Scaled affine quantization of ultralocal φ_2^4 a comparative path integral Monte Carlo study with scaled canonical quantization

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After the success of affine quantization in proving through Monte Carlo analysis that the covariant euclidean scalar field theory, φ_n^r, where r denotes the power of the interaction term and $n = s + 1$ with s the spatial dimension and 1 adds imaginary time, such that $r \geq 2n/(n-2)$ can be acceptably quantized and the resulting theory is nontrivial, unlike what happens using canonical quantization, we show here that the same has to be expected for $r > 2$ and any n even for the ultralocal field theory. In particular we consider the ultralocal φ_2^4 model and study its renormalized properties for both the scaled canonical quantization version and the scaled affine quantization version through path integral Monte Carlo.

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I. INTRODUCTION

Ultralocal euclidean scalar field quantization, henceforth denoted φ_n^r, where r is the power of the interaction term and $n = s + 1$ where s is the spatial dimension and 1 adds imaginary time, such that $r < 2$ can be treated by canonical quantization (CQ), while models such that $r > 2$ and any $n \geq 2$ are trivial [1–5]. However, there exists a different approach called *affine quantization* (AQ) [1,6,7] that promotes a different set of classical variables to become the basic quantum operators and it offers different results, such as models for which $r > 2$. In particular one can show that while the Fubini-Study metric for the canonical coherent states that evaluates the distance-squared between two infinitesimally close ray vectors (minimized over any simple phase) leads to a *flat space* that already involves Cartesian coordinates, in the affine case the Fubini-Study metric describes a *Poincaré half plane* [7,8], has a constant negative curvature [9], and is geodesically complete. Unlike a flat plane, or a constant positive curvature surface (which holds the metric of three-dimensional spin coherent states), a space of constant negative curvature can not be visualized in a three-dimensional flat space. At every point in this space the negative curvature appears like a saddle having an "up curve" in the direction of the rider's chest and a "down curve" in the direction of the rider's legs.[1]

In the present work we show, with the aid of a path integral Monte Carlo (PIMC) analysis, that $r = 4$ and $n = 2$ can be acceptably quantized using scaled affine quantization which had been previously successfully used for the covariant case [12–19]

Being the current study in a lower, therefore unphysical, spacial dimension it nonetheless allowed us to get closer to the continuum limit than it was feasible for the physically relevant four-dimensional case on the computer due to the rapid increase of necessary lattice points as dimensionality is increased. Therefore, this work can indirectly give us a better understanding of the physically relevant case which had been already preliminarily studied by us in its covariant version [13]. Interestingly, the triviality of the scaling limits of the canonical Ising and covariant self-interacting scalar field models in four dimensions has been rigorously demonstrated recently [20].

II. A COMPARISON BETWEEN CANONICAL QUANTIZATION AND AFFINE QUANTIZATION FOR ULTRALOCAL FIELDS

A. Canonical quantization of scalar fields

Let us begin with the classical Hamiltonian for a single ultralocal field $\varphi(x)$

$$H(\pi, \varphi) = \int \left\{ \frac{1}{2}[\pi(x)^2 + m^2\varphi(x)^2] + g|\varphi(x)|^r \right\} d^s x, \quad (1)$$

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[1] Of course other types of more complex quantizations may still be possible which involve nonconstant curvature surfaces [10,11].

where $n = s + 1$ is the number of spacetime variables, and r is any real number. When g is zero, the remaining expression involves a domain in which a full set of variables, i.e., $\pi(x)$ and $\varphi(x)$, lead to a finite Hamiltonian value. If $g = 0 \to g > 0$, there are two possible results. If $r < 2$, then the domain remains the same. However, if $r > 2$, then there is a new domain that is smaller than the original domain because the interaction term $\int |\varphi(x)|^r d^s x = \infty$ leads to a reduction of certain fields. Since we work in a finite volume region, the fields that cause that divergence are not $\varphi(x) = \infty$, because that would have eliminated the original domain when $g = 0$. The only way for $\int |\varphi(x)|^r d^s x = \infty$ is, for example, given by $\varphi(x) = 1/|x - c|^k$ where k is small enough so that $\int \varphi(x)^2 d^s x < \infty$, while $r > 2$ is big enough so that $\int |\varphi(x)|^r d^s x = \infty$ for example for $r = 5/2$. Such behavior leads to immediate results in perturbation infinities in a power series of g, leading to a nonrenormalizable process, for which quantum efforts, using canonical quantization, collapse to "free" results, despite that $g > 0$, as all that is continuously connected to the original free theory where $g = 0$. Here we should be more precise since the relevant quantity to look at is the action rather than the Hamiltonian, so we should really compare the interaction term $\int |\varphi(x)|^r d^n x$ and the kinetic term $\int [\partial\varphi(x)/\partial x_0]^2 d^n x$ or the mass term $\int \varphi(x)^2 d^n x$. If we consider stationary fields as particular cases then the relevant integral is the mass term and we immediately see that we may have triviality for $r > 2$. But the same remains true also for space independent fields.[2] As we will see in Sec. IV, our numerical results give evidence for a "free" behavior of the CQ theory in this case.

