Steady-state entanglement for rotating Unruh-DeWitt detectors

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(Received 27 June 2022; accepted 15 August 2022; published 30 November 2022)

The Unruh-DeWitt model of particle detector is widely used to probe quantum effects in noninertial frames and curved spacetimes. In this paper, we investigate the entanglement dynamics of a quantum system composed of two rotating Unruh-DeWitt detectors in interaction with a fluctuating massless scalar field in vacuum. We obtain the necessary and sufficient condition for the detectors to achieve steady-state entanglement, and systematically investigate the steady-state entanglement of two Unruh-DeWitt detectors rotating in coaxial orbits with the same orbital radius and angular velocity. When the separation between the detectors is vanishing, the detectors can obtain steady-state entanglement dependent on the initial state, as the detectors in uniform acceleration and static detectors in a thermal bath do. Remarkably, however, when the separation between the detectors is nonvanishing but small compared with the transition wavelength of detectors, the detectors can obtain steady-state entanglement irrespectively of what the initial state is. This is the unique phenomenon caused by the centripetal acceleration and is not present in the uniform acceleration case and the case of static detectors in a thermal bath.

DOI: 10.1103/PhysRevD.106.105028

I. INTRODUCTION

A uniformly accelerated observer perceives the vacuum state defined by an inertial observer as a thermal bath at a temperature proportional to its proper acceleration. This is known as the Unruh effect [1–4]. In the investigation leading to the discovery of the Unruh effect as well as other quantum effects in curved spacetimes, the Unruh-DeWitt model of particle detector [1,5] is usually exploited, which is a pointlike two-level quantum system interacting with fluctuating quantum fields, e.g., massless scalar fields in the original work of Unruh [1]. For a single uniformly accelerated Unruh-DeWitt detector, its radiative properties, such as the transition rates [6–9] and the Lamb shift [10,11], have been extensively studied.

The physics is richer when two Unruh-DeWitt detectors are concerned since nonlocal quantum correlations between detectors, quantum entanglement for instance, may emerge. Quantum entanglement is one of the key concepts in quantum theory, and is at the core of quantum information science [12]. However, the fragility of quantum entanglement [13,14] due to inevitable environmental noises has become one of the main obstacles to the realization of entanglement-based quantum technologies. Therefore, understanding the effects of environmental noises on quantum entanglement and searching for possible steady-state entanglement which is robust against environmental noises are important in the application of quantum information science. In recent years, relevant investigations have been generalized to noninertial frames. Benatti and Floreanini showed that two uniformly accelerated Unruh-DeWitt detectors with vanishing separation coupled with fluctuating scalar fields in vacuum can obtain steady-state entanglement when an appropriate initial state is chosen, and the steady-state entanglement is the same as that of static atoms in a thermal bath at the Unruh temperature [15]. Later, the entanglement dynamics of uniformly accelerated detectors with nonvanishing separation coupled with various kinds of fluctuating quantum fields in vacuum have been studied in Refs. [16–18]. However, all these results show that uniformly accelerated Unruh-DeWitt detectors can only obtain steady-state entanglement, which is initial-state dependent, when the spatial separation between the two detectors is vanishing.

Besides the uniform acceleration, another common accelerated motion is the circular motion with a constant centripetal acceleration [19]. The quantization of scalar fields in rotating frames was first investigated by Letaw and Pfautch in 1980 [20]. Later, some interesting topics, such as the energy spectrum of the vacuum field seen by a rotating observer and the transition rate of a rotating detector, have been studied [21–29]. These studies are partly motivated by the fact that a large acceleration, which is required but still

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remains a challenging task for an observation of the Unruh effect, is easier to be achieved in the circular case compared with the linear case. Noteworthy, an interesting feature of the circularly accelerated detectors in vacuum is that the environmental noise they perceive is non-Planckian [21,22] as opposed to the Planckian one for uniformly accelerated detectors. Therefore, it is of interest to study how the behavior of entanglement dynamics for circularly moving Unruh-DeWitt detectors differs from that for the uniformly accelerated ones. In this regard, let us note that the entanglement dynamics for two rotating Unruh-DeWitt detectors has been investigated in the ultrarelativistic limit [30–32]. It has been found that, similar to the uniformly accelerated case, steady-state entanglement, which is initial-state dependent, is possible only in the limit of a vanishing spatial separation.

Recently, we have developed a physical-scenario-independent approach to studying the general properties of the entanglement dynamics of an open quantum system and obtained the necessary and sufficient condition for the steady-state entanglement in some circumstances [33]. In this paper, we systematically investigate, with the help of the approach introduced in Ref. [33], the steady-state entanglement of two Unruh-DeWitt detectors rotating in coaxial orbits with the same orbital radius and angular velocity in vacuum, which are coupled with a fluctuating massless scalar field. Different from Refs. [30–32], we are not restricted to the ultrarelativistic limit, and we discover that very interesting physical phenomena remarkably emerge when the limit is relaxed. Hereafter, natural units with $\hbar = c = k_B = 1$ are used, where c is the speed of light, $\hbar$ the reduced Planck constant, and k_B the Boltzmann constant.

