BRST formalism of Weyl conformal gravity

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(Received 30 September 2022; accepted 24 October 2022; published 14 November 2022)

We present the Becchi-Rouet-Stora-Tyupin (BRST) formalism of a Weyl conformal gravity in Weyl geometry. Choosing the extended de Donder gauge-fixing condition (or harmonic gauge condition) for the general coordinate invariance and the new scalar gauge fixing for the Weyl invariance, we find that there is a Poincaré-like $IOSp(10|10)$ supersymmetry as in a Weyl invariant scalar-tensor gravity in Riemann geometry. We also point out that there is a gravitational conformal symmetry in quantum gravity although there is a massive Weyl gauge field as a result of spontaneous symmetry breakdown of Weyl gauge symmetry, and we account for how the gravitational conformal symmetry is spontaneously broken to the Poincaré symmetry. The corresponding massless Nambu-Goldstone bosons are the graviton and the dilaton. We also show the unitarity of the physical S-matrix on the basis of the BRST quartet mechanism in the case of an absence of anomalies.

DOI: 10.1103/PhysRevD.106.106007

I. INTRODUCTION

More than 100 years ago, Weyl advocated a new theory to unify all the interactions known at that time, namely gravitational interaction and electromagnetic force, within the framework of a newly established geometry which is nowadays called "Weyl geometry" [1,2]. In Riemann geometry both length and angle are preserved under parallel transport while in Weyl geometry, only angle, but not length, is preserved by the Weyl gauge field. Soon after the advent of Weyl's idea, Einstein criticized that regarding the spacing of atomic spectral lines, the prediction obtained from Weyl's theory and the experimental observations were in contradiction (this problem is sometimes called the second clock problem [3]); thus, the Weyl theory has been buried in oblivion for a long time.[1]

However, in recent years a considerate interest has been developed for Weyl conformal geometry. This is because it was found that the Weyl gauge field acquires a huge mass around the Planck scale and decouples at low energies, thereby avoiding the second clock problem [12–15]. In addition, we have noticed the importance of global scale invariance and also local scale invariance, which is also called Weyl invariance, in formulating a theory beyond the Standard Model [16] and quantum gravity. Hence, Weyl geometry provides us with a natural playground for describing Weyl symmetry.

The study of Weyl conformal gravity in Weyl geometry has been mainly limited to a classical analysis thus far.[2] One of the motivations behind the present article is to present a quantum theory of Weyl conformal gravity. To this end, we construct a Becchi-Rouet-Stora-Tyupin (BRST) formalism of the theory from which we can shed some light on important features of quantum aspects of Weyl conformal gravity in Weyl geometry. For instance, as has already been shown in the case of Weyl invariant scalar-tensor gravity [17–19], there is an extended $IOSp(10|10)$ choral symmetry compared with the $IOSp(8|8)$ choral symmetry in Einstein's gravity [20,21]. This extended symmetry is not confined to the sector of the Nakanishi-Lautrup auxiliary fields and the Faddeev-Popov (FP) (anti)ghosts but relevant to a classical theory. Moreover, it can be shown that we have a gravitational analog of conformal algebra as a subalgebra of the $IOSp(10|10)$ choral symmetry. That algebra then gives rise to a spontaneous symmetry breakdown to the Poincaré symmetry, by which we can prove that the graviton [22] and the dilaton [18,19] are exactly massless since they are the Nambu-Goldstone particles.

The paper is organized as follows. In Sec. II, we give a brief review of Weyl geometry. In Sec. III, we consider a

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[1]Even during this period, there were some papers dealing with Weyl theory [4–11].

[2]At the one-loop level, the effective potential has already been calculated in [14,15].

classical theory that is not only invariant under Weyl gauge transformation but also is free of ghosts, which generally exist in the higher-derivative gravities. Based on the classical theory in Sec. III, we fix the gauge symmetries by the extended de Donder gauge and new scalar gauge conditions and construct a BRST invariant quantum Lagrangian in Sec. IV. In Sec. V, we perform the canonical quantization of the quantum Lagrangian where we meet primary and secondary constraints associated with Weyl symmetry. They are the second-class constraints and hence are treated by applying the Dirac brackets. In Sec. VI, we prove the unitarity of the physical S-matrix on the basis of the BRST quartet mechanism. We find that physical modes are the two polarizations of the massless graviton and the three modes of the massive Weyl gauge fields. Furthermore, it is shown that the massless dilaton, which is eaten by the Weyl gauge field via the Higgs mechanism, belongs to the unphysical sector. In Sec. VII, we show that the quantum Lagrangian of Weyl conformal gravity possesses the huge global $IOSp(10|10)$ choral symmetry. In Sec. VIII, we point out that there exists a gravitational analog of conformal symmetry in quantum gravity and explain the spontaneous symmetry breaking. We find that the graviton and the dilaton are massless Nambu-Goldstone bosons. The final section is devoted to discussion.

II. REVIEW OF WEYL CONFORMAL GEOMETRY

In this section, we briefly review the basic concepts and definitions of Weyl conformal geometry [15].[3] In Weyl geometry, the Weyl gauge transformation, which is a combination of a local scale transformation for a generic field $\Phi(x)$ and a gauge transformation for the Weyl gauge field $S_\mu(x)$, is defined as

$$\begin{aligned}\Phi(x) &\to \Phi'(x) = e^{w\Lambda(x)}\Phi(x),\\ S_\mu(x) &\to S'_\mu(x) = S_\mu(x) - \frac{1}{f}\partial_\mu\Lambda(x),\end{aligned} \tag{2.1}$$

where w is called the "Weyl weight," or simply "weight" henceforth; f is the coupling constant for the noncompact Abelian gauge group; and $\Lambda(x)$ is a local parameter for the Weyl transformation. The Weyl gauge transformation for various fields is explicitly given by

$$\begin{aligned}&g_{\mu\nu}(x)\to g'_{\mu\nu}(x)=e^{2\Lambda(x)}g_{\mu\nu}(x),\quad \phi(x)\to\phi'(x)=e^{-\Lambda(x)}\phi(x),\\ &\psi(x)\to\psi'(x)=e^{-\frac{3}{2}\Lambda(x)}\psi(x),\quad A_\mu(x)\to A'_\mu(x)=A_\mu(x),\end{aligned} \tag{2.2}$$

where $g_{\mu\nu}(x)$, $\phi(x)$, $\psi(x)$, and $A_\mu(x)$ are the metric tensor, scalar, spinor, and electromagnetic gauge fields, respectively. The covariant derivative D_μ for the Weyl gauge transformation for a generic field $\Phi(x)$ of weight w is defined as

$$D_\mu\Phi \equiv \partial_\mu\Phi + wfS_\mu\Phi, \tag{2.3}$$

which transforms covariantly under the Weyl transformation:

$$D_\mu\Phi \to (D_\mu\Phi)' = e^{w\Lambda(x)}D_\mu\Phi. \tag{2.4}$$

The Weyl geometry is defined as a geometry with a real symmetric metric tensor $g_{\mu\nu}(=g_{\nu\mu})$ and a symmetric connection $\tilde{\Gamma}^\lambda_{\mu\nu}(=\tilde{\Gamma}^\lambda_{\nu\mu})$ which is defined as[4]

$$\begin{aligned}\tilde{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(D_\mu g_{\nu\rho} + D_\nu g_{\mu\rho} - D_\rho g_{\mu\nu})\\ &= \Gamma^\lambda_{\mu\nu} + f(S_\mu\delta^\lambda_\nu + S_\nu\delta^\lambda_\mu - S^\lambda g_{\mu\nu}),\end{aligned} \tag{2.5}$$

where $\Gamma^\lambda_{\mu\nu}$ is the standard Christoffel symbol in Riemann geometry. The most important difference between Riemann geometry and Weyl geometry lies in the fact that in Riemann geometry the metric condition is satisfied,

$$\nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu}g_{\rho\nu} - \Gamma^\rho_{\lambda\nu}g_{\mu\rho} = 0, \tag{2.6}$$

while in Weyl geometry we have

$$\tilde{\nabla}_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho_{\lambda\mu}g_{\rho\nu} - \tilde{\Gamma}^\rho_{\lambda\nu}g_{\mu\rho} = -2fS_\lambda g_{\mu\nu}, \tag{2.7}$$

where ∇_μ and $\tilde{\nabla}_\mu$ are covariant derivatives for diffeomorphisms in Riemann and Weyl geometries, respectively. Since the metric condition (2.6) implies that both length and angle are preserved under parallel transport, Eq. (2.7) shows that only angle, but not length, is preserved by the Weyl connection.

The general covariant derivative for both diffeomorphisms and Weyl gauge transformation, for instance, for a covariant vector of weight w, is defined as

$$\begin{aligned}\mathcal{D}_\mu V_\nu &\equiv D_\mu V_\nu - \tilde{\Gamma}^\rho_{\mu\nu}V_\rho\\ &= \tilde{\nabla}_\mu V_\nu + wfS_\mu V_\nu\\ &= \nabla_\mu V_\nu + wfS_\mu V_\nu - f(S_\mu\delta^\rho_\nu + S_\nu\delta^\rho_\mu - S^\rho g_{\mu\nu})V_\rho\\ &= \partial_\mu V_\nu + wfS_\mu V_\nu - \Gamma^\rho_{\mu\nu}V_\rho\\ &\quad - f(S_\mu\delta^\rho_\nu + S_\nu\delta^\rho_\mu - S^\rho g_{\mu\nu})V_\rho.\end{aligned} \tag{2.8}$$

One can verify that using the general covariant derivative, the following metric condition is satisfied:

[3]We follow the notation and conventions of Misner-Thorne-Wheeler (MTW) textbook [23]. Lowercase Greek letters $\mu, \nu, \ldots$ and Latin ones $i, j, \ldots$ are used for spacetime and spatial indices, respectively; for instance, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. The Riemann curvature tensor and the Ricci tensor are, respectively, defined by $R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\sigma\nu} - \partial_\nu\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\sigma\mu}$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. The Minkowski metric tensor is denoted by $\eta_{\mu\nu}$; $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = -1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$.

[4]We often use the tilde characters to express quantities belonging to Weyl geometry.

$$\mathcal{D}_\lambda g_{\mu\nu} = 0. \tag{2.9}$$

Moreover, under Weyl gauge transformation the general covariant derivative for a generic field Φ of weight w transforms in a covariant manner as desired:

$$\mathcal{D}_\mu \Phi \to (\mathcal{D}_\mu \Phi)' = e^{w\Lambda(x)} \mathcal{D}_\mu \Phi, \tag{2.10}$$

because the Weyl connection is invariant under Weyl gauge transformation, i.e., $\tilde{\Gamma}'^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu}$.

As in Riemann geometry, in Weyl geometry one can also construct a Weyl invariant curvature tensor $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ via a commutator of the covariant derivative $\tilde{\nabla}_\mu$:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] V_\rho = \tilde{R}_{\mu\nu\rho}{}^{\sigma} V_\sigma. \tag{2.11}$$

Calculating this commutator, one finds that

$$\begin{aligned}\tilde{R}_{\mu\nu\rho}{}^{\sigma} &= \partial_\nu \tilde{\Gamma}^{\sigma}_{\mu\rho} - \partial_\mu \tilde{\Gamma}^{\sigma}_{\nu\rho} + \tilde{\Gamma}^{\alpha}_{\mu\rho}\tilde{\Gamma}^{\sigma}_{\alpha\nu} - \tilde{\Gamma}^{\alpha}_{\nu\rho}\tilde{\Gamma}^{\sigma}_{\alpha\mu} \\ &= R_{\mu\nu\rho}{}^{\sigma} + 2f(\delta^{\sigma}_{[\mu}\nabla_{\nu]}S_\rho - \delta^{\sigma}_{\rho}\nabla_{[\mu}S_{\nu]} - g_{\rho[\mu}\nabla_{\nu]}S^\sigma) \\ &\quad + 2f^2(S_{[\mu}\delta^{\sigma}_{\nu]}S_\rho - S_{[\mu}g_{\nu]\rho}S^\sigma + \delta^{\sigma}_{[\mu}g_{\nu]\rho}S_\alpha S^\alpha),\end{aligned} \tag{2.12}$$

where $R_{\mu\nu\rho}{}^{\sigma}$ is the curvature tensor in Riemann geometry and we have defined the antisymmetrization by the square bracket, i.e., $A_{[\mu}B_{\nu]} \equiv \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)$. Then, it is straightforward to prove the following identities:

$$\tilde{R}_{\mu\nu\rho}{}^{\sigma} = -\tilde{R}_{\nu\mu\rho}{}^{\sigma}, \quad \tilde{R}_{[\mu\nu\rho]}{}^{\sigma} = 0, \quad \tilde{\nabla}_{[\lambda}\tilde{R}_{\mu\nu]\rho}{}^{\sigma} = 0. \tag{2.13}$$

From $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ one can define a Weyl invariant Ricci tensor:

$$\begin{aligned}\tilde{R}_{\mu\nu} &\equiv \tilde{R}_{\mu\rho\nu}{}^{\rho} \\ &= R_{\mu\nu} + f(-2\nabla_\mu S_\nu - H_{\mu\nu} - g_{\mu\nu}\nabla_\alpha S^\alpha) \\ &\quad + 2f^2(S_\mu S_\nu - g_{\mu\nu}S_\alpha S^\alpha),\end{aligned} \tag{2.14}$$

where $H_{\mu\nu}$ is the field strength of S_μ defined as

$$H_{\mu\nu} \equiv \nabla_\mu S_\nu - \nabla_\nu S_\mu = \partial_\mu S_\nu - \partial_\nu S_\mu. \tag{2.15}$$

Let us note that

$$\tilde{R}_{[\mu\nu]} \equiv \frac{1}{2}(\tilde{R}_{\mu\nu} - \tilde{R}_{\nu\mu}) = -2fH_{\mu\nu}. \tag{2.16}$$

Similarly, one can define not a Weyl invariant but a Weyl covariant scalar curvature:

$$\tilde{R} \equiv g^{\mu\nu}\tilde{R}_{\mu\nu} = R - 6f\nabla_\mu S^\mu - 6f^2 S_\mu S^\mu. \tag{2.17}$$

One finds that under Weyl gauge transformation, $\tilde{R} \to \tilde{R}' = e^{-2\Lambda(x)}\tilde{R}$ while $\tilde{\Gamma}^{\lambda}_{\mu\nu}$, $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$, and $\tilde{R}_{\mu\nu}$ are all invariant.