Having seen what CQ can show us, now let us turn to AQ.

B. Affine quantization of scalar fields

The classical affine variables are the *dilation* $\kappa(x) \equiv \pi(x)\varphi(x)$ and the field $\varphi(x) \neq 0$. The reason we insist that $\varphi(x) \neq 0$ is because if $\varphi(x) = 0$ and $\kappa(x) = 0$ then $\pi(x)$ is not well-defined.

We next introduce the same classical Hamiltonian we chose before now expressed in affine variables. This leads us to

$$H'(\kappa,\varphi) = \int \left\{ \frac{1}{2}[\kappa(x)^2\varphi(x)^{-2} + m^2\varphi(x)^2] + g|\varphi(x)|^r \right\} d^s x, \tag{2}$$

in which $\varphi(x) \neq 0$ is an important fact. With these variables we see that $\pi = k/\varphi$ so we should not let neither $\varphi = 0$ nor $\varphi = \pm\infty$ otherwise in either cases we could find a form of indecision ($0/0$ or ∞/∞) for the dilation k which would then be not well defined. The essential result $0 < |\varphi(x)| < \infty$, leads to the fact that these AQ bounds on $\varphi(x)$ *forbid any nonrenormalizability*, a 'disease' which plagues the CQ analysis. With AQ, this new insight implies that any model φ_n^r does not become a "free" result, but leads to an appropriate "nonfree" result.

What follows in the coming sections is additional PIMC studies using AQ and CQ procedures.

III. LATTICE FORMULATION OF THE FIELD THEORY

The quantum affine operators are the scalar field $\hat{\varphi}(x) = \varphi(x)$ and the *dilation* operator[3] $\hat{\kappa}(x) = [\hat{\varphi}(x)\hat{\pi}(x) + \hat{\pi}(x)\hat{\varphi}(x)]/2$ where the momentum operator is $\hat{\pi}(x) = -i\hbar\delta/\delta\varphi(x)$. Accordingly for the self-adjoint kinetic term $\hat{\kappa}(x)\hat{\varphi}(x)^{-2}\hat{\kappa}(x) = \hat{\pi}(x)^2 + (3/4)\hbar\delta(0)^{2s}\varphi(x)^{-2}$ and one finds for the quantum Hamiltonian operator

$$\hat{H}'(\hat{\kappa},\hat{\varphi}) = \int \left\{ \frac{1}{2}[\hat{\pi}(x)^2 + m^2\varphi(x)^2] + g|\varphi(x)|^r + \frac{3}{8}\hbar^2 \frac{\delta(0)^{2s}}{\varphi(x)^2} \right\} d^s x. \tag{3}$$

As in previous works [16,17,19] we use the scaling $\pi \to a^{-s/2}\pi, \varphi \to a^{-s/2}\varphi, g \to a^{s(r-2)/2}g$, which is necessary[4] to eliminate the Dirac delta factor $\delta(0) = a^{-1}$ divergent in the continuum limit $a \to 0$. Of course for $r > 2$ the rescaled coupling constant, g, vanishes in the continuum limit since $a \to 0$, therefore we expect no difference between the interacting and the free model in such a limit. The theory considers a real scalar field φ taking the value $\varphi(x)$ on each site of a periodic, hypercubic, n-dimensional

[2] If we consider $\varphi(x) = 1/|x_0 - c|^k$ then in order to have $\mathcal{D}_{g>0} \subset \mathcal{D}_{g=0}$, where $\mathcal{D}_g$ is the domain of those $\varphi(x)$, in the ultralocal theory, where the action is not divergent, we require a divergent interaction term but a convergent kinetic term or $rk > 1 > 2(k+1)$ that is possible if $r > -2$. Additionally, since we always want a convergent mass term we must also have $0 < k < 1/2$ which again requires $r > 2$ for triviality. This means that for $r > 2$ the domains change because of reducing g back to zero will only retain the smallest version of the domain by continuity, and that will not be the theory you started out with. For space independent fields the ultralocal theory is the same as the covariant theory which is trivial for $r > 2n/(n-2) > 2$. This is due to the fact that $\mathcal{D}_g^{\text{covariant}} \neq \mathcal{D}_g^{\text{ultralocal}}$.