II. THE BASIC FORMALISM

We consider a quantum system composed of a pair of rotating Unruh-DeWitt detectors coupled with a fluctuating massless scalar field in vacuum. As shown in Fig. 1, the two detectors rotate around a common axis (z-axis) with the same orbital radius R and angular velocity Ω. We denote the separation between their rotation planes and the phase angle difference as L and φ $(0 \leq \varphi < 2\pi)$, respectively. Then, in the laboratory frame, the trajectories of the two detectors can be written as

$$x_1(t) = R\cos(\Omega t + \varphi), \quad y_1(t) = R\sin(\Omega t + \varphi), \quad z_1(t) = L,$$
$$x_2(t) = R\cos(\Omega t), \quad y_2(t) = R\sin(\Omega t), \quad z_2(t) = 0, \quad (1)$$

respectively, where t is the coordinate time in the laboratory frame. Moreover, the linear velocity v, the proper acceleration a, and the distance between the two detectors d can be expressed as

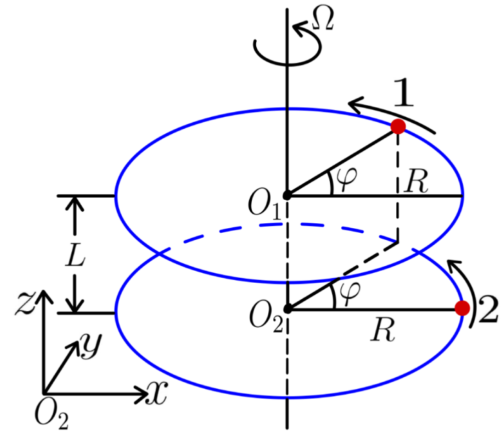

FIG. 1. Two Unruh-DeWitt detectors rotate in coaxial orbits with the same orbital radius R and angular velocity Ω.

$$v = R\Omega, \quad a = \gamma^2 R\Omega^2, \quad d = \sqrt{4R^2\sin^2(\varphi/2) + L^2}, \quad (2)$$

respectively, with $\gamma = 1/\sqrt{1 - R^2\Omega^2}$ being the Lorentz factor.

The Hamiltonian of the total system takes the form

$$H = H_S + H_F + H_I, \quad (3)$$

with H_S, H_F, and H_I being the Hamiltonian of the detectors, the Hamiltonian of the scalar field, and the Hamiltonian describing the detector-field interaction, respectively. The Hamiltonian of the two-detector system H_S can be written as

$$H_S = \frac{\omega}{2}\sigma_3^{(1)} + \frac{\omega}{2}\sigma_3^{(2)}, \quad (4)$$

where $\omega = \omega_0/\gamma$ is the energy level spacing of the detector observed in the laboratory frame, with ω_0 being the energy level spacing in the proper frame of the detector. Here, $\sigma_i^{(1)} = \sigma_i \otimes \sigma_0$, $\sigma_i^{(2)} = \sigma_0 \otimes \sigma_i$, with σ_i $(i = 1, 2, 3)$ being the Pauli matrices and σ_0 the 2×2 unit matrix. The interaction Hamiltonian H_I can be written in analogy to the electric dipole interaction as [7]

$$H_I = \mu[\sigma_2^{(1)}\phi(x^{(1)}(t)) + \sigma_2^{(2)}\phi(x^{(2)}(t))], \quad (5)$$

where μ is the coupling constant which is assumed to be small, and $\phi(x(t))$ is the field operator. Note that $(x^{(\alpha)}(t))$ is the abbreviation of the spacetime coordinates $(t, \mathbf{x}_\alpha(t))$ for the α-th detector in the laboratory frame.

We assume that initially the detectors are switched off, i.e., the initial state of the total system can be written as $\rho_{\rm tot}(0) = \rho(0) \otimes |0\rangle_{MM}\langle 0|$, where $|0\rangle_M$ is the vacuum state of the massless scalar field, and $\rho(0)$ denotes the initial state of the detectors. The density matrix of the total system satisfies the Liouville equation

$$\frac{\partial\rho_{\rm tot}(t)}{\partial t} = -i[H,\rho_{\rm tot}(t)]. \tag{6}$$

Under the Born-Markov approximation, the reduced density matrix of the two-detector system $\rho(t) = {\rm Tr}_F[\rho_{\rm tot}(t)]$ can be described by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation [34,35], i.e.,

$$\frac{\partial\rho(t)}{\partial t} = -i[H_{\rm eff},\rho(t)] + \mathcal{D}[\rho(t)]. \tag{7}$$

Here $H_{\rm eff}$ and $\mathcal{D}[\rho(t)]$ in the equation above are the effective Hamiltonian and the dissipator respectively, whose explicit forms can be written as

$$H_{\rm eff} = H_S - \frac{i}{2}\sum_{\alpha,\varrho=1}^{2}(H_+^{(\alpha\varrho)}\sigma_+^{(\alpha)}\sigma_-^{(\varrho)} + H_-^{(\alpha\varrho)}\sigma_-^{(\alpha)}\sigma_+^{(\varrho)}),$$

and

$$\mathcal{D}[\rho(t)] = \frac{1}{2}\sum_{\alpha,\varrho=1}^{2}[D_+^{(\alpha\varrho)}(2\sigma_-^{(\varrho)}\rho\sigma_+^{(\alpha)} - \{\sigma_+^{(\alpha)}\sigma_-^{(\varrho)},\rho\}) + D_-^{(\alpha\varrho)}(2\sigma_+^{(\varrho)}\rho\sigma_-^{(\alpha)} - \{\sigma_-^{(\alpha)}\sigma_+^{(\varrho)},\rho\})], \tag{8}$$

where $\sigma_\pm^{(1)} = \sigma_\pm \otimes \sigma_0$ and $\sigma_\pm^{(2)} = \sigma_0 \otimes \sigma_\pm$ with $\sigma_- = |0\rangle\langle 1|$, $\sigma_+ = |1\rangle\langle 0|$. Here, $|0\rangle$ and $|1\rangle$ are the ground and excited states of the detectors, respectively. $D_\pm^{(\alpha\varrho)}$ and $H_\pm^{(\alpha\varrho)}$ are determined by the Fourier and Hilbert transforms of the field correlation functions

$$G^{(\alpha\varrho)}(\Delta t) = {}_M\langle 0|\phi(x^{(\alpha)}(t))\phi(x^{(\varrho)}(t'))|0\rangle_M, \tag{9}$$

respectively. Moreover, since the environment perceived by the detectors is stationary, the correlation functions of the field Eq. (9) possess temporal translation symmetry and are functions of $\Delta t = t - t'$. Then, the dissipation coefficients $D_\pm^{(\alpha\varrho)}$ can be expressed as

$$D_\pm^{(\alpha\varrho)}(\omega) = \mu^2\int_{-\infty}^{\infty} G^{(\alpha\varrho)}(\Delta t)e^{\pm i\omega\Delta t}d\Delta t. \tag{10}$$

