Permutation symmetry in large-N matrix quantum mechanics and partition algebras

George Barnes,[1,*] Adrian Padellaro,[1,†] and Sanjaye Ramgoolam[1,2,‡]

[1]*Centre for Theoretical Physics, School of Physical and Chemical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom*

[2]*School of Physics and Mandelstam Institute for Theoretical Physics, University of Witwatersrand, Wits 2050, South Africa*

(Received 6 October 2022; accepted 22 October 2022; published 29 November 2022)

We describe the implications of permutation symmetry for the state space and dynamics of quantum mechanical systems of matrices of general size N. We solve the general 11-parameter permutation invariant quantum matrix harmonic oscillator Hamiltonian and calculate the canonical partition function. The permutation invariant sector of the Hilbert space, for general Hamiltonians, can be described using partition algebra diagrams forming the bases of a tower of partition algebras $P_k(N)$. The integer k is interpreted as the degree of matrix oscillator polynomials in the quantum mechanics. Families of interacting Hamiltonians are described which are diagonalized by a representation theoretic basis for the permutation invariant subspace which we construct for $N \geq 2k$. These include Hamiltonians for which the low-energy states are permutation invariant and can give rise to large ground state degeneracies related to the dimensions of partition algebras. A symmetry-based mechanism for quantum many body scars discussed in the literature can be realized in these matrix systems with permutation symmetry. A mapping of the matrix index values to lattice sites allows a realization of the mechanism in the context of modified Bose-Hubbard models. Extremal correlators analogous to those studied in AdS/CFT are shown to obey selection rules based on Clebsch-Gordan multiplicites (Kronecker coefficients) of symmetric groups.

DOI: 10.1103/PhysRevD.106.106020

I. INTRODUCTION

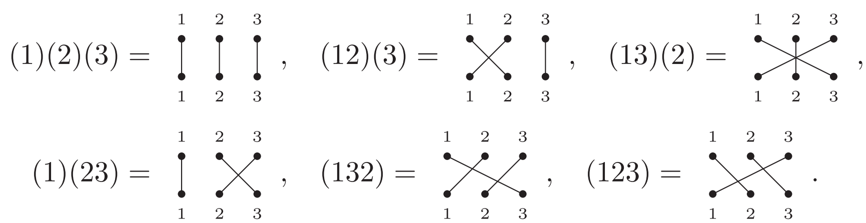
Systems with matrix degrees of freedom transforming in the adjoint or bifundamental representation of a group G, such as $U(N)$, $SU(N)$, $SO(N)$, $SP(N)$, are ubiquitous in physics. The group G is often a gauge symmetry and physical states or operators of interest are G invariant. The large N limit has been known, since the work of 't Hooft [1], to exhibit important simplifications related to the combinatorics of string worldsheets. Notable examples of gauge-string duality based on such large N properties include: the duality between low-dimensional noncritical strings and matrix models [2–4], between two-dimensional Yang-Mills theories and Hurwitz spaces [5–13]; the AdS/CFT correspondence [14–16]; the correspondence between Gaussian matrix theories and Belyi maps [17–20]. Random matrix theories have also been used to model statistical properties of complex systems [21–26]. In zero-dimensional matrix models, invariance is not forced upon us by any gauge symmetry. However, it is still a fruitful perspective to consider the invariant sectors as computationally tractable sectors which encode significant properties of complex systems. This was the perspective taken in [27–30], which used zero-dimensional matrix models with permutation symmetry to model the statistics of words in computational linguistics [31–34].

Large discrete groups, e.g. the symmetric groups S_N of all permutations of N objects, also play a central role in holography. Two-dimensional conformal field theories (CFTs) for orbifolds M^N/S_N, for some CFT M, provide the CFTs in $\mathrm{AdS}_3/\mathrm{CFT}_2$ dualities [35]. These orbifold CFTs have recently provided the setting for a derivation of holographic duality [36]. It is natural to ask if matrix systems with discrete symmetries such as S_N have holographic duals. Recent results on large N factorization in permutation invariant matrix models [37] are encouraging for this prospect. By regarding matrix models as zero-dimensional quantum field theories (QFTs), in this paper we take the natural next step of considering one-dimensional QFTs, i.e. matrix quantum mechanical systems with permutation symmetry. We pay particular attention to methods which are applicable for general N and allow large N expansions. We give a general description of the

[*]g.barnes@qmul.ac.uk
[†]a.k.s.padellaro@qmul.ac.uk
[‡]s.ramgoolam@qmul.ac.uk

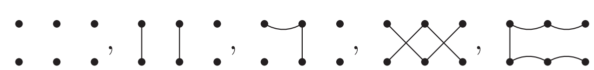
permutation invariant subspace in matrix quantum mechanical systems, drawing on relevant results from the mathematical literature on partition algebras. This is followed by a discussion of interesting Hamiltonians for many-body quantum physics. This is motivated by the vibrant interplay between holography and many-body quantum mechanical systems which manifests itself, for example, in the connection between free fermions and large N two-dimensional Yang Mills theory [38]; free fermions and the half-Bogomol'nyi–Prasad–Sommerfield (BPS) sector of $\mathcal{N}=4$ Super Yang-Mills (SYM) [39,40]; free fermions and supersymmetric indices [41], bosons in a 3D harmonic oscillator and eighth BPS states in $\mathcal{N}=4$ SYM [42–44]; quantum mechanical spin matrix theory which is used as a simplified setup to study the emergence mechanisms of AdS/CFT [45,46]. This interplay is also visible in the prominent role of coherent states, a technique widely used in many body quantum physics, in the study of large N systems. This theme appears in early work on large N (e.g. [47,48]) as well as more recent developments (e.g. [49–51]).

Many aspects of large N simplifications in matrix systems are consequences of Schur-Weyl duality. The standard instance of Schur-Weyl duality [52] concerns the tensor product $V^{\otimes k}$ of the fundamental representation V of $U(N)$. The symmetric group S_k of all permutations of k objects acts on $V^{\otimes k}$ by permuting the factors of the tensor product. Schur-Weyl duality states that the algebra of operators commuting with the standard $U(N)$ action on the tensor product $V^{\otimes k}$ is the group algebra $\mathbb{C}[S_k]$. This has important implications for the classification of $U(N)$ gauge invariant polynomial functions of matrix variables, where a matrix X transforms as $X \to UXU^\dagger$ for $U \in U(N)$. Schur-Weyl duality relates this problem to the rich combinatorics and representation theory of symmetric groups (see e.g. [53]). For example, the gauge invariant polynomial functions of degree k for one matrix of size N, taking $N > k$ for simplicity, are labeled by conjugacy classes of S_k. Finite N effects are captured with the use of Young diagrams. Schur-Weyl duality has been used as a powerful tool in the construction of gauge invariant observables in one-matrix and multimatrix systems in connection with the AdS/CFT correspondence. This played an important role in identifying the CFT duals [39,40,54] of giant gravitons [55–57] in the AdS/CFT correspondence. The Schur-Weyl duality framework has been further applied to the computation of one-matrix and multimatrix correlators [39,58–70]. A short review is [71]. These multimatrix applications involve dual algebras beyond the symmetric group algebras. For example Brauer algebras, which have a basis of diagrams, are used in [58]. The symmetric group algebra $\mathbb{C}[S_k]$ can also be viewed as a diagram algebra with multiplication given by the composition of diagrams. For example the following six diagrams give a basis of $\mathbb{C}[S_3]$, the corresponding permutations are given in cycle notation



(1.1)

The same general philosophy can be applied to the case where we are considering polynomial functions of a matrix X invariant under the transformation $X \to M_\sigma X M_\sigma^T$, where M_σ is a matrix representing the permutation $\sigma \in S_N$ in the N-dimensional natural representation of S_N, satisfying $M_\sigma^T = M_\sigma^{-1}$. This problem in the invariant theory of matrices arises in the application of permutation invariant matrix models to language data [27,28] and Schur-Weyl duality was used to study these invariants in [37]. The algebra dual to S_N acting on $V_N^{\otimes k}$, where V_N is the natural representation of the symmetric group S_N, is called the partition algebra $P_k(N)$. Partition algebras were first introduced in [72–74] in application to the statistical mechanics of Potts models (see [75] for a survey of partition algebras). Partition algebras $P_k(N)$ are diagram algebras with a basis labeled by diagrams corresponding to set partitions of $2k$ objects. These include the diagrams corresponding to elements of $\mathbb{C}[S_k]$ as well as more general diagrams. For example, in addition to the diagrams in (1.1), the following are elements of $P_3(N)$



(1.2)

We will discuss these diagrams in more detail in Sec. III. Matrix systems with S_N symmetry together with partition algebras allow us to study large N simplifications in the case of discrete (finite) groups. Partition algebras and their relation to the representation theory of symmetric groups is an active area of mathematical research [76–79].

An algebraic description of permutation invariant matrix polynomials of degree k was given in [37] using symmetrized partition algebras $SP_k(N)$. $SP_k(N)$ consists of equivalence classes of elements in $P_k(N)$. The equivalence is defined using the $\mathbb{C}[S_k]$ subalgebra of $P_k(N)$ and accounts for the commuting nature of matrix variables. The work in [37] showed that distinct permutation invariant matrix polynomials in the diagram basis satisfy a factorization property at large N. The diagram basis is the analog of the trace basis for $U(N)$ invariant matrix polynomials. Partition algebras have also been used to study permutation invariant random matrix distributions from the point of view of mathematical statistics [80–82].

Polynomials in matrix variables M^i_j are closely related to quantum mechanical states constructed from matrix oscillators $(a^\dagger)^i_j$. This allows us to translate the technology developed for zero-dimensional matrix models [27,28,37,83] to the setting of matrix quantum mechanics. We will give a detailed description of the space of S_N invariant states constructed from matrix oscillators. Polynomials in matrix oscillators can be organized by the degree of the polynomials. At degree k, the state space is isomorphic to an S_k symmetric subspace $\mathcal{H}^{(k)}$ of $\mathrm{End}(V_N^{\otimes k})$:

$$\mathcal{H}^{(k)} \to \mathrm{End}(V_N^{\otimes k}). \tag{1.3}$$

There is a one-to-one correspondence between tensors

$$\langle e^{i_1} \cdots e^{i_k}|T|e_{j_1} \cdots e_{j_k}\rangle = T^{i_1 \cdots i_k}_{j_1 \cdots j_k} \tag{1.4}$$

and elements in $\mathrm{End}(V_N^{\otimes k})$. The bosonic symmetry of the oscillators imposes an invariance under simultaneous reordering of the upper and lower indices. Commuting with the S_k action is the S_N action on $V_N^{\otimes k}$ which we denote $\mathcal{L}(\sigma)$. The S_N permutation invariance translates to an invariance of T under an adjoint action

$$\mathrm{Ad}(\sigma)[T] = \mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1}). \tag{1.5}$$

Many of our results on the S_N invariant state space of matrix oscillators, particularly in Secs. III and IV are independent of the Hamiltonian. They can be viewed as a detailed account of the S_N invariant subspace in matrix quantum mechanics using partition algebras and representation theory. The use of the partition algebra $P_k(N)$ to study operators and quantum states in $\mathcal{H}^{(k)}$ allows us to take advantage of simplifications in the limit where k is kept fixed as $N \to \infty$.

The representation theoretic approach allows the construction of solvable algebraic Hamiltonians where the S_N invariant states are resolved according to representation theoretic characteristics. Sections V and VI discuss different classes of solvable S_N invariant Hamiltonians obeying

$$\mathrm{Ad}(\sigma)H = H\mathrm{Ad}(\sigma). \tag{1.6}$$

We build on this discussion in Sec. VII, using Hamitonians of the form $(H + H_s)$: H obeys (1.6) while H_s is subject to a restriction defined in terms of permutation invariant states.

The paper is organized as follows. For concreteness, Sec. II contains a review of the simplest quantum mechanical model with matrix degrees of freedom. This is the free matrix quantum harmonic oscillator. It is a model containing N^2 decoupled harmonic oscillators $X_{ij}, i, j = 1, \ldots, N$ with a global $U(N^2)$ symmetry. The Hilbert space of this model is a Fock space $\mathcal{H}$ of states constructed using matrix oscillators $(a^\dagger)^i_j$. This model also serves as a good place to introduce the diagram notation that we will use in the rest of the paper.

In Sec. III we consider the S_N invariant subspace $\mathcal{H}_{\mathrm{inv}}$ of the total Hilbert space $\mathcal{H}$ of a general quantum mechanics matrix system. This is the subspace of states invariant under $a^\dagger \to M_\sigma a^\dagger M_\sigma^T$, where M_σ is a permutation matrix of size N. We explain the correspondence between permutation invariant matrix states of degree k and partition algebras $P_k(N)$. The partition algebras have three natural bases, and each one gives rise to a different basis for $\mathcal{H}_{\mathrm{inv}}$. The diagram basis is natural when discussing inner and outer products. The factorization property in [37] translates to orthogonality of the diagram basis at large N. The so-called orbit basis gives rise to an orthogonal basis for all N. We call the third basis the representation basis. In the mathematical literature, the representation basis is called a complete set of matrix units. The product in the matrix unit basis is a generalization of the product for elementary matrices for matrix algebras. The representation basis can be constructed using Fourier transformation on $P_k(N)$ and is a direct analog of the Schur basis for $U(N)$ invariants. Appendix A gives the necessary background for Fourier transforms on semisimple algebras, closely following [84] but with some modifications that are important for our application. Physically, the representation basis can be understood as a basis that diagonalizes a set of algebraic commuting charges.

Section IV is devoted to the construction and diagonalization of these charges, which can be used to give the explicit transformation from the diagram basis to the representation basis at large N. We illustrate the method for small k and large N. These are tabulated in Appendix C. The representation basis forms an energy eigenbasis for the Hamiltonian of the free matrix quantum harmonic oscillator presented in Sec. II.

In Sec. V we introduce an 11 parameter family of exactly solvable quantum matrix systems. The potential in these systems is the most general permutation invariant quadratic function of the matrix variables. These quantum systems can therefore be viewed as general matrix harmonic oscillator systems compatible with permutation symmetry. We find the spectrum for general choices of the parameters by adapting the representation theoretic techniques which have been used to compute correlators in permutation invariant Gaussian matrix models [28]. Further, we write the canonical partition function in a simple closed form. The representation basis states from Sec. III do not form an eigenbasis for the general Hamiltonians considered here. The action of the Hamiltonians on the representation basis states leads to a mixing which is constrained by Clebsch-Gordan multiplicities for the symmetric groups. We briefly discuss this mixing.

In Sec. VI we discuss interacting Hamiltonians, parametrized by a positive integer K, constructed using partition algebra elements, with the property that the ground states are all permutation invariant states and have degeneracies controlled by a sequence of partition algebras $P_k(N)$ for $k \in \{0, 1, \ldots, K\}$. The energy gap between the ground states and the lowest excited state is also determined by K. By deforming these Hamiltonians with other partition algebra elements, we design Hamiltonians where the degeneracy of the ground states is broken by small amounts—these two scenarios are illustrated in Fig. 1. We also include a general description of permutation invariant Hamiltonians, finding an interesting relation to the counting of 2-matrix permutation invariants of the kind considered in [83]. We conclude this section with an interpretation of the oscillators $(a^\dagger)^j_i$ as creation operators on a square lattice with sites labeled (i, j).

Subspaces of invariant states play an important role in the group-theoretic proposal [85,86] for a mechanism of weak ergodicity breaking, experimentally discovered in [87], now known as quantum many-body scars [88]. In Sec. VII we discuss how the permutation invariant state space in this paper can be turned into a scar subspace. Adapting the ideas in [85,86] for the realization of group-theoretic scar states, we describe Hamiltonians which exhibit the revival properties characteristic of scars. The lattice interpretation of the matrix oscillators from Sec. VI allows us to interpret these Hamiltonians as deformations of Bose Hubbard models.

We compute a set of two- and three-point correlators of invariant operators in Sec. VIII. The two-point correlators have a large N factorization property described in the context of matrix models in [37]. The three-point functions are similar to extremal correlators, which are relevant to quantum mechanical models considered in AdS/CFT. The extremal correlators are shown to obey selection rules based on Clebsch-Gordan multiplicities (Kronecker coefficients) of symmetric groups.

II. REVIEW: MATRIX HARMONIC OSCILLATOR

This section is a review of the simplest matrix quantum harmonic oscillator. The Lagrangian (2.1) describes N^2 free harmonic oscillators. The corresponding Hamiltonian has a global $U(N^2)$ symmetry. This has a $U(N) \times U(N)$ subgroup of unitary matrices acting by left and right multiplication. There is also a smaller $S_N \times S_N$ subgroup of the $U(N) \times U(N)$ which plays an important role in subsequent sections. The simplest, noninteracting $U(N^2)$ invariant model will serve as a very good setup to introduce the notation used in the rest of the paper. In particular, we describe how to construct states and operators in $\mathcal{H}$, the Hilbert space of the theory, by considering the oscillators $a_{ij}, a^\dagger_{ij}$ as endomorphisms on V_N (an N-dimensional vector space). We will frequently have this view in mind when manipulating states and operators and it is often practical to employ diagrammatic notation in order to do so. The basics of this diagrammatic notation we introduce at the end of this section.

The simplest matrix harmonic oscillator is described by the Lagrangian

$$L_0 = \frac{1}{2}\left(\sum_{i,j=1}^{N} \partial_t X_{ij} \partial_t X_{ij} - X_{ij} X_{ij}\right). \tag{2.1}$$

It describes N^2 decoupled oscillators. The conjugate momenta are

$$\Pi_{ij} = \frac{\partial L_0}{\partial(\partial_t X_{ij})} = \frac{\partial}{\partial t} X_{ij}. \tag{2.2}$$

FIG. 1. The figure illustrates the type of spectra that can be engineered using the algebraic Hamiltonians discussed in this section. Blue lines correspond to states that are invariant under the adjoint action of S_N. Black lines are noninvariant states. (a) Illustrates the splitting of the ground state degeneracy achieved by adding a term to the Hamiltonian involving central algebraic charges (b) a spectrum in which the degeneracy associated with multiplicity labels has been lifted.

The Hamiltonian corresponding to L_0 is

$$H_0 = \frac{1}{2}\left(\sum_{i,j=1}^{N} \Pi_{ij}\Pi_{ij} + X_{ij}X_{ij}\right). \tag{2.3}$$

The canonical commutation relations are

$$[X_{ij}, \Pi_{kl}] = i\delta_{ik}\delta_{jl}. \tag{2.4}$$

The Hamiltonian given in (2.3) is diagonalized in the usual way—introducing oscillators $a_{ij}^{\dagger}, a_{ij}$ defined by

$$X_{ij} = \sqrt{\frac{1}{2}}(a_{ij}^{\dagger} + a_{ij}),$$
$$\Pi_{ij} = i\sqrt{\frac{1}{2}}(a_{ij}^{\dagger} - a_{ij}), \tag{2.5}$$

with commutation relations

$$[a_{ij}, a_{kl}^{\dagger}] = \delta_{ik}\delta_{jl}. \tag{2.6}$$

Normal ordering H_0 gives

$$H_0 = \sum_{i,j=1}^{N} a_{ij}^{\dagger} a_{ij}, \tag{2.7}$$

which is just a number operator. We now show that H_0 is invariant under a $U(N^2)$ symmetry that acts on oscillators as

$$a_{ij} \to \sum_{k,l=1}^{N} U_{ij;kl} a_{kl}, \tag{2.8}$$

$$a_{ij}^{\dagger} \to \sum_{k,l=1}^{N} U_{kl;ij}^{\dagger} a_{kl}^{\dagger}, \tag{2.9}$$

with $U_{ij;kl}$ an $N^2 \times N^2$ unitary matrix satisfying

$$\sum_{k,l=1}^{N} U_{ij;kl} U_{kl;mn}^{\dagger} = \delta_{im}\delta_{jn}. \tag{2.10}$$

Under the $U(N^2)$ transformation H_0 is invariant,

$$\begin{aligned} H_0 \to& \sum_{i,j,k,l,m,n} U_{kl;ij}^{\dagger} U_{ij;mn} a_{kl}^{\dagger} a_{mn} \\ &= \sum_{k,l,m,n} \delta_{km}\delta_{ln} a_{kl}^{\dagger} a_{mn} \\ &= \sum_{k,l} a_{kl}^{\dagger} a_{kl}. \end{aligned} \tag{2.11}$$

The oscillator states

$$\prod_{i,j} \frac{(a_{ij}^{\dagger})^{k_{ij}}}{\sqrt{k_{ij}!}} |0\rangle \tag{2.12}$$

labeled by non-negative integers k_{ij} with $i, j = 1, \ldots, N$ are energy eigenstates of H_0. The total Hilbert (Fock) space $\mathcal{H}$ decomposes into subspaces $\mathcal{H}^{(k)}$ with fixed number of oscillators (degree) k,

$$\mathcal{H} \cong \bigoplus_{k=0}^{\infty} \mathcal{H}^{(k)}. \tag{2.13}$$

The subset of states with $k = \sum_{i,j} k_{ij}$ form an eigenbasis for the subspace $\mathcal{H}^{(k)}$ and have energy k. In general the spectrum is highly degenerate. The number of states with energy k is

$$\mathrm{Dim}\mathcal{H}^{(k)} = \binom{N^2 + k - 1}{k} = \frac{N^2(N^2+1)\ldots(N^2+k-1)}{k!}. \tag{2.14}$$

This is the number of ways to choose k elements from a set of N^2 when repetition is allowed. It is also the dimension of the symmetric part of a k-fold tensor product of a vector space with dimension N^2. Equivalently, it is the dimension of the vector space of states composed of k bosonic oscillators $a_{ij}^{\dagger}$. For fixed k and $N \gg 2k$ the dimension grows as N^{2k}.

A. Diagram notation

Throughout this paper we will use diagrammatic notation to describe states and operators in $\mathcal{H}^{(k)}$. For this purpose, it is useful to introduce the following matrices of oscillators $(a^{\dagger})^i_j = a_{ji}^{\dagger}$ and $a^i_j = a_{ij}$ which satisfy

$$[a^i_j, (a^{\dagger})^l_k] = \delta^i_k \delta^l_j. \tag{2.15}$$

Let V_N be an N-dimensional vector space with basis $\{e_1, \ldots, e_N\}$. The matrices of oscillators can be viewed as (operator-valued) elements in $\mathrm{End}(V_N)$, where $\mathrm{End}(V_N)$ is the set of all linear maps $V_N \to V_N$. In this language, the above oscillators are matrix elements,

$$a^{\dagger}(e_i) = \sum_{j=1}^{N} (a^{\dagger})^j_i e_j \quad \text{and} \quad a(e_i) = \sum_{j=1}^{N} a^j_i e_j. \tag{2.16}$$

Consequently, a general degree one state in $\mathcal{H}$ can be written as

$$\mathrm{Tr}_{V_N}(Ta^\dagger)|0\rangle = \sum_{i,j=1}^{N} T^i_j (a^\dagger)^j_i |0\rangle \equiv |T\rangle, \qquad (2.17)$$

where $T \in \mathrm{End}(V_N)$ (an N-by-N matrix) and the last equality is a definition of $|T\rangle$.

The degree k subspace is given by

$$\mathcal{H}^{(k)} \cong \mathrm{Span}_{\mathbb{C}}\,\{(a^\dagger)^{i_1}_{j_1}\ldots(a^\dagger)^{i_k}_{j_k}|0\rangle\}, \qquad (2.18)$$

therefore general states are parametrized by tensors $T^{j_1\ldots j_k}_{i_1\ldots i_k}$. It is convenient to view these tensors as elements of $\mathrm{End}(V_N^{\otimes k})$, where $V_N^{\otimes k}$ is the kth tensor product of V_N. That is, in the usual basis for tensor product spaces

$$T(e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_k}) = \sum_{j_1,j_2,\ldots,j_k=1}^{N} T^{j_1\ldots j_k}_{i_1\ldots i_k} e_{j_1}\otimes e_{j_2}\otimes\ldots\otimes e_{j_k}. \qquad (2.19)$$

Analogous to the degree one case, a general state $|T\rangle \in \mathcal{H}^{(k)}$ can be written as a trace

$$\boxed{|T\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(T(a^\dagger)^{\otimes k})|0\rangle = \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k}} T^{j_1\ldots j_k}_{i_1\ldots i_k}(a^\dagger)^{i_1}_{j_1}\ldots(a^\dagger)^{i_k}_{j_k}|0\rangle,} \qquad (2.20)$$

for $T \in \mathrm{End}(V_N^{\otimes k})$ and $(a^\dagger)^{\otimes k} = a^\dagger\otimes\ldots\otimes a^\dagger$ with matrix elements

$$(a^\dagger)^{\otimes k}(e_{j_1}\otimes\ldots\otimes e_{j_k}) = \sum_{i_1,\ldots,i_k}(a^\dagger)^{i_1}_{j_1}\ldots(a^\dagger)^{i_k}_{j_k} e_{i_1}\otimes\ldots\otimes e_{i_k}. \qquad (2.21)$$

It should be emphasized that, due to the bosonic symmetry of the oscillators, $T^{j_1\ldots j_k}_{i_1\ldots i_k}$ is a symmetric tensor (under simultaneous permutations of upper and lower indices), for example $T^{j_1 j_2\ldots j_k}_{i_1 i_2\ldots i_k} = T^{j_2 j_1\ldots j_k}_{i_2 i_1\ldots i_k}$.

It is useful to formulate this restriction in terms of S_k invariance. An element $\tau \in S_k$, viewed as a bijective map $\tau\colon\{1,\ldots,k\}\to\{1,\ldots,k\}$, defines a linear operator $\mathcal{L}_{\tau^{-1}}$ which acts on $V_N^{\otimes k}$ as

$$\mathcal{L}_{\tau^{-1}}(e_{i_1}\otimes\ldots\otimes e_{i_k}) = e_{i_{\tau(1)}}\otimes\ldots\otimes e_{i_{\tau(k)}}. \qquad (2.22)$$

The symmetry of T is equivalent to the statement

$$\boxed{\mathcal{L}_\tau T \mathcal{L}_{\tau^{-1}} = T, \quad \forall\ \tau\in S_k,} \qquad (2.23)$$

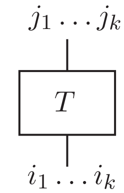
or in index notation

$$T^{j_{\tau(1)}\ldots j_{\tau(k)}}_{i_{\tau(1)}\ldots i_{\tau(k)}} = T^{j_1\ldots j_k}_{i_1\ldots i_k}, \quad \forall\ \tau\in S_k. \qquad (2.24)$$

Therefore, states in $\mathcal{H}^{(k)}$ are in one-to-one correspondence with elements $T\in\mathrm{End}_{S_k}(V_N^{\otimes k})$, the subspace of linear maps that commute with the action of S_k.

We introduce diagrammatic notation to simplify manipulations involving tensor equations. A map $T\in\mathrm{End}(V_N^{\otimes k})$ is represented by a box

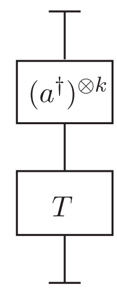
$$T^{j_1\ldots j_k}_{i_1\ldots i_k} =$$



, (2.25)

where the edges correspond to states in $V_N^{\otimes k}$. Internal lines in a diagram correspond to contracted indices. For example, the state $|T\rangle\in\mathcal{H}^{(k)}$ can be represented diagrammatically as

$|T\rangle =$



$|0\rangle$. (2.26)

The horizontal lines identify the top edge with the bottom edge to give a trace, and the line between the $(a^\dagger)^{\otimes k}$ and T boxes signifies that the corresponding indices are identified and summed over. This diagram should be compared to (2.20).

III. PERMUTATION INVARIANT SECTORS FOR QUANTUM MATRIX SYSTEMS

In this section we consider the action of S_N on the subspace $\mathcal{H}^{(k)}$, spanned by degree k polynomials in matrix oscillators $(a^\dagger)^i_j$ acting on the vacuum. The adjoint action of permutations $\sigma\in S_N$ on the quantum mechanical matrix variables

$$\sigma\colon X^i_j \to (M_\sigma X M_{\sigma^{-1}})^i_j = X^{\sigma(i)}_{\sigma(j)} \qquad (3.1)$$

translates into action on oscillators

$$\sigma\colon\ (a^\dagger)^i_j \to (a^\dagger)^{\sigma(i)}_{\sigma(j)}. \qquad (3.2)$$

We turn our attention to the subspace $\mathcal{H}^{(k)}_{\mathrm{inv}}\subset\mathcal{H}^{(k)}$ of S_N invariant states constructed from polynomials in these oscillators. We will construct bases for ${\mathcal{H}_{\mathrm{inv}}}^{(k)}$, for general k, taking inspiration from [37]. There, a basis for the space of S_N invariant polynomials in matrix indeterminates X^i_j of degree k was given in terms of elements of the diagrammatic partition algebra $P_k(N)$ [75]. With the identification

$$X^i_j \leftrightarrow (a^\dagger)^i_j, \tag{3.3}$$

we can employ these techniques to construct S_N invariant states in matrix quantum mechanics.

The algebra $\mathrm{End}_{S_N}(V_N^{\otimes k})$, of linear operators on $V_N^{\otimes k}$ that commute with S_N, is of central importance in understanding the implications of permutation invariance in quantum mechanical matrix systems. For $N \geq 2k$ this algebra is isomorphic to the partition algebra $P_k(N)$ [75]:

$$\mathrm{End}_{S_N}(V_N^{\otimes k}) \cong P_k(N). \tag{3.4}$$

The Hilbert space $\mathcal{H}^{(k)}_{\mathrm{inv}}$ spanned by degree k polynomials in the oscillators is isomorphic to an S_k invariant subalgebra of $P_k(N)$:

$$\mathcal{H}^{(k)}_{\mathrm{inv}} \cong \mathrm{End}_{S_N\times S_k}(V_N^{\otimes k}) \subseteq \mathrm{End}_{S_N}(V_N^{\otimes k}). \tag{3.5}$$

The partition algebras are finite-dimensional associative algebras with dimension $B(2k)$, the Bell numbers. The Bell numbers $B(k)$ count the number of possible set partitions of a set of k elements. Notably, $B(2k)$ does not depend on N. Consequently, $\mathrm{Dim}\mathcal{H}^{(k)}_{\mathrm{inv}}$ does not grow with N for $N \geq 2k$. This is in contrast to $\mathrm{Dim}\mathcal{H}^{(k)}$, which grows like N^{2k} for $N \gg 2k$.

We have chosen to construct states using the oscillators $(a^\dagger)^i_j$. This produces a basis for $\mathcal{H}_{\mathrm{inv}}$ that is simultaneously an energy eigenbasis of H_0. However, it is worth emphasizing that the resulting description of $\mathcal{H}_{\mathrm{inv}}$ is applicable to any quantum matrix system, not only the system with Hamiltonian H_0. For example, the description of $\mathcal{H}_{\mathrm{inv}}$ in terms of partition algebras holds equally well if the Hamiltonian is a perturbation of H_0 by a polynomial in the matrix creation and annihilation operators.

We begin this Section in III A by reviewing the connection between partition algebras and states in $\mathcal{H}_{\mathrm{inv}}$. The basic algebraic structure of partition algebras is reviewed in Sec. III B. The partition algebras are introduced in the most geometrical basis, the diagram basis, where multiplication is given by diagram concatenation. In Sec. III C we introduce the representation basis, so called because it is labeled by a set of representation theoretic data. This basis uses Fourier transforms [84] on $P_k(N)$ to construct an all-orders orthogonal basis for $N \geq 2k$, which diagonalizes a set of algebraic charges. These charges are discussed in detail in Sec. IV and used in Sec. VI to construct algebraic Hamiltonians with interesting spectra.

A. Partition algebras and invariant tensors

For any $\sigma \in S_N$ we have a linear operator $\mathcal{L}(\sigma) \in \mathrm{End}(V_N^{\otimes k})$ defined by

$$\begin{aligned}\mathcal{L}(\sigma^{-1})&(e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_k})\\ &= e_{\sigma(i_1)}\otimes e_{\sigma(i_2)}\otimes\ldots\otimes e_{\sigma(i_k)}.\end{aligned} \tag{3.6}$$

Here $\sigma \in S_N$ is a bijective map $\{1,\ldots,N\} \to \{1,\ldots,N\}$. Group multiplication is given by composition of maps $\sigma_1\sigma_2(i) = \sigma_2(\sigma_1(i))$ for $\sigma_1,\sigma_2 \in S_N$. This is used to define the adjoint action $\mathrm{Ad}(\sigma)$ of $\sigma \in S_N$ on states $|T\rangle \in \mathcal{H}^{(k)}$,

$$\begin{aligned}\mathrm{Ad}(\sigma)|T\rangle &= \mathrm{Tr}_{V_N^{\otimes k}}[\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k}]|0\rangle,\\ &= \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k}} T^{j_1\ldots j_k}_{i_1\ldots i_k}(a^\dagger)^{\sigma^{-1}(i_1)}_{\sigma^{-1}(j_1)}\ldots(a^\dagger)^{\sigma^{-1}(i_k)}_{\sigma^{-1}(j_k)}|0\rangle,\\ &= \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k}} T^{\sigma(j_1)\ldots\sigma(j_k)}_{\sigma(i_1)\ldots\sigma(i_k)}(a^\dagger)^{i_1}_{j_1}\ldots(a^\dagger)^{i_k}_{j_k}|0\rangle.\end{aligned} \tag{3.7}$$

This adjoint action on the tensor coefficients of the oscillators corresponds to the adjoint action on the oscillators which follows from (3.2). States $|T\rangle \in \mathcal{H}^{(k)}_{\mathrm{inv}}$ are called S_N invariant because they satisfy

$$\mathrm{Ad}(\sigma)|T\rangle = |T\rangle. \tag{3.8}$$

That is, all states in $\mathcal{H}^{(k)}_{\mathrm{inv}}$ can be constructed from tensors satisfying

$$T^{\sigma(j_1)\ldots\sigma(j_k)}_{\sigma(i_1)\ldots\sigma(i_k)} = T^{j_1\ldots j_k}_{i_1\ldots i_k}, \quad \forall\ \sigma \in S_N, \tag{3.9}$$

or

$$\boxed{\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1}) = T.} \tag{3.10}$$

The vector space of S_N invariant linear maps on $V_N^{\otimes k}$ is denoted $\mathrm{End}_{S_N}(V_N^{\otimes k})$. For $N \geq 2k$, $\mathrm{End}_{S_N}(V_N^{\otimes k})$ is isomorphic to the partition algebra $P_k(N)$:

$$\begin{aligned}\mathrm{End}_{S_N}(V_N^{\otimes k}) = \mathrm{Span}_{\mathbb{C}}\{T \in \mathrm{End}(V_N^{\otimes k})\colon\ &\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})\\ &= T,\ \forall\ \sigma\in S_N\} \cong P_k(N).\end{aligned} \tag{3.11}$$

For tensors labeling states we have further S_k invariance. The vector space of $S_N\times S_k$ invariant linear maps is denoted

$$\begin{aligned}\mathrm{End}_{S_N\times S_k}(V_N^{\otimes k}) = \mathrm{Span}_{\mathbb{C}}\{T \in \mathrm{End}(V_N^{\otimes k})\colon\ &\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})\\ &= \mathcal{L}_\tau T\mathcal{L}_{\tau^{-1}} = T, \quad \forall\ \sigma\in S_N, \tau\in S_k\}.\end{aligned} \tag{3.12}$$

and we have the correspondence

$$\mathcal{H}^{(k)}_{\mathrm{inv}} \cong \mathrm{End}_{S_N\times S_k}(V_N^{\otimes k}). \tag{3.13}$$

The partition algebra $P_k(N)$ contains a subalgebra $SP_k(N)$, spanned by elements that commute with $\mathbb{C}[S_k] \subset P_k(N)$, called a symmetrized partition algebra. For $N \geq 2k$, $SP_k(N)$ is isomorphic to $\mathrm{End}_{S_N \times S_k}(V_N^{\otimes k})$, and by extension $\mathcal{H}_{\mathrm{inv}}{}^{(k)}$:

$$\boxed{\mathcal{H}_{\mathrm{inv}}^{(k)} \cong \mathrm{End}_{S_N \times S_k}(V_N^{\otimes k}) \cong SP_k(N).} \tag{3.14}$$

This motivates the next subsection, where we study $P_k(N)$ and its symmetrized subalgebra $SP_k(N)$.

To summarize the above steps in words, we are investigating the adjoint action of permutations in S_N on $N \times N$ quantum mechanical matrix variables X^i_j. The corresponding oscillators inherit the adjoint S_N action. Oscillator states with k oscillators correspond to tensors T with k upper and lower indices, subject to an S_k symmetry permuting the k upper-lower index pairs along the tensor. This S_k symmetry arises from the bosonic nature of the oscillators. The S_N invariant k-oscillator states correspond to tensors having k upper and k lower indices, subject to an $S_N \times S_k$ invariance. This subspace of tensors can be described as a symmetrized subalgebra $SP_k(N)$ of the partition algebra $P_k(N)$.

B. Diagram basis

We introduce the partition algebras in the diagram basis following the treatment in [75]. This is a nice starting point because it gives the most straightforward description of multiplication in $P_k(N)$. As we will see in Sec. VIII, the diagram basis also gives a simple description of an outer product in $P_k(N)$, which is relevant to the discussion of extremal correlators.

The partition algebra $P_k(N)$ is an algebra of dimension $B(2k)$. The Bell number $B(2k)$ is the number of possible partitions of a set with $2k$ distinct elements. Bell numbers can be computed from the generating function

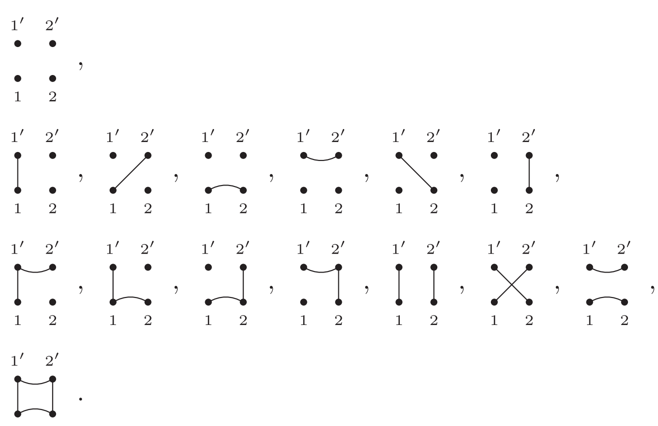
$$\sum_{k=0}^{\infty} \frac{B(k)}{k!} x^k = e^{e^x - 1}, \tag{3.15}$$

from which one finds $B(2k) = 2, 15, 203, 4140$ for $k = 1, 2, 3, 4$.

