each pattern, we focus on the structures of $\hat{Y}^{ij(g_H-1)} = U^{ii'}_{\psi_L}(p)U^{jj'}_{\psi_R}(p)Y^{i'j'k}(U^{(g_H-1)k}_{\psi_H}(p))^*$, where g_H denotes the number of Higgs fields, because we use the Higgs mode basis such that the wave function for the (g_H-1)th Higgs mode is nonvanishing at p and the others vanish.

1. $p \notin P_{\psi_L}, p \notin P_{\psi_R}, p \notin P_{\psi_H}$

Table V shows the zero points of redefined zero modes. In this case, the product expansion in Eq. (53) at $z = p$ leads to

$$\underbrace{\hat{\psi}^i_L(p)}_{\propto\delta^{i,2}}\cdot\underbrace{\hat{\psi}^j_R(p)}_{\propto\delta^{j,2}} = \hat{y}^{ijk}\underbrace{\hat{\psi}^k_H(p)}_{\propto\delta^{k,(g_H-1)}} \Leftrightarrow \hat{Y}^{ij(g_H-1)} \propto \delta^{i,2}\delta^{j,2} \quad \text{(rank one matrix)} \tag{59}$$

$$\Leftrightarrow Y^{ijk}(U^{(g_H-1)k}_{\psi_H}(p))^* = M_{\text{rank}-1}. \tag{60}$$

2. $p \in P_{\psi_L}, p \in P_{\psi_R}, p \notin P_{\psi_H}$

Table VI shows the zero points of redefined zero modes. In this case, the product expansion in Eq. (53) at $z = p$ leads to

TABLE V. Zero points of redefined zero modes in pattern (1).

	$j=0$	1	2	3	$\cdots$	g_H-2	g_H-1
$\hat{\psi}^j_L = U^{jk}_{\psi_1}\psi^k_L$	P_{ψ_1}, p	P_{ψ_L}, p	P_{ψ_L}	/	$\cdots$	/	/
$\hat{\psi}^j_R = U^{jk}_{\psi_2}\psi^k_R$	P_{ψ_R}, p	P_{ψ_R}, p	P_{ψ_R}	/	$\cdots$	/	/
$\hat{\psi}^j_H = U^{jk}_{\psi_H}\psi^k_H$	P_{ψ_H}, p	P_{ψ_H}, p	P_{ψ_H}, p	P_{ψ_H}, p	$\cdots$	P_{ψ_H}, p	P_{ψ_H}

TABLE VI. Zero points of redefined zero modes in pattern (2).

	$j=0$	1	2	3	$\cdots$	g_H-2	g_H-1
$\hat{\psi}^j_L = U^{jk}_{\psi_L}\psi^k_L$	P_{ψ_L}	P_{ψ_L}	P_{ψ_L}	/	$\cdots$	/	/
$\hat{\psi}^j_2 = U^{jk}_{\psi_R}\psi^k_R$	P_{ψ_R}	P_{ψ_R}	P_{ψ_R}	/	$\cdots$	/	/
$\hat{\psi}^j_H = U^{jk}_{\psi_H}\psi^k_H$	P_{ψ_H}, p	P_{ψ_H}, p	P_{ψ_H}, p	P_{ψ_H}, p	$\cdots$	P_{ψ_H}, p	P_{ψ_H}

$$\underbrace{\hat{\psi}^i_L(p)}_{=0}\cdot\underbrace{\hat{\psi}^j_R(p)}_{=0} = \hat{y}^{ijk}\underbrace{\hat{\psi}^k_H(p)}_{\propto\delta^{k,(g_H-1)}} \Leftrightarrow \hat{Y}^{ij(g_H-1)} = 0 \qquad (61)$$

$$\Leftrightarrow Y^{ijk}(U^{(g_H-1)k}_{\psi_H}(p))^* = 0. \qquad (62)$$

3. $p\in P_{\psi_L}, p\notin P_{\psi_R}, p\in P_{\psi_H}$ $(p\notin P'_{\psi_L}, p\notin P'_{\psi_H})$

Table VII shows the zero points of redefined zero-mode wave functions and their derivatives. In this case, the product expansion in Eq. (53) at $z=p$ give no information for $\hat{Y}^{ij(g_H-1)}$. Instead of Eq. (53) we consider the derivative of Eq. (53). At $z=p$, it leads to

$$\underbrace{\hat{\psi}^i_L(p)}_{=0}\cdot\underbrace{\hat{\psi}^j_R(p)}_{\propto\delta^{j,2}} = \hat{y}^{ijk}\underbrace{\hat{\psi}^k_H(p)}_{=0}$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial z}\hat{\psi}^i_L(p)}_{\propto\delta^{i,2}}\cdot\underbrace{\hat{\psi}^j_R(p)}_{\propto\delta^{j,2}} + \underbrace{\hat{\psi}^i_L(p)}_{=0}\cdot\frac{\partial}{\partial z}\hat{\psi}^j_R(p) = \hat{y}^{ijk}\underbrace{\frac{\partial}{\partial z}\hat{\psi}^k_H(p)}_{\propto\delta^{k,(g_H-1)}} \qquad (63)$$

$$\Leftrightarrow \hat{Y}^{ij(g_H-1)} \propto \delta^{i,2}\delta^{j,2} \quad \text{(rank one matrix)} \qquad (64)$$

$$\Leftrightarrow Y^{ijk}(U^{(g_H-1)k}_{\psi_H}(p))^* = M_{\text{rank}-1}. \qquad (65)$$

4. $p\notin P_{\psi_L}, p\in P_{\psi_R}, p\in P_{\psi_H}$ $(p\notin P'_{\psi_R}, p\notin P'_{\psi_H})$

This case is flipping between ψ_L and ψ_R in the pattern (3); therefore it gives the same result as the pattern (3),

$$Y^{ijk}(U^{(g_H-1)k}_{\psi_H}(p))^* = M_{\text{rank}-1}. \qquad (66)$$

In Eqs. (60) and (65), note that unitary transformations U_{ψ_L} and U_{ψ_R} do not change the rank of the matrix. Thus we can obtain Higgs VEV directions $h^k_{u,d} = v_{u,d}(U^{(g_H-1)k}_{H_{u,d}})^*$ leading to rank one fermion mass matrices in three patterns (1), (3), and (4). On the other hand, the pattern (2) gives vanishing mass matrices.

C. Classification of models

In this subsection, we classify all of the quark and lepton flavor models on the magnetized orbifold model which satisfy the conditions I, II, III, and IV. In what follows we denote sets of the zero points at the fixed points of each field f as P_f for

$$f\in\{Q=(u_L,d_L),u_R,d_R|L=(\nu_L,e_L),\nu_R,e_R|H_u,H_d\}. \qquad (67)$$

First, we show the constraints of P_f to satisfy each condition.

1. Condition I

The condition I is that the up-type Higgs VEV direction leading to the up-sector quarkmass matrix with the rank one must exist. Hence,

$$Y^{ijk}_u h^k_u = M_{\text{rank}-1} \qquad (68)$$

is required. As shown in the previous subsection, this requirement means that the following point p_u must exist:

TABLE VII. Zero points of redefined zero modes and their derivatives in pattern (3).

	$j=0$	1	2	3	$\cdots$	g_H-2	g_H-1
$\hat{\psi}^j_L = U^{jk}_{\psi_L}\psi^k_L$	P_{ψ_L}	P_{ψ_L}	P_{ψ_L}	/	$\cdots$	/	/
$\hat{\psi}^j_R = U^{jk}_{\psi_R}\psi^k_R$	P_{ψ_R}, p	P_{ψ_R}, p	P_{ψ_R}	/	$\cdots$	/	/
$\hat{\psi}^j_H = U^{jk}_{\psi_H}\psi^k_H$	P_{ψ_H}	P_{ψ_H}	P_{ψ_H}	P_{ψ_H}	$\cdots$	P_{ψ_H}	P_{ψ_H}
$\frac{\partial}{\partial z}\hat{\psi}^j_L = U^{jk}_{\psi_L}\frac{d}{dz}\psi^k_L$	P'_{ψ_L}, p	P'_{ψ_L}, p	P'_{ψ_L}	/	$\cdots$	/	/
$\frac{\partial}{\partial z}\hat{\psi}^j_H = U^{jk}_{\psi_H}\frac{d}{dz}\psi^k_H$	P'_{ψ_H}, p	$P'_{\psi_H} p$	P'_{ψ_H}, p	P'_{ψ_H}, p	$\cdots$	P'_{ψ_H}, p	P'_{ψ_H}

$$
{}^{\exists}p_u = p\,\text{s.t.}\begin{cases}(1)\ p\notin P_{\psi_L}, p\notin P_{\psi_R}, p\notin P_{\psi_H}\\ (3)\ p\in P_{\psi_L}, p\notin P_{\psi_R}, p\in P_{\psi_H}(p\notin P'_{\psi_L}, p\notin P'_{\psi_H}),\\ (4)\ p\notin P_{\psi_L}, p\in P_{\psi_R}, p\in P_{\psi_H}(p\notin P'_{\psi_R}, p\notin P'_{\psi_H})\end{cases} \tag{69}
$$

for $(\psi_L, \psi_R, \psi_H) = (Q, u_R, H_u)$, i.e., constraint I.

2. Condition II

The condition II is that the down-type Higgs VEV direction leading to the down-sector quark and charged lepton mass matrices with rank one must exist. Hence,

$$
Y_d^{ijk} h_d^k = M_{\text{rank}-1}, \qquad Y_e^{ijk} h_d^k = M_{\text{rank}-1} \tag{70}
$$

are required. As shown in the previous subsection, this requirement means that the following point p_d must exist:

$$
{}^{\exists}p_d = p\,\text{s.t.}\begin{cases}(1)\ p\notin P_{\psi_L}, p\notin P_{\psi_R}, p\notin P_{\psi_H}\\ (3)\ p\in P_{\psi_L}, p\notin P_{\psi_R}, p\in P_{\psi_H}(p\notin P'_{\psi_L}, p\notin P'_{\psi_H}),\\ (4)\ p\notin P_{\psi_L}, p\in P_{\psi_R}, p\in P_{\psi_H}(p\notin P'_{\psi_R}, p\notin P'_{\psi_H})\end{cases} \tag{71}
$$

for $(\psi_L, \psi_R, \psi_H) = (Q, d_R, H_d)$, i.e., constraint II_1 and for $(\psi_L, \psi_R, \psi_H) = (L, e_R, H_d)$, i.e., constraint II_2.

