NS three-form flux deformation for the critical non-Abelian vortex string

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It has been shown that the non-Abelian solitonic vortex string supported in four-dimensional (4D) $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the U(2) gauge group and $N_f = 4$ quark flavors becomes a critical superstring. This string propagates in the ten-dimensional space formed by a product of the flat 4D space and an internal space given by a Calabi-Yau noncompact threefold, namely, the conifold. The spectrum of low-lying closed string states in the associated type-IIA string theory was found and interpreted as a spectrum of hadrons in 4D $\mathcal{N} = 2$ SQCD. In particular, the lowest string state appears to be a massless BPS baryon associated with the deformation of the complex structure modulus b of the conifold. In the previous work the deformation of the ten-dimensional background with nonzero Neveu-Schwarz 3-form flux was considered and interpreted as a switching on a particular choice of quark masses in 4D SQCD. This deformation was studied to leading order at small 3-form flux. In this paper we study the back reaction of the nonzero 3-form flux on the metric and the dilaton introducing the ansatz with several warp factors and solving the gravity equations of motion. We show that 3-form flux produces a potential for the conifold complex structure modulus b, which leads to the runaway vacuum. At the runaway vacuum warp factors disappear, while the conifold degenerates. In 4D SQCD we relate this to the flow to the U(1) gauge theory upon switching on quark masses and decoupling of two flavors.

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I. INTRODUCTION

Non-Abelian vortices were first found in 4D $\mathcal{N} = 2$ SQCD with the gauge group $\mathrm{U}(N)$ and $N_f \geq N$ flavors of quarks [1–4]. The non-Abelian vortex string is 1/2 Bogomolny-Prasad-Sommerfeld (BPS) saturated and, therefore, has $\mathcal{N} = (2,2)$ supersymmetry on its world sheet. In addition to four translational moduli of the Abrikosov-Nielsen-Olesen (ANO) strings [5], the non-Abelian string carries orientational moduli, as well as the size moduli if $N_f > N$ [1–4] (see [6–9] for reviews).

It was shown in [10] that the non-Abelian solitonic vortex string in $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the $\mathrm{U(N = 2)}$ gauge group and $N_f = 4$ flavors of quark hypermultiplets becomes a critical superstring. The dynamics of the internal orientational and size moduli of the non-Abelian vortex string for the case $N = 2$, $N_f = 4$ is described by the so-called two-dimensional (2D) weighted CP sigma model, which we denote as $\mathbb{WCP}(N = 2, N_f - N = 2)$.

For $N_f = 2N$ this world sheet sigma model becomes conformal. Moreover, for $N = 2$ the number of the orientational and size moduli is six and they can be combined with four translational moduli to form a ten-dimensional (10D) space required for a superstring to become critical [10,11]. In this case the target space of the world sheet sigma model on the non-Abelian vortex string is $\mathbb{R}^4 \times Y_6$, where Y_6 is a noncompact six dimensional Calabi-Yau (CY) manifold, the conifold [12,13]. Moreover, the theory of the critical vortex string at hand was identified as the superstring theory of type IIA [11]. This allows one to apply the string theory for the calculation of the spectrum of string states and identify it with a spectrum of hadrons in 4D $\mathcal{N} = 2$ SQCD [11]. Since non-Abelian vortex strings are topologically stable and cannot be broken (see [8] for a review) we focus on the closed strings and consider Kaluza-Klein reduction of 10D string theory associated with the non-Abelian vortex to 4D.

A version of the string-gauge duality for 4D SQCD was proposed [10]: at weak coupling this theory is in the Higgs phase and can be described in terms of quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as closed string states formed by the non-Abelian vortex string. We call this approach "solitonic string-gauge duality".

The first step of the above program, namely, finding massless string states was carried out in [11,14] using supergravity approximation. It turns out that most of massless modes have non-normalizable wave functions over the non-compact conifold Y_6, i.e., they are not localized in 4D and, hence, cannot be interpreted as dynamical states in 4D SQCD. In particular, the 4D graviton and unwanted vector multiplet associated with deformations of the Kähler form of the conifold are absent. However, a single massless BPS hypermultiplet was found at the self-dual point at strong coupling. It is associated with deformations of a complex structure of the conifold and was interpreted as a composite 4D baryon b.[1] Later low-lying massive non-BPS 4D states were found in [15,16] using the little string theory approach, see [17] for a review.

In the previous work [18] a study of possible flux deformations of the 10D background for non-Abelian vortex string was initiated. The goal is to look for flux deformations of the string background which do not destroy $\mathcal{N} = 2$ supersymmetry in 4D and interpret them in terms of certain deformations in SQCD. Fluxes generically induce a potential for CY moduli lifting flat directions (see for example, [19] for a review). It is known that for type-IIA CY compactifications the potential for the Kähler form moduli arise from Ramond-Ramond (RR) even-form fluxes, while the potential for complex structure moduli is induced by the Neveu-Schwarz (NS) 3-form flux H_3 [20,21]. Since for the conifold case at hand the only modulus associated with a physical state is the complex structure modulus b we focus on the NS 3-form flux. It does not break $\mathcal{N} = 2$ supersymmetry in 4D theory [20].

In [18] the NS 3-form flux H_3 was interpreted as switching on quark masses in 4D SQCD. The reason is that the only scalar potential deformation, which is allowed in SQCD by $\mathcal{N} = 2$ supersymmetry is the mass term for quarks. Field theory arguments were used to find a particular choice of nonzero quark masses associated with H_3.

The flux deformation was studied in [18] to the leading order at small H_3 which translates into small values of quark masses. In this paper we study the back reaction of the nonzero 3-form flux on the metric and dilaton. We introduce the ansatz with several warp factors and solve the gravity equations of motion for an arbitrary value of H_3. This allows us to switch on large masses for certain flavors in 4D SQCD and consider the decoupling limit.

Note that there is one puzzling feature of the solitonic string-gauge duality. If $N = N_f/2 \neq 2$ the dimension of the target space of the string sigma model is not equal to ten and it is not clear how one can quantize a non-Abelian string in these cases. The program initiated in [18] and continued in the present paper is an attempt to resolve this puzzle. Upon switching on quark masses and decoupling certain quark flavors one can obtain $\mathcal{N} = 2$ SQCD with different gauge groups and matter content. Studying the string theory response to mass deformations may shed light on the above mentioned puzzle.

We show that 3-form flux produces a potential for the conifold complex structure modulus b, which leads to the runaway vacuum. At the runaway vacuum warp factors disappear, while the deformed conifold degenerates. In 4D SQCD we relate this to the flow to U(1) gauge theory upon switching on quark masses and the decoupling of two flavors.

Note that we assume that the conifold complex structure modulus b is large enough to make sure that the curvature of the conifold is small everywhere. This justifies the gravity approximation.

The paper is organized follows. In Sec. II we briefly review 4D $\mathcal{N} = 2$ SQCD and the world sheet sigma model on the non-Abelian string. Next we review the massless baryon b as a deformation of the complex structure of the conifold. In Sec. III we introduce the metric ansatz and solve gravity equations of motion with nonzero 3-form H_3 in the limit of large radial coordinate of the conifold. In Sec. IV we solve gravity equations for the deformed conifold and calculate the potential for the complex structure modulus b in the large b limit. In Sec. V we interpret H_3-form in terms of quark masses in 4D SQCD. We also discuss the degeneration of the conifold at the runaway vacuum as a flow of 4D SQCD to $\mathcal{N} = 2$ supersymmetric QED (SQED) upon the decoupling of two quark flavors. Section VI summarizes our conclusions.

II. NON-ABELIAN CRITICAL VORTEX STRING

A. Four-dimensional $\mathcal{N} = 2$ SQCD

As was already mentioned, non-Abelian vortex strings were first found in 4D $\mathcal{N} = 2$ SQCD with the gauge group $\mathrm{U}(N)$ and $N_f \geq N$ quark flavors supplemented by the Fayet-Iliopoulos (FI) term [22] with parameter ξ [1–4] (see for example, [8] for a detailed review of this theory). Here, we just mention that at weak coupling $g^2 \ll 1$, this theory is in the Higgs phase in which the scalar components of the quark multiplets (squarks) develop vacuum expectation values (VEVs). These VEVs break the $\mathrm{U}(N)$ gauge group Higgsing all gauge bosons. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N} = 2$ multiplets with mass $m_G \sim g\sqrt{\xi}$.

The global flavor $\mathrm{SU}(N_f)$ is broken down to the so called color-flavor locked group. The resulting global symmetry is

$$\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(N_f - N) \times \mathrm{U}(1)_B, \tag{2.1}$$

(see [8] for more details).

The unbroken global $\mathrm{U}(1)_B$ factor above is identified with a baryonic symmetry. Note that what is usually

[1]The definition of the baryonic charge is nonstandard and will be given below in Sec. II.

identified as the baryonic U(1) charge is a part of our 4D theory gauge group. "Our" $U(1)_B$ is an unbroken by the squark VEVs combination of two U(1) symmetries; the first is a subgroup of the flavor $SU(N_f)$ and the second is the global U(1) subgroup of $U(N)$ gauge symmetry.