A set partition π of a set S is a set of disjoint subsets of S such that their union is all of S. The diagram basis for $P_k(N)$ is labeled by set partitions of the set $\{1, \ldots, k, 1', \ldots, k'\}$. The set of all set partitions of $\{1, \ldots, k, 1', \ldots, k'\}$ is denoted Π_{2k}. For example, the set Π_4 contains the following $B(4) = 15$ set partitions (subsets are separated by a vertical bar)

$$\begin{aligned} &1|2|1'|2', \\ &11'|2|2', \quad 12'|1'|2, \quad 12|1'|2', \quad 1'2'|1|2, \quad 1'2|1|2', \quad 22'|1'|1, \\ &11'2'|2, \quad 121'|2', \quad 122'|1', \quad 1'2'2|1, \quad 11'|22', \quad 12'|1'2, \quad 12|1'2', \\ &121'2'. \end{aligned} \tag{3.16}$$

Each $\pi \in \Pi_{2k}$ labels an element of the diagram basis of $P_k(N)$. We write d_π for the diagram basis element corresponding to $\pi \in \Pi_{2k}$. As the name suggests, d_π should be thought of as a diagram. It is a diagram with $2k$ vertices divided into two rows. The bottom vertices are labeled $1, \ldots, k$ from left to right and the vertices of the top row are labeled $1', \ldots, k'$ from left to right. Two vertices are connected by an edge if they belong to the same subset of π. The diagrams corresponding to the set partitions in (3.16) are

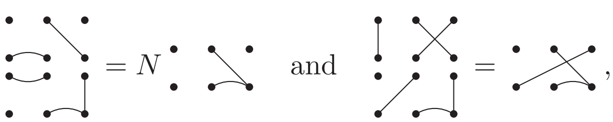


(3.17)

There is a redundancy in the diagram picture. The redundancy arises from the fact that we are free to choose any set of edges, as long as every vertex in a subset of the set partition can be reached from any other vertex in the same subset, by a path along the edges. For example, the following pairs of diagrams correspond to the same element in $P_3(N)$

(3.18)

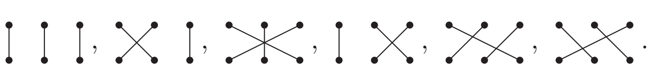
The partition algebras are so-called diagram algebras because multiplication can be defined through diagram concatenation (in the diagram basis). The product in $P_k(N)$ is independent of the choice of representative diagram. Let d_π and $d_{\pi'}$ be two diagrams in $P_k(N)$. The composition $d_{\pi''} = d_\pi d_{\pi'}$ is constructed by placing d_π above $d_{\pi'}$ and identifying the bottom vertices of d_π with the top vertices of $d_{\pi'}$. The diagram is simplified by following the edges connecting the bottom vertices of $d_{\pi'}$ to the top vertices of d_π. Any connected components within the middle rows are removed and we multiply by N^c, where c is the number of these complete blocks removed. For example,



(3.19)

where the factor of N in the first equation comes from removing the middle component at vertex 1 and 2. For linear combinations of diagrams, multiplication is defined by linear extension.

The subset of diagrams with k edges, each connecting a vertex at the top to a vertex at the bottom, where every vertex has exactly one incident edge, span a subalgebra. This subalgebra is isomorphic to the symmetric group algebra $\mathbb{C}[S_k]$. For example, there is a one-to-one correspondence between permutations in S_3 and the following set of diagrams in $P_3(N)$



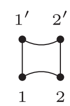
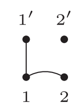
(3.20)

In the language of set partitions, these diagrams correspond to set partitions with subsets of the form $\{ij'\}$ for $i, j \in 1, \ldots, k$. We denote the diagrams forming a basis for $\mathbb{C}[S_k]$ by τ.

The diagram $d_\pi \in P_k(N)$ corresponds to an element of $\mathrm{End}(V_N^{\otimes k})$ through the following action

$$d_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{i_{1'},\ldots,i_{k'}} (d_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} e_{i_{1'}} \otimes \ldots \otimes e_{i_{k'}}. \tag{3.21}$$

The matrix elements $(d_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}$ correspond to the diagram representation by associating a Kronecker delta to every edge connecting a pair of vertices. For example,

 $= \delta_{i_1 i_2} \delta^{i_{2'}}_{i_2} \delta^{i_{2'} i_{1'}}$ and  $= \delta_{i_1 i_2} \delta^{i_{1'}}_{i_1}$. (3.22)

Every diagram corresponds to an S_N invariant tensor in the sense of equation (3.10). As mentioned previously, this gives a basis for $\mathrm{End}_{S_N}(V_N^{\otimes k})$ for $N \geq 2k$ [75].

Due to the bosonic symmetry of the oscillators, the invariant states are not in one-to-one correspondence with elements in $P_k(N)$. Instead, every state in $\mathcal{H}_{\mathrm{inv}}{}^{(k)}$ corresponds to an element in the S_k invariant subalgebra of $P_k(N)$, which we call the symmetrized partition algebra and denote $SP_k(N)$. Consider the action of S_k on the diagrams given, for any $\tau \in S_k, d_\pi \in P_k(N)$, by

$$\tau : d_\pi \to \tau d_\pi \tau^{-1}. \tag{3.23}$$

A basis for $SP_k(N)$ is labeled by distinct orbits under this action. We denote as $[d_\pi] \in SP_k(N)$ the invariant element obtained by averaging over the S_k orbit of d_π:

$$[d_\pi] = \frac{1}{k!} \sum_{\tau \in S_k} \tau d_\pi \tau^{-1} = \frac{1}{|[d_\pi]|} \sum_{d_{\pi'} \in [d_\pi]} d_{\pi'}, \tag{3.24}$$

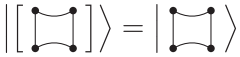
where $|[d_\pi]|$ is the size of the orbit. The equality follows because $|[d_\pi]|$ is equal to $k!$ divided by the number of permutations τ leaving d_π fixed (orbit stabilizer theorem). It follows that a basis for $\mathcal{H}_{\mathrm{inv}}{}^{(k)}$ is labeled by $[d_\pi] \in SP_k(N)$ through the correspondence

$$\begin{aligned} |[d_\pi]\rangle &= \mathrm{Tr}_{V_N^{\otimes k}}([d_\pi](a^\dagger)^{\otimes k})|0\rangle \\ &= \sum_{\substack{i_1,\ldots,i_k \\ i_{1'},\ldots,i_{k'}}} ([d_\pi])^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} (a^\dagger)^{i_1}_{i_{1'}} \ldots (a^\dagger)^{i_k}_{i_{k'}} |0\rangle. \end{aligned} \tag{3.25}$$

Note that

$$|d_\pi\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(d_\pi(a^\dagger)^{\otimes k})|0\rangle = |[d_\pi]\rangle, \tag{3.26}$$

for the sake of notational efficiency we will often label states with d_π instead of $[d_\pi]$. Examples are



$$= \sum_i (a^\dagger)^i_i (a^\dagger)^i_i \,|0\rangle\,, \tag{3.27}$$

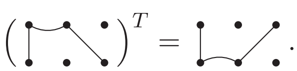
and

$$= \sum_{i,j} (a^\dagger)^i_i (a^\dagger)^i_j \,|0\rangle\,. \tag{3.28}$$

States obtained by acting with the annihilation operators a^i_j on the dual vacuum $\langle 0|$ can also be labeled by partition algebra diagrams as follows:

$$\begin{aligned} \langle d_\pi| &= \langle 0|\mathrm{Tr}_{V_N^{\otimes k}}(d_\pi^T a^{\otimes k}) \\ &= \langle 0|\mathrm{Tr}_{V_N^{\otimes k}}([d_\pi^T] a^{\otimes k}) \\ &= \langle 0| \sum_{\substack{i_1,\ldots,i_k \\ i_{1'},\ldots,i_{k'}}} ([d_\pi])^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}\, a^{i_{1'}}_{i_1}\ldots a^{i_{k'}}_{i_k} \\ &= \langle 0| \sum_{\substack{i_1,\ldots,i_k \\ i_{1'},\ldots,i_{k'}}} ([d_\pi^T])^{i_1\ldots i_k}_{i_{1'}\ldots i_{k'}}\, a^{i_{1'}}_{i_1}\ldots a^{i_{k'}}_{i_k}, \end{aligned} \tag{3.29}$$

where d_π^T is the transpose of d_π. As a diagram, d_π^T is the reflection of d_π across a horizontal line, for example



(3.30)

The use of the transpose in this definition is motivated by the orthonormality property below (3.33). Using the commutation relations in Eq. (2.15), the inner product can be written as a trace of products of elements in $SP_k(N)$,

$$\langle d_\pi||d_{\pi'}\rangle = \sum_{\tau\in S_k} (d_\pi^T \tau d_{\pi'} \tau^{-1})^{i_1\ldots i_k}_{i_1\ldots i_k} = \sum_{\tau\in S_k} \mathrm{Tr}_{V_N^{\otimes k}}(d_\pi^T \tau d_{\pi'}\tau^{-1}). \tag{3.31}$$

The large N factorization result in [37] implies that the normalized states

$$|\hat{d}_\pi\rangle = \frac{1}{\sqrt{\langle d_\pi||d_\pi\rangle}}|d_\pi\rangle, \tag{3.32}$$

are orthonormal at large N (to leading order in $1/\sqrt{N}$)

$$\langle \hat{d}_\pi||\hat{d}_{\pi'}\rangle = \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_\pi] = [d_{\pi'}] \\ 0 + O(1/\sqrt{N}) & \text{otherwise} \end{cases}. \tag{3.33}$$

C. Representation basis

The connection between S_N invariant states and partition algebras gives rise to a natural basis, labeled by representation theoretic data. The representation basis diagonalizes a set of commuting algebraic charges that we introduce in Sec. IV. This observation gives a concrete construction algorithm for the change of basis matrix (from diagram basis to representation basis). We now describe how the representation theoretic basis for $SP_k(N)$ arises using Schur-Weyl duality between S_N and $P_k(N)$, along with the implementation of the invariance under the S_k action of (3.23) in the representation theoretic basis. The transition from a combinatorial basis of diagrams in an algebra defined by physical constraints (in this case a bosonic symmetry of matrix oscillators) to a representation theoretic basis is an example of Fourier transformation which has been useful in a multimatrix as well as tensor systems of interest in AdS/CFT and holography (a short review of these applications is in [71]). The proofs of some statements quoted here are in Appendix A.

From the point of view of representation theory, the correspondence between permutation invariant states and partition algebras should be understood as a consequence of Schur-Weyl duality. In particular, Schur-Weyl duality says that the decomposition (see Sec. II. 5 in [79])

$$V_N^{\otimes k} \cong \bigoplus_{l=0}^{k} \bigoplus_{\Lambda_1^\# \vdash l} V^{S_N}_{[N-l,\Lambda_1^\#]} \otimes V^{P_k(N)}_{[N-l,\Lambda_1^\#]} \tag{3.34}$$

is multiplicity free in terms of irreducible representations of S_N and $P_k(N)$. The Young diagram $\Lambda_1 = [N-l, \Lambda_1^\#]$, which is an integer partition of N, is constructed by placing the diagram $\Lambda_1^\#$ (having l boxes) below a first row of $N-l$ boxes. Requiring Λ_1 to be a valid Young diagram imposes a condition on the first row length of $r_1(\Lambda_1^\#) \leq N - l$. This condition is nontrivial for $N < 2k$, while it is trivially satisfied for all $\Lambda_1^\#$ having up to k boxes for $N \geq 2k$. The latter is called the stable limit. In this limit we can write the decomposition (3.34) in a simplified form

$$V_N^{\otimes k} = \bigoplus_{\Lambda_1 \in \mathcal{Y}_S(k)} V^{S_N}_{\Lambda_1} \otimes V^{P_k(N)}_{\Lambda_1}, \tag{3.35}$$

in which the sum can be labeled by the set of all Young diagrams $\Lambda_1^\#$ having up to k boxes: these are inserted below a first row to form Young diagrams with N boxes. This stable set of Young diagrams having N boxes is denoted $\mathcal{Y}_S(k)$. With the exception of Appendix B this is the limit within which we will work.

Equation (3.7) implies that we can identify

$$\mathrm{End}(V_N^{\otimes k}) \cong V_N^{\otimes k} \otimes V_N^{\otimes k}, \tag{3.36}$$

as a representation of S_N. We use Schur-Weyl duality (3.35) to decompose each factor on the rhs as

$$V_N^{\otimes k} \otimes V_N^{\otimes k} = \left(\bigoplus_{\Lambda_1 \in \mathcal{Y}_S(k)} V_{\Lambda_1}^{S_N} \otimes V_{\Lambda_1}^{P_k(N)} \right) \otimes \left(\bigoplus_{\Lambda_1' \in \mathcal{Y}_S(k)} V_{\Lambda_1'}^{S_N} \otimes V_{\Lambda_1'}^{P_k(N)} \right), \tag{3.37}$$

where we are assuming the stable limit. Projecting to S_N invariants on both sides gives

$$P_k(N) \cong \mathrm{End}_{S_N}(V_N^{\otimes k}) \cong \bigoplus_{\Lambda_1 \in \mathcal{Y}_S(k)} V_{\Lambda_1}^{P_k(N)} \otimes V_{\Lambda_1}^{P_k(N)}. \tag{3.38}$$

This follows because the decomposition of $V_{\Lambda_1}^{S_N} \otimes V_{\Lambda_1'}^{S_N}$ contains an invariant if and only if $\Lambda_1 = \Lambda_1'$.

The rhs of (3.38) reflects a decomposition of $P_k(N)$ into a direct sum of matrix algebras. Such a decomposition always exists for a semisimple algebra by the Artin-Wedderburn theorem. This implies that there exists a basis of generalized elementary matrices (also called a complete set of matrix units) for $P_k(N)$. A complete set of matrix units is a basis

$$Q_{\alpha\beta}^{\Lambda_1}, \quad \Lambda_1 \in \mathcal{Y}_S(k), \quad \alpha, \beta \in \{1, \ldots, \mathrm{Dim}(V_{\Lambda_1}^{P_k(N)})\}, \tag{3.39}$$

with the property

$$Q_{\alpha\beta}^{\Lambda_1} Q_{\alpha'\beta'}^{\Lambda_1'} = \delta^{\Lambda_1 \Lambda_1'} \delta_{\beta\alpha'} Q_{\alpha\beta'}^{\Lambda_1}. \tag{3.40}$$

In other words, $P_k(N)$ can be realized as block-diagonal matrices, with each block labeled by an irreducible representation Λ_1 of $P_k(N)$. The Artin-Wedderburn decomposition implies

$$\mathrm{Dim}(P_k(N)) = B(2k) = \sum_{\Lambda_1 \in \mathcal{Y}_S(k)} (\mathrm{Dim} V_{\Lambda_1}^{P_k(N)})^2, \tag{3.41}$$

which is analogous to the expression

$$|G| = \sum_{R \in \mathrm{Rep}(G)} (\mathrm{Dim} V_R^G)^2 \tag{3.42}$$

for the order of a finite group G in terms of its irreducible representations R.

As we prove in Appendix A 3, the following set of linear combinations of elements in $P_k(N)$ form a complete set of matrix units for $P_k(N)$,

$$\boxed{Q_{\alpha\beta}^{\Lambda_1} = \sum_{i=1}^{B(2k)} \mathrm{Dim}(V_{\Lambda_1}^{S_N}) D_{\beta\alpha}^{\Lambda_1}((b_i^*)^T) b_i.} \tag{3.43}$$

The coefficients $D_{\beta\alpha}^{\Lambda_1}(d)$ are matrix elements of the representation of $P_k(N)$, labeled by $\Lambda_1 \vdash N$. The sum is over a basis $b_i, i \in \{1, \ldots, B(2k)\}$ for $P_k(N)$ (for example the diagram basis). The element b_i^* is called the dual of b_i. It has an explicit construction in terms of the inverse of the Gram matrix defined by

$$g_{ij} = \mathrm{Tr}_{V_N^{\otimes k}}(b_i b_j^T). \tag{3.44}$$

The dual of b_i is

$$b_i^* = \sum_{j=1}^{B(2k)} g_{ij}^{-1} b_j, \tag{3.45}$$

and the inverse of the Gram matrix in the diagram basis can be written as a series expansion in N [see Eq. (D6)].

To construct a representation basis for $\mathcal{H}_{\mathrm{inv}}{}^{(k)}$, we need to construct matrix units for $SP_k(N)$. They can be constructed from matrix units for $P_k(N)$ as follows. The partition algebra $P_k(N)$ contains a subalgebra $\mathbb{C}[S_k]$. Consequently, we can restrict an irreducible representation $V_{\Lambda_1}^{P_k(N)}$ to a representation of $\mathbb{C}[S_k]$, which in general is reducible. Letting $V_{\Lambda_2}^{\mathbb{C}[S_k]}$ be an irreducible representation of $\mathbb{C}[S_k]$ labeled by a Young diagram Λ_2 with k boxes, we have

$$V_{\Lambda_1}^{P_k(N)} \cong \bigoplus_{\Lambda_2 \vdash k} V_{\Lambda_2}^{\mathbb{C}[S_k]} \otimes V_{\Lambda_1 \Lambda_2}^{P_k(N) \to \mathbb{C}[S_k]}. \tag{3.46}$$

The dimension of $V_{\Lambda_1 \Lambda_2}^{P_k(N) \to \mathbb{C}[S_k]}$ is the branching multiplicity

$$\mathrm{Dim}(V_{\Lambda_1 \Lambda_2}^{P_k(N) \to \mathbb{C}[S_k]}) = \mathrm{Mult}(V_{\Lambda_1}^{P_k(N)} \to V_{\Lambda_2}^{\mathbb{C}[S_k]}). \tag{3.47}$$

In the rest of the paper we will use Λ_1 to label irreducible representations of S_N and $P_k(N)$. Irreducible representations of S_k are denoted by Λ_2. Inserting the decomposition (3.46) into Eq. (3.38) and projecting to S_k invariants gives

$$\boxed{\mathcal{H}_{\mathrm{inv}}{}^{(k)} \cong \mathrm{End}_{S_N \times S_k}(V_N^{\otimes k}) \cong \bigoplus_{\substack{\Lambda_1 \in \mathcal{Y}_S(k) \\ \Lambda_2 \vdash k}} V_{\Lambda_1 \Lambda_2}^{P_k(N) \to \mathbb{C}[S_k]} \otimes V_{\Lambda_1 \Lambda_2}^{P_k(N) \to \mathbb{C}[S_k]}.} \tag{3.48}$$

This should be understood as an Artin-Wedderburn decomposition of $SP_k(N)$.

Equation (3.46) points us towards a construction of matrix units for $SP_k(N)$ from matrix units of $P_k(N)$. On the lhs we have a basis

$$E_\alpha^{\Lambda_1}, \qquad \alpha \in \{1, \ldots \mathrm{Dim}(V_{\Lambda_1}^{P_k(N)})\}, \tag{3.49}$$

where the representation of $d \in P_k(N)$ is irreducible,

$$d(E_\alpha^{\Lambda_1}) = \sum_\beta D_{\beta\alpha}^{\Lambda_1}(d) E_\beta^{\Lambda_1}. \tag{3.50}$$

The rhs has a basis

$$E_{\Lambda_2,p}^{\Lambda_1,\mu}, \qquad p \in \left\{1, \ldots, \mathrm{Dim}\left(V_{\Lambda_1}^{\mathbb{C}[S_k]}\right)\right\}, \quad \mu \in \left\{1, \ldots, \mathrm{Dim}\left(V_{\Lambda_1\Lambda_2}^{P_k(N)\to\mathbb{C}[S_k]}\right)\right\}, \tag{3.51}$$

where μ is a multiplicity label for $V_{\Lambda_2}^{\mathbb{C}[S_k]}$ in the decomposition. We demand that the representation of $\tau \in \mathbb{C}[S_k]$ is irreducible in this basis,

$$\tau(E_{\Lambda_2,p}^{\Lambda_1,\mu}) = \sum_q D_{qp}^{\Lambda_2}(\tau) E_{\Lambda_2,q}^{\Lambda_1,\mu}, \tag{3.52}$$

where $D_{qp}^{\Lambda_2}(\tau)$ is an irreducible representation of $\tau \in \mathbb{C}[S_k]$. The change of basis coefficients are called branching coefficients

$$E_{\Lambda_2,p}^{\Lambda_1,\mu} = \sum_\alpha B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]} E_\alpha^{\Lambda_1}, \tag{3.53}$$

or in bracket notation

$$B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]} = \langle E_\alpha^{\Lambda_1} | E_{\Lambda_2,p}^{\Lambda_1,\mu} \rangle. \tag{3.54}$$

The elements

$$\boxed{Q_{\Lambda_2,\mu\nu}^{\Lambda_1} = \sum_{\alpha,\beta,p} Q_{\alpha\beta}^{\Lambda_1} B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]} B_{\Lambda_1,\beta\to\Lambda_2,p;\nu}^{P_k(N)\to\mathbb{C}[S_k]}} \tag{3.55}$$

form a complete set of matrix units for $SP_k(N)$. The sum over p implements the projection to S_k invariants. The above elements satisfy [see Eq. (A62)]

$$Q_{\Lambda_2,\mu\nu}^{\Lambda_1} Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'} = \delta^{\Lambda_1\Lambda_1'} \delta_{\Lambda_2\Lambda_2'} \delta_{\nu\mu'} Q_{\Lambda_2\mu\nu'}^{\Lambda_1}, \tag{3.56}$$

and orthogonality of states

$$|Q_{\Lambda_2,\mu\nu}^{\Lambda_1}\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\nu}^{\Lambda_1} (a^\dagger)^{\otimes k}) \tag{3.57}$$

follows from the form of the inner product (3.31). The proof goes as follows

$$\begin{aligned} \langle Q_{\Lambda_2,\mu\nu}^{\Lambda_1} | Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'} \rangle &= \sum_{\tau\in S_k} \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\nu}^{\Lambda_1} \tau (Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'})^T \tau^{-1}), \\ &= \sum_{\tau\in S_k} \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\nu}^{\Lambda_1} \tau Q_{\Lambda_2',\nu'\mu'}^{\Lambda_1'} \tau^{-1}), \\ &= k! \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\nu}^{\Lambda_1} Q_{\Lambda_2',\nu'\mu'}^{\Lambda_1'}), \\ &= k! \delta^{\Lambda_1\Lambda_1'} \delta_{\Lambda_2\Lambda_2'} \delta_{\nu\nu'} \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\mu'}^{\Lambda_1}). \end{aligned} \tag{3.58}$$

In the second equality we used $(Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'})^T = Q_{\Lambda_2',\nu'\mu'}^{\Lambda_1'}$ which follows from Eq. (A25). Note that

$$\begin{aligned} \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu\mu'}^{\Lambda_1}) &= \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,\mu 1}^{\Lambda_1} Q_{\Lambda_2,1\mu'}^{\Lambda_1}), \\ &= \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,1\mu'}^{\Lambda_1} Q_{\Lambda_2,\mu 1}^{\Lambda_1}), \\ &= \delta_{\mu\mu'} \mathrm{Tr}_{V_N^{\otimes k}}(Q_{\Lambda_2,11}^{\Lambda_1}), \\ &= \delta_{\mu\mu'} \mathcal{N}_{\Lambda_1\Lambda_2}, \end{aligned} \tag{3.59}$$

such that the normalization [see Eq. (A66)]

$$\mathcal{N}_{\Lambda_1\Lambda_2} = \mathrm{Dim} V_{\Lambda_1}^{S_N} \mathrm{Dim} V_{\Lambda_2}^{S_k}, \tag{3.60}$$

only depends on irreducible representations Λ_1, Λ_2, which proves orthogonality.

To summarize, we have shown that there exists an orthogonal basis for $\mathcal{H}_{\mathrm{inv}}{}^{(k)}$ labeled by representation theoretic data, for arbitrary $N \geq 2k$, using Fourier transforms on semisimple algebras. In Appendix A we provide the detailed proofs of these results. In the next section we will provide explicit formulas for the change of basis from the diagram basis to the basis of matrix units. We leave the elucidation of finite N effects (the case $N < 2k$ which lies beyond the stable limit) in the representation basis for future work.

IV. REPRESENTATION BASIS AND ALGEBRAIC CHARGES

In this section we discuss the construction of the representation basis elements $Q_{\Lambda_2,\mu\nu}^{\Lambda_1}$ as linear combinations of diagrams in $P_k(N)$. These can, in principle, be computed using Eq. (3.55) by first computing the branching coefficients. The computation of these requires explicit choices of basis in the representations $V_{\Lambda_1}^{P_k(N)}$ and $V_{\Lambda_2}^{\mathbb{C}[S_k]}$. Such choices can be bypassed. The basic idea is to find the $Q_{\Lambda_2,\mu\nu}^{\Lambda_1}$ as eigenvectors of appropriate elements of $P_k(N)$ which can be viewed as operators on $P_k(N)$ acting by the algebra multiplication. The subspaces labeled by Λ_1, Λ_2, associated with irreducible representations of S_N and S_k, respectively, are identified using central elements (Casimirs) in the group algebras $\mathbb{C}[S_N]$ and $\mathbb{C}[S_k]$. These Casimirs can be expressed as elements of $P_k(N)$ using Schur-Weyl duality. This is particularly useful in the large N limit where k is kept fixed and $N \gg k$, since the dimension of $P_k(N)$ does not grow with N. The more refined determination of subspaces labeled by μ and ν is achieved by picking noncentral elements of $P_k(N)$ which nevertheless generate a maximally commuting subalgebra.

We explicitly construct the change of basis for the special cases of degree $k = 1, 2, 3$. Tables of these basis elements are found in Appendix C. The expansion coefficients are given as functions of N and are therefore valid for all $N \geq 2k$.

Analogous constructions in multimatrix systems with continuous gauge symmetry, relevant to AdS/CFT, are given in [62,89]. They also played a role, using developments in tensor models with $U(N)$ symmetries, in [90] in giving a combinatorial interpretation of Kronecker coefficients.

A. Central elements in the partition algebra

For a fixed pair Λ_1, Λ_2, the linear span of $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ for $\mu, \nu = 1, \ldots, \mathrm{Dim}V^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1\Lambda_2}$ forms a subspace of $SP_k(N)$. We will now describe how this subspace can be identified with simultaneous eigenspaces of Casimirs associated with $\mathbb{C}[S_N]$ and $\mathbb{C}[S_k]$.

First, we will define Casimirs of $\mathbb{C}[S_N]$, and explain their relation to $P_k(N)$. The center $\mathcal{Z}(\mathbb{C}[S_N])$ of $\mathbb{C}[S_N]$ consists of elements

$$\mathcal{Z}(\mathbb{C}[S_N]) = \{z \in \mathbb{C}[S_N] : z\sigma = \sigma z, \quad \forall\ \sigma \in \mathbb{C}[S_N]\}. \tag{4.1}$$

Elements in the center are called central elements. For a central element z, the homomorphism property of representations implies

$$\mathcal{L}(z)\mathcal{L}(\sigma) = \mathcal{L}(\sigma)\mathcal{L}(z), \quad \forall\ \sigma \in S_N, \tag{4.2}$$

and it follows that $\mathcal{L}(z)$ is an element of the algebra of operators acting on $V_N^{\otimes k}$ which commutes with S_N. This algebra is denoted $\mathrm{End}_{S_N}(V_N^{\otimes k})$.

As we reviewed in the previous section, $P_k(N) \cong \mathrm{End}_{S_N}(V_N^{\otimes k})$ for $N \geq 2k$. This establishes a connection between $\mathcal{Z}(\mathbb{C}[S_N])$ and $P_k(N)$ as linear operators acting on $V_N^{\otimes k}$. In particular, for every $z \in \mathcal{Z}(\mathbb{C}[S_N])$, there exists an element $\bar{z} \in P_k(N)$ defined by

$$\boxed{\bar{z}(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \mathcal{L}(z)(e_{i_1} \otimes \ldots \otimes e_{i_k}).} \tag{4.3}$$

Note that the definition of $\bar{z}$ depends on k. Further, observe that

$$\mathcal{L}(z)d(e_{i_1} \otimes \ldots \otimes e_{i_k}) = d\mathcal{L}(z)(e_{i_1} \otimes \ldots \otimes e_{i_k}), \tag{4.4}$$

for all $d \in P_k(N)$ because $P_k(N)$ and $\mathbb{C}[S_N]$ are mutual commutants in $\mathrm{End}(V_N^{\otimes k})$. This implies that $\bar{z}$ is automatically in the center of $P_k(N)$, which we denote $\mathcal{Z}(P_k(N))$. In other words, Eq. (4.3) defines a homomorphism from $\mathcal{Z}(\mathbb{C}[S_N])$ to $\mathcal{Z}(P_k(N))$. As a particular case of being central in $P_k(N)$, $\bar{z}$ commutes with $\mathbb{C}[S_k] \subset P_k(N)$.

Central elements play a special role in representation theory. Schur's lemma implies that an irreducible matrix representation of a central element is proportional to the identity matrix. The proportionality constant is a normalized character. In particular we have

$$D^{\Lambda_1}_{ab}(z) = \hat{\chi}^{\Lambda_1}(z)\delta_{ab}, \tag{4.5}$$

where we have introduced the short-hand

$$\hat{\chi}^{\Lambda_1}(z) = \frac{\chi^{\Lambda_1}(z)}{\mathrm{Dim}V^{S_N}_{\Lambda_1}} \tag{4.6}$$

for normalized characters. In this sense central elements are Casimirs, they act by constants on irreducible subspaces, and the constants can be used to determine the particular representation.

The element of $\mathbb{C}[S_N]$ formed by summing over all elements in a distinct conjugacy class of S_N is central. For example, we define the element $T_2 \in \mathcal{Z}(\mathbb{C}[S_N])$ as

$$T_2 = \sum_{1\leq i<j\leq N} (ij), \tag{4.7}$$

where the sum is over all transpositions. By the argument in the previous paragraph, there exists an element $\bar{T}_2^{(k)} \in \mathcal{Z}(P_k(N))$ such that

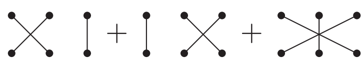
$$\begin{array}{ccc} \bar{T}_2^{(k)}(e_{i_1} \otimes \cdots \otimes e_{i_k}) & = & \mathcal{L}(T_2)(e_{i_1} \otimes \cdots \otimes e_{i_k}) \\ \| & & \| \\ \displaystyle\sum_{i_{1'}\ldots i_{k'}} (\bar{T}_2^{(k)})^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} e_{i_{1'}} \otimes \cdots \otimes e_{i_{k'}} & = & \displaystyle\sum_{\substack{\sigma=(ij)\\ 1\leq i<j\leq N}} e_{\sigma^{-1}(i_1)} \otimes \cdots \otimes e_{\sigma^{-1}(i_k)}. \end{array} \tag{4.8}$$

As we will explain, the eigenvalues of the central element $\bar{T}_2^{(k)}$ can be used to distinguish the label Λ_1 on matrix units. Since $\bar{T}_2^{(k)}$ is an element of $SP_k(N)$, it has an expansion in terms of diagrams [see Eq. (3.32), Theorem 3.35 in [75]]

$$\boxed{\bar{T}_2^{(k)} = \sum_{\pi \in \Pi_{2k}} (\bar{T}_2^{(k)})^\pi d_\pi.} \tag{4.9}$$

The equality in (4.8) implies a radical simplification for large N. The element T_2 contains order N^2 transpositions, while $\bar{T}_2^{(k)}$ contains at most $B(2k)$ diagrams. The dependence on N is incorporated in the coefficients $(\bar{T}_2^{(k)})^\pi$, which are polynomial functions of N. Explicit examples are in (4.34), (4.39), and (4.47).

There exist similar elements $t_2^{(k)} \in \mathcal{Z}(\mathbb{C}[S_k]) \subset \mathcal{Z}(P_k(N))$ defined by summing over transposition diagrams. For example,

$t_2^{(2)} =$  , $t_2^{(3)} =$  . (4.10)

The eigenvalues of $t_2^{(k)}$ will be used to distinguish the label Λ_2.

Equation (4.8) together with Eq. (4.5) gives

$$D^{\Lambda_1}_{\alpha\beta}(\bar{T}_2^{(k)}) = \frac{\chi^{\Lambda_1}(\bar{T}_2^{(k)})}{\mathrm{Dim} V^{P_k(N)}_{\Lambda_1}} \delta_{\alpha\beta} = \hat{\chi}^{\Lambda_1}(T_2)\delta_{\alpha\beta}, \tag{4.11}$$

where the distinction between the two characters is

$$\chi^{\Lambda_1}(\bar{T}_2^{(k)}) = \sum_{\alpha=1}^{\mathrm{Dim} V^{P_k(N)}_{\Lambda_1}} D^{\Lambda_1}_{\alpha\alpha}(\bar{T}_2^{(k)}), \quad \text{and}$$
$$\chi^{\Lambda_1}(T_2) = \sum_{a=1}^{\mathrm{Dim} V^{S_N}_{\Lambda_1}} D^{\Lambda_1}_{aa}(T_2). \tag{4.12}$$

That is, the first character is a character of $P_k(N)$, the second is a character of $\mathbb{C}[S_N]$. Similarly,

$$D^{\Lambda_2}_{pq}(t_2^{(k)}) = \hat{\chi}^{\Lambda_2}(t_2^{(k)})\delta_{pq}, \tag{4.13}$$

where

$$\hat{\chi}^{\Lambda_2}(t_2^{(k)}) = \frac{\chi^{\Lambda_2}(t_2^{(k)})}{\mathrm{Dim} V^{S_k}_{\Lambda_2}}. \tag{4.14}$$

Normalized characters of T_2 and $t_2^{(k)}$ can be expressed in terms of combinatorial quantities (known as the contents) of boxes of Young diagrams (see example 7 in Sec. I. 7 of [91]). Let $Y_{\Lambda_1}, Y_{\Lambda_2}$ be the Young diagrams corresponding to integer partitions $\Lambda_1 \in \mathcal{Y}_{\mathcal{S}}(k), \Lambda_2 \vdash k$. Then

$$\hat{\chi}^{\Lambda_1}(T_2) = \sum_{(i,j)\in Y_{\Lambda_1}} (j-i), \quad \hat{\chi}^{\Lambda_2}(t_2^{(k)}) = \sum_{(i,j)\in Y_{\Lambda_2}} (j-i), \tag{4.15}$$

where (i, j) corresponds to the cell in the ith row and jth column of the Young diagram [the top left box has coordinate (1,1)].

With the above facts at hand, we can understand how the Λ_1, Λ_2 labels correspond to eigenvalues of $\bar{T}_2^{(k)}, t_2^{(k)}$. As we prove in Appendix A 3 the $P_k(N)$ matrix units have the property

$$dQ^{\Lambda_1}_{\alpha\beta} = \sum_\gamma D^{\Lambda_1}_{\gamma\alpha}(d) Q^{\Lambda_1}_{\gamma\beta}, \quad \text{for } d \in P_k(N), \tag{4.16}$$

and therefore

$$Q^{\Lambda_1}_{\alpha\beta}\bar{T}_2^{(k)} = \bar{T}_2^{(k)} Q^{\Lambda_1}_{\alpha\beta} = \sum_\gamma D^{\Lambda_1}_{\gamma\alpha}(\bar{T}_2^{(k)}) Q^{\Lambda_1}_{\gamma\beta} = \hat{\chi}^{\Lambda_1}(T_2) Q^{\Lambda_1}_{\alpha\beta}. \tag{4.17}$$

We derive a similar equation for $t_2^{(k)}$ acting on $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ using the definition in (3.55). From the definition we have

$$\begin{aligned} t_2^{(k)} Q^{\Lambda_1}_{\Lambda_2,\mu\nu} &= \sum_{\alpha,\beta,p} t_2^{(k)} Q^{\Lambda_1}_{\alpha\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}, \\ &= \sum_{\alpha,\beta,\gamma,\gamma',p} D^{\Lambda_1}_{\gamma\alpha}(t_2^{(k)}) \delta_{\gamma\gamma'} Q^{\Lambda_1}_{\gamma'\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}. \end{aligned} \tag{4.18}$$

We rewrite the Kronecker delta using the completeness relation

$$\sum_{\Lambda_2',p',\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2',p';\mu'} = \delta_{\gamma\gamma'}. \tag{4.19}$$

Inserting this into (4.18) gives

$$\sum_{\alpha,\beta,\gamma,\gamma',p} D^{\Lambda_1}_{\gamma\alpha}(t_2^{(k)})\delta_{\gamma\gamma'}Q^{\Lambda_1}_{\gamma'\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}$$
$$= \sum_{\alpha,\beta,\gamma,\gamma',p}\sum_{\Lambda_2',p',\mu'} D^{\Lambda_1}_{\gamma\alpha}(t_2^{(k)})B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2',p';\mu'}Q^{\Lambda_1}_{\gamma'\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}. \tag{4.20}$$

Now note that

$$\sum_{\gamma,\alpha}D^{\Lambda_1}_{\gamma\alpha}(t_2^{(k)})B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} = \delta_{\Lambda_2\Lambda_2'}\delta_{\mu'\mu}D^{\Lambda_2}_{p'p}(t_2^{(k)}) = \delta_{\Lambda_2\Lambda_2'}\delta_{\mu'\mu}\delta_{p'p}\hat{\chi}^{\Lambda_2}(t_2^{(k)}). \tag{4.21}$$

We substitute this into (4.20) and find

$$\sum_{\alpha,\beta,\gamma,\gamma',p}\sum_{\Lambda_2',p',\mu'} D^{\Lambda_1}_{\gamma\alpha}(t_2^{(k)})B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2',p';\mu'}Q^{\Lambda_1}_{\gamma'\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}$$
$$= \sum_{\beta,\gamma',p}\sum_{\Lambda_2',p',\mu'}\delta_{\Lambda_2\Lambda_2'}\delta_{\mu'\mu}\delta_{p'p}\hat{\chi}^{\Lambda_2}(t_2^{(k)})B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2',p';\mu'}Q^{\Lambda_1}_{\gamma'\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}$$
$$= \sum_{\beta,\gamma',p}\hat{\chi}^{\Lambda_2}(t_2^{(k)})Q^{\Lambda_1}_{\gamma'\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2,p;\mu}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu} = \hat{\chi}^{\Lambda_2}(t_2^{(k)})Q^{\Lambda_1}_{\Lambda_2,\mu\nu}, \tag{4.22}$$

which proves the analog of (4.17) in the case of $t_2^{(k)}$.