We close this section by discussing a spinor field as an example of matter fields in Weyl geometry. As is well known, to describe a spinor field it is necessary to introduce the vierbein e^a_μ, which is defined as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{2.18}$$

where $a, b, \ldots$ are local Lorentz indices taking 0,1,2,3 and $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Now the metric condition (2.9) takes the form

$$\mathcal{D}_\mu e^a_\nu \equiv D_\mu e^a_\nu + \tilde{\omega}^a{}_{b\mu} e^b_\nu - \tilde{\Gamma}^{\rho}_{\mu\nu} e^a_\rho = 0, \tag{2.19}$$

where the general covariant derivative is extended to include the local Lorentz transformation whose gauge connection is the spin connection $\tilde{\omega}^a{}_{b\mu}$ of weight 0 in Weyl geometry, and $D_\mu e^a_\nu = \partial_\mu e^a_\nu + f S_\mu e^a_\nu$ since the vierbein e^a_μ has weight 1. Solving the metric condition (2.19) leads to the expression of the spin connection in Weyl geometry,

$$\tilde{\omega}_{ab\mu} = \omega_{ab\mu} + f e^c_\mu(\eta_{ac}S_b - \eta_{bc}S_a), \tag{2.20}$$

where $\omega_{ab\mu}$ is the spin connection in Riemann geometry and we have defined $S_a \equiv e^\mu_a S_\mu$. Then, the general covariant derivative for a spinor field Ψ of weight $-\frac{3}{2}$ reads

$$\mathcal{D}_\mu \Psi = D_\mu \Psi + \frac{i}{2}\tilde{\omega}_{ab\mu} S^{ab} \Psi, \tag{2.21}$$

where $D_\mu \Psi = \partial_\mu \Psi - \frac{3}{2} f S_\mu \Psi$ and the Lorentz generator S^{ab} for a spinor field is defined as $S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$. Here we define the gamma matrices to satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. Since the spin connection $\tilde{\omega}^a{}_{b\mu}$ has weight 0, the covariant derivative $\mathcal{D}_\mu \Psi$ transforms covariantly under Weyl gauge transformation:

$$\mathcal{D}_\mu \Psi \to (\mathcal{D}_\mu \Psi)' = e^{-\frac{3}{2}\Lambda(x)} \mathcal{D}_\mu \Psi. \tag{2.22}$$

Then, the Lagrangian density for a massless Dirac spinor field is of the form

$$\mathcal{L} = \frac{i}{2} e\, e^\mu_a (\bar{\Psi}\gamma^a \mathcal{D}_\mu \Psi - \mathcal{D}_\mu \bar{\Psi}\gamma^a \Psi), \tag{2.23}$$

where $e \equiv \sqrt{-g}$, $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$, and $\mathcal{D}_\mu \bar{\Psi}$ is given by

$$\mathcal{D}_\mu \bar{\Psi} = D_\mu \bar{\Psi} - \bar{\Psi}\frac{i}{2}\tilde{\omega}_{ab\mu} S^{ab}. \tag{2.24}$$

Inserting Eqs. (2.21) and (2.24) into the Lagrangian density (2.23), we find that

$$\begin{aligned}\mathcal{L} = \frac{i}{2} e \Bigg[e^\mu_a \Bigg(&\bar{\Psi}\gamma^a \partial_\mu \Psi - \partial_\mu \bar{\Psi}\gamma^a \Psi + \frac{i}{2}\omega_{bc\mu}\bar{\Psi}\{\gamma^a, S^{bc}\}\Psi \Bigg) \\ &+ \frac{i}{2} f(\eta_{ab}S_c - \eta_{ac}S_b)\bar{\Psi}\{\gamma^a, S^{bc}\}\Psi \Bigg].\end{aligned} \tag{2.25}$$

The last term identically vanishes owing to the relation

$$\{\gamma^a, S^{bc}\} = -\varepsilon^{abcd}\gamma_5\gamma_d, \tag{2.26}$$

where we have defined $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon^{0123} = +1$. Thus, as is well known, the Weyl gauge field S_μ does not couple minimally to a spinor field Ψ. Technically speaking, it is the absence of imaginary unit i in the covariant derivative $D_\mu\Psi = \partial_\mu\Psi - \frac{3}{2}fS_\mu\Psi$ that induced this decoupling of the Weyl gauge field from the spinor field. Without the imaginary unit, the terms including the Weyl gauge field cancel out each other in Eq. (2.23). In a similar manner, we can prove that the Weyl gauge field does not couple to a gauge field, i.e., the electromagnetic potential A_μ either. On the other hand, the Weyl gauge field can couple to a scalar field such as the Higgs field as well as a graviton.

III. CLASSICAL THEORY

We wish to consider a model of Weyl conformal gravity in Weyl geometry. It is of interest to recall that without matter fields we have a unique classical Lagrangian which is invariant under the Weyl gauge transformation; the Lagrangian must be of form of quadratic gravity:

$$\mathcal{L}_{\rm QG} = \sqrt{-g}\left(-\frac{1}{2\xi^2}\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} + \alpha\tilde{R}^2\right), \tag{3.1}$$

where ξ and α are dimensionless coupling constants, and $\tilde{C}_{\mu\nu\rho\sigma}$ and $\tilde{R}$ are a generalization of conformal tensor and scalar curvature in Weyl geometry, respectively. Note that the Lagrangian of the Einstein-Hilbert type or the higher-derivative terms involving more than quadratic terms are prohibited to be present by Weyl gauge symmetry. The fatal defect of the Lagrangian (3.1), however, is the existence of a massless ghost that breaks unitarity in the quantum regime. Another unsatisfactory feature of the Lagrangian (3.1) is that it does not reduce to Einstein's general relativity at low energies which is known to be a good description of the physics relevant to gravitational phenomena at such long range scales.

Provided that we are allowed to use matter fields,[5] the situation changes and we can construct a scalar-tensor gravity of the Einstein-Hilbert type which includes at most the second-order derivatives of the metric tensor [4]:

$$\mathcal{L}_{ST} = \sqrt{-g}\frac{1}{2}\xi\phi^2\tilde{R}, \tag{3.2}$$

where ϕ is a real scalar field.[6] The most general classical Lagrangian, which is invariant under Weyl gauge transformation and is free of the massless ghost, reads

$$\mathcal{L}_G = \sqrt{-g}\left[\frac{1}{2}\xi\phi^2\tilde{R} - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon g^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{\lambda}{4!}\phi^4 + \eta\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\right], \tag{3.3}$$

where ξ, λ, η are all dimensionless constants and $\epsilon = \pm 1$ depending on a normal field $\epsilon = 1$ or a ghost field $\epsilon = -1$. In this article, we limit ourselves to the case $6\xi + \epsilon \neq 0$ since the specific case $6\xi + \epsilon = 0$ leads to the same expression as the last term with the constant η, which is called "Weyl invariant scalar-tensor gravity," when surface terms are ignored. Finally, the scalar field ϕ has the weight -1 so the Weyl covariant derivative in (3.3) takes the form[7]

$$D_\mu\phi = \partial_\mu\phi - S_\mu\phi. \tag{3.4}$$

Since we have already analyzed the Weyl invariant scalar-tensor gravity in Riemann geometry [18] and the quartic potential term has no essential role in the BRST formalism, we will put $\lambda = \eta = 0$. Thus, the classical Lagrangian that is treated in this article reads

$$\begin{aligned}\mathcal{L}_c &= \sqrt{-g}\left[\frac{1}{2}\xi\phi^2\tilde{R} - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon g^{\mu\nu}D_\mu\phi D_\nu\phi\right] \\ &= \sqrt{-g}\left[\frac{1}{2}\xi\phi^2(R - 6\nabla_\mu S^\mu - 6S_\mu S^\mu) - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon g^{\mu\nu}(\partial_\mu\phi - S_\mu\phi)(\partial_\nu\phi - S_\nu\phi)\right].\end{aligned} \tag{3.5}$$

IV. QUANTUM THEORY

The classical Lagrangian (3.5) is invariant under both general coordinate transformation (GCT) and Weyl gauge transformation. For a quantum theory we have to fix such gauge symmetries by introducing suitable gauge-fixing conditions. After introducing the gauge-fixing conditions, the quantum Lagrangian is no longer invariant under the gauge transformations, but as residual global symmetries the quantum Lagrangian is invariant under two BRST transformations, one of which is denoted as δ_B, corresponding to the GCT, and is defined as

$$\begin{aligned}&\delta_B g_{\mu\nu} = -(\nabla_\mu c_\nu + \nabla_\nu c_\mu) = -(c^\alpha\partial_\alpha g_{\mu\nu} + \partial_\mu c^\alpha g_{\alpha\nu} + \partial_\nu c^\alpha g_{\mu\alpha}), \\ &\delta_B\phi = -c^\lambda\partial_\lambda\phi, \quad \delta_B S_\mu = -c^\lambda\nabla_\lambda S_\mu - \nabla_\mu c^\lambda S_\lambda, \\ &\delta_B c^\rho = -c^\lambda\partial_\lambda c^\rho, \quad \delta_B\bar{c}_\rho = iB_\rho, \quad \delta_B B_\rho = 0,\end{aligned} \tag{4.1}$$

where c^ρ and $\bar{c}_\rho$ are, respectively, the FP ghost and antighost, and B_ρ is the Nakanishi-Lautrup (NL) field. For convenience, in place of the NL field B_ρ we will introduce a new NL field defined as [24]

[5] As explained in the previous section, fermions and the conventional gauge fields do not couple to the Weyl gauge field, but only the scalar field does.

[6] The extension to a complex scalar field or multiple scalar fields is straightforward.

[7] In what follows, we will set $f = 1$ for the coupling constant for the noncompact Abelian gauge group.

$$b_\rho = B_\rho - ic^\lambda \partial_\lambda \bar{c}_\rho, \tag{4.2}$$

and its BRST transformation reads

$$\delta_B b_\rho = -c^\lambda \partial_\lambda b_\rho. \tag{4.3}$$

The other BRST transformation, which is denoted as $\bar{\delta}_B$, corresponding to the Weyl transformation is defined as

$$\bar{\delta}_B g_{\mu\nu} = 2cg_{\mu\nu}, \qquad \bar{\delta}_B \phi = -c\phi, \qquad \bar{\delta}_B S_\mu = -\partial_\mu c,$$
$$\bar{\delta}_B \bar{c} = iB, \qquad \bar{\delta}_B c = \bar{\delta}_B B = 0, \tag{4.4}$$

where c and $\bar{c}$ are, respectively, the FP ghost and FP antighost, and B is the NL field. Note that the two BRST transformations are nilpotent, i.e.,

$$\delta_B^2 = \bar{\delta}_B^2 = 0. \tag{4.5}$$

To complete the two BRST transformations, we have to fix not only the GCT BRST transformation δ_B on $c, \bar{c}$, and B but also the Weyl BRST transformation δ_B on $c^\rho, \bar{c}_\rho$, and b_ρ. The BRST transformations on these fields are fixed by requiring that the two BRST transformations anticommute with each other [18], that is,

$$\{\delta_B, \bar{\delta}_B\} \equiv \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B = 0. \tag{4.6}$$

Then, the resultant BRST transformations take the form

$$\delta_B B = -c^\lambda \partial_\lambda B, \qquad \delta_B c = -c^\lambda \partial_\lambda c, \qquad \delta_B \bar{c} = -c^\lambda \partial_\lambda \bar{c},$$
$$\bar{\delta}_B b_\rho = \bar{\delta}_B c^\rho = \bar{\delta}_B \bar{c}_\rho = 0. \tag{4.7}$$

In this context, it is worthwhile to recall that the gauge condition for the GCT must be invariant under Weyl gauge transformation while the one for Weyl transformation must be invariant under GCT in order for the two BRST transformations to anticommute. In that case we can consider the two BRST transformations separately. The suitable gauge condition for the GCT is almost unique and is called "the extended de Donder gauge" [18][8]:

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \tag{4.8}$$

where we have defined $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$.

On the other hand, we have a few candidates for the gauge-fixing condition for the Weyl transformation, which must be invariant under the GCT, i.e., a scalar quantity. The first one is the well-known "unitary gauge," $\phi = \text{const}$, which is taken to show that Weyl invariant scalar-tensor gravity is equivalent to the Einstein-Hilbert term. The other gauge condition is the Lorenz gauge, $\nabla_\mu S^\mu = 0$, which is usually adopted in quantum field theories. However, it turns out that these gauge conditions are not so interesting in the present context since they do not allow for conformal symmetry to remain. Hence, we shall choose what we call "the scalar gauge condition" [18]:

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi) = 0, \tag{4.9}$$

which can alternatively be written as

$$\Box\phi^2 = 0. \tag{4.10}$$

After taking the extended de Donder gauge condition (4.8) for the GCT and the scalar gauge condition (4.9) for the Weyl transformation, the gauge-fixed and BRST invariant quantum Lagrangian is given by

$$\begin{aligned}\mathcal{L}_q &= \mathcal{L}_c + \mathcal{L}_{GF+FP} + \bar{\mathcal{L}}_{GF+FP} \\ &= \mathcal{L}_c + i\delta_B(\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\nu) + i\bar{\delta}_B[\bar{c}\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)] \\ &= \sqrt{-g}\left[\frac{1}{2}\xi\phi^2(R - 6\nabla_\mu S^\mu - 6S_\mu S^\mu)\right. \\ &\quad \left. -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon g^{\mu\nu}D_\mu\phi D_\nu\phi\right] \\ &\quad - \tilde{g}^{\mu\nu}\phi^2(\partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda) \\ &\quad + \tilde{g}^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c,\end{aligned} \tag{4.11}$$

where surface terms are dropped.

From the Lagrangian $\mathcal{L}_q$, it is straightforward to derive the field equations by taking the variation with respect to $g_{\mu\nu}, S_\mu, \phi, b_\nu, B, c^\rho, \bar{c}_\rho, c$, and $\bar{c}$ in order:

$$\begin{aligned}&\frac{1}{2}\xi\phi^2 G_{\mu\nu} - \frac{1}{2}\xi(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi^2 - 3\xi\phi^2\left(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S_\alpha S^\alpha\right) + 3\xi\phi(S_\mu\partial_\nu\phi + S_\nu\partial_\mu\phi - g_{\mu\nu}S^\alpha\partial_\alpha\phi) - \frac{1}{2}H_{\mu\alpha}H_\nu{}^\alpha + \frac{1}{8}g_{\mu\nu}H_{\alpha\beta}^2 \\ &-\frac{1}{2}\epsilon\left[D_\mu\phi D_\nu\phi - \frac{1}{2}g_{\mu\nu}(D_\alpha\phi)^2\right] - \frac{1}{2}\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) = 0, \quad (6\xi + \epsilon)g^{\mu\nu}\phi D_\nu\phi - \nabla_\nu H^{\mu\nu} = 0, \\ &\xi\phi^2(R - 6\nabla_\mu S^\mu - 6S_\mu S^\mu) + \epsilon\frac{1}{\sqrt{-g}}\phi D_\mu(\tilde{g}^{\mu\nu}\phi D_\nu\phi) - E - 2g^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - \phi^2\Box B = 0, \\ &\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \quad \partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi) = 0, \quad g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho = g^{\mu\nu}\partial_\mu\partial_\nu c^\rho = g^{\mu\nu}\partial_\mu\partial_\nu\bar{c} = g^{\mu\nu}\partial_\mu\partial_\nu c = 0,\end{aligned} \tag{4.12}$$

[8]Let us note that this gauge condition breaks the general coordinate invariance, but it is invariant under the general linear transformation $GL(4)$. Thus, the quantum Lagrangian that is obtained shortly is also invariant under the $GL(4)$.