Similarly, $H_\pm^{(\alpha\varrho)}$ can be obtained by replacing $G^{(\alpha\varrho)}(\Delta t)$ with $G^{(\alpha\varrho)}(\Delta t){\rm sign}(\Delta t)$, where ${\rm sign}(\Delta t)$ is the sign function which equals to -1, 0, 1 when $\Delta t <,=,> 0$, respectively.

The Wightman function of the massless scalar field in vacuum is

$$G^+(x,x') = -\frac{1}{4\pi^2}\frac{1}{(t-t'-i\varepsilon)^2-(x-x')^2-(y-y')^2-(z-z')^2}. \tag{11}$$

Here ε is a positive infinitesimal. Taking the trajectories (1) into the correlation functions (9), one obtains

$$G^{(11)}(\Delta t) = G^{(22)}(\Delta t) = P(\Delta t,0,0),$$
$$G^{(12)}(\Delta t) = P(\Delta t,\varphi,L), \quad G^{(21)}(\Delta t) = P(\Delta t,-\varphi,L), \tag{12}$$

where

$$P(\Delta t,\varphi,L) = -\frac{1}{4\pi^2}\frac{1}{(\Delta t-i\varepsilon)^2-4R^2\sin^2(\Omega\Delta t/2+\varphi/2)-L^2}. \tag{13}$$

Then, the dissipation coefficient $D_\pm^{(\alpha\varrho)}$ can be further written as

$$D_\pm^{(11)}(\omega) = D_\pm^{(22)}(\omega) = \mu^2\mathcal{P}(\pm\omega,0,0),$$
$$D_\pm^{(12)}(\omega) = D_\pm^{(21)*}(\omega) = \mu^2\mathcal{P}(\pm\omega,\varphi,L), \tag{14}$$

where $\mathcal{P}(\omega,\varphi,L)$ is the Fourier transform of $P(\Delta t,\varphi,L)$, i.e.,

$$\mathcal{P}(\pm\omega,\varphi,L) = \int_{-\infty}^{\infty} P(\Delta t,\varphi,L)e^{\pm i\omega\Delta t}d\Delta t. \tag{15}$$

Substituting Eq. (13) into Eq. (15), we obtain

$$\mathcal{P}(\pm\omega,\varphi,L) = \sum_{n=n_0^\pm}^{+\infty}\frac{e^{in\varphi}R^{2|n|}(n\Omega\pm\omega)^{2|n|+1}}{2\pi(2|n|+1)!}F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix} -R^2(n\Omega\pm\omega)^2 \\ -L^2(n\Omega\pm\omega)^2/4 \end{pmatrix}\right], \tag{16}$$

where $n_0^\pm = \lfloor \mp\omega/\Omega \rfloor + 1$. Here $\lfloor\zeta\rfloor$ is the largest integer less than or equal to ζ. The function $F^{0:1;0}_{1:1;0}\left[\begin{smallmatrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{smallmatrix}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right]$, which is a generalized Kampé De Fériet's double hypergeometric series (function) [36–38], can be expressed as

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\frac{(|n|+\frac{1}{2})_k}{(|n|+\frac{3}{2})_{k+l}(2|n|+1)_k}\frac{x^k y^l}{k!l!}. \tag{17}$$

Here, $(a)_b \equiv \Gamma(a+b)/\Gamma(a)$ is the Pochhammer sign, with $\Gamma(x)$ being the Euler Gamma function. See the Appendix for the derivation of Eq. (16), and the numerical method used for its calculation.

III. THE NECESSARY AND SUFFICIENT CONDITION FOR STEADY-STATE ENTANGLEMENT

In this section, we give the necessary and sufficient condition for steady-state entanglement following the methods we recently introduced in Ref. [33]. We first define a set of dimensionless parameters describing the entanglement dynamics with the transition coefficients of the two-detector system, and then express the condition for steady-state entanglement with these parameters.

A. The definition of the entanglement dynamics parameters

First, we define six parameters following Ref. [33] as:

$$\eta=\frac{D_+^{(\alpha\alpha)}+D_-^{(\alpha\alpha)}}{D_+^{(\alpha\alpha)}-D_-^{(\alpha\alpha)}},\quad \lambda_1=\frac{|D_+^{(12)}-D_-^{(21)}|}{D_+^{(\alpha\alpha)}-D_-^{(\alpha\alpha)}},\quad \lambda_2=\frac{|D_-^{(12)}|}{D_-^{(\alpha\alpha)}},$$
$$\Gamma=D_+^{(\alpha\alpha)}-D_-^{(\alpha\alpha)},\ \theta_1=\arg[D_+^{(12)}-D_-^{(21)}],\ \theta_2=\arg[D_-^{(21)}], \tag{18}$$

where $\arg[z]$ gives the argument of the complex number z. Using the following conclusions: *1)* The transition rate between any two collective states is non-negative; *2)* The downward transition rate is not smaller than the upward one for two arbitrary energy eigenstates, it can be proved that $\Gamma\geq 0$, $\eta\geq 1$, $0\leq\lambda_1\leq 1$, and $0\leq\lambda_2\leq 1$ [33]. Then, the dissipation coefficients $D_\pm^{(\alpha\varrho)}$ (14) can be reexpressed as

$$D_\pm^{(11)}=D_\pm^{(22)}=\frac{1}{2}(\eta\pm 1)\Gamma, \tag{19}$$

$$D_+^{(12)}=D_+^{(21)*}=\frac{1}{2}[(\eta-1)\lambda_2 e^{i\theta_2}+2\lambda_1 e^{i\theta_1}]\Gamma, \tag{20}$$

$$D_-^{(12)}=D_-^{(21)*}=\frac{1}{2}(\eta-1)\lambda_2 e^{-i\theta_2}\Gamma. \tag{21}$$