We define linear operators on $SP_k(N)$ using multiplication by $\bar{T}_2^{(k)}, t_2^{(k)}$

$$\boxed{\bar{T}_2^{(k)}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}) = \bar{T}_2^{(k)}Q^{\Lambda_1}_{\Lambda_2,\mu\nu} = \hat{\chi}^{\Lambda_1}(T_2)Q^{\Lambda_1}_{\Lambda_2,\mu\nu},} \tag{4.23}$$

and

$$\boxed{t_2^{(k)}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}) = t_2^{(k)}Q^{\Lambda_1}_{\Lambda_2,\mu\nu} = \hat{\chi}^{\Lambda_2}(t_2^{(k)})Q^{\Lambda_1}_{\Lambda_2,\mu\nu}.} \tag{4.24}$$

That is, the matrix units for $SP_k(N)$ are eigenvectors of the linear operators associated with $\bar{T}_2^{(k)}, t_2^{(k)}$. The eigenvalues are sufficient to determine the subspaces labeled by irreducible representations Λ_1, Λ_2 for $k = 1$, 2, 3 and general N. As discussed in detail in [89], a larger set of central elements is needed to distinguish different pairs Λ_1, Λ_2 for general k and N.

B. Multiplicity labels and maximal commuting subalgebras

In the previous subsection we described how the subspace spanned by $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ for fixed Λ_1, Λ_2 is a simultaneous eigenspace of central elements $\bar{T}_2^{(k)}, t_2^{(k)}$. The subspaces labeled by fixed μ, ν are not eigenspaces of any central elements of $SP_k(N)$. Nevertheless, they are eigenspaces of elements that (multiplicatively) generate a maximal commutative subalgebra of $SP_k(N)$.

We illustrate this in the simple case of a single matrix algebra. This is directly relevant, because the matrix units $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ form (are isomorphic to) a matrix algebra M_n with $n = \mathrm{Dim}V^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1\Lambda_2}$, for fixed Λ_1, Λ_2. The matrix algebra M_n has a basis of matrix units E_{rs} for $r, s = 1, \ldots, n$. These are just the elementary matrices with zeroes everywhere except in row r, column s where there is a one. In this explicitly realized algebra, it is straightforward to verify that

$$E_{rs}E_{r's'} = \delta_{sr'}E_{rs'}. \tag{4.25}$$

It follows from Eq. (4.25) that

$$E_{tt}E_{rs} = \delta_{tr}E_{ts} = \begin{cases} E_{rs} & \text{if } r = t \\ 0 & \text{otherwise} \end{cases}. \tag{4.26}$$

This fact will be useful in what follows.

We will now define a pair of linear operators acting on M_n whose eigenvalues uniquely determine the indices r, s on E_{rs}. Let

$$T = 1E_{11} + 2E_{22} + \cdots + nE_{nn}, \tag{4.27}$$

and T^L, T^R be the linear operators on M_n defined by left and right action of T, respectively,

$$T^L(E_{rs}) = TE_{rs}, \qquad T^R(E_{rs}) = E_{rs}T. \tag{4.28}$$

The $n^2 \times n^2$ matrix $(T^L)^{tu}_{rs}$ associated with the linear operator T^L has eigenvalues $\{1, 2, \ldots, n\}$ (each one is n-fold degenerate) with eigenvectors E_{rs},

$$\sum_{t,u}(T^L)^{tu}_{rs}E_{tu} = T^L(E_{rs}) = rE_{rs}. \tag{4.29}$$

Similarly for the matrix $(T^R)^{tu}_{rs}$ associated with the linear operator T^R,

$$\sum_{t,u}(T^R)^{tu}_{rs}E_{tu} = T^R(E_{rs}) = sE_{rs}. \tag{4.30}$$

The operators T^L and T^R commute, and their simultaneous eigenvectors E_{rs} have eigenvalues r and s, respectively.

The algebra spanned by $\{E_{11}, E_{22}, \ldots, E_{nn}\}$ is a maximal commuting subalgebra of M_n. It is multiplicatively generated by T. In particular (see Lemma 3.3.1 of [89] or Lemma 2.1 of [92])

$$E_{rr} = \prod_{s\neq r}\frac{(T-s)}{(r-s)}. \tag{4.31}$$

These ideas generalize to $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$, and in the next section we will give the appropriate operators corresponding to T^L, T^R for $SP_2(N)$.

C. Construction of low degree representation bases

We now use the tools presented in this section to explicitly construct the representation basis elements as sums of diagrams, for $k = 1, 2, 3$ and large N. Tables of the representation basis elements expanded in terms of diagrams are found in Appendix C. The associated Sage code can be found together with the arXiv version of this paper.

1. Degree one basis

For $k = 1$ it is enough to use $\bar{T}_2{}^{(1)}$ to distinguish the irreducible representations. We expect to find matrix units

$$Q^{[N]}_{[1]}, Q^{[N-1,1]}_{[1]}, \tag{4.32}$$

since S_1 only has the trivial representation and the decomposition in (3.35) only contains irreducible representations $[N]$ and $[N-1, 1]$ of $P_1(N)$.

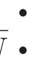
The map

$$T_2 \mapsto \bar{T}^{(1)}_2 \tag{4.33}$$

is given by [see the section called Murphy elements for $\mathbb{C}A_k(N)$ in [75]]

$$\bar{T}^{(1)}_2 = \frac{N(N-3)}{2}$$  $+$  . (4.34)

It is straightforward to diagonalize $\bar{T}^{(1)}_2$ acting on $P_1(N)$ from the left. We define

$$Q^{[N]}_{[1]} = \frac{1}{N}$$  , $Q^{[N-1,1]}_{[1]} =$ $- \frac{1}{N}$  , (4.35)

and they satisfy

$$Q^{[N]}_{[1]}Q^{[N-1,1]}_{[1]} = 0, \quad Q^{[N]}_{[1]}Q^{[N]}_{[1]} = Q^{[N]}_{[1]}, \quad Q^{[N-1,1]}_{[1]}Q^{[N-1,1]}_{[1]} = Q^{[N-1,1]}_{[1]} \tag{4.36}$$

and have eigenvalues

$$\bar{T}^{(1)}_2 Q^{[N]}_{[1]} = \frac{N(N-1)}{2}Q^{[N]}_{[1]}, \quad \bar{T}^{(1)}_2 Q^{[N-1,1]}_{[1]} = \frac{N(N-3)}{2}Q^{[N-1,1]}_{[1]}, \tag{4.37}$$

which are exactly equal to the normalized characters. Note that S_1 has no nontrivial representations, and $t^{(1)}_2 = 0$, which is consistent with the normalized character $\frac{k(k-1)}{2} = 0$ of the trivial representation.

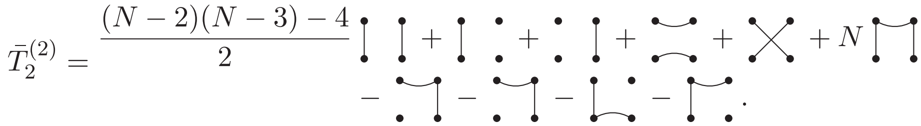
The orthogonal basis elements for $\mathcal{H}_{\mathrm{inv}}{}^{(1)}$, corresponding to these matrix units, are

$$|Q^{[N]}_{[1]}\rangle = \frac{1}{N}\sum_{i_1,i_{1'}}(a^\dagger)^{i_{1'}}_{i_1}|0\rangle \quad \text{and}$$
$$|Q^{[N-1,1]}_{[1]}\rangle = \sum_{i_1}(a^\dagger)^{i_1}_{i_1} - \frac{1}{N}\sum_{i_1,i_{1'}}(a^\dagger)^{i_{1'}}_{i_1}|0\rangle. \tag{4.38}$$

2. Degree two basis

The procedure was particularly easy at degree one because S_1 is trivial, and there are no multiplicities appearing. For $k = 2$ we have the sign representation [1, 1] and the trivial representation [2] of S_2, and pairs of irreducible representations Λ_1, Λ_2 appear with multiplicity larger than one. To distinguish multiplicities we will have to introduce noncentral elements, as discussed in Sec. IV B.

At degree two, the partition algebra element we use to distinguish Λ_1 is [75]

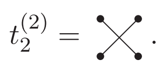


(4.39)

As a linear map (acting on the left or right) on $P_2(N)$, it has eigenvalues

$$\bar{T}_2^{(2)}(Q^{[N]}_{\Lambda_2,\mu\nu}) = \frac{N(N-1)}{2} Q^{[N]}_{\Lambda_2,\mu\nu},$$
$$\bar{T}_2^{(2)}(Q^{[N-1,1]}_{\Lambda_2,\mu\nu}) = \frac{N(N-3)}{2} Q^{[N-1,1]}_{\Lambda_2,\mu\nu},$$
$$\bar{T}_2^{(2)}(Q^{[N-2,2]}_{\Lambda_2,\mu\nu}) = \frac{(N-1)(N-4)}{2} Q^{[N-2,2]}_{\Lambda_2,\mu\nu},$$
$$\bar{T}_2^{(2)}(Q^{[N-2,1,1]}_{\Lambda_2,\mu\nu}) = \frac{N(N-5)}{2} Q^{[N-2,1,1]}_{\Lambda_2,\mu\nu}. \tag{4.40}$$

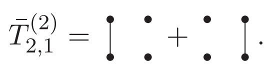
The element we use to distinguish Λ_2 is



(4.41)

The eigenvalues of the corresponding linear map are 1 for [2] and −1 for [1, 1].

The noncentral element we will use to distinguish multiplicities is



(4.42)

It is closely related to $\bar{T}_2^{(1)} \in \mathcal{Z}(P_1)$ in Eq. (4.34) because

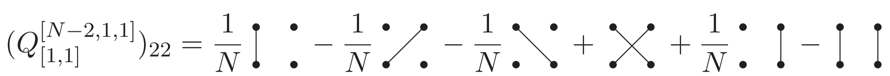
(4.43)

Roughly speaking, $\bar{T}_{2,1}^{(2)}$ comes from the embedding of $\bar{T}_2^{(1)}$ into $SP_2(N)$ by adding strands. Symmetrization has been used to ensure that we have an element in $SP_2(N)$.

To determine the multiplicity labels we need to act from the left as well as the right using $\bar{T}_{2,1}^{(2)}$. We define $\bar{T}_{2,1}^{(2),L}$ and $\bar{T}_{2,1}^{(2),R}$ acting on $d \in P_2(N)$ by

$$\bar{T}_{2,1}^{(2),L} d = \bar{T}_{2,1}^{(2)} d, \qquad \bar{T}_{2,1}^{(2),R} d = d\bar{T}_{2,1}^{(2)}. \tag{4.44}$$

Appendix C gives a representation theoretic argument for why these operators fully distinguish all labels on matrix units, together with a complete table of all $k=2$ matrix units. As an example, we find a matrix unit [see (C30)]



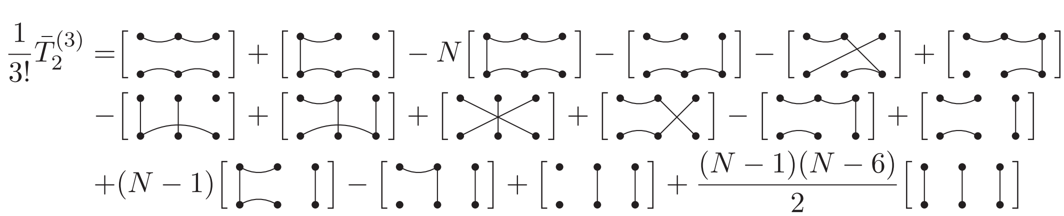
(4.45)

which corresponds to the (unnormalized) S_N invariant state

$$|(Q^{[N-2,1,1]}_{[1,1]})_{22}\rangle = \frac{2}{N}\left(\sum_{i,j,k=1}^{N}[(a^\dagger)^i_i(a^\dagger)^j_k - (a^\dagger)^j_i(a^\dagger)^i_k] + \sum_{i,j=1}^{N}[(a^\dagger)^i_j(a^\dagger)^j_i - (a^\dagger)^i_i(a^\dagger)^j_j]\right)|0\rangle. \tag{4.46}$$

3. Degree three basis

The multiplicity free matrix units for $k=3$ have $\Lambda_1 = [N-3,3], [N-3,2,1], [N-3,1,1,1]$. To find the corresponding linear combinations of diagrams it is sufficient to find eigenvectors of $\bar{T}_2^{(3)}$ defined by



(4.47)

with eigenvalues

$$\bar{T}_2^{(3)}(Q_{\Lambda_2,\mu\nu}^{[N-3,3]}) = \frac{(N-3)(N-4)}{2} Q_{\Lambda_2,\mu\nu}^{[N-3,3]},$$
$$\bar{T}_2^{(3)}(Q_{\Lambda_2,\mu\nu}^{[N-3,2,1]}) = \frac{(N-1)(N-6)}{2} Q_{\Lambda_2,\mu\nu}^{[N-3,2,1]},$$
$$\bar{T}_2^{(3)}(Q_{\Lambda_2,\mu\nu}^{[N-3,1,1,1]}) = \frac{N(N-7)}{2} Q_{\Lambda_2,\mu\nu}^{[N-3,1,1,1]}. \tag{4.48}$$

The square brackets in (4.47) denote S_3 symmetrization as in Eq. (3.24). Explicit expansions of these matrix units in terms of diagrams are given in Appendix C.

V. EXACTLY SOLVABLE PERMUTATION INVARIANT MATRIX HARMONIC OSCILLATOR

The simplest quantum mechanical matrix Hamiltonian we considered in Sec. II is invariant under the symmetric group action

$$\sigma: X_{ij} \to X_{\sigma(i)\sigma(j)}, \quad \forall\ \sigma \in S_N. \tag{5.1}$$

It is also invariant under the much larger symmetry of continuous transformations by $U(N^2)$. In this section we generalize the quadratic potential to the most general quadratic function $V(X)$ invariant under the above permutation symmetry. We will thus present a quantum mechanical model of N^2 matrix variables X_{ij} in a permutation invariant quadratic potential $V(X)$. The most general permutation invariant quadratic action in a zero-dimensional matrix model was constructed in [28] using representation theory. Borrowing these techniques, we explicitly construct an 11 parameter family of permutation invariant quadratic potentials. The corresponding Hamiltonian is exactly diagonalizable. In general, the diagonalization only involves diagonalizing a 3×3 symmetric matrix and a 2×2 symmetric matrix. We describe the spectrum of the full Hamiltonian and discuss the degeneracy when the quanta of energy are generic, and when they satisfy integrality properties. In the former case we are able to give a lower bound on the order of the degeneracy, this is given in Eq. (5.35). In the latter case, the degeneracy can be phrased in terms of an integer partition problem. The integer partition problem has a solution in terms of a canonical partition function (generating function) given by Eq. (5.38). We end this section in V D with a brief discussion of the role that the representation basis could play in simplifying the diagonalization of H, given in Eq. (5.27), on $\mathcal{H}_{\rm inv}$.

A. Construction

A matrix harmonic oscillator in a potential is described by the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{N} \partial_t X_{ij}\partial_t X_{ij} - \frac{1}{2}V(X). \tag{5.2}$$

We take the potential to be a general quadratic S_N (permutation) invariant potential

$$V(X_{ij}) = V(X_{\sigma(i)\sigma(j)}). \tag{5.3}$$

The action of S_N on X_{ij} defined in (5.3) corresponds to the diagonal action on the tensor product $V_N\otimes V_N$. This is given in (3.6) for general k, for the $k=2$ case at hand we have

$$\mathcal{L}(\sigma^{-1})(e_i\otimes e_j) = e_{\sigma(i)}\otimes e_{\sigma(j)}. \tag{5.4}$$

The vector space $V_N\otimes V_N$ is reducible with respect to the diagonal action. There exists an isomorphism

$$V_N\otimes V_N \cong 2V_{[N]}^{S_N} \oplus 3V_{[N-1,1]}^{S_N} \oplus V_{[N-2,2]}^{S_N} \oplus V_{[N-2,1,1]}^{S_N} \tag{5.5}$$

into irreducible subspaces. The representation $V_{[N]}^{S_N}$ is the one-dimensional trivial representation of S_N. The representations $V_{[N-1,1]}^{S_N}, V_{[N-2,2]}^{S_N}, V_{[N-2,1,1]}^{S_N}$ are nontrivial irreducible representations of S_N, labeled by integer partitions of N. Detailed descriptions, including explicit constructions of irreducible representations of S_N can be found in [53,93]. The dimensions of the nontrivial irreducible representations in (5.5) are, respectively,

$$N-1, \qquad (N-1)(N-2)/2, \qquad N(N-3)/2. \tag{5.6}$$

We take the rhs of the isomorphism (5.5) to be a vector space with orthonormal basis $X_a^{\Lambda,\alpha}$ labeled by

$$\Lambda \in \{[N],[N-1,1],[N-2,2],[N-2,1,1]\},$$
$$a \in \{1,\ldots,\mathrm{Dim}V_\Lambda^{S_N}\},$$
$$\alpha \in \{1,\ldots,\mathrm{Mult}(V_N\otimes V_N \to V_\Lambda^{S_N})\}. \tag{5.7}$$

By definition the Clebsch-Gordan coefficients $C_{a,ij}^{\Lambda,\alpha}$ are the matrix elements of the equivariant map between the two sides of Eq. (5.5),

$$X_a^{\Lambda,\alpha} = \sum_{i,j} C_{a,ij}^{\Lambda,\alpha} X_{ij}. \tag{5.8}$$

As a consequence, they have the following property

$$\sum_{i,j} C_{a,ij}^{\Lambda,\alpha} X_{\sigma^{-1}(i)\sigma^{-1}(j)} = \sum_b D_{ba}^{\Lambda}(\sigma) X_b^{\Lambda,\beta}, \tag{5.9}$$

where $D^\Lambda(\sigma)$ is an irreducible (unitary and real) matrix representation of $\sigma \in S_N$.

In the representation basis the potential has a simple form,

$$V(X) = \sum_{\Lambda,\alpha,\beta,a} X_a^{\Lambda,\alpha} g_{\alpha\beta}^\Lambda X_a^{\Lambda,\beta}, \tag{5.10}$$

where $g_{\alpha\beta}^\Lambda$ are symmetric matrices. To define a system with energy bounded from below they are required to have non-negative eigenvalues. Translating back to the original basis gives

$$V(X) = \sum_{\Lambda,\alpha,\beta,a} \sum_{i,j,k,l} C_{a,ij}^{\Lambda,\alpha} g_{\alpha\beta}^\Lambda C_{a,kl}^{\Lambda,\beta} X_{ij} X_{kl}. \tag{5.11}$$

We define the tensors

$$Q_{ijkl}^{\Lambda,\alpha\beta} = \sum_a C_{a,ij}^{\Lambda,\alpha} C_{a,kl}^{\Lambda,\beta} \tag{5.12}$$

and write the potential $V(X)$ as

$$V(X) = \sum_{\Lambda,\alpha,\beta} \sum_{i,j,k,l} Q_{ijkl}^{\Lambda,\alpha\beta} g_{\alpha\beta}^\Lambda X_{ij} X_{kl}. \tag{5.13}$$

The tensors $Q_{ijkl}^{\Lambda,\alpha\beta}$ are known explicitly [28]. For example,

$$Q_{ijkl}^{[N],11} = \frac{1}{N^2}, \tag{5.14}$$

$$Q_{ijkl}^{[N],22} = \frac{1}{N-1}\left(\delta_{ij}\delta_{kl} - \frac{1}{N}\delta_{ij} - \frac{1}{N}\delta_{kl} - \frac{1}{N^2}\right). \tag{5.15}$$

Their construction using Clebsch-Gordan coefficients means that they satisfy

$$Q_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)}^{\Lambda,\alpha\beta} = Q_{ijkl}^{\Lambda,\alpha\beta}. \tag{5.16}$$

This follows from the equivariance property (5.9)

$$\begin{aligned} Q_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)}^{\Lambda,\alpha\beta} &= \sum_a C_{a,\sigma(i)\sigma(j)}^{\Lambda,\alpha} C_{a,\sigma(k)\sigma(l)}^{\Lambda,\beta} \\ &= \sum_{a,b,c} C_{b,ij}^{\Lambda,\alpha} C_{c,kl}^{\Lambda,\beta} D_{ab}^\Lambda(\sigma) D_{ac}^\Lambda(\sigma), \\ &= \sum_{b,c} C_{b,ij}^{\Lambda,\alpha} C_{c,kl}^{\Lambda,\beta} \delta_{bc} = Q_{ijkl}^{\Lambda,\alpha\beta}. \end{aligned} \tag{5.17}$$

Going to the second line uses $D_{ab}^\Lambda(\sigma) = D_{ba}^\Lambda(\sigma^{-1})$ which follows from the fact that representation matrices for S_N can be chosen to real and unitary, i.e. orthogonal matrices.

B. Spectrum

The full Hamiltonian with quadratic potential given in (5.13) can be diagonalized using oscillators. We will see that diagonalizing the Hamiltonian only requires the diagonalization of a set of small parameter matrices (one 3×3 and another 2×2), despite having a potentially large number of harmonic oscillators (N^2).

The full Lagrangian in the representation basis is

$$L = \sum_{\Lambda,\alpha,\beta,a} \delta_{\alpha\beta} \partial_t X_a^{\Lambda,\alpha} \partial_t X_a^{\Lambda,\beta} - X_a^{\Lambda,\alpha} g_{\alpha\beta}^\Lambda X_a^{\Lambda,\beta}. \tag{5.18}$$

It describes a set of coupled harmonic oscillators. We write the Lagrangian in decoupled form in the usual way. Let $\Omega_{\alpha\beta}^\Lambda = (\omega_\alpha^\Lambda)^2 \delta_{\alpha\beta}$ be the diagonal matrix[1] such that

$$g_{\alpha\beta}^\Lambda = \sum_{\gamma,\delta} U_{\alpha\gamma}^\Lambda \Omega_{\gamma\delta}^\Lambda U_{\beta\delta}^\Lambda, \tag{5.19}$$

where U^Λ is orthogonal change of basis matrices. In the decoupled basis

$$S_a^{\Lambda,\alpha} = \sum_\beta X_a^{\Lambda,\beta} U_{\beta\alpha}^\Lambda, \tag{5.20}$$

we have

$$L = \sum_{\Lambda,\alpha,a} \frac{1}{2} \partial_t S_a^{\Lambda,\alpha} \partial_t S_a^{\Lambda,\alpha} - \frac{1}{2} (\omega_\alpha^\Lambda)^2 S_a^{\Lambda,\alpha} S_a^{\Lambda,\alpha}. \tag{5.21}$$

The canonical momenta are given by

$$\Sigma_a^{\Lambda,\alpha} = \partial_t S_a^{\Lambda,\alpha}. \tag{5.22}$$

The new canonical coordinates satisfy

$$[\Sigma_a^{\Lambda,\alpha}, S_b^{\Lambda',\beta}] = i\delta^{\Lambda\Lambda'} \delta^{\alpha\beta} \delta_{ab}, \tag{5.23}$$

since U^Λ are orthogonal matrices.

The corresponding Hamiltonian,

$$H = \frac{1}{2} \sum_{\Lambda,\alpha,a} \Sigma_a^{\Lambda,\alpha} \Sigma_a^{\Lambda,\alpha} + (\omega_\alpha^\Lambda)^2 S_a^{\Lambda,\alpha} S_a^{\Lambda,\alpha}, \tag{5.24}$$

is diagonalized by introducing oscillators

[1]We assume the eigenvalues are positive such that the spectrum of the Hamiltonian is bounded from below. Therefore, we may write the eigenvalues as squares without loss of generality.

$$S_a^{\Lambda,\alpha} = \sqrt{\frac{1}{2\omega_\alpha^\Lambda}}((A^\dagger)_a^{\Lambda,\alpha} + A_a^{\Lambda,\alpha}),$$

$$\Sigma_a^{\Lambda,\alpha} = i\sqrt{\frac{\omega_\alpha^\Lambda}{2}}((A^\dagger)_a^{\Lambda,\alpha} - A_a^{\Lambda,\alpha}), \qquad (5.25)$$

which satisfy

$$[A_a^{\Lambda,\alpha}, (A^\dagger)_{a'}^{\Lambda',\alpha'}] = \delta^{\Lambda\Lambda'}\delta^{\alpha\alpha'}\delta_{aa'}. \qquad (5.26)$$

In the oscillator basis, the normal ordered Hamiltonian has the form

$$\boxed{H = \sum_{\Lambda,\alpha,a}\omega_\alpha^\Lambda (A^\dagger)_a^{\Lambda,\alpha} A_a^{\Lambda,\alpha}.} \qquad (5.27)$$

Defining number operators $\hat{N}_a^{\Lambda,\alpha}$ and $\hat{N}^{\Lambda,\alpha}$:

$$\hat{N}_a^{\Lambda,\alpha} = (A^\dagger)_a^{\Lambda,\alpha} A_a^{\Lambda,\alpha}, \qquad (5.28)$$

$$\hat{N}^{\Lambda,\alpha} = \sum_a \hat{N}_a^{\Lambda,\alpha}, \qquad (5.29)$$

we may write

$$H = \sum_{\Lambda,\alpha,a}\hat{N}_a^{\Lambda,\alpha} = \sum_{\Lambda,\alpha}\hat{N}^{\Lambda,\alpha}. \qquad (5.30)$$

The energy quanta ω_α^Λ do not depend on the oscillator state index a. This is a manifestation of the S_N invariance of the Hamiltonian H.

The Hilbert space $\mathcal{H}^{(k)}$ has a basis of energy eigenstates

$$\prod_{\substack{\Lambda\in\{[N],[N-1,1],[N-2,2],[N-2,1,1]\} \\ \alpha\in\{1,\ldots\mathrm{Mult}(V_N\otimes V_N\to V_\Lambda^{S_N})\} \\ a\in\{1,\ldots,\mathrm{Dim}V_\Lambda^{S_N}\}}} \frac{[(A^\dagger)_a^{\Lambda,\alpha}]^{N_a^{\Lambda,\alpha}}}{\sqrt{N_a^{\Lambda,\alpha}!}}|0\rangle, \qquad (5.31)$$

where $k = \sum_{\Lambda,\alpha,a} N_a^{\Lambda,\alpha}$ is the eigenvalue of the (total) number operator

$$\hat{N} = \sum_{\Lambda,\alpha,a}\hat{N}_a^{\Lambda,\alpha}, \qquad (5.32)$$

and $N_a^{\Lambda,\alpha}$ is the eigenvalue of $\hat{N}_a^{\Lambda,\alpha}$.

Since the Hamiltonian (5.27) is a linear combination of number operators $\hat{N}^{\Lambda,\alpha}$, it is natural to organize $\mathcal{H}^{(k)}$ into eigenspaces of $\hat{N}^{\Lambda,\alpha}$ with eigenvalues $N^{\Lambda,\alpha} = \sum_a N_a^{\Lambda,\alpha}$ satisfying $k = \sum_{\Lambda,\alpha} N^{\Lambda,\alpha}$. Diagonalizing the number operators $\hat{N}^{\Lambda,\alpha}$ organizes $\mathcal{H}^{(k)}$ into subspaces

$$\mathcal{H}^{(k)} \cong \bigoplus_{\sum N^{\Lambda,\alpha}=k}\bigotimes_{\Lambda,\alpha}\mathcal{H}^{[N^{\Lambda,\alpha}]}, \qquad (5.33)$$

where

$$\mathcal{H}^{[N^{\Lambda,\alpha}]} \cong \mathrm{Sym}^{N^{\Lambda,\alpha}}(V_\Lambda^{S_N}). \qquad (5.34)$$

Each summand in (5.33) is a vector space of dimension

$$\begin{aligned}
\mathrm{Dim}\Big(\bigotimes_{\Lambda,\alpha}\mathcal{H}^{[N^{\Lambda,\alpha}]}\Big) &= \prod_{\Lambda,\alpha}\binom{\mathrm{Dim}V_\Lambda^{S_N} + N^{\Lambda,\alpha} - 1}{N^{\Lambda,\alpha}} \\
&= \binom{1 + N^{[N],1} - 1}{N^{[N],1}}\binom{1 + N^{[N],2} - 1}{N^{[N],2}} \\
&\quad\times\binom{N-1+N^{[N-1,1],1}-1}{N^{[N-1,1],1}}\binom{N-1+N^{[N-1,1],2}-1}{N^{[N-1,1],2}}\binom{N-1+N^{[N-1,1],3}-1}{N^{[N-1,1],3}} \\
&\quad\times\binom{(N-1)(N-2)/2 + N^{[N-2,2]} - 1}{N^{[N-2,2]}}\binom{N(N-3)/2 + N^{[N-2,1,1]} - 1}{N^{[N-2,2]}} \\
&= \binom{N-2+N^{[N-1,1],1}}{N^{[N-1,1],1}}\binom{N-2+N^{[N-1,1],2}}{N^{[N-1,1],2}}\binom{N-2+N^{[N-1,1],3}}{N^{[N-1,1],3}} \\
&\quad\times\binom{N(N-3)/2 + N^{[N-2,2]}}{N^{[N-2,2]}}\binom{N(N-3)/2 - 1 + N^{[N-2,1,1]}}{N^{[N-2,2]}}.
\end{aligned} \qquad (5.35)$$

The vectors in $\bigotimes_{\Lambda,\alpha}\mathcal{H}^{[N^{\Lambda,\alpha}]}$ have energy

$$E(\{N^{\Lambda,\alpha}\}) = \sum_{\Lambda,\alpha} N^{\Lambda,\alpha}\omega_\alpha^\Lambda. \tag{5.36}$$

Equation (5.35) thus gives the degeneracy of energy eigenstates for the specified integers $\{N^{\Lambda,\alpha}\}$, associated with Λ, α as given in (5.7). This puts a lower bound on the degeneracy of energy eigenstates. Further degeneracy may occur for particular choices of the constants ω_α^Λ, which can lead to the same numerical value of $E(\{N^{\Lambda,\alpha}\})$ for different choices of $\{N^{\Lambda,\alpha}\}$.

C. Canonical partition function

The canonical partition function is defined as

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H} = \sum_{\mathcal{E}} N(\mathcal{E}) e^{-\beta\mathcal{E}}, \tag{5.37}$$

where $N(\mathcal{E})$ is the degeneracy of eigenstates at energy $\mathcal{E}$ and β is the inverse temperature.

The binomial factors in (5.35) arise in the expansion of simple rational functions. Defining $x = e^{-\beta}$ for convenience, we can therefore write

$$\begin{aligned} Z(\beta) = {} & \frac{1}{(1-x^{\omega_1^{[N]}})(1-x^{\omega_2^{[N]}})} \\ & \times \frac{1}{(1-x^{\omega_1^{[N-1,1]}})^{N-1}(1-x^{\omega_2^{[N-1,1]}})^{N-1}(1-x^{\omega_3^{[N-1,1]}})^{N-1}} \\ & \times \frac{1}{(1-x^{\omega^{[N-2,2]}})^{(N-1)(N-2)/2}(1-x^{\omega^{[N-2,1,1]}})^{N(N-3)/2}}. \end{aligned} \tag{5.38}$$

When the quanta of energy (ω_α^Λ) in (5.27) are integers, the possible state energies $\mathcal{E}$ are integers and $N(\mathcal{E})$ is related to what we refer to as an integer partition problem. The integer partition problem is the following: Pick any integer $\mathcal{E}$, enumerate the set of solutions (choices of $N_a^{\Lambda,\alpha}$) to

$$\mathcal{E} = \sum_{\Lambda,\alpha,a} N_a^{\Lambda,\alpha}\omega_\alpha^\Lambda. \tag{5.39}$$

The number of solutions is equal to $N(\mathcal{E})$ and a single solution is denoted $N_a^{\Lambda,\alpha}(\mathcal{E})$. This problem depends on N because the state label a ranges over $\{1,\ldots,\mathrm{Dim}V_\Lambda^{S_N}\}$. Fortunately the N dependence can be factorized, due to the S_N symmetry of the problem, thus greatly simplifying the problem.

To see this, consider the N-independent integer partition problem

$$\mathcal{E} = \sum_{\Lambda,\alpha} N^{\Lambda,\alpha}\omega_\alpha^\Lambda, \tag{5.40}$$

where a solution is given by a list of seven integers $N^{\Lambda,\alpha}(\mathcal{E})$. For every solution $N^{\Lambda,\alpha}(\mathcal{E})$ to (5.40) the number of solutions to the integer partition problem in (5.39) is given by

$$\mathrm{Dim}\left(\bigotimes_{\Lambda,\alpha}\mathcal{H}^{[N^{\Lambda,\alpha}(\mathcal{E})]}\right). \tag{5.41}$$

In this sense, the N dependence in the problem has factorized: we only need to find solutions to the N-independent equation (5.40) and multiply each solution by a known N-dependent factor. The total number of solutions to (5.39) is given by

$$\sum_{N^{\Lambda,\alpha}(\mathcal{E})} \mathrm{Dim}\left(\bigotimes_{\Lambda,\alpha}\mathcal{H}^{[N^{\Lambda,\alpha}(\mathcal{E})]}\right), \tag{5.42}$$

where the sum is over the set of solutions to (5.40).

D. Energy eigenbases

We have observed that the oscillator states constructed using partition algebra diagram operators in tensor space contracted with oscillators $(a^\dagger)_i^j$ obeying (2.6) are eigenstates of the simplest matrix Hamiltonian H_0 in (2.7). By contracting the representation basis elements in the partition algebra with the oscillators we produce quantum states

$$|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}(a^\dagger)^{\otimes k})|0\rangle, \tag{5.43}$$

which are eigenstates of H_0 and also diagonalize algebraic conserved charges.

The representation basis states are not eigenstates of the general permutation invariant harmonic oscillator Hamiltonians H in (5.24). There is mixing of the representation basis labels $(\Lambda_1,\Lambda_2,\mu,\nu)$ caused by the different weights for the representations Λ appearing in the expansion of the S_N invariant harmonic oscillator Hamiltonian defined in Eq. (5.27). We expect this mixing of the labels in the $(\Lambda_1,\Lambda_2,\mu,\nu)$ basis to be constrained, for example by the S_N Clebsch-Gordan decompositions of $\Lambda\otimes\Lambda_1$. Such constrained mixing of representation theory bases for matrix systems arises in Hamiltonians of interest in AdS/CFT. A number of representation theory bases for $U(N)$ invariant multimatrix systems have been described which capture information about finite N effects and are eigenstates of the Hamiltonian (in radial quantization) in the free Yang-Mills limit [58–63]. However, the one-loop dilatation operator defines a nontrivial Hamiltonian which is, in general, not diagonalized by these representation theoretic bases (although there are some interesting exceptions to this statement, see [94]). Representation theoretic constraints on the mixing caused by the one-loop dilatation operator are described in [94–98], following earlier work on one-loop mixings related to strings attached to giant gravitons, e.g. [99,100].

VI. ALGEBRAIC HAMILONIANS AND PERMUTATION INVARIANT GROUND STATES

So far our discussion of S_N invariant subspaces in quantum mechanical matrix systems has largely (with the exception of the previous section) been independent of any choice of Hamiltonian acting on the Hilbert space. It can be viewed as a general description of the kinematics of S_N invariance, independent of the dynamics determined by the Hamiltonian. In this section we present Hamiltonians H which realize the eigenspectrum scenarios depicted in Fig. 1, this includes Hamiltonians for which the low energy eigenstates are permutation invariant states.

The Hamiltonians we consider here preserve the S_N invariant subspace $\mathcal{H}_{\rm inv}$ defined as

$$\mathcal{H}_{\rm inv} = \{|T\rangle \in \mathcal{H} \text{ s.t. } \mathrm{Ad}(\sigma)|T\rangle = |T\rangle,\ \ \forall\, \sigma \in S_N\}. \tag{6.1}$$

The adjoint action of permutations $\sigma \in S_N$ on the tensors T labeling the states simultaneously transforms the upper and lower indices of T according to (3.7). For any state $|T\rangle \in \mathcal{H}_{\rm inv}$ the Hamiltonians H obey the condition

$$H|T\rangle \in \mathcal{H}_{\rm inv}. \tag{6.2}$$

A sufficient condition for H to satisfy (6.2) is for H itself to be S_N invariant or $[\mathrm{Ad}(\sigma), H] = 0$ for all $\sigma \in S_N$.

We will show how to construct Hamiltonians H_K of this type, depending on an integer parameter K, with a finite-dimensional space of S_N invariant ground states. Both the energy gap between the ground states and the lowest nonzero energy level, and the ground state degeneracy depend on K in a way that is determined by the algebraic construction. As sketched in the left-hand side of Fig. 1(a), H_K has an energy gap of order K. The construction of H_K can be viewed as including, in the Hamiltonian, central elements in $\mathbb{C}[S_N]$ acting on $\mathcal{H}^{(k)}$ using $\mathrm{Ad}(\sigma)$ for $k \leq K$. This can be related to the action of elements of $P_{2k}(N)$ acting on $\mathcal{H}^{(k)}$ for $k \leq K$. We will briefly mention some analogies between the present construction and the phenomenon of topological degeneracy which is widely studied in condensed matter physics.

The ground state degeneracy of H_K can be resolved by adding a term $H_{\rm res}$, made from the central algebraic charges discussed in Sec. IV. This breaks the degeneracy of the invariant ground states as illustrated in the spectrum on the right of Fig. 1(a). The representation basis $|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle$ presented in Sec. III C diagonalizes these Hamiltonians in the invariant subspace, and the state energies depend on labels Λ_1, Λ_2.

Multiplicity labels μ, ν are not distinguished by the central algebraic charges. Distinguishing multiplicity labels requires more general elements of $P_k(N)$, as discussed in Sec. IV B. Generalizing the construction of $H_{\rm res}$ naturally leads to a large class of S_N invariant Hamiltonians related to the left action of elements of $P_k(N)$, which can be used to break the degeneracy associated with multiplicity labels. Hamiltonians of this type can have nontrivial spectra, in which invariant states are distributed across the energy spectrum, with no discernible pattern of difference compared to noninvariant states, as illustrated in Fig. 1(b).

The 11-parameter Hamiltonians in Sec. V typically have such nontrivial spectra. Given the nontrivial index contractions in (5.13),

$$\sum_{i,j,k,l} Q^{\Lambda,\alpha\beta}_{ijkl} X_{ij} X_{kl} \to (a^\dagger)^i_j a^k_l Q^{\Lambda,\alpha\beta}_{ijkl}, \tag{6.3}$$

these Hamiltonians are not of the kind involving only the left action of $P_k(N)$. Similarly, H_K is not of this kind. This implies that a more general construction of S_N invariant Hamiltonians exists. We give a description of this more general construction, which involves elements of $P_{2k}(N)$. We end the section with a lattice interpretation of the matrix oscillators. This sets us up for Sec. VII which concerns the nontrivial interplay between the invariant sector and the Hamiltonian and includes realizations based on the lattice interpretation.