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ denotes the Einstein tensor, while $E_{\mu\nu}$ and E are defined as

$$\begin{aligned} E_{\mu\nu} &= \phi^2(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda) - \phi \partial_\mu B \partial_\nu \phi \\ &\quad + i\phi^2 \partial_\mu \bar{c} \partial_\nu c + (\mu \leftrightarrow \nu), \\ E &= g^{\mu\nu} E_{\mu\nu}. \end{aligned} \tag{4.13}$$

Moreover, since $\tilde{g}^{\mu\nu} D_\nu \phi$ has the weight 1, the Weyl covariant derivative is defined as

$$D_\mu(\tilde{g}^{\mu\nu} D_\nu \phi) = \partial_\mu(\tilde{g}^{\mu\nu} D_\nu \phi) + S_\mu \tilde{g}^{\mu\nu} D_\nu \phi. \tag{4.14}$$

When we introduce the dilaton $\sigma(x)$ by defining

$$\phi(x) \equiv e^{\sigma(x)}, \tag{4.15}$$

the two gauge-fixing conditions in (4.12), or equivalently, Eqs. (4.8) and (4.9), lead to a very simple d'Alembert-like equation for the dilaton:

$$g^{\mu\nu} \partial_\mu \partial_\nu \sigma = 0. \tag{4.16}$$

It is worthwhile to notice that it is not the scalar field ϕ but the dilaton σ that satisfies this type of equation.

To show that the auxiliary field B also obeys the same type of equation, let us take into account the trace part of the Einstein equation, i.e., the first field equation in (4.12), which gives us the equation

$$\xi \phi^2 R - 6\xi \phi^2 S_\alpha S^\alpha + 12\xi \phi S^\alpha \partial_\alpha \phi - \epsilon (D_\alpha \phi)^2 - E = 0. \tag{4.17}$$

Next, we can rewrite the field equation for ϕ, the third equation in (4.12), as

$$\begin{aligned} &\xi \phi^2 R - 6\xi \phi^2 S_\alpha S^\alpha + 12\xi \phi S^\alpha \partial_\alpha \phi - \epsilon (D_\alpha \phi)^2 - E \\ &- (6\xi + \epsilon)\phi^2 g^{\mu\nu} \partial_\mu S_\nu - 2 g^{\mu\nu} \phi \partial_\mu B \partial_\nu \phi - \phi^2 \Box B = 0. \end{aligned} \tag{4.18}$$

Using Eqs. (4.17) and (4.18), we can obtain the equation

$$g^{\mu\nu} \partial_\mu \partial_\nu B + (6\xi + \epsilon) g^{\mu\nu} \partial_\mu S_\nu = 0. \tag{4.19}$$

Now we are ready to prove

$$g^{\mu\nu} \partial_\mu S_\nu = 0. \tag{4.20}$$

To do that, let us consider the field equation for S_μ in (4.12), multiply by $\sqrt{-g}$, and then operate the covariant derivative consequently leading to

$$\sqrt{-g} \nabla_\mu \nabla_\nu H^{\mu\nu} = (6\xi + \epsilon) \nabla_\mu (\tilde{g}^{\mu\nu} \phi D_\nu \phi). \tag{4.21}$$

The left-hand side (LHS) of Eq. (4.21) is identically zero and $6\xi + \epsilon \neq 0$ by our assumption, and we find that

$$\nabla_\mu (\tilde{g}^{\mu\nu} \phi D_\nu \phi) = 0. \tag{4.22}$$

Using the formula

$$\nabla_\mu (\tilde{g}^{\mu\nu} A_\nu) = \partial_\mu (\tilde{g}^{\mu\nu} A_\nu), \tag{4.23}$$

which holds for an arbitrary covariant vector A_μ, Eq. (4.22) is reduced to the form

$$\partial_\mu (\tilde{g}^{\mu\nu} \phi \partial_\nu \phi - \tilde{g}^{\mu\nu} \phi^2 S_\nu) = 0. \tag{4.24}$$

Then, using the gauge conditions (4.8) and (4.9), we can reach Eq. (4.20). Hence, Eq. (4.19) implies that the auxiliary field B obeys the equation

$$g^{\mu\nu} \partial_\mu \partial_\nu B = 0. \tag{4.25}$$

Surprisingly enough, using the Weyl BRST transformation, we can show Eq. (4.25) in the simplest way. For this aim, let us start with the field equation for $\bar{c}$ in (4.12):

$$g^{\mu\nu} \partial_\mu \partial_\nu \bar{c} = 0. \tag{4.26}$$

Operating $\bar{\delta}_B$ on this equation leads to

$$-2c g^{\mu\nu} \partial_\mu \partial_\nu \bar{c} + i g^{\mu\nu} \partial_\mu \partial_\nu B = 0. \tag{4.27}$$

The first term on the LHS is vanishing owing to (4.26), so we can arrive at Eq. (4.25).

In a perfectly similar manner, we can show that the Nakanishi-Lautrup auxiliary field b_ρ satisfies the d'Alembert-like equation by either an explicit calculation or using the BRST transformation for the GCT. Here we present only the latter proof since the former one was given in our previous paper [17].

Let us start with the field equation for $\bar{c}_\rho$ in (4.12):

$$g^{\mu\nu} \partial_\mu \partial_\nu \bar{c}_\rho = 0. \tag{4.28}$$

Taking the GCT BRST transformation of this equation yields

$$(-\partial_\lambda g^{\mu\nu} c^\lambda + g^{\mu\alpha} \partial_\alpha c^\nu + g^{\nu\alpha} \partial_\alpha c^\mu) \partial_\mu \partial_\nu \bar{c}_\rho + i g^{\mu\nu} \partial_\mu \partial_\nu B_\rho = 0, \tag{4.29}$$

where we have used the GCT BRST transformation (4.1). Substituting the definition of b_ρ in Eq. (4.2) into (4.29), we have the equation for b_ρ:

$$ig^{\mu\nu}\partial_\mu\partial_\nu b_\rho = g^{\mu\nu}\partial_\mu\partial_\nu(c^\lambda\partial_\lambda\bar{c}_\rho) - (-\partial_\lambda g^{\mu\nu}c^\lambda + 2g^{\mu\alpha}\partial_\alpha c^\nu)\partial_\mu\partial_\nu\bar{c}_\rho. \quad (4.30)$$

With the help of Eq. (4.28) and the field equation for c^ρ in (4.12), the right-hand side (RHS) is found to be vanishing so we have the desired equation

$$g^{\mu\nu}\partial_\mu\partial_\nu b_\rho = 0. \quad (4.31)$$

In other words, setting $X^M = \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$,[9] X^M turns out to obey the very simple equation

$$g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0. \quad (4.32)$$

This fact, together with the gauge condition $\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0$ produces the two kinds of conserved currents:

$$\mathcal{P}^{\mu M} \equiv \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2(1\overleftrightarrow{\partial}_\nu X^M),$$
$$\mathcal{M}^{\mu MN} \equiv \tilde{g}^{\mu\nu}\phi^2(X^M\overleftrightarrow{\partial}_\nu Y^N), \quad (4.33)$$

where we have defined $X^M\overleftrightarrow{\partial}_\mu Y^N \equiv X^M\partial_\mu Y^N - (\partial_\mu X^M)Y^N$. These conserved currents constitute a Poincaré-like $IOSp(10|10)$ supersymmetry as will be shown later.

V. CANONICAL QUANTIZATION AND EQUAL-TIME COMMUTATION RELATIONS

In this section, after introducing the Poisson brackets, we will evaluate various equal-time commutation relations (ETCRs) among fundamental variables. To simplify various expressions, we will obey the following abbreviations adopted in the textbook of Nakanishi and Ojima [21]:

$$[A, B'] = [A(x), B(x')]|_{x^0=x'^0}, \qquad \delta^3 = \delta(\vec{x} - \vec{x}'),$$
$$\tilde{f} = \frac{1}{\tilde{g}^{00}} = \frac{1}{\sqrt{-g}g^{00}}, \quad (5.1)$$

where we assume that $\tilde{g}^{00}$ is invertible. Here the above brackets $[A, B']$ symbolically describe the Poisson brackets and the ETCRs.

First of all, let us set up the Poisson brackets of canonical variables:

$$\{g_{\mu\nu}, \pi_g^{\rho\lambda\prime}\}_P = \frac{1}{2}(\delta^\rho_\mu\delta^\lambda_\nu + \delta^\lambda_\mu\delta^\rho_\nu)\delta^3, \quad \{\phi, \pi'_\phi\}_P = \delta^3,$$
$$\{S_\mu, \pi_S^{\nu\prime}\}_P = \delta^\nu_\mu\delta^3, \quad \{c^\sigma, \pi'_{c\lambda}\}_P = \{\bar{c}_\lambda, \pi^{\sigma\prime}_{\bar{c}}\}_P = \delta^\sigma_\lambda\delta^3,$$
$$\{B, \pi'_B\}_P = \{c, \pi'_c\}_P = \{\bar{c}, \pi'_{\bar{c}}\}_P = \delta^3, \quad (5.2)$$

where the other Poisson brackets vanish. Here the canonical variables are $g_{\mu\nu}, \phi, S_\mu, B, c^\rho, \bar{c}_\rho, c, \bar{c}$ and the corresponding canonical conjugate momenta are $\pi_g^{\mu\nu}, \pi_\phi, \pi_S^\mu$, $\pi_B, \pi_{c\rho}, \pi^\rho_{\bar{c}}, \pi_c, \pi_{\bar{c}}$, respectively, and the b_μ field is regarded as not a canonical variable but a conjugate momentum of $\tilde{g}^{0\mu}$.[10]

To remove second-order derivatives of the metric involved in R, we perform the integration by parts once and rewrite the Lagrangian (4.11) as

$$\mathcal{L}_q = -\frac{1}{2}\xi\tilde{g}^{\mu\nu}\phi^2(\Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\alpha} - \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\sigma\nu} + 6S_\mu S_\nu) - \xi\phi\partial_\mu\phi(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) + 6\xi\tilde{g}^{\mu\nu}\phi S_\mu\partial_\nu\phi$$
$$-\frac{1}{4}\sqrt{-g}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon\tilde{g}^{\mu\nu}D_\mu\phi D_\nu\phi + \partial_\mu(\tilde{g}^{\mu\nu}\phi^2)b_\nu - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\rho\partial_\nu c^\rho$$
$$+\tilde{g}^{\mu\nu}\partial_\mu B\phi\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c + \partial_\mu\mathcal{V}^\mu, \quad (5.3)$$

where we have also integrated by parts two terms with the linear S_μ and b_μ, and a surface term $\mathcal{V}^\mu$ is thus given by

$$\mathcal{V}^\mu = \frac{1}{2}\xi\phi^2(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) - 3\xi\tilde{g}^{\mu\nu}\phi^2 S_\nu - \tilde{g}^{\mu\nu}\phi^2 b_\nu. \quad (5.4)$$

Using this Lagrangian, the concrete expressions for canonical conjugate momenta become

[9]The conventional de Donder condition, $\partial_\mu\tilde{g}^{\mu\nu} = 0$, can be rewritten as $\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu x^\lambda) = 0$. This can also be described as $g^{\mu\nu}\partial_\mu\partial_\nu x^\lambda = 0$ by means of the de Donder condition. In other words, by adding the spacetime coordinates x^μ to X^M, we can consider the de Donder condition as well.

[10]Taking the variation of (5.3) with respect to $\partial_0\tilde{g}^{0\mu}$ leads to the expression $\phi^2 b_\mu + \cdots$ where $\cdots$ does not involve the time derivative of fields, which means that the b_μ field is in essence a conjugate momentum of $\tilde{g}^{0\mu}$ up to a factor of ϕ^2.

$$
\begin{aligned}
\pi_g^{\mu\nu} &= \frac{\partial \mathcal{L}_q}{\partial \dot{g}_{\mu\nu}} \\
&= -\frac{1}{4}\sqrt{-g}\xi\phi^2\Big[-g^{0\lambda}g^{\mu\nu}g^{\sigma\tau} - g^{0\tau}g^{\mu\lambda}g^{\nu\sigma} - g^{0\sigma}g^{\mu\tau}g^{\nu\lambda} \\
&\quad + g^{0\lambda}g^{\mu\tau}g^{\nu\sigma} + g^{0\tau}g^{\mu\nu}g^{\lambda\sigma} + \frac{1}{2}(g^{0\mu}g^{\nu\lambda} + g^{0\nu}g^{\mu\lambda})g^{\sigma\tau}\Big]\partial_\lambda g_{\sigma\tau} \\
&\quad - \sqrt{-g}\left[\frac{1}{2}(g^{0\mu}g^{\rho\nu} + g^{0\nu}g^{\rho\mu}) - g^{\mu\nu}g^{\rho 0}\right]\xi\phi\partial_\rho\phi \\
&\quad - \frac{1}{2}\sqrt{-g}(g^{0\mu}g^{\nu\rho} + g^{0\nu}g^{\mu\rho} - g^{0\rho}g^{\mu\nu})\phi^2 b_\rho, \\
\pi_\phi &= \frac{\partial \mathcal{L}_q}{\partial \dot{\phi}} = -\epsilon\tilde{g}^{0\mu}D_\mu\phi + 2\tilde{g}^{0\mu}\phi b_\mu + \xi\phi(-\tilde{g}^{\alpha\beta}\Gamma^0_{\alpha\beta} + \tilde{g}^{0\alpha}\Gamma^\beta_{\alpha\beta}) \\
&\quad + 6\xi\tilde{g}^{0\mu}\phi S_\mu + \tilde{g}^{0\mu}\phi\partial_\mu B, \\
\pi_S^\mu &= \frac{\partial \mathcal{L}_q}{\partial \dot{S}_\mu} = -\sqrt{-g}H^{0\mu}, \quad \pi_B = \frac{\partial \mathcal{L}_q}{\partial \dot{B}} = \tilde{g}^{0\mu}\phi\partial_\mu\phi, \\
\pi_{c\sigma} &= \frac{\partial \mathcal{L}_q}{\partial \dot{c}^\sigma} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}_\sigma, \quad \pi_{\bar{c}}^\sigma = \frac{\partial \mathcal{L}_q}{\partial \dot{\bar{c}}_\sigma} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c^\sigma, \\
\pi_c &= \frac{\partial \mathcal{L}_q}{\partial \dot{c}} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}, \quad \pi_{\bar{c}} = \frac{\partial \mathcal{L}_q}{\partial \dot{\bar{c}}} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c, \qquad (5.5)
\end{aligned}
$$

where we have defined the time derivative, such as $\dot{g}_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial t} \equiv \partial_0 g_{\mu\nu}$, and differentiation of ghosts is taken from the right.