As was already noted, we consider $\mathcal{N}=2$ SQCD in the Higgs phase; N squarks condense. Therefore, non-Abelian vortex strings confine monopoles. In the $\mathcal{N}=2$ 4D theory these strings are $1/2$ BPS-saturated; hence, their tension is determined exactly by the FI parameter,

$$T = 2\pi\xi. \tag{2.2}$$

However, as we already mentioned, non-Abelian strings cannot be broken, therefore monopoles cannot be attached to the string end points. In fact, in the $U(N)$ theories confined monopoles are junctions of two distinct elementary non-Abelian strings [3,4,23] (see [8] for a review). As a result, in four-dimensional $\mathcal{N}=2$ SQCD we have monopole-antimonopole mesons in which the monopole and antimonopole are connected by two confining strings. In addition, in the $U(N)$ gauge theory we can have baryons appearing as a closed "necklace" configurations of $N\times(\text{integer})$ monopoles [8]. For the U(2) gauge group the massless BPS baryon b found from string theory in [11] consists of four monopoles [24].

Below we focus on the particular case $N=2$ and $N_f=4$ because, as was mentioned in the Introduction, in this case 4D $\mathcal{N}=2$ SQCD supports non-Abelian vortex strings which behave as critical superstrings [10]. Also, for $N_f=2N$ the gauge coupling g^2 of the 4D SQCD does not run; the β function vanishes. However, the conformal invariance of the 4D theory is explicitly broken by the FI parameter ξ, which defines the VEVs of quarks. The FI parameter is not renormalized.

Both stringy monopole-antimonopole mesons and monopole baryons with spins $J\sim 1$ have masses determined by the string tension ($\sim\sqrt{\xi}$) and are heavier at weak coupling $g^2\ll 1$ than perturbative states with masses $m_G\sim g\sqrt{\xi}$. Thus, they can decay into perturbative states,[2] and in fact at weak coupling we do not expect them to appear as stable states.

Only in the strong coupling domain $g^2\sim 1$ we expect that (at least some of) stringy mesons and baryons become stable. These expectations were confirmed in [11,15] where low-lying string states in the string theory for the critical non-Abelian vortex were found at the self-dual point at strong coupling.

In this paper we introduce quark masses m_A, $A=1,\dots 4$ assuming that two first squark flavors with masses m_1 and m_2 develop VEVs.

[2]Their quantum numbers with respect to the global group (2.1) allows these decays, see [8].

B. World sheet sigma model

The presence of the color-flavor locked group $SU(N)_{C+F}$ is the reason for the formation of non-Abelian vortex strings [1–4]. The most important feature of these vortices is the presence of the orientational zero modes. As was already mentioned, in $\mathcal{N}=2$ SQCD these strings are $1/2$ BPS saturated and preserve $\mathcal{N}=(2,2)$ supersymmetry on the world sheet.

Let us briefly review the model emerging on the world sheet of the non-Abelian string [8].

The translational moduli fields are described by the Nambu-Goto action and decouple from all other moduli. Below we focus on internal moduli.

If $N_f=N$, the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, are described by the 2D $\mathcal{N}=(2,2)$ supersymmetric $\mathbb{CP}(N-1)$ model.

If one adds additional quark flavors, the non-Abelian vortices become semilocal—they acquire size moduli [25]. In particular, for the non-Abelian semilocal vortex in U(2) $\mathcal{N}=2$ SQCD with four flavors, in addition to the complex orientational moduli n^P (here $P=1,2$), we must add two complex size moduli ρ^K (where $K=3,4$), see [1,4,25–28].

The effective theory on the string world sheet is a two-dimensional $\mathcal{N}=(2,2)$ supersymmetric $\mathbb{WCP}(2,2)$ model (see review [8] for details). This model can be defined as a low-energy limit of the U(1) gauge theory [29]. The fields n^P and ρ^K have charges $+1$ and -1 with respect to the U(1) gauge field. The target space of the $\mathbb{WCP}(2,2)$ model is defined by the D-term condition

$$|n^P|^2-|\rho^K|^2=\mathrm{Re}\beta, \qquad P=1,2, \qquad K=3,4. \tag{2.3}$$

The number of real bosonic degrees of freedom in the model $\mathbb{WCP}(2,2)$ is $8-1-1=6$. Here 8 is the number of real degrees of freedom of n^P and ρ^K fields and we subtracted one real constraint imposed by the D term condition in (2.3) and one gauge phase eaten by the Higgs mechanism. As we already mentioned, these six internal degrees of freedom in the massless limit can be combined with four translational moduli to form a 10D space needed for a superstring to be critical.

The global symmetry of the world sheet $\mathbb{WCP}(2,2)$ model is

$$SU(2)\times SU(2)\times U(1)_B, \tag{2.4}$$

i.e., exactly the same as the unbroken global group in the 4D theory at $N=2$ and $N_f=4$. The fields n and ρ transform in the following representations,

$$n:\ \left(\mathbf{2},\mathbf{1},\frac{1}{2}\right), \qquad \rho:\ \left(\mathbf{1},\mathbf{2},\frac{1}{2}\right). \tag{2.5}$$

Here the global "baryonic" $U(1)_B$ group rotates n and ρ fields with the same phase (see [11] for details).

Twisted masses of n^P and ρ^K fields coincide with quark masses of 4D SQCD and are given respectively by m_P and m_K, $P = 1, 2$ and $K = 3, 4$, see [8]. Nonzero twisted masses m_A break each of the SU(2) factors in (2.4) down to U(1).

The 2D coupling constant Reβ can be naturally complexified to the complex coupling constant β if we include the θ term in the action [29]. At the quantum level, the coupling β does not run in this theory. Thus, the $\mathbb{WCP}(2,2)$ model is superconformal at zero masses $m_A = 0$. Therefore, its target space is Ricci flat and [being Kähler due to $\mathcal{N} = (2,2)$ supersymmetry] represents a noncompact Calabi-Yau manifold, namely the conifold Y_6 (see [13] for a review).

The $\mathbb{WCP}(2,2)$ model with $m_A = 0$ was used in [10,11] to define the critical string theory for the non-Abelian vortex at hand.

Typically, solitonic strings are "thick" and the effective world sheet theory has a series of unknown high-derivative corrections in powers of ∂/m_G. The string transverse size is given by $1/m_G$, where $m_G \sim g\sqrt{\xi}$ is a mass scale of the gauge bosons and quarks forming the string. The string cannot be thin in a weakly coupled 4D SQCD because at weak coupling $m_G \sim g\sqrt{T}$ and m_G^2 is always small in the units of the string tension T, see (2.2).

A conjecture was put forward in [10] that at strong coupling in the vicinity of a critical value $g_c^2 \sim 1$ the non-Abelian string in the theory at hand becomes thin, and higher-derivative corrections in the world sheet theory are absent. This is possible because the low energy $\mathbb{WCP}(2,2)$ model already describes a critical string and higher-derivative corrections are not required to improve its ultraviolet behavior (see [30] for the discussion of this problem). The above conjecture implies that $m_G(g^2) \to \infty$ at $g^2 \to g_c^2$. As expected the thin string produces linear Regge trajectories even for small spins [16].

It was also conjectured in [11] that g_c corresponds to the value of the 2D coupling constant $\beta = 0$. The motivation for this conjecture is that this value is a self-dual point for the $\mathbb{WCP}(2,2)$ model. Also $\beta = 0$ is a natural choice because at this point we have a regime change in the $\mathbb{WCP}(2,2)$ model. The resolved conifold defined by the D term condition (2.3) develops a conical singularity at this point. The point $\beta = 0$ corresponds to $\tau_{SW} = 1$ in the 4D SQCD, where τ_{SW} is the complexified inverse coupling, $\tau_{SW} = i\frac{8\pi}{g^2} + \frac{\theta_{4D}}{\pi}$, where θ_{4D} is the 4D θ angle [24].

The above conjecture cannot be proven at the moment because we deal with a strong coupling regime. However, it passes very general important tests [11]. In particular, 4D gravitons are absent after "compactification" due to the non-normalizability of its wave function over the noncompact conifold. This result matches our expectations since we started with $\mathcal{N} = 2$ SQCD in the flat four-dimensional space without gravity. Also, the fact that non-Abelian vortex strings in $U(N)$ theories are closed strings leads to type-II string theory. Upon compactification on 6D CY space this ensures required $\mathcal{N} = 2$ supersymmetry in 4D SQCD.

Moreover, in [24] we showed the presence of the massless BPS baryon b in 4D SQCD at strong coupling at $\beta = 0$ using purely field theory arguments in the agreement with the result found from the string theory [11]. These successful tests strongly support the string theory on the conifold as a dual theory for our 4D SQCD.

As we already mentioned in the Introduction a solitonic string-gauge duality proposed in [10,11] for 4D SQCD implies that at weak coupling this theory is in the Higgs phase and can be described in terms of quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as closed string states in the string theory on $\mathbb{R}^4 \times Y_6$.

Nonzero twisted masses $m_A \neq 0$ define a mass deformation of the superconformal CY theory on the conifold. Generically, quark masses break the world sheet conformal invariance. The $\mathbb{WCP}(2,2)$ model with nonzero m_A can no longer be used to define a string theory for the non-Abelian vortex in the massive 4D SQCD.

C. Massless 4D baryon

In this section we briefly review the only 4D massless state found in the string theory of the critical non-Abelian vortex in the massless limit [11]. It is associated with the deformation of the conifold complex structure. As was already mentioned, all other massless string modes have non-normalizable wave functions over the conifold. In particular, 4D graviton associated with a constant wave function over the conifold Y_6 is absent as expected [11].

We can construct the U(1) gauge-invariant "mesonic" variables

$$w^{PK} = n^P \rho^K. \tag{2.6}$$

These variables are subject to the constraint

$$\det w^{PK} = 0. \tag{2.7}$$

Equation (2.7) defines the conifold Y_6. It has the Kähler Ricci-flat metric and represents a noncompact Calabi-Yau manifold [12,13,29]. It is a cone which can be parametrized by the noncompact radial coordinate

$$\tilde{r}^2 = \mathrm{Tr}\bar{w}w \tag{2.8}$$

and five angles (see [12]). Its section at fixed $\tilde{r}$ is $S_2 \times S_3$.