For the model considered in the present paper, the parameters defined in Eq. (18) can be explicitly worked out:

$$\eta=1+2N,\qquad \lambda_1=\sqrt{B_\mathcal{R}^2+(2A_\mathcal{I}+B_\mathcal{I})^2},\qquad \lambda_2=\frac{\sqrt{A_\mathcal{R}^2+A_\mathcal{I}^2}}{N},$$
$$\Gamma=\frac{\mu^2\gamma^2\omega}{2\pi},\qquad \theta_1=\arg[B_\mathcal{R}+i(2A_\mathcal{I}+B_\mathcal{I})],\qquad \theta_2=\arg[A_\mathcal{R}+iA_\mathcal{I}], \tag{22}$$

where

$$A_\mathcal{R}=\frac{\Omega}{4\pi\gamma^2\omega}\int_0^{+\infty}[f_+(x)+f_-(x)]\cos(2\omega x/\Omega)dx, \tag{23}$$

$$A_\mathcal{I}=\frac{\Omega}{4\pi\gamma^2\omega}\int_0^{+\infty}[g_+(x)-g_-(x)]\sin(2\omega x/\Omega)dx, \tag{24}$$

$$B_\mathcal{R}=\frac{\Omega}{2\gamma^2\omega}\left[\frac{\sin(2\omega r_+/\Omega)}{2r_+-R^2\Omega^2\sin(2r_++\varphi)}+\frac{\sin(2\omega r_-/\Omega)}{2r_--R^2\Omega^2\sin(2r_--\varphi)}\right], \tag{25}$$

$$B_\mathcal{I}=\frac{\Omega}{2\gamma^2\omega}\left[\frac{\cos(2\omega r_-/\Omega)}{2r_--R^2\Omega^2\sin(2r_--\varphi)}-\frac{\cos(2\omega r_+/\Omega)}{2r_+-R^2\Omega^2\sin(2r_++\varphi)}\right], \tag{26}$$

$$N=\frac{\Omega}{2\pi\omega}\int_0^{+\infty}\frac{R^2\Omega^2(x^2-\sin^2x)}{x^2(x^2-R^2\Omega^2\sin^2x)}\cos(2\omega x/\Omega)dx. \tag{27}$$

Here, functions $f_\pm(x)$ and $g_\pm(x)$, both of which have only two removable singularities at $x=r_\pm$, are, respectively, defined as

$$f_\pm(x)=\frac{R^2\Omega^2\left[\sin(2r_\pm\pm\varphi)-2r_\pm\frac{\sin(x-r_\pm)\sin(x+r_\pm\pm\varphi)}{(x-r_\pm)(x+r_\pm)}\right]}{[x^2-R^2\Omega^2\sin^2(x\pm\varphi/2)-\Omega^2L^2/4][2r_\pm-R^2\Omega^2\sin(2r_\pm\pm\varphi)]}, \tag{28}$$

$$g_\pm(x)=\frac{2(x-r_\pm)+R^2\Omega^2\left[\sin(2r_\pm\pm\varphi)-2r_\pm\frac{\sin(x-r_\pm)\sin(x+r_\pm\pm\varphi)}{(x-r_\pm)(x+r_\pm)}\right]}{[x^2-R^2\Omega^2\sin^2(x\pm\varphi/2)-\Omega^2L^2/4][2r_\pm-R^2\Omega^2\sin(2r_\pm\pm\varphi)]}, \tag{29}$$

where $r_\pm$ is the root of the following equation:

$$r_\pm = \sqrt{R^2\Omega^2 \sin^2\left(r_\pm \pm \frac{\varphi}{2}\right) + \frac{\Omega^2 L^2}{4}}. \tag{30}$$

Moreover, one can find from Eqs. (22)–(30) that, if $\varphi = 0$ or π, then $r_+ = r_-$, $A_\mathcal{I} = B_\mathcal{I} = 0$, and thus $\lambda_1 = |B_\mathcal{R}|$, $\lambda_2 = |A_\mathcal{R}|/N$.

B. The necessary and sufficient condition of steady-state entanglement

We measure the degree of entanglement by concurrence C [39], which is

$$C[\rho(t)] = \max\{0, K(t)\}, \tag{31}$$

where $K(t) = \kappa_1 - \kappa_2 - \kappa_3 - \kappa_4$, with κ_i $(i = 1,2,3,4)$ being the square roots of the eigenvalues of the matrix $\rho(\sigma_2 \otimes \sigma_2)\rho^T(\sigma_2 \otimes \sigma_2)$ in decreasing order. Here ρ is the density matrix in the decoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, and ρ^T is its transpose.

In order to obtain the density matrix $\rho(\infty)$ characterizing the steady state of the two-detector system, we substitute Eqs. (19)–(21) into the master equation (7), and take the time derivative in the master equation to be zero. Then, taking the resulting explicit form of $\rho(\infty)$ into Eq. (31), and with the help of $\Gamma > 0$, $\eta \geq 1$, $0 \leq \lambda_1 \leq 1$, $0 \leq \lambda_2 \leq 1$, we can obtain the necessary and sufficient condition for the two rotating Unruh-DeWitt detectors to be entangled in the asymptotic steady state, which can be categorized into the following two cases. For convenience, "&" denotes the logical AND, and "||" denotes the logical OR.