[3] Since $\varphi(x) \neq 0$, that means $\pi^\dagger \neq \pi$ so, to make that clear we should say that $\hat{\kappa}(x) \equiv [\hat{\varphi}(x)\hat{\pi}(x) + \hat{\pi}^\dagger(x)\hat{\varphi}(x)]/2$ to make sure that $\hat{k}^\dagger = \hat{k}$. But $\hat{\pi}^\dagger\hat{\varphi} = \hat{\pi}\hat{\varphi}$ because in that case $\hat{\pi}^\dagger$ acts like $\hat{\pi}$ thanks to having $\hat{\pi}$ acting on $\hat{\varphi}$.

[4] Note that from a physical point of view one never has to worry about the mathematical divergence since the lattice spacing will necessarily have a lower bound. For example at an atomic level one will have $a \gtrsim 1$ Å. In other words the continuum limit will never be a mathematical one.

lattice of lattice spacing a, our ultraviolet cutoff, and periodicity $L = Na$. The affine action for the field, $\mathcal{S}' = \int \overline{H'} dx_0$ (with $x_0 = ct$ where c is the speed of light constant and t is imaginary time), with $\overline{H'}$ the semiclassical Hamiltonian corresponding to the one of Eq. (3), is then approximated by

$$\mathcal{S}'[\varphi]/a \approx \frac{1}{2}\left\{\sum_x a^{-2}[\varphi(x) - \varphi(x+e_0)]^2 + m^2 \sum_x \varphi(x)^2\right\} + \sum_x g|\varphi(x)|^r + \frac{3}{8}\sum_x \hbar^2 \frac{1}{\varphi(x)^2}, \tag{4}$$

where e_μ is a vector of length a in the $+\mu$ direction. Respect to the previously considered covariant case [12–19], being now absent derivatives with respect to space, the field is allowed to be discontinuous in space (but will still be continuous in time).

The corresponding canonical action, $\mathcal{S} = \int \bar{H} dx_0$, is then approximated by

$$\mathcal{S}[\varphi]/a \approx \frac{1}{2}\left\{\sum_x a^{-2}[\varphi(x) - \varphi(x+e_0)]^2 + m^2 \sum_x \varphi(x)^2\right\} + \sum_x g|\varphi(x)|^r. \tag{5}$$

In this work we are interested in reaching the continuum limit by taking Na fixed and letting $N \to \infty$ at fixed volume L^s and absolute temperature $T = 1/k_B L$ with k_B the Boltzmann's constant.

The vacuum expectation value of an observable $\mathcal{O}[\varphi]$ will then be given by the following expression

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}[\varphi] \exp(-\mathcal{S}[\varphi]) \mathcal{D}\varphi(x)}{\int \exp(-\mathcal{S}[\varphi]) \mathcal{D}\varphi(x)}, \tag{6}$$

where the functional integrals will be calculated on a lattice using the path integral Monte Carlo method as explained further on.

IV. PATH INTEGRAL MONTE CARLO SIMULATION

We performed PIMC [21–24] for the action of Eq. (4) with $r = 4$ and $n = 2$. In particular we calculated the renormalized coupling constant g_R and mass m_R defined in Eqs. (11) and (13) of [13] respectively, measuring them in the path integral MC through vacuum expectation values like in Eq. (6). In particular $m_R^2 = p_0^2 \langle |\tilde{\varphi}(p_0)|^2 \rangle / [\langle \tilde{\varphi}(0)^2 \rangle - \langle |\tilde{\varphi}(p_0)|^2 \rangle]$ and at zero momentum $g_R = [3\langle \tilde{\varphi}(0)^2 \rangle^2 - \langle \tilde{\varphi}(0)^4 \rangle] / \langle \tilde{\varphi}(0)^2 \rangle^2$, where $\tilde{\varphi}(p) = \int d^n x e^{ip\cdot x} \varphi(x)$ is the Fourier transform of the field and we choose the 2-momentum p_0 with the zero component equal to $2\pi/Na$ and the other component equal to zero. Since the integration variables in Eq. (6) are N^n, being able to choose $n = 2$ allowed us to greatly speed up the calculations compared to our previous covariant studies for $n > 2$ [12–19] and this made possible to push ourselves closer to the continuum limit, to bigger N.