At $\beta = 0$ the conifold develops a conical singularity, so both spheres S_2 and S_3 can shrink to zero. The conifold singularity can be smoothed out in two distinct ways; by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to keeping a nonzero value of β in (2.3). This resolution preserves the Kähler structure and Ricci-flatness of the metric. If we put $\rho^K = 0$ in (2.3) we get the $\mathbb{CP}(1)$ model with the sphere S_2 as a target space (with the radius $\sqrt{\beta}$). The resolved conifold has no normalizable zero modes. In particular, the modulus β which becomes a scalar field in four dimensions has a non-normalizable wave function over the Y_6 and therefore is not dynamical [11].

If $\beta = 0$ another option exists, namely a deformation of the complex structure [13]. It preserves the Kähler structure and Ricci-flatness of the conifold and is usually referred to as the deformed conifold. It is defined by the deformation of Eq. (2.7); namely,

$$\det w^{PK} = b, \tag{2.9}$$

where b is a complex parameter. Now the sphere S_3 cannot shrink to zero, its minimal size is determined by b.

The modulus b becomes a 4D complex scalar field. The effective action for this field was calculated in [11] using the explicit metric on the deformed conifold [12,31,32],

$$S_{\rm kin}(b) = T\int d^4x |\partial_\mu b|^2 \log\frac{\tilde{R}^2_{\rm IR}}{|b|}, \tag{2.10}$$

where $\tilde{R}_{\rm IR}$ is the maximal value of the radial coordinate $\tilde{r}$ introduced as an infrared regularization of the logarithmically divergent b-field norm. Here the logarithmic integral at small $\tilde{r}$ is cut off by the minimal size of S_3, which is equal to $|b|$.

To avoid confusion we note that in AdS/CFT correspondence the radial coordinate of internal dimensions has an interpretation of energy. The large values of this coordinate correspond to the ultraviolet region. In our approach it is vice versa. The radial coordinate $\tilde{r}$ measures absolute values of products $n^P\rho^K$ and since ρ's are vortex string size moduli [25] $\tilde{r}$ has a 4D interpretation as a distance from the string axis. In particular, large $\tilde{r}$ corresponds to the infrared region.

We see that the norm of the modulus b turns out to be logarithmically divergent in the infrared. The modes with the logarithmically divergent norm are at the borderline between normalizable and non-normalizable modes. Usually such states are considered as "localized" in 4D. We follow this rule. This scalar mode is localized near the conifold singularity in the same sense as the orientational and size zero modes are localized on the vortex string solution (see [28]).

The field b being massless can develop a VEV. Thus, we have a new Higgs branch in 4D $\mathcal{N} = 2$ SQCD which is developed only for the critical value of the 4D coupling constant $\tau_{SW} = 1$ associated with $\beta = 0$.

In [11] the massless state b was interpreted as a baryon of 4D $\mathcal{N} = 2$ QCD. Let us explain this. From Eq. (2.9) we see that the complex parameter b (which is promoted to a 4D scalar field) is a singlet with respect to both SU(2) factors in (2.4), i.e., the global world sheet group.[3] What about its baryonic charge? From (2.5) and (2.9) we see that the b state transforms as

$$(\mathbf{1}, \mathbf{1}, 2). \tag{2.11}$$

In particular it has the baryon charge $Q_B(b) = 2$.

In type-IIA superstring compactifications the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters as a 4D $\mathcal{N} = 2$ BPS hypermultiplet (see [19] for a review).

On the field theory side we know that if we switch on generic quark masses in 4D SQCD the b-baryon becomes massive. Since it is a BPS state its mass is dictated by its baryonic charge [24],

$$m_b = |m_1 + m_2 - m_3 - m_4|. \tag{2.12}$$

To conclude this section let us present the explicit metric of the singular conifold (with both β and b equal to zero), which will be used in the next section. It has the form [12]

$$ds_6^2 = dr^2 + \frac{r^2}{6}(e^2_{\theta_1} + e^2_{\varphi_1} + e^2_{\theta_2} + e^2_{\varphi_2}) + \frac{r^2}{9}e^2_\psi, \tag{2.13}$$

where

$$\begin{aligned} e_{\theta_1} &= d\theta_1, \qquad e_{\varphi_1} = \sin\theta_1 d\varphi_1, \\ e_{\theta_2} &= d\theta_2, \qquad e_{\varphi_2} = \sin\theta_2 d\varphi_2, \\ e_\psi &= d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2. \end{aligned} \tag{2.14}$$

Here r is another radial coordinate on the cone while the angles above are defined at $0 \le \theta_{1,2} < \pi$, $0 \le \varphi_{1,2} < 2\pi$, and $0 \le \psi < 4\pi$.

The volume integral associated with this metric is

$$({\rm Vol})_{Y_6} = \frac{1}{108}\int r^5 dr\, d\psi d\theta_1 \sin\theta_1 d\varphi_1 d\theta_2 \sin\theta_2 d\varphi_2. \tag{2.15}$$

The radial coordinate, $\tilde{r}$ defined in terms of matrix w^{PK}, [see (2.8)] is related to r in (2.13) via [12]

$$r^2 = \frac{3}{2}\tilde{r}^{4/3}. \tag{2.16}$$

[3]This is isomorphic to the 4D global group (2.1) for $N = 2$, $N_f = 4$.

III. GRAVITY EQUATIONS IN THE LARGE r LIMIT

Below we switch on NS 3-form flux H_3 and study its backreaction on the metric and the dilaton solving the gravity equations of motion. As we already mentioned in the Introduction H_3 flux produces a potential lifting the flat direction associated with the conifold complex structure modulus b. We confirm the result obtained in [18] for this potential.

In this section we start with the large r limit and show that the geometry is smooth and that metric warp factors do not develop singularities at $r \to \infty$. Large r limit means that $r \gg |b|^{1/3}$ [see (2.16)] so for $H_3 = 0$ we can use the metric of the singular conifold (2.13).

A. The setup

The bosonic part of the action of the type-IIA supergravity in the Einstein frame is given by

$$S_{10D} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{R - \frac{1}{2}G^{MN}\partial_M\Phi\partial_N\Phi - \frac{e^{-\Phi}}{12}H_{MNL}H^{MNL}\right\}, \tag{3.1}$$

where G_{MN} and Φ are 10D metric and dilaton, the string coupling $g_s = e^{\Phi}$. We also keep only NS 2-form B_2 with the field strength $H_3 = dB_2$. We do not consider RR forms here, in particular, the RR 3-form potential C_3. For compact CYs the mass term for complex structure moduli can be generated via topological term $\int \frac{1}{2} H_3 \wedge C_3 \wedge dC_3$ in the action [20]. However, it was shown in [18] that for the noncompact case of the conifold this mechanism does not work due to the non-normalizability of the 4D part of C_3.

Einstein's equations of motion following from the action (3.1) have the form

$$R_{MN} = \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{e^{-\Phi}}{4}H_{MAB}H_N^{AB} - \frac{e^{-\Phi}}{48}G_{MN}H_3^2, \tag{3.2}$$

while the equation for the dilaton reads

$$G^{MN}D_M D_N\Phi + \frac{e^{-\Phi}}{12}H_3^2 = 0. \tag{3.3}$$

Finally the equation for the NS 3-form is

$$d(e^{-\Phi} * H_3) = 0, \tag{3.4}$$

where $*$ denotes the Hodge star.

We will see below that we need to introduce four warp factors to solve the Einstein equations. Our ansatz for the metric is

$$ds_{10}^2 = T h_4^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}dx^m dx^n, \tag{3.5}$$

where $\mu, \nu = 0, \ldots, 3$ are indices of the 4D space and $\eta_{\mu\nu}$ is the flat Minkowski metric with signature $(-1, 1, 1, 1)$, while $m, n = 5, \ldots 10$ are indices of the 6D internal space. Here, internal coordinates x^m are defined to be dimensionless to match the dimension of scalar fields in the world sheet $\mathbb{WCP}(2,2)$ model. We also introduced the string tension T [see (2.2)] in (3.5) to fix dimensions.

The internal space has a conifold metric deformed by three warp factors

$$g_{mn}dx^m dx^n = h_6^{1/2}(r)\left\{a(r)dr^2 + \frac{r^2}{6}(e_{\theta_1}^2 + e_{\varphi_1}^2 + e_{\theta_2}^2 + e_{\varphi_2}^2) + \frac{r^2}{9}\omega(r)e_\psi^2\right\}, \tag{3.6}$$

[see (2.13)] and we assume that warp factors h_4, h_6, a, and ω depend only on the radial coordinate r. If $H_3 = 0$ all warp factors are equal to unity and the 10D space has the structure $\mathbb{R}^4 \times Y_6$.

B. NS 3-form at large r

We will see below that the solution of the gravity equations of motion in the large r limit can be expanded in powers of μ^2/r^4, where μ parametrizes the H_3 flux. To find its behavior we can use a perturbation theory in powers of the above parameter. At the first step we solve equations of motion for H_3 form using the undeformed conifold metric. This was done in [18].