3. Condition III

The condition III is that unitary matrices $u_L^{u,d}$ which diagonalize rank one matrices $Y_{u,d}^{ijk} h_{u,d}^k$ must satisfy $u_L^u = u_L^d$. When the condition I is satisfied, that is, the constraint I is satisfied, we can find the up-sector quark mass matrix with the rank one,

$$
Y_u^{ijk} h_u^k = v_u Y_u^{ijk} (U_{\psi_H}^{(g_H-1)k}(p_u))^*, \tag{72}
$$

and they are diagonalized as

$$
U_{\psi_L}^{ii'}(p_u) U_{\psi_R}^{jj'}(p_u) v_u Y_u^{i'j'k} (U_{\psi_H}^{(g_H-1)k}(p_u))^* \propto \delta^{i,2}\delta^{j,2} \tag{73}
$$

for $(\psi_L, \psi_R, \psi_H) = (Q, u_R, H_u)$. Note that U_{ψ_L}, U_{ψ_R}, and U_{ψ_H} are defined by Eq. (55) and the sentence below. Similarly, when the condition II is satisfied, that is, the constraints II_1 and II_2 are satisfied, we can find the down-sector quark mass matrix with the rank one,

$$
Y_d^{ijk} h_d^k = v_d Y_d^{ijk} (U_{\psi_H}^{(g_H-1)k}(p_d))^*, \tag{74}
$$

and they are diagonalized as

$$
U_{\psi_L}^{ii'}(p_d) U_{\psi_R}^{jj'}(p_d) v_d Y_d^{i'j'k} (U_{\psi_H}^{(g_H-1)k}(p_d))^* \propto \delta^{i,2}\delta^{j,2} \tag{75}
$$

for $(\psi_L, \psi_R, \psi_H) = (Q, d_R, H_d)$. Then $u_L^u = u_L^d$ is equivalent to the equation

$$
U_{\psi_L}^{ij}(p_u) = U_{\psi_L}^{ij}(p_d) \quad \text{for } \psi_L = Q. \tag{76}
$$

Obviously this can be satisfied by

$$
p_u = p_d \quad (\text{constraint III}). \tag{77}
$$

4. Condition IV

The condition IV is that when the up-sector quark mass matrix is a rank one matrix, the neutrino Dirac mass matrix must vanish. Hence,

$$
Y_u^{ijk} h_u^k = M_{\text{rank}-1} \Rightarrow Y_\nu^{ijk} h_u^k = 0, \tag{78}
$$

is required. As shown in the previous subsection, this requirement means that the following point p_u must exist:

$$
{}^{\exists}p_u = p\,\text{s.t.}(2)\ p\in P_{\psi_L}, p\in P_{\psi_R}, p\notin P_{\psi_H}, \tag{79}
$$

for $(\psi_L, \psi_R, \psi_H) = (L, \nu_R, H_u)$, i.e., constraint IV.

In Table VIII, we summarize all constraints.

Next, we classify all possible flavor models satisfying the above constraints. See Table VIII. From the constraint III, $p_u = p_d \equiv p$ must consist. Furthermore, from the constraint IV, p must be in P_F and satisfy

$$
p \in P_L, \qquad p \in P_{\nu_R}, \qquad p \notin P_{H_u}. \tag{80}
$$

From the constraint I, this makes p be the pattern (1) for $(\psi_L, \psi_R, \psi_H) = (Q, u_R, H_u)$ in Eq. (69), i.e.,

$$
p \notin P_Q, \qquad p \notin P_{u_R}, \qquad p \notin P_{H_u}. \tag{81}
$$

TABLE VIII. The constraints I, II_1, II_2, III, and IV. For example, if p_u corresponds to (1), it is not included in P_Q. If p_u corresponds to (3), it is included in P_Q. The bold texts denote the choices in Eq. (84) which are consistent with all constraints.

	P_Q	P_{u_R}	P_{d_R}	P_L	P_{ν_R}	P_{e_R}	P_{H_u}	P_{H_d}
I: p_u is (1)	**not in**	**not in**	⋯	⋯	⋯	⋯	**not in**	⋯
I: p_u is (3)	in	not in	⋯	⋯	⋯	⋯	in	⋯
I: p_u is (4)	not in	in	⋯	⋯	⋯	⋯	in	⋯
II_1: p_d is (1)	not in	⋯	not in	⋯	⋯	⋯	⋯	not in
II_1: p_d is (3)	in	⋯	not in	⋯	⋯	⋯	⋯	in
II_1: p_d is (4)	**not in**	⋯	**in**	⋯	⋯	⋯	⋯	**in**
II_2: p_d is (1)	⋯	⋯	⋯	not in	⋯	not in	⋯	not in
II_2: p_d is (3)	⋯	⋯	⋯	**in**	⋯	**not in**	⋯	**in**
II_2: p_d is (4)	⋯	⋯	⋯	not in	⋯	in	⋯	in
III: $p_u = p_d$	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
IV: p_u is	⋯	⋯	⋯	**in**	**in**	⋯	**not in**	⋯

Similarly, from the constraint II_2, p must be the pattern (3) for $(\psi_L, \psi_R, \psi_H) = (L, e_R, H_d)$ in Eq. (71), i.e.,

$$p \in P_L, \qquad p \notin P_{e_R}, \qquad p \in P_{H_d}. \tag{82}$$

Finally, from the constraint II_1, p must be the pattern (4) for $(\psi_L, \psi_R, \psi_H) = (Q, d_R, H_d)$ in Eq. (71), i.e.,

$$p \notin P_Q, \qquad p \in P_{d_R}, \qquad p \in P_{H_d}. \tag{83}$$

Thus, the point p consistent with all conditions must satisfy

$$\begin{aligned} &p \in P_L \cup P_{d_R} \cup P_{\nu_R} \cup P_{H_d} \subset P_F, \\ &p \notin P_Q \cup P_{u_R} \cup P_{e_R} \cup P_{H_u} \subset P_F. \end{aligned} \tag{84}$$

Now we are ready to classify all possible flavor models satisfying the conditions I, II, III, and IV. We again note that $p_u = p_d = p \in P_F$ and therefore we can find flavor models with consistent p by checking the zero points of zero modes of each field from Table II. Flavor models are picked up by Eq. (84) in addition to the nonvanishing Yukawa coupling conditions, Eqs. (31) and (32) and the anomaly cancellation condition which makes the number of up- and down-type Higgs fields the same. The results are shown in Appendix D of Ref. [78]. There are 408 flavor models in total.

IV. MODULAR SYMMETRIC MODELS

In this section, we classify the flavor models, which have a specific property under the S transformation. To calculate fermion flavors, we need to identify two types of VEVs; one is the VEV of modulus and another one is the VEVs of Higgs fields. In the former, we consider the vacuum where the modulus lies on either of three modular fixed points; (i) $\tau = i$ is invariant under S transformation; (ii) $\tau = e^{2\pi i/3} \equiv \omega$ is invariant under ST transformation; (iii) $\tau = i\infty$ is invariant under T transformation. In the latter, we consider Higgs VEVs aligned in eigenbasis of the modular transformation corresponding to each fixed point. We will show that some flavor models have the possibility to lead to realistic flavor observations in a vicinity of S-symmetric vacuum but there are no consistent flavor models for ST- and T-symmetric vacua.

A. Higgs μ term

First, we start from assuming that the value of modulus is fixed at either of $\tau = i$, ω, and $i\infty$. In this subsection, we study which direction Higgs VEVs are aligned at these three modular fixed points.

Higgs VEVs are aligned in the lightest mass direction. Supersymmetric mass term (μ term) of Higgs fields can be generated by D-brane instanton effects [71–75]. As shown in Appendix B, actually in a leading order, D-brane instanton effects give the following Higgs μ term:

$$\mu^{ij}\varepsilon_{nm}H^i_{um}H^j_{dn} = \Lambda e^{-S_{\mathrm{inst}}}(2\mathrm{Im}\tau)^{-1}(Y^i_u Y^j_d)\varepsilon_{nm}H^i_{um}H^j_{dn}, \tag{85}$$

where Λ denotes a typical scale such as the compactification scale and S_{inst} denotes the instanton action. Here, Y^i_u (Y^j_d) are the 3-point couplings among instanton zero modes α, β (γ), and Higgs fields H^i_u (H^j_d) given by

$$\begin{aligned} Y^i_u &= g(\mathrm{Im}\tau)^{1/2}\int d^2z\psi_\alpha(z)\cdot\psi_\beta(z)\cdot(\psi^i_{H_u}(z))^*, \\ Y^j_d &= g(\mathrm{Im}\tau)^{1/2}\int d^2z\psi_\alpha(z)\cdot\psi_\gamma(z)\cdot(\psi^j_{H_d}(z))^*, \end{aligned} \tag{86}$$

where ψs are the zero-mode wave functions on $T^2/\mathbb{Z}_2$ corresponding to instanton zero modes α, β (γ), and Higgs fields H_u (H_d).