Following Freedman *et al.* [2], for each N and g, we adjusted the bare mass m in such a way to maintain the renormalized mass approximately constant, $m_R \approx 3$, to within a few percent (in all cases less than 20%), and we measured the renormalized coupling constant g_R for various values of the bare coupling constant g at a given small value of the lattice spacing $a = 1/N$ (this corresponds to choosing an absolute temperature $k_B T = 1$ and a fixed length $L = 1$). Note that in the CQ case it was necessary to choose imaginary bare masses for $g > 0$. With Na and m_R fixed, as a was made smaller, whatever change we found in $g_R m_R^n$ as a function of g could only be due to the change in a. We generally found that a depression in m_R produced an elevation in the corresponding value of g_R and viceversa; for this reason it is convenient to define an alternative renormalized coupling constant less sensitive to small variations of m_R, namely $g_R(m_R)^n$ (see Ref. [2]). The results are shown in Fig. 1 for the scaled canonical action (5) and the scaled affine action (4) in natural units $c = \hbar = k_B = 1$, where, following Freedman *et al.* [2] we decided to compress the range of g for display, by choosing the horizontal axis to be $g/(50 + g)$. The constraint $m_R \approx 3$ was not easy to implement since for each N and g we had to run the simulation several times with different values of the bare mass m in order to determine the value which would satisfy the constraint $m_R \approx 3$. This was the main source of uncontrolled uncertainty in the data.

In our simulations we used 10^8 MC steps where in each step we attempt to move once all the N^n fields variables of integration through the Metropolis algorithm [21]. We used block averages and estimated the statistical errors using the jackknife method (described in Sec. 3.6 of [25]) to take into account of the correlation time of the simulations. We always adjusted the field displacement in the random walk so to keep the acceptance ratios as close as possible to 1/2.

Comparing the results for the scaled canonical and affine action we can see how the renormalized coupling constant of the two approaches behaves very similarly at $g \neq 0$,[5] but in a neighborhood of $g = 0$ the affine version remains far from zero in the continuum limit when the ultraviolet cutoff is removed ($Na = 1$ and $N \to \infty$). The decrease of the renormalized coupling g_R for increasing N has to be expected, both for the CQ and the AQ cases, due to the use we made of the scaling $g \to a^{s(r-2)/2} g$ which makes the model a "free" one in the continuum limit, $a \to 0$, when $r > 2$. Of course the scaling we used has just a mathematical

[5]Comparing with the previous covariant studies [12–19] we can now say that removing the gradient term leads to a wilder behavior of the paths which could complicate finding any difference between CQ and AQ.