Let us define two real 3-forms on Y_6,

$$\alpha_3 \equiv \frac{dr}{r} \wedge (e_{\theta_1} \wedge e_{\varphi_1} - e_{\theta_2} \wedge e_{\varphi_2}) \tag{3.7}$$

and

$$\beta_3 \equiv e_\psi \wedge (e_{\theta_1} \wedge e_{\varphi_1} - e_{\theta_2} \wedge e_{\varphi_2}). \tag{3.8}$$

They are both closed [33,34],

$$d\alpha_3 = 0, \qquad d\beta_3 = 0, \tag{3.9}$$

Moreover, using the conifold metric (2.13) to the leading order one can check that their 10D duals are given by

$$*\alpha_3 \approx -\frac{T^2}{3}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \beta_3,$$
$$*\beta_3 \approx 3T^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \alpha_3. \tag{3.10}$$

The above relations ensure that both 10D dual forms are also closed.

$$d * \alpha_3 = 0, \qquad d * \beta_3 = 0. \tag{3.11}$$

Two solutions for H_3 form found in [18] are

$$H_3 \approx \mu_1\alpha_3 + \frac{\mu_2}{3}\beta_3, \tag{3.12}$$

where μ_1 and μ_2 are two independent real parameters, while the factor $\frac{1}{3}$ is introduced for convenience. This H_3 form satisfy both the Bianchi identity and the equations of motion (3.4), where the dilaton is considered as a constant to the leading order in μ^2/r^4.

3-Forms (3.7) and (3.8) form a basis similar to the simplectic basis of harmonic α and β3-forms for compact CYs (see for example review [19]). In particular,

$$\int_{Y_6} \alpha_3 \wedge \alpha_3 = \int_{Y_6} \beta_3 \wedge \beta_3 = 0, \qquad (3.13)$$

while

$$\int_{Y_6} \alpha_3 \wedge \beta_3 \sim -\int \frac{dr}{r} \sim -\log \frac{R_{\rm IR}^3}{|b|}. \qquad (3.14)$$

Here $R_{\rm IR}$ is the maximal value of the radial coordinate r introduced to regularize the infrared logarithmic divergence, while at small r the integral is cut off by the minimal size of S_3 which is equal to $|b|$. Note that this logarithm is similar to the one, which determines the metric for the b-baryon in (2.10).[4]

C. Warp factors at large r

For Minkowski indices μ, $\nu = 0, 1, 2, 3$ the Einstein's equations (3.2) read

$$R_{\mu\nu} = -\frac{\eta_{\mu\nu}}{48} \frac{e^{-\Phi}}{h_4^{1/2}} H_3^2, \qquad (3.15)$$

where the Ricci components for the ansatz (3.5), (3.6) can be calculated using results of [35]

$$R_{\mu\nu} = \frac{\eta_{\mu\nu}}{4ah_4^{1/2}h_6^{1/2}} \left\{ \frac{1}{h_4}\Delta h_4 + \frac{h_6' h_4'}{h_6 h_4} - 2\frac{(h_4')^2}{h_4^2} - \frac{1}{2}\frac{a' h_4'}{a h_4} + \frac{1}{2}\frac{\omega' h_4'}{\omega h_4} \right\}. \qquad (3.16)$$

Here, prime denotes the derivative with respect to r and Δ is the Laplacian calculated using the conifold metric (2.13).

Using expression in (3.16) we can compare Einstein's equations for Minkowski indices (3.15) with the dilaton equation (3.3). Rewriting the latter one as

$$\Delta\Phi + \left(\frac{h_6'}{h_6} - \frac{h_4'}{h_4} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{\omega'}{\omega}\right)\Phi' = -\frac{e^{-\Phi}}{12} a h_6^{1/2} H_3^2 \qquad (3.17)$$

it is easy to see that it is identical to Eq. (3.15) upon substitution

$$\Phi = \Phi_0 + \ln h_4, \qquad (3.18)$$

where Φ_0 is a constant value of the dilaton present at $H_3 = 0$.

Let us now continue studying Eq. (3.15). At the first nontrivial order in the parameter μ^2/r^4 all nonlinearities in the expression in (3.16) can be neglected and it reduces simply to

$$R_{\mu\nu} \approx \frac{\eta_{\mu\nu}}{4} \Delta h_4. \qquad (3.19)$$

This gives (for the Minkowski part of Einstein's equations)

$$\Delta h_4 \approx -\frac{e^{-\Phi_0}}{12} H_3^2, \qquad (3.20)$$

where H_3^2 can be calculated using the conifold metric and we used only the constant part of the dilaton Φ_0 at this order. We have

$$e^{-\Phi_0} H_3^2 = 3!72 \frac{\mu_1^2 + \mu_2^2}{g_s} \frac{1}{r^6} = 2^4 3^3 \frac{\mu_1^2 + \mu_2^2}{g_s} \frac{1}{r^6} \qquad (3.21)$$

where $g_s = e^{\Phi_0}$, while $72/r^4$ say, for the first solution for H_3 [proportional to μ_1 in (3.12)] comes from $g^{\theta_1\theta_1} g^{\varphi_1\varphi_1}$ and $g^{\theta_2\theta_2} g^{\varphi_2\varphi_2}$.

Then Eq. (3.20) gives

$$h_4 = 1 + \frac{9}{g_s} \frac{\mu_1^2 + \mu_2^2}{r^4} \log \frac{r}{|b|^{1/3}} + O(\mu^4/r^8), \qquad (3.22)$$

up to a nonlogarithmic term proportional to μ^2/r^4 which we set to zero.

Consider now Einstein's equations with internal indices. Let index α (β) denote differentials $e_{\theta_1}, e_{\varphi_1}, e_{\theta_2}, e_{\varphi_2}$. Then we can calculate Christoffel symbols with r indices, namely

$$\Gamma^r_{\alpha\beta} = -\frac{g^{(c)}_{\alpha\beta}}{a}\left(\frac{1}{r} + \frac{1}{4}\frac{h_6'}{h_6}\right), \quad \Gamma^r_{\psi\psi} = -\frac{g^{(c)}_{\psi\psi}}{a}\left(\frac{1}{r} + \frac{1}{4}\frac{h_6'}{h_6} + \frac{1}{2}\omega'\right),$$
$$\Gamma^\beta_{r\alpha} = \Gamma^\beta_{\alpha r} = \delta^\beta_\alpha \left(\frac{1}{r} + \frac{1}{4}\frac{h_6'}{h_6}\right), \quad \Gamma^\psi_{r\psi} = \Gamma^\psi_{\psi r} = \frac{1}{r} + \frac{1}{4}\frac{h_6'}{h_6} + \frac{1}{2}\omega',$$
$$\Gamma^r_{rr} = \frac{1}{2}\frac{a'}{a} + \frac{1}{4}\frac{h_6'}{h_6}, \quad \Gamma^n_{rr} = \Gamma^r_{rn} = \Gamma^r_{nr} = 0, \quad n \neq r, \qquad (3.23)$$

where $g^{(c)}_{\alpha\beta}$ and $g^{(c)}_{\psi\psi}$ denote the conifold metric (2.13).

Using these formulas we find nonzero Ricci components at the leading order in μ^2/r^4. We have

[4] Note that $R_{\rm IR}^3 \sim \tilde{R}_{\rm IR}^2$ [see (2.16)].

$$R_{\alpha\beta} \approx g^{(c)}_{\alpha\beta}\left\{\frac{4}{r^2}(a-1) - \frac{1}{4}\Delta h_6 - \frac{1}{r}h_6' + \frac{1}{r}h_4' + \frac{1}{2r}a' - \frac{1}{2r}\omega'\right\},$$
$$R_{\psi\psi} \approx g^{(c)}_{\psi\psi}\left\{\frac{4}{r^2}(a-1) - \frac{1}{4}\Delta h_6 - \frac{1}{r}h_6' + \frac{1}{r}h_4' + \frac{1}{2r}a' - \frac{2}{r}\omega' - \frac{1}{2}\omega''\right\},$$
$$R_{rr} \approx -\frac{1}{4}\Delta h_6 - h_6'' + h_4'' + \frac{5}{2r}a' - \frac{1}{r}\omega' - \frac{1}{2}\omega''. \quad (3.24)$$

Here we used that Ricci tensor is zero if all warp factors are equal to unity and the dependence on h_4 can be found using formulas in [35].

Now calculating rhs for Einstein's equations (3.2) we get for the first solution proportional to μ_1 in (3.12)

$$R_{\alpha\beta} = \frac{1}{48} g^{(c)}_{\alpha\beta} e^{-\Phi_0}(H_3^{(1)})^2,$$
$$R_{\psi\psi} = -\frac{g^{(c)}_{\psi\psi}}{48} e^{-\Phi_0}(H_3^{(1)})^2,$$
$$R_{rr} = \frac{1}{16} e^{-\Phi_0}(H_3^{(1)})^2, \quad (3.25)$$

where $(H_3^{(1)})^2$ is given by (3.21) with $\mu_2 = 0$.

For the second solution in (3.12) (proportional to μ_2) we have

$$R_{\alpha\beta} = \frac{1}{48} g^{(c)}_{\alpha\beta} e^{-\Phi_0}(H_3^{(2)})^2,$$
$$R_{\psi\psi} = \frac{g^{(c)}_{\psi\psi}}{16} e^{-\Phi_0}(H_3^{(2)})^2,$$
$$R_{rr} = -\frac{1}{48} e^{-\Phi_0}(H_3^{(2)})^2, \quad (3.26)$$

where $(H_3^{(2)})^2$ is given by (3.21) with $\mu_1 = 0$.