Here, let us consider the modular transformation of this leading mass term. Under modular transformation, the zero modes of α, β, γ, and Higgs fields behave as the modular forms of weight 1/2:

$$\psi_{\alpha,\beta,\gamma} \to \tilde{J}_{1/2}(\tilde{\gamma},\tau)\tilde{\rho}_{\alpha,\beta,\gamma}(\tilde{\gamma})\psi_{\alpha,\beta,\gamma}, \\ \psi^i_{H_{u,d}} \to \tilde{J}_{1/2}(\tilde{\gamma},\tau)\tilde{\rho}_{H_{u,d}}(\tilde{\gamma})_{ij}\psi^j_{H_{u,d}}, \tag{87}$$

where $\tilde{J}_{1/2}(\tilde{\gamma},\tau)$ is the automophy factor given by

$$\tilde{J}_{1/2}(\tilde{S},\tau) = (-\tau)^{1/2}, \qquad \tilde{J}_{1/2}(\tilde{T},\tau) = 1, \\ \tilde{J}_{1/2}(\widetilde{ST},\tau) = (-(\tau+1))^{1/2}, \tag{88}$$

and $\tilde{\rho}_{\alpha,\beta,\gamma}$ and $\tilde{\rho}_{H_{u,d}}$ denote 1×1 and $g_H \times g_H$ unitary matrices for α, β, γ, and $H_{u,d}$. Then, the modular transformations of 3-point couplings Y^i_u and Y^j_d are obtained as

$$Y^i_u \to \tilde{J}_{1/2}(\tilde{\gamma},\tau)\tilde{\rho}_\alpha(\tilde{\gamma})\cdot\tilde{\rho}_\beta(\tilde{\gamma})\cdot(\tilde{\rho}_{H_u}(\tilde{\gamma})_{ik})^* Y^k_u, \tag{89}$$

$$Y^j_d \to \tilde{J}_{1/2}(\tilde{\gamma},\tau)\tilde{\rho}_\alpha(\tilde{\gamma})\cdot\tilde{\rho}_\gamma(\tilde{\gamma})\cdot(\tilde{\rho}_{H_d}(\tilde{\gamma})_{jk})^* Y^k_d, \tag{90}$$

and it follows from these that the μ matrix is transformed as

$$\mu^{ij}(\tau) \to \tilde{J}_{1/2}(\tilde{\gamma},\tau)^4\tilde{J}^*_{1/2}(\tilde{\gamma},\tau)^2[\tilde{\rho}_\alpha(\tilde{\gamma})]^2\tilde{\rho}_\beta(\tilde{\gamma})\tilde{\rho}_\gamma(\tilde{\gamma}) \\ \cdot(\tilde{\rho}_{H_u}(\tilde{\gamma})_{ii'})^*(\tilde{\rho}_{H_d}(\tilde{\gamma})_{jj'})^*\mu^{i'j'}(\tau). \tag{91}$$

At the modular fixed points $\tau = i$, ω, and $i\infty$, the leading mass term becomes invariant under S, ST, and T transformations respectively since $S: \tau = -1/\tau$, $ST: \tau = -1/(\tau+1)$ and $T: \tau = \tau+1$. That is, the μ^{ij} matrix obeys the following modular invariance relations:

$$\mu^{ij}(i) = (\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_{H_u}(\tilde{S})_{ii'})^*\cdot[\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\alpha(\tilde{S})]^2\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\beta(\tilde{S})\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\gamma(\tilde{S})\mu^{ij}(i)\cdot(\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_{H_d}(\tilde{S})_{j'j})^\dagger, \tag{92}$$

$$\mu^{ij}(\omega) = (\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_{H_u}(\widetilde{ST})_{ii'})^*\cdot[\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_\alpha(\widetilde{ST})]^2\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_\beta(\widetilde{ST})\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_\gamma(\widetilde{ST})\mu^{ij}(\omega) \\ \cdot(\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_{H_d}(\widetilde{ST})_{j'j})^\dagger, \tag{93}$$

$$\mu^{ij}(i\infty) = (\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_{H_u}(\tilde{T})_{ii'})^*\cdot[\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_\alpha(\tilde{T})]^2\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_\beta(\tilde{T})\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_\gamma(\tilde{T})\mu^{ij}(i\infty) \\ \cdot(\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_{H_d}(\tilde{T})_{j'j})^\dagger, \tag{94}$$

where we have used Eq. (88). These relations mean that the mass eigenbasis of the leading mass term at each modular fixed point $\tau = i$, ω, and $i\infty$ is also S, ST, and T transformation eigenbasis, respectively. Let us check this conclusion at $\tau = i$ as an example. Let us consider the simple case that $\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\alpha(\tilde{S}) = \tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\beta(\tilde{S}) = \tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_\gamma(\tilde{S}) = 1$ and $\mathrm{diag}(\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_{H_{u,d}}(\tilde{S})) = (1,-1)$, hence two pairs of Higgs fields. Then the relation Eq. (92) in S eigenbasis is given by

$$\begin{pmatrix}\mu^{00}(i) & \mu^{01}(i)\\ \mu^{10}(i) & \mu^{11}(i)\end{pmatrix} \\ = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mu^{00}(i) & \mu^{01}(i)\\ \mu^{10}(i) & \mu^{11}(i)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{95}$$

and obviously the leading mass matrix is diagonalized due to $\mu^{01}(i) = \mu^{10}(i) = 0$. In a similar way, we can show the leading mass eigenbasis at each modular fixed point is also eigenbasis of each residual symmetry of the modular transformation.

In general, there would exist some configurations giving a single instanton zero mode; therefore the leading mass term should be rewritten by the liner combination of them as

$$\mu^{ij}(\tau) = \sum_a d_a Y^{ia}_u Y^{ja}_d \equiv \sum_a c_a \mu^{ij}_a(\tau), \tag{96}$$

where a runs all possible instanton zero-mode configurations and Y^{ia}_u (Y^{ja}_d) denotes 3-point couplings among instanton zero modes α_a, β_a (γ_a) and Higgs fields H_u (H_d). Under modular transformation, μ^{ij}_a is transformed as Eq. (91) and obeys the same modular invariance relations in Eqs. (92)–(94). Hence, the general leading mass eigenbasis at each modular fixed point is also eigenbasis of corresponding modular transformation. Thus, at the leading order, Higgs VEVs which are aligned in the lightest mass direction at $\tau = i$, ω, and $i\infty$ must be eigenbasis of S, ST, and T transformations, respectively.

Unfortunately, on the magnetized $T^2/\mathbb{Z}_2$ orbifold models, we cannot find the leading order Higgs μ term being able to determine the lightest mass direction uniquely because of the shortage of number of instanton zero-mode configurations which couple to Higgs fields. In what follows, we assume that Higgs VEVs are aligned along eigenvectors of residual modular symmetry as the leading order although we do not know the full order μ-term structure. Such Higgs VEVs can be realized as long as the μ term transforms under modular transformation as

$$\sum_a c_a \mu_a^{ij} H_u^i H_d^j \xrightarrow{\gamma} \sum_a A(\tilde{\gamma},\tau,a)\cdot c_a \mu_a^{ij} H_u^i H_d^j, \tag{97}$$

where $A(\tilde{\gamma},\tau,a)$ means a modular symmetry anomaly on a μ term. In fact, the leading mass term, Eq. (96), is transformed as

$$\sum_a c_a \mu_a^{ij}(\tau) H_u^i H_d^j \xrightarrow{\gamma} \tilde{J}_{1/2}(\tilde{\gamma},\tau)^4 \sum_a [\tilde{\rho}_{\alpha_a}(\tilde{\gamma})]^2 \tilde{\rho}_{\beta_a}(\tilde{\gamma})\tilde{\rho}_{\gamma_a}(\tilde{\gamma})\cdot c_a \mu_a^{ij} H_u^i H_d^j, \tag{98}$$

under modular transformation since $H_{u,d}$ behaves as the modular form of weight $-1/2$:

$$H_{u,d}^i \xrightarrow{\gamma} \tilde{J}^*_{-1/2}(\tilde{\gamma},\tau)\tilde{\rho}_{H_{u,d}}(\tilde{\gamma})_{ij} H_{u,d}^j. \tag{99}$$

The modular transformation Eq. (97) ensures the modular invariances of μ_a^{ij} at the fixed points:

$$\mu_a^{ij}(i) = A(\tilde{S},\tau,a)(\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_{H_u}(\tilde{S})_{ii'})^*(\tilde{J}_{1/2}(\tilde{S},i)\tilde{\rho}_{H_d}(\tilde{S})_{jj'})^*\mu_a^{i'j'}(i), \tag{100}$$

$$\mu_a^{ij}(\omega) = A(\widetilde{ST},\tau,a)(\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_{H_u}(\widetilde{ST})_{ii'})^*(\tilde{J}_{1/2}(\widetilde{ST},\omega)\tilde{\rho}_{H_d}(\widetilde{ST})_{jj'})^*\mu_a^{i'j'}(\omega), \tag{101}$$

$$\mu_a^{ij}(i\infty) = A(\tilde{T},\tau,a)(\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_{H_u}(\tilde{T})_{ii'})^*(\tilde{J}_{1/2}(\tilde{T},i\infty)\tilde{\rho}_{H_d}(\tilde{T})_{jj'})^*\mu_a^{i'j'}(i\infty). \tag{102}$$

Thus, there is no mixing between Higgs modes with different eiganvalues of residual symmetry in the μ matrix as seen in Eq. (95). Therefore, the Higgs VEVs are aligned along eigenvectors of residual modular symmetry.

B. Classification of the modular symmetric models

In this subsection, we investigate the conditions to realize the Higgs VEVs which correspond to the eigenvectors of the residual modular transformation at the modular fixed points. Note that we ignore T-symmetric vacuum because the values of elements of Yukawa matrices at $\tau = i\infty$ are strictly restricted by T symmetry and it is difficult to realize realistic flavor observations. In addition, the fixed point, $\tau = i\infty$, corresponds the decompactification limit, and it is not valid from the viewpoint of four-dimensional effective theory.