Case I: $\{\lambda_1 = \lambda_2 = 1 \,\&\, \theta_1 = \theta_2\} || \{\lambda_1 = 1 \,\&\, \eta = 1\}$

When the condition $\{\lambda_1 = \lambda_2 = 1 \,\&\, \theta_1 = \theta_2\} || \{\lambda_1 = 1 \,\&\, \eta = 1\}$ is satisfied, the final state $\rho(\infty)$ is related to the initial state $\rho(0)$. The necessary and sufficient condition for steady-state entanglement is

$$\rho_{aa}(0) > \frac{3(\eta^2 - 1)}{2(3\eta^2 - 1)}, \tag{32}$$

and the corresponding steady-state concurrence can be written as

$$C(\infty) = \frac{2(3\eta^2 - 1)\rho_{aa}(0) - 3(\eta^2 - 1)}{3\eta^2 + 1} > 0. \tag{33}$$

Case II: $\{\lambda_1 \neq 1 || \lambda_2 \neq 1 || \theta_1 \neq \theta_2\} \,\&\, \{\lambda_1 \neq 1 || \eta \neq 1\}$

In this case, the final state $\rho(\infty)$ is independent of the initial state $\rho(0)$, and it can be obtained that

$$K(\infty) = \frac{(\eta - 1)(\sqrt{\eta^2(\lambda_1 - \lambda_2)^2 + \Phi_1} - \sqrt{\Phi_2 \Phi_3})}{\eta[\lambda_1^2 - \eta + (\eta - 1)\lambda_1\lambda_2\cos(\Delta\theta)] + (\eta - 1)^2\Phi_2 + \Phi_3}, \tag{34}$$

where

$$\begin{aligned}\Phi_1 = {} & 4\lambda_1\lambda_2 \sin^2(\Delta\theta/2)\{\eta^2 + \lambda_1\lambda_2\cos^2(\Delta\theta/2) \\ & \times [(\lambda_1 - (\eta - 1)\lambda_2)^2 - 2\eta^2 \\ & + 4(\eta - 1)\lambda_1\lambda_2\cos^2(\Delta\theta/2)]\},\end{aligned} \tag{35}$$

$$\begin{aligned}\Phi_2 = {} & \frac{1}{2}\eta[\eta(1 - \lambda_2^2) - 2\lambda_2(\lambda_1 - \lambda_2)] \\ & + 2\lambda_1\lambda_2\sin^2(\Delta\theta/2)[\eta - \lambda_1\lambda_2\cos^2(\Delta\theta/2)],\end{aligned} \tag{36}$$

$$\begin{aligned}\Phi_3 = {} & (\eta - 1)^2\Phi_2 + 2\eta^3 - 2\eta[\lambda_1 + (\eta - 1)\lambda_2]^2 \\ & + 8\eta(\eta - 1)\lambda_1\lambda_2\sin^2(\Delta\theta/2),\end{aligned} \tag{37}$$

with $\Delta\theta = \theta_1 - \theta_2$. From Eq. (34), the necessary and sufficient condition for $K(\infty) > 0$ is found to be

$$\{\lambda_1 > \lambda_c\} \,\&\, \{\eta \neq 1\}, \tag{38}$$

where

$$\lambda_c(\eta, \lambda_2, \Delta\theta) = \frac{\sqrt{\eta^2(\eta^2\lambda_2^2 + 4\eta + 4) - \mathcal{Z}} + (4 - \eta^2 - 2\eta)\lambda_2\cos(\Delta\theta)}{4 + (\eta - 1)\lambda_2^2\sin^2(\Delta\theta)}, \tag{39}$$

with $\mathcal{Z} = \lambda_2^2\{(\eta^2 - 4)[(\eta - 1)^2\lambda_2^2 + 4\eta - 3] + 4\}\sin^2(\Delta\theta)$. When condition (38) is satisfied, the steady-state concurrence can be written as $C(\infty) = K(\infty) > 0$, which is independent of the initial state.

Moreover, with the help of $\eta \geq 1$ and $0 \leq \lambda_{1,2} \leq 1$, we can also obtain some necessary conditions for steady-state entanglement. It can be proved that $\partial\lambda_c/\partial\eta \geq 0$ and $\partial\lambda_c(1, \lambda_2, \Delta\theta)/\partial(\lambda_2\cos(\Delta\theta)) > 0$ [33]. Then,

$$1 \geq \lambda_1 > \lambda_c(\eta, \lambda_2, \Delta\theta) \geq \lambda_c(1, \lambda_2, \Delta\theta) \geq \lambda_c(1, 1, \pm\pi) = \frac{1}{2}. \tag{40}$$

Here, $\lambda_1 > \lambda_c(1, \lambda_2, \Delta\theta)$ can be equivalently expressed as $\lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] > 1$. Moreover, it can be found from $\lambda_c(\eta, \lambda_2, \Delta\theta) < 1$ that the upper limit value of η is $\sqrt{2}$. Therefore, we obtain the following necessary conditions for steady-state entanglement independent of the initial state, i.e.:

$$(1)\ \lambda_1 > 1/2; \quad (2)\ \eta < \sqrt{2}; \quad (3)\ \lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] > 1. \tag{41}$$

The necessary conditions shown in Eq. (41) are helpful for later discussions in some specific cases, such as the ultrarelativistic limit case.

IV. THE STEADY-STATE ENTANGLEMENT FOR TWO ROTATING UNRUH-DEWITT DETECTORS

With the help of the necessary and sufficient condition obtained in Sec. III, we study whether two rotating Unruh-DeWitt detectors in vacuum can be entangled in the asymptotic steady state, and, if so, what conditions are required for the parameters $\{R,\Omega,\varphi,L\}$.