The above equations together with expressions (3.24) and solution for h_4 (3.22) determine three warp factors h_6, a, and ω at the leading order. For the first solution for H_3 we have

$$h_6^{(1)} = 1 + \frac{9}{g_s}\frac{\mu_1^2}{r^4}\log\frac{r}{|b|^{1/3}} - \frac{9}{5}\frac{1}{g_s}\frac{\mu_1^2}{r^4} + \cdots,$$
$$a^{(1)} = 1 - \frac{9}{10}\frac{1}{g_s}\frac{\mu_1^2}{r^4} + \cdots,$$
$$\omega^{(1)} = 1 + \frac{9}{2}\frac{1}{g_s}\frac{\mu_1^2}{r^4} + \cdots, \quad (3.27)$$

where dots stand for subleading terms of order of μ^4/r^8. Warp factors for the second solution for H_3 have the form

$$h_6^{(2)} = 1 + \frac{9}{g_s}\frac{\mu_2^2}{r^4}\log\frac{r}{|b|^{1/3}} + \frac{9}{5}\frac{1}{g_s}\frac{\mu_2^2}{r^4} + \cdots,$$
$$a^{(2)} = 1 + \frac{9}{10}\frac{1}{g_s}\frac{\mu_2^2}{r^4} + \cdots,$$
$$\omega^{(2)} = 1 - \frac{9}{2}\frac{1}{g_s}\frac{\mu_2^2}{r^4} + \cdots. \quad (3.28)$$

Finally, solutions (3.18) and (3.22) give for the dilaton

$$e^{(\Phi-\Phi_0)} = 1 + \frac{9}{g_s}\frac{\mu_1^2+\mu_2^2}{r^4}\log\frac{r}{|b|^{1/3}} + \cdots. \quad (3.29)$$

We see that warp factors and the dilaton have smooth behavior at large r and can be found order by order in the parameter μ^2/r^4 using perturbation theory in the gravity equations. The region of validity of the above solutions is

$$r \gg |b|^{1/3} \gg \mu^{1/2}. \quad (3.30)$$

To conclude this section we would like to comment on a subtlety in solving Eqs. (3.25) and (3.26). In fact, these equations do not determine coefficients in nonlogarithmic terms proportional to $1/r^4$ for h_6 and a separately. Denoting these coefficients χ and A respectively we find that the first and the third equations in (3.25) and (3.26) give the same conditions for them, namely

$$\chi^{(1)} + \frac{A^{(1)}}{2} = -\frac{9}{4}\frac{\mu_1^2}{g_s}, \qquad \chi^{(2)} + \frac{A^{(2)}}{2} = \frac{9}{4}\frac{\mu_2^2}{g_s}, \quad (3.31)$$

for (3.25) and (3.26), respectively. The resolution of this puzzle is related to the possibility of redefinition of the conifold radial coordinate r. Let us put $H_3 = 0$ so the metric is reduced to the conifold one in (2.13). However, we can redefine r at the relevant order,

$$r = f(r') = r'\left(1 + \frac{\alpha}{r'^4}\right), \quad (3.32)$$

where α is a constant. This gives

$$r^2 \approx r'^2\left(1 + \frac{2\alpha}{r'^4}\right), \qquad dr^2 \approx dr'^2\left(1 - \frac{6\alpha}{r'^4}\right), \quad (3.33)$$

which in terms of the new coordinate r' imply nontrivial warp factors

$$h_6 = 1 + \frac{4\alpha}{r'^4}, \qquad a = 1 - \frac{8\alpha}{r'^4}, \quad (3.34)$$

or nonzero coefficients

$$\chi = 4\alpha, \qquad A = -8\alpha. \tag{3.35}$$

Now we see that the combination which enters Eqs. (3.31) is zero on this solution,

$$\chi + \frac{A}{2} = 0. \tag{3.36}$$

Thus, nontrivial solutions of the above equation are related to the possibility of r redefinition.

To fix the definition of r we require that the combination orthogonal to the one which enters (3.36) should be zero, namely

$$\frac{\chi}{2} - A = 0. \tag{3.37}$$

This condition together with Eq. (3.31) gives coefficients

$$\chi^{(1)} = -\frac{9}{5}\frac{\mu_1^2}{g_s}, \qquad A^{(1)} = -\frac{9}{10}\frac{\mu_1^2}{g_s},$$
$$\chi^{(2)} = \frac{9}{5}\frac{\mu_2^2}{g_s}, \qquad A^{(2)} = \frac{9}{10}\frac{\mu_2^2}{g_s}, \tag{3.38}$$

for two solutions for H_3 respectively, which we presented in (3.27) and (3.28).

D. The scalar potential

To find the scalar potential induced by 3-form flux H_3 we substitute the solution of the gravity equations found above into the 10D action (3.1). The trace of the Einstein's equations (3.2) reads

$$R - \frac{1}{2}G^{MN}\partial_M\Phi\partial_N\Phi - \frac{e^{-\Phi}}{12}H_3^2 = 0. \tag{3.39}$$

Substituting this into Eq. (3.1) we get the action calculated on the solution,

$$S_{10D} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{-\frac{e^{-\Phi}}{24}H_3^2\right\}, \tag{3.40}$$

where $2\kappa^2 = (2\pi)^3 g_s^2$ in our convention.

This leads to the potential for the b-baryon (complex structure modulus b of the conifold) in 4D SQCD,

$$V(b) = \frac{T^2}{(2\pi)^3 g_s^2}\int d^6x\sqrt{g_6}\frac{e^{-\Phi}}{24}H_3^2, \tag{3.41}$$

where the string tension T appears due to our normalization of the Minkowski part of the metric [see (3.5)]. Here the integral is taken over the internal 6D space and g_6 is the determinant of the 6D metric. To the leading order we can neglect warp factors and calculate the above integral using the conifold metric (2.13). Using Eqs. (2.15) and (3.21) we get

$$V(b) = \frac{4}{3}\frac{T^2}{g_s^3}(\mu_1^2 + \mu_2^2)\int\frac{dr}{r}$$
$$= \frac{4}{9}\frac{T^2}{g_s^3}(\mu_1^2 + \mu_2^2)\log\frac{R_{\rm IR}^3}{|b|}, \tag{3.42}$$

where $R_{\rm IR}$ is the infrared cutoff for the radial coordinate r, while modulus b plays the role of the ultraviolet cutoff at small r, cf. (3.14). This potential was calculated in [18]. Note, that the same infrared logarithm determines the metric (2.10) for the b-baryon. If we take into account warp factors in the integrand in (3.41) this would give finite corrections to the potential of order of

$$T^2\frac{\mu^4}{|b|^{4/3}}, \tag{3.43}$$

which are negligible compared to the logarithmic term.

We see that the Higgs branch for b is lifted by H_3 flux deformation and we have a runaway vacuum with VEV

$$\langle|b|\rangle \to R_{\rm IR}^3 \to \infty. \tag{3.44}$$

However, our solution of gravity equations is found in this section using the metric of the singular conifold and therefore is valid at $r \gg |b|^{1/3}$. Thus, the potential (3.42) cannot be trusted at $|b| \sim R_{\rm IR}^3$ where the logarithm becomes small. In the next section we consider the region of $r \sim |b|^{1/3}$ and confirm our conclusion in (3.44) that the VEV of the baryon b tends to infinity.

IV. GRAVITY EQUATIONS FOR THE DEFORMED CONIFOLD

The result for the potential (3.42) suggests that we have a runaway vacuum and VEV of b becomes infinitely large. To confirm this, in this section we study gravity equations with nonzero H_3-flux on the deformed conifold assuming that the radial coordinate $r \sim |b|^{1/3}$. Anticipating the runaway behavior (3.44) we still keep the second condition in (3.30),

$$\mu \ll |b|^{2/3}. \tag{4.1}$$

A. Metric of the deformed conifold

In this section we briefly review the metric of the deformed conifold. It has the form [12,31,32]

$$ds_6^2 = \frac{1}{2}|b|^{2/3}K(\tau)\left\{\frac{1}{3K^3(\tau)}(d\tau^2 + e_\psi^2) + \cosh^2\frac{\tau}{2}(g_3^2 + g_4^2) + \sinh^2\frac{\tau}{2}(g_1^2 + g_2^2)\right\}, \tag{4.2}$$

where the angle differentials are defined as

$$g_1 = -\frac{1}{\sqrt{2}}(e_{\phi_1} + e_3), \qquad g_2 = \frac{1}{\sqrt{2}}(e_{\theta_1} - e_4),$$
$$g_3 = -\frac{1}{\sqrt{2}}(e_{\phi_1} - e_3), \qquad g_4 = \frac{1}{\sqrt{2}}(e_{\theta_1} + e_4), \tag{4.3}$$

while

$$e_3 = \cos\psi \sin\theta_2 d\varphi_2 - \sin\psi d\theta_2,$$
$$e_4 = \sin\psi \sin\theta_2 d\varphi_2 + \cos\psi d\theta_2, \tag{4.4}$$

—see also (2.14).

Here

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \tag{4.5}$$

and the new radial coordinate τ is defined as

$$\tilde{r}^2 = |b|\cosh\tau = \left(\frac{2}{3}\right)^{\frac{3}{2}} r^3. \tag{4.6}$$

In the limit of large τ the metric (4.2) reduces to the metric (2.13) of the singular conifold.