A. Partition algebra elements as quantum mechanical operators

We now translate much of the discussion in Sec. IV into the language of quantum mechanical operators on $\mathcal{H}$. Finding representation bases corresponds to the diagonalization of commuting operators on $\mathcal{H}$. Notably, elements of $SP_k(N)$ naturally correspond to operators for fixed k, or maps $\mathcal{H}^{(k)} \to \mathcal{H}^{(k)}$. However, it will be useful to have expressions for these fixed k operators in terms of oscillators, which act on the entire Hilbert space $\mathcal{H}$. These two kinds of operators are related by projectors $\mathcal{P}_k\colon \mathcal{H} \to \mathcal{H}^{(k)}$ to fixed k subspaces. We use this in the construction of Hamiltonians in the remainder of Sec. VI.

For a general state $|T\rangle \in \mathcal{H}^{(k)}$ [see (2.20)] and element $[d] \in SP_k(N)$ there is a corresponding operator defined as

$$[d]^L|T\rangle = |[d]T\rangle = |dT\rangle, \tag{6.4}$$

where the superscript L stands for left action, and

$$(dT)^{i_1\ldots i_k}_{i_{1'}\ldots i_{k'}} = \sum_{j_1,\ldots,j_k} d^{i_1\ldots i_k}_{j_1\ldots j_k} T^{j_1\ldots j_k}_{i_{1'}\ldots i_{k'}}. \tag{6.5}$$

The second equality in (6.4) follows since

$$\begin{aligned} |[d]T\rangle &= \mathrm{Tr}_{V_N^{\otimes k}}([d]T(a^\dagger)^{\otimes k})|0\rangle \\ &= \frac{1}{k!}\sum_{\gamma\in S_k} \mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}_\gamma d \mathcal{L}_{\gamma^{-1}} T (a^\dagger)^{\otimes k})|0\rangle, \\ &= \mathrm{Tr}_{V_N^{\otimes k}}(dT(a^\dagger)^{\otimes k})|0\rangle = |dT\rangle, \end{aligned} \tag{6.6}$$

where $\mathcal{L}(\sigma)$ is defined in Eq. (3.6). We have used $\mathcal{L}_\gamma T = T\mathcal{L}_\gamma$ together with $\mathcal{L}_\gamma (a^\dagger)^{\otimes k} = (a^\dagger)^{\otimes k}\mathcal{L}_\gamma$ to go to the second line. We may also define operators corresponding to right action,

$$[d]^R|T\rangle = |Td\rangle. \tag{6.7}$$

We extend $[d]^L$ to an operator on $\mathcal{H}$, expressible in terms of oscillators and projectors $\mathcal{P}_k\colon \mathcal{H} \to \mathcal{H}^{(k)}$ as

$$\boxed{[d]^L = \frac{1}{k!}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}((a^\dagger)^{\otimes k} d a^{\otimes k})\mathcal{P}_k.} \tag{6.8}$$

Similarly, we can extend $[d]^R$ to an operator on $\mathcal{H}$,

$$\boxed{[d]^R = \frac{1}{k!}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}(d(a^\dagger)^{\otimes k} a^{\otimes k})\mathcal{P}_k.} \tag{6.9}$$

In what follows we will prove results explicitly for the left action. For the sake of brevity we omit the analogous proofs for the right action.

The definition of $\mathcal{P}_k$ in the oscillator basis is

$$\mathcal{P}_{k'}(a^\dagger)^{i_1}_{j_1}\dots(a^\dagger)^{i_k}_{j_k}|0\rangle = \delta_{kk'}(a^\dagger)^{i_1}_{j_1}\dots(a^\dagger)^{i_k}_{j_k}|0\rangle. \tag{6.10}$$

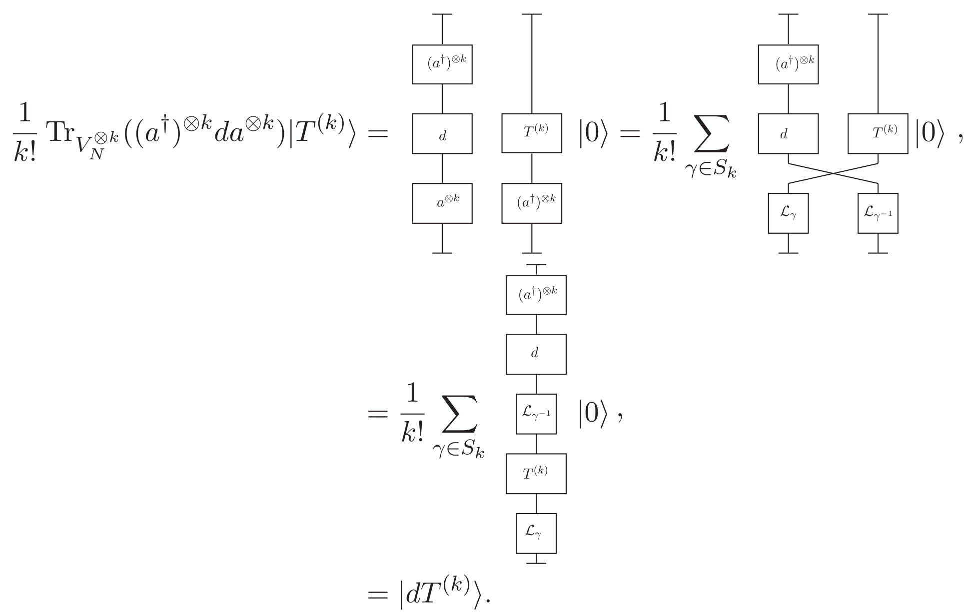
We now prove

$$\frac{1}{k!}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}((a^\dagger)^{\otimes k} d a^{\otimes k})\mathcal{P}_k|T\rangle = |dT^{(k)}\rangle, \tag{6.11}$$

where $|T\rangle = \sum_{k=0}^{\infty}|T^{(k)}\rangle$ and $|T^{(k)}\rangle \in \mathcal{H}^{(k)}$. The projector immediately gives $\mathcal{P}_k|T\rangle = |T^{(k)}\rangle$. It remains to prove

$$\frac{1}{k!}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}((a^\dagger)^{\otimes k} d a^{\otimes k})|T^{(k)}\rangle = |dT^{(k)}\rangle. \tag{6.12}$$

We prove this diagrammatically, using the state definition in terms of diagrams (2.26)



(6.13)

In the second equality we have moved all annihilation operators past the creation operators, giving a sum over contractions. The sum over $\gamma \in S_k$ encodes the contractions and in the second line we have straightened the diagram. The last identification follows since $\mathcal{L}_{\gamma^{-1}}T^{(k)}\mathcal{L}_\gamma = T^{(k)}$. Because $|dT^{(k)}\rangle \in \mathcal{H}^{(k)}$ we have $\mathcal{P}_k|dT^{(k)}\rangle = |dT^{(k)}\rangle$, which establishes the equality in (6.11).

As we now show, the Hermitian conjugate of the operator $[d_\pi]^L$ is $[d_\pi^T]^L$, where d_π^T is the element obtained by flipping the diagram d_π horizontally. This follows from the inner product

$$\langle T'|T\rangle = \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}((T')^T\gamma T\gamma^{-1}), \tag{6.14}$$

defined in (3.31) and

$$\begin{aligned}\langle T'|d_\pi T\rangle &= \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}((T')^T\gamma d_\pi T\gamma^{-1})\\ &= k!\mathrm{Tr}_{V_N^{\otimes k}}((T')^T d_\pi T),\\ &= \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}((d_\pi^T T')^T\gamma^{-1}T\gamma)\\ &= \langle d_\pi^T T'|T\rangle. \end{aligned} \tag{6.15}$$

As operators on $\mathcal{H}$, $T_2\in\mathcal{Z}(\mathbb{C}[S_N])$, $\bar{T}_2\in\mathcal{Z}(P_k(N))$ and $t_2\in\mathcal{Z}(\mathbb{C}[S_k])$ can be written as oscillators. From the definition of the action of T_2 in (4.8) we have

$$\begin{aligned}T_2^{(k),L} &\equiv \frac{1}{k!}\mathcal{P}_k\mathrm{Tr}_{V_N^{\otimes k}}[(a^\dagger)^{\otimes k}\mathcal{L}(T_2)a^{\otimes k}]\mathcal{P}_k,\\ &= \frac{1}{k!}\mathcal{P}_k\sum_{\substack{\sigma=(ij)\\1\le i<j\le N}}\mathrm{Tr}_{V_N^{\otimes k}}[(a^\dagger)^{\otimes k}\mathcal{L}(\sigma)a^{\otimes k}]\mathcal{P}_k,\\ &= \frac{1}{k!}\mathcal{P}_k\sum_{\substack{\sigma=(ij)\\1\le i<j\le N}}\sum_{\substack{i_1\ldots i_k\\ i_{1'}\ldots i_{k'}}}(a^\dagger)^{i_{1'}}_{\sigma^{-1}(i_1)}\ldots(a^\dagger)^{i_{k'}}_{\sigma^{-1}(i_k)}a^{i_1}_{i_{1'}}\ldots a^{i_k}_{i_{k'}}\mathcal{P}_k.\end{aligned} \tag{6.16}$$

Similarly, the fixed k operators corresponding to $\bar{T}_2$ are

$$\begin{aligned}\bar{T}_2{}^{(k),L} &= \frac{1}{k!}\mathcal{P}_k\mathrm{Tr}_{V_N^{\otimes k}}[(a^\dagger)^{\otimes k}\overline{T_2}a^{\otimes k}]\mathcal{P}_k,\\ &= \frac{1}{k!}\mathcal{P}_k\sum_{\substack{i_1\ldots i_k\\ j_1\ldots j_k\\ j_{1'}\ldots j_{k'}}}(a^\dagger)^{i_1}_{i_{1'}}\ldots(a^\dagger)^{i_k}_{i_{k'}}(\overline{T_2})^{i_{1'}\ldots i_{k'}}_{j_1\ldots j_k}a^{j_1}_{i_1}\ldots a^{j_k}_{i_k}\mathcal{P}_k,\end{aligned} \tag{6.17}$$

where $\overline{T_2}$ can be expanded in in the diagram basis as in (4.9). Finally, the fixed k operators corresponding to t_2 are

$$\begin{aligned}t_2^{(k),L} &= \frac{1}{k!}\mathcal{P}_k\sum_{\substack{\tau=(ij)\\1\le i<j\le k}}\mathrm{Tr}_{V_N^{\otimes k}}[(a^\dagger)^{\otimes k}\mathcal{L}_{\tau^{-1}}a^{\otimes k}]\mathcal{P}_k,\\ &= \frac{1}{k!}\mathcal{P}_k\sum_{\substack{\tau=(ij)\\1\le i<j\le k}}\sum_{\substack{i_1\ldots i_k\\ i_{1'}\ldots i_{k'}}}(a^\dagger)^{i_{1'}}_{i_{\tau(1)}}\ldots(a^\dagger)^{i_{k'}}_{i_{\tau(k)}}a^{i_1}_{i_{1'}}\ldots a^{i_k}_{i_{k'}}\mathcal{P}_k.\end{aligned} \tag{6.18}$$

These operators are Hermitian, because $(T_2)^T=T_2$ and $(t_2)^T=t_2$, and consequently their eigenvectors with distinct eigenvalues are orthogonal. They are difficult to diagonalize over the entirety of $\mathcal{H}^{(k)}$, since the dimension grows as N^{2k} for $N\gg k$. But the diagonalization over $\mathcal{H}_{\mathrm{inv}}{}^k$ is feasible since the dimension is bounded by $B(2k)$, which does not scale with N. Further simplification arises when acting on states $|d\rangle\in\mathcal{H}_{\mathrm{inv}}{}^{(k)}$, since the action can be formulated as multiplication in $SP_k(N)$, thus bypassing the computation of large index contractions. That is, for $|d\rangle\in\mathcal{H}_{\mathrm{inv}}{}^k$

$$\overline{T_2}^{(k),L}|d\rangle = |\overline{T_2}d\rangle, \tag{6.19}$$

where the product $\overline{T_2}^{(k)}d$ can be taken in $P_k(N)$. It follows that,

$$\overline{T_2}^{(k),L}|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = |\overline{T_2}Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \hat{\chi}^{\Lambda_1}(T_2)|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle, \tag{6.20}$$

and similarly for $t_2^{(k),L}$.

The free Hamiltonian H_0 in Eq. (2.3) is just the number operator. The above operators conserve the number of particles. Consequently,

$$[H_0,T_2^{(k),L}] = [H_0,\overline{T_2}^{(k),L}] = [H_0,t_2^{(k),L}] = 0, \tag{6.21}$$

and the corresponding charges are conserved.

B. Decoupling invariant sectors and invariant ground states

We now present a Hermitian operator with algebraic origin that can be used to control the energies of states invariant under the adjoint action of S_N on $\mathcal{H}^{(k)}$. We use the operator to construct a Hamiltonian with a large number of invariant ground states.

The adjoint action of $\sigma\in S_N$ on $\mathcal{H}^{(k)}$ is defined in Eq. (3.7) as

$$\begin{aligned}\mathrm{Ad}(\sigma)|T\rangle &= \mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k})|0\rangle\\ &= \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k}}T^{\sigma(j_1)\ldots\sigma(j_k)}_{\sigma(i_1)\ldots\sigma(i_k)}(a^\dagger)^{i_1}_{j_1}\ldots(a^\dagger)^{i_k}_{j_k}|0\rangle.\end{aligned} \tag{6.22}$$

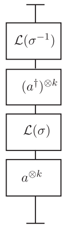
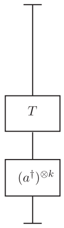
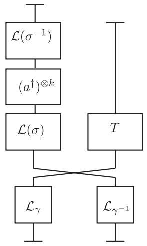
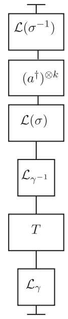
We may write $\mathrm{Ad}(\sigma)$ in terms of oscillators and projectors $\mathcal{P}_k\colon\mathcal{H}\to\mathcal{H}^{(k)}$ defined in Eq. (6.10). For $|T\rangle\in\mathcal{H}^{(k)}$,

$$\mathrm{Ad}(\sigma)|T\rangle = \frac{1}{k!}\mathcal{P}_k\mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k}\mathcal{L}(\sigma)a^{\otimes k})\mathcal{P}_k|T\rangle. \tag{6.23}$$

We note that the ordering of $a^\dagger$ relative to a is understood to be as shown in the above equation. To understand the equality in (6.23), we evaluate

$$\mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k}\mathcal{L}(\sigma)a^{\otimes k})|T\rangle, \tag{6.24}$$

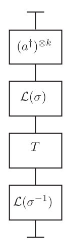
where we take $|T\rangle\in\mathcal{H}^{(k)}$ (there is no loss of generality since $\mathcal{P}_k$ projects to $\mathcal{H}^{(k)}$). Diagrammatically we have



$|0\rangle = \sum_{\gamma\in S_k}$

$|0\rangle = \sum_{\gamma\in S_k}$

$|0\rangle\,,$

$$= k!\,\mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k})\,|0\rangle\,. \tag{6.25}$$

The first equality follows by encoding the contraction of annihilation/creation operators in a sum over $\gamma\in S_k$, and the last equality follows by $\mathcal{L}_\gamma T = T\mathcal{L}_\gamma$. This establishes the equality (6.23).

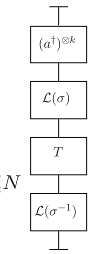
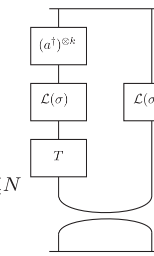
We are now in a position to define the Hermitian operator of interest. Let $C_3^{(k)}$ be the operator defined to act on $|T\rangle\in\mathcal{H}^{(k)}$ as

$$C_3^{(k)}\,|T\rangle = \frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}\mathrm{Ad}(\sigma)\,|T\rangle = \frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}$$

$$|0\rangle\,, \tag{6.26}$$

where the sum is over all 3-cycles. It commutes with the adjoint action of S_N,

$$\mathrm{Ad}(\gamma)C_3^{(k)} = C_3^{(k)}\mathrm{Ad}(\gamma), \quad \forall\ \gamma\in S_N, \tag{6.27}$$

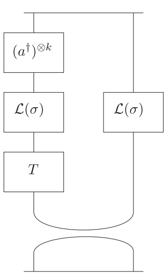
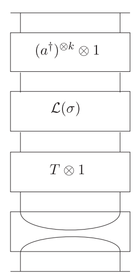
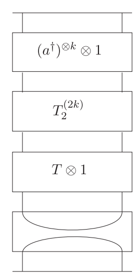
because $C_3^{(k)}$ is a sum over an entire conjugacy class. We now use a sequence of diagrammatic manipulations to show that the action of $C_3^{(k)}$ can equivalently be expressed using an element $\bar{T}_3^{(2k)}\in P_{2k}(N)$. A useful way to rewrite the diagram in (6.26) is

$$\frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}$$

$$|0\rangle = \frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}$$

$$|0\rangle\,, \tag{6.28}$$

where we have gone from a trace in $V_N^{\otimes k}$ to a trace in $V_N^{\otimes 2k}$. By arguments analogous to those in Sec. IV A, the action of

$$\frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}\mathcal{L}(\sigma), \tag{6.29}$$

on $V_N^{\otimes 2k}$ is related to an element in $P_{2k}(N)$, which we call $\bar{T}_3^{(2k)}$. Diagrammatically, this is understood from the following sequence of identifications,

$$\frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}$$

$$|0\rangle = \frac{1}{3}\sum_{\substack{\sigma=(ijk)\\1\le i\neq j\neq k\le N}}$$

$|0\rangle =$

$$|0\rangle\,. \tag{6.30}$$

That is, we have

$$C_3^{(k)}|T\rangle = \mathrm{Tr}_{V_N^{\otimes 2k}}(c(T\otimes 1)\bar{T}_3^{(2k)}((a^\dagger)^{\otimes k}\otimes 1)))|0\rangle, \tag{6.31}$$

where $c\in P_{2k}(N)$ is the bottom box in the diagram on the rhs of (6.30) and

$$(c)^{i_1\ldots i_{2k}}_{j_1\ldots j_{2k}} = \delta^{i_1 i_{k+1}}\ldots\delta^{i_k i_{2k}}\delta_{j_1 j_{k+1}}\ldots\delta_{j_k j_{2k}}. \tag{6.32}$$

The explicit formula for $\bar{T}_3^{(2k)}$ could be derived using steps similar to the derivation of the relation between $\bar{T}_2^{(k)}$ and $T_2^{(k)}$ in Sec. IV A. Relating $C_3^{(k)}$ to an element $\bar{T}_3^{(2k)}$ using $P_{2k}(N)$ allows for two kinds of large N simplification.

First, in place of $N!/(N-3)!3!$ terms in $C_3^{(k)}$ we have no more than $B(2k)$ terms in $\bar{T}_3^{(2k)}$, where $B(2k)$ are the Bell numbers. Additionally, index contractions ranging over N can be replaced by multiplication in the partition algebra $P_{2k}(N)$ when $|T\rangle \in \mathcal{H}_{\rm inv}$, the complexity of this multiplication scales with k.

We now move on to discuss the spectrum of $C_3^{(k)}$. Since $\mathcal{H}^{(k)}$ is reducible with respect to the adjoint action of S_N, it decomposes into irreducible representations of S_N, labeled by Young diagrams Y with N boxes. By Schur's lemma the action of $C_3^{(k)}$ on each irreducible subspace of this decomposition is proportional to the identity. The constant of proportionality is the normalized character of $C_3^{(k)}$ in the irreducible representation Y,

$$\hat{\chi}_Y(C_3^{(k)}) = \frac{\chi_Y(C_3^{(k)})}{{\rm Dim} V_Y^{S_N}}. \tag{6.33}$$

Normalized characters of $C_3^{(k)}$ are known (Theorem 4 of [101]) to equal

$$\hat{\chi}_Y(C_3^{(k)}) = \sum_{(p,q)\in Y} (q-p)^2 - \frac{N(N-1)}{2}, \tag{6.34}$$

where the sum is over all cells in the Young diagram Y, using coordinates (p,q) for rows and columns, respectively. For example, the largest eigenvalue of $C_3^{(k)}$ corresponds to the trivial representation (Young diagram with all N boxes in the first row) where

$$\sum_{(p,q)\in Y} (q-p)^2 = 0^2 + 1^2 + 2^2 + \cdots + (N-1)^2 = \frac{N(N-1)(2N-1)}{6}, \tag{6.35}$$

which gives the eigenvalue $\frac{N(N-1)(N-2)}{3}$ in (6.34). In what follows it will be useful to shift the eigenvalue of the trivial representation to zero by considering the operator

$$\hat{C}_3^{(k)} = \frac{N(N-1)(N-2)}{3} - C_3^{(k)}. \tag{6.36}$$

In terms of oscillators and projectors, $\hat{C}_3^{(k)}$ is written as

$$\hat{C}_3^{(k)} = \frac{1}{k!}\mathcal{P}_k \left[\frac{N(N-1)(N-2)}{3} - \sum_{\substack{\sigma=(ijk)\\ 1\le i\neq j\neq k\le N}} {\rm Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k}\mathcal{L}(\sigma)a^{\otimes k})\right]\mathcal{P}_k. \tag{6.37}$$

We can use $\hat{C}_3^{(k)}$ to construct Hamiltonians with interesting spectra. Consider the family of Hamiltonians (depending on K)

$$\boxed{H_K = \sum_{k=0}^{K} \hat{C}_3^{(k)} H_0 + \sum_{k=K+1}^{\infty} \mathcal{P}_k H_0,} \tag{6.38}$$

where H_0 is the free Hamiltonian (number operator) defined in (2.3). In this model, all invariant states of degree $k \le K$ have zero energy, while noninvariant states have energies that scale with N. For example, degree $k\le K$ states in the representation $[N-1, 1]$ (a Young diagram with $N-1$ boxes in the first row and a single box in the second row) of S_N have energies $kN(N-2)$. More generally, degree $k\le K$ states in the representation $[N-a, a]$ for $1\le a < \lfloor N/2 \rfloor$ have energy $k(N-a+1)(N-2)a$. States of degree $k > K$ have energy k. The spectrum of H_K is illustrated on the left-hand side of Fig. 1(a). Taking $N \gg K$, there is a K-dependent degeneracy of invariant ground states and a gap of order K. In this scenario, the subspace of ground states has dimension

$$\sum_{k=0}^{K} {\rm Dim}\mathcal{H}_{\rm inv}{}^{(k)} = 1 + \sum_{k=1}^{K} {\rm Dim} SP_k(N), \tag{6.39}$$

where $\mathcal{H}_{\rm inv}{}^{(k)}$ is the degree k subspace of $\mathcal{H}_{\rm inv}$ [see Eq. (B.11) in [27] for explicit formulas computing ${\rm Dim}\mathcal{H}_{\rm inv}{}^{(k)}$]. By taking $N \gg K \gg 1$, we can have a large degeneracy of ground states alongside the interesting correlations between the degeneracy of ground states and the energy gap. A large ground state degeneracy associated with elements of a diagrammatic algebra, in this case the partition algebras $SP_k(N)$ for $k\le K$, is reminiscent of topological degeneracy and its links to anyons [102,103]. We leave a more detailed investigation of the analogies between the present algebraic constructions and topological degeneracy for the future.

C. Resolving the invariant spectrum

In the previous section we discussed a Hamiltonian (6.38) with degenerate ground state. We will now use the commuting algebraic charges $\bar{T}_2^{(k)}, t_2^{(k)} \in P_k(N)$, constructed in Sec. IV, to resolve this degeneracy. Note that the charges commute with ${\rm Ad}(\sigma)$ and in particular they commute with $\hat{C}_3^{(k)}$. We prove this in the next subsection, where we consider more general operators coming from elements of $P_k(N)$. Note that because $\bar{T}_2^{(k)}$ and $t_2^{(k)}$ are central elements of $P_k(N)$, and the representation basis states $|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle$ correspond to elements in $P_k(N)$, the charge's left and right actions are equivalent on these basis states.

The algebraic charges can be written in terms of oscillators and projectors as in (6.17) and (6.18). Importantly, the representation basis states $|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle$ are eigenstates of $\bar{T}^{(k),L}_2, t^{(k),L}_2$. The eigenvalues are normalized characters of the representations Λ_1 of S_N and Λ_2 of S_k, respectively [see (6.20)]. That is

$$\bar{T}^{(k),L}_2|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = |\bar{T}^{(k)}_2 Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \bar{T}^{(k),R}_2|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \hat{\chi}^{\Lambda_1}(T_2)|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle, \quad (6.40)$$

$$t^{(k),L}_2|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = |t^{(k)}_2 Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = t^{(k),R}_2|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \hat{\chi}^{\Lambda_2}(t_2)|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle, \quad (6.41)$$

where the normalized characters $\hat{\chi}$ are defined in (4.15). Note that the eigenvalues of the operator $t^{(k),L}_2$ range between $\pm\frac{k(k-1)}{2}$ and those of $\bar{T}^{(k),L}_2$ between $\pm\frac{N(N-1)}{2}$, including an infinite number of such operators in a Hamiltonian may result in a spectrum that is not bounded from below. By adding these algebraic charges to the Hamiltonian (6.38) the energy of the states $|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle$ labeled by distinct pairs Λ_1, Λ_2 will split. As discussed in Sec. IV B, the multiplicity labels μ, ν are not distinguished by these central algebraic charges. Hamiltonians that resolve more detailed information such as multiplicity labels are discussed in the next subsection.

For concreteness we consider the spectrum of the Hamiltonian

$$\begin{aligned} H'_K &= H_K + H_{\rm res} \\ &= H_K - \frac{2}{N(N-1)}\sum_{k=1}^{K}\bar{T}^{(k),L}_2, \\ &= \sum_{k=0}^{K}\hat{C}^{(k)}_3 H_0 + \sum_{k=K+1}^{\infty}\mathcal{P}_k H_0 - \frac{2}{N(N-1)}\sum_{k=1}^{K}\bar{T}^{(k),L}_2. \end{aligned} \quad (6.42)$$

The ground state degeneracy is reduced compared to H_K. The lowest energy states are degree $k \le K$ states $|Q^{[N]}_{\Lambda_2,\mu\nu}\rangle$ with energy -1. The highest energy state with degree $k \le K$ is $|Q^{[N-K,1^K]}_{[1^K]}\rangle$, it has degree K and energy $-\frac{(N-2K-1)}{(N-1)}$. The gap of order K remains, as illustrated on the right of Fig. 1(a). The label Λ_2 can be resolved by including $t^{(k),L}_2$ in the Hamiltonian.

To fully resolve the labels Λ_1, Λ_2 for general k and N, new charges are necessary. Detailed discussions of the problem of using such charges in the center of the symmetric group algebra $\mathbb{C}[S_n]$, with motivations coming from a model for information loss in AdS/CFT [104], are given in [89,105]. It can be proved that $\{T_2, T_3, \ldots, T_n\}$ provide an adequate set of charges and these also provide a multiplicative generating set for the center of the group algebra. Typically, a smaller set $\{T_2, T_3, \ldots, T_{k_*(n)}\}$ suffices. For example $k_*(5) = 2, k_*(14) = 3, k_*(80) = 6$. In the present discussion these results can be applied by choosing $n = k$ and $n = N$, respectively.

D. Precision resolution of the invariant spectrum

In the previous section we presented Hamiltonians involving commuting algebraic charges, constructed from central elements in $P_k(N)$, that resolve the representation labels Λ_1, Λ_2 of representation basis elements $|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle$. As discussed in Sec. IV B, and illustrated in an explicit example in Sec. IV C 2, more general elements of $SP_k(N)$ are necessary to resolve the multiplicity labels μ, ν. We will use this observation to construct S_N invariant Hamiltonians, involving operators $[d]^L$ and $[d]^R$ constructed from noncentral elements $[d] \in SP_k(N)$, with nondegenerate eigenvalues.

Since we want to construct Hamiltonians H satisfying $[\mathrm{Ad}(\sigma), H] = 0$, built from operators $[d]^L$, $[d]^R$, we will now prove that $[\mathrm{Ad}(\sigma),[d]^L] = [\mathrm{Ad}(\sigma),[d]^R] = 0$. To show that $[d]^L\mathrm{Ad}(\sigma) = \mathrm{Ad}(\sigma)[d]^L$ we combine Eq. (6.4) with (6.22)

$$\begin{aligned} \mathrm{Ad}(\sigma)[d]^L|T\rangle &= \mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma)dT\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k})|0\rangle, \\ &= \mathrm{Tr}_{V_N^{\otimes k}}(d\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})(a^\dagger)^{\otimes k})|0\rangle, \\ &= [d]^L\mathrm{Ad}(\sigma)|T\rangle, \end{aligned} \quad (6.43)$$

where the second line follows since $\mathcal{L}(\sigma)d = d\mathcal{L}(\sigma)$ as elements of $\mathrm{End}(V_N^{\otimes k})$ (linear maps $V_N^{\otimes k} \to V_N^{\otimes k}$). The argument is identical for $[d]^R\mathrm{Ad}(\sigma) = \mathrm{Ad}(\sigma)[d]^R$.

To construct Hamiltonians H, using the above operators, we need to ensure that any operator we include in H is Hermitian. The operators $[d]^L$, $[d]^R$ are not Hermitian in general, unless $[d^T] = [d]$. Taking this into account, we can parametrize a large family of S_N invariant Hamiltonians using the diagram basis for $P_k(N)$. We write

$$\begin{aligned} H = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{[d_\pi]}(&L_{k,\pi}[d_\pi]^L + L^*_{k,\pi}[d^T_\pi]^L \\ &+ R_{k,\pi}[d_\pi]^R + R^*_{k,\pi}[d^T_\pi]^R), \end{aligned} \quad (6.44)$$

where the sum over $[d_\pi]$ runs over a basis for $SP_k(N)$ and $L_{k,\pi}, R_{k,\pi}$ are complex parameters with the constraint $L^*_{k,\pi} = L_{k,\pi'}$ and $R^*_{k,\pi} = R_{k,\pi'}$ if $d^T_\pi = d_{\pi'}$. The equivalent expression for H in terms of oscillators and projectors is

$$H = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{[d_\pi]}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}\left((a^\dagger)^{\otimes k}\frac{L_{k,\pi}d_\pi + L^*_{k,\pi}d^T_\pi}{k!}a^{\otimes k}\right)\mathcal{P}_k + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{[d_\pi]}\mathcal{P}_k \mathrm{Tr}_{V_N^{\otimes k}}\left(\frac{R_{k,\pi}d_\pi + R^*_{k,\pi}d^T_\pi}{k!}(a^\dagger)^{\otimes k}a^{\otimes k}\right)\mathcal{P}_k. \tag{6.45}$$

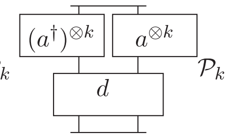
Progressively turning on parameters in Eq. (6.44) will tend to break degeneracy in the spectrum. Eventually, the spectrum may take the form in Fig. 1(b) where invariant and noninvariant states are mixed, and most of the degeneracy is broken.

E. General invariant Hamiltonians from partition algebras

The Hamiltonian H in (6.44) is not the most general Hamiltonian satisfying $[H, \mathrm{Ad}(\sigma)] = 0$. For example, it does not include the Hamiltonian (5.27) constructed in Sec. V nor H_K in (6.38). As we noticed in (6.30), $C_3^{(k)}$ is related to an element in $P_{2k}(N)$. We now generalize this observation to give a construction of general S_N invariant operators from elements in $P_{2k}(N)$.

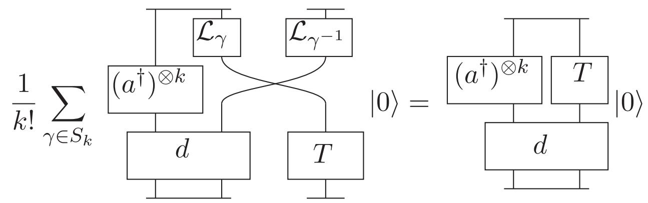
General degree preserving operators that commute with $\mathrm{Ad}(\sigma)$ can be constructed from elements $d \in P_{2k}(N)$ as

$$\frac{1}{k!}\mathcal{P}_k \,\mathrm{Tr}_{V_N^{\otimes 2k}}(d(a^\dagger)^{\otimes k}\otimes a^{\otimes k})\mathcal{P}_k \leftrightarrow \frac{1}{k!}\mathcal{P}_k$$



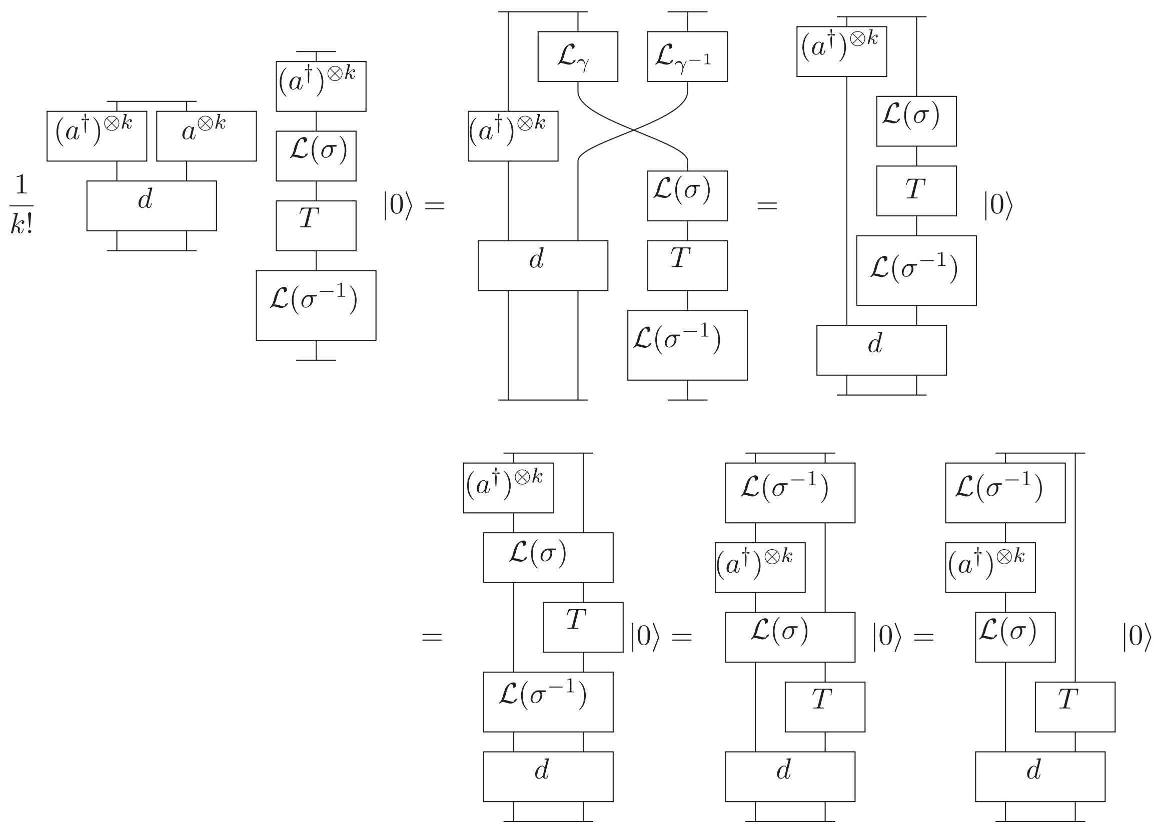
(6.46)

The action of these operators on $|T\rangle \in \mathcal{H}^{(k)}$ is



(6.47)

Commutativity with $\mathrm{Ad}(\sigma)$ follows from the following diagrammatic manipulations



(6.48)

where the first equality uses Eq. (6.47). The second line introduces an identity operator of the form $\mathcal{L}(\sigma^{-1})\mathcal{L}(\sigma)$ acting on the left-hand vector space $V_N^{\otimes k}$. The third equality follows from $\mathcal{L}(\sigma^{-1})d = d\mathcal{L}(\sigma^{-1})$ and the cyclicity of the trace. The last line removes the identity operator $\mathcal{L}(\sigma)\mathcal{L}(\sigma^{-1})$ acting on the right-hand vector space $V_N^{\otimes k}$. The last diagram is equal to

$$\mathrm{Ad}(\sigma)\frac{1}{k!}\mathrm{Tr}_{V_N^{\otimes 2k}}(d(a^\dagger)^{\otimes k}\otimes a^{\otimes k})|T\rangle, \qquad (6.49)$$

which proves that they commute.

The construction readily generalizes to operators that do not preserve the degree of states. Consider

$$\frac{1}{k_1!}\mathcal{P}_{k_2}\mathrm{Tr}_{V_N^{\otimes 2k}}(d(a^\dagger)^{\otimes k_2}\otimes a^{\otimes k_1})\mathcal{P}_{k_1}, \qquad (6.50)$$

this gives a map $d\colon \mathcal{H}^{(k_1)} \to \mathcal{H}^{(k_2)}$ labeled by elements $d \in P_{k_1+k_2}(N)$. Note that these operators have a $S_{k_2} \times S_{k_1}$ symmetry, which permutes the creation operators and annihilation operators separately. Therefore, the dimension of the space of these operators is related to the counting of 2-matrix permutation invariants, which was studied in Sec. 2 of [83].

F. Bosons on a lattice

The Fock space of matrix oscillators can be interpreted as the Fock space of bosons on a two-dimensional lattice of size N^2. The lattice is parametrized by ordered pairs (i, j) for $i, j = 1, \ldots, N$ which label the site in the ith row, jth column as in Fig. 2. The creation operator $(a^\dagger)^j_i$ creates a quantum of excitation at the site (i, j). In our conventions, a^i_j annihilates a quantum at site (i, j). Permutation invariant states naturally contain excitations spread throughout the entire lattice. For example, the state

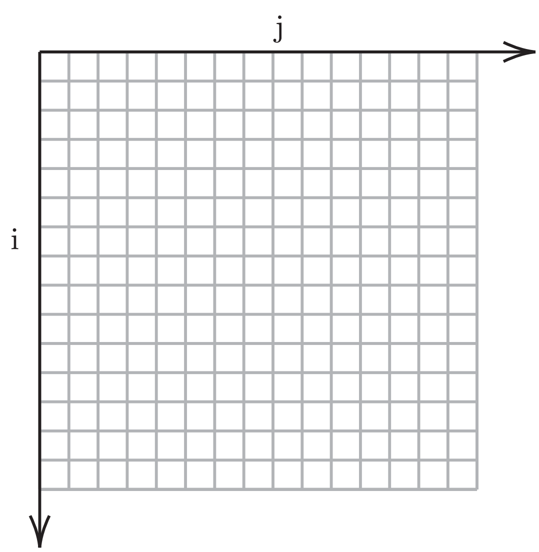


FIG. 2. Matrix oscillators are naturally associated with a N-by-N square lattice. The creation operator $(a^\dagger)^j_i$ creates a quanta of excitation at row i column j in the lattice.

$$\Big|\, \vdots \,\Big\rangle = \sum_{i=1}^{N} (a^\dagger)^j_j\,|0\rangle\,, \qquad (6.51)$$

contains an excitation of every site on the diagonal, and the state

$$\Big|\, : \,\Big\rangle - \Big|\, \vdots \,\Big\rangle = \sum_{i\neq j} (a^\dagger)^i_j\,|0\rangle\,, \qquad (6.52)$$

contains an excitation on every off-diagonal site.