First, when the separation between the rotating planes of the two detectors is vanishing $\{L=0\}$, and the orbital radius or the phase angle difference of the trajectories of the two detectors is vanishing $\{R=0||\varphi=0\}$, the distance between the two detectors $d=\sqrt{4R^2\sin^2(\varphi/2)+L^2}$ is vanishing. According to Eqs. (14) and (16), when $\{\varphi=0\}\,\&\,\{L=0\}$, one obtains $D_\pm^{(\alpha\varrho)}=D_\pm^{(11)}$, and thus $\{\lambda_1=\lambda_2=1\}\,\&\,\{\theta_1=\theta_2=0\}$ according to Eq. (18). Similarly, when $\{R=0\}\,\&\,\{L=0\}$, one obtains $\{\lambda_1=1\}\,\&\,\{\eta=1\}$. Therefore, when the separation between the two detectors is vanishing, as long as the initial state satisfies the necessary and sufficient condition Eq. (32), the detectors can obtain initial-state dependent steady-state entanglement. This property is the same as that in the uniform acceleration case and in the case of static detectors in a thermal bath [15].

Second, for uniformly accelerated detectors in vacuum and static ones in a thermal bath, it is straightforward to verify that

$$\frac{D_+^{(\alpha\varrho)}+D_-^{(\varrho\alpha)}}{D_+^{(\alpha\varrho)}-D_-^{(\varrho\alpha)}}=\eta. \tag{42}$$

Taking Eq. (42) into Eqs. (19)–(21), one obtains

$$\{\lambda_1=\lambda_2\}\,\&\,\{\theta_1=\theta_2\}, \tag{43}$$

which has been verified in Ref. [18]. Then, $\lambda_1[2\lambda_1-\lambda_2\cos(\Delta\theta)]=\lambda_1^2\leq 1$, so the necessary condition *(3)* in Eq. (41) cannot be satisfied. That is, initial-state independent steady-state entanglement is impossible for uniformly accelerated detectors and static ones in a thermal bath. However, for rotating detectors, Eq. (42) is not satisfied, so, in the following, we focus on whether initial-state independent steady-state entanglement can be obtained for rotating detectors. We first give an analytical investigation for some special cases, followed by a numerical analysis for general cases.

A. The analytical investigation

1. The ultrarelativistic limit

To begin with, we consider the ultrarelativistic limit case, i.e., $v=R\Omega\to 1$, or equivalently the Lorentz factor $\gamma\gg 1$, and focus on whether the detectors can obtain steady-state entanglement.

a. Case I. When the orbital radius R is finite, the proper acceleration $a=\gamma^2v^2/R$ tends to infinity in the ultrarelativistic limit $v\to 1$ and $\gamma\to\infty$. According to Eqs. (22) and (27), we obtain

$$\eta\to 1+\frac{\gamma\Omega}{\pi\omega_0}\int_0^{+\infty}\frac{1}{x^2}dx\to+\infty, \tag{44}$$

which means that the necessary condition $\eta<\sqrt{2}$ cannot be satisfied. Therefore, initial-state independent steady-state entanglement is impossible in this case.

b. Case II. When the orbital radius $R\to\infty$, while the proper acceleration $a=\gamma^2v^2/R$ is finite, η can be approximated as

$$\eta=1+\frac{ae^{-\frac{2\sqrt{3}\omega_0}{a}}}{2\sqrt{3}\omega_0}+O[1-v], \tag{45}$$

where $O[\varepsilon^n]$ denotes that infinitesimals of the nth and higher orders of ε are omitted. That is, as long as the proper acceleration a is small enough, the necessary condition $\eta<\sqrt{2}$ can always be satisfied. This case can be further divided into two subcases.

Subcase 1. When $\varphi\neq 0$, i.e., the phase difference φ of the trajectories of the two detectors is nonvanishing, the roots of Eq. (30) satisfy $0<r_\pm=\sqrt{\sin[r_\pm\pm\varphi/2]^2}\leq 1$. Then it can be proved that λ_1 given in Eq. (22) approaches 0 when $v\to 1$ and the acceleration a is finite, which indicates that the detectors cannot obtain initial-state independent steady-state entanglement since the necessary condition $\lambda_1>1/2$ is not satisfied.

Subcase 2. This is the case considered in Refs. [31,32]. When $\varphi=0$, i.e., the phase difference φ of the trajectories of the two detectors is vanishing, the roots of Eq. (30) in the neighborhood of $v=1$ can be expressed as

$$r_+=r_-\equiv r=\sqrt{\sqrt{3a^2L^2+9}-3}(1-v)^{1/2}+O[(1-v)^{3/2}]. \tag{46}$$

Substituting Eq. (46) into Eq. (22), and keeping the zeroth-order terms of $(1-v)$, we obtain

$$\lambda_1=\frac{(1-\mathcal{V}^2)^2}{(1+6\mathcal{V}^2+\mathcal{V}^4)}\frac{|\sin\mathcal{U}|}{\mathcal{U}},\quad \lambda_2=\frac{(1-\mathcal{V}^2)^3e^{-\mathcal{U}\mathcal{V}}}{(1+6\mathcal{V}^2+\mathcal{V}^4)(1+\mathcal{V}^2)},$$
$$\cos(\Delta\theta)=\mathrm{sign}[\sin\mathcal{U}], \tag{47}$$

where

$$\mathcal{U}=\frac{\omega_0\sqrt{2\sqrt{3a^2L^2+9}-6}}{a},$$
$$\mathcal{V}=\frac{\sqrt{\sqrt{3a^2L^2+9}+3}-\sqrt{6}}{\sqrt{\sqrt{3a^2L^2+9}-3}}.$$