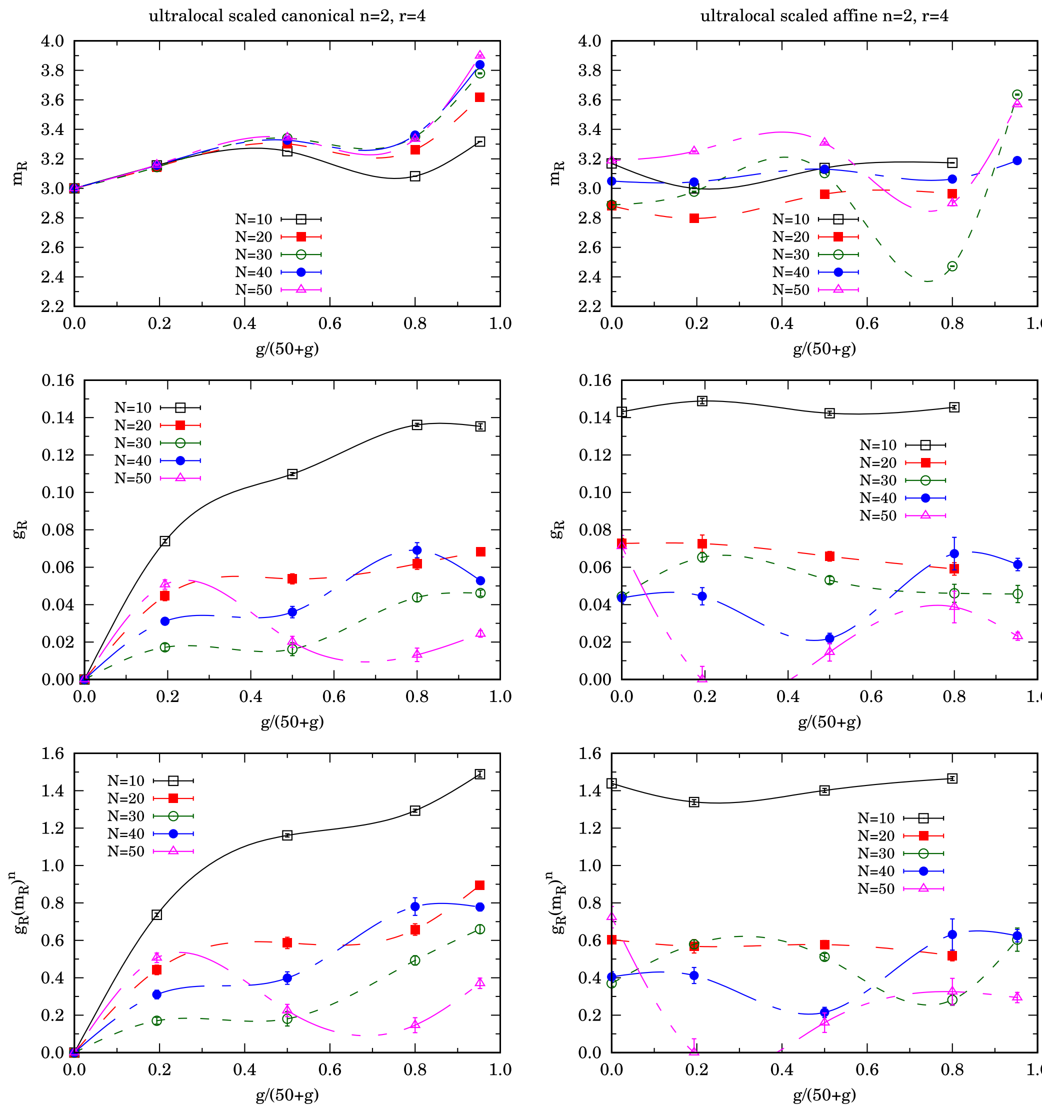

FIG. 1. For the scaled *canonical* (left panels) and scaled *affine* (right panels) φ_2^4 ultralocal euclidean scalar field theory, we show the renormalized mass m_R (top panels), the renormalized coupling constants g_R (central panels), and $g_R m_R^n$ (bottom panels) for various values of the bare coupling constant g at decreasing values of the lattice spacing $a = 1/N$ ($N \to \infty$ continuum limit). The statistical errors in the Monte Carlo were smaller than the symbols used. The main source of uncertainty is nonetheless the indirect one stemming from the unavoidable difficulty of keeping the renormalized mass constant throughout all cases. The lines connecting the simulation points are just a guide for the eye.

and not a physical justification (see footnote 4). In particular the scaling permits to have nondiverging field expectation values in the AQ approach. Therefore, as already observed in several previous covariant studies [12–15,18], we expect that also in this ultralocal case the AQ approach gives rise to a "nonfree" field theory contrary to what happens for the CQ approach for $r > 2$. This success of affine quantization to produce a well-defined, renormalizable, nontrivial, "nonfree" quantum field theory is one of its merits and benefits.

During our simulations we kept under control also the vacuum expectation value of the field which in all cases was found to vanish in agreement with the fact that the symmetry $\varphi \to -\varphi$ of the scaled canonical action is preserved in the scaled affine case. The random walk in the field is always able to tunnel through the barrier at $\varphi = 0$ due to the affine effective term, $\frac{3}{8}(\hbar/\varphi)^2$, in the interaction. This is a consequence of working at finite N and we expect the symmetry to be spontaneously broken in the continuum $N \to \infty$ limit[6] when the point at $\varphi = 0$ is excluded. Our results also show how the sum rules $g_R \to 0$ and $m_R \to m$ for $g \to 0$ are satisfied for CQ as it should for any gaussian weighting factor $\exp(-\mathcal{S}[\varphi])$ for any N.

[6]Once again this is only possible in a mathematical world but not in the physical (see footnote 4).

V. CONCLUSIONS

In conclusion we performed a path integral Monte Carlo study of the properties (mass and coupling constant) of the renormalized ultralocal euclidean scalar field theory φ_2^4 quantized through scaled affine and canonical quantization. Our results confirm the theoretical expectation for a "free" theory in the continuum limit. This is merely a consequence of the chosen scaling. As in previous works on covariant theories we expect that also in this ultralocal case the un-scaled AQ approach gives rise to a "nonfree" field theory contrary to what happens for the unscaled CQ approach for $r > 2$. Indeed already for the scaled version in the AQ theory the renormalized coupling does not seem to go towards zero at least when the bare coupling is zero when one approaches the continuum limit, as stems from our path integral Monte Carlo results. This means that a "free" scaled AQ theory is profoundly different from a "free" scaled CQ one; the former is therefore nontrivial and renormalizable and the latter is trivial and nonrenormalizable.

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