Results of the previous section show that we have a runaway vacuum with $|b| \sim R_{\rm IR}^3$ so we are interested in the metric (4.2) in the limit of small τ, $\tau \ll 1$. In this limit the metric of the deformed conifold takes the form

$$ds_6^2|_{\tau\to 0} = \frac{1}{2}|b|^{2/3}\left(\frac{2}{3}\right)^{\frac{1}{3}}\left\{\frac{1}{2}d\tau^2 + \frac{1}{2}e_\psi^2 + g_3^2 + g_4^2 + \frac{\tau^2}{4}(g_1^2 + g_2^2)\right\}. \tag{4.7}$$

The last term here corresponds to the collapsing sphere S_2, while the sphere S_3 associated with three angular terms in the first line has a fixed radius in the limit $\tau \to 0$ [12,32]. The radial coordinate r approaches its minimal value with

$$r^3|_{\min} = \left(\frac{3}{2}\right)^{\frac{3}{2}}|b| \tag{4.8}$$

at $\tau = 0$.

The square root of the determinant of the metric

$$\sqrt{g_6} \sim |b|^2 \cosh^2\frac{\tau}{2}\sinh^2\frac{\tau}{2}\Big|_{\tau\to 0} \sim |b|^2\tau^2 \tag{4.9}$$

vanishes at $\tau = 0$, which shows the degeneration of the conifold metric.

B. NS 3-form at small τ

We will see below that leading nontrivial contributions to warp factors are proportional to $\mu^2\tau^2/|b|^{4/3}$. At the first step of the perturbation theory we can neglect them and look for solutions for H_3 flux using the metric of the deformed conifold (summarized in the previous section) and a constant dilaton, $\Phi \approx \Phi_0$.

One solution was found in [18] using the ansatz suggested in [32] for the type-IIB flux compactification on the deformed conifold. The ansatz reads

$$H_3 = p' d\tau \wedge g_1 \wedge g_2 + k' d\tau \wedge g_3 \wedge g_4 - \frac{1}{2}(p - k)e_\psi \wedge (g_1 \wedge g_3 + g_2 \wedge g_4), \tag{4.10}$$

where p and k are functions of the radial coordinate τ. Here primes denote derivatives with respect to τ. The 3-form above is closed so the Bianchi identity is satisfied.

At large $\tau p' \approx k' \to \mu_1/3$ and using the identity [32]

$$e_{\theta_1} \wedge e_{\varphi_1} - e_{\theta_2} \wedge e_{\varphi_2} = g_1 \wedge g_2 + g_3 \wedge g_4, \tag{4.11}$$

it is easy to show that this solution tends to the first solution for H_3 (proportional to μ_1) in (3.12).

For small τ equation of motion (3.4) for H_3 was solved in [18] at leading order using the metric of the deformed conifold and a constant dilaton. The result is $k \approx \mu_1\tau$ and $p \approx -\mu_1\tau^5/80$ so the solution takes the form

$$H_3^{(1)} \approx \mu_1\gamma_3 \tag{4.12}$$

up to an overall constant, where we introduced a 3-form

$$\gamma_3 = d\tau \wedge g_3 \wedge g_4 - \frac{\tau^4}{16}d\tau \wedge g_1 \wedge g_2 + \frac{\tau}{2}e_\psi \wedge (g_1 \wedge g_3 + g_2 \wedge g_4). \tag{4.13}$$

Now let us find another solution which at large τ tends to the second solution in (3.12) (proportional to μ_2). To do so we use the ansatz,

$$H_3 = l(\tau)e_\psi \wedge g_1 \wedge g_2 + n(\tau)e_\psi \wedge g_3 \wedge g_4 + q(\tau)d\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4), \tag{4.14}$$

where l, n, and q are functions of τ. Using identity (4.11) and [32]

$$d(g_1 \wedge g_3 + g_2 \wedge g_4) = e_\psi \wedge (g_1 \wedge g_2 - g_3 \wedge g_4) \tag{4.15}$$

we calculate

$$dH_3 = l' d\tau \wedge e_\psi \wedge g_1 \wedge g_2 + n' d\tau \wedge e_\psi \wedge g_3 \wedge g_4 - q(\tau)d\tau \wedge e_\psi \wedge (g_1 \wedge g_2 - g_3 \wedge g_4). \tag{4.16}$$

Now Bianchi identity $dH_3 = 0$ leads to

$$l' - q = 0, \qquad n' + q = 0. \tag{4.17}$$

A solution to these equations with $q = 0$, $l = n = \mu_2/3$ corresponds to the second solution in (3.12) at large τ. Let us find the extrapolation of this solution to small τ. For nonzero q we have $l' = -n'$ and setting the integration constant to zero we get $l = -n$. The ansatz for H_3 acquires the form

$$H_3 = l(e_\psi \wedge g_1 \wedge g_2 - e_\psi \wedge g_3 \wedge g_4) + l' d\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4). \tag{4.18}$$

Calculating the 10D dual of (4.18) using metric in (4.7) we get

$$*H_3 = -T^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \left\{ \frac{4l}{\tau^2} d\tau \wedge g_3 \wedge g_4 - \frac{l\tau^2}{4} d\tau \wedge g_1 \wedge g_2 + l' e_\psi \wedge (g_1 \wedge g_3 + g_2 \wedge g_4) \right\}. \tag{4.19}$$

Then the equation of motion (3.4) reads

$$d * H_3 = T^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \left\{ -\left(\frac{2l}{\tau^2} + \frac{l\tau^2}{8} - l'' \right) d\tau \wedge e_\psi \wedge (g_1 \wedge g_3 + g_2 \wedge g_4) \right\} = 0, \tag{4.20}$$

where we used the identity [32]

$$d(g_1 \wedge g_2 - g_3 \wedge g_4) = -e_\psi \wedge (g_1 \wedge g_3 + g_2 \wedge g_4). \tag{4.21}$$

Equation (4.20) gives

$$l'' - \frac{2l}{\tau^2} = 0, \tag{4.22}$$

where we neglect the τ^2 term at small τ.

Eq. (4.22) gives $l \approx \mu_2 \tau^2/4$ up to a constant and we write down the second solution for H_3 in the form

$$H_3^{(2)} \approx \mu_2 \delta_3, \tag{4.23}$$

where

$$\delta_3 = \frac{\tau^2}{4} e_\psi \wedge (g_1 \wedge g_2 - g_3 \wedge g_4) + \frac{\tau}{2} d\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4). \tag{4.24}$$

Both 3-forms γ_3 and δ_3 are closed. Moreover, their 10D duals are given by [see [18] and (4.19)]

$$*\gamma_3 \approx T^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \delta_3,$$
$$*\delta_3 \approx -T^2 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \gamma_3. \tag{4.25}$$

The above relations ensure that both 10D dual forms are also closed,

$$d * \gamma_3 = 0, \qquad d * \delta_3 = 0. \tag{4.26}$$

Much in the same way as forms (3.7) and (3.8) 3-forms (4.13) and (4.24) satisfy relations

$$\int_{Y_6} \gamma_3 \wedge \gamma_3 = \int_{Y_6} \delta_3 \wedge \delta_3 = 0, \tag{4.27}$$

while

$$\int_{Y_6} \gamma_3 \wedge \delta_3 \sim \int d\tau \tau^2 \tag{4.28}$$

at small τ.

To conclude this section, we note that at $\tau = 0$ the first solution (4.12) tends to a constant

$$H_3^{(1)}(\tau = 0) = \mu_1 d\tau \wedge g_3 \wedge g_4, \tag{4.29}$$

which we impose as boundary conditions at S_3, which does not shrinks at $\tau = 0$. These boundary conditions ensure a nonzero solution for $H_3^{(1)}$.

Similarly for the second solution (4.23) we fix its derivative with respect to τ as boundary conditions at S_3 at $\tau = 0$,

$$\frac{\partial}{\partial \tau} H_3^{(2)}(\tau = 0) = \frac{\mu_2}{2} d\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4). \tag{4.30}$$

C. Warp factors at small τ

In this section we study the backreaction of the two solutions for H_3 flux found above on the metric and dilaton to the leading order in $\mu^2\tau^2/|b|^{4/3}$. Our ansatz for the metric is given by (3.5) where h_4 now is a function of τ, while

$$g_{mn} dx^m dx^n = \frac{1}{2}|b|^{2/3} \left(\frac{2}{3}\right)^{\frac{1}{3}} \left\{ h_1^{1/2}(\tau) \times \left(\frac{1}{2} a(\tau) d\tau^2 + \frac{1}{2} e_\psi^2 + g_3^2 + g_4^2 \right) + h_2^{1/2}(\tau) \frac{\tau^2}{4} (g_1^2 + g_2^2) \right\}, \tag{4.31}$$

where the metric of the deformed conifold (4.7) is further deformed with another three warp factors h_1, h_2, and a, which are assumed to be functions of τ. Here we also assume the limit of small τ, $\tau \ll 1$.

For Minkowski indices μ, $\nu = 0, 1, 2, 3$ Einstein's equations (3.2) has the form (3.15), where using results from [35] we calculate

$$R_{\mu\nu} = \frac{\eta_{\mu\nu} g_c^{\tau\tau}}{4ah_4^{1/2}h_1^{1/2}} \left\{ \frac{1}{h_4}\Delta h_4 + \frac{1}{2}\frac{h_1' h_4'}{h_1 h_4} + \frac{1}{2}\frac{h_2' h_4'}{h_2 h_4} - 2\frac{(h_4')^2}{h_4^2} - \frac{1}{2}\frac{a' h_4'}{a h_4} \right\}, \tag{4.32}$$

where Δ is the Laplacian calculated using metric (4.7). Here and below g_c^{mn}, g_{mn}^c denote the deformed conifold metric (4.7), for example

$$g_c^{\tau\tau} \approx \frac{2^{5/3}3^{1/3}}{|b|^{2/3}}. \tag{4.33}$$

At first order all nonlinearities in (4.32) can be neglected and Einstein's equations (3.15) reduce to

$$\Delta h_4 \approx -\frac{e^{-\Phi_0} g_{\tau\tau}^c}{12} H_3^2, \tag{4.34}$$

where H_3^2 can be calculated using the deformed conifold metric and we used only the constant part of the dilaton Φ_0 at this order. We have

$$e^{-\Phi_0} H_3^2 \approx 2^4 3^3 \frac{\mu_1^2 + \mu_2^2}{g_s} \frac{1}{|b|^2} [1 + O(\tau^2)], \tag{4.35}$$

where, say, for the first solution for H_3 in (4.12) only first and the last terms in γ_3 contribute at the leading order in τ.