Under S and T transformations, the complex coordinate on $T^2/\mathbb{Z}_2$, z, and the modulus, τ, are transformed as

$$(z,\tau) \xrightarrow{S} \left(-\frac{z}{\tau}, -\frac{1}{\tau}\right), \qquad (z,\tau) \xrightarrow{T} (z,\tau+1). \tag{103}$$

This gives the following S and T transformations of zero modes:

$$\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(z,\tau) \xrightarrow{S} \psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(S:(z,\tau)) = (-\tau)^{1/2}\rho(S)^{jk\alpha_1\alpha_1'\alpha_2\alpha_2'}\psi_{T^2/\mathbb{Z}_2^m}^{(k+\alpha_1,\alpha_2),M}(z,\tau), \tag{104}$$

$$\psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(z,\tau) \xrightarrow{T} \psi_{T^2/\mathbb{Z}_2^m}^{(j+\alpha_1,\alpha_2),M}(T:(z,\tau)) = \rho(T)^{jk\alpha_1\alpha_1'\alpha_2\alpha_2'}\psi_{T^2/\mathbb{Z}_2^m}^{(k+\alpha_1,\alpha_2),M}(z,\tau), \tag{105}$$

where

$$\rho(S)^{jk\alpha_1\alpha_1'\alpha_2\alpha_2'} = \begin{cases} \mathcal{N}^{(j+\alpha_1)}\mathcal{N}^{(k+\alpha_1')}\frac{4e^{\pi i/4}}{\sqrt{M}}\cos\left(\frac{2\pi(j+\alpha_1)(k+\alpha_1')}{M}\right)\delta_{(\alpha_2,\alpha_1),(\alpha_1',\alpha_2')} & (m=0), \\ \mathcal{N}^{(j+\alpha_1)}\mathcal{N}^{(k+\alpha_1')}\frac{4ie^{\pi i/4}}{\sqrt{M}}\sin\left(\frac{2\pi(j+\alpha_1)(k+\alpha_1')}{M}\right)\delta_{(\alpha_2,\alpha_1),(\alpha_1',\alpha_2')} & (m=1), \end{cases} \tag{106}$$

$$\rho(T)^{jk\alpha_1\alpha_1'\alpha_2\alpha_2'} = e^{\frac{\pi i(j+\alpha_1)^2}{M}}\delta_{j,k}\delta_{(\alpha_1,\alpha_2-\alpha_1+\frac{M}{2}),(\alpha_1',\alpha_2')}. \tag{107}$$

Obviously zero modes are mapped into ones with the same SS phases only if $\alpha_1 = \alpha_2$ under S transformation and only if $\alpha_1 = \alpha_2 = M/2$ (mod 1) under ST transformation. Therefore, modular symmetric Higgs VEVs at least have the following SS phases:

$$\begin{cases} (\alpha_1,\alpha_2) = (0,0)\ \text{or}\ (1/2,1/2) & \text{for } S\text{-symmetric vacuum}, \\ (\alpha_1,\alpha_2) = (M/2,M/2)\ (\text{mod } 1) & \text{for } ST\text{-symmetric vacuum}. \end{cases} \tag{108}$$

In each case, we can find Higgs fields which are eigenbasis of S and ST transformations, respectively. On the other hand, it is not clear whether the realistic flavor structure is realized in these vacua or not.

Next we study the condition to realize the direction $h_{u,d}^k = v_{u,d}(U_{H_{u,d}}^{(g_H-1)k}(p))^*$ which are eigenvectors of residual symmetries at modular fixed points. We have the possibilities of realizing realistic flavor structure by assuming the vicinity of such an eigenvector directions since fermion mass hierarchies can be realized near $h_{u,d}^k$ as described in the end of Sec. III B. The conditions for the modular eigenvectors $h_{u,d}^k$ are given by

$$\begin{cases} p = 0\ \text{or}\ \frac{1+i}{2} & \text{for } S\text{-symmetric vacuum}, \\ p = 0 & \text{for } ST\text{-symmetric vacuum}. \end{cases} \tag{109}$$

Let us prove the condition for S-symmetric vacuum. To make the direction $h_{u,d}^k = v_{u,d}(U_{H_{u,d}}^{(g_H-1)k}(p))^*$ S eigenstate at $\tau = i$, the nonvanishing redefined zero modes of Higgs fields defined in Eq. (55), $\hat{\psi}_{H_{u,d}}^{(g_H-1)}(z,\tau) = U_{H_{u,d}}^{(g_H-1)k}(p)\psi_{H_{u,d}}^k(z,\tau)$, must be eigenbasis of S transformation. We will check this by calculating S transformation of $\hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i)$ for $p \notin P_{H_{u,d}}$ and $\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i)$ for $p \in P_{H_{u,d}}$. Note that the redefined zero modes satisfy

$$\begin{cases} \hat{\psi}^{j\neq(g_H-1)}(p,\tau) = 0, \quad \hat{\psi}^{(g_H-1)}(p,\tau) \neq 0 & \text{for } p \notin P_{H_{u,d}}, \\ \frac{\partial}{\partial z}\hat{\psi}^{j\neq(g_H-1)}(p,\tau) = 0, \quad \frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}(p,\tau) \neq 0, \quad (\hat{\psi}^j(p,\tau) = 0) & \text{for } p \in P_{H_{u,d}}, \end{cases} \tag{110}$$

as defined in Eq. (55). For $p = 0$ or $\frac{1+i}{2}$ and $p \notin P_{H_{u,d}}$, S transformation of nonvanishing mode $\hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i)$ is given by

$$\begin{aligned} \hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i) \xrightarrow{S} \hat{\psi}_{H_{u,d}}^{(g_H-1)}(S:(p,i)) &= \begin{cases} \hat{\psi}_{H_{u,d}}^{(g_H-1)}(0,i) & \text{for } p = 0 \\ \hat{\psi}_{H_{u,d}}^{(g_H-1)}\left(\frac{-1+i}{2}, i\right) & \text{for } p = \frac{1+i}{2} \end{cases} \\ &= \begin{cases} \hat{\psi}_{H_{u,d}}^{(g_H-1)}(0,i) & \text{for } p = 0 \\ e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \hat{\psi}_{H_{u,d}}^{(g_H-1)}\left(\frac{1+i}{2}, i\right) & \text{for } p = \frac{1+i}{2} \end{cases}, \end{aligned} \tag{111}$$

from the boundary condition in Eq. (2). Similarly, for $p = 0$ or $\frac{1+i}{2}$ and $p \in P_{H_{u,d}}$, S transformation of $\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i)$ is given by

$$\begin{aligned} \frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(p,i) \xrightarrow{S} (-i)\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(S:(p,i)) &= \begin{cases} (-i)\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(0,i) & \text{for } p = 0 \\ (-i)\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}\left(\frac{-1+i}{2}, i\right) & \text{for } p = \frac{1+i}{2} \end{cases} \\ &= \begin{cases} (-i)\frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}(0,i) & \text{for } p = 0 \\ (-i)e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \frac{\partial}{\partial z}\hat{\psi}_{H_{u,d}}^{(g_H-1)}\left(\frac{1+i}{2}, i\right) & \text{for } p = \frac{1+i}{2} \end{cases}, \end{aligned} \tag{112}$$

from the boundary condition in Eq. (18). As shown in Eq. (104), the transformation law is independent of z; therefore the same relations consist on $z \neq p$,

$$\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) \xrightarrow{S} \begin{cases} \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=0 \notin P_{H_{u,d}} \\ e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=\frac{1+i}{2} \notin P_{H_{u,d}} \end{cases}, \tag{113}$$

$$\frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) \xrightarrow{S} \begin{cases} (-i)\frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=0 \in P_{H_{u,d}} \\ (-i)e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=\frac{1+i}{2} \in P_{H_{u,d}} \end{cases}. \tag{114}$$

Using $\frac{\partial}{\partial z}\xrightarrow{S}(-i)\frac{\partial}{\partial z}$, finally we obtain

$$\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) \xrightarrow{S} \begin{cases} \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=0 \notin P_{H_{u,d}} \\ e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=\frac{1+i}{2} \notin P_{H_{u,d}} \\ \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=0 \in P_{H_{u,d}} \\ e^{-2\pi i\alpha_{1H_{u,d}}} e^{-\pi i\frac{M_{H_{u,d}}}{2}} \hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i) & \text{for } p=\frac{1+i}{2} \in P_{H_{u,d}} \end{cases}. \tag{115}$$

This means that $\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i)$ with $p=0$ or $\frac{1+i}{2}$ becomes eigenbasis of S transformation. This is because $z=0$ and $\frac{1+i}{2}$ are invariant under S transformation up to lattice translations of torus.

On the other hand, the cases $p=\frac{1}{2}$ and $\frac{i}{2}$ are complicated. The boundary conditions Eqs. (13) and (18) may give the relations

$$\hat{\psi}^{(g_H-1)}_{H_{u,d}}\left(\frac{1}{2},i\right) \propto \hat{\psi}^{(g_H-1)}_{H_{u,d}}\left(\frac{i}{2},i\right),$$
$$\frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}_{H_{u,d}}\left(\frac{1}{2},i\right) \propto \frac{\partial}{\partial z}\hat{\psi}^{(g_H-1)}_{H_{u,d}}\left(\frac{i}{2},i\right) \tag{116}$$

in certain patterns of flux, SS phases and $\mathbb{Z}_2$ parity. If these exist, $\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i)$ with $p=\frac{1}{2}$ or $\frac{i}{2}$ can be eigenbasis of S transformation since $S: z = S: \frac{1}{2} = \frac{i}{2}$ at $\tau = i$. However it is unclear and difficult to show whether the relations in Eq. (116) exist or not. Instead, we directly calculate whether the directions $\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i)$ in each model is eigenbasis of S transformation or not by using Eq. (104). As a result, there are no models where $\hat{\psi}^{(g_H-1)}_{H_{u,d}}(z,i)$ with $p=\frac{1}{2}$ or $\frac{i}{2}$ is S eigenbasis. Thus, the direction $h^k_{u,d} = v_{u,d}(U^{(g_H-1)k}_{H_{u,d}}(p))^*$ with S-invariant points $p=0$ and $\frac{1+i}{2}$ is S-symmetric vacuum.

In a similar way, we can check the condition for ST-symmetric vacuum in Eq. (109). The direction $h^k_{u,d}$ with ST-invariant point $p=0$ is ST eigenstate but other points $p \in P_F$ which are not ST invariant lead to not ST-symmetric vacuum.

Now, we are ready to classify the flavor models whose $h^k_{u,d}$ is modular symmetric. The conditions are Eqs. (108) and (109). As a result, we cannot find the flavor models satisfying conditions for ST-symmetric vacuum but we can find models for S-symmetric vacuum. The results for S-symmetric vacuum are shown in Table IX. There are 24 flavor models in total.

V. NUMERICAL EXAMPLE

In this section, we study a flavor model shown in Table X which can be realistic in the vicinity of S eigenvector, and derive a realistic quark and lepton flavor structure. In this model, quark doublets Q have (flux, $\mathbb{Z}_2$ parity, SS phases α_1, α_2) $= (6,0,0,\frac{1}{2})$; right-handed up-sector quarks u_R have $(5,0,0,\frac{1}{2})$; right-handed down-sector quarks d_R have $(6,0,\frac{1}{2},0)$; lepton doublets L have $(6,0,\frac{1}{2},0)$; right-handed neutrinos ν_R have $(5,0,\frac{1}{2},0)$; right-handed charged leptons e_R have $(6,0,0,\frac{1}{2})$; up-type Higgs fields H_u have (11,0,0,0); down-type Higgs fields H_d have $(12,0,\frac{1}{2},\frac{1}{2})$. The number of both up- and down-types Higgs fields are six.