It can easily be seen that we have a primary constraint

$$\Psi_1 \equiv \pi_S^0 = 0. \qquad (5.6)$$

Let us recall that a secondary constraint comes from the consistency under time evolution of the primary constraint:

$$\Psi_2 \equiv \dot{\pi}_S^0 = \{\pi_S^0, H_T\}_P \approx 0, \qquad (5.7)$$

where H_T is the Hamiltonian of the system at hand, which is defined as

$$
\begin{aligned}
H_T &\equiv \int d^3x \mathcal{H}_T \\
&= \int d^3x(\pi_g^{\mu\nu}\dot{g}_{\mu\nu} + \pi_\phi\dot{\phi} + \pi_S^\mu\dot{S}_\mu + \pi_B\dot{B} + \pi_{c\mu}\dot{c}^\mu \\
&\quad + \pi_{\bar{c}}^\mu\dot{\bar{c}}_\mu + \pi_c\dot{c} + \pi_{\bar{c}}\dot{\bar{c}} - \mathcal{L}_q). \qquad (5.8)
\end{aligned}
$$

To obtain the Hamiltonian, we have to express the time derivatives of the canonical variables in terms of the canonical conjugate momenta in (5.5). To do that, let us first consider π_B, which gives us the expression of $\dot{\phi}$ as

$$\dot{\phi} = \tilde{f}\left(\frac{1}{\phi}\pi_B - \tilde{g}^{0i}\partial_i\phi\right). \qquad (5.9)$$

Next, let us turn our attention to the (kl) components of $\pi_g^{\mu\nu}$, which take the form

$$\pi_g^{kl} = \hat{A}^{kl} + \hat{B}^{kl\rho}b_\rho + \hat{C}^{klmn}\dot{g}_{mn} + \hat{D}^{kl}\dot{\phi}, \qquad (5.10)$$

where $\hat{A}^{kl}$, $\hat{B}^{kl\rho}$, $\hat{C}^{klmn}$, and $\hat{D}^{kl}$ commute with g_{mn} and are defined as

$$
\begin{aligned}
\hat{A}^{kl} &= -\frac{1}{4}\sqrt{-g}\phi^2\Big[-g^{0m}g^{kl}g^{\sigma\tau} - g^{0\tau}g^{km}g^{l\sigma} - g^{0\sigma}g^{k\tau}g^{lm} + g^{0m}g^{k\tau}g^{l\sigma} + g^{0\tau}g^{kl}g^{m\sigma} \\
&\quad + \frac{1}{2}(g^{0k}g^{lm} + g^{0l}g^{km})g^{\sigma\tau}\Big]\partial_m g_{\sigma\tau} - \sqrt{-g}\xi\phi\left[\frac{1}{2}(g^{0k}g^{lm} + g^{0l}g^{km}) - g^{kl}g^{0m}\right]\partial_m\phi, \\
\hat{B}^{kl\rho} &= -\frac{1}{2}\sqrt{-g}\phi^2(g^{0k}g^{l\rho} + g^{0l}g^{k\rho} - g^{0\rho}g^{kl}), \\
\hat{C}^{klmn} &= -\frac{1}{4}\sqrt{-g}\xi\phi^2(-g^{00}g^{kl}g^{mn} - g^{0n}g^{0k}g^{lm} - g^{0m}g^{kn}g^{0l} + g^{00}g^{kn}g^{lm} \\
&\quad + g^{0n}g^{kl}g^{0m} + g^{0k}g^{0l}g^{mn}), \\
\hat{D}^{kl} &= \sqrt{-g}\xi\phi(g^{00}g^{kl} - g^{0k}g^{0l}). \qquad (5.11)
\end{aligned}
$$

Solving (5.10) with respect to $\dot{g}_{kl}$ together with Eq. (5.9) leads to

$$\dot{g}_{kl} = \hat{C}^{-1}_{klmn}\left[\pi_g^{mn} - \hat{A}^{mn} - \hat{B}^{mn\rho}b_\rho - \hat{D}^{mn}\tilde{f}\left(\frac{1}{\phi}\pi_B - \tilde{g}^{0i}\partial_i\phi\right)\right], \qquad (5.12)$$

where $\hat{C}^{-1}_{klmn}$ is the inverse matrix of $\hat{C}^{klmn}$ given by

$$
\begin{aligned}
\hat{C}^{-1}_{klmn} &= \frac{2}{\xi\phi^2}\tilde{f}(g_{kl}g_{mn} - g_{km}g_{ln} - g_{kn}g_{lm}), \\
\hat{C}^{klmn}\hat{C}^{-1}_{mnij} &= \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k). \qquad (5.13)
\end{aligned}
$$

Using the extended de Donder gauge condition (4.8), $\dot{g}_{00}$ and $\dot{g}_{0k}$ are described as

$$\dot{g}_{00} = \frac{1}{g^{00}}\left(g^{ij}\dot{g}_{ij} - 2g^{\alpha i}\partial_i g_{0\alpha} + \frac{4}{\phi}\dot{\phi}\right)$$
$$= \frac{1}{g^{00}}\left\{g^{kl}\hat{C}^{-1}_{klmn}\left[\pi_g^{mn} - \hat{A}^{mn} - \hat{B}^{mn\rho}b_\rho - \hat{D}^{mn}\tilde{f}\left(\frac{1}{\phi}\pi_B - \tilde{g}^{0i}\partial_i\phi\right)\right] - 2g^{\alpha i}\partial_i g_{0\alpha} + \frac{4}{\phi}\tilde{f}\left(\frac{1}{\phi}\pi_B - \tilde{g}^{0i}\partial_i\phi\right)\right\},$$
$$\dot{g}_{0k} = \frac{1}{g^{00}}\left(-g^{0j}\dot{g}_{jk} - g^{\alpha i}\partial_i g_{\alpha k} + \frac{1}{2}g^{\alpha\beta}\partial_k g_{\alpha\beta} + \frac{2}{\phi}\partial_k\phi\right)$$
$$= \frac{1}{g^{00}}\left\{-g^{0j}\hat{C}^{-1}_{jkmn}\left[\pi_g^{mn} - \hat{A}^{mn} - \hat{B}^{mn\rho}b_\rho - \hat{D}^{mn}\tilde{f}\left(\frac{1}{\phi}\pi_B - \tilde{g}^{0i}\partial_i\phi\right)\right] - g^{\alpha i}\partial_i g_{\alpha k} + \frac{1}{2}g^{\alpha\beta}\partial_k g_{\alpha\beta} + \frac{2}{\phi}\partial_k\phi\right\}. \quad (5.14)$$

In a similar manner, based on $\pi_\phi, \pi_S^\mu, \pi_{c\sigma}, \pi_{\bar{c}}^\sigma, \pi_c$, and $\pi_{\bar{c}}$ in Eq. (5.5), the time derivatives $\dot{B}, \dot{S}_k, \dot{S}_0, \dot{\bar{c}}_\sigma, \dot{c}^\sigma, \dot{\bar{c}}$, and $\dot{c}$ can be expressed in terms of the canonical conjugate momenta as follows:

$$\dot{B} = \tilde{f}\frac{1}{\phi}\left[\pi_\phi + \epsilon\frac{1}{\phi}\pi_B - (6\xi+\epsilon)\tilde{g}^{0\mu}\phi S_\mu - 2\tilde{g}^{0\mu}\phi b_\mu - \tilde{g}^{0i}\phi\partial_i B\right]$$
$$-\xi\tilde{f}\{(\tilde{g}^{00}g^{ij} - \tilde{g}^{0i}g^{0j})\hat{C}^{-1}_{ijmn}[\pi_g^{mn} - \hat{A}^{mn} - \hat{B}^{mn\rho}b_\rho$$
$$-\xi\tilde{f}(\tilde{g}^{00}g^{mn} - \tilde{g}^{0m}g^{0n})(\pi_B - \tilde{g}^{0k}\phi\partial_k\phi)]$$
$$+(\tilde{g}^{0i}g^{\alpha\beta} - \tilde{g}^{0\alpha}g^{i\beta})\partial_i g_{\alpha\beta}\},$$
$$\dot{S}_k = \partial_k S_0 + \tilde{f}(-g_{kj}\pi_S^j + \tilde{g}^{0j}H_{kj}),$$
$$\dot{S}_0 = -\tilde{f}\{\tilde{g}^{0i}[2\partial_i S_0 + \tilde{f}(-g_{ij}\pi_S^j + \tilde{g}^{0j}H_{ij})] + \tilde{g}^{ij}\partial_i S_j\},$$
$$\dot{\bar{c}}_\sigma = i\tilde{f}\phi^{-2}\pi_{c\sigma} - \tilde{f}\tilde{g}^{0i}\partial_i\bar{c}_\sigma,$$
$$\dot{c}^\sigma = -i\tilde{f}\phi^{-2}\pi_{\bar{c}}^\sigma - \tilde{f}\tilde{g}^{0i}\partial_i c^\sigma,$$
$$\dot{\bar{c}} = i\tilde{f}\phi^{-2}\pi_c - \tilde{f}\tilde{g}^{0i}\partial_i\bar{c},$$
$$\dot{c} = -i\tilde{f}\phi^{-2}\pi_{\bar{c}} - \tilde{f}\tilde{g}^{0i}\partial_i c, \quad (5.15)$$

where we have used Eq. (4.20) in deriving $\dot{S}_0$.

Finally, we can also express the b_μ field in terms of canonical conjugate momenta. Since the b_μ field is regarded as a conjugate momentum of $\tilde{g}^{0\mu}$, we begin with $\pi_g^{\alpha 0}$, which has a structure

$$\pi_g^{\alpha 0} = A^\alpha + B^{\alpha\beta}\partial_\beta\phi + C^{\alpha\beta}b_\beta, \quad (5.16)$$

where $A^\alpha, B^{\alpha\beta}$, and $C^{\alpha\beta} = -\frac{1}{2}\tilde{g}^{00}g^{\alpha\beta}\phi^2$ do not include $\dot{g}_{\mu\nu}$, and $B^{\alpha\beta}\partial_\beta\phi$ does not have $\dot{\phi}$. Solving this equation with respect to b_μ leads to

$$b_\mu = -2\tilde{f}\phi^{-2}g_{\mu\alpha}\pi_g^{\alpha 0} - \frac{1}{2}\xi\tilde{f}\left[\delta_\mu^0\left(\tilde{g}^{0\tau}g^{\lambda\sigma} - \frac{1}{2}\tilde{g}^{0\lambda}g^{\sigma\tau}\right)\partial_\lambda g_{\sigma\tau}\right.$$
$$\left. - \left(\tilde{g}^{0\tau}g^{0\sigma} - \frac{1}{2}\tilde{g}^{00}g^{\sigma\tau}\right)\partial_\mu g_{\sigma\tau}\right]$$
$$- \xi\phi^{-1}(\partial_\mu\phi - \delta_\mu^0\tilde{f}\tilde{g}^{0\alpha}\partial_\alpha\phi). \quad (5.17)$$

Note that the RHS of this equation does not involve $\dot{g}_{\mu\nu}$ and $\dot{\phi}$ as can be verified explicitly. Incidentally, the relation (5.16) is utilized to derive some useful Poisson brackets such as $\{g_{\mu\nu}, b'_\rho\}_P$.

Using the Hamiltonian H_T, Eq. (5.7) provides us with a secondary constraint:

$$\Psi_2 = \partial_i\pi_S^i + (6\xi+\epsilon)(\pi_B - \tilde{g}^{0\mu}\phi^2 S_\mu) \approx 0, \quad (5.18)$$

which is just the same as the (0μ) components of the field equation for S_μ in (4.12) and there are no more constraints since we can show that[11]

$$\dot{\Psi}_2 = \{\Psi_2, H_T\}_P = 0. \quad (5.19)$$

The Poisson bracket between the constraints is evaluated to be

$$\{\Psi_1, \Psi'_2\}_P = (6\xi+\epsilon)\tilde{g}^{00}\phi^2\delta^3 = (6\xi+\epsilon)\frac{1}{\tilde{f}}\phi^2\delta^3, \quad (5.20)$$

which implies that the constraints are the second-class constraint so that they can be treated by means of the Dirac bracket defined as

$$\{A, B'\}_D \equiv \{A, B'\}_P - \{A, \Psi''_a\}_P C^{-1}_{ab}\{\Psi''_b, B'\}_P, \quad (5.21)$$

where Ψ_a $(a = 1, 2)$ are the second-class constraints and C^{-1}_{ab} is the inverse matrix of $C_{ab} = \{\Psi_a, \Psi'_b\}_P$. Concretely, the matrix elements, C^{-1}_{ab}, are given by

$$C^{-1}_{12} = -C^{-1}_{21} = -\frac{1}{6\xi+\epsilon}\tilde{f}\phi^{-2}\delta^3, \quad C^{-1}_{11} = C^{-1}_{22} = 0. \quad (5.22)$$

As is well known, the canonical quantization can be carried out by replacing $i\{A, B'\}_D$ with the equal-time commutation relation $[A, B']$. After some calculations, we can write down several important ETCRs, which are needed for later calculations:

[11] A derivation of constraints is exhibited in Appendix.