Then Eq. (4.34) gives

$$h_4 = 1 - \frac{3^{2/3}}{2^{2/3}} \frac{\mu_1^2 + \mu_2^2}{g_s} \frac{\tau^2}{|b|^{4/3}} [1 + O(\tau^2)]. \tag{4.36}$$

Much in the same way as in the large r limit it is easy to see that the dilaton equation reduces to the Eq. (3.15) on the solution (3.18).

Consider now Einstein's equations with internal indices. Let index a (b) denote differentials e_ψ, g_3, g_4, while index i (j) denote g_1, g_2. Then we can calculate leading contributions to Christoffel symbols with τ indices at small τ, namely

$$\Gamma^\tau_{ij} = -g_{ij}^{(c)} \frac{g_c^{\tau\tau} h_2^{1/2}}{a h_1^{1/2}} \left(\frac{1}{\tau} + \frac{1}{4}\frac{h_2'}{h_2} \right), \quad \Gamma^\tau_{ab} = -g_{ab}^{(c)} \frac{g_c^{\tau\tau}}{a} \frac{1}{4}\frac{h_1'}{h_1},$$
$$\Gamma^j_{\tau i} = \Gamma^j_{i\tau} = \delta_i^j \left(\frac{1}{\tau} + \frac{1}{4}\frac{h_2'}{h_2} \right), \quad \Gamma^b_{\tau a} = \Gamma^b_{a\tau} = \delta_a^b \frac{1}{4}\frac{h_1'}{h_1},$$
$$\Gamma^\tau_{\tau\tau} = \frac{1}{2}\frac{a'}{a} + \frac{1}{4}\frac{h_1'}{h_1}, \quad \Gamma^n_{\tau\tau} = \Gamma^\tau_{\tau n} = \Gamma^\tau_{n\tau} = 0, \quad n \neq r. \tag{4.37}$$

Then nonzero components of the Ricci tensor to the leading order in τ take the form

$$R_{ij} \approx g_{ij}^{(c)} g_c^{\tau\tau} \left\{ \frac{1}{\tau^2} \left(\frac{a h_1^{1/2}}{h_2^{1/2}} - 1 \right) - \frac{1}{4}\Delta h_2 - \frac{1}{2\tau} h_2' - \frac{1}{2\tau} h_1' + \frac{1}{\tau} h_4' + \frac{1}{2\tau} a' \right\},$$
$$R_{ab} \approx g_{ab}^{(c)} g_c^{\tau\tau} \left\{ -\frac{1}{4}\Delta h_1 \right\},$$
$$R_{\tau\tau} \approx -\frac{1}{2}\Delta h_2 + \frac{1}{4}\Delta h_1 - h_1'' + h_4'' + \frac{1}{\tau} a'. \tag{4.38}$$

Here again we used that Ricci tensor is zero if all warp factors are equal to unity and the dependence on h_4 is found using formulas in [35].

For the first solution in (4.12) rhs of Einstein's equations (3.2) take the form

$$R_{ij} = \frac{1}{2^4 3^2} g_{ij}^{(c)} e^{-\Phi_0} (H_3^{(1)})^2,$$
$$R_{ab} = \frac{5}{2^4 3^2} g_{ab}^{(c)} e^{-\Phi_0} (H_3^{(1)})^2,$$
$$R_{\tau\tau} = \frac{1}{2^4 3^2} g_{\tau\tau}^{(c)} e^{-\Phi_0} (H_3^{(1)})^2, \tag{4.39}$$

where $(H_3^{(1)})^2$ is given by (4.35) with $\mu_2 = 0$.

Solutions to these equations are given by

$$h_1^{(1)} = 1 - \frac{5}{2^{2/3}3^{1/3}} \frac{\mu_1^2}{g_s} \frac{\tau^2}{|b|^{4/3}} + \cdots,$$
$$h_2^{(1)} = 1 - \frac{5}{13}\frac{3^{2/3}}{2^{2/3}} \frac{\mu_1^2}{g_s} \frac{\tau^2}{|b|^{4/3}} + \cdots,$$
$$a^{(1)} = 1 + \frac{5}{13}\frac{2^{1/3}}{3^{1/3}} \frac{\mu_1^2}{g_s} \frac{\tau^2}{|b|^{4/3}} + \cdots, \tag{4.40}$$

where dots stand for corrections in powers of τ and powers of $\mu^2/|b|^{4/3}$.

For the second solution (4.23) the rhs of Einstein's equations (3.2) has the form

$$R_{ij} = \frac{5}{2^4 3^2} g_{ij}^{(c)} e^{-\Phi_0} (H_3^{(2)})^2,$$
$$R_{ab} = \frac{1}{2^4 3^2} g_{ab}^{(c)} e^{-\Phi_0} (H_3^{(2)})^2,$$
$$R_{\tau\tau} = \frac{5}{2^4 3^2} g_{\tau\tau}^{(c)} e^{-\Phi_0} (H_3^{(2)})^2, \tag{4.41}$$

where $(H_3^{(2)})^2$ is given by (4.35) with $\mu_1 = 0$.

For this case solutions take the form

$$h_1^{(2)} = 1 - \frac{1}{2^{2/3}3^{1/3}}\frac{\mu_2^2}{g_s}\frac{\tau^2}{|b|^{4/3}} + \cdots,$$
$$h_2^{(2)} = 1 - \frac{3^{2/3}}{2^{2/3}}\frac{\mu_2^2}{g_s}\frac{\tau^2}{|b|^{4/3}} + \cdots,$$
$$a^{(2)} = 1 + \frac{2^{1/3}}{3^{1/3}}\frac{\mu_2^2}{g_s}\frac{\tau^2}{|b|^{4/3}} + \cdots. \quad (4.42)$$

Note that much in the same way as for the large r case the first and the third Einstein's equations in (4.39) and (4.41) coincides and give rise to conditions for the same combination $(3c_2 - 2\tilde{A})$, where c_2 and $\tilde{A}$ are coefficients in front of τ^2 for h_2 and a. As we explained before this is due to the possibility of redefinition of the radial coordinate (τ in the present case)—see Sec. III C. To fix the definition of τ we require that the orthogonal combination to the one above is zero, $(2c_2 + 3\tilde{A}) = 0$. This gives warp factors h_2 and a presented in (4.40) and (4.40).

We see that warp factors in (4.40) and (4.42) as well as h_4 (4.36) and the dilaton (3.18) have smooth behavior at small τ and do not develop singularities provided $\mu^2 \ll |b|^{4/3}$. They can be found order by order at $\mu^2 \ll |b|^{4/3}$ using perturbation theory in gravity equations.

D. The scalar potential at large $|b|$

To find the scalar potential for the complex structure modulus b we substitute solutions found above in this section into Eq. (3.41). At the leading order in $\mu^2/|b|^{4/3}$ we can neglect warp factors and use the metric of the deformed conifold together with the leading-order expression (4.35). Using (4.9) at small τ we get

$$V(b) = \mathrm{const}(\mu_1^2 + \mu_2^2)\frac{T^2}{g_s^3}\tau_{\max}^3, \quad (4.43)$$

where $\tau_{\max}$ is the infrared cutoff with respect to the radial coordinate τ related to R_{IR} as follows:

$$|b|\cosh(\tau_{\max}) = \left(\frac{2}{3}\right)^{\frac{3}{2}} R_{\mathrm{IR}}^3, \quad (4.44)$$

[see (4.6)]. This potential was obtained in [18] for the first solution for H_3 proportional to μ_1.

As we already explained, we expect that in our runaway vacuum b is large, close to R_{IR}, therefore $\tau_{\max}$ is small. Expanding $\cosh\tau$ at small τ we get

$$\tau_{\max} \sim \sqrt{\frac{(\frac{2}{3})^{\frac{3}{2}} R_{\mathrm{IR}}^3 - |b|}{|b|}}. \quad (4.45)$$

This gives the potential for the baryon b at large $|b|$

$$V(b) = \mathrm{const}(\mu_1^2 + \mu_2^2)\frac{T^2}{g_s^3}\left[\frac{(\frac{2}{3})^{\frac{3}{2}} R_{\mathrm{IR}}^3 - |b|}{|b|}\right]^{\frac{3}{2}}. \quad (4.46)$$

We see that to minimize the potential above $|b|$ becomes large and approaches the infrared cutoff,

$$\langle |b| \rangle = \left(\frac{2}{3}\right)^{\frac{3}{2}} R_{\mathrm{IR}}^3 \to \infty. \quad (4.47)$$

As we expected earlier in Sec. III D, we get a runaway vacuum.

The corrections to the potential (4.46) arise from taking into account higher powers of τ in the deformed conifold metric as well as from warp factors and go in powers of $\tau_{\max}$ and in powers of $\mu^2/|b|^{4/3}$, respectively. Both type of corrections disappear at the runaway vacuum (4.47).