Yukawa couplings Y^{ijk}_u, Y^{ijk}_d, Y^{ijk}_ν, and Y^{ijk}_e appearing in this model are summarized in Appendix C 1; the Majorana mass matrix of the right-handed neutrinos induced by D-brane instanton effect is shown in Appendix C 2.

TABLE IX. All quark and lepton flavor models satisfying S-symmetric vacuum conditions in Eqs. (108) and (109). The first to eighth rows show the flux M, $\mathbb{Z}_2$ parity m (even, odd = 0, 1), and SS phases (α_1, α_2) of the zero modes of the fields. g_H denotes the number of Higgs fields.

B_Q	B_{u_R}	B_{d_R}	B_L	B_{ν_R}	B_{e_R}	B_{H_u}	B_{H_d}	g_H	p
5,0,0,0	$7,0,\frac{1}{2},\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$5,0,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	$11,1,\frac{1}{2},\frac{1}{2}$	6	0
5,0,0,0	$7,0,\frac{1}{2},\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$5,0,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$11,1,\frac{1}{2},\frac{1}{2}$	6	0
$5,0,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	$5,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	11,0,0,0	$11,1,\frac{1}{2},\frac{1}{2}$	6	0
$5,0,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	$5,1,\frac{1}{2},\frac{1}{2}$	$11,1,\frac{1}{2},\frac{1}{2}$	11,0,0,0	6	$\frac{1+i}{2}$
$5,0,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,1,\frac{1}{2},\frac{1}{2}$	$5,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	11,0,0,0	$11,1,\frac{1}{2},\frac{1}{2}$	6	0
$5,0,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	$5,1,\frac{1}{2},\frac{1}{2}$	$11,1,\frac{1}{2},\frac{1}{2}$	11,0,0,0	6	$\frac{1+i}{2}$
$5,1,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$6,1,\frac{1}{2},\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$5,0,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	11,0,0,0	6	$\frac{1+i}{2}$
$5,1,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$6,1,\frac{1}{2},\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$5,0,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	11,0,0,0	6	$\frac{1+i}{2}$
$6,0,\frac{1}{2},0$	$5,0,\frac{1}{2},0$	$7,1,\frac{1}{2},0$	$6,1,\frac{1}{2},\frac{1}{2}$	$5,1,\frac{1}{2},\frac{1}{2}$	$7,0,\frac{1}{2},\frac{1}{2}$	11,0,0,0	13,1,0,0	6	0
$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	0
$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	0
$6,0,0,\frac{1}{2}$	$5,0,0,\frac{1}{2}$	$7,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	$5,1,\frac{1}{2},\frac{1}{2}$	$7,0,\frac{1}{2},\frac{1}{2}$	11,0,0,0	13,1,0,0	6	0
$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	0
$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	0
$6,0,\frac{1}{2},\frac{1}{2}$	$7,0,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$7,1,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	13,0,0,0	$13,1,\frac{1}{2},\frac{1}{2}$	7	0
$6,0,\frac{1}{2},\frac{1}{2}$	$7,0,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$7,1,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	13,0,0,0	$13,1,\frac{1}{2},\frac{1}{2}$	7	0
$6,0,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$7,0,\frac{1}{2},\frac{1}{2}$	$7,1,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$13,1,\frac{1}{2},\frac{1}{2}$	13,0,0,0	7	$\frac{1+i}{2}$
$6,0,\frac{1}{2},\frac{1}{2}$	7,1,0,0	$7,0,\frac{1}{2},\frac{1}{2}$	$7,1,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$13,1,\frac{1}{2},\frac{1}{2}$	13,0,0,0	7	$\frac{1+i}{2}$
$6,1,\frac{1}{2},0$	$5,0,0,\frac{1}{2}$	$7,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	7,1,0,0	$11,1,\frac{1}{2},\frac{1}{2}$	$13,0,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$
$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$
$6,1,\frac{1}{2},0$	$6,1,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$
$6,1,0,\frac{1}{2}$	$5,0,\frac{1}{2},0$	$7,1,\frac{1}{2},0$	$6,1,\frac{1}{2},\frac{1}{2}$	5,0,0,0	7,1,0,0	$11,1,\frac{1}{2},\frac{1}{2}$	$13,0,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$
$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$
$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,0,\frac{1}{2},0$	$6,1,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6	$\frac{1+i}{2}$

In our numerical study, we fix the value of modulus by $\tau = i$ and use Higgs VEV directions as parameters. Higgs VEV directions satisfying conditions I–IV, $h^k_{u,d}$, in this model are given by

$$h_u^k = v_u(0.8464, 0.5014, 0.1759, 0.03657, 0.004504, 0.0003144), \tag{117}$$

$$h_d^k = v_d(0.4330, 0.7696, 0.4501, 0.1310, 0.02074, 0.001945), \tag{118}$$

where h_u^k and h_d^k are S-eigenbasis directions with eigenvalues $+1$ and $+i$, respectively. Thus the modulus is S-symmetric vacuum, while these Higgs VEV directions correspond to S eigenstates. First, we try to realize flavor observations in exact S-eigenstate directions in the Higgs VEV directions. Six pairs of up- (down-)type Higgs fields include three S eigenstates with eigenvalue $+1$ $(+i)$ in total. We use these three eigenstates as parameters for up- and down-type Higgs VEVs, respectively. To obtain realistic flavors, let us choose the following Higgs VEV directions:

TABLE X. Flux, $\mathbb{Z}_2$ parity (even, odd = 0, 1), SS phases (α_1, α_2) of quarks, leptons, and Higgs fields in the model. g_H denotes the number of Higgs fields.

B_Q	B_{u_R}	B_{d_R}	B_L	B_{ν_R}	B_{e_R}	B_{H_u}	B_{H_d}	g_H
$6,0,\frac{1}{2},0$	$6,0,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,0,\frac{1}{2}$	$6,1,\frac{1}{2},0$	$6,0,\frac{1}{2},0$	$12,0,\frac{1}{2},\frac{1}{2}$	$12,1,\frac{1}{2},\frac{1}{2}$	6

$$\langle H_u^k \rangle = v_u(0.8466, 0.5009, 0.1762, 0.03715, 0.004794, 0.0003797), \tag{119}$$

$$\langle H_d^k \rangle = v_d(0.5006, 0.7890, 0.3521, 0.05382, -0.003787, -0.003709). \tag{120}$$

We note that again these directions are eigenbasis of S transformation. They lead to the following up quark, down quark, and charged lepton mass ratios:

$$(m_u, m_c, m_t)/m_t = (2.96 \times 10^{-5}, 5.35 \times 10^{-4}, 1), \tag{121}$$

$$(m_d, m_s, m_b)/m_b = (4.36 \times 10^{-4}, 1.17 \times 10^{-2}, 1), \tag{122}$$

$$(m_e, m_\mu, m_\tau)/m_\tau = (4.36 \times 10^{-4}, 1.17 \times 10^{-2}, 1), \tag{123}$$

and a ratio of the differences of the squares of the neutrino masses,

$$\sqrt{\frac{\Delta m_{\nu 12}^2}{\Delta m_{\nu 13}^2}} = \sqrt{\frac{|m_{\nu_1}^2 - m_{\nu_2}^2|}{|m_{\nu_1} - m_{\nu_3}^2|}} = 0.179, \tag{124}$$

for normal ordering (NO), $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$. Also the absolute values of the CKM matrix, $|V_{\mathrm{CKM}}|$, and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, $|V_{\mathrm{PMNS}}|$, are obtained as follows:

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.972 & 0.235 & 0.00134 \\ 0.233 & 0.964 & 0.126 \\ 0.0309 & 0.122 & 0.992 \end{pmatrix},$$
$$|V_{\mathrm{PMNS}}| = \begin{pmatrix} 0.990 & 0.137 & 0.0134 \\ 0.129 & 0.957 & 0.261 \\ 0.0487 & 0.257 & 0.965 \end{pmatrix}. \tag{125}$$

The mass ratios of quarks and leptons, and the absolute values of the CKM matrix are roughly realized, but the absolute values of the PMNS matrix are not realistic. As a result, in this model it is difficult to realize both quark and lepton flavors in the exact S eigenvectors of the Higgs VEV directions.

Next, we consider the vicinity of above S eigenvector of Higgs VEV directions. We use all six pairs of Higgs VEVs as parameters for both up and down types but fix the modulus at $\tau = i$ to simplify the analysis. To obtain realistic flavors, in the vicinity of $h_{u,d}^k$, we have chosen the following Higgs VEV directions:

$$\langle H_u^k \rangle = v_u(0.8509, 0.4970, 0.1679, 0.02805, -0.006762, -0.003731), \tag{126}$$

$$\langle H_d^k \rangle = v_d(0.4340, 0.7688, 0.4499, 0.1283, 0.02538, 0.03302). \tag{127}$$

The norm of h_u^k in $\langle H_u^k \rangle$ is 0.9998 and one of h_d^k in $\langle H_d^k \rangle$ is 0.9995. In these directions, the mass matrices for quarks and leptons are given by

$$M_u/m_t = \begin{pmatrix} 0.7202 & 0.5992 & 0.1214 \\ 0.2492 & 0.2063 & 0.03922 \\ 0.03057 & 0.02249 & -0.002550 \end{pmatrix}, \qquad M_d/m_b = \begin{pmatrix} 0.8675 & 0.3620 & 0.05514 \\ 0.3053 & 0.1303 & 0.02287 \\ 0.03861 & 0.03580 & 0.03967 \end{pmatrix}, \tag{128}$$

$$M_\nu/m_{\nu_3} = \begin{pmatrix} -0.3614 & -0.09456 & -0.3323 \\ -0.09456 & -0.1345 & -0.4077 \\ -0.3323 & -0.4077 & -0.5819 \end{pmatrix}, \qquad M_e/m_\tau = \begin{pmatrix} 0.8675 & 0.3053 & 0.03861 \\ 0.3620 & 0.1303 & 0.03580 \\ 0.05514 & 0.02287 & 0.03967 \end{pmatrix}. \tag{129}$$

Then they lead to the following up quark, down quark, and charged lepton mass ratios:

$$(m_u, m_c, m_t)/m_t = (3.13 \times 10^{-5}, 8.14 \times 10^{-3}, 1), \tag{130}$$

$$(m_d, m_s, m_b)/m_b = (8.46 \times 10^{-4}, 4.10 \times 10^{-2}, 1), \tag{131}$$

$$(m_e, m_\mu, m_\tau)/m_\tau = (8.46 \times 10^{-4}, 4.10 \times 10^{-2}, 1), \tag{132}$$

and a ratio of the differences of the squares of the neutrino masses,

TABLE XI. The mass ratios of the quarks and leptons, and the absolute values of the CKM matrix and the PMNS matrix elements at $\tau = i$ under the vacuum alignments of Higgs fields in Eqs. (126) and (127). Reference values of mass ratios are shown in Ref. [79]. Those of the CKM matrix and PMNS matrix elements are shown in Refs. [80,81].