$$
\begin{aligned}
&[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = -\frac{2}{\xi} i\tilde{f}\phi^{-2}[g_{\rho\sigma}g_{\mu\nu} - g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} + \sqrt{-g}\tilde{f}(\delta^0_\rho\delta^0_\mu g_{\sigma\nu} \\
&\quad + \delta^0_\rho\delta^0_\nu g_{\sigma\mu} + \delta^0_\sigma\delta^0_\mu g_{\rho\nu} + \delta^0_\sigma\delta^0_\nu g_{\rho\mu})]\delta^3, \\
&[\dot{g}_{\rho\sigma}, \phi'] = 0, \qquad [\dot{g}_{\rho\sigma}, B'] = 2i\tilde{f}\phi^{-2}g_{\rho\sigma}\delta^3, \\
&[\phi, b'_\rho] = [B, b'_\rho] = [B, \dot{B}'] = [\dot{\phi}, \phi'] = [\dot{\phi}, S'_\mu] = 0, \qquad [\dot{\phi}, B'] = -i\tilde{f}\phi^{-1}\delta^3, \\
&[S_0, S'_k] = -\frac{1}{6\xi+\epsilon} i\tilde{f}\phi^{-2}\partial_k\delta^3, \qquad [S_k, S'_l] = 0, \\
&[S_0, \dot{S}'_k] = -i\tilde{f}g_{0k}\delta^3, \qquad [S_k, \dot{S}'_l] = -i\tilde{f}g_{kl}\delta^3 + \frac{1}{6\xi+\epsilon} i\partial_k(\tilde{f}\phi^{-2}\partial_l\delta^3), \\
&[S_0, b'_\mu] = -i\tilde{f}\phi^{-2}S_\mu\delta^3, \qquad [S_0, B'] = -i\tilde{f}\phi^{-2}\delta^3, \qquad [S_k, b'_\rho] = [S_k, B'] = 0, \\
&[\dot{S}_0, B'] = 2i\tilde{f}\tilde{g}^{0i}\partial_i(\tilde{f}\phi^{-2}\delta^3), \qquad [\dot{S}_k, B'] = -i\partial_k(\tilde{f}\phi^{-2}\delta^3), \\
&\{\dot{\bar{c}}_\lambda, c^{\sigma\prime}\} = -\{\dot{c}^\sigma, \bar{c}'_\lambda\} = -\tilde{f}\phi^{-2}\delta^\sigma_\lambda\delta^3, \qquad \{\dot{\bar{c}}, c'\} = -\{\dot{c}, \bar{c}'\} = -\tilde{f}\phi^{-2}\delta^3, \\
&[g_{\mu\nu}, b'_\rho] = -i\tilde{f}\phi^{-2}(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})\delta^3, \qquad [\tilde{g}^{\mu\nu}, b'_\rho] = i\tilde{f}\phi^{-2}(\tilde{g}^{\mu 0}\delta^\nu_\rho + \tilde{g}^{\nu 0}\delta^\mu_\rho - \tilde{g}^{\mu\nu}\delta^0_\rho)\delta^3, \\
&[g_{\mu\nu}, \dot{b}'_\rho] = i\{[\tilde{f}\phi^{-2}\partial_\rho g_{\mu\nu} - \partial_0(\tilde{f}\phi^{-2})(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})]\delta^3 + [(\delta^k_\mu - 2\delta^0_\mu\tilde{f}\tilde{g}^{0k})g_{\rho\nu} \\
&\quad + (\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\}, \\
&[\dot{\phi}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\phi\delta^3, \qquad [\dot{B}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho B\delta^3, \\
&[\dot{S}_0, b'_\mu] = -i\tilde{f}^2\phi^{-2}[\tilde{g}^{0\nu}(\partial_\mu S_\nu + \partial_\nu S_\mu) - \tilde{g}^{0i}H_{\mu i}]\delta^3 + 2i\tilde{f}\tilde{g}^{0i}\partial_i(\tilde{f}\phi^{-2}S_\mu\delta^3), \\
&[\dot{S}_k, b'_\mu] = i\tilde{f}\phi^{-2}H_{k\mu}\delta^3 - i\partial_k(\tilde{f}\phi^{-2}S_\mu\delta^3), \\
&[b_\mu, b'_\nu] = 0, \qquad [b_\mu, \dot{b}'_\nu] = i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3, \\
&[b_\rho, c^{\sigma\prime}] = [b_\rho, \bar{c}'_\lambda] = [b_\rho, c'] = [b_\rho, \bar{c}'] = 0, \\
&[\dot{\bar{c}}_\lambda, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}_\lambda\delta^3, \qquad [\dot{c}^\sigma, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c^\sigma\delta^3, \\
&[\dot{\bar{c}}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}\delta^3, \qquad [\dot{c}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c\delta^3.
\end{aligned} \tag{5.23}
$$

These ETCRs can be obtained from the explicit calculations and/or the BRST transformations. For instance, we will present a derivation of $[B, \dot{B}'] = 0$ by both methods. First, let us focus on the explicit calculation via the Dirac bracket:

$$
\{B, \dot{B}'\}_D = \{B, \dot{B}'\}_P - \{B, \Psi''_2\}_P C^{-1}_{21}\{\Psi''_1, \dot{B}'\}_P. \tag{5.24}
$$

Since we can easily evaluate each Poisson bracket whose result reads

$$
\begin{aligned}
&\{B, \dot{B}'\}_P = \{B, (6\xi+\epsilon)\tilde{f}\phi^{-2}\pi'_B\}_P = (6\xi+\epsilon)\tilde{f}\phi^{-2}\delta^3, \\
&\{B, \Psi'_2\}_P = \{B, (6\xi+\epsilon)\pi'_B\}_P = (6\xi+\epsilon)\delta^3, \\
&\{\Psi_1, \dot{B}'\}_P = \{\pi^0_S, -(6\xi+\epsilon)\tilde{f}\tilde{g}^{0\mu}S'_\mu\} = (6\xi+\epsilon)\delta^3,
\end{aligned} \tag{5.25}
$$

the Dirac bracket becomes

$$
\{B, \dot{B}'\}_D = (6\xi+\epsilon)\tilde{f}\phi^{-2}\delta^3 - (6\xi+\epsilon)C^{-1}_{21}(6\xi+\epsilon)\delta^3 = 0. \tag{5.26}
$$

Second, we will present a derivation by means of the BRST transformation that is more general and elegant than the above explicit calculation. The ETCR, $[B, \pi'_c] = 0$, leads to $[B, \dot{\bar{c}}'] = 0$. Taking the Weyl BRST transformation of this ETCR yields the equation

$$
\{[i\bar{Q}_B, B], \dot{\bar{c}}'\} + [B, \{i\bar{Q}_B, \dot{\bar{c}}'\}] = 0. \tag{5.27}
$$

Then, the Weyl BRST transformation (4.4) immediately leads to $[B, \dot{B}'] = 0$.

VI. UNITARITY ANALYSIS

As in the conventional BRST formalism, the physical state $|\text{phys}\rangle$ is defined by imposing two subsidiary conditions [25]

$$
Q_B|\text{phys}\rangle = \bar{Q}_B|\text{phys}\rangle = 0. \tag{6.1}
$$

It is then well known that the physical S-matrix is unitary under the assumption that all BRST singlet states have positive norm. In this section, we would like to prove the unitarity of the physical S-matrix in the present theory. From the classical analysis we know that the gauge field becomes massive via the Higgs mechanism. Thus, we wish to understand how the Higgs mechanism is described in terms of the BRST formalism.

In analyzing the unitarity, it is enough to take into account the asymptotic fields of all the fundamental fields and the free part of the Lagrangian. Let us first assume the asymptotic fields as

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}, \qquad \phi = \phi_0 + \tilde{\phi}, \qquad S_\mu = s_\mu,$$
$$b_\mu = \beta_\mu, \qquad B = \beta, \qquad c^\mu = \gamma^\mu, \qquad \bar{c}_\mu = \bar{\gamma}_\mu,$$
$$c = \gamma, \qquad \bar{c} = \bar{\gamma}, \tag{6.2}$$

where $\eta_{\mu\nu}(=\eta^{\mu\nu})$ is the flat Minkowski metric with the mostly positive signature and ϕ_0 is a nonzero constant. In this section, the Minkowski metric is used to lower or raise the Lorentz indices. Using these asymptotic fields, the free part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_q = {}& \frac{1}{2}\xi\phi_0^2\left(\frac{1}{4}\varphi_{\mu\nu}\Box\varphi^{\mu\nu} - \frac{1}{4}\varphi\Box\varphi - \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\rho\varphi_\nu{}^\rho + \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\nu\varphi\right) + \xi\phi_0\tilde{\phi}(-\Box\varphi + \partial_\mu\partial_\nu\varphi^{\mu\nu}) \\ & + (6\xi+\epsilon)\left(\phi_0 s_\mu\partial^\mu\tilde{\phi} - \frac{1}{2}\phi_0^2 s_\mu s^\mu\right) - \frac{1}{4}h_{\mu\nu}^2 - \frac{1}{2}\epsilon\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \left(2\eta^{\mu\nu}\phi_0\tilde{\phi} - \phi_0^2\varphi^{\mu\nu} + \frac{1}{2}\phi_0^2\eta^{\mu\nu}\varphi\right)\partial_\mu\beta_\nu \\ & - i\phi_0^2\partial_\mu\bar{\gamma}_\rho\partial^\mu\gamma^\rho + \phi_0\partial_\mu\beta\partial^\mu\tilde{\phi} - i\phi_0^2\partial_\mu\bar{\gamma}\partial^\mu\gamma,\end{aligned} \tag{6.3}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\varphi \equiv \eta^{\mu\nu}\varphi_{\mu\nu}$, and $h_{\mu\nu} \equiv \partial_\mu s_\nu - \partial_\nu s_\mu$. Based on this Lagrangian, it is easy to derive the linearized field equations:

$$\frac{1}{2}\xi\phi_0^2\left(\frac{1}{2}\Box\varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box\varphi - \partial_\rho\partial_{(\mu}\varphi_{\nu)}{}^\rho + \frac{1}{2}\partial_\mu\partial_\nu\varphi + \frac{1}{2}\eta_{\mu\nu}\partial_\rho\partial_\sigma\varphi^{\rho\sigma}\right)$$
$$+\xi\phi_0(-\eta_{\mu\nu}\Box + \partial_\mu\partial_\nu)\tilde{\phi} + \phi_0^2\partial_{(\mu}\beta_{\nu)} - \frac{1}{2}\phi_0^2\eta_{\mu\nu}\partial_\rho\beta^\rho = 0, \tag{6.4}$$

$$\epsilon\Box\tilde{\phi} + \xi\phi_0(-\Box\varphi + \partial_\mu\partial_\nu\varphi^{\mu\nu}) - (6\xi+\epsilon)\phi_0\partial_\mu s^\mu - 2\phi_0\partial_\rho\beta^\rho - \phi_0\Box\beta = 0, \tag{6.5}$$

$$\partial^\nu h_{\mu\nu} + (6\xi+\epsilon)\phi_0^2\left(s_\mu - \frac{1}{\phi_0}\partial_\mu\tilde{\phi}\right) = 0, \tag{6.6}$$

$$\partial_\mu\tilde{\phi} - \frac{1}{2}\phi_0\left(\partial^\nu\varphi_{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\right) = 0, \tag{6.7}$$

$$\Box\tilde{\phi} = \Box\gamma^\mu = \Box\bar{\gamma}_\mu = \Box\gamma = \Box\bar{\gamma} = 0. \tag{6.8}$$

Here we have introduced the symmetrization notation $A_{(\mu}B_{\nu)} \equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$.

Now, operating ∂^μ on Eq. (6.7) and using Eq. (6.8), we obtain

$$\partial_\mu\partial_\nu\varphi^{\mu\nu} - \frac{1}{2}\Box\varphi = 0. \tag{6.9}$$

Next, taking the trace of Eq. (6.4) with the help of Eqs. (6.8) and (6.9) leads to

$$\Box\varphi + \frac{4}{\xi}\partial_\rho\beta^\rho = 0. \tag{6.10}$$

Moreover, operating ∂^μ on Eq. (6.6), and using the identity $\partial^\mu\partial^\nu h_{\mu\nu} = 0$ and Eq. (6.8), yields the Lorenz condition

$$\partial_\mu s^\mu = 0. \tag{6.11}$$

As can been seen in Eq. (6.6), it is more convenient to introduce $\hat{s}_\mu$ defined as

$$\hat{s}_\mu = s_\mu - \frac{1}{\phi_0}\partial_\mu\tilde{\phi}, \tag{6.12}$$

which also obeys the Lorenz condition owing to Eqs. (6.8) and (6.11):

$$\partial_\mu\hat{s}^\mu = 0. \tag{6.13}$$

With the new gauge field $\hat{s}_\mu$ and the corresponding field strength $\hat{h}_{\mu\nu} \equiv \partial_\mu\hat{s}_\nu - \partial_\nu\hat{s}_\mu$, the "Maxwell equation" (6.6) can be cast to the form

$$\partial^\nu\hat{h}_{\mu\nu} + (6\xi+\epsilon)\phi_0^2\hat{s}_\mu = 0, \tag{6.14}$$

which clearly shows that the Weyl gauge field absorbs the Nambu-Goldstone boson $\tilde{\phi}$ associated with the spontaneous symmetry breakdown of the Weyl gauge symmetry, thereby becoming massive with the mass squared $(6\xi+\epsilon)\phi_0^2$. (Here we assume $6\xi+\epsilon > 0$, which is consistent with the positive Newton constant $\xi > 0$.) To put it differently, after spontaneous symmetry breakdown of the Weyl gauge symmetry, the Weyl gauge field $\hat{s}_\mu$ satisfies not

only the Lorenz condition (6.13) but also the massive Klein-Gordon equation

$$(\Box - m^2)\hat{s}_\mu = 0, \tag{6.15}$$

where m^2 is defined by

$$m^2 \equiv (6\xi + \epsilon)\phi_0^2. \tag{6.16}$$

Furthermore, with the help of Eqs. (6.8)–(6.11) and (6.5) can be rewritten as

$$\Box\beta = 0. \tag{6.17}$$

Moreover, acting ∂^μ on Eq. (6.4) yields

$$\Box\beta_\mu = 0. \tag{6.18}$$

Finally, using various equations obtained thus far, the "Einstein equation" (6.4) is reduced to the form

$$\Box\varphi_{\mu\nu} + \frac{4}{\xi}\partial_{(\mu}\beta_{\nu)} = 0, \tag{6.19}$$

which means that the field $\varphi_{\mu\nu}$ is not a simple pole field but a dipole field:

$$\Box^2\varphi_{\mu\nu} = 0. \tag{6.20}$$

On the other hand, in addition to Eq. (6.15), the other fields are all simple pole fields:

$$\Box\tilde{\phi} = \Box\beta_\mu = \Box\beta = \Box\gamma^\mu = \Box\bar{\gamma}_\mu = \Box\gamma = \Box\bar{\gamma} = 0. \tag{6.21}$$

Note that Eq. (6.21) corresponds to Eq. (4.32) in a curved spacetime.