Most choices of S_N invariant Hamiltonians constructed in Eq. (6.44) contain nonlocal interactions, connecting sites at opposite sides of the lattice. Note that the left acting terms in the Hamiltonian (6.44) leave the columns fixed while the right acting terms fix the rows. An example of the nonlocality is seen by considering

$$H = P_1\,\mathrm{Tr}_{V_N}(a^\dagger \,:\, a)P_1 = P_1 \sum_{i,j,k=1}^{N} (a^\dagger)^i_j (a)^k_i P_1. \qquad (6.53)$$

This interaction moves a single excitation at site (i, j) to every row in column j. In particular,

$$H(a^\dagger)^1_1|0\rangle = \sum_{i=1}^{N} (a^\dagger)^1_i|0\rangle, \qquad (6.54)$$

contains the state $(a^\dagger)^1_N$.

We can enumerate a set of diagrams that give local S_N invariant terms, through left and right action, as follows. First note that the identity element in $P_k(N)$ gives a local term. For example, in $k = 2$

$$\mathrm{Tr}_{V_N^{\otimes 2}}((a^\dagger)^{\otimes 2}\, \vdots\;\vdots\, a^{\otimes 2}) = \sum_{i_1,i_2,j_1,j_2=1}^{N} (a^\dagger)^{i_1}_{j_1}(a^\dagger)^{i_2}_{j_2}(a)^{j_1}_{i_1}(a)^{j_2}_{i_2}. \qquad (6.55)$$

It follows that any diagram that can be constructed from the identity element by adding additional edges is local. For example

$$\mathrm{Tr}_{V_N^{\otimes 2}}((a^\dagger)^{\otimes 2}\, \square\, a^{\otimes 2}) = \sum_{i_1,i_2,j=1}^{N} (a^\dagger)^{i_1}_j (a^\dagger)^{i_2}_j (a)^j_{i_1} (a)^j_{i_2}, \qquad (6.56)$$

which is still local.

VII. PERMUTATION INVARIANT QUANTUM SCARS

Energy eigenstates in quantum ergodic many-body systems are expected to thermalize, in accordance with the eigenstate thermalization hypothesis [106,107], which says that such systems are well described by statistical mechanical ensembles. Integrable systems, and systems which exhibit many-body localization [108] are known exceptions to the eigenstate thermalization hypothesis. This is a consequence of the existence of a large number of conserved quantities, which leads to nonergodicity. A weak form of nonergodicity was recently observed in experiments involving Rydberg-atom quantum simulators [87]. For some initial states, the behavior was as expected from an ergodic system, while other states would exhibit periodic revival. This is unexpected, since the experiment is described by a system without any conserved charges or disorder [109]. The term "quantum many-body scars" was coined in [88] to describe these nonergodic states embedded in a large space of ergodic states. Many mechanisms and construction schemes for systems and states that exhibit quantum many-body scars have been discussed in the theoretical literature [110–114].

In this section, we will follow the group theoretic scheme invented in [85,86] for constructing Hamiltonians which have many-body scars. Two basic ingredients are required in this scheme: a group G acting on a Hilbert space $\mathcal{H}$, and a subspace $\mathcal{H}_{\rm inv} \subset \mathcal{H}$ of states that are invariant under the action of G. To promote $\mathcal{H}_{\rm inv}$ to a space of many-body scars, the prescription is as follows. First, find a Hamiltonian H such that for all states $|d\rangle \in \mathcal{H}_{\rm inv}$

$$H|d\rangle \in \mathcal{H}_{\rm inv}, \tag{7.1}$$

and the time-evolution of $|d\rangle$ using H is periodic. This condition is discussed in Sec. VII A. Note that H commuting with the action of $g \in G$ is sufficient to satisfy (7.1). Now we break the symmetry of H, while retaining the many-body scars, by constructing a total Hamiltonian

$$H_{\rm tot} = H + H_{\rm s}. \tag{7.2}$$

The new term will completely break the symmetry of H but is required to satisfy

$$H_{\rm s}|d\rangle = 0 \quad \text{for all} \quad |d\rangle \in \mathcal{H}_{\rm inv}. \tag{7.3}$$

This ensures that the time evolution of $|d\rangle$ using $H_{\rm tot}$ is equivalent to time evolution using H, which was periodic by construction. Since $H_{\rm tot}$ has no remaining symmetry the noninvariant states in $\mathcal{H}$, which are not annihilated by $H_{\rm s}$, will be ergodic and therefore thermalize. The group theoretic construction of $H_{\rm s}$ is reviewed in Sec. VII B.

By combining the technology presented in this paper with the above scheme, we can construct models with many-body scars for $G = S_N$ acting on the Fock space $\mathcal{H}$ of matrix oscillators. In particular, Sec. III A contains a detailed description of the S_N invariant subspace $\mathcal{H}_{\rm inv} \subset \mathcal{H}$ and the Hamiltonians in Sec. VI can be used for H in (7.2). We gave a lattice interpretation of the matrix oscillators in Sec. VI F, which we will use to construct a lattice model with many-body scars. The model will be a modified version of the Bose-Hubbard model [115], which is relevant for physics of cold atoms in an optical lattice [116].

A. Periodic time evolution and revival

The Hamiltonian in (7.2) contains two pieces, but the dynamics (time evolution) of invariant states is governed by H alone. In this subsection we will focus on H, and give a sufficient condition for it to give rise to periodic time evolution in the invariant subspace, turning the subspace into a many-body scar space.

Let $|d\rangle$ be a (normalized) state in $\mathcal{H}_{\rm inv}$. Since H is S_N invariant we have $H|d\rangle \in \mathcal{H}_{\rm inv}$ and we can construct an orthonormal energy eigenbasis $|e_i\rangle$ for $\mathcal{H}_{\rm inv}$ with eigenvalues E_i,

$$H|e_i\rangle = E_i|e_i\rangle. \tag{7.4}$$

The state $|d\rangle$ exhibits revival with periodicity T if the quantum fidelity (return probability) [117]

$$f(t) = |\langle d|e^{-iHt}|d\rangle|^2, \tag{7.5}$$

satisfies $f(mT) = 1$ for $m = 0, 1, \ldots$. Expanding $|d\rangle$ in the eigenbasis

$$|d\rangle = \sum_i d_i|e_i\rangle, \tag{7.6}$$

and computing $f(t)$ gives

$$f(t) = |\langle d|e^{-iHt}|d\rangle|^2 = \sum_{i,j}|d_i|^2|d_j|^2 e^{-i(E_i-E_j)t}. \tag{7.7}$$

If all energy differences $\Delta E_{ij} = E_i - E_j$ have a greatest common divisor E, that is

$$\Delta E_{ij} = E_i - E_j = E(\varepsilon_i - \varepsilon_j) \tag{7.8}$$

and $\varepsilon_i - \varepsilon_j$ is an integer for all i, j, then $f(mT) = 1$ for $T = 2\pi/E$. Note that trading H for $H_{\rm tot}$ in (7.7) does not change the argument above since $H_{\rm s}|d\rangle = 0$ by construction. That is, the time evolution of states in $\mathcal{H}_{\rm inv}$ is determined by H. As a special case, $f(t)$ is periodic if the energies E_i of the states $|e_i\rangle$ relevant to the expansion of $|d\rangle$ are integers.

B. Scar Hamiltonians

We now turn to the construction of the second part of the Hamiltonian (7.2), using the group theoretic scheme introduced in [85,86]. In order to implement this scheme in the present setup we observe

$$(1-\mathrm{Ad}(\sigma))|d\rangle = 0, \qquad \forall\ \sigma\in S_N, \qquad |d\rangle\in\mathcal{H}_{\mathrm{inv}}. \quad (7.9)$$

This follows from $\mathrm{Ad}(\sigma)|d\rangle = |d\rangle$, and we will use it to construct H_{s}.

As we show below, the Hermitian conjugate of $\mathrm{Ad}(\sigma)$ is $\mathrm{Ad}(\sigma^{-1})$. This will be important because, at the end of the day, we want H_{s} to be Hermitian. Starting from the definition of the inner product we have

$$\begin{aligned}\langle T'||\mathrm{Ad}(\sigma)T\rangle &= \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}((T')^T\mathcal{L}_\gamma\mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1})\mathcal{L}_{\gamma^{-1}}),\\ &= \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}(\mathcal{L}(\sigma^{-1})(T')^T\mathcal{L}(\sigma)\mathcal{L}_\gamma T\mathcal{L}_{\gamma^{-1}}),\\ &= \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}((\mathcal{L}(\sigma^{-1})T'\mathcal{L}(\sigma))^T\mathcal{L}_\gamma T\mathcal{L}_{\gamma^{-1}}),\\ &= \langle\mathrm{Ad}(\sigma^{-1})T'||T\rangle, \end{aligned} \quad (7.10)$$

where the second equality uses $\mathcal{L}_\gamma\mathcal{L}(\sigma) = \mathcal{L}(\sigma)\mathcal{L}_\gamma$ and the third equality follows from

$$(\mathcal{L}(\sigma^{-1})T'\mathcal{L}(\sigma))^T = \mathcal{L}(\sigma^{-1})(T')^T\mathcal{L}(\sigma). \quad (7.11)$$

Consequently, an operator of the form

$$H_\sigma = (1-\mathrm{Ad}(\sigma^{-1}))h_\sigma(1-\mathrm{Ad}(\sigma)), \quad (7.12)$$

where h_σ is any Hermitian operator, is itself Hermitian and satisfies $H_\sigma|d\rangle = 0$, in accordance with the setup in (7.2). In general, we can write H_{s} in the form

$$H_{\mathrm{s}} = \sum_{\sigma\in S_N} c_\sigma H_\sigma, \quad (7.13)$$

where $c_\sigma\in\mathbb{R}$ is a parameter for every $\sigma\in S_N$.

The real dimension of the space of independent Hermitian operators (candidate choices for h_σ) can be counted as follows. We organize general (normal ordered) k-oscillator operators in terms of the number of creation and annihilation operators (k_1, k_2, respectively). They have the form

$$O = O^{i_1\ldots i_k}_{j_1\ldots j_k}(a^\dagger)^{j_1}_{i_1}\ldots(a^\dagger)^{j_{k_1}}_{i_{k_1}}a^{j_{k_1+1}}_{i_{k_1+1}}\ldots a^{j_k}_{i_k}. \quad (7.14)$$

Their adjoints contain k_2 creation and k_1 annihilation operators

$$O^\dagger = (O^*)^{i_1\ldots i_{k_1}i_{k_1+1}\ldots i_k}_{j_1\ldots j_{k_1}j_{k_1+1}\ldots j_k}(a^\dagger)^{i_{k_1+1}}_{j_{k_1+1}}\ldots(a^\dagger)^{i_k}_{j_k}a^{i_1}_{j_1}\ldots a^{i_{k_1}}_{j_{k_1}}. \quad (7.15)$$

For every $k = k_1+k_2$ oscillator operator O with $k_1 > k_2$ there is a Hermitian operator $O + O^\dagger$. The real dimension of the independent Hermitian operators of this form can be counted in terms of the dimensions of symmetric tensor product spaces

$$\begin{aligned}&2\ \mathrm{Dim}(\mathrm{Sym}^{k_1}(V_N\otimes V_N)\otimes\mathrm{Sym}^{k_2}(V_N\otimes V_N))\\ &\quad = 2\binom{N^2+k_1-1}{k_1}\binom{N^2+k_2-1}{k_2},\end{aligned} \quad (7.16)$$

which follows by identifying operators $(a^\dagger)^{j_1}_{i_1}$ and $a^{j_1}_{i_1}$ with the vector space $V_N\otimes V_N$. The factor of 2 comes from the fact that $O^{i_1\ldots i_k}_{j_1\ldots j_k}$ are complex numbers. The Hermitian operators associated with $k_2 < k_1$ oscillator operators are accounted for in (7.16) as their conjugates are the $k_1 < k_2$ oscillator operators.

The remaining Hermitian operators to count are those with equal numbers of creation and annihilation oscillators, i.e., those with $k_1 = k_2$. Some of these will be self-adjoint, while the remaining operators can be paired with their adjoints to construct Hermitian operators as before. Inspecting Eqs. (7.14) and (7.15) we see that for an operator to be equal to its own adjoint it must be real, with $k_1 = k_2$ and

$$O^{i_1\ldots i_{k_1}i_{k_1+1}\ldots i_k}_{j_1\ldots j_{k_1}j_{k_1+1}\ldots j_k} = (O^*)^{j_{k_1+1}\ldots j_k j_1\ldots j_{k_1}}_{i_{k_1+1}\ldots i_k i_1\ldots i_{k_1}}. \quad (7.17)$$

As they are real, the number of these operators is equal to their real dimension

$$\mathrm{Dim}(\mathrm{Sym}^{k_1}(V_N\otimes V_N)) = \binom{N^2+k_1-1}{k_1}. \quad (7.18)$$

This counting can be understood as there being exactly one choice of $\{\{i_{k_1+1},\ldots,i_{2k_1}\},\{j_{k_1+1},\ldots,j_{2k_1}\}\}$ for which each choice of $\{\{i_1,\ldots i_{k_1}\},\{j_1,\ldots j_{k_1}\}\}$ satisfies (7.17). The remaining number of operators is

$$\begin{aligned}&\mathrm{Dim}(\mathrm{Sym}^{k_1}(V_N\otimes V_N))\times[\mathrm{Dim}(\mathrm{Sym}^{k_1}(V_N\otimes V_N))-1]\\ &\quad = \binom{N^2+k_1-1}{k_1}\times\left[\binom{N^2+k_1-1}{k_1}-1\right].\end{aligned} \quad (7.19)$$

The factor of 2 due to $O^{i_1\ldots i_k}_{j_1\ldots j_k}$ being complex is canceled by the factor of a half introduced when forming Hermitian operators. The real dimension of Hermitian operators of type $k_1 = k_2$ is then

$$\binom{N^2+k_1-1}{k_1}\times\binom{N^2+k_1-1}{k_1}. \quad (7.20)$$

C. Modified Bose-Hubbard on a square lattice

Having discussed the general setup, we will now implement the scheme in a specific example. In Sec. VI F we gave a lattice interpretation of the matrix oscillators $(a^\dagger)^j_i$, where they played the role of bosonic creation operators on the lattice site labeled (i, j). For simplicity, we will consider a square lattice of dimension N. A simple model of interacting bosons on a lattice is the Bose-Hubbard model [115], relevant for the physics of cold atoms in an optical lattice [116]. The Bose-Hubbard Hamiltonian H_{BH} contains hopping (kinetic) terms, on-site interactions and chemical potentials. In the simplest case, the model contains three parameters t, U, μ, and in terms matrix of oscillators it takes the form

$$H_{\mathrm{BH}} = -t \sum_{\langle (i,j),(k,l) \rangle} (a^\dagger)^j_i a^k_l + \frac{U}{2} \sum_{i,j=1}^{N} (a^\dagger)^j_i a^i_j ((a^\dagger)^j_i a^i_j - 1) - \mu \sum_{i,j=1}^{N} (a^\dagger)^j_i a^i_j, \tag{7.21}$$

where the sum in the first term is over neighboring sites. For a square lattice this implies the restriction $(i, j) = (k \pm 1, l)$ or $(i, j) = (k, l \pm 1)$. On-site interactions are implemented using the operators $N_{ij} = (a^\dagger)^j_i a^i_j$, which count the number of excitations on site (i, j).

Our aim is to construct a modified Bose-Hubbard Hamiltonian H'_{BH} such that

$$H'_{\mathrm{BH}} = H + H_{\mathrm{s}}, \tag{7.22}$$

where $[\mathrm{Ad}(\sigma), H] = 0$ for all $\sigma \in S_N$, and $H_{\mathrm{s}}|d\rangle = 0$ for all $|d\rangle \in \mathcal{H}_{\mathrm{inv}}$, as per the construction in (7.2). To this end, we observe that the second and third term in (7.21) are S_N invariant. That is,

$$H = \frac{U}{2} \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i ((a^\dagger)^i_j a^j_i - 1) - \mu \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i, \tag{7.23}$$

satisfies $[\mathrm{Ad}(\sigma), H] = 0$.

The hopping term

$$h_t = -t \sum_{\langle (i,j),(k,l) \rangle} (a^\dagger)^j_i a^k_l \tag{7.24}$$

is not S_N invariant, but the combination

$$H_{\mathrm{s}} = (1 - \mathrm{Ad}(\sigma^{-1})) h_t (1 - \mathrm{Ad}(\sigma)) \tag{7.25}$$

satisfies $H_{\mathrm{s}}|d\rangle = 0$ for any choice of $\sigma \in S_N$ by the construction in (7.12).

To keep H_{s} as local as possible, σ should not permute distant sites. With this restriction in mind, a simple choice is $\sigma = (23)$ [for the choice $\sigma = (12)$ we have to consider additional complications from being near the boundary of the square lattice]. This defines our modified Bose-Hubbard Hamiltonian H'_{BH},

$$\boxed{H'_{\mathrm{BH}} = \frac{U}{2} \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i ((a^\dagger)^i_j a^j_i - 1) - \mu \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i \quad - t \sum_{\langle (i,j),(k,l) \rangle} (1 - \mathrm{Ad}((23))) (a^\dagger)^j_i a^k_l (1 - \mathrm{Ad}((23))).} \tag{7.26}$$

It can be written as

$$H'_{\mathrm{BH}} = H_{\mathrm{BH}} - t \sum_{\langle (i,j),(k,l) \rangle} \mathrm{Ad}((23)) (a^\dagger)^j_i a^k_l \mathrm{Ad}((23)) + t \sum_{\langle (i,j),(k,l) \rangle} \mathrm{Ad}((23)) (a^\dagger)^j_i a^k_l + t \sum_{\langle (i,j),(k,l) \rangle} (a^\dagger)^j_i a^k_l \mathrm{Ad}((23)). \tag{7.27}$$

We now investigate the conditions on U and μ for which H'_{BH} exhibits revival. It is useful to rewrite Eq. (7.23) as

$$H = \frac{U}{2} \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i \left[(a^\dagger)^i_j a^j_i - 1 - \frac{2\mu}{U} \right]. \tag{7.28}$$

For integer values of $\frac{2\mu}{U}$ the eigenvalues of H are integer multiples of $\frac{U}{2}$, and similarly for differences of eigenvalues. By the argument given in Sec. VII A we therefore expect H'_{BH} to have many-body scars that revive with period $T = \frac{4\pi}{U}$. Similarly, we may write (7.23) as

$$H = \frac{\mu}{2} \sum_{i,j=1}^{N} (a^\dagger)^i_j a^j_i \left[\frac{U}{\mu} (a^\dagger)^i_j a^j_i - \frac{U}{\mu} - 2 \right], \tag{7.29}$$

from which we conclude that revival is possible when $\frac{U}{\mu}$ is an integer as well, with an expected revival time of $\frac{4\pi}{\mu}$. In the special case $U = 2\mu$ where both integrality conditions are satisfied, the revival time is $T = \min(\frac{4\pi}{\mu}, \frac{4\pi}{U}) = \frac{4\pi}{U}$.

In the subspace where there is a single excitation, the operator Ad((23)) takes the form

$$\mathrm{Ad}((23)) = (a^\dagger)^3_3 a^2_2 + (a^\dagger)^2_2 a^3_3 + \sum_{i\neq 2,3}((a^\dagger)^i_3 a^2_i + (a^\dagger)^i_2 a^3_i + (a^\dagger)^3_i a^i_2 + (a^\dagger)^2_i a^i_3) + \sum_{i,j\neq 2,3}(a^\dagger)^j_i a^i_j. \tag{7.30}$$

The first term is a diagonal hopping term, between sites (2,2) and (3,3). Consequently, the modified Bose-Hubbard Hamiltonian (7.26) will contain some hopping terms beyond nearest neighbors.

VIII. AdS/CFT INSPIRED EXTREMAL CORRELATORS IN MATRIX QUANTUM MECHANICS

Extremal correlators in $\mathcal{N} = 4$ SYM form interesting sectors having nonrenormalization properties [118]. They are closely connected to representation theoretic quantities such as Littlewood-Richardson coefficients, and form a crucial set of examples for checking the AdS/CFT correspondence. In the quantum mechanical model presented in this paper, vacuum expectation values similar to extremal correlators can be computed exactly. In this section we make use of a recent factorization result concerning the two-point function of permutation invariant matrix observables [37]—this is used to demonstrate that a similar factorization property holds for quantum mechanical permutation invariant states. We then compute an expression for extremal three-point correlators associated with S_N invariant states, which are simple in the diagram basis and obey representation theoretic selection rules.

A. Two-point correlators

The Eq. (3.25) can be interpreted as a quantum mechanical operator-state correspondence for S_N invariant states labeled by $[d_\pi] \in SP_k(N)$,

$$|d_\pi\rangle \leftrightarrow \mathcal{O}_\pi = \mathrm{Tr}_{V_N^{\otimes k}}([d_\pi](a^\dagger)^{\otimes k}). \tag{8.1}$$

From Eq. (3.29) we have

$$\mathcal{O}_\pi^\dagger = \mathrm{Tr}_{V_N^{\otimes k}}([d_\pi^T]a^{\otimes k}), \tag{8.2}$$

where the transpose d_π^T is the diagram obtained by reflecting d_π across a horizontal line, as illustrated in (3.30). The time-dependent operators are given by

$$\mathcal{O}_\pi(t) = \mathrm{e}^{-iH_0 t}\mathcal{O}_\pi \mathrm{e}^{iH_0 t} = \mathrm{e}^{-ikt}\mathcal{O}_\pi, \tag{8.3}$$

where H_0 is the free Hamiltonian, defined in Eq. (2.7).

In [37] the two-point function of permutation invariant matrix observables was shown to factorize in the large N limit. Here we use this result to show an equivalent factorization property for the two-point function of permutation invariant quantum mechanical states. Let $[d_{\pi_1}] \in SP_{k_1}(N), [d_{\pi_2}] \in SP_{k_2}(N)$, and define the two-point correlator to be the vacuum expectation value

$$\langle 0|\mathcal{O}^\dagger_{\pi_1}(t_1)\mathcal{O}_{\pi_2}(t_2)|0\rangle = \mathrm{e}^{ik_1t_1 - ik_2t_2}\langle 0|\mathrm{Tr}_{V_N^{\otimes k_1}}([d^T_{\pi_1}]a^{\otimes k_1})\mathrm{Tr}_{V_N^{\otimes k_2}}([d_{\pi_2}](a^\dagger)^{\otimes k_2})|0\rangle. \tag{8.4}$$

Ignoring the trivial time dependence and taking normalized operators $[\hat{d}_{\pi_1}], [\hat{d}_{\pi_2}]$, as defined in (3.32), in the large N limit we have

$$\begin{aligned}\langle 0|\mathrm{Tr}_{V_N^{\otimes k_1}}([\hat{d}^T_{\pi_1}]a^{\otimes k_1})\mathrm{Tr}_{V_N^{\otimes k_2}}([\hat{d}_{\pi_2}](a^\dagger)^{\otimes k_2})|0\rangle &= \delta_{k_1k_2}\sum_{\gamma\in S_{k_1}}\mathrm{Tr}_{V_N^{\otimes k_2}}(\gamma^{-1}\hat{d}^T_{\pi_1}\gamma\hat{d}_{\pi_2}),\\ &= \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_{\pi_1}] = [d_{\pi_2}] \\ 0 + O(1/\sqrt{N}) & \text{otherwise}\end{cases}.\end{aligned} \tag{8.5}$$

In the first line we have absorbed the S_{k_1} averaging into the sum over $\gamma \in S_{k_1}$ arising from the Wick contractions of a and $a^\dagger$. In the second line we have used the factorization result of [37].

B. Three-point correlators

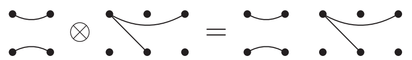
Let $[d_{\pi_1}] \in SP_{k_1}(N), [d_{\pi_2}] \in SP_{k_2}(N), [d_\pi] \in SP_k(N)$, and define the extremal three-point correlator to be the vacuum expectation value

$$\langle 0|\mathcal{O}^\dagger_{\pi_1}(t_1)\mathcal{O}^\dagger_{\pi_2}(t_2)\mathcal{O}_\pi(t)|0\rangle = \mathrm{e}^{ik_1t_1 + ik_2t_2 - ikt}\langle 0|\mathrm{Tr}_{V_N^{\otimes k_1}}([d^T_{\pi_1}]a^{\otimes k_1})\mathrm{Tr}_{V_N^{\otimes k_2}}([d^T_{\pi_2}]a^{\otimes k_2})\mathrm{Tr}_{V_N^{\otimes k}}([d_\pi](a^\dagger)^{\otimes k})|0\rangle, \tag{8.6}$$

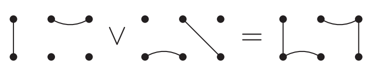
with the constraint that $k = k_2 + k_1$. As we now show, extremal correlators are simple in the diagram basis. We compute (8.6) by Wick contractions, which are encoded in a sum over $\gamma \in S_k$. Ignoring the trivial time dependence we have

$$\langle 0|\mathrm{Tr}_{V_N^{\otimes k_1}}([d_{\pi_1}^T]a^{\otimes k_1})\mathrm{Tr}_{V_N^{\otimes k_2}}([d_{\pi_2}^T]a^{\otimes k_2})\mathrm{Tr}_{V_N^{\otimes k}}([d_\pi](a^\dagger)^{\otimes k})|0\rangle = \sum_{\gamma\in S_k}\mathrm{Tr}_{V_N^{\otimes k}}(\gamma^{-1}(d_{\pi_1}^T\otimes d_{\pi_2}^T)\gamma d_\pi), = \sum_{\gamma\in S_k} N^{c(\gamma^{-1}(d_{\pi_1}\otimes d_{\pi_2})\gamma\vee d_\pi)}. \tag{8.7}$$

The tensor product $d_{\pi_1}\otimes d_{\pi_2}$ is the diagram obtained by horizontal concatenation of d_{π_1} and d_{π_2}, for example


(8.8)

This can be viewed as an outer product on partition algebra diagrams which maps $P_{k_1}(N)\times P_{k_2}(N)$ to $P_{k_1+k_2}(N)$. It is a diagram with $2k_1 + 2k_2$ vertices. The join $d_{\pi_1}\vee d_{\pi_2}$ of two diagrams, each with $2k$ vertices, is obtained by adding all the edges of d_{π_1} to the edges of d_{π_2} (or vice versa), for example


(8.9)

The resulting diagram also has $2k$ vertices. For general elements (linear combinations of diagram basis elements) the two operations are defined by linear extension.

We will now derive a set of representation theoretic selection rules for the extremal correlators. To state the result we are going to prove, we define the operators

$$\mathcal{O}^{\Lambda_1}_{\Lambda_2,\mu\nu} = \mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}(a^\dagger)^{\otimes k}), \tag{8.10}$$

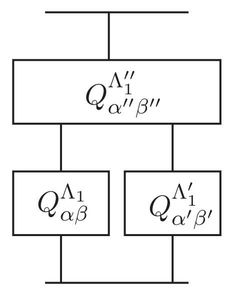
associated with representation basis elements $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\in SP_k(N)$. Consider the extremal correlator (time-independent part)

$$\langle 0|(\mathcal{O}^{\Lambda_1}_{\Lambda_2,\mu\nu})^\dagger(\mathcal{O}^{\Lambda_1'}_{\Lambda_2',\mu'\nu'})^\dagger\mathcal{O}^{\Lambda_1''}_{\Lambda_2'',\mu''\nu''}|0\rangle = k!\mathrm{Tr}_{V_N^{\otimes k}}((Q^{\Lambda_1}_{\Lambda_2,\nu\mu}\otimes Q^{\Lambda_1'}_{\Lambda_2',\nu'\mu'})Q^{\Lambda_1''}_{\Lambda_2'',\mu''\nu''}), \tag{8.11}$$

for $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\in SP_{k_1}(N), Q^{\Lambda_1'}_{\Lambda_2',\mu'\nu'}\in SP_{k_2}(N), Q^{\Lambda_1''}_{\Lambda_2'',\mu''\nu''}\in SP_k(N)$. The factor of $k!$ follows since the matrix units for $SP_k(N)$ are invariant under conjugation by S_k. Note that the multiplicity labels are exchanged under diagram transposition, which follows from (A25). The selection rule that we will find says that the trace in (8.11) vanishes if $C(\Lambda_1,\Lambda_1',\Lambda_1'')=0$, where $C(\Lambda_1,\Lambda_1',\Lambda_1'')$ is the Kronecker coefficient for tensor products of irreducible representations of S_N.

We start with the simpler but analogous expression for matrix units of $P_k(N)$,

$$\mathrm{Tr}_{V_N^{\otimes k}}((Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})Q^{\Lambda_1''}_{\alpha''\beta''}) =$$



(8.12)

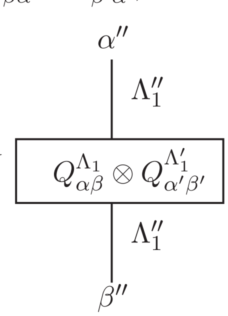
Using [see e.g. Eq. (4.16)]

$$(Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})Q^{\Lambda_1''}_{\alpha''\beta''} = \sum_{\gamma''} D^{\Lambda_1''}_{\gamma''\alpha''}(Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})Q^{\Lambda_1''}_{\gamma''\beta''}, \tag{8.13}$$

we have

$$\mathrm{Tr}_{V_N^{\otimes k}}((Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})Q^{\Lambda_1''}_{\alpha''\beta''}) = \sum_{\gamma''} D^{\Lambda_1''}_{\gamma''\alpha''}(Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})\,\mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1''}_{\gamma''\beta''}), = D^{\Lambda_1''}_{\beta''\alpha''}(Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})\,\mathrm{Dim}\,V^{S_N}_{\Lambda_1''},$$

$$= \mathrm{Dim}\,V^{S_N}_{\Lambda_1''}$$



(8.14)

The second equality uses (A65).

To further simplify, we want to turn the rhs into a product of matrix elements. This is achieved by inserting a resolution of the identity using representations of $P_{k_1}(N)\otimes P_{k_2}(N)$. This resolves to a set of branching coefficients for $P_k(N)\to P_{k_1}(N)\otimes P_{k_2}(N)$. We denote these by

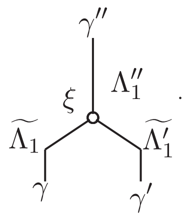
$$B^{\Lambda_1''\to\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1',\xi}_{\gamma''\to\gamma\gamma'}, \tag{8.15}$$

where it is implicit that $k=k_1+k_2$. The ranges of the labels are

$$\begin{aligned}
&\gamma\in[1,\dots,\mathrm{Dim}(V^{P_{k_1}(N)}_{\tilde{\Lambda}_1})],\\
&\gamma'\in[1,\dots,\mathrm{Dim}(V^{P_{k_2}(N)}_{\tilde{\Lambda}_1'})],\\
&\gamma''\in[1,\dots,\mathrm{Dim}(V^{P_{k}(N)}_{\Lambda_1''})],\\
&\xi\in[1,\dots,\mathrm{Mult}(V^{P_k(N)}_{\Lambda_1''}\to V^{P_{k_1}(N)}_{\tilde{\Lambda}_1}\otimes V^{P_{k_2}(N)}_{\tilde{\Lambda}_1'})],
\end{aligned}\tag{8.16}$$

the final label, ξ, gives the multiplicity of Λ_1'' in the decomposition. Branching coefficients are represented by the following diagrams

$$B^{\Lambda_1''\to\widetilde{\Lambda}_1\otimes\widetilde{\Lambda}_1',\xi}_{\gamma''\to\gamma\gamma'} =$$



(8.17)

It is worth noting that by Schur-Weyl duality the branching multiplicities for partition algebras are related to the multiplicities $C(\tilde{\Lambda}_1,\tilde{\Lambda}_1',\Lambda_1'')$, known as Kronecker coefficients, of irreducible representations Λ_1'' in tensor products of S_N representations $\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1'$ [see Eq. (3.1.3) of [119]]

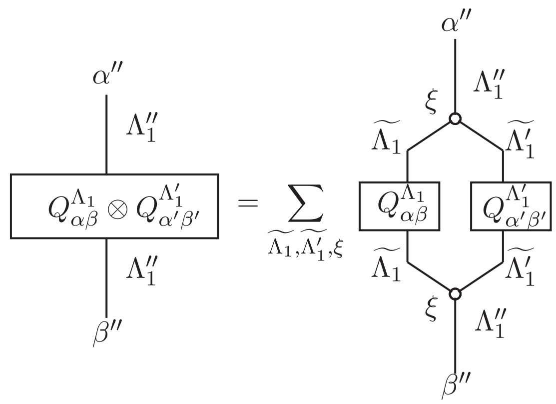
$$\mathrm{Mult}(V^{P_k(N)}_{\Lambda_1''}\to V^{P_{k_1}(N)}_{\tilde{\Lambda}_1}\otimes V^{P_{k_2}(N)}_{\tilde{\Lambda}_1'})=C(\tilde{\Lambda}_1,\tilde{\Lambda}_1',\Lambda_1''). \tag{8.18}$$

For simiplicity we are assuming $N\geq(2k_1+2k_2)$. For comparison, in Schur-Weyl duality between $U(N)$ and $\mathbb{C}[S_k]$, Littlewood-Richardson coefficients are branching multiplicities for $S_{k_1+k_2}\to S_{k_1}\times S_{k_2}$ but correspond to decomposition of tensor products of $U(N)$ representations.

Branching coefficients are equivariant:

$$\begin{aligned}
&D^{\Lambda_1''}_{\gamma''\delta''}(d_{\pi_1}\otimes d_{\pi_2})\\
&=\sum_{\tilde{\Lambda}_1,\tilde{\Lambda}_1',\gamma,\delta,\gamma',\delta',\xi} B^{\Lambda_1''\to\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1',\xi}_{\gamma''\to\gamma\gamma'} D^{\tilde{\Lambda}_1}_{\gamma\delta}(d_{\pi_1}) D^{\tilde{\Lambda}_1'}_{\gamma'\delta'}(d_{\pi_2}) B^{\Lambda_1''\to\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1',\xi}_{\delta''\to\delta\delta'},
\end{aligned}\tag{8.19}$$

for $d_{\pi_1}\in P_{k_1}(N), d_{\pi_2}\in P_{k_2}(N)$. Setting $d_{\pi_1}=Q^{\Lambda_1}_{\alpha\beta}$, $d_{\pi_2}=Q^{\Lambda_1'}_{\alpha'\beta'}$, Eq. (8.19) corresponds to the diagram identity



(8.20)

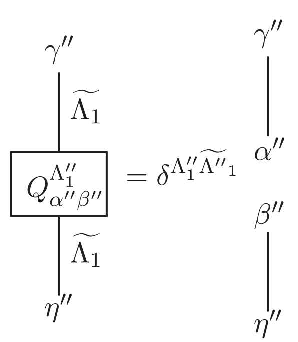
Inserting this into Eq. (8.14) gives

$$\begin{aligned}
&\mathrm{Tr}_{V_N^{\otimes k}}((Q^{\Lambda_1}_{\beta\alpha}\otimes Q^{\Lambda_1'}_{\beta'\alpha'})Q^{\Lambda_1''}_{\alpha''\beta''})\\
&\quad=\mathrm{Dim}V^{S_N}_{\Lambda_1''}\sum_{\tilde{\Lambda}_1,\tilde{\Lambda}_1',\gamma,\eta,\gamma',\eta',\xi} B^{\Lambda_1''\to\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1',\xi}_{\gamma''\to\gamma\gamma'} D^{\tilde{\Lambda}_1}_{\gamma\eta}(Q^{\Lambda_1}_{\alpha\beta})\\
&\qquad\times D^{\tilde{\Lambda}_1'}_{\gamma'\eta'}(Q^{\Lambda_1'}_{\alpha'\beta'}) B^{\Lambda_1''\to\tilde{\Lambda}_1\otimes\tilde{\Lambda}_1',\xi}_{\eta''\to\eta\eta'}.
\end{aligned}\tag{8.21}$$

Matrix elements of irreducible representations are orthogonal [see Eq. (A42)]. This implies

$$D^{\tilde{\Lambda}_1''}_{\eta''\gamma''}(Q^{\Lambda_1''}_{\alpha''\beta''})=\delta^{\tilde{\Lambda}_1''\Lambda_1''}\delta_{\eta''\beta''}\delta_{\gamma''\alpha''} \tag{8.22}$$

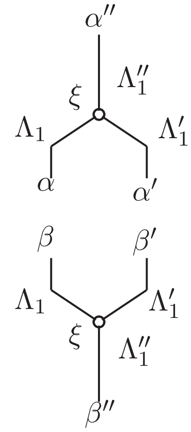
or the equivalent diagrammatic expression



(8.23)

Substituting this identity into (8.21) reduces it to

$$\sum_{\xi} \mathrm{Dim}\, V^{S_N}_{\Lambda_1''} B^{\Lambda_1''\to\Lambda_1\otimes\Lambda_1',\xi}_{\alpha''\to\alpha\alpha'} B^{\Lambda_1''\to\Lambda_1\otimes\Lambda_1',\xi}_{\beta''\to\beta\beta'} = \sum_{\xi} \mathrm{Dim}\, V^{S_N}_{\Lambda_1''} \quad (8.24)$$



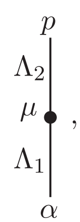
This gives the final result for matrix units of $P_k(N)$.