	Obtained values	Reference values
$(m_u, m_c, m_t)/m_t$	$(3.13\times10^{-5}, 8.14\times10^{-3}, 1)$	$(5.58\times10^{-6}, 2.69\times10^{-3}, 1)$
$(m_d, m_s, m_b)/m_b$	$(8.46\times10^{-4}, 4.10\times10^{-2}, 1)$	$(6.86\times10^{-4}, 1.37\times10^{-2}, 1)$
$\lvert V_{\rm CKM}\rvert$	$\begin{pmatrix} 0.973 & 0.232 & 0.00234 \\ 0.232 & 0.973 & 0.0162 \\ 0.00603 & 0.0152 & 1.00 \end{pmatrix}$	$\begin{pmatrix} 0.974 & 0.227 & 0.00361 \\ 0.226 & 0.973 & 0.0405 \\ 0.00854 & 0.0398 & 0.999 \end{pmatrix}$
$\sqrt{\Delta m^2_{\nu 12}/\Delta m^2_{\nu 13}}$	0.162 (NO)	0.173
$(m_e, m_\mu, m_\tau)/m_\tau$	$(8.46\times10^{-4}, 4.10\times10^{-2}, 1)$	$(2.78\times10^{-4}, 5.88\times10^{-2}, 1)$
$\lvert V_{\rm PMNS}\rvert$	$\begin{pmatrix} 0.841 & 0.522 & 0.147 \\ 0.246 & 0.608 & 0.755 \\ 0.483 & 0.598 & 0.639 \end{pmatrix}$	$\begin{pmatrix} 0.801-0.845 & 0.513-0.579 & 0.143-0.156 \\ 0.232-0.507 & 0.459-0.694 & 0.629-0.779 \\ 0.260-0.526 & 0.470-0.702 & 0.609-0.763 \end{pmatrix}$

$$\sqrt{\frac{\Delta m^2_{\nu 12}}{\Delta m^2_{\nu 13}}} = \sqrt{\frac{|m^2_{\nu_1} - m^2_{\nu_2}|}{|m_{\nu_1} - m^2_{\nu_3}|}} = 0.162 \qquad (133)$$

for NO. For inverted ordering (IO), $m_{\nu_3} < m_{\nu_1} < m_{\nu_2}$, it is difficult to realize realistic the flavor structure. Also the absolute values of the CKM matrix, $|V_{\rm CKM}|$, and the PMNS matrix, $|V_{\rm PMNS}|$, are obtained as follows:

$$|V_{\rm CKM}| = \begin{pmatrix} 0.973 & 0.232 & 0.00234 \\ 0.232 & 0.973 & 0.0162 \\ 0.00603 & 0.0152 & 1.00 \end{pmatrix},$$

$$|V_{\rm PMNS}| = \begin{pmatrix} 0.841 & 0.522 & 0.147 \\ 0.246 & 0.608 & 0.755 \\ 0.483 & 0.598 & 0.639 \end{pmatrix}. \qquad (134)$$

The results are summarized in Table XI. As a result, in this model we could realize quark and lepton flavor structure in the vicinity of the S eigenvector of Higgs VEV direction.

VI. CONCLUSION

In this paper, we have investigated the conditions to realize the quark and lepton flavor structure in magnetized orbifold models. We have found four conditions I, II, III, and IV. The condition I demands the directions of up type Higgs VEVs h_u^k leading to rank one mass matrix for up quark to realize its mass hierarchy. The condition II demands the directions of down-type Higgs VEVs h_d^k leading to rank one mass matrices for both down-sector quarks and charged leptons to realize their mass hierarchies. The condition III demands the equivalence between u_L^u and u_L^d which are unitary matrices diagonalizing rank one mass matrices to realize small quark mixing. The condition IV demands that h_u^k is also the direction leading to vanishing neutrino Dirac mass matrix to realize not small lepton mixing. Note that the rank one mass matrices are favorable in the limit that we neglect masses of the first and second generations. Through zero points analysis for zero modes of each field, we could check whether the flavor models can satisfy these four conditions or not. Consequently we have found the 408 flavor models which are consistent with the conditions I–IV. In such models it is possible to realize the large hierarchy of up quark, down quark, and charged lepton masses and realistic mixings of quark and lepton in the vicinity of $h_{u,d}^k$.

Also we have classified the flavor models which can be realistic in the vicinity of specific points under S symmetry, where VEV of modulus lies on the fixed point of S transformation, $\tau = i$, and Higgs VEVs are aligned in eigenbasis of S transformation. Indeed Higgs VEVs led by the leading μ term generated by D-brane instanton effects at $\tau = i$ are generally aligned in eigenbasis of S transformation. In this paper, we have classified the flavor models whose $h_{u,d}^k$ becomes eigenbasis of S transformation. As a result we have found 24 flavor models, and they have the possibilities to realize realistic flavor observations in the vicinity of S eigenvector of Higgs VEV direction.

Here, we have given numerical studies on the model shown in Table X in the exact and the vicinity of S eigenvector of Higgs VEV direction. In the exact S-eigenvector direction, we could roughly realize the values of the quark and lepton mass ratios and the CKM matrix but the PMNS matrix was not realistic. In the vicinity of S eigenvector of Higgs VEV direction, we could realize the values of quark and lepton mass ratios as well as the CKM and PMNS matrices.

Similar classifications through the zero-point analysis can be applied for the flavor models in other orbifold models such as $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$, and $T^2/\mathbb{Z}_6$. It would be possible for magnetized T^4 and its orbifold models. Also we need to study Higgs μ term through D-brane instanton effects to check the direction of Higgs VEVs. We would study them and examine the possibilities of realization of quark and lepton flavor structure elsewhere.

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APPENDIX A: MAJORANA NEUTRINO MASSES BY D-BRANE INSTANTON EFFECTS

Here we give a brief review of Majorana neutrino mass terms induced by D-brane instanton effects [71–75].

We consider two stacks of D-branes, D_{N_1} and D_{N_2}, and assume D-brane instanton $D_{\rm inst}$ with magnetic fluxes. Right-handed neutrinos ν_R appear as zero modes of open strings between D_{N_1} and D_{N_2}; instanton zero modes β (γ) appear between D_{N_1} (D_{N_2}) and $D_{\rm inst}$. Then D-brane instanton effects give the following term:

$$\int d^2\beta d^2\gamma e^{-({\rm Im}\tau)^{-1/2} d_a^{ij}\beta^i\gamma^j\nu_R^a}, \tag{A1}$$

where β and γ are Grasmannian, and d_a^{ij} are the 3-point couplings among β, γ, and ν_R given by

$$d_a^{ij} = g({\rm Im}\tau)^{1/2}\int d^2z\psi_\beta^i(z)\cdot\psi_\gamma^j(z)\cdot(\psi_{\nu_R}^a(z))^*, \tag{A2}$$

where ψs are the zero-mode wave functions on $T^2/\mathbb{Z}_2$ corresponding to instanton zero modes β, γ and right-handed neutrinos ν_R. Mass terms can be generated only if each of β and γ has two zero modes. By Grasmannian integral, we obtain Majorana mass term,

$$\begin{aligned}&\Lambda e^{-S_{\rm inst}}\int d^2\beta d^2\gamma e^{-({\rm Im}\tau)^{-1/2} d_a^{ij}\beta^i\gamma^j\nu_R^a}\\ &\quad = \Lambda e^{-S_{\rm inst}}({\rm Im}\tau)^{-1}\varepsilon_{ij}\varepsilon_{k\ell}d_a^{ik}d_b^{j\ell}\nu_R^a\nu_R^b = M_{RR}^{ab}\nu_R^a\nu_R^b,\end{aligned} \tag{A3}$$

where $S_{\rm inst}$ denotes the instanton action and Λ denotes a typical sale as the compactification scale. The possible instanton zero-modes configurations are given by the following nonvanishing conditions for 3-point couplings:

$$\begin{aligned}&M_\beta \pm M_\gamma = \pm M_{\nu_R}, \qquad m_\beta + m_\gamma = m_{\nu_R},\\ &\quad(\alpha_1,\alpha_2)_\beta + (\alpha_1,\alpha_2)_\gamma = (\alpha_1,\alpha_2)_{\nu_R},\end{aligned} \tag{A4}$$

where M_f, m_f and $(\alpha_1,\alpha_2)_f$, $f=\beta,\gamma,\nu_R$ denote the magnetic fluxes, $\mathbb{Z}_2$ twist parities, and SS phases for zero modes of β, γ, and ν_R.

APPENDIX B: HIGGS μ TERM BY D-BRANE INSTANTON EFFECTS

Here we give a brief review of Higgs μ terms induced by D-brane instanton effects [71–75].