Following the standard technique, we can calculate the four-dimensional (anti)commutation relations (4D CRs) between asymptotic fields. The point is that the simple pole fields, for instance, the Nakanishi-Lautrup field $\beta_\mu(x)$, can be expressed in terms of the invariant delta function $D(x)$ as

$$\beta_\mu(x) = -\int d^3z D(x-z)\overset{\leftrightarrow}{\partial}{}^z_0\beta_\mu(z), \tag{6.22}$$

whereas the dipole field $\varphi_{\mu\nu}(x)$ takes the form

$$\begin{aligned}\varphi_{\mu\nu}(x) &= -\int d^3z [D(x-z)\overset{\leftrightarrow}{\partial}{}^z_0\varphi_{\mu\nu}(z) + E(x-z)\overset{\leftrightarrow}{\partial}{}^z_0\Box\varphi_{\mu\nu}(z)] \\ &= -\int d^3z \left[D(x-z)\overset{\leftrightarrow}{\partial}{}^z_0\varphi_{\mu\nu}(z) - \frac{4}{\xi}E(x-z)\overset{\leftrightarrow}{\partial}{}^z_0\partial_{(\mu}\beta_{\nu)}(z)\right],\end{aligned} \tag{6.23}$$

where in the last equality we have used Eq. (6.19). Here the invariant delta function $D(x)$ for massless simple pole fields and its properties are described as

$$\begin{aligned}&D(x) = -\frac{i}{(2\pi)^3}\int d^4k\epsilon(k_0)\delta(k^2)e^{ikx}, \quad \Box D(x) = 0, \\ &D(-x) = -D(x), \quad D(0,\vec{x}) = 0, \quad \partial_0 D(0,\vec{x}) = \delta^3(x),\end{aligned} \tag{6.24}$$

where $\epsilon(k_0) \equiv \frac{k_0}{|k_0|}$. Similarly, the invariant delta function $E(x)$ for massless dipole fields and its properties are given by

$$\begin{aligned}&E(x) = -\frac{i}{(2\pi)^3}\int d^4k\epsilon(k_0)\delta'(k^2)e^{ikx}, \quad \Box E(x) = D(x), \\ &E(-x) = -E(x), \quad E(0,\vec{x}) = \partial_0 E(0,\vec{x}) = \partial_0^2 E(0,\vec{x}) = 0, \\ &\partial_0^3 E(0,\vec{x}) = -\delta^3(x),\end{aligned} \tag{6.25}$$

where $\delta'(k^2) \equiv \frac{d\delta(k^2)}{dk^2}$.

On the other hand, the Weyl gauge field $\hat{s}(x)$ obeys the massive Klein-Gordon equation (6.15), so it needs to be described in terms of the invariant delta function $\Delta(x;m^2)$ for massive simple pole fields as

$$\hat{s}_\mu(x) = -\int d^3z\Delta(x-z;m^2)\overset{\leftrightarrow}{\partial}{}^z_0\hat{s}_\mu(z), \tag{6.26}$$

where $\Delta(x;m^2)$ is defined as

$$\begin{aligned}&\Delta(x;m^2) = -\frac{i}{(2\pi)^3}\int d^4k\epsilon(k_0)\delta(k^2+m^2)e^{ikx}, \\ &(\Box - m^2)\Delta(x;m^2) = 0, \\ &\Delta(-x;m^2) = -\Delta(x;m^2), \qquad \Delta(0,\vec{x};m^2) = 0, \\ &\partial_0\Delta(0,\vec{x};m^2) = \delta^3(x), \qquad \Delta(x;0) = D(x).\end{aligned} \tag{6.27}$$

It is easy to show that the RHS of Eqs. (6.22), (6.23), and (6.26) is independent of z^0. Thus, for instance, when we evaluate the four-dimensional commutation relation $[\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)]$, we can put $z^0 = y^0$ and use the three-dimensional commutation relations among asymptotic fields. After some manipulation, we find that the 4D CRs are given by

$$\begin{aligned}&[\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)] \\ &= -\frac{2}{\xi}i\phi_0^{-2}[(\eta_{\mu\nu}\eta_{\sigma\tau} - \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma})D(x-y) \\ &\quad + (\eta_{\mu\sigma}\partial_\nu\partial_\tau + \eta_{\nu\sigma}\partial_\mu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\sigma + \eta_{\nu\tau}\partial_\mu\partial_\sigma)E(x-y)],\end{aligned} \tag{6.28}$$

$$[\varphi_{\mu\nu}(x), \beta_\rho(y)] = -i\phi_0^{-2}(\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu)D(x-y), \tag{6.29}$$

$$[\varphi_{\mu\nu}(x),\beta(y)] = 2i\phi_0^{-1}\eta_{\mu\nu}D(x-y), \tag{6.30}$$

$$[\tilde{\phi}(x),\beta(y)] = -i\phi_0^{-1}D(x-y), \tag{6.31}$$

$$[\hat{s}_\mu(x),\hat{s}_\nu(y)] = i\left(\eta_{\mu\nu}-\frac{1}{m^2}\partial_\mu\partial_\nu\right)\Delta(x-y;m^2), \tag{6.32}$$

$$\{\gamma^\sigma(x),\bar{\gamma}_\tau(y)\} = \phi_0^{-2}\delta^\sigma_\tau D(x-y), \tag{6.33}$$

$$\{\gamma(x),\bar{\gamma}(y)\} = \phi_0^{-2}D(x-y). \tag{6.34}$$

The other 4D CRs vanish identically.

Now we would like to discuss the issue of the unitarity of the physical S-matrix. To do that, it is convenient to perform the Fourier transformation of Eqs. (6.28)–(6.34). However, for the dipole field we cannot use the three-dimensional Fourier expansion to define the creation and annihilation operators. We therefore make use of the four-dimensional Fourier expansion [21][12]

$$\varphi_{\mu\nu}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\varphi_{\mu\nu}(p)e^{ipx}+\varphi^\dagger_{\mu\nu}(p)e^{-ipx}], \tag{6.35}$$

where $\theta(p_0)$ is the step function. For any simple pole fields, we adopt the same Fourier expansion, for instance,

$$\beta_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\beta_\mu(p)e^{ipx}+\beta^\dagger_\mu(p)e^{-ipx}]. \tag{6.36}$$

Thus, using Eqs. (6.22), (6.23), (6.35), and (6.36), for instance, the Fourier transforms of, e.g., $\varphi_{\mu\nu}(x)$ and $\beta_\mu(x)$, take the following expressions:

$$\varphi_{\mu\nu}(p) = \frac{i}{(2\pi)^{\frac{3}{2}}}\theta(p_0)\int d^3z e^{-ipz}\overset{\leftrightarrow}{\partial}{}^z_0[\delta(p^2)\varphi_{\mu\nu}(z) + \delta'(p^2)\Box\varphi_{\mu\nu}(z)],$$

$$\beta_\mu(p) = \frac{i}{(2\pi)^{\frac{3}{2}}}\theta(p_0)\delta(p^2)\int d^3z e^{-ipz}\overset{\leftrightarrow}{\partial}{}^z_0\beta_\mu(z). \tag{6.37}$$

Incidentally, for a generic simple pole field Φ with a mass m, the three-dimensional Fourier expansion is defined as

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p\,\frac{1}{\sqrt{2\omega_p}}[\Phi(\vec{p})e^{ipx}+\Phi^\dagger(\vec{p})e^{-ipx}], \tag{6.38}$$

with $\omega_p = \sqrt{\vec{p}^2+m^2}$, whereas the four-dimensional Fourier expansion reads

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\Phi(p)e^{ipx}+\Phi^\dagger(p)(p)e^{-ipx}]. \tag{6.39}$$

Thus, the annihilation operator $\Phi(p)$ in the four-dimensional Fourier expansion has a connection with the annihilation operator $\Phi(\vec{p})$ in the three-dimensional Fourier expansion via

$$\Phi(p) = \theta(p_0)\delta(p^2+m^2)\sqrt{2\omega_p}\Phi(\vec{p}). \tag{6.40}$$

Based on these Fourier expansions, we can calculate the Fourier transforms of Eqs. (6.28)–(6.34):

$$[\varphi_{\mu\nu}(p),\varphi^\dagger_{\sigma\tau}(q)] = -\frac{2}{\xi}\phi_0^{-2}\theta(p_0)\delta^4(p-q)[\delta(p^2)(\eta_{\mu\nu}\eta_{\sigma\tau}-\eta_{\mu\sigma}\eta_{\nu\tau}-\eta_{\mu\tau}\eta_{\nu\sigma}) - 3\delta'(p^2)(\eta_{\mu\sigma}p_\nu p_\tau+\eta_{\nu\sigma}p_\mu p_\tau+\eta_{\mu\tau}p_\nu p_\sigma+\eta_{\nu\tau}p_\mu p_\sigma)], \tag{6.41}$$

$$[\varphi_{\mu\nu}(p),\beta^\dagger_\rho(q)] = -i\phi_0^{-2}(\eta_{\mu\rho}p_\nu+\eta_{\nu\rho}p_\mu)\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.42}$$

$$[\varphi_{\mu\nu}(p),\beta^\dagger(q)] = 2\phi_0^{-1}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.43}$$

$$[\tilde{\phi}(p),\beta^\dagger(q)] = -\phi_0^{-1}\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.44}$$

$$[\hat{s}_\mu(p),\hat{s}^\dagger_\nu(q)] = +\left(\eta_{\mu\nu}-\frac{1}{m^2}p_\mu p_\nu\right)\times\theta(p_0)\delta(p^2+m^2)\delta^4(p-q), \tag{6.45}$$

$$\{\gamma^\sigma(p),\bar{\gamma}^\dagger_\tau(q)\} = -i\phi_0^{-2}\delta^\sigma_\tau\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.46}$$

$$\{\gamma(p),\bar{\gamma}^\dagger(q)\} = -i\phi_0^{-2}\theta(p_0)\delta(p^2)\delta^4(p-q). \tag{6.47}$$

Next, let us turn our attention to the linearized field equations. After Fourier transformation, Eq. (6.7) takes the form

$$p^\nu\varphi_{\mu\nu}-\frac{1}{2}p_\mu\varphi = 2\phi_0^{-1}p_\mu\tilde{\phi}. \tag{6.48}$$

If we fix the degree of freedom associated with $\tilde{\phi}$, which will be discussed later, this equation gives us four independent relations on ten components of $\varphi_{\mu\nu}(p)$, thereby reducing the independent components of $\varphi_{\mu\nu}(p)$ to be six.

[12]For simplicity, the Fourier transform of a field is denoted by the same field except for the argument p instead of x.

To deal with six independent components of $\varphi_{\mu\nu}(p)$, it is convenient to take a specific Lorentz frame such that $p_1 = p_2 = 0$ and $p_3 > 0$, and choose the six components as follows:

$$\varphi_1(p) = \frac{1}{2}[\varphi_{11}(p) - \varphi_{22}(p)], \quad \varphi_2(p) = \varphi_{12}(p),$$
$$\omega_0(p) = -\frac{1}{2p_0}\varphi_{00}(p), \quad \omega_I(p) = -\frac{1}{p_0}\varphi_{0I}(p),$$
$$\omega_3(p) = -\frac{1}{2p_3}\varphi_{33}(p), \tag{6.49}$$

where the index I takes the transverse components $I = 1, 2$.

In this respect, it is worthwhile to consider the GCT BRST transformation for these components. First, let us write down the GCT BRST transformation for the Fourier expansion of the asymptotic fields, which reads

$$\delta_B\varphi_{\mu\nu}(p) = -i[p_\mu\gamma_\nu(p) + p_\nu\gamma_\mu(p)], \quad \delta_B\gamma^\mu(p) = 0,$$
$$\delta_B\bar{\gamma}_\mu(p) = i\beta_\mu(p),$$
$$\delta_B\tilde{\phi}(p) = \delta_B\beta_\mu(p) = \delta_B\beta(p) = \delta_B\gamma(p) = \delta_B\bar{\gamma}(p) = 0. \tag{6.50}$$

Using this BRST transformation, the GCT BRST transformation for the components in (6.49) takes the form

$$\delta_B\varphi_I(p) = 0, \qquad \delta_B\omega_\mu(p) = i\gamma_\mu(p),$$
$$\delta_B\bar{\gamma}_\mu(p) = i\beta_\mu(p), \qquad \delta_B\gamma_\mu(p) = \delta_B\beta_\mu(p) = 0, \tag{6.51}$$

where $p_1 = p_2 = 0$ was used. This BRST transformation implies that $\varphi_I(p)$ could be the physical observable while a set of fields, $\{\omega_\mu(p), \beta_\mu(p), \gamma_\mu(p), \bar{\gamma}_\mu(p)\}$, might belong to the BRST quartet and thus are dropped from the physical state by the Kugo-Ojima subsidiary condition, $Q_B|\text{phys}\rangle = 0$ [25].[13]

Next, let us move on to the other BRST transformation, which is the BRST transformation for the Weyl transformation. The Weyl BRST transformation for the asymptotic fields is of the form

$$\bar{\delta}_B\varphi_{\mu\nu} = 2c\eta_{\mu\nu}, \quad \bar{\delta}_B\tilde{\phi} = -\phi_0\gamma, \quad \bar{\delta}_B\gamma = 0, \quad \bar{\delta}_B\bar{\gamma} = i\beta,$$
$$\bar{\delta}_B\beta = \bar{\delta}_B\beta_\mu = \bar{\delta}_B\gamma_\mu = \bar{\delta}_B\bar{\gamma}_\mu = 0. \tag{6.52}$$

The Weyl BRST transformation of φ_I is vanishing,

$$\bar{\delta}_B\varphi_I = 0, \tag{6.53}$$

which means that together with $\delta_B\varphi_I = 0$, φ_I is truly the physical observable. The four-dimensional commutation relations among the fields $\{\tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ read

$$[\tilde{\phi}(p), \tilde{\phi}^\dagger(q)] = 0,$$
$$[\tilde{\phi}(p), \beta^\dagger(q)] = -\phi_0^{-1}\theta(p_0)\delta(p^2)\delta^4(p-q),$$
$$\{\gamma(p), \bar{\gamma}^\dagger(q)\} = -i\phi_0^{-2}\theta(p_0)\delta(p^2)\delta^4(p-q). \tag{6.54}$$

As can also be seen in these 4D CRs, all the fields $\{\varphi_I, \tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ are massless simple pole fields. Via relation (6.40) the three-dimensional commutation relations $[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\}$ with $\Phi(\vec{p}) \equiv \{\varphi_I(\vec{p}), \tilde{\phi}(\vec{p}), \beta(\vec{p}), \gamma(\vec{p}), \bar{\gamma}(\vec{p})\}$ are of form

$$[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\} = \left(\begin{array}{c|cc|c} \frac{2}{\xi}\phi_0^{-2}\delta_{IJ} & & & \\ \hline & 0 & -\phi_0^{-1} & \\ & -\phi_0^{-1} & 0 & \\ \hline & & & -i\phi_0^{-2} \\ & & & +i\phi_0^{-2} \end{array}\right) \times \delta(\vec{p} - \vec{q}). \tag{6.55}$$

Thus, φ_I is the physical observable while the set of fields $\{\tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ consists of the BRST quartet and is the unphysical mode by Kugo-Ojima's subsidiary condition [25]. Here it is worth mentioning that the Nambu-Goldstone boson $\tilde{\phi}$ associated with spontaneous symmetry breaking of the Weyl gauge symmetry is an unphysical particle. In this context, let us recall that the Nambu-Goldstone theorem never tells us whether the Nambu-Goldstone boson is physical or unphysical. From our analysis at hand, we can conclude that the Nambu-Goldstone boson $\tilde{\phi}$ is the unphysical mode, which is absorbed into the longitudinal mode of the Weyl gauge field $s_\mu(x)$, and thereby the gauge field becomes massive.