The full expression for (8.11)—extremal three-point correlators in the representation basis—is given by (8.24) together with branching coefficients from the partition algebras to symmetric group algebras [see (3.53)],

$$\begin{aligned} \mathrm{Tr}_{V_N^{\otimes k}}((Q^{\Lambda_1}_{\Lambda_2,\nu\mu}\otimes Q^{\Lambda_1'}_{\Lambda_2',\nu'\mu'})Q^{\Lambda_1''}_{\Lambda_2'',\mu''\nu''}) = \mathrm{Dim} V^{S_N}_{\Lambda_1''} \sum_{\substack{\alpha,\beta,\alpha',\beta',\alpha'',\beta'',\\ p,p',p'',\xi}} B^{\Lambda_1''\to\Lambda_1\otimes\Lambda_1',\xi}_{\alpha''\to\alpha\alpha'} B^{\Lambda_1''\to\Lambda_1\otimes\Lambda_1',\xi}_{\beta''\to\beta\beta'} B^{P_{k_1}(N)\to\mathbb{C}[S_{k_1}]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_{k_1}(N)\to\mathbb{C}[S_{k_1}]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu} \\ \times B^{P_{k_2}(N)\to\mathbb{C}[S_{k_2}]}_{\Lambda_1',\alpha'\to\Lambda_2',p';\mu'} B^{P_{k_2}(N)\to\mathbb{C}[S_{k_2}]}_{\Lambda_1',\beta'\to\Lambda_2',p';\nu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1'',\alpha''\to\Lambda_2'',p'';\mu''} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1'',\beta''\to\Lambda_2'',p'';\nu''}. \end{aligned} \quad (8.25)$$

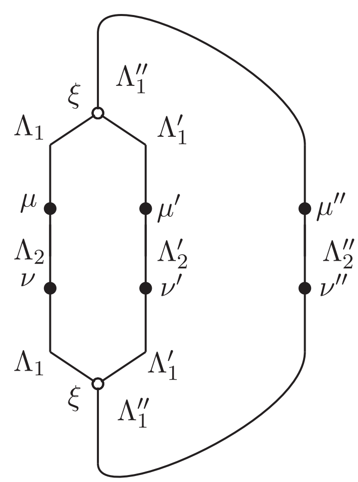
Introducing the following diagram representation of these branching coefficients,

$$B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} = \quad , \quad (8.26)$$



we can write (8.25) as the following diagram

$$\mathrm{Tr}_{V_N^{\otimes k}}\left((Q^{\Lambda_1}_{\Lambda_2,\nu\mu}\otimes Q^{\Lambda_1'}_{\Lambda_2',\nu'\mu'})Q^{\Lambda_1''}_{\Lambda_2'',\mu''\nu''}\right) = \quad . \quad (8.27)$$



From the above formula we see that the extremal correlator vanishes if the Kronecker coefficient $C(\Lambda_1, \Lambda_1', \Lambda_1'') = 0$. Analogous results for extremal correlators in general quiver gauge theories are described in [64].

IX. SUMMARY AND OUTLOOK

In this paper we investigated the effects of permutation symmetry on the state space and dynamics of quantum mechanical systems of $N \times N$ matrix variables. After a brief review of the matrix harmonic oscillator and introduction of notation in Sec. II, we began in Sec. III by investigating the S_N invariant Hilbert space $\mathcal{H}_{\rm inv}$ of generic matrix quantum mechanics systems at large N. We found that there is a one-to-one correspondence between S_N invariant states of degree k and elements in the symmetrized partition algebra $SP_k(N)$. Two bases of $SP_k(N)$ were discussed: the diagram basis and the representation basis. A construction of the latter was explained in Sec. IV in terms of diagonalizing commuting algebraic charges. Having discussed the S_N invariant state space, we moved on to interesting invariant Hamiltonians. The general permutation invariant harmonic matrix oscillator was described and solved (diagonalized) in Sec. V. This was achieved with the introduction of oscillators labeled by representation theoretic quantities, as in (5.27). In Sec. VI we described a set of algebraic Hamiltonians for matrix quantum mechanics that preserve the S_N invariant subspace of the Hilbert space. These Hamiltonians, given by Eqs. (6.38), (6.42), and (6.44) realize the three dynamical scenarios illustrated on the left-hand side of Fig. 1(a), the right-hand side of Figs. 1(a) and 1(b), respectively. The representation basis introduced in Sec. III C diagonalizes all of these algebraic Hamiltonians. We provided a lattice interpretation of the matrix oscillators in Sec. VI F. In Sec. VII we constructed Hamiltonians which turn the S_N invariant state space into quantum many-body scars. Following the ideas in [85,86], we gave Hamiltonians (7.2) of the form $H + H_s$ where H is S_N invariant and H_s annihilates states in the S_N invariant subspace. We noted that the Hamiltonians in Sec. VI are suitable candidates for H if their energies satisfy an integrality condition. As an example, we used the lattice interpretation to give a modified Bose-Hubbard Hamiltonian which exhibits S_N invariant quantum many-body scars. The diagram basis is the most efficient basis for describing inner and outer products. As a consequence extremal correlators, defined in (8.6), which are analogs of three-point extremal correlators in $\mathcal{N} = 4$ SYM are simple in the diagram basis. The extremal correlators satisfy representation theoretic selection rules, based on Kronecker coefficients, which were derived in the representation basis. The selection rules are based on exact expressions for extremal correlators, involving Kronecker coefficients and Littlewood-Richardson coefficients, given in Eq. (8.25).

The representation theoretic basis $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ for the S_N invariant Hilbert space $\mathcal{H}_{\rm inv}$ constructed as linear combinations of symmetrized partition algebra elements in $SP_k(N)$ in Sec. III C is an eigenstate basis for the free Hamiltonian H_0 of Sec. II as well as the algebraic Hamiltonians constructed in Sec. VI. The action of the general permutation invariant harmonic oscillator Hamiltonian given in (5.27) of Sec. V however causes a nontrivial mixing of the representation labels. This mixing was discussed briefly in Sec. V D. Diagonalizing the general harmonic oscillator Hamiltonians in $\mathcal{H}_{\rm inv}$ is an interesting, unsolved problem of finding appropriate linear combinations of the $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$ which are invariant, up to scaling, under the mixing.

With the exception of the $P_k(N)$ orbit basis discussion in Appendix B we have assumed $N \geq 2k$, known as the stable limit. This simplified the construction of a basis for the S_N invariant subspace $\mathcal{H}_{\rm inv}$, a simplification related to the existence of a kernel free map from $P_k(N)$ to $\mathrm{End}(V_N^{\otimes k})$. However, it would be interesting to uncover any finite N effects appearing in these permutation invariant quantum mechanical matrix systems. At finite N the diagrams in $P_k(N)$ provide an over complete basis of operators. That is, there are some linear relations between operators. The precise form of these relations can be found using the orbit basis. The question remains of how to use this knowledge in order to construct a representation theoretic basis for $2k < N$. We leave this for future work, but note here that it would involve a detailed study of the Artin-Wedderburn decomposition in (3.48) below the stable limit. The detailed study includes putting constraints on the irreducible representations appearing in the decomposition below the stable limit, as well as computing the dimension of the multiplicity spaces.

In Sec. VI F we gave one interpretation of our model in terms of bosonic excitations $(a^\dagger)^j_i$ on an N-by-N lattice with sites labeled by (i, j). It is natural to ask if the S_N invariant Hamiltonians described by (6.44) interpreted in this way can be realized in experiments. In the real world interactions tend to be local. The demand for these Hamiltonians to be local places restrictions on the sets of permissible terms. In Sec. VII C we used this interpretation to construct a modified Bose-Hubbard Hamiltonian exhibiting S_N invariant quantum many-body scars. More generally, combining the lattice interpretation of matrix oscillators with the group theoretic scheme given in [85,86], as was done in Secs. VII A and VII B, provides a useful framework for describing systems with many-body scars in $2 + 1$ dimensions.

A very interesting avenue towards applications of the Hilbert spaces and Hamiltonians considered here is to find systems where the permutation invariant sectors described using partition algebras are naturally selected by the physics. For example, in a Bose-Einstein condensate composed of N identical bosons, excited by vibrational modes between pairs of particles, oscillators $(a^\dagger)^j_i$ exciting the pair (i, j) of particles with $i, j \in \{1, \ldots, N\}$ would naturally be subject to the kind of S_N invariance we have considered here. This would provide links between the theoretical application of partition algebras as considered here with the phenomenological modeling of Bose-Einstein physics, e.g. along the lines of [120].

As a closing remark, we note that much of the initial study of the representation theory of partition algebras $P_k(N)$ was done with physical motivations coming from classical statistical models (Potts models) where k is the number of lattice sites and N is the number of classical states for each lattice site. The transfer matrix of the classical statistical model plays a crucial role in these studies [72–74]. The present application of partition algebras looks substantially different: we have quantum mechanical matrix oscillators, with matrix size N possible values and k specifies the sector of quantum states with k oscillators acting on the vacuum. Exploring potential dualities relating systems of the kind studied earlier and the matrix quantum systems defined here is a fascinating future direction.

ACKNOWLEDGMENTS

S. R. is supported by the STFC consolidated Grant No. ST/T000686/1 “Amplitudes, Strings and Duality” and a Visiting Professorship at the University of the Witwatersrand. We are pleased to acknowledge useful conversations on the subject of this paper with Matthew Buican, Robert de Mello Koch, Adam Denchfield, Masanori Hanada, Costis Papageorgakis, Rajath Radhakrishnan.

APPENDIX A: MATRIX UNITS AND FOURIER INVERSION FROM INNER PRODUCTS

In this appendix we prove the results in Sec. III C on representation bases. We review the construction of matrix units for semisimple algebras closely following [84]. We focus in particular on the partition algebras $P_k(N)$ and the symmetrized partition algebras $SP_k(N)$. The appendix is divided into four subsections. We start by discussing nondegenerate bilinear forms on algebras and how they define dual elements through (A19). The existence of dual elements allows us to prove orthogonality of matrix elements of irreducible representations of $P_k(N)$, as stated in (A42). Orthogonality is essential for the construction of matrix units of $P_k(N)$ using the Fourier inversion formula (A43). Matrix units for $SP_k(N)$ are constructed using branching coefficients, as in (A54). Minor modifications to the construction in [84], which defines a nondegenerate bilinear using the trace in the regular representation of $P_k(N)$, are necessary. In this paper, the physical trace relevant to the inner product (3.31) and two point function, is a trace in $V_N^{\otimes k}$. This induces minor changes to the basic formulas. The two traces are related in (A13), through a so-called Ω factor.

1. Schur-Weyl duality and nondegenerate bilinear forms

The construction of matrix units for $P_k(N)$ relies on the existence of a nondegenerate bilinear form on $P_k(N)$. The bilinear form used in [84] is defined using the trace in the regular representation of $P_k(N)$. In this paper the physical trace, associated with inner products, is a trace in $V_N^{\otimes k}$ including a transposition as in Eq. (3.31). The aim of this subsection is to prove that this trace defines a nondegenerate bilinear form as well. The outline of the proof is as follows. The trace in the regular representation is related to the trace on $V_N^{\otimes k}$ by the insertion of a central element. Given this relation, nondegeneracy of the bilinear form defined by the trace on $V_N^{\otimes k}$ follows by the nondegeneracy of the bilinear form defined by the trace in the regular representation.

Let $\mathcal{B} = \{b_1, \ldots, b_{B(2k)}\}$ be a basis for $P_k(N)$. The regular representation of $P_k(N)$ is defined by the left action of $P_k(N)$ on itself. The matrix representation of b_i is defined by the structure constants C_{ij}^k

$$b_i b_j = \sum_{k=1}^{B(2k)} C_{ij}^k b_k. \tag{A1}$$

Consequently, the trace in the regular representation can be written as

$$\mathrm{tr}(b_i) = \sum_{j=1}^{B(2k)} C_{ij}^j = \sum_{j=1}^{B(2k)} \mathrm{Coeff}(b_j, b_i b_j), \tag{A2}$$

where $\mathrm{Coeff}(b_j, d)$ is the coefficient of b_j in the expansion of $d \in P_k(N)$ in the basis $\mathcal{B}$.

For $N \geq 2k$, $P_k(N)$ is semisimple (see Theorem 3.27 in [75]) and therefore,

$$G_{ij} \equiv \mathrm{tr}(b_i b_j) \tag{A3}$$

is an invertible matrix. We say that the trace in the regular representation defines a nondegenerate bilinear form on $P_k(N)$ [see Eq. (5.9) in [75]]. It will be useful to use the following equivalent definition of nondegeneracy in what follows. A bilinear form on $P_k(N)$ is nondegenerate if there exists no nonzero element $d \in P_k(N)$ such that

$$\mathrm{tr}(b_i d) = 0 \quad \forall \ i = 1, \ldots, B(2k). \tag{A4}$$

The regular representation of $P_k(N)$ has a decomposition (see statements in proof of Proposition 5.7 in [75])

$$V^{\mathrm{reg}} = \bigoplus_{\Lambda_1} V_{\Lambda_1}^{P_k(N)} \otimes V_{\Lambda_1}^{P_k(N)}. \tag{A5}$$

It follows that the trace of $d \in P_k(N)$ in the regular representation can be decomposed as

$$\mathrm{tr}(d) = \sum_{\Lambda_1} \mathrm{tr}_{\Lambda_1}(d)\mathrm{tr}_{\Lambda_1}(1) = \sum_{\Lambda_1} \mathrm{Dim}V_{\Lambda_1}^{P_k(N)} \chi^{\Lambda_1}(d), \quad \text{(A6)}$$

where the sum is over all irreducible representations of $P_k(N)$, $\mathrm{Dim}V_{\Lambda_1}^{P_k(N)}$ is the dimension of the representation Λ_1 and χ^{Λ_1} is the corresponding character.

The characters can be extracted from the trace by means of projection operators $p_{\Lambda_1} \in P_k(N)$,

$$\mathrm{tr}(p_{\Lambda_1} d) = \mathrm{Dim}V_{\Lambda_1}^{P_k(N)} \chi^{\Lambda_1}(d). \quad \text{(A7)}$$

This can be seen as a consequence of character orthogonality (see Theorems 3.8 and 3.9 in [84])

$$\sum_{i,j=1}^{B(2k)} \mathrm{Dim}V_{\Lambda_1}^{P_k(N)} \chi^{\Lambda_1}(b_i)(G^{-1})_{ij}\chi^{\Lambda_1'}(b_j) = \delta^{\Lambda_1\Lambda_1'}, \quad \text{(A8)}$$

and the fact that projectors can be written as

$$p_{\Lambda_1} = \sum_{i,j=1}^{B(2k)} \mathrm{Dim}V_{\Lambda_1}^{P_k(N)} \chi^{\Lambda_1}(b_i)(G^{-1})_{ij} b_j, \quad \text{(A9)}$$

where $(G^{-1})_{ij}$ is the inverse of the matrix G_{ij} in (A3). Alternatively, it follows from the decomposition (A5).

We now move on to the trace in $V_N^{\otimes k}$. As we have reviewed in Sec. III A, $P_k(N) \cong \mathrm{End}_{S_N}(V_N^{\otimes k})$ when $N \geq 2k$, where $\mathrm{End}(V_N^{\otimes k})$ is the vector space of linear maps $V_N^{\otimes k} \to V_N^{\otimes k}$ and $\mathrm{End}_{S_N}(V_N^{\otimes k})$ is the subspace of maps that commute with the action of S_N. Note that we use the same symbol for elements $d \in P_k(N)$ and the corresponding element in $d \in \mathrm{End}_{S_N}(V_N^{\otimes k})$ in what follows. It will be clear from context if d is acting on $V_N^{\otimes k}$.

By Schur-Weyl duality (3.34), the trace in $V_N^{\otimes k}$ decomposes as

$$\mathrm{Tr}_{V_N^{\otimes k}}(d) = \sum_{\Lambda_1} \mathrm{Dim}V_{\Lambda_1}^{S_N} \chi^{\Lambda_1}(d), \quad \text{(A10)}$$

where the sum is over the irreducible representations that appear in Eq. (3.34). Consequently, we can relate the two traces by substituting (A7) into each summand of (A10)

$$\mathrm{Tr}_{V_N^{\otimes k}}(d) = \sum_{\Lambda_1} \mathrm{Dim}V_{\Lambda_1}^{S_N} \chi^{\Lambda_1}(d) = \sum_{\Lambda_1} \frac{\mathrm{Dim}V_{\Lambda_1}^{S_N}}{\mathrm{Dim}V_{\Lambda_1}^{P_k(N)}} \mathrm{tr}(p_{\Lambda_1} d). \quad \text{(A11)}$$

It is convenient to define

$$\Omega = \sum_{\Lambda_1} \frac{\mathrm{Dim}V_{\Lambda_1}^{S_N}}{\mathrm{Dim}V_{\Lambda_1}^{P_k(N)}} p_{\Lambda_1}, \quad \text{(A12)}$$

such that Eq. (A11) becomes

$$\boxed{\mathrm{Tr}_{V_N^{\otimes k}}(d) = \mathrm{tr}(\Omega d).} \quad \text{(A13)}$$

We can now prove that the bilinear form $(-,-)$: $P_k(N) \times P_k(N) \to \mathbb{C}$ given by

$$(b_i, b_j) = \mathrm{Tr}_{V_N^{\otimes k}}(b_i b_j) \quad \text{(A14)}$$

is nondegenerate. We give a proof by contradiction. Suppose there exists a nonzero $d \in P_k(N)$ such that

$$(b_i, d) = 0, \quad \forall \ i = 1, \ldots, B(2k). \quad \text{(A15)}$$

From above, it follows that d is such that

$$(b_i, d) = \mathrm{Tr}_{V_N^{\otimes k}}(b_i d) = \mathrm{tr}(\Omega b_i d) = 0, \quad \forall \ i = 1, \ldots, B(2k). \quad \text{(A16)}$$

However, this implies that the element $d' = d\Omega \in P_k(N)$ is such that

$$\mathrm{tr}(b_i d') = 0, \quad \forall \ i = 1, \ldots, B(2k), \quad \text{(A17)}$$

which contradicts the fact that the trace in the regular representation of $P_k(N)$ defines a nondegenerate bilinear form.

It immediately follows (use proof by contradiction again) that the bilinear form given by

$$\langle b_i, b_j \rangle = \mathrm{Tr}_{V_N^{\otimes k}}(b_i b_j^T) \equiv g_{ij}, \quad \text{(A18)}$$

is nondegenerate and g_{ij} is invertible. The inverse matrix is used to define elements dual to b_i which we denote b_i^*

$$\boxed{b_i^* = \sum_{j=1}^{B(2k)} (g^{-1})_{ij} b_j.} \quad \text{(A19)}$$

Dual elements satisfy

$$\langle b_i^*, b_j \rangle = \delta_{ij}. \quad \text{(A20)}$$

The dual elements are essential for proving orthogonality of matrix elements. The proof also uses the following property of the bilinear form

$$\langle b_i, b_j b_k \rangle = \langle b_i b_k^T, b_j \rangle = \langle b_j^T b_i, b_k \rangle. \quad \text{(A21)}$$

The first step uses $(b_j b_k)^T = b_k^T b_j^T$ and the second step uses cyclicity of the trace.

2. Orthogonality of matrix elements

The matrix elements $D^{\Lambda_1}_{\alpha\beta}(b_i)$ of irreducible representations of $P_k(N)$ are orthogonal. This is a generalization of the corresponding orthogonality theorem for group algebras (see Sec. III. 15 in [53]). As we will now prove, the definition of dual elements given in the previous subsection is such that

$$\sum_{i=1}^{B(2k)} D^{\Lambda_1}_{\alpha\beta}(b_i) D^{\Lambda_1'}_{\rho\sigma}((b_i^*)^T) \propto \delta_{\alpha\sigma}\delta_{\beta\rho}\delta^{\Lambda_1\Lambda_1'}. \quad (A22)$$

As we now prove, we can always choose irreducible representations satisfying

$$D^{\Lambda_1}_{\alpha\beta}(d^T) = D^{\Lambda_1}_{\beta\alpha}(d), \qquad \text{for } d \in P_k(N), \quad (A23)$$

where d^T is as in (3.30). Starting from the Clebsch-Gordan coefficients $C^{\Lambda_1,\alpha}_{a,i_1\ldots i_k}$ for the decomposition of $V_N^{\otimes k}$ and using Schur-Weyl duality, we identify the multiplicity index α with an orthogonal basis for $V^{P_k(N)}_{\Lambda_1}$. Specifically, define

$$D^{\Lambda_1}_{\alpha\beta}(d) = \sum_a C^{\Lambda_1,\alpha}_{a,i_{1'}\ldots i_{k'}} C^{\Lambda_1,\beta}_{a,i_1\ldots i_k} (d)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}. \quad (A24)$$

Here we are using the fact that Clebsch-Gordan coefficients for S_N can be chosen real (see Section 7.14 of [53]). It follows that

$$D^{\Lambda_1}_{\alpha\beta}(d^T) = \sum_a C^{\Lambda_1,\alpha}_{a,i_{1'}\ldots i_{k'}} C^{\Lambda_1,\beta}_{a,i_1\ldots i_k} (d^T)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}$$
$$= \sum_a C^{\Lambda_1,\alpha}_{a,i_{1'}\ldots i_{k'}} C^{\Lambda_1,\beta}_{a,i_1\ldots i_k} (d)^{i_1\ldots i_k}_{i_{1'}\ldots i_{k'}} = D^{\Lambda_1}_{\beta\alpha}(d). \quad (A25)$$

Because the above bilinear form (A18) includes a transpose, the symmetrization theorem (Proposition 2.6 in [84]) is modified accordingly. Let C be a $\mathrm{Dim}V^{P_k(N)}_{\Lambda_1}$ by $\mathrm{Dim}V^{P_k(N)}_{\Lambda_1'}$ matrix, and $D^{\Lambda_1}(d), D^{\Lambda_1'}(d)$ be two irreducible matrix representations of $P_k(N)$ with dimension $\mathrm{Dim}V^{P_k(N)}_{\Lambda_1}, \mathrm{Dim}V^{P_k(N)}_{\Lambda_1'}$, respectively. We have the following version of the symmetrization theorem. The matrix

$$[C] = \sum_{i=1}^{B(2k)} D^{\Lambda_1}(b_i) C D^{\Lambda_1'}((b_i^*)^T) \quad (A26)$$

satisfies

$$D^{\Lambda_1}(d)[C] = [C]D^{\Lambda_1'}(d), \quad (A27)$$

for all $d \in P_k(N)$. The proof is essentially identical to the original case,

$$D^{\Lambda_1}(d)[C] = \sum_i D^{\Lambda_1}(db_i) C D^{\Lambda_1'}((b_i^*)^T)$$
$$= \sum_i D^{\Lambda_1}\left(\sum_j \langle b_j^*, db_i\rangle b_j\right) C D^{\Lambda_1'}((b_i^*)^T),$$
$$= \sum_j D^{\Lambda_1}(b_j) C D^{\Lambda_1'}(\sum_i (b_i^*)^T \langle b_j^*, db_i\rangle),$$
$$= \sum_j D^{\Lambda_1}(b_j) C D^{\Lambda_1'}(\sum_i (b_i^*)^T \langle d^T b_j^*, b_i\rangle),$$
$$= \sum_j D^{\Lambda_1}(b_j) C D^{\Lambda_1'}((d^T b_j^*)^T),$$
$$= [C]D^{\Lambda_1'}(d), \quad (A28)$$

where in the third line we used the modified Frobenius associativity in Eq. (A21).

By Schur's lemma, $[C]$ is proportional to the identity matrix if and only if $\Lambda_1 = \Lambda_1'$ and zero otherwise. For some constant c^{Λ_1},

$$[C]_{\alpha\sigma} = \delta^{\Lambda_1\Lambda_1'} c^{\Lambda_1} \delta_{\alpha\sigma}. \quad (A29)$$

The lhs of Eq. (A22) is equal to

$$\sum_i (D^{\Lambda_1}(b_i) E_{\beta\rho} D^{\Lambda_1'}((b_i^*)^T))_{\alpha\sigma} = [E_{\beta\rho}]_{\alpha\sigma}, \quad (A30)$$

where $E_{\beta\rho}$ is the elementary matrix with 0 everywhere except in row β, column ρ which has a 1. It follows from the symmetrization theorem (A27) that

$$\sum_i D^{\Lambda_1}_{\alpha\beta}(b_i) D^{\Lambda_1'}_{\rho\sigma}((b_i^*)^T) = [E_{\beta\rho}]_{\alpha\sigma} = C^{\Lambda_1}_{\beta\rho}\delta_{\alpha\sigma}\delta^{\Lambda_1\Lambda_1'}, \quad (A31)$$

where $C^{\Lambda_1}_{\beta\rho}$ is a constant that *a priori* depends on the choice of elementary matrix. We now show that it only depends on Λ_1.

Using the property

$$D^{\Lambda_1}_{\alpha\beta}(d^T) = D^{\Lambda_1}_{\beta\alpha}(d), \quad (A32)$$

we derive

$$C^{\Lambda_1}_{\beta\rho}\delta_{\alpha\sigma}\delta^{\Lambda_1\Lambda_1'} = [E_{\beta\rho}]_{\alpha\sigma} = \sum_i D^{\Lambda_1}_{\alpha\beta}(b_i) D^{\Lambda_1'}_{\rho\sigma}((b_i^*)^T),$$
$$= \sum_i D^{\Lambda_1}_{\beta\alpha}(b_i^T) D^{\Lambda_1'}_{\sigma\rho}(b_i^*)$$
$$= \sum_i D^{\Lambda_1}_{\beta\alpha}(b_i) D^{\Lambda_1'}_{\sigma\rho}((b_i^*)^T),$$
$$= [E_{\alpha\sigma}]_{\beta\rho} = C^{\Lambda_1}_{\alpha\sigma}\delta_{\beta\rho}\delta^{\Lambda_1\Lambda_1'}. \quad (A33)$$

Going from the first line to the second uses (A32). The rhs of the second line follows by summing over b_i^T instead of b_i (transposition maps $\mathcal{B}$ to $\mathcal{B}$ bijectively, this is particularly clear in the diagram basis). Comparing the lhs of the first line to the rhs of the last line gives

$$C^{\Lambda_1}_{\beta\rho} = C^{\Lambda_1}\delta_{\beta\rho}, \tag{A34}$$

which proves Eq. (A22).

The normalization constant C^{Λ_1} is important for constructing matrix units. We will prove that

$$C^{\Lambda_1} = \frac{1}{\mathrm{Dim}V^{S_N}_{\Lambda_1}}. \tag{A35}$$

The normalization constant is determined by contracting all the indices in Eq. (A22). Set $\Lambda_1 = \Lambda_1'$, then

$$(\mathrm{Dim}V^{P_k(N)}_{\Lambda_1})^2 C^{\Lambda_1} = \sum_\alpha D^{\Lambda_1}_{\alpha\alpha}\Big(\sum_i b_i(b_i^*)^T\Big), \tag{A36}$$

and all that remains is to compute the element $\sum_i b_i(b_i^*)^T$.

As we will now see,

$$\sum_{i=1}^{B(2k)} b_i(b_i^*)^T = \Omega^{-1} = \sum_{\Lambda_1}\frac{\mathrm{Dim}V^{P_k(N)}_{\Lambda_1}}{\mathrm{Dim}V^{S_N}_{\Lambda_1}}p_{\Lambda_1}, \tag{A37}$$

where Ω^{-1} is the inverse of the element defined in Eq. (A12). Using the relationship (A13) between the two traces we have that

$$\begin{aligned}\mathrm{tr}\Big(d\sum_i b_i(b_i^*)^T\Big) &= \mathrm{Tr}_{V_N^{\otimes k}}\Big(\Omega^{-1}d\sum_i b_i(b_i^*)^T\Big),\\ &= \sum_i\langle\Omega^{-1}db_i, b_i^*\rangle,\\ &= \sum_i \mathrm{Coeff}(b_i,\Omega^{-1}db_i),\\ &= \mathrm{tr}(\Omega^{-1}d),\end{aligned} \tag{A38}$$

holds for all $d\in P_k(N)$, from which it follows that

$$\mathrm{tr}\Big(d\Big(\sum_i b_i(b_i^*)^T - \Omega^{-1}\Big)\Big) = 0 \tag{A39}$$

holds for all $d\in P_k(N)$. Since the trace in the regular representation is nondegenerate we must have

$$\sum_{i=1}^{B(2k)} b_i(b_i^*)^T - \Omega^{-1} = 0. \tag{A40}$$

Inserting this expression into Eq. (A36) gives

$$\begin{aligned}(\mathrm{Dim}V^{P_k(N)}_{\Lambda_1})^2 C^{\Lambda_1} &= \sum_\alpha D^{\Lambda_1}_{\alpha\alpha}(\Omega^{-1})\\ &= \sum_\alpha\sum_{\Lambda_1'\vdash N}\frac{\mathrm{Dim}V^{P_k(N)}_{\Lambda_1'}}{\mathrm{Dim}V^{S_N}_{\Lambda_1'}}D^{\Lambda_1}_{\alpha\alpha}(p_{\Lambda_1'}),\\ &= \sum_{\Lambda_1'\vdash N}\frac{\mathrm{Dim}V^{P_k(N)}_{\Lambda_1'}}{\mathrm{Dim}V^{S_N}_{\Lambda_1'}}\delta^{\Lambda_1\Lambda_1'}\mathrm{Dim}V^{P_k(N)}_{\Lambda_1},\\ &= \frac{\mathrm{Dim}V^{P_k(N)}_{\Lambda_1'}}{\mathrm{Dim}V^{S_N}_{\Lambda_1'}}\mathrm{Dim}V^{P_k(N)}_{\Lambda_1'},\end{aligned} \tag{A41}$$

which gives Eq. (A35). To summarize, we have proven that

$$\boxed{\sum_{i=1}^{B(2k)} D^{\Lambda_1}_{\alpha\beta}(b_i)D^{\Lambda_1'}_{\rho\sigma}((b_i^*)^T) = \frac{1}{\mathrm{Dim}V^{S_N}_{\Lambda_1}}\delta_{\beta\rho}\delta_{\alpha\sigma}\delta^{\Lambda_1\Lambda_1'}.} \tag{A42}$$

3. Matrix units for $P_k(N)$

In this subsection we want to use orthogonality of matrix elements to show that

$$\boxed{Q^{\Lambda_1}_{\alpha\beta} = \sum_i \mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)b_i} \tag{A43}$$

multiply like a generalized matrix algebra. That is,

$$Q^{\Lambda_1}_{\alpha\beta}Q^{\Lambda_1'}_{\rho\sigma} = \delta^{\Lambda_1\Lambda_1'}\delta_{\beta\rho}Q^{\Lambda_1}_{\alpha\sigma}. \tag{A44}$$

This is straightforward given the results in the previous subsections. The proof goes as follows. By definition we have

$$\begin{aligned}Q^{\Lambda_1}_{\alpha\beta}Q^{\Lambda_1'}_{\rho\sigma} &= \sum_{i,j}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}((b_j^*)^T)b_ib_j,\\ &= \sum_{i,j,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}((b_j^*)^T)\langle b_ib_j, b_k^*\rangle b_k,\end{aligned} \tag{A45}$$

where the second equality expands the product b_ib_j in the basis b_k using Eq. (A20). Using the modified Frobenius associativity (A21) we have

$$\begin{aligned}
&\sum_{i,j,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}((b_j^*)^T)\langle b_ib_j, b_k^*\rangle b_k\\
&\quad=\sum_{i,j,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}((b_j^*)^T)\langle b_j, b_i^Tb_k^*\rangle b_k,\\
&\quad=\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}\Big(\sum_j(b_j^*)^T\langle b_j, b_i^Tb_k^*\rangle\Big)b_k, \qquad \text{(A46)}
\end{aligned}$$

where in the last line we have pulled the coefficient $\langle b_j, b_i^Tb_k^*\rangle$ inside the matrix representation (using linearity). This prepares us for the next step, where we use the fact that

$$\sum_j(b_j^*)^T\langle b_j, b_i^Tb_k^*\rangle=(b_i^Tb_k^*)^T, \qquad \text{(A47)}$$

which follows from (A20),

$$\begin{aligned}
&\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}\Big(\sum_j(b_j^*)^T\langle b_j, b_i^Tb_k^*\rangle\Big)b_k\\
&\quad=\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\rho}((b_i^Tb_k^*)^T)b_k,\\
&\quad=\sum_{i,k}\sum_{\gamma}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\gamma}((b_k^*)^T)D^{\Lambda_1'}_{\gamma\rho}(b_i)b_k. \qquad \text{(A48)}
\end{aligned}$$

In the last line we can use orthogonality of matrix elements (A42) to find

$$\begin{aligned}
\sum_{i,k}\sum_{\gamma}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\mathrm{Dim}(V^{S_N}_{\Lambda_1'})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)D^{\Lambda_1'}_{\sigma\gamma}((b_k^*)^T)D^{\Lambda_1'}_{\gamma\rho}(b_i)b_k&=\sum_{k}\sum_{\gamma}\mathrm{Dim}(V^{S_N}_{\Lambda_1})\delta^{\Lambda_1\Lambda_1'}\delta_{\rho\beta}\delta_{\gamma\alpha}D^{\Lambda_1'}_{\sigma\gamma}((b_k^*)^T)b_k,\\
&=\delta^{\Lambda_1\Lambda_1'}\delta_{\beta\rho}Q^{\Lambda_1}_{\alpha\sigma}, \qquad \text{(A49)}
\end{aligned}$$

which concludes the proof.

Equipped with a matrix unit basis of $P_k(N)$ we use this to show

$$\boxed{dQ^{\Lambda_1}_{\alpha\beta}=D^{\Lambda_1}_{\alpha\sigma}(d^T)Q^{\Lambda_1}_{\sigma\beta}.} \qquad \text{(A50)}$$

Expanding $Q^{\Lambda_1}_{\alpha\beta}$ on the rhs of this expression as per (A43) we find

$$\begin{aligned}
dQ^{\Lambda_1}_{\alpha\beta}&=\sum_i\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)db_i,\\
&=\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}((b_i^*)^T)\langle db_i, b_k\rangle b_k^*,\\
&=\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}(\langle db_i, b_k\rangle(b_i^*)^T)b_k^*,\\
&=\sum_{i,k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}(\langle b_k^Td, b_i^T\rangle(b_i^*)^T)b_k^*,\\
&=\sum_{k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\alpha}(b_k^Td)b_k^*,\\
&=\sum_{k}\mathrm{Dim}(V^{S_N}_{\Lambda_1})D^{\Lambda_1}_{\beta\sigma}(b_k^T)D^{\Lambda_1}_{\sigma\alpha}(d)b_k^*,\\
&=D^{\Lambda_1}_{\alpha\sigma}(d^T)Q^{\Lambda_1}_{\sigma\beta}. \qquad \text{(A51)}
\end{aligned}$$

In the third line we have used (A21), and in the fourth line we have used the following property of the bilinear form

$$\langle db_i, b_k\rangle=\mathrm{Tr}_{V_N^{\otimes k}}(db_ib_k^T)=\mathrm{Tr}_{V_N^{\otimes k}}(b_i^T(b_k^Td)^T)=\langle b_k^Td, b_i^T\rangle. \qquad \text{(A52)}$$

For the sake of brevity we omit the analogous proof of the action of d on the rhs,

$$\boxed{Q^{\Lambda_1}_{\alpha\beta}d=Q^{\Lambda_1}_{\alpha\sigma}D^{\Lambda_1}_{\sigma\beta}(d^T).} \qquad \text{(A53)}$$

4. Matrix units for $SP_k(N)$ and normalization constants

The matrix units for $SP_k(N)$ are constructed from $Q^{\Lambda_1}_{\alpha\beta}$ using branching coefficients.

$$\boxed{Q^{\Lambda_1}_{\Lambda_2,\mu\nu}=\sum_{\alpha,\beta,p}Q^{\Lambda_1}_{\alpha\beta}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}.} \qquad \text{(A54)}$$

Branching coefficients are understood as follows. The partition algebra $P_k(N)$ has a subalgebra (isomorphic to)

$\mathbb{C}[S_k]$ [for example, see Eq. (3.20)]. For any given irreducible representation $V_{\Lambda_1}^{P_k(N)}$ there exists a basis where the action of $\mathbb{C}[S_k]\subset P_k(N)$ is manifest and irreducible. That is, we consider the decomposition

$$V_{\Lambda_1}^{P_k(N)} \cong \bigoplus_{\Lambda_2\vdash k} V_{\Lambda_2}^{\mathbb{C}[S_k]} \otimes V_{\Lambda_1\Lambda_2}^{P_k(N)\to\mathbb{C}[S_k]}. \tag{A55}$$

On the lhs we have a basis

$$E_\alpha^{\Lambda_1}, \qquad \alpha\in\{1,\ldots\mathrm{Dim}(V_{\Lambda_1}^{P_k(N)})\}, \tag{A56}$$

where the representation of $d\in P_k(N)$ is irreducible,

$$d(E_\alpha^{\Lambda_1}) = \sum_\beta D_{\beta\alpha}^{\Lambda_1}(d)E_\beta^{\Lambda_1}. \tag{A57}$$

The rhs has a basis

$$\begin{aligned} E_{\Lambda_2,p}^{\Lambda_1,\mu}, \qquad & p\in\{1,\ldots,\mathrm{Dim}(V_{\Lambda_1}^{\mathbb{C}[S_k]})\},\\ & \mu\in\{1,\ldots,\mathrm{Dim}(V_{\Lambda_1\Lambda_2}^{P_k(N)\to\mathbb{C}[S_k]})\}, \end{aligned} \tag{A58}$$

where μ is a multiplicity label for $V_{\Lambda_2}^{\mathbb{C}[S_k]}$ in the decomposition. We demand that the representation of $\tau\in\mathbb{C}[S_k]$ is irreducible in this basis,

$$\tau(E_{\Lambda_2,p}^{\Lambda_1,\mu}) = \sum_q D_{qp}^{\Lambda_2}(\tau)E_{\Lambda_2,q}^{\Lambda_1,\mu}, \tag{A59}$$

where $D_{qp}^{\Lambda_2}(\tau)$ is an irreducible representation of $\tau\in\mathbb{C}[S_k]$. The change of basis coefficients are called branching coefficients

$$E_{\Lambda_2,p}^{\Lambda_1,\mu} = \sum_\alpha B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}E_\alpha^{\Lambda_1}. \tag{A60}$$

The matrix unit property

$$Q_{\Lambda_2,\mu\nu}^{\Lambda_1}Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'} = \delta^{\Lambda_1\Lambda_1'}\delta^{\Lambda_2\Lambda_2'}\delta_{\nu\mu'}Q_{\Lambda_2,\mu\nu'}^{\Lambda_1}, \tag{A61}$$

of the $SP_k(N)$ basis follows from that of the $P_k(N)$ units together with orthogonality of $E_{\Lambda_2,p}^{\Lambda_1,\mu}$,

$$\begin{aligned} Q_{\Lambda_2,\mu\nu}^{\Lambda_1}Q_{\Lambda_2',\mu'\nu'}^{\Lambda_1'} &= \sum_{\substack{\alpha,\beta,p\\ \alpha',\beta',p'}} B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1,\beta\to\Lambda_2,p;\nu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\alpha'\to\Lambda_2',p';\mu'}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\beta'\to\Lambda_2',p';\nu'}^{P_k(N)\to\mathbb{C}[S_k]}Q_{\alpha\beta}^{\Lambda_1}Q_{\alpha'\beta'}^{\Lambda_1'},\\ &= \sum_{\substack{\alpha,\beta,p\\ \alpha',\beta',p'}} B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1,\beta\to\Lambda_2,p;\nu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\alpha'\to\Lambda_2',p';\mu'}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\beta'\to\Lambda_2',p';\nu'}^{P_k(N)\to\mathbb{C}[S_k]}\delta^{\Lambda_1\Lambda_1'}\delta_{\beta\alpha'}Q_{\alpha\beta'}^{\Lambda_1},\\ &= \sum_{\substack{\alpha,p\\ \beta',p'}} B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\beta'\to\Lambda_2',p';\nu'}^{P_k(N)\to\mathbb{C}[S_k]}\delta^{\Lambda_1\Lambda_1'}\delta^{\Lambda_2\Lambda_2'}\delta_{pp'}\delta_{\nu\mu'}Q_{\alpha\beta'}^{\Lambda_1},\\ &= \sum_{\alpha,\beta',p} B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1',\beta'\to\Lambda_2',p;\nu'}^{P_k(N)\to\mathbb{C}[S_k]}\delta^{\Lambda_1\Lambda_1'}\delta^{\Lambda_2\Lambda_2'}\delta_{\nu\mu'}Q_{\alpha\beta'}^{\Lambda_1},\\ &= \delta^{\Lambda_1\Lambda_1'}\delta^{\Lambda_2\Lambda_2'}\delta_{\nu\mu'}Q_{\Lambda_2,\mu\nu'}^{\Lambda_1}. \end{aligned} \tag{A62}$$

Going from the first line to the second we used the matrix unit property of $Q_{\alpha\beta}^{\Lambda_1}$. Going from the second line to the third line uses orthogonality

$$\sum_\alpha B_{\Lambda_1,\alpha\to\Lambda_2,p;\mu}^{P_k(N)\to\mathbb{C}[S_k]}B_{\Lambda_1,\alpha\to\Lambda_2',q;\nu}^{P_k(N)\to\mathbb{C}[S_k]} = \delta^{\Lambda_2\Lambda_2'}\delta_{pq}\delta_{\mu\nu}. \tag{A63}$$

Further, we prove Eq. (3.60) for the normalization of the two-point function. The orthogonality of matrix elements implies

$$\chi^{\Lambda_1'}(Q_{\alpha\beta}^{\Lambda_1}) = \sum_i D_{\beta\alpha}^{\Lambda_1}((b_i^*)^T)\sum_\gamma D_{\gamma\gamma}^{\Lambda_1'}(b_i) = \delta^{\Lambda_1\Lambda_1'}\delta_{\alpha\beta}. \tag{A64}$$

We use this fact together with Schur-Weyl duality to compute $\mathrm{Tr}_{V_N^{\otimes k}}(Q_{\alpha\beta}^{\Lambda_1})$

$$\mathrm{Tr}_{V_N^{\otimes k}}(Q_{\alpha\beta}^{\Lambda_1}) = \sum_{\Lambda_1'\vdash N}\mathrm{Dim}V_{\Lambda_1'}^{S_N}\chi^{\Lambda_1'}(Q_{\alpha\beta}^{\Lambda_1}) = \sum_{\Lambda_1'\vdash N}\mathrm{Dim}V_{\Lambda_1'}^{S_N}\delta_{\alpha\beta}\delta^{\Lambda_1\Lambda_1'} = \mathrm{Dim}V_{\Lambda_1}^{S_N}\delta_{\alpha\beta}. \tag{A65}$$

Consequently,

$$\boxed{\begin{aligned}\mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}) &= \sum_{\alpha,\beta,p} B^{P_k(N)\to S_k}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to S_k}_{\Lambda_1,\beta\to\Lambda_2,p;\nu} \mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1}_{\alpha\beta}),\\ &= \sum_{\alpha,\beta,p} B^{P_k(N)\to S_k}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to S_k}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}\delta_{\alpha\beta}\mathrm{Dim}V^{S_N}_{\Lambda_1},\\ &= \sum_p \delta_{pp}\delta_{\mu\nu}\mathrm{Dim}V^{S_N}_{\Lambda_1} = \delta_{\mu\nu}\mathrm{Dim}V^{S_N}_{\Lambda_1}\mathrm{Dim}V^{S_k}_{\Lambda_2},\end{aligned}} \tag{A66}$$

where the last two equalities hold if and only if the branching coefficients are nonzero.