We consider three stacks of D-branes, D_a, D_b, and D_c with magnetic fluxes. The D-brane D_b is parallel to D_c. Up- (down-)type Higgs fields, H_u (H_d), appear as zero modes of open strings between D_a and D_b (D_c). To generate μ terms, we also assume the D-brane instanton $D_{\rm inst}$ with magnetic flux which has a single zero mode with each of the other branes. The instanton zero modes α, β, and γ appear as zero modes of open strings between D_a and $D_{\rm inst}$, D_b and $D_{\rm inst}$, and D_c and $D_{\rm inst}$. Then D-brane instanton effects give the following term:

$$\int d^2\alpha d\beta d\gamma e^{({\rm Im}\tau)^{-1/2}(Y_u^i\alpha\cdot H_u^i\beta + Y_d^j\alpha\cdot H_d^j\gamma)}, \tag{B1}$$

where α, β, and γ are Grasmannian and Y_u^i (Y_d^j) are the 3-point couplings among α, β (γ), and H_u^i (H_d^j) given by

$$\begin{aligned}Y_u^i &= g({\rm Im}\tau)^{1/2}\int d^2z\psi_\alpha(z)\cdot\psi_\beta(z)\cdot(\psi_{H_u}^i(z))^*,\\ Y_d^j &= g({\rm Im}\tau)^{1/2}\int d^2z\psi_\alpha(z)\cdot\psi_\gamma(z)\cdot(\psi_{H_d}^j(z))^*,\end{aligned} \tag{B2}$$

where ψs are the zero-mode wave functions on $T^2/\mathbb{Z}_2$ corresponding to instanton zero modes α, $\beta(\gamma)$ and Higgs fields H_u (H_d). Mass terms can be generated only if each of α, β, and γ has a single zero mode. By Grasmannian integral, we obtain the Higgs μ term,

$$\begin{aligned}&\Lambda e^{-S_{\rm inst}}\int d^2\alpha d\beta d\gamma e^{({\rm Im}\tau)^{-1/2}(Y_u^i\alpha\cdot H_u^i + Y_d^j\alpha\cdot H_d^j\gamma)}\\ &\quad = \Lambda e^{-S_{\rm inst}}({\rm Im}\tau)^{-1}(Y_u^iY_d^j)\varepsilon_{nm}H_{um}^iH_{dn}^j = \mu^{ij}\varepsilon_{nm}H_{um}^iH_{dn}^j,\end{aligned} \tag{B3}$$

where $m,n\in\{1,2\}$ denote components of $SU(2)_L$ doublet. The possible instanton zero-mode configurations are given by the following nonvanishing conditions for 3-point couplings:

$$\begin{aligned}&M_\alpha \pm M_\beta = \pm M_{H_u},\\ &m_\alpha + m_\beta = m_{H_u}, (\alpha_1,\alpha_2)_\alpha + (\alpha_1,\alpha_2)_\beta = (\alpha_1,\alpha_2)_{H_u},\end{aligned} \tag{B4}$$

$$\begin{aligned}&M_\alpha \pm M_\gamma = \pm M_{H_d},\\ &m_\alpha + m_\gamma = m_{H_d}, (\alpha_1,\alpha_2)_\alpha + (\alpha_1,\alpha_2)_\gamma = (\alpha_1,\alpha_2)_{H_d},\end{aligned} \tag{B5}$$

where M_f, m_f and $(\alpha_1, \alpha_2)_f$, $f = \alpha, \beta, \gamma, H_u, H_d$, denote the magnetic fluxes, $\mathbb{Z}_2$ twist parities, and SS phases for zero modes of α, β, γ, H_u, and H_d.

APPENDIX C: THE MODEL IN THE NUMERICAL EXAMPLE

1. Yukawa couplings

Here we summarize Yukawa couplings of up-sector quarks, down-sector quarks, neutrinos, and charged leptons, Y_u^{ijk}, Y_d^{ijk}, Y_ν^{ijk}, and Y_e^{ijk} in our model.

a. Up quark: $B_Q - B_{u_R} - B_{H_u} = (6,0,\frac{1}{2},0) - (6,0,0,\frac{1}{2}) - (12,0,\frac{1}{2},\frac{1}{2})$

Table XII shows the zero-mode assignments for quark doublets Q^i, right-handed up-sector quarks u_R^j, and up-type Higgs fields H_u^k. Yukawa couplings are given by

$$Y_u^{ijk} H_u^k = Y_u^{ij0} H_u^0 + Y_u^{ij1} H_u^1 + Y_u^{ij2} H_u^2 + Y_u^{ij3} H_u^3 + Y_u^{ij4} H_u^4 + Y_u^{ij5} H_u^5, \tag{C1}$$

with

$$Y_u^{ij0} = c_{(6-6-12)} \begin{pmatrix} X_0 & \frac{1}{\sqrt{2}}X_1 & 0 \\ 0 & \frac{1}{\sqrt{2}}X_2 & \frac{1}{\sqrt{2}}X_3 \\ 0 & 0 & \frac{1}{\sqrt{2}}X_4 \end{pmatrix}, \qquad Y_u^{ij1} = c_{(6-6-12)} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}X_0 & \frac{1}{\sqrt{2}}X_2 \\ X_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}X_3 & \frac{1}{\sqrt{2}}X_5 \end{pmatrix},$$

$$Y_u^{ij2} = c_{(6-6-12)} \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}X_1 \\ 0 & \frac{1}{\sqrt{2}}X_0 & -\frac{1}{\sqrt{2}}X_5 \\ X_2 & \frac{1}{\sqrt{2}}X_4 & 0 \end{pmatrix}, \qquad Y_u^{ij3} = c_{(6-6-12)} \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}}X_4 \\ 0 & \frac{1}{\sqrt{2}}X_5 & \frac{1}{\sqrt{2}}X_0 \\ X_3 & \frac{1}{\sqrt{2}}X_1 & 0 \end{pmatrix},$$

$$Y_u^{ij4} = c_{(6-6-12)} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}X_5 & -\frac{1}{\sqrt{2}}X_3 \\ X_4 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}X_2 & \frac{1}{\sqrt{2}}X_0 \end{pmatrix}, \qquad Y_u^{ij5} = c_{(6-6-12)} \begin{pmatrix} X_5 & -\frac{1}{\sqrt{2}}X_4 & 0 \\ 0 & \frac{1}{\sqrt{2}}X_3 & -\frac{1}{\sqrt{2}}X_2 \\ 0 & 0 & \frac{1}{\sqrt{2}}X_1 \end{pmatrix},$$

where

$$X_N \equiv \sum_{n=0}^{5} (-1)^n \eta_{6(N+1/2)+72n}, \qquad \eta_N \equiv \vartheta \begin{bmatrix} \frac{N}{432} \\ 0 \end{bmatrix} (0, 432\tau). \tag{C2}$$

b. Down quark: $B_Q - B_{d_R} - B_{H_d} = (6,0,\frac{1}{2},0) - (6,1,0,\frac{1}{2}) - (12,1,\frac{1}{2},\frac{1}{2})$

Table XIII shows the zero-mode assignments for quark doublets Q^i, right-handed down-sector quarks d_R^j, and down-type Higgs fields H_d^k. Yukawa couplings are given by

$$Y_d^{ijk} H_d^k = Y_d^{ij0} H_d^0 + Y_d^{ij1} H_d^1 + Y_d^{ij2} H_d^2 + Y_d^{ij3} H_d^3 + Y_d^{ij4} H_d^4 + Y_d^{ij5} H_d^5, \tag{C3}$$

with

TABLE XII. Zero-mode wave functions in "$(6,0,\frac{1}{2},0) - (6,0,0,\frac{1}{2}) - (12,0,\frac{1}{2},\frac{1}{2})$" model.

	Q^i	u_R^j	H_u^k
0	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,0),6} + \psi_{T^2}^{(11/2,0),6})$	$\psi_{T^2}^{(0,1/2),6}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,1/2),12} - \psi_{T^2}^{(23/2,1/2),12})$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,0),6} + \psi_{T^2}^{(9/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1,1/2),6} - \psi_{T^2}^{(5,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,1/2),12} - \psi_{T^2}^{(21/2,1/2),12})$
2	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,0),6} + \psi_{T^2}^{(7/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(2,1/2),6} - \psi_{T^2}^{(4,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,1/2),12} - \psi_{T^2}^{(19/2,1/2),12})$
3			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(7/2,1/2),12} - \psi_{T^2}^{(17/2,1/2),12})$
4			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(9/2,1/2),12} - \psi_{T^2}^{(15/2,1/2),12})$
5			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(11/2,1/2),12} - \psi_{T^2}^{(13/2,1/2),12})$

TABLE XIII. Zero-mode wave functions in "$(6,0,\frac{1}{2},0)-(6,1,0,\frac{1}{2})-(12,1,\frac{1}{2},\frac{1}{2})$" model.

	Q^i	d_R^j	H_d^k
0	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,0),6}+\psi_{T^2}^{(11/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1,1/2),6}+\psi_{T^2}^{(5,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,1/2),12}+\psi_{T^2}^{(23/2,1/2),12})$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,0),6}+\psi_{T^2}^{(9/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(2,1/2),6}+\psi_{T^2}^{(4,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,1/2),12}+\psi_{T^2}^{(21/2,1/2),12})$
2	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,0),6}+\psi_{T^2}^{(7/2,0),6})$	$\psi_{T^2}^{(3,1/2),6}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,1/2),12}+\psi_{T^2}^{(19/2,1/2),12})$
3			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(7/2,1/2),12}+\psi_{T^2}^{(17/2,1/2),12})$
4			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(9/2,1/2),12}+\psi_{T^2}^{(15/2,1/2),12})$
5			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(11/2,1/2),12}+\psi_{T^2}^{(13/2,1/2),12})$

$$Y_d^{ij0}=c_{(6-6-12)}\begin{pmatrix}\frac{1}{\sqrt{2}}X_1 & 0 & 0\\ -\frac{1}{\sqrt{2}}X_2 & \frac{1}{\sqrt{2}}X_3 & 0\\ 0 & -\frac{1}{\sqrt{2}}X_4 & X_5\end{pmatrix},\qquad Y_d^{ij1}=c_{(6-6-12)}\begin{pmatrix}\frac{1}{\sqrt{2}}X_0 & \frac{1}{\sqrt{2}}X_2 & 0\\ 0 & 0 & X_4\\ -\frac{1}{\sqrt{2}}X_3 & -\frac{1}{\sqrt{2}}X_5 & 0\end{pmatrix},$$

$$Y_d^{ij2}=c_{(6-6-12)}\begin{pmatrix}0 & \frac{1}{\sqrt{2}}X_1 & X_3\\ \frac{1}{\sqrt{2}}X_0 & \frac{1}{\sqrt{2}}X_5 & 0\\ -\frac{1}{\sqrt{2}}X_4 & 0 & 0\end{pmatrix},\qquad Y_d^{ij3}=c_{(6-6-12)}\begin{pmatrix}0 & \frac{1}{\sqrt{2}}X_4 & X_2\\ -\frac{1}{\sqrt{2}}X_5 & \frac{1}{\sqrt{2}}X_0 & 0\\ \frac{1}{\sqrt{2}}X_1 & 0 & 0\end{pmatrix},$$

$$Y_d^{ij4}=c_{(6-6-12)}\begin{pmatrix}\frac{1}{\sqrt{2}}X_5 & \frac{1}{\sqrt{2}}X_3 & 0\\ 0 & 0 & X_1\\ \frac{1}{\sqrt{2}}X_2 & \frac{1}{\sqrt{2}}X_0 & 0\end{pmatrix},\qquad Y_d^{ij5}=c_{(6-6-12)}\begin{pmatrix}\frac{1}{\sqrt{2}}X_4 & 0 & 0\\ \frac{1}{\sqrt{2}}X_3 & \frac{1}{\sqrt{2}}X_2 & 0\\ 0 & \frac{1}{\sqrt{2}}X_1 & X_0\end{pmatrix},$$

where

$$X_N\equiv\sum_{n=0}^{5}(-1)^n\eta_{6(N+1/2)+72n},\qquad \eta_N\equiv\vartheta\begin{bmatrix}\frac{N}{432}\\ 0\end{bmatrix}(0,432\tau). \tag{C4}$$

c. Neutrino: $B_L-B_{\nu_R}-B_{H_u}=(6,1,0,\frac{1}{2})-(6,1,\frac{1}{2},0)-(12,0,\frac{1}{2},\frac{1}{2})$