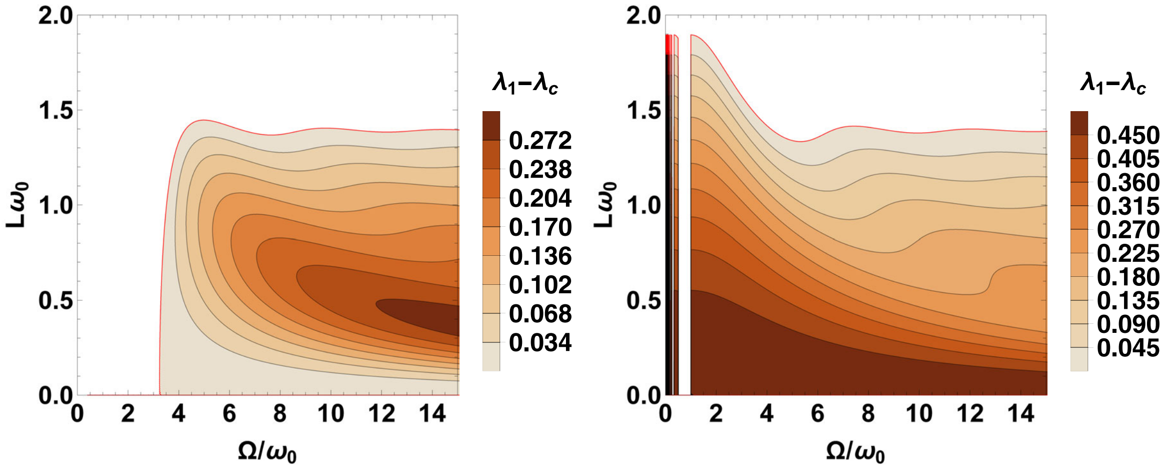

FIG. 2. The contour map of $\lambda_1 - \lambda_c$ in parameter space $(\Omega/\omega_0, L\omega_0)$ when $R\omega_0 \to 0$, with $\varphi = 0$ (left), and $\varphi = \pi$ (right).

Then, it is obvious that $\mathcal{U} \geq 0$ and $0 \leq \mathcal{V} \leq 1$. According to Eq. (47), it can be verified that $\lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] \leq 1$, which means that the necessary condition *(3)* in Eq. (41) cannot be satisfied. This explains why initial-state independent steady-state entanglement has not been found in Refs. [31,32].

2. The nonrelativistic limit

In the following, we consider the nonrelativistic case, i.e., $v = R\Omega \ll 1$, or equivalently the Lorentz factor $\gamma \to 1$. This can be divided into the following two cases.

a. Case I. The dimensionless orbital radius $R\omega_0$ is finite, while the dimensionless angular velocity Ω/ω_0 approaches zero. This reduces to the static case, in which the only way to obtain steady-state entanglement is when the two detectors are infinitely close, i.e., $d \to 0$. In this case, the steady-state concurrence has been given in Eq. (33), when the initial state of the two-detector system satisfies the necessary and sufficient condition Eq. (32).

b. Case II. The dimensionless angular velocity Ω/ω_0 is finite, while the dimensionless orbital radius $R\omega_0 \ll 1$. This can be further divided into two subcases.

Subcase 1. When the separation between the orbits of the two detectors is nonvanishing, i.e., $L \neq 0$, substituting Eq. (14) into Eq. (18), we obtain the approximate expressions of the parameters η, λ_1, and λ_2 as

$$\begin{aligned}\eta &= 1 + H(m)(R\omega_0)^{2m} + O[(R\omega_0)^{2m+2}],\\ \lambda_1 &= |p| + O[(R\omega_0)^2], \qquad \lambda_2 = |q| + O[(R\omega_0)^2],\\ \cos(\Delta\theta) &= \mathrm{Sign}[pq]\cos(m\varphi),\end{aligned} \tag{48}$$

where

$$H(m) = \frac{2(m\Omega/\omega_0 - 1)^{2m+1}}{\Gamma(2m+2)}, \tag{49}$$

$$p = \frac{\sin(L\omega_0)}{L\omega_0}, \qquad q = {}_0F_1\left[\frac{3}{2} + m; -\frac{1}{4}L^2\Omega^2\xi^2\right], \tag{50}$$

with $m = \lfloor \omega_0/\Omega \rfloor + 1$, and $\xi = m - \omega_0/\Omega$. Obviously, $m \geq 1$ and $0 < \xi \leq 1$. Plugging Eq. (48) into Eqs. (39) and (34), we obtain

$$\lambda_1 - \lambda_c = |p| - \frac{pq\cos(m\varphi) + |p|\sqrt{8 + q^2\cos^2(m\varphi)}}{4|p|} + O[(R\omega_0)^2], \tag{51}$$

and

$$K(\infty) = \left(1 - \sqrt{1 - \frac{[1 - pq\cos(m\varphi)][2p^2 - pq\cos(m\varphi) - 1]}{[p - q\cos(m\varphi)]^2 + q^2(1-p^2)^2\sin^2(m\varphi)}}\right)\sqrt{q^2\sin^2(m\varphi) + \left[\frac{p - q\cos(m\varphi)}{1 - p^2}\right]^2} H(m)(R\omega_0)^{2m} + O[(R\omega_0)^{2m+2}]. \tag{52}$$

According to the necessary and sufficient condition Eq. (38), when $\lambda_1 - \lambda_c$ shown in Eq. (51) is positive, the detectors can obtain initial-state independent steady-state entanglement when the dimensionless orbital radius $R\omega_0 \ll 1$. Due to the complexity of Eq. (51), we show numerically its dependence on the dimensionless angular velocity Ω/ω_0 and the dimensionless orbit separation $L\omega_0$ when the dimensionless orbital radius tends to zero (i.e., $R\omega_0 \to 0$) for two special cases, i.e., the phase angle difference of the trajectories $\varphi = 0$ and $\varphi = \pi$. When

$\varphi = 0$, the separation of the two Unruh-DeWitt detectors is parallel to the rotating axis, while, when $\varphi = \pi$, the separation of the detectors intersects with the rotating axis. From Fig. 2, we can draw the following conclusions.