Finally, we check that this construction gives S_k invariant elements:

$$\boxed{\tau Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\tau^{-1} = Q^{\Lambda_1}_{\Lambda_2,\mu\nu} \quad \text{for } \tau\in S_k.} \tag{A67}$$

From the definition (A54) and (A50) we have

$$\tau Q^{\Lambda_1}_{\Lambda_2,\mu\nu} = \sum_{\alpha,\beta,\gamma,p} D^{\Lambda_1}_{\gamma\alpha}(\tau) Q^{\Lambda_1}_{\gamma\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}. \tag{A68}$$

We rewrite $D^{\Lambda_1}_{\gamma\alpha}(\tau)$ by inserting the completeness relation

$$\sum_{\Lambda_2',p',\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma'\to\Lambda_2',p';\mu'} = \delta_{\gamma\gamma'} \tag{A69}$$

on both sides. This gives

$$D^{\Lambda_1}_{\gamma\alpha}(\tau) = \sum_{\Lambda_2',p',p'',\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'} D^{\Lambda_2'}_{p'p''}(\tau) B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2',p'';\mu'}. \tag{A70}$$

Inserting this into (A68) gives

$$\begin{aligned}\sum_{\alpha,\beta,\gamma,p} D^{\Lambda_1}_{\gamma\alpha}(\tau) Q^{\Lambda_1}_{\gamma\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu} &= \sum_{\alpha,\beta,\gamma,p}\sum_{\Lambda_2',p',p'',\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'} D^{\Lambda_2'}_{p'p''}(\tau) B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2',p'';\mu'} Q^{\Lambda_1}_{\gamma\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu},\\ &= \sum_{\beta,\gamma,p}\sum_{\Lambda_2',p',p'',\mu'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2',p';\mu'} D^{\Lambda_2'}_{p'p''}(\tau)\delta_{\Lambda_2\Lambda_2'}\delta_{p''p}\delta_{\mu'\mu} Q^{\Lambda_1}_{\gamma\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu},\end{aligned} \tag{A71}$$

where we used completeness in the last line. Eliminating the Kronecker deltas by carrying out the sums gives

$$\tau Q^{\Lambda_1}_{\Lambda_2,\mu\nu} = \sum_{\beta,\gamma,p,p'} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2,p';\mu} D^{\Lambda_2}_{p'p}(\tau) Q^{\Lambda_1}_{\gamma\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}. \tag{A72}$$

To finish the proof we follow the same steps for the right action: using (A53) gives

$$Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\tau = \sum_{\alpha,\beta,\gamma,p} Q^{\Lambda_1}_{\alpha\gamma} D^{\Lambda_1}_{\beta\gamma}(\tau) B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu}. \tag{A73}$$

Inserting (A69) and carrying out the sums yields

$$Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\tau = \sum_{\alpha,\gamma,p,p''} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} D^{\Lambda_2}_{pp''}(\tau) Q^{\Lambda_1}_{\alpha\gamma} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\gamma\to\Lambda_2,p'';\nu} = \tau Q^{\Lambda_1}_{\Lambda_2,\mu\nu}, \tag{A74}$$

which immediately leads to (A67).

APPENDIX B: ORBIT BASIS

In Sec. III we described two bases for the partition algebra $P_k(N)$: a diagram basis and a representation basis. Here we describe another basis, in terms of combinatorially explicit linear combination of the diagrams from Sec. III B. This basis is called the orbit basis [73]. In this appendix we describe this basis and show that it is orthogonal for any N and k. This makes it a suitable basis to describe permutation invariant matrix quantum mechanics in the $N < 2k$ regime, a preliminary discussion of which concludes this subsection. A possible future direction is to use the orbit basis to describe how the representation basis is modified in this regime.

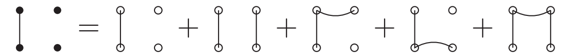
As in the diagram basis, this basis is indexed by the set partitions Π_{2k} of $\{1, \ldots, k, 1', \ldots, k'\}$. These are partially ordered under the relation

$$\pi \preccurlyeq \pi' \quad \text{if every block of } \pi \text{ is contained within a block of } \pi', \tag{B1}$$

in this case we say that π is a refinement of π' or equivalently that π' is a coarsening of π. Since we are already familiar with the diagram basis of $P_k(N)$ we express the orbit basis in terms of the diagram basis using the above partial ordering

$$d_\pi = \sum_{\pi \preccurlyeq \pi'} x_{\pi'}, \tag{B2}$$

with $\{x_\pi | \pi \in \Pi_{2k}\}$. The diagram basis element d_π is a sum of all orbit basis elements labeled by set partitions equal to or coarser than π, for example


(B3)

We will continue to distinguish the diagram and orbit bases by drawing diagram basis elements with black vertices and labeling them with the letter d, and drawing orbit basis elements with white vertices and labeling them with the letter x. The transition matrix determined by (B2) is ζ_{2k} and is called the zeta matrix of the partially ordered set Π_{2k}. It is upper triangular, with ones on the diagonal and hence invertible.

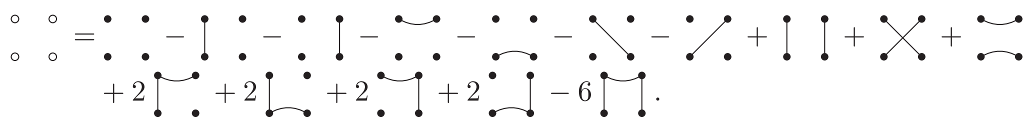
The inverse of ζ_{2k} is given in [78]. It is the matrix μ_{2k}

$$x_\pi = \sum_{\pi \preccurlyeq \pi'} \mu_{2k}(\pi, \pi') d_{\pi'}. \tag{B4}$$

If $\pi \preccurlyeq \pi'$ and π' consists of l blocks such that the ith block of π' is the union of b_i blocks of π, then

$$\mu_{2k}(\pi, \pi') = \prod_{i=1}^{l} (-1)^{b_i - 1} (b_i - 1)!. \tag{B5}$$

For example, this gives the following expansion of the orbit basis element labeled by $\pi = \{1|2|3|4\}$


(B6)

The orbit basis is orthogonal with respect to the inner product (3.31). We will prove,

$$\boxed{\langle x_\pi | x_{\pi'} \rangle = \begin{cases} |G_\pi| N_{(|\pi|)} & \text{if } [x_{\pi'}] = [x_\pi] \\ 0 & \text{otherwise} \end{cases},} \tag{B7}$$

where π, π' are set partitions of $\{1, \ldots, k, 1', \ldots, k'\}$, $N_{(l)} = N(N-1)\ldots(N-l+1)$ is the falling factorial, $|\pi|$ is the number of blocks in π, and $|G_\pi|$ is the order of the subgroup of S_k that leaves x_π invariant. As was the case in the diagram basis, we note that

$$|[x_{\pi'}]\rangle = |x_{\pi'}\rangle \tag{B8}$$

and use the rhs ket labels for the sake of notational efficiency.

First consider the simpler proposition

$$\boxed{\mathrm{Tr}_{V_N^{\otimes k}}(x_\pi x_{\pi'}^T) = N_{(|\pi|)} \delta_{\pi\pi'}.} \tag{B9}$$

The proof of this follows from the definition (see Sec. 5. 2 in [78]) of x_π acting on $V_N^{\otimes k}$

$$(x_\pi)_{i_1 \ldots i_k}^{i_{1'} \ldots i_{k'}} = \begin{cases} 1 & \text{if } i_a = i_b \text{ if and only if a and b are in the same block of } \pi \\ 0 & \text{otherwise} \end{cases}. \tag{B10}$$

The trace is equal to

$$\mathrm{Tr}_{V_N^{\otimes k}}(x_\pi x_{\pi'}^T) = \sum_{\substack{i_1 \ldots i_k \\ i_{1'} \ldots i_{k'}}} (x_\pi)_{i_1 \ldots i_k}^{i_{1'} \ldots i_{k'}} (x_{\pi'})_{i_1 \ldots i_k}^{i_{1'} \ldots i_{k'}}. \tag{B11}$$

Equation (B10) implies

$$(x_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}(x_{\pi'})^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} = \begin{cases} 1 & \text{if } i_a = i_b \text{ if and only if } a \text{ and } b \text{ are in the same block of } \pi \text{ and the same block of } \pi' \\ 0 & \text{otherwise} \end{cases}. \quad \text{(B12)}$$

If $\pi \neq \pi'$ two situations exist. Consider the set of all pairs (a, b) for $a, b = 1, \ldots, k, 1', \ldots, k'$ such that a and b are in the same block of π. Since $\pi \neq \pi'$ at least one of these pairs are such that a and b are in different blocks of π'. The second case is the reverse. Consider the set of all (a, b) such that a and b are in the same block of π'. Then $\pi' \neq \pi$ implies that there exists at least one pair such that a and b are not in the same block of π. In that case, there are no choices of i_a, i_b which satisfy the first criteria in (B12). For example, take a, b to be in the same block of π but different blocks of π'. The matrix elements $(x_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}$ vanish if $i_a \neq i_b$ while the matrix elements $(x_{\pi'})^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}$ vanish unless $i_a \neq i_b$. Therefore, the product identically vanishes,

$$(x_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}(x_{\pi'})^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} = \delta_{\pi\pi'}(x_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} \quad \text{(B13)}$$

and

$$\begin{aligned} \mathrm{Tr}_{V_N^{\otimes k}}(x_\pi x_{\pi'}^T) &= \sum_{\substack{i_1\ldots i_k \\ i_{1'}\ldots i_{k'}}} (x_\pi)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}\delta_{\pi\pi'} \\ &= \delta_{\pi\pi'}N(N-1)\ldots(N-|\pi|+1). \end{aligned} \quad \text{(B14)}$$

The last equality is a consequence of (B10). For example, consider the set partition $12|1'2'$. The trace of $x_{12|1'2'}$ is

$$\mathrm{Tr}_{V_N^{\otimes 2}}(x_{12|1'2'}) = \sum_{i_1=i_2\neq i_3, i_3=i_4} = N(N-1), \quad \text{(B15)}$$

since we have N choices of indices for i_1 and $(N-1)$ choices for i_3 (for every choice of i_1). The general case is analogous,

$$\mathrm{Tr}_{V_N^{\otimes k}}(x_\pi) = N_{(|\pi|)}. \quad \text{(B16)}$$

We have N choices for the indices of the first block of π, $N-1$ choices for the indices of the second block and so on.

The inner product of two orbit basis elements of $SP_k(N)$ is given by (3.31)

$$\langle x_\pi | x_{\pi'}\rangle = \sum_{\gamma\in S_k} \mathrm{Tr}_{V_N^{\otimes k}}(\gamma x_\pi \gamma^{-1} x_{\pi'}^T). \quad \text{(B17)}$$

We rewrite

$$\sum_{\gamma\in S_k} \gamma x_\pi \gamma^{-1} = |G_\pi| \sum_{\lambda\in[\pi]} x_\lambda, \quad \text{(B18)}$$

where the sum on the rhs is over the distinct elements in the S_k orbit of x_π. Substituting this into the trace gives

$$\begin{aligned} \langle x_\pi | x_{\pi'}\rangle &= |G_\pi| \sum_{\lambda\in[\pi]} \mathrm{Tr}_{V_N^{\otimes k}}(x_\lambda x_{\pi'}) \\ &= |G_\pi| \sum_{\lambda\in[\pi]} N_{(|\pi|)}\delta_{\lambda\pi'} \\ &= \begin{cases} |G_\pi| N_{(|\pi|)} & \text{if } [x_{\pi'}] = [x_\pi] \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \quad \text{(B19)}$$

where $[x_\pi]$ denotes S_k symmetrization as in Eq. (3.24).

For the majority of this paper we assume $N \geq 2k$ in order to take advantage of the many simplifications that occur in this limit. However, utilizing results from the partition algebra literature we are able to say something about what happens below this limit, in which we expect to encounter finite N effects.

In the limit $N \geq 2k$ the map from the partition algebra to $\mathrm{End}_{S_N}(V_N^{\otimes k})$ is bijective. When $N < 2k$ this map acquires a nontrivial kernel (but remains surjective). Accordingly, we expect a reduction in the size of the state space $\mathcal{H}_{\mathrm{inv}}$. This reduction is most easily expressed in the orbit basis of $P_k(N)$. Theorem 5.17(a) in [78] states that if $N \in \mathbb{Z}_{\geq 1}$ and $\{x_\pi | \pi \in \Pi_{2k}\}$ is the orbit basis for $P_k(N)$, then for $k \in \mathbb{Z}_{\geq 1}$ the representation $\Phi_{k,N}\colon P_k(N) \to \mathrm{End}(V_N^{\otimes k})$ has the following image and kernel:

$$\begin{aligned} \mathrm{im}(\Phi_{k,N}) &= \mathrm{End}_{S_N}(V_N^{\otimes k}) \\ &= \mathrm{span}_{\mathbb{C}}\{\Phi_{k,N}(x_\pi) | \pi \in \Pi_{2k} \text{ has} \leq N \text{ blocks}\}, \\ \ker(\Phi_{k,N}) &= \mathrm{span}_{\mathbb{C}}\{x_\pi | \pi \in \Pi_{2k} \text{ has} > N \text{ blocks}\}. \end{aligned} \quad \text{(B20)}$$

Due to the bosonic symmetry of our theory we are actually interested in the map from the symmetrized partition algebra $SP_k(N)$, defined in (3.24), to $\mathrm{End}(V_N^{\otimes k})$. To this end we note that the definition of the kernel of $\Phi_{k,N}$ given in (B20) is S_k invariant. If one element of an S_k orbit is in the kernel then (B20) tells us that the entire orbit belongs to the kernel—the action of S_k does not change the number of blocks in a partition π. The image and kernel of this map are the following:

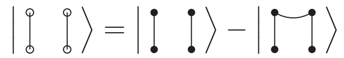
$$\mathrm{im}(\Phi_{k,N}) = \mathrm{span}_{\mathbb{C}}\left\{[b] = \frac{1}{k!}\sum_{\gamma\in S_k}\gamma b\gamma^{-1} | b = \Phi_{k,N}(x_\pi),\ \forall\ \pi\in\Pi_{2k}\ \text{with}\ \leq N\,\text{blocks}\right\},$$
$$\ker(\Phi_{k,N}) = \mathrm{span}_{\mathbb{C}}\{[x_\pi] | \pi\in\Pi_{2k}, \pi\,\text{has} > N\,\text{blocks}\}. \tag{B21}$$

Therefore a state basis is given by $|[x_\pi]\rangle$ for π having N or fewer blocks, this basis is orthogonal for all N, including for $N < 2k$.

The original statement (B20) applies to multimatrix theories in which observables are constructed from distinct matrices—in this case there is no bosonic S_k symmetry to account for. If a state in this theory is null then all states generated by the action of S_k on this state will also be null. The equivalent of (3.25) for the multimatrix case is

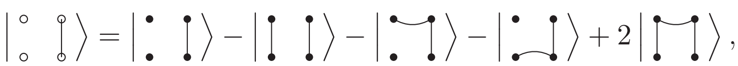
$$|d\rangle = \sum_{\substack{i_1,\ldots,i_k\\ i_{1'},\ldots,i_{k'}}} (d)^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k}(a_1^\dagger)^{i_1}_{i_{1'}}\ldots(a_k^\dagger)^{i_k}_{i_{k'}}|0\rangle$$
$$= \mathrm{Tr}_{V_N^{\otimes k}}[d(a_1^\dagger\otimes\ldots\otimes a_k^\dagger)]|0\rangle, \tag{B22}$$

in which we have k distinct oscillators and each element d in the full partition algebra $P_k(N)$ corresponds to a unique state, instead of S_k equivalence classes $[d]\in SP_k(N)$. We illustrate with the following examples that under the map (B22) elements $d\in P_k(N)$ that are in the kernel of $\Phi_{k,N}$ label zero vectors in the Hilbert space $\mathcal{H}$. For $k=2$ and $N=1$ we see



$$= \left[\sum_{i,j}(a_1^\dagger)^i_i(a_2^\dagger)^j_j - \sum_i(a_1^\dagger)^i_i(a_2^\dagger)^i_i\right]|0\rangle,$$
$$= \left[(a_1^\dagger)^1_1(a_2^\dagger)^1_1 - (a_1^\dagger)^1_1(a_2^\dagger)^1_1\right]|0\rangle,$$
$$= 0, \tag{B23}$$

in the first line we have used (B4) to express the orbit basis element in terms of the diagram basis. Similarly, taking $k=2$ and $N=2$ we have



$$= \left[\sum_{i,j,k}(a_1^\dagger)^i_j(a_2^\dagger)^k_k - \sum_{i,j}(a_1^\dagger)^i_i(a_2^\dagger)^j_j - \sum_{i,j}(a_1^\dagger)^i_j(a_2^\dagger)^j_j - \sum_{i,j}(a_1^\dagger)^i_j(a_2^\dagger)^i_i + 2\sum_i(a_1^\dagger)^i_i(a_2^\dagger)^i_i\right]|0\rangle,$$
$$= 0. \tag{B24}$$

We can split the first term by imposing different restrictions on the ranges of the sum

$$\sum_{i,j,k} = \sum_{i=j=k} + \sum_{\substack{i=j\\ j\neq k}} + \sum_{\substack{i=k\\ k\neq j}} + \sum_{\substack{j=k\\ k\neq i}} + \sum_{i\neq j\neq k}. \tag{B25}$$

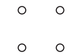
Similarly, we can split the second, third, and fourth terms

$$\sum_{i,j} = \sum_{i=j} + \sum_{i\neq j}. \tag{B26}$$

The terms in (B24) cancel due to the equivalence of coarsening diagrams and restricting summation ranges—adding edges to a diagram $d\in P_k(N)$ is equivalent to evaluating the original diagram d over a restricted summation range. Another way of saying this is that (B2) and (B25) encode identical expansions, in fact the five terms in each expansion give equivalent contributions. Orbit basis elements label states in which the oscillator indices are summed over the restricted range $i_1\neq i_2\neq\cdots\neq i_m$ where m is the number of blocks in the orbit basis element. From this perspective it is easy to see that these states must be zero when $N < m$ as there are not enough distinct values in $[1, N]$ to satisfy the inequality defining the summation range. Contrastingly, the diagram basis produces states corresponding to sums with unrestricted indices. Although at finite N there is a stark difference between states in the orbit and diagram bases, at large N the two descriptions are equivalent.

Elements of $SP_k(N)$ are S_k orbits on $P_k(N)$ and so states in $\mathcal{H}^{(k)}_{\mathrm{inv}}$ are linear combinations of states in $\mathcal{H}$. If a state in $\mathcal{H}$ is labeled by a partition algebra element in the kernel of $\Phi_{k,N}$, the state in $\mathcal{H}^{(k)}_{\mathrm{inv}}$ generated by the action of S_k on this zero $\mathcal{H}$ state will also be zero. It is clear that if an element $d\in P_k(N)$ produces a zero vector under (B22) then the equivalence class $[d]\in SP_k(N)$ containing that element $d\in P_k(N)$ also produces a zero vector under the map to $\mathcal{H}^{(k)}_{\mathrm{inv}}$

$$|d\rangle = \sum_{\substack{i_1,\ldots,i_k \\ i_{1'},\ldots,i_{k'}}} ([d])^{i_{1'}\ldots i_{k'}}_{i_1\ldots i_k} (a^\dagger)^{i_1}_{i_{1'}} \ldots (a^\dagger)^{i_k}_{i_{k'}} |0\rangle = \mathrm{Tr}_{V_N^{\otimes k}}([d](a^\dagger)^{\otimes k})|0\rangle. \tag{B27}$$

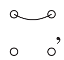
We can also check that for suitably low values of N the norm of the orbit basis states vanishes. For $x_{\pi_1} =$  we expect

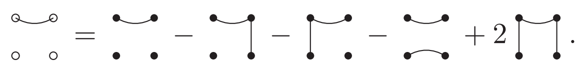
$$\langle x_{\pi_1}|x_{\pi_1}\rangle|_{N<4} = g_{x_{\pi_1}x_{\pi_1}}|_{N<4} = 0. \tag{B28}$$

Indeed, substituting (B6) into this expression gives

$$\begin{aligned}\langle x_{\pi_1}|x_{\pi_1}\rangle &= \langle d_{\pi_1}|d_{\pi_1}\rangle - \langle d_{\pi_1}|d_{\pi_2}\rangle - \langle d_{\pi_2}|d_{\pi_1}\rangle + \langle d_{\pi_2}|d_{\pi_2}\rangle + \cdots - 12\langle d_{\pi_{14}}|d_{\pi_{15}}\rangle + 36\langle d_{\pi_{15}}|d_{\pi_{15}}\rangle, \\ &= N(N-1)(N-2)(N-3),\end{aligned}$$

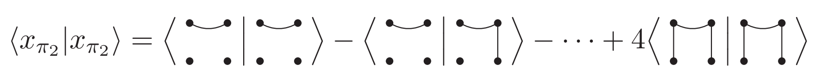
which is zero for $N < 4$.

Similarly, we consider $x_{\pi_2} =$ , which we expect to vanish for $N < 3$. This has a diagram basis expansion



(B29)

The norm of this state is



$$= N(N-1)(N-2), \tag{B30}$$

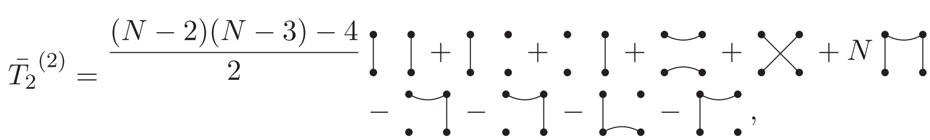
which does vanish for $N < 3$. For a general orbit basis state x_π we expect the norm to be some polynomial in N, which vanishes for any $N < |\pi|$.

APPENDIX C: COMPUTING LOW DEGREE MATRIX UNITS

In this appendix we find the full set of matrix units for $k = 2$ and the subset of multiplicity free matrix units for $k = 3$. These results can be reproduced using the accompanying Sage code.

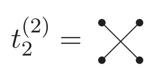
1. Degree two

As discussed in Sec. IV C 2, we use the following elements of $SP_2(N)$ to distinguish the full set of labels on matrix units $Q^{\Lambda_1}_{\Lambda_2,\mu\nu}$. The irreducible representation $\Lambda_1 \vdash N$ is distinguished by



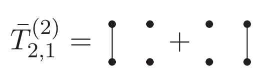
(C1)

while $\Lambda_2 \vdash k$ is distinguished by



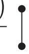
(C2)

and multiplicity labels μ, ν are distinguished by acting with

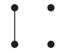
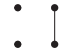
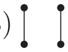


(C3)

on the left and right. It will be useful to know that $\bar{T}^{(2)}_{2,1}$ is related to

$$\bar{T}_2^{(1)} = \frac{N(N-3)}{2}$$  $$+$$, (C4)

since

$$\bar{T}_2^{(1)} \otimes 1 + 1 \otimes \bar{T}_2^{(1)} =$$  $$+$$  $$+ N(N-3)$$  . (C5)

As we will now explain, the eigenvalues of $\bar{T}_{2,1}^{(2)}$ uniquely determine the labels μ, ν by left and right action, respectively. For fixed Λ_1, Λ_2 the multiplicity labels correspond to basis elements for $V_{\Lambda_1,\Lambda_2}^{P_2(N)\to\mathbb{C}[S_2]}$, appearing in the decomposition

$$\begin{aligned} V_N \otimes V_N \cong{}& (V_{[N]}^{S_N} \otimes V_{[2]}^{S_2} \otimes V_{[N],[2]}^{P_2(N)\to\mathbb{C}[S_2]}) \oplus (V_{[N-1,1]}^{S_N} \otimes V_{[2]}^{S_2} \otimes V_{[N-1,1],[2]}^{P_2(N)\to\mathbb{C}[S_2]}) \\ &\oplus (V_{[N-1,1]}^{S_N} \otimes V_{[1,1]}^{S_2} \otimes V_{[N-1,1],[1,1]}^{P_2(N)\to\mathbb{C}[S_2]}) \oplus (V_{[N-2,2]}^{S_N} \otimes V_{[2]}^{S_2} \otimes V_{[N-2,2],[2]}^{P_2(N)\to\mathbb{C}[S_2]}) \\ &\oplus (V_{[N-2,1,1]}^{S_N} \otimes V_{[1,1]}^{S_2} \otimes V_{[N-2,1,1],[1,1]}^{P_2(N)\to\mathbb{C}[S_2]}). \end{aligned} \tag{C6}$$

On the right-hand side, $\bar{T}_{2,1}^{(2)}$ acts on the vector spaces $V_{\Lambda_1\Lambda_2}^{P_2(N)\to\mathbb{C}[S_2]}$ with dimensions

$$\begin{aligned} &\mathrm{Dim} V_{[N],[2]}^{P_2(N)\to\mathbb{C}[S_2]} = 2, \qquad \mathrm{Dim} V_{[N-1,1],[2]}^{P_2(N)\to\mathbb{C}[S_2]} = 2, \qquad \mathrm{Dim} V_{[N-1,1],[1,1]}^{P_2(N)\to\mathbb{C}[S_2]} = 1 \\ &\mathrm{Dim} V_{[N-2,2],[2]}^{P_2(N)\to\mathbb{C}[S_2]} = 1, \qquad \mathrm{Dim} V_{[N-2,1,1],[1,1]}^{P_2(N)\to\mathbb{C}[S_2]} = 1. \end{aligned} \tag{C7}$$

We will find that $\bar{T}_{2,1}^{(2)}$ has precisely as many distinct eigenvalues (in each subspace) as the corresponding dimension.

To confirm that this is the case, note that $\bar{T}_{2,1}^{(2)}$ acts on $V_N^{\otimes 2}$ as

$$\bar{T}_{2,1}^{(2)}(e_{i_1} \otimes e_{i_2}) = \bar{T}_2^{(1)} e_{i_1} \otimes e_{i_2} + e_{i_1} \otimes \bar{T}_2^{(1)} e_{i_2} - N(N-3) e_{i_1} \otimes e_{i_2}. \tag{C8}$$

It follows that the eigenvalues are directly related to the eigenvalues of $\bar{T}_2^{(1)}$ defined in (4.34). These are known by the decomposition

$$V_N \cong V_{[N]}^{S_N} \oplus V_{[N-1,1]}^{S_N}, \tag{C9}$$

where $\bar{T}_2^{(1)}$ acts on each summand by a normalized character. Using this on the left-hand side of (C6) gives

$$V_N \otimes V_N \cong (V_{[N]}^{S_N} \otimes V_{[N]}^{S_N}) \oplus (V_{[N]}^{S_N} \otimes V_{[N-1,1]}^{S_N}) \oplus (V_{[N-1,1]}^{S_N} \otimes V_{[N]}^{S_N}) \oplus (V_{[N-1,1]}^{S_N} \otimes V_{[N-1,1]}^{S_N}). \tag{C10}$$

Consequently, the three distinct eigenvalues of $\bar{T}_{2,1}^{(2)}$ are (one for each summand, but the vectors in the second and third space have the same eigenvalue)

$$2\frac{\chi^{[N]}(T_2)}{\mathrm{Dim} V_{[N]}^{S_N}} - N(N-3) = N(N-1) - N(N-3) = 2N, \tag{C11}$$

$$2\frac{\chi^{[N-1,1]}(T_2)}{\mathrm{Dim} V_{[N-1,1]}^{S_N}} - -N(N-3) = N(N-3) - N(N-3) = 0, \tag{C12}$$

$$\frac{\chi^{[N]}(T_2)}{\mathrm{Dim} V_{[N]}^{S_N}} + \frac{\chi^{[N-1,1]}(T_2)}{\mathrm{Dim} V_{[N-1,1]}^{S_N}} - N(N-3) = \frac{1}{2}N(N-1) + \frac{1}{2}N(N-3) - N(N-3) = N. \tag{C13}$$

By decomposing (C10) into $S_N \times S_k$ representations we will see that the multiplicities in (C6) are uniquely associated with one of the above eigenvalues. We start by considering the multiplicities of $V^{S_N}_{[N]} \otimes V^{S_2}_{[2]}$. The representation $V^{S_N}_{[N]}$ occurs in the decomposition (C10) as subspaces

$$V^{S_N}_{[N]} \cong V^{S_N}_{[N]} \otimes V^{S_N}_{[N]} \quad \text{and} \quad V^{S_N}_{[N]} \subset V^{S_N}_{[N-1,1]} \otimes V^{S_N}_{[N-1,1]}. \tag{C14}$$

The first subspace has eigenvalue $2N$, while the second subspace has eigenvalue 0 with respect to $\bar{T}^{(2)}_{2,1}$. Therefore, the two multiplicities are distinguished. Next we consider multiple occurrences of $V^{S_N}_{[N-1,1]}$. The two spaces

$$(V^{S_N}_{[N]} \otimes V^{S_N}_{[N-1,1]}) \oplus (V^{S_N}_{[N-1,1]} \otimes V^{S_N}_{[N]}) \tag{C15}$$

combine into representations of $S_N \times S_2$ as

$$(V^{S_N}_{[N-1,1]} \otimes V^{S_2}_{[2]}) \oplus (V^{S_N}_{[N-1,1]} \otimes V^{S_2}_{[1,1]}). \tag{C16}$$

Both of these spaces have an eigenvalue N with respect to $\bar{T}^{(2)}_{2,1}$, but they are distinguished by their S_2 representation (or equivalently eigenvalue of $t^{(2)}_2$). The symmetric part of $V^{S_N}_{[N-1,1]} \otimes V^{S_N}_{[N-1,1]}$ has a subspace

$$V^{S_N}_{[N-1,1]} \otimes S^{S_2}_{[2]} \subset V^{S_N}_{[N-1,1]} \otimes V^{S_N}_{[N-1,1]}, \tag{C17}$$

with eigenvalue 0. We have found that the two subspaces $V^{S_N}_{[N-1,1]} \otimes V^{S_2}_{[2]}$ are distinguished by the eigenvalues N and 0 with respect to $\bar{T}^{(2)}_{2,1}$. The last two terms in (C6) are multiplicity free and uniquely determined by their eigenvalue with respect to $T^{(2)}_2$.

In the Sage code, we simultaneously diagonalized all the operators by considering a linear combination

$$T = a\bar{T}_2{}^{(2)} + bt^{(2)}_2 + c\bar{T}^{(2),L}_{2,1} + f\bar{T}^{(2),R}_{2,1}, \tag{C18}$$

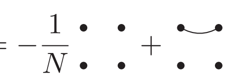
with $a, b, c, f \in \mathbb{R}$ such that there is no eigenvalue degeneracy in T. The superscript L means left action and R means right action. An eigenbasis for T will be a simultaneous eigenbasis for $\{\overline{T_2}^{(2)}, t^{(2)}_2, \bar{T}^{(2),L}_{2,1}, \bar{T}^{(2),R}_{2,1}\}$, which corresponds to a basis of matrix units. In the implementation, these operators act on $P_2(N)$, as opposed to $SP_2(N)$. The projection to $SP_2(N)$ was implemented by adding a fifth operator $P^{SP_2(N)}$ to T. The action of $P^{SP_2(N)}$ on $d \in P_2(N)$ is $d \mapsto [d]$. It commutes with all of the previous operators. This was useful in practice, since elements in $SP_2(N)$ will have eigenvalue 1 with respect to $P^{SP_2(N)}$ (the orthogonal complement has eigenvalue 0).

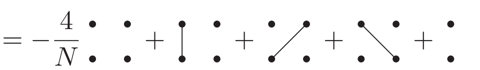
The matrix units for $k = 2$ are given below. The multiplicity labels have been chosen to correspond to eigenvalues of $\bar{T}^{(2),L}_{2,1}$ and $\bar{T}^{(2),R}_{2,1}$ as follows:

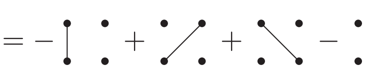
$$\begin{aligned} 1 &\leftrightarrow 2N, \\ 2 &\leftrightarrow 0, \\ 3 &\leftrightarrow N. \end{aligned} \tag{C19}$$

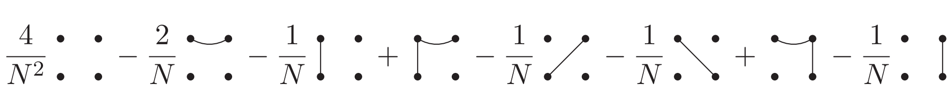
The elements below have not gone through the final step of being normalized.