[13]The situation is, in fact, a bit complicated since $\beta_\mu(p), \gamma_\mu(p)$, and $\bar{\gamma}_\mu(p)$ are simple pole fields obeying $p^2\beta_\mu(p) = p^2\gamma_\mu(p) = p^2\bar{\gamma}_\mu(p) = 0$, while $\varphi_{\mu\nu}(p)$ is a dipole field satisfying $(p^2)^2\varphi_{\mu\nu}(p) = 0$, so that a naive Kugo-Ojima's quartet mechanism does not work in a direct way. But this problem can be remedied by introducing an operator that takes out a simple pole from a dipole field. The detail can be shown in Ref. [18].

Finally, let us focus on the Weyl gauge field $\hat{s}_\mu$, which satisfies the Lorenz condition (6.13) and the massive Klein-Gordon equation (6.15). In a specific Lorentz frame

$$p_\mu = (m, 0, 0, 0), \tag{6.56}$$

the Lorenz condition (6.13) produces

$$\hat{s}_0(p) = 0. \tag{6.57}$$

With the Lorentz frame (6.56), it turns out that the spatial components of $\hat{s}_\mu$ are invariant under both GCT and Weyl BRST transformations:

$$\delta_B \hat{s}_i(p) = \bar{\delta}_B \hat{s}_i(p) = 0. \tag{6.58}$$

Moreover, using the relation (6.40) and Eq. (6.45), the commutation relation between the three-dimensional annihilation and creation operators reads

$$[\hat{s}_i(\vec{p}), \hat{s}_j^\dagger(\vec{q})] = \delta_{ij}\delta^3(\vec{p} - \vec{q}). \tag{6.59}$$

Together with the BRST invariance in Eq. (6.58), this equation clearly shows that the spatial components $\hat{s}_i(x)$ are really genuine physical massive modes belonging to BRST singlets with positive norm.

VII. CHORAL SYMMETRY

In the previous article [18], we have clarified the existence of a huge global symmetry called "choral symmetry," which is the $IOSp(10|10)$ symmetry, in Weyl invariant scalar-tensor gravity in Riemann geometry. We will show that the choral symmetry also exists in the theory at hand. The existence of the choral symmetry is expected from the fact that as shown in Sec. IV, a set of fields (including the spacetime coordinates x^μ) $X^M \equiv \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$ obeys a very simple equation,

$$g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0. \tag{7.1}$$

It is worthwhile to note that this equation holds if and only if we adopt the extended de Donder gauge condition (4.8) for the GCT and the scalar gauge condition (4.9) for the Weyl gauge transformation. Furthermore, Eq. (7.1) implies that there should be many conserved currents defined in Eq. (4.33) in the theory under consideration. In this section, along the same line of argument as that in the previous article [18,19], we will explicitly prove that there is the choral symmetry $IOSp(10|10)$ in Weyl conformal gravity in Weyl geometry.

Let us start with the Lagrangian (4.11), which can be cast to the form

$$\mathcal{L}_q = \sqrt{-g}\left[\frac{1}{2}\xi\phi^2(R - 6\nabla_\mu S^\mu - 6S_\mu S^\mu) - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon g^{\mu\nu}(-2\phi\partial_\mu\phi S_\nu + S_\mu S_\nu \phi^2)\right] - \frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\hat{E}_{\mu\nu}, \tag{7.2}$$

where we have defined $\hat{E}_{\mu\nu}$ as

$$\hat{E}_{\mu\nu} = \frac{1}{2}\epsilon\partial_\mu\sigma\partial_\nu\sigma + \partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda - \partial_\mu B\partial_\nu\sigma + i\partial_\mu\bar{c}\partial_\nu c + (\mu \leftrightarrow \nu) \tag{7.3}$$

and used the relation (4.15) between the scalar field ϕ and the dilaton σ.

Next, let us focus our attention on the last term in Eq. (7.2) and rewrite it into a more compact form:

$$\mathcal{L}_q^{(E)} \equiv -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\hat{E}_{\mu\nu} = -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\eta_{NM}\partial_\mu X^M\partial_\nu X^N = -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu X^M\tilde{\eta}_{MN}\partial_\nu X^N. \tag{7.4}$$

Here we have introduced an $IOSp(10|10)$ metric $\eta_{NM} = \eta_{MN}^T \equiv \tilde{\eta}_{MN}$ defined as [26]

$$\eta_{NM} = \tilde{\eta}_{MN} = \begin{array}{c} \\ x^\nu \\ b_\nu \\ \sigma \\ B \\ c^\nu \\ \bar{c}_\nu \\ c \\ \bar{c} \end{array} \begin{array}{c} \begin{array}{cccccccc} x^\mu & b_\mu & \sigma & B & c^\mu & \bar{c}_\mu & c & \bar{c} \end{array} \\ \left(\begin{array}{cccccccc} & \delta_\mu^\nu & & & & & & \\ \delta_\nu^\mu & & & & & & & \\ & & \epsilon & -1 & & & & \\ & & -1 & 0 & & & & \\ & & & & & -i\delta_\mu^\nu & & \\ & & & & i\delta_\nu^\mu & & & \\ & & & & & & & -i \\ & & & & & & i & \end{array}\right) \end{array}. \tag{7.5}$$

Let us note that this $IOSp(10|10)$ metric η_{NM}, which is a c-number quantity, has the symmetry property such that

$$\eta_{MN} = (-)^{|M|\cdot|N|}\eta_{NM} = (-)^{|M|}\eta_{NM} = (-)^{|N|}\eta_{NM}, \tag{7.6}$$

where the statistics index $|M|$ is 0 or 1 when X^M is Grassmann-even or Grassmann-odd, respectively. This property comes from the fact that η_{MN} is "diagonal" in the sense that its off-diagonal, Grassmann-even and Grassmann-odd, and vice versa, matrix elements vanish, i.e., $\eta_{MN} = 0$ when $|M| \neq |N|$, thereby being $|M| = |N| = |M| \cdot |N|$ in front of η_{MN} [26].

Now that (7.4) is expressed in a manifestly $IOSp(10|10)$ invariant form except for the Weyl invariant metric $\tilde{g}^{\mu\nu}\phi^2$, which will be discussed later, there could exist an $IOSp(10|10)$ as a global symmetry in our theory. Note that the infinitesimal OSp rotation is defined by

$$\delta X^M = \eta^{ML}\varepsilon_{LN}X^N \equiv \varepsilon^M{}_N X^N, \tag{7.7}$$

where η^{MN} is the inverse matrix of η_{MN}, and the infinitesimal parameter ε_{MN} has the following properties:

$$\varepsilon_{MN} = (-)^{1+|M|\cdot|N|}\varepsilon_{NM}, \quad \varepsilon_{MN}X^L = (-)^{|L|(|M|+|N|)}X^L\varepsilon_{MN}. \tag{7.8}$$

To find the conserved current, we assume that the infinitesimal parameter ε_{MN} depends on the spacetime coordinates x^μ, i.e., $\varepsilon_{MN} = \varepsilon_{MN}(x^\mu)$.

Assuming for a while that the metric $\tilde{g}^{\mu\nu}\phi^2$ is invariant under the OSp rotation (7.7), we find that (7.4) is transformed as

$$\delta\mathcal{L}_q^{(E)} = -\tilde{g}^{\mu\nu}\phi^2(\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N + \varepsilon_{NM}\partial_\mu X^M\partial_\nu X^N). \tag{7.9}$$

It is easy to prove that the second term on the RHS vanishes owing to the first property in Eq. (7.8). Thus, $\mathcal{L}_q^{(E)}$ is invariant under the infinitesimal OSp rotation. The conserved current is then calculated to be

$$\begin{aligned}\delta\mathcal{L}_q^{(E)} &= -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}[X^M\partial_\nu X^N - (-)^{|M|\cdot|N|}X^N\partial_\nu X^M] \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}(X^M\partial_\nu X^N - \partial_\nu X^M X^N) \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\overleftrightarrow{\partial}_\nu X^N \\ &\equiv -\frac{1}{2}\partial_\mu\varepsilon_{NM}\mathcal{M}^{\mu MN},\end{aligned} \tag{7.10}$$

with the conserved current $\mathcal{M}^{\mu MN}$ for the OSp rotation taking the form

$$\mathcal{M}^{\mu MN} = \tilde{g}^{\mu\nu}\phi^2 X^M\overleftrightarrow{\partial}_\nu X^N. \tag{7.11}$$

The above proof makes sense only under the assumption that the metric $\tilde{g}^{\mu\nu}\phi^2$ and the other terms except for the last term in (7.2) are invariant under the OSp rotation, but it is obviously not the case. However, this problem is cured by noticing that the OSp rotation includes a Weyl transformation on the dilaton,

$$\delta\sigma = \eta^{\sigma L}\varepsilon_{LN}X^N = -\varepsilon_{BN}X^N \equiv -\varepsilon(x), \tag{7.12}$$

where we have used (7.5) and

$$\begin{pmatrix}\epsilon & -1\\ -1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & -1\\ -1 & -\epsilon\end{pmatrix}, \tag{7.13}$$

where we recall that the matrix η^{ML} is the inverse matrix of η_{ML}. As for the scalar field $\phi(x)$, this transformation for the dilaton can be interpreted as a Weyl transformation,

$$\phi \to \phi' = e^{-\varepsilon(x)}\phi. \tag{7.14}$$

Thus, simultaneously with the OSp rotation, if we perform a Weyl transformation, which is denoted as δ', given by

$$\delta' g_{\mu\nu} = 2\varepsilon(x)g_{\mu\nu}, \qquad \delta' S_\mu = -\partial_\mu\varepsilon(x), \tag{7.15}$$

and a local shift for the Nakanishi-Lautrup field B, which is also denoted as δ',[14]

$$\delta' B = \epsilon\varepsilon(x), \tag{7.16}$$

it turns out that under the (local) OSp rotation δ in (7.7) and the additional transformation δ', the quantum Lagrangian $\mathcal{L}_q$ transforms as

$$(\delta+\delta')\mathcal{L}_q = -\frac{1}{2}\partial_\mu\varepsilon_{NM}\mathcal{M}^{\mu MN}. \tag{7.17}$$

As a result, the conserved current $\mathcal{M}^{\mu MN}$ for the OSp rotation takes the form (7.11).

In a similar way, we can derive the conserved current for the infinitesimal translation,

$$\delta X^M = \varepsilon^M, \tag{7.18}$$

and it turns out that the conserved current $\mathcal{P}^{\mu M}$ for the translation reads

$$\mathcal{P}^{\mu M} = \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2(1\overleftrightarrow{\partial}_\nu X^M). \tag{7.19}$$

From the conserved currents (7.11) and (7.19), the corresponding conserved charges are given by

$$\begin{aligned}M^{MN} &\equiv \int d^3x\mathcal{M}^{0MN} = \int d^3x\tilde{g}^{0\nu}\phi^2 X^M\overleftrightarrow{\partial}_\nu X^N, \\ P^M &\equiv \int d^3x\mathcal{P}^{0M} = \int d^3x\tilde{g}^{0\nu}\phi^2\partial_\nu X^M.\end{aligned} \tag{7.20}$$

For instance, the BRST charges for the GCT and Weyl transformation are, respectively, expressed as

$$\begin{aligned}Q_B &\equiv M(b_\rho, c^\rho) = \int d^3x\tilde{g}^{0\nu}\phi^2 b_\rho\overleftrightarrow{\partial}_\nu c^\rho, \\ \bar{Q}_B &\equiv M(B, c) = \int d^3x\tilde{g}^{0\nu}\phi^2 B\overleftrightarrow{\partial}_\nu c.\end{aligned} \tag{7.21}$$

We can then verify that using various ETCRs obtained so far, the $IOSp(10|10)$ generators $\{M^{MN}, P^M\}$ generate an $IOSp(10|10)$ algebra:

[14]Under the OSp rotation, the B field is transformed as $\delta B = \eta^{BL}\varepsilon_{LN}X^N = -\varepsilon_{\sigma N}X^N + \epsilon\varepsilon$. The transformation (7.16) is carried out independently of this OSp rotation.

$$
\begin{aligned}
&[P^M, P^N\} = 0,\\
&[M^{MN}, P^R\} = i[P^M\tilde{\eta}^{NR} - (-)^{|N||R|}P^N\tilde{\eta}^{MR}],\\
&[M^{MN}, M^{RS}\} = i[M^{MS}\tilde{\eta}^{NR} - (-)^{|N||R|}M^{MR}\tilde{\eta}^{NS}\\
&\quad - (-)^{|N||R|}M^{NS}\tilde{\eta}^{MR} + (-)^{|M||R|+|N||S|}M^{NR}\tilde{\eta}^{MS}], \qquad (7.22)
\end{aligned}
$$

where $[A, B\}$ denotes the graded commutator, which is the anticommutator if both A and B are fermionic and the commutator otherwise.

Finally, it is useful to compare our extended choral symmetry $IOSp(10|10)$ with the original choral symmetry $IOSp(8|8)$ in Einstein's general relativity [21]. In our case, the choral symmetry is extended in the sense that the GCT is replaced with a larger symmetry, which consists of both the GCT and the Weyl gauge transformation. Accordingly the dilaton σ, the Nakanishi-Lautrup field B, ghost c, and antighost $\bar{c}$ are joined in the algebra. The choral symmetry $IOSp(10|10)$ therefore includes the dilaton or, equivalently, the scalar field, which exists in the classical Lagrangian and is closely related to a classical theory. In contrast, the original $IOSp(8|8)$ symmetry is purely a symmetry among quantum fields, which are the NL field and ghosts, so the symmetry is limited to the sector related to the gauge-fixing procedure. From this viewpoint, we expect that the extended $IOSp(10|10)$ choral symmetry might play an important role in clarifying the dynamics peculiar to the classical theory.