Table XIV shows the zero-mode assignments for lepton doublets L^i, right-handed neutrinos ν_R^j and up-type Higgs fields H_u^k. Yukawa couplings are given by

TABLE XIV. Zero-mode wave functions in "$(6,1,0,\frac{1}{2})-(6,1,\frac{1}{2},0)-(12,0,\frac{1}{2},\frac{1}{2})$" model.

	L^i	ν_R^j	H_u^k
0	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1,1/2),6}+\psi_{T^2}^{(5,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,0),6}-\psi_{T^2}^{(11/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,1/2),12}-\psi_{T^2}^{(23/2,1/2),12})$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(2,1/2),6}+\psi_{T^2}^{(4,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,0),6}-\psi_{T^2}^{(9/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,1/2),12}-\psi_{T^2}^{(21/2,1/2),12})$
2	$\psi_{T^2}^{(3,1/2),6}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,0),6}-\psi_{T^2}^{(7/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,1/2),12}-\psi_{T^2}^{(19/2,1/2),12})$
3			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(7/2,1/2),12}-\psi_{T^2}^{(17/2,1/2),12})$
4			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(9/2,1/2),12}-\psi_{T^2}^{(15/2,1/2),12})$
5			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(11/2,1/2),12}-\psi_{T^2}^{(13/2,1/2),12})$

$$Y_\nu^{ijk} H_u^k = Y_\nu^{ij0} H_u^0 + Y_\nu^{ij1} H_u^1 + Y_\nu^{ij2} H_u^2 + Y_\nu^{ij3} H_u^3 + Y_\nu^{ij4} H_u^4 + Y_\nu^{ij5} H_u^5, \tag{C5}$$

with

$$Y_\nu^{ij0} = c_{(6-6-12)} \begin{pmatrix} -\frac{1}{\sqrt{2}}X_1 & -\frac{1}{\sqrt{2}}X_2 & 0 \\ 0 & -\frac{1}{\sqrt{2}}X_3 & -\frac{1}{\sqrt{2}}X_4 \\ 0 & 0 & -X_5 \end{pmatrix}, \qquad Y_\nu^{ij1} = c_{(6-6-12)} \begin{pmatrix} \frac{1}{\sqrt{2}}X_0 & 0 & -\frac{1}{\sqrt{2}}X_3 \\ -\frac{1}{\sqrt{2}}X_2 & 0 & \frac{1}{\sqrt{2}}X_5 \\ 0 & -X_4 & 0 \end{pmatrix},$$
$$Y_\nu^{ij2} = c_{(6-6-12)} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}X_0 & \frac{1}{\sqrt{2}}X_4 \\ \frac{1}{\sqrt{2}}X_1 & -\frac{1}{\sqrt{2}}X_5 & 0 \\ -X_3 & 0 & 0 \end{pmatrix}, \qquad Y_\nu^{ij3} = c_{(6-6-12)} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}X_5 & \frac{1}{\sqrt{2}}X_1 \\ -\frac{1}{\sqrt{2}}X_4 & \frac{1}{\sqrt{2}}X_0 & 0 \\ X_2 & 0 & 0 \end{pmatrix},$$
$$Y_\nu^{ij4} = c_{(6-6-12)} \begin{pmatrix} -\frac{1}{\sqrt{2}}X_5 & 0 & -\frac{1}{\sqrt{2}}X_2 \\ \frac{1}{\sqrt{2}}X_3 & 0 & \frac{1}{\sqrt{2}}X_0 \\ 0 & X_1 & 0 \end{pmatrix}, \qquad Y_\nu^{ij5} = c_{(6-6-12)} \begin{pmatrix} \frac{1}{\sqrt{2}}X_4 & -\frac{1}{\sqrt{2}}X_3 & 0 \\ 0 & \frac{1}{\sqrt{2}}X_2 & -\frac{1}{\sqrt{2}}X_1 \\ 0 & 0 & X_0 \end{pmatrix}, \tag{C6}$$

where

$$X_N \equiv \sum_{n=0}^{5} (-1)^n \eta_{6(N+1/2)+72n}, \qquad \eta_N \equiv \vartheta \begin{bmatrix} \frac{N}{432} \\ 0 \end{bmatrix} (0, 432\tau).$$

d. Charged lepton: $B_L - B_{e_R} - B_{H_d} = (6,1,0,\frac{1}{2}) - (6,0,\frac{1}{2},0) - (12,1,\frac{1}{2},\frac{1}{2})$

Table XV shows the zero-mode assignments for lepton doublets L^i, right-handed charged leptons e_R^j, and down-type Higgs fields H_d^k. Yukawa couplings are given by

$$Y_e^{ijk} = Y_d^{jik}. \tag{C7}$$

2. Majorana mass of right-handed neutrino

Majorana masses of right-handed neutrinos can be induced by D-brane instanton effects as shown in Appendix A. For the right-handed neutrinos in our model, there are two possible instanton zero-mode configurations, β_1, γ_1 and β_2, γ_2,

$$B_{\beta_1} - B_{\gamma_1} - B_{\nu_R} = \left(3, 0, 0, \frac{1}{2}\right) - \left(3, 1, \frac{1}{2}, \frac{1}{2}\right) - \left(6, 1, \frac{1}{2}, 0\right),$$
$$B_{\beta_2} - B_{\gamma_2} - B_{\nu_R} = (2, 0, 0, 0) - \left(4, 1, \frac{1}{2}, 0\right) - \left(6, 1, \frac{1}{2}, 0\right). \tag{C8}$$

TABLE XV. Zero-mode wave functions in "$(6,1,0,\frac{1}{2}) - (6,0,\frac{1}{2},0) - (12,1,\frac{1}{2},\frac{1}{2})$" model.

	L^i	e_R^j	H_d^k
0	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1,1/2),6} + \psi_{T^2}^{(5,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,0),6} + \psi_{T^2}^{(11/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,1/2),12} + \psi_{T^2}^{(23/2,1/2),12})$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(2,1/2),6} + \psi_{T^2}^{(4,1/2),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,0),6} + \psi_{T^2}^{(9/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,1/2),12} + \psi_{T^2}^{(21/2,1/2),12})$
2	$\psi_{T^2}^{(3,1/2),6}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,0),6} + \psi_{T^2}^{(7/2,0),6})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,1/2),12} + \psi_{T^2}^{(19/2,1/2),12})$
3			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(7/2,1/2),12} + \psi_{T^2}^{(17/2,1/2),12})$
4			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(9/2,1/2),12} + \psi_{T^2}^{(15/2,1/2),12})$
5			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(11/2,1/2),12} + \psi_{T^2}^{(13/2,1/2),12})$

TABLE XVI. Zero-mode wave functions in "$(3,0,0,\frac{1}{2})-(3,1,\frac{1}{2},\frac{1}{2})-(6,1,\frac{1}{2},0)$" model.

	β_1^i	γ_1^j	ν_R^a
0	$\psi_{T^2}^{(0,1/2),3}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,1/2),3}+\psi_{T^2}^{(5/2,1/2),3})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1/2,0),6}-\psi_{T^2}^{(11/2,0),6})$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(1,1/2),3}-\psi_{T^2}^{(2,1/2),3})$	$\psi_{T^2}^{(3/2,1/2),3}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(3/2,0),6}-\psi_{T^2}^{(9/2,0),6})$
2			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{(5/2,0),6}-\psi_{T^2}^{(7/2,0),6})$

However, these two configurations give the same Majorana mass matrix up to overall factor; therefore we concentrate on the former configuration, β_1, γ_1. Table XVI shows the zero-mode assignments for two zero modes of them. The 3-point couplings d_a^{ij} are given by

$$d_0^{ij}=c_{(3-3-6)}\begin{pmatrix}\eta_{1.5}+\eta_{16.5}+\eta_{19.5} & 0\\ -\frac{1}{\sqrt{2}}(\eta_{4.5}+\eta_{13.5}+\eta_{22.5}) & \eta_{7.5}+\eta_{10.5}+\eta_{25.5}\end{pmatrix}, \tag{C9}$$

$$d_1^{ij}=c_{(3-3-6)}\begin{pmatrix}0 & \sqrt{2}(\eta_{4.5}+\eta_{13.5}+\eta_{22.5})\\ \frac{1}{\sqrt{2}}(\eta_{1.5}+\eta_{16.5}+\eta_{19.5}+\eta_{7.5}+\eta_{10.5}+\eta_{25.5}) & 0\end{pmatrix}, \tag{C10}$$

$$d_2^{ij}=c_{(3-3-6)}\begin{pmatrix}\eta_{7.5}+\eta_{10.5}+\eta_{25.5} & 0\\ -\frac{1}{\sqrt{2}}(\eta_{4.5}+\eta_{13.5}+\eta_{22.5}) & \eta_{1.5}+\eta_{16.5}+\eta_{19.5}\end{pmatrix}, \tag{C11}$$

where

$$\eta_N\equiv\vartheta\begin{bmatrix}\frac{N}{54}\\ 0\end{bmatrix}(0,54\tau). \tag{C12}$$

Using above d_a^{ij}, Majorana masses of right-handed neutrinos can be calculated by Eq. (A3).

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