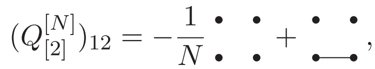
$(Q^{[N]}_{[2]})_{11} =$  , (C20)

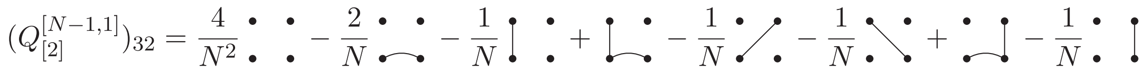
$(Q^{[N]}_{[2]})_{21} =$  , (C21)

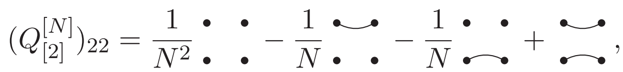
$(Q^{[N-1,1]}_{[2]})_{33} =$  , (C22)

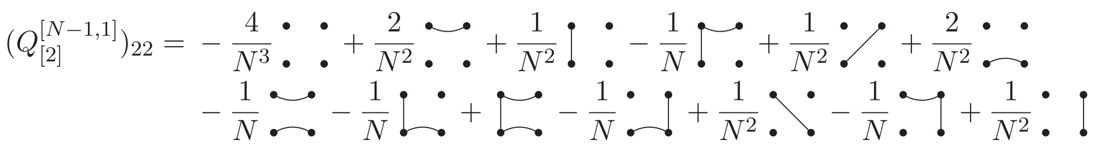
$(Q^{[N-1,1]}_{[1,1]})_{33} =$  , (C23)

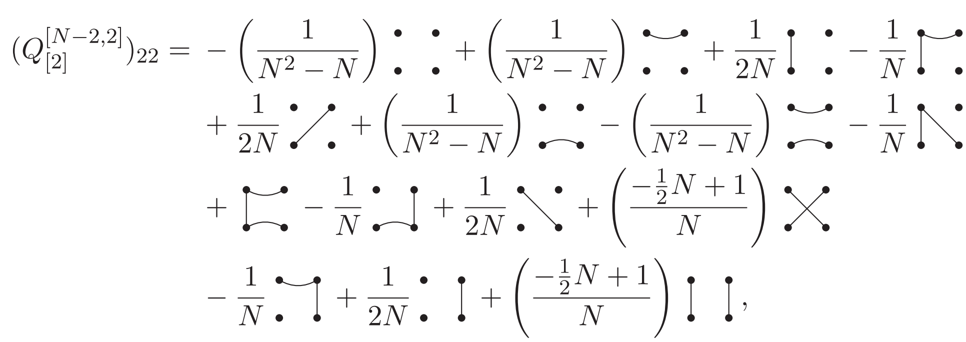
$(Q^{[N-1,1]}_{[2]})_{23} =$  , (C24)

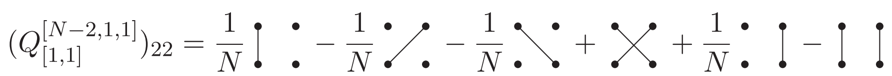

(C25)


(C26)


(C27)


(C28)


(C29)


(C30)

For example,

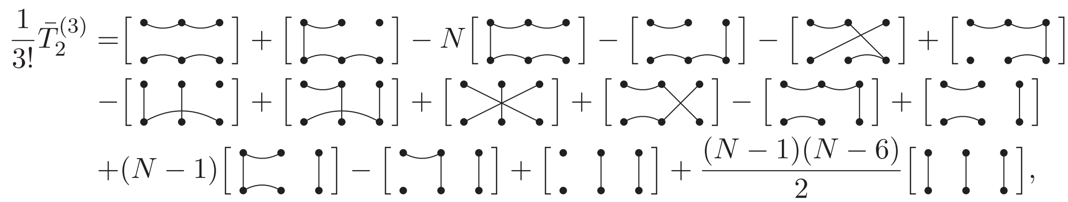
$$(Q_{[2]}^{[N]})_{11}(Q_{[2]}^{[N]})_{11} = N^2(Q_{[2]}^{[N]})_{11}, \tag{C31}$$

and the properly normalized matrix unit is given by

$$\frac{(Q_{[2]}^{[N]})_{11}}{N^2}. \tag{C32}$$

2. Degree three

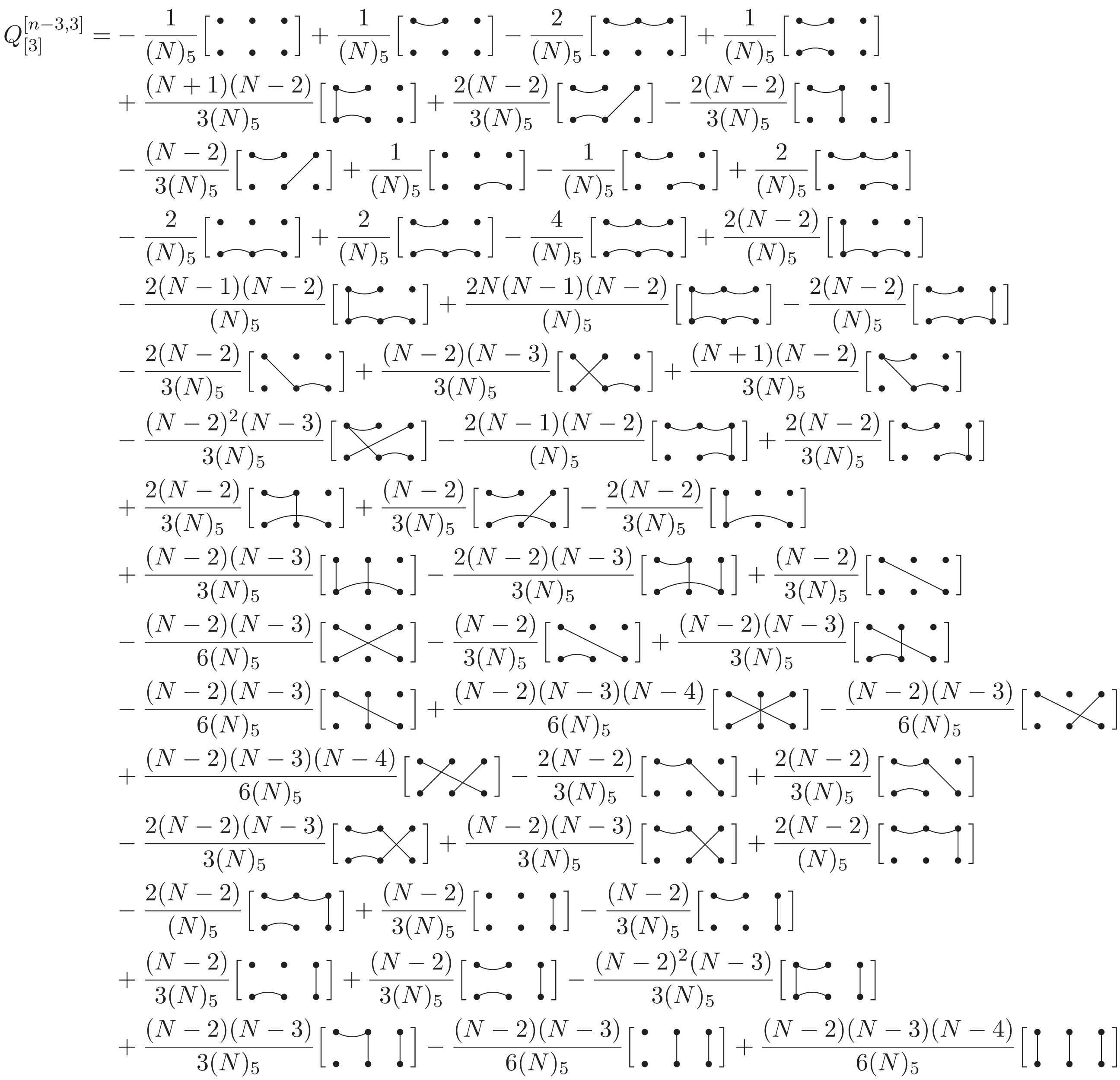
For degree $k = 3$, we find the multiplicity free matrix units $Q^{\Lambda_1}_{\Lambda_2}$, where $\Lambda_1 = [N-3,3], [N-3,2,1], [N-3,1,1,1]$. It is sufficient to use



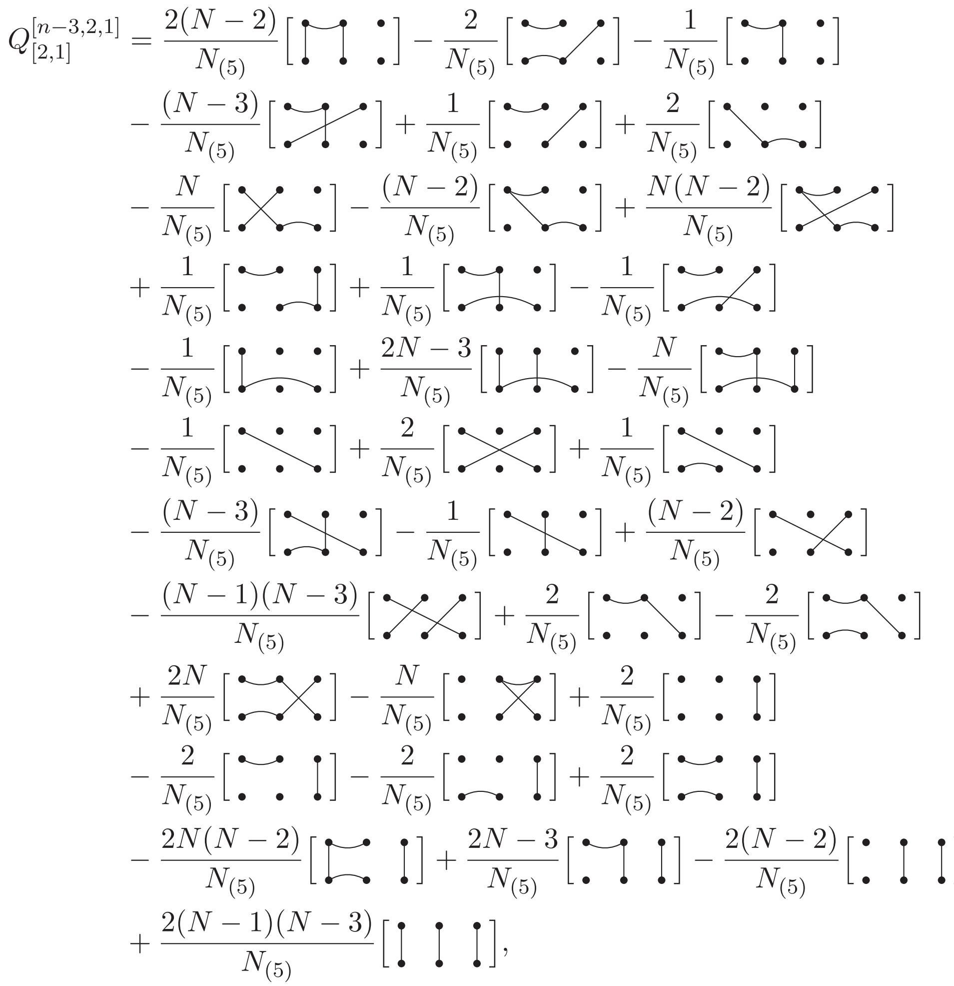
(C33)

which distinguishes Λ_1. The square brackets denote S_3 symmetrization as in (3.24).

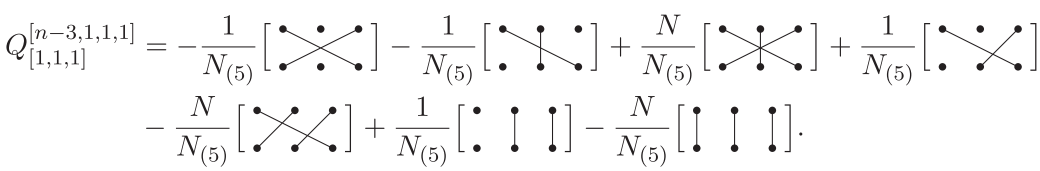
The multiplicity free matrix units for $k = 3$ are given below, where $(N)_m$ is the falling factorial $N(N-1)\ldots(N-m+1)$,



(C34)



(C35)



(C36)

APPENDIX D: THE METRIC AND ITS INVERSE

We would like to be more explicit about the form of the metric on $P_k(N)$ as well as its inverse, defined by our inner product on observables

$$\langle \mathcal{O}_{d_1} \mathcal{O}_{d_2} \rangle = \sum_{\gamma \in S_k} \mathrm{Tr}_{V_N^{\otimes k}} (d_1 \gamma d_2^T \gamma^{-1}). \tag{D1}$$

We note that similar results hold for the metric on $SP_k(N)$. First of all we write an explicit form for the metric. It was shown in [37] that in the large N limit the inner product on normalized PIMOs

$$\hat{\mathcal{O}}_d = \frac{\mathcal{O}_d}{\sqrt{\langle \mathcal{O}_d \mathcal{O}_d \rangle_{\rm con}}}, \tag{D2}$$

factorizes, and so the metric is given by a delta function at leading order

$$\hat{g}_{d_1 d_2} = \langle \hat{\mathcal{O}}_{d_1} \hat{\mathcal{O}}_{d_2} \rangle_{\rm con} = \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_1] = [d_2] \\ 0 + O(1/\sqrt{N}) & \text{if } [d_1] \neq [d_2] \end{cases}. \tag{D3}$$

Furthermore, it was shown that the $\frac{1}{\sqrt{N}}$ corrections are given by the inclusion of a second term

$$\hat{g} = \mathbb{1} + \sum_{d_1 \neq d_2} N^{c(d_1 \vee d_2) - \frac{1}{2}(c(d_1) + c(d_2))} E_{d_1 d_2}, \tag{D4}$$

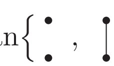
with $E_{d_1 d_2}$ the matrix consisting of a 1 in the (d_1, d_2) position and zeros elsewhere. Setting

$$X = \sum_{d_1 \neq d_2} N^{c(d_1 \vee d_2) - \frac{1}{2}(c(d_1) + c(d_2))} E_{d_1 d_2} \tag{D5}$$

we have

$$\hat{g}^{-1} = (\mathbb{1} + X)^{-1} = \mathbb{1} - X + X^2 - X^3 + \ldots. \tag{D6}$$

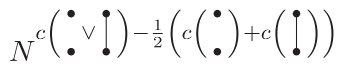
We now calculate the inverse metric for $P_1(N)$ explicitly. $P_1(N)$ is spanned by just two diagrams

$P_1(N) = \text{Span}\{$  $\}$. (D7)

Using our expression for off-diagonal elements of the metric

$$X_{P_1(N)} = \sum_{d_1 \neq d_2} N^{c(d_1 \vee d_2) - \frac{1}{2}(c(d_1) + c(d_2))} E_{d_1 d_2} \tag{D8}$$

we find the one independent element

 $= N^{(1-1-\frac{1}{2})} = N^{-\frac{1}{2}}$ (D9)

and therefore

$$\hat{g}_{P_1(N)} = \mathbb{1} + X_{P_1(N)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & N^{-\frac{1}{2}} \\ N^{-\frac{1}{2}} & 0 \end{bmatrix}. \tag{D10}$$

Substituting this into our equation for the inverse metric (D6) we find that the inverse is given by

$$\begin{aligned} \hat{g}^{-1}_{P_1(N)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - N^{-\frac{1}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + N^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad - N^{-\frac{3}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \ldots, \\ &= \sum_{i=0}^{\infty} \left(N^{-i} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - N^{-(i+\frac{1}{2})} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right), \\ &= \frac{N}{N-1} \begin{bmatrix} 1 & -N^{-\frac{1}{2}} \\ -N^{-\frac{1}{2}} & 1 \end{bmatrix}, \end{aligned} \tag{D11}$$

as

$$\begin{aligned} \sum_{i=0}^{\infty} N^{-i} &= \frac{1}{1 - \frac{1}{N}} = \frac{N}{N-1}, \\ \sum_{i=0}^{\infty} N^{-(i+\frac{1}{2})} &= N^{-\frac{1}{2}} \sum_{i=0}^{\infty} N^{-i} = N^{-\frac{1}{2}} \frac{1}{1 - \frac{1}{N}} = \frac{N^{\frac{1}{2}}}{N-1}. \end{aligned} \tag{D12}$$

As a further example we calculate the inverse metric for $P_2(N)$, which is spanned by 15 diagrams. We are interested in the off-diagonal elements of our metric and so we have $\frac{15(15-1)}{2} = 105$ independent elements. Using Sage we find the metric is given by

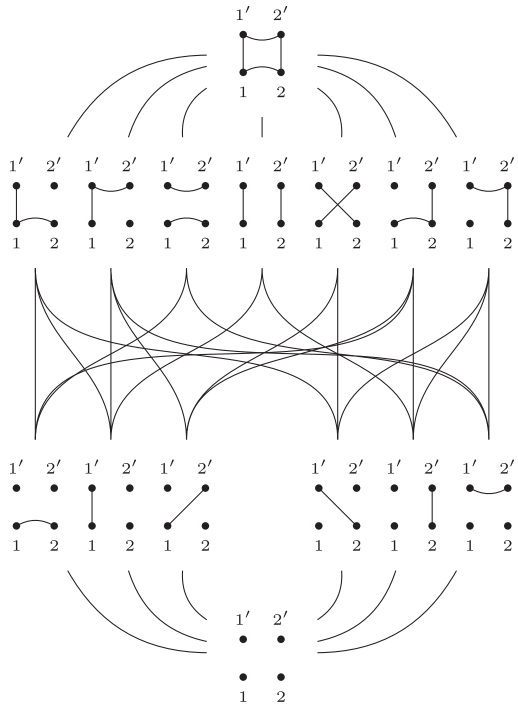
$$
\begin{aligned}
\hat{g}_{P_2(N)} &= \mathbb{1} + X_{P_2(N)} \\
&= \begin{bmatrix}
1 & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{3}{2}} \\
N^{-\frac{1}{2}} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-1} \\
N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-1} \\
N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} \\
N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} \\
N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} \\
N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-1} & N^{-\frac{1}{2}} \\
N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-1} & N^{-\frac{1}{2}} \\
N^{-1} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{3}{2}} & N^{-\frac{3}{2}} & N^{-\frac{1}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & 1 & N^{-\frac{1}{2}} \\
N^{-\frac{3}{2}} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-1} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & N^{-\frac{1}{2}} & 1
\end{bmatrix}.
\end{aligned}
\tag{D13}
$$

Inverting this metric directly gives

$$
\begin{aligned}
\hat{g}^{-1} &= \frac{N}{(N-1)(N-2)(N-3)} \\
&\times \begin{bmatrix}
N^2+N & ((-N)-1)N^{\frac{1}{2}} & ((-N)-1)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & ((-N)-1)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & ((-N)-1)N^{\frac{1}{2}} & 2N & 2N & 2N & 2N & 2N & 2N & -6N^{\frac{1}{2}} \\
((-N)-1)N^{\frac{1}{2}} & (N-1)^2 & N+1 & N-1 & N+1 & N-1 & N-1 & N+1 & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & 2N \\
((-N)-1)N^{\frac{1}{2}} & N+1 & (N-1)^2 & N-1 & N+1 & N-1 & N-1 & N+1 & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & 2N \\
(1-N)N^{\frac{1}{2}} & N-1 & N-1 & N^2-3N+1 & N-1 & 1 & 1 & N-1 & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & N \\
((-N)-1)N^{\frac{1}{2}} & N+1 & N+1 & N-1 & (N-1)^2 & N-1 & N-1 & N+1 & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & 2N \\
(1-N)N^{\frac{1}{2}} & N-1 & N-1 & 1 & N-1 & N^2-3N+1 & 1 & N-1 & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & N \\
(1-N)N^{\frac{1}{2}} & N-1 & N-1 & 1 & N-1 & 1 & N^2-3N+1 & N-1 & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & N \\
((-N)-1)N^{\frac{1}{2}} & N+1 & N+1 & N-1 & N+1 & N-1 & N-1 & (N-1)^2 & -2N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & 2N \\
2N & (1-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & N^2-2N & N & N & N & N & N & -N^{\frac{3}{2}} \\
2N & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & N & N^2-2N & N & N & N & N & -N^{\frac{3}{2}} \\
2N & (1-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & N & N & N^2-2N & N & N & N & -N^{\frac{3}{2}} \\
2N & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & N & N & N & N^2-2N & N & N & -N^{\frac{3}{2}} \\
2N & -2N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & N & N & N & N & N^2-2N & N & -N^{\frac{3}{2}} \\
2N & -2N^{\frac{1}{2}} & -2N^{\frac{1}{2}} & (2-N)N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & -N^{\frac{1}{2}} & -N^{\frac{1}{2}} & (1-N)N^{\frac{1}{2}} & N & N & N & N & N & N^2-2N & -N^{\frac{3}{2}} \\
-6N^{\frac{1}{2}} & 2N & 2N & N & 2N & N & N & 2N & -N^{\frac{3}{2}} & -N^{\frac{3}{2}} & -N^{\frac{3}{2}} & -N^{\frac{3}{2}} & -N^{\frac{3}{2}} & -N^{\frac{3}{2}} & N^2
\end{bmatrix}.
\end{aligned}
\tag{D14}
$$

1. First and second order corrections

The leading corrections to this are of order $N^{-\frac{1}{2}}$ and occur precisely when d_2 is obtained from d_1 by removing a single edge (or vice versa). We recognize this as a method of constructing the Hasse diagram for $P_k(N)$. Accordingly the leading order corrections to the metric are given precisely by the elements $d_1, d_2 \in P_k(N)$ that share a connection in the relevant Hasse diagram. For example, the following is the Hasse diagram for $k = 2$



(D15)

Indeed, every connection in this diagram corresponds to an $N^{\frac{1}{2}}$ element in $\hat{g}_{P_2(N)}$ and all $N^{\frac{1}{2}}$ elements of $\hat{g}_{P_2(N)}$ are given by a connection in the diagram. We call each row in the Hasse diagram a level L_i and index them by i—the number of connected components in the partition diagrams on that level. For example  $\in L_1$,  $\in L_2$,  $\in L_3$, and  $\in L_4$.

Ordering our basis according to the levels in the Hasse diagram we see that the metric has block diagonal contributions from within any given level of the Hasse diagram. As the leading order corrections are generated by d_1 and d_2 in different levels these occur outside of the diagonal blocks. Everything we have said here about the metric applies equally well to the inverse metric, as to first order this is given by

$$\hat{g}^{-1} \sim \mathbb{1} - X. \tag{D16}$$

The N^{-1} corrections to the metric are again easily described with reference to the Hasse diagram. There are two ways in which we can get N^{-1} contributions:

(1) For any $d_1, d_2 \in L_i$ if $d_1 \vee d_2 \in L_{i-1}$, then $\hat{g}_{d_1,d_2} = N^{-1}$.

(2) For and $d_1 \in L_i$, $d_2 \in L_{i-2}$ if $d_1 < d_2$, that is if d_1 is contained within d_2, then $\hat{g}_{d_1,d_2} = N^{-1}$. If d_1 and d_2 are incomparable then their inner product will be a larger negative power of N as this incomparability will only reduce the number of connected components of the merge of d_1 and d_2.

More generally for $d_1 \in L_i$, $d_2 \in L_{i-2n}$ for $n \in \mathbb{Z}^+$ and $d_1 < d_2$ we have

$$\hat{g}_{d_1,d_2} = N^{c(d_1 \vee d_2) - \frac{1}{2}(c(d_1)+c(d_2))} = N^{(i-2n)-\frac{1}{2}(i+i-2n)} = N^{-n}. \tag{D17}$$

[1] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. **B72**, 461 (1974).
[2] M. R. Douglas and S. H. Shenker, Strings in less than one-dimension, Nucl. Phys. **B335**, 635 (1990).
[3] E. Brezin and V. A. Kazakov, Exactly solvable field theories of closed strings, Phys. Lett. B **236**, 144 (1990).
[4] D. J. Gross and A. A. Migdal, Nonperturbative Two-Dimensional Quantum Gravity, Phys. Rev. Lett. **64**, 127 (1990).
[5] D. J. Gross, Two-dimensional QCD as a string theory, Nucl. Phys. **B400**, 161 (1993).
[6] J. A. Minahan, Summing over inequivalent maps in the string theory interpretation of two-dimensional QCD, Phys. Rev. D **47**, 3430 (1993).
[7] D. J. Gross and W. Taylor, Two-dimensional QCD is a string theory, Nucl. Phys. **B400**, 181 (1993).
[8] S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, Two-dimensional yang-mills theories are string theories, Mod. Phys. Lett. A **08**, 2223 (1993).
[9] D. J. Gross and W. Taylor, Twists and wilson loops in the string theory of twodimensional QCD, Nucl. Phys. **B403**, 395 (1993).
[10] J. A. Minahan and A. P. Polychronakos, Equivalence of two-dimensional QCD and the $C = 1$ matrix model, Phys. Lett. B **312**, 155 (1993).
[11] P. Hořava, Topological rigid string theory and two-dimensional QCD, Nucl. Phys. **B463**, 238 (1996).
[12] S. Cordes, G. W. Moore, and S. Ramgoolam, Large N 2-D Yang-Mills theory and topological string theory, Commun. Math. Phys. **185**, 543 (1997).
[13] Y. Kimura and S. Ramgoolam, Holomorphic maps and the complete $1/N$ expansion of $2DSU(N)$ Yang-Mills, J. High Energy Phys. 06 (2008) 015.
[14] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).
[15] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[16] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[17] M. Bauer and C. Itzykson, Triangulations (in French), Recherche Coopérative sur Programme n25 **44**, 39 (1993).
[18] R. de Mello Koch and S. Ramgoolam, From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group, arXiv:1002.1634.
[19] R. Gopakumar, What is the simplest gauge-string duality?, arXiv:1104.2386.
[20] R. de Mello Koch and L. Nkumane, Topological string correlators from matrix models, J. High Energy Phys. 03 (2015) 004.
[21] E. P. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, Ann. Math. **62**, 548 (1955).
[22] F. J. Dyson, A brownian-motion model for the eigenvalues of a random matrix, J. Math. Phys. (N.Y.) **3**, 1191 (1962).
[23] M. L. Mehta, *Random Matrices* (Elsevier, New York, 2004).
[24] C. W. J. Beenakker, Random-matrix theory of quantum transport, Rev. Mod. Phys. **69**, 731 (1997).
[25] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Random-matrix theories in quantum physics: Common concepts, Phys. Rep. **299**, 189 (1998).
[26] A. Edelman and Y. Wang, Random matrix theory and its innovative applications, in *Advances in Applied Mathematics, Modeling, and Computational Science* (Springer US, Boston, MA, 2013), pp. 91–116.
[27] D. Kartsaklis, S. Ramgoolam, and M. Sadrzadeh, Linguistic matrix theory, Ann. Inst. H. Poincare D Comb. Phys. Interact. **6**, 385 (2019).
[28] S. Ramgoolam, Permutation invariant gaussian matrix models, Nucl. Phys. **B945**, 114682 (2019).
[29] S. Ramgoolam, M. Sadrzadeh, and L. Sword, Gaussianity and typicality in matrix distributional semantics, Ann. Inst. H. Poincare D Comb. Phys. Interact **9**, 1 (2022).
[30] M. A. Huber, A. Correia, S. Ramgoolam, and M. Sadrzadeh, Permutation invariant matrix statistics and computational language tasks, arXiv:2202.06829.
[31] B. Coecke, M. Sadrzadeh, and S. Clark, Mathematical foundations for a compositional distributional model of meaning, arXiv:1003.4394.
[32] J. Maillard, S. Clark, and E. Grefenstette, A type-driven tensor-based semantics for CCG, in *Proceedings of the EACL 2014 Workshop on Type Theory and Natural Language Semantics (TTNLS)* (Association for Computational Linguistics, Gothenburg, Sweden, 2014), pp. 46–54.
[33] M. Baroni, R. Bernardi, and R. Zamparelli, Frege in space: A program of compositional distributional semantics, Linguist. Issues Lang. Technol. **9**, 241 (2014).
[34] E. Grefenstette and M. Sadrzadeh, Concrete models and empirical evaluations for the categorical compositional distributional model of meaning, Comput. Linguist. **41**, 71 (2015).
[35] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Large N field theories, string theory and gravity, Phys. Rep. **323**, 183 (2000).
[36] L. Eberhardt, M. R. Gaberdiel, and R. Gopakumar, Deriving the AdS3/CFT2 correspondence, J. High Energy Phys. 02 (2020) 136.
[37] G. Barnes, A. Padellaro, and S. Ramgoolam, Hidden symmetries and Large N factorisation for permutation invariant matrix observables, J. High Energy Phys. 08 (2022) 090.
[38] M. R. Douglas, Conformal field theory techniques in large N Yang-Mills theory, in *NATO Advanced Research Workshop on New Developments in String Theory, Conformal Models and Topological Field Theory* (1993), arXiv: hep-th/9311130.
[39] S. Corley, A. Jevicki, and S. Ramgoolam, Exact correlators of giant gravitons from dual $N = 4$ SYM theory, Adv. Theor. Math. Phys. **5**, 809 (2001).
[40] D. Berenstein, A Toy model for the AdS/CFT correspondence, J. High Energy Phys. 07 (2004) 018.
[41] S. Murthy, Unitary matrix models, free fermion ensembles, and the giant graviton expansion, arXiv:2202.06897.
[42] D. Berenstein, Large N BPS states and emergent quantum gravity, J. High Energy Phys. 01 (2006) 125.
[43] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, An index for 4 dimensional super conformal theories, Commun. Math. Phys. **275**, 209 (2007).
[44] C. Lewis-Brown and S. Ramgoolam, Quarter-BPS states, multi-symmetric functions and set partitions, J. High Energy Phys. 03 (2021) 153.
[45] T. Harmark and M. Orselli, Spin matrix theory: A quantum mechanical model of the AdS/CFT correspondence, J. High Energy Phys. 11 (2014) 134.
[46] S. Baiguera, T. Harmark, and Y. Lei, Spin matrix theory in near $\frac{1}{8}$-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills, J. High Energy Phys. 02 (2022) 191.
[47] A. Jevicki and B. Sakita, The quantum collective field method and its application to the planar limit, Nucl. Phys. **B165**, 511 (1980).
[48] L. G. Yaffe, Large N limits as classical mechanics, Rev. Mod. Phys. **54**, 407 (1982).
[49] D. Berenstein and S.Wang, BPS coherent states and localization, J. High Energy Phys. 08 (2022) 164.
[50] A. Holguin and S. Wang, Giant gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal $\mathcal{N} = 4$ SYM, J. High Energy Phys. 10 (2022) 078.
[51] H. Lin, Coherent state excitations and string-added coherent states in gauge-gravity correspondence, arXiv:2206.06524.
[52] W. Fulton and J. Harris, *Representation Theory: A First Course* (Springer Science and Business Media, New York, 2013), Vol. 129.
[53] Hamermesh, *Group Theory and Its Application to Physical Problem* (Addison-Wesley, Reading, MA, 1962).
[54] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, Giant gravitons in conformal field theory, J. High Energy Phys. 04 (2002) 034.
[55] J. McGreevy, L. Susskind, and N. Toumbas, Invasion of the giant gravitons from Anti-de Sitter space, J. High Energy Phys. 06 (2000) 008.
[56] A. Hashimoto, S. Hirano, and N. Itzhaki, Large branes in AdS and their field theory dual, J. High Energy Phys. 08 (2000) 051.
[57] M. T. Grisaru, R. C. Myers, and O. Tafjord, SUSY and goliath, J. High Energy Phys. 08 (2000) 040.

[58] Y. Kimura and S. Ramgoolam, Branes, anti-branes and brauer algebras in gaugegravity duality, J. High Energy Phys. 11 (2007) 078.
[59] T. W. Brown, P. J. Heslop, and S. Ramgoolam, Diagonal multi-matrix correlators and BPS operators in $N = 4$ SYM, J. High Energy Phys. 02 (2008) 030.
[60] R. Bhattacharyya, S. Collins, and R. de Mello Koch, Exact multi-matrix correlators, J. High Energy Phys. 03 (2008) 044.
[61] R. Bhattacharyya, R. de Mello Koch, and M. Stephanou, Exact multi-restricted schur polynomial correlators, J. High Energy Phys. 06 (2008) 101.
[62] Y. Kimura and S. Ramgoolam, Enhanced symmetries of gauge theory and resolving the spectrum of local operators, Phys. Rev. D **78**, 126003 (2008).
[63] T. W. Brown, P. J. Heslop, and S. Ramgoolam, Diagonal free field matrix correlators, global symmetries and giant gravitons, J. High Energy Phys. 04 (2009) 089.
[64] J. Pasukonis and S. Ramgoolam, Quivers as calculators: Counting, correlators and riemann surfaces, J. High Energy Phys. 04 (2013) 094.
[65] P. Caputa, R. de Mello Koch, and P. Diaz, A basis for large operators in $N = 4$ SYM with orthogonal gauge group, J. High Energy Phys. 03 (2013) 041.
[66] D. Berenstein, Extremal chiral ring states in the AdS/CFT correspondence are described by free fermions for a generalized oscillator algebra, Phys. Rev. D **92**, 046006 (2015).
[67] Y. Kimura, S. Ramgoolam, and R. Suzuki, Flavour singlets in gauge theory as Permutations, J. High Energy Phys. 12 (2016) 142.
[68] C. Lewis-Brown and S. Ramgoolam, BPS operators in $\mathcal{N} = 4SO(N)$ super Yang- Mills theory: Plethysms, dominoes and words, J. High Energy Phys. 11 (2018) 035.
[69] F. Aprile, J. M. Drummond, P. Heslop, H. Paul, F. Sanfilippo, M. Santagata, and A. Stewart, Single particle operators and their correlators in free $N = 4$ SYM, J. High Energy Phys. 11 (2020) 072.
[70] H. Lin and Y. Zhu, Entanglement and mixed states of Young tableau states in gauge/gravity correspondence, Nucl. Phys. **B972**, 115572 (2021).
[71] S. Ramgoolam, Permutations and the combinatorics of gauge invariants for general N, Proc. Sci. CORFU2015 (**2016**) 107 [arXiv:1605.00843].
[72] P. Martin, Temperley-lieb algebras for non-planar statistical mechanics—the partition algebra construction, J. Knot Theory Ramif. **03**, 51 (1994).
[73] V. Jones, The Potts model and the symmetric group, in *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras* (World Scientific Publishing, River Edge, NJ, 1994), pp. 259–267.
[74] P. Martin, The structure of the partition algebras, J. Algebra **183**, 319 (1996).
[75] T. Halverson and A. Ram, Partition algebras, arXiv:math/0401314v2.
[76] T. Halverson, Characters of the partition algebras, J. Algebra **238**, 502 (2001).
[77] J. Enyang, Jucys-murphy elements and a presentation for partition algebras, J. Algebraic Combinatorics **37**, 401 (2013).
[78] G. Benkart and T. Halverson, Partition algebras and the invariant theory of the symmetric group, arXiv:1709.07751v1.
[79] T. Halverson and T. N. Jacobson, Set-partition tableaux and representations of diagram algebras, arXiv:1808.08118v2.
[80] F. Gabriel, Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization, arXiv:1503.02792v1.
[81] F. Gabriel, Combinatorial theory of permutation-invariant random matrices II: Cumulants, freeness and levy processes, arXiv:1507.02465.
[82] F. Gabriel, A Combinatorial Theory of Random Matrices III: Random Walks on $\mathfrak{S}(N)$, Ramified Coverings and the $\mathfrak{s}(\infty)$ Yang-Mills Measure, arXiv:1510.01046.
[83] G. Barnes, A. Padellaro, and S. Ramgoolam, Permutation invariant Gaussian 2-matrix models, J. Phys. A **55**, 145202 (2022).
[84] A. Ram, Chapter 1 representation theory, Dissertation, 1990.
[85] K. Pakrouski, P. N. Pallegar, F. K. Popov, and I. R. Klebanov, Many Body Scars as a Group Invariant Sector of Hilbert Space, Phys. Rev. Lett. **125**, 230602 (2020).
[86] K. Pakrouski, P. N. Pallegar, F. K. Popov, and I. R. Klebanov, Group theoretic approach to many-body scar states in fermionic lattice models, Phys. Rev. Res. **3**, 043156 (2021).
[87] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. Zibrov, M. Endres, M. Greiner, V. Vuletic, and M. Lukin, Probing many-body dynamics on a 51-atom quantum simulator, Nature (London) **551**, 579 (2017).
[88] C. J. Turner, A. Michailidis, D. Abanin, M. Serbyn, and Z. Papic, Weak ergodicity breaking from quantum many-body scars, Nat. Phys. **14**, 745 (2018).
[89] G. Kemp and S. Ramgoolam, BPS states, conserved charges and centres of symmetric group algebras, J. High Energy Phys. 01 (2020) 146.
[90] J. B. Geloun and S. Ramgoolam, Quantum Mechanics of Bipartite Ribbon Graphs: Integrality, Lattices and Kronecker Coefficients, arXiv:2010.04054.
[91] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxford, 1998).
[92] S. Doty, A. Lauve, and G. H. Seelinger, Canonical idempotents of multiplicity-free families of algebras, Enseign. Math. **64**, 23 (2018).
[93] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions* (Springer Science and Business Media, New York, 2013).
[94] Y. Kimura, Quarter BPS classified by brauer algebra, J. High Energy Phys. 05 (2010) 103.
[95] R. de Mello Koch, M. Dessein, D. Giataganas, and C. Mathwin, Giant graviton oscillators, J. High Energy Phys. 10 (2011) 009.
[96] V. D. Comarmond, R. de Mello Koch, and K. Jefferies, Surprisingly simple spectra, J. High Energy Phys. 02 (2011) 006.
[97] T. Brown, Permutations and the loop, J. High Energy Phys. 06 (2008) 008.

[98] R. de Mello Koch and S. Ramgoolam, A double coset ansatz for integrability in AdS/CFT, J. High Energy Phys. 06 (2012) 083.

[99] R. de Mello Koch, J. Smolic, and M. Smolic, Giant gravitons with strings attached (II), J. High Energy Phys. 09 (2007) 049.

[100] V. Balasubramanian, D. Berenstein, B. Feng, and M. xin Huang, D-branes in yangmills theory and emergent gauge symmetry, J. High Energy Phys. 03 (2005) 006.

[101] M. Lassalle, An explicit formula for the characters of the symmetric group, Math. Ann. **340**, 383 (2008).

[102] X. G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus riemann surfaces, Phys. Rev. B **41**, 9377 (1990).

[103] A. Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. (Amsterdam) **321**, 2 (2006).

[104] V. Balasubramanian, B. Czech, K. Larjo, and J. Simón, Integrability vs. information loss: A simple example, J. High Energy Phys. 11 (2006) 001.

[105] S. Ramgoolam and E. Sharpe, Combinatoric topological string theories and group theory algorithms, arXiv:2204.02266.

[106] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A **43**, 2046 (1991).

[107] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).

[108] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, Rev. Mod. Phys. **91**, 021001 (2019).

[109] M. Serbyn, D. A. Abanin, and Z. Papic, Quantum many-body scars and weak breaking of ergodicity, Nat. Phys. **17**, 675 (2021).

[110] N. Shiraishi and T. Mori, Systematic Construction of Counterexamples to the Eigenstate Thermalization Hypothesis, Phys. Rev. Lett. **119**, 030601 (2017).

[111] S. Moudgalya, N. Regnault, and B. A. Bernevig, Entanglement of exact excited states of Affleck-Kennedy-Lieb-Tasaki models: Exact results, many-body scars, and violation of the strong eigenstate thermalization hypothesis, Phys. Rev. B **98**, 235156 (2018).

[112] M. Schecter and T. Iadecola, Weak Ergodicity Breaking and Quantum Many-Body Scars in Spin-1 XY Magnets, Phys. Rev. Lett. **123**, 147201 (2019).

[113] N. O'Dea, F. Burnell, A. Chandran, and V. Khemani, From tunnels to towers: Quantum scars from lie algebras and q-deformed lie algebras, Phys. Rev. Res. **2**, 043305 (2020).

[114] J. Ren, C. Liang, and C. Fang, Quasisymmetry Groups and Many-Body Scar Dynamics, Phys. Rev. Lett. **126**, 120604 (2021).

[115] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Boson localization and the superfluid-insulator transition, Phys. Rev. B **40**, 546 (1989).

[116] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Cold Bosonic Atoms in Optical Lattices, Phys. Rev. Lett. **81**, 3108 (1998).

[117] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papic, Quantum scarred eigenstates in a rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations, Phys. Rev. B **98**, 155134 (2018).

[118] B Eden, P. S. Howe, E Sokatchev, and P. C. West, Extremal and next-to-extremal n-point correlators in four-dimensional SCFT, Phys. Lett. B **494**, 141 (2000).

[119] C. Bowman, M. De Visscher, and R. Orellana, The partition algebra and the Kronecker coefficients, Trans. Am. Math. Soc. **367**, 3647 (2015).

[120] P. C. Diniz, E. A. B. Oliveira, A. R. P. Lima, and E. A. L. Henn, Ground state and collective excitations of a dipolar bose-einstein condensate in a bubble trap, Sci. Rep. **10**, 4831 (2020).