In fact, $\tau_{\max}^3$ which enters (4.43) is the volume of the three-dimensional cone bounded by the sphere S_2 of the conifold with maximum radius $\tau_{\max}$. It shrinks to zero as b tends to its VEV (4.47). To avoid singularities we can regularize the size of S_2 introducing small nonzero β, which makes the conifold "slightly resolved" [see (2.3)]. We take the limit $\beta \to 0$ at the last step. Then the value of the potential and all its derivatives vanish in the vacuum (4.47) at $|b| = \langle |b| \rangle$, for example

$$V(b)|_{|b|=\langle|b|\rangle} = \mathrm{const}(\mu_1^2 + \mu_2^2)\frac{T^2}{g_s^3}\frac{\beta^3}{R_{\mathrm{IR}}^{9/2}} \to 0. \quad (4.48)$$

In particular, the mass term for b is zero.

Absence of warp factors and the vanishing of the potential $V(b)$ together with all its derivatives at the runaway vacuum confirms that $\mathcal{N} = 2$ supersymmetry is not broken in 4D SQCD.

To summarize, the H_3-form flux produces following effects:

(i) The Higgs branch of the baryon b in 4D SQCD is lifted.
(ii) The vacuum is of the runaway type $\langle |b| \rangle \to \infty$.
(iii) At the runaway vacuum warp factors tend to unity and the geometry becomes that of the deformed conifold.
(iv) At the runaway vacuum the radial coordinate τ and the sphere S_2 of the conifold degenerates, while the radius of the sphere S_3 tends to infinity.

We will interpret this degeneration in terms of $\mathcal{N} = 2$ SQCD in the next section.

V. INTERPRETATION IN TERMS OF 4D SQCD

A. 3-form flux in terms of quark masses

As we already mentioned in the Introduction, H_3-form flux was interpreted in terms of 4D SQCD in [18] as switching on quark masses. The motivation is that the only scalar potential deformation allowed in 4D SQCD by

$\mathcal{N}=2$ supersymmetry is the mass term for quarks. Field theory arguments were used in [18] to find a particular choice of nonzero quark masses associated with H_3. In this section we briefly review this interpretation. For $N_f=4$ we have four complex mass parameters. However, a shift of the complex scalar a, a superpartner of the U(1) gauge field, produces an overall shift of quark masses. Thus, in fact we have three independent complex mass parameters in our 4D SQCD. For example, we can choose three mass differences

$$m_1-m_2, \qquad m_3-m_4, \qquad m_1-m_3, \tag{5.1}$$

as independent parameters.

On the string theory side our solution (3.12) for the 3-form H_3 is parametrized by two real parameters μ_1 and μ_2. Thus, we expect that nonzero H_3-flux can be interpreted in terms of a particular choice of quark masses, subject to two complex constraints.

One constraint follows from (2.12). We have seen in the previous subsection that H_3 does not produces a mass term for the b-baryon. This ensures that

$$m_1+m_2-m_3-m_4=0. \tag{5.2}$$

Another constraint is

$$m_1m_2-m_3m_4=0, \tag{5.3}$$

which is imposed to avoid infinite VEV of σ (a scalar superpartner of the U(1) gauge field), which would costs an infinite energy in the world sheet $\mathbb{WCP}(2,2)$ model at large b, see [18] for details.

Solving two constraints above leads to two options for the choice of the quark masses

$$m_3=m_1, \qquad m_4=m_2, \tag{5.4}$$

and

$$m_3=m_2, \qquad m_4=m_1. \tag{5.5}$$

These two options are essentially the same, up to permutation of quarks q^3 and q^4. Let us choose the first option in (5.4).

The arguments above lead to the conclusion that the H_3-flux can be interpreted in terms of the single mass difference (m_1-m_2). We define a complex parameter μ and identify [18]

$$\mu\equiv\mu_1+i\mu_2=\mathrm{const}\sqrt{\frac{g_s^3}{T}}(m_1-m_2), \\ m_3=m_1, \qquad m_4=m_2. \tag{5.6}$$

The potential (3.42) calculated at large r, $r\gg|b|^{1/3}$ takes the form

$$V(b)=\mathrm{const}T|m_1-m_2|^2\log\frac{R_{\mathrm{IR}}^3}{|b|}. \tag{5.7}$$

Similar substitution can be done for the large-b potential (4.46).

B. Degeneration of the conifold and flow to SQED

Since our solution to the gravity equations is valid at

$$\frac{|\mu|^2}{|b|^{4/3}}\sim\frac{|m_1-m_2|^2}{T|b|^{4/3}}\ll 1 \tag{5.8}$$

and VEV of b goes to infinity [see (4.47)] we can use our solution at arbitrary large fixed values of (m_1-m_2). In particular, if we take $|m_1-m_2|\gg\sqrt{\xi}$ in 4D SQCD keeping the constraint (5.4) non-Abelian degrees of freedom decouple and U(2) gauge theory flows to $\mathcal{N}=2$ supersymmetric QED with the gauge group U(1) and $N_f=2$ quark flavors. Off-diagonal gauge fields together with two quark flavors acquire large masses $\sim|m_1-m_2|$[5] and decouple.

What happens to the non-Abelian vortex string upon this decoupling? The string survives, but transforms into an Abelian string. To see this note, that if we say, increase masses $m_2=m_4$ keeping $m_1=m_3=0$ fields n^2 and ρ^4 decouple in the world sheet $\mathbb{WCP}(2,2)$ model on the string and it flows into $\mathbb{WCP}(1,1)$ model. The D-term condition (2.3) now reads

$$|n^1|^2-|\rho^3|^2=\mathrm{Re}\beta. \tag{5.9}$$

The number of real degrees of freedom in $\mathbb{WCP}(1,1)$ model is $4-1-1=2$ where 4 is the number of real degrees of freedom of n^1 and ρ^3 and we subtract 2 due to the D-term constraint (5.9) and the U(1) phase eaten by the Higgs mechanism.

Physically $\mathbb{WCP}(1,1)$ model describes an Abelian semilocal vortex string supported in $\mathcal{N}=2$ supersymmetric U(1) gauge theory with $N_f=2$ quark flavors. This vortex has no orientational moduli, but it has one complex size modulus ρ^3, see [25–27]. Thus, we see that upon switching on (m_1-m_2) a non-Abelian string flows to an Abelian one.

The low-energy $\mathbb{WCP}(1,1)$ model is also conformal. Moreover, it was shown in [36] that in the nonlinear sigma model formulation it flows to a free theory on $\mathbb{R}^2$ in the infrared. Thus, in fact, switching on (m_1-m_2) with constraint (5.4) does not break the conformal invariance on the world sheet. It just reduces the number of degrees of freedom transforming a non-Abelian string into an Abelian one. The string theory which one would associate with the $\mathbb{WCP}(1,1)$ model is noncritical.

[5]In addition to masses $m_G\sim g\sqrt{\xi}$ due to the Higgs mechanism, see [8] for a review.

The field theory physics described above supports our interpretation of the H_3-form flux on the conifold in terms of quark masses. On the string theory side switching on $(m_1 - m_2)$ is reflected in the degeneration of the conifold, which effectively reduces its dimension. Also, in the limit $|b| \to \infty$ the radius of the sphere S_3 of the conifold becomes infinite and it tends to a flat three-dimensional space. This matches the field theory result [36] that $\mathbb{WCP}(1,1)$ model flows to a free theory in the infrared. It would be tempting to interpret the extra coordinate of the sphere S_3 of the conifold in the limit $|b| \to \infty$ as a Liouville coordinate for a noncritical string associated with the $\mathbb{WCP}(1,1)$ model. This is left for future work.

We also note that Eq. (2.10) suggests that massless stringy baryon b acquires infinitely strong interactions at the runaway vacuum (4.47) and the associated physics is no longer under analytic control.

VI. CONCLUSIONS

In this paper we considered a deformation of the string theory for the critical non-Abelian vortex supported in $\mathcal{N} = 2$ SQCD with gauge group U(2) and $N_f = 4$ quark flavors with NS 3-form flux building on the results of our previous paper [18]. Using the supergravity approach we found a solution for the 3-form H_3 and its backreaction on the conifold metric and the dilaton at the first nontrivial order in the parameter $\mu^2/|b|^{4/3}$. The nonzero 3-form H_3 generates a potential for the complex structure modulus b of the conifold, which is interpreted as a massless BPS baryonic hypermultiplet in 4D SQCD at strong coupling. This potential lifts the Higgs branch formed by VEVs of b and leads to a runaway vacuum for b, $\langle |b| \rangle \to \infty$. The warp factors disappear at this runaway vacuum.

Following [18] we interpret the 3-form H_3 as a quark mass deformation of 4D SQCD. Field theory arguments are used to relate the 3-form H_3 to the quark mass difference $(m_1 - m_2)$, subject to the constraint (5.4) [see (5.6)].

At the runaway vacuum the conifold degenerates to lower dimensions. This qualitatively matches with a flow to the $\mathbb{WCP}(1,1)$ model on the string world sheet, which is expected if one switches on the mass difference $(m_1 - m_2)$ and decouples one n-field and one ρ-field. In 4D SQCD this corresponds to a flow to $\mathcal{N} = 2$ supersymmetric QED with two charged flavors.

As one of the directions of a future work one can look for a physically observable effect of decoupling of certain massive hadron states in 4D SQCD upon increasing $(m_1 - m_2)$. These hadrons on the string theory side should be visible as massive string states and it would be interesting to study their decoupling as $\langle |b| \rangle \to \infty$.

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