VIII. GRAVITATIONAL CONFORMAL SYMMETRY AND SPONTANEOUS SYMMETRY BREAKDOWN

One of the most interesting features in the formalism at hand is that as an analog of the well-known conformal symmetry in a flat Minkowski spacetime, there is a gravitational conformal symmetry which is a subgroup of the choral symmetry, and its spontaneous symmetry breakdown down to the Poincaré symmetry guarantees that the graviton and the dilaton are exactly massless Nambu-Goldstone particles [18]. This feature is so important for future developments of quantum gravity that we would like to explain the gravitational conformal symmetry and its spontaneous symmetry breakdown in detail.

In particular, as already shown in Sec. VI, there is a *massive* Weyl gauge field in the spectrum, so at first sight it appears to be strange that there is a conformal symmetry in the present theory since it is usually thought that conformal or scale symmetry exists in the theories with only massless particles. With regard to this, it is worthwhile to recall that the massless Weyl gauge field acquires the mass via spontaneous symmetry breakdown (SSB) of Weyl gauge symmetry and the SSB is the breakdown of symmetry at the level of not field operators but the representation of field operators in the sense that the symmetry cannot be realized by a unitary transformation in the state vector space. Thus, it is not strange that there is a conformal symmetry in the present theory with the massive gauge field if the mass is generated through the SSB. Moreover, this physical situation is also supported by the Zumino theorem [27] to some degree since the theorem insists that theories invariant under general coordinate transformation and Weyl transformation at the same time should possess conformal symmetry in a flat Minkowski background at least classically.

As clarified in the previous paper [18], the extended de Donder gauge condition (4.8) and the scalar gauge condition (4.9) have a residual symmetry that corresponds to the dilatation and the special conformal transformation in a flat Minkowski spacetime. Indeed, the quantum Lagrangian (4.11) is still invariant under the restricted Weyl transformation [28]:

$$
\begin{aligned}
&\delta g_{\mu\nu} = 2\Lambda g_{\mu\nu}, \qquad \delta\phi = -\Lambda\phi,\\
&\delta S_\mu = -\partial_\mu\Lambda, \qquad \delta b_\mu = -\partial_\mu\Lambda B, \qquad (8.1)
\end{aligned}
$$

where the infinitesimal transformation parameter Λ takes the form

$$
\Lambda = \lambda - 2k_\mu x^\mu, \qquad (8.2)
$$

with λ and k_μ being infinitesimal constants corresponding to a global scale transformation and the special conformal transformation, respectively [18]. Note that Λ obeys the equation $g^{\mu\nu}\partial_\mu\partial_\nu\Lambda = 0$, which is a characteristic feature of the restricted Weyl transformation. The whole global symmetry in the theory under consideration should be included in the extended $IOSp(10|10)$ choral symmetry. Actually, we can construct the generators corresponding to the transformation parameters λ and k_μ out of those of the choral symmetry as

$$
\begin{aligned}
&D_0 \equiv -P(B) = -\int d^3x\tilde{g}^{0\nu}\phi^2\partial_\nu B,\\
&K^\mu \equiv 2M^\mu(x, B) = 2\int d^3x\tilde{g}^{0\nu}\phi^2 x^\mu \overleftrightarrow{\partial}_\nu B. \qquad (8.3)
\end{aligned}
$$

It is easy to verify that these generators generate the symmetry (8.1) in terms of the ETCRs in (5.23).

Our theory is also invariant under the translation and the general linear transformation $GL(4)$. Actually, we can make the translation generator P_μ and $GL(4)$ generator $G^\mu{}_\nu$ from the choral symmetry as

$$
\begin{aligned}
&P_\mu \equiv P_\mu(b) = \int d^3x\tilde{g}^{0\nu}\phi^2\partial_\nu b_\mu,\\
&G^\mu{}_\nu \equiv M^\mu{}_\nu(x, b) - iM^\mu{}_\nu(c^\tau, \bar{c}_\tau)\\
&\quad = \int d^3x\tilde{g}^{0\lambda}\phi^2(x^\mu \overleftrightarrow{\partial}_\lambda b_\nu - ic^\mu \overleftrightarrow{\partial}_\lambda \bar{c}_\nu). \qquad (8.4)
\end{aligned}
$$

For instance, based on the ETCRs in (5.23), we can check that the $GL(4)$ generator $G^{\mu}{}_{\nu}$ correctly generates the $GL(4)$ transformation on the fields ϕ, S_ρ, and $g_{\sigma\tau}$:

$$[iG^{\mu}{}_{\nu}, \phi] = x^\mu \partial_\nu \phi, \qquad [iG^{\mu}{}_{\nu}, S_\rho] = x^\mu \partial_\nu S_\rho + \delta^\mu_\rho S_\nu,$$
$$[iG^{\mu}{}_{\nu}, g_{\sigma\tau}] = x^\mu \partial_\nu g_{\sigma\tau} + \delta^\mu_\sigma g_{\nu\tau} + \delta^\mu_\tau g_{\nu\sigma}. \tag{8.5}$$

Finally, we can build a generator corresponding to the dilatation in a flat Minkowski spacetime, which is closely related to the generator D_0 of the scale transformation in (8.3). The construction of such a dilatation generator has been explained in Ref. [18], so the result is given by

$$D \equiv G^{\mu}{}_{\mu} - D_0. \tag{8.6}$$

It turns out that the algebra among the generators $\{P_\mu, G^{\mu}{}_{\nu}, K^\mu, D\}$ closes and takes the form

$$[P_\mu, P_\nu] = 0, \qquad [P_\mu, G^{\rho}{}_{\sigma}] = iP_\sigma \delta^\rho_\mu,$$
$$[P_\mu, K^\nu] = -2i(G^{\rho}{}_{\rho} - D)\delta^\nu_\mu, \qquad [P_\mu, D] = iP_\mu,$$
$$[G^{\mu}{}_{\nu}, G^{\rho}{}_{\sigma}] = i(G^{\mu}{}_{\sigma}\delta^\rho_\nu - G^{\rho}{}_{\nu}\delta^\mu_\sigma),$$
$$[G^{\mu}{}_{\nu}, K^\rho] = iK^\mu \delta^\rho_\nu, \qquad [G^{\mu}{}_{\nu}, D] = [K^\mu, K^\nu] = 0,$$
$$[K^\mu, D] = -iK^\mu, \qquad [D, D] = 0. \tag{8.7}$$

To extract the gravitational conformal algebra in quantum gravity, it is necessary to introduce the "Lorentz" generator. It can be constructed from the $GL(4)$ generator and the flat Minkowski metric to be

$$M_{\mu\nu} \equiv -\eta_{\mu\rho} G^{\rho}{}_{\nu} + \eta_{\nu\rho} G^{\rho}{}_{\mu}. \tag{8.8}$$

In terms of the generator $M_{\mu\nu}$, the algebra (8.7) can be cast to the form

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = i(P_\rho \eta_{\mu\sigma} - P_\sigma \eta_{\mu\rho}),$$
$$[P_\mu, K^\nu] = -2i(G^{\rho}{}_{\rho} - D)\delta^\nu_\mu, \quad [P_\mu, D] = iP_\mu,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\sigma}\eta_{\mu\rho} + M_{\rho\mu}\eta_{\sigma\nu} - M_{\rho\nu}\eta_{\sigma\mu}),$$
$$[M_{\mu\nu}, K^\rho] = i(-K_\mu \delta^\rho_\nu + K_\nu \delta^\rho_\mu), \quad [M_{\mu\nu}, D] = [K^\mu, K^\nu] = 0,$$
$$[K^\mu, D] = -iK^\mu, \quad [D, D] = 0, \tag{8.9}$$

where we have defined $K_\mu \equiv \eta_{\mu\nu} K^\nu$. It is of interest that the algebra (8.9) in quantum gravity, which we call "gravitational conformal algebra," formally resembles conformal algebra in the flat Minkowski spacetime except for the expression of $[P_\mu, K^\nu]$ [29,30].[15] This difference stems from the difference of the definition of conformal dimension (or weight) in both gravity and conformal field theory, for which the metric tensor field $g_{\mu\nu}$ has weight 2 in gravity while it has weight 0 in conformal field theory.

On the basis of the gravitational conformal symmetry, we are able to show that the $GL(4)$ symmetry is spontaneously broken to the Poincaré symmetry whose Nambu-Goldstone boson is the graviton [18]. The dilatation symmetry and the special conformal symmetry are also spontaneously broken and the corresponding Nambu-Goldstone bosons are the dilaton and the derivative of the dilaton, respectively [18]. Interest here is that the Nambu-Goldstone boson associated with the special conformal symmetry is not an independent field in quantum gravity as in conformal field theory [31].

IX. CONCLUSION

In this article, we have presented a BRST formalism of a Weyl conformal gravity in Weyl geometry. The essential ingredient in our formalism is choosing suitable gauge conditions for the general coordinate invariance and the Weyl invariance. To implement two independent BRST transformations δ_B and $\bar{\delta}_B$ corresponding to the GCT and the Weyl transformation, respectively, i.e., $\{\delta_B, \bar{\delta}_B\} = 0$, one has to select the gauge conditions in such a way that the gauge condition for the GCT must be invariant under the Weyl transformation and that for the Weyl transformation must be so under the GCT [18].

In addition, both gauge conditions must give us a gauge invariant measure in place of the conventional measure $\sqrt{-g}$ and ensure the masslessness of the dilaton. Interestingly enough, such gauge conditions are almost uniquely determined by the extended de Donder gauge condition (4.8) for the GCT and the scalar gauge condition (4.9) for the Weyl transformation. With the other gauge conditions, we cannot construct the conserved currents for the extended choral symmetry, and without the choral symmetry we cannot ensure the gravitational conformal algebra such that we cannot prove the masslessness of the graviton and the dilaton. It is usually said that the gauge conditions do not change the physical content of a theory, but it is true that the existence of global symmetries seems to critically depend on the gauge choice as seen in the present study of Weyl conformal gravity.[16]

As for the future works, we would like to present a BRST formalism of quadratic conformal gravity (3.1) since this theory is the unique theory that is invariant under Weyl gauge transformation without matter fields.[17] However, it is known that higher-derivative gravities such as quadratic gravity generally suffer from the existence of a massless or

[15] In the case of conformal algebra in the flat spacetime, the expression is given by $[P_\mu, K^\nu] = -2i(\delta^\nu_\mu D + M_\mu{}^\nu)$.

[16] Independence of the gauge conditions holds only in the case of the absence of gauge anomalies, which might stem from the Abelian gauge field S_μ. The issue of anomalies is one of the future problems.

[17] Recently, spontaneous symmetry breakdown of conformal symmetry in quantum quadratic gravity in Riemann geometry has been investigated in [32].

massive ghost which prevents a lower bound of energy at the classical level and violates the unitarity at the quantum level. Thus, we have to provide a recipe for nullifying such a ghost. Since our choral symmetry is a huge global symmetry including the gravitational conformal symmetry, it might give us a useful tool for attacking various important problems such as the ghost and renormalizability. The work is currently in progress with partial affirmative results.

ACKNOWLEDGMENTS

This work is supported in part by the JSPS Kakenhi Grant No. 21K03539, and P. S. is supported by the IMPRS-PTFS.

APPENDIX: A DERIVATION OF CONSTRAINTS

In this appendix we present a derivation of the secondary constraint (5.18) and show that the tertiary constraint is vanishing as seen in Eq. (5.19).

Since the primary constraint (5.6) is given by $\pi_S^0 \approx 0$, the terms involving S_0 in the Hamiltonian density $\mathcal{H}_T$ contribute in the calculation of the secondary constraint. Thus, the relevant part in $\mathcal{H}_T$ is given by[18]

$$\mathcal{H}_T \sim \pi_S^\mu \dot{S}_\mu + \pi_B \dot{B} - \mathcal{L}_q, \tag{A1}$$

where the relevant part in $\mathcal{L}_q$ reads

$$\mathcal{L}_q = -3\xi\tilde{g}^{\mu\nu}\phi^2 S_\mu S_\nu + 6\xi\tilde{g}^{\mu\nu}\phi S_\mu \partial_\nu\phi - \frac{1}{4}\sqrt{-g}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}\epsilon\tilde{g}^{\mu\nu}D_\mu\phi D_\nu\phi + \tilde{g}^{\mu\nu}\partial_\mu B\phi\partial_\nu\phi. \tag{A2}$$

Note that S_0 in $-\frac{1}{4}\sqrt{-g}H_{\mu\nu}H^{\mu\nu}$ can be expressed in terms of π_S^i so this term can be ignored.

Using Eqs. (5.5), (5.9), and (5.15), the Hamiltonian density takes the form

$$\mathcal{H}_T \sim \pi_S^i\partial_i S_0 + 3\xi\tilde{g}^{\mu\nu}\phi^2 S_\mu S_\nu - 6\xi\pi_B S_0 + \frac{1}{2}\epsilon\tilde{g}^{\mu\nu}D_\mu\phi D_\nu\phi, \tag{A3}$$

where $\dot{\phi}$ is defined by (5.9). Then, a straightforward calculation of (5.7) gives us the secondary constraint (5.18).

Next, let us evaluate the tertiary constraint which comes from the time development of the secondary constraint (5.18). For this purpose, let us calculate the Poisson bracket between the total Hamiltonian and each term in the secondary constraint (5.18). The method of the calculation is similar to that of the derivation of the secondary constraint; just write out the relevant terms in the Hamiltonian density and then evaluate the Poisson bracket. The results are presented in what follows:

$$\begin{aligned}\{H_T, \partial_i\pi_S^i\}_P &= -(6\xi+\epsilon)\partial_i(\tilde{g}^{i\mu}\phi D_\mu\phi),\\ \{H_T, \pi_B\}_P &= \partial_i(\tilde{g}^{i\mu}\phi\partial_\mu\phi),\\ \{H_T, \tilde{g}^{0\mu}\phi^2 S_\mu\}_P &= \partial_i(\tilde{g}^{i\mu}\phi^2 S_\mu).\end{aligned} \tag{A4}$$

From these expressions, it is easy to see that the tertiary constraint identically vanishes. As a remark, a direct calculation of the last Poisson bracket is a bit complicated, but instead we can make use of the relation between the Hamiltonian and time derivative and the extended de Donder gauge condition (4.8) and Eq. (4.20) as follows:

$$\begin{aligned}\{H_T, \tilde{g}^{0\mu}\phi^2 S_\mu\}_P &= -\partial_0(\tilde{g}^{0\mu}\phi^2 S_\mu)\\ &= -\partial_0(\tilde{g}^{0\mu}\phi^2)S_\mu - \tilde{g}^{0\mu}\phi^2\partial_0 S_\mu\\ &= \partial_i(\tilde{g}^{i\mu}\phi^2)S_\mu + \tilde{g}^{i\mu}\phi^2\partial_i S_\mu\\ &= \partial_i(\tilde{g}^{i\mu}\phi^2 S_\mu).\end{aligned} \tag{A5}$$

[18] We use the symbol "∼" to denote the relevant terms.

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