(1) In either case, a sufficiently small dimensionless orbital separation $L\omega_0$ is required for initial-state independent steady-state entanglement (i.e., $\lambda_1 - \lambda_c > 0$) in the neighborhood of a vanishing orbital radius R.

(2) In the parallel case (i.e., $\varphi = 0$), a sufficiently large angular velocity Ω (larger than the transition frequency of the detectors ω_0) is required for initial-state independent steady-state entanglement in the neighborhood of a vanishing orbital radius R. In contrast, for the intersectant case (i.e., $\varphi = \pi$) in which the separation intersects with the rotational axis, there is no such restriction.

Subcase 2. Now, we discuss the case when the separation between the orbits of the two detectors is vanishing, i.e., $L = 0$. First of all, if $\varphi = 0$, which means that the separation between the detectors is vanishing, only steady-state entanglement related to the initial state is possible, as shown in Eq. (33). In the following, we discuss the case when $\varphi \neq 0$. In the limit $R\omega_0 \ll 1$, i.e., the orbital radius is much smaller than the transition wavelength of the detector, we obtain

$$\begin{aligned} \eta &= 1 + H(m)(R\omega_0)^{2m} + O[(R\omega_0)^{2m+2}], \\ \lambda_1 &= 1 - \frac{2}{3}\left(1 + \frac{3\Omega^2}{\omega_0^2}\right)\sin^2\left(\frac{\varphi}{2}\right)(R\omega_0)^2 + O[(R\omega_0)^4], \\ \lambda_2 &= 1 - \frac{2H(m+1)}{H(m)}\sin^2\left(\frac{\varphi}{2}\right)(R\omega_0)^2 + O[(R\omega_0)^4], \end{aligned} \tag{53}$$

and $\cos\Delta\theta = \cos(m\varphi)$, where $H(m)$ has been defined in Eq. (49), and $m = \lfloor \omega_0/\Omega \rfloor + 1$. Substituting Eq. (53) into Eqs. (39) and (34), we obtain

$$\lambda_1 - \lambda_c = \begin{cases} 1 - \frac{1}{4}\left[\sqrt{\cos^2(m\varphi) + 8} + \cos(m\varphi)\right] + O[(R\omega_0)^2], & \frac{m\varphi}{\pi} \notin \text{Evens}, \\ \left\{\frac{2}{3}\left[\frac{H(m+1)}{H(m)} - \left(1 + \frac{3\Omega^2}{\omega_0^2}\right)\right]\sin^2\left(\frac{\varphi}{2}\right)\right\}(R\omega_0)^2 + O[(R\omega_0)^4], & \frac{m\varphi}{\pi} \in \text{Evens}, \end{cases} \tag{54}$$

and

$$K(\infty) = \begin{cases} \frac{(\Omega/\omega_0 - 1)^3}{(\Omega/\omega_0 - 1)^3 + 2(\Omega^2/\omega_0^2 + 3)\Omega/\omega_0}\left\{1 - 2\sin\left(\frac{\varphi}{2}\right)\sqrt{\frac{\Omega}{\omega_0} + \frac{\Omega^3}{3\omega_0^3}}(R\omega_0) + O[(R\omega_0)^2]\right\}, & m = 1, \\ \frac{3H(m)}{2(1 + 3\Omega^2/\omega_0^2)}\frac{\sin^2(m\varphi/2)}{\sin^2(\varphi/2)}(R\omega_0)^{2m-2} + O[(R\omega_0)^{2m-1}], & m > 1\ \&\ \frac{m\varphi}{\pi} \notin \text{Evens}, \\ \frac{3}{2}\left[\frac{H(m+1)}{1 + 3\Omega^2/\omega_0^2} - H(m)\right](R\omega_0)^{2m} + O[(R\omega_0)^{2m+2}], & m > 1\ \&\ \frac{m\varphi}{\pi} \in \text{Evens}. \end{cases} \tag{55}$$

According to Eqs. (54) and (55), we obtain the following conclusions.

(1) In the limit of a vanishing dimensionless orbital radius, i.e., $R\omega_0 \to 0$, the steady-state entanglement independent of the initial state measured by concurrence is

$$\lim_{R\omega_0 \to 0} C(\infty) = \begin{cases} \frac{(\Omega/\omega_0 - 1)^3}{(\Omega/\omega_0 - 1)^3 + 2(\Omega^2/\omega_0^2 + 3)\Omega/\omega_0}, & \Omega > \omega_0, \\ 0, & \Omega \leq \omega_0. \end{cases} \tag{56}$$

That is, when the angular velocity is larger than the energy level spacing, the steady-state concurrence which is independent of the initial state will be nonzero. Otherwise, it will be zero.

(2) Although the steady-state concurrence tends to zero in the limit of a vanishing dimensionless orbital radius ($R\omega_0 \to 0$) when the angular velocity Ω is less than the energy level spacing of detectors ω_0, it can be seen from Eqs. (54) and (55) that the steady-state concurrence is certainly nonzero in the neighborhood of $R\omega_0 = 0$ if $\frac{m\varphi}{\pi}$ is not an even number. If $\frac{m\varphi}{\pi}$ is even, whether the detectors can obtain initial-state independent steady-state entanglement in the neighborhood of $R\omega_0 = 0$ depends on whether $\frac{H(m+1)}{1+3\Omega^2/\omega_0^2} - H(m)$ is positive. For example, if $\varphi = \pi$ (i.e., when the detectors rotate at opposite ends of the diameter of a circular trajectory) and $m = 2$ (i.e., the angular velocity satisfies $\frac{1}{2} < \Omega/\omega_0 \leq 1$), one can find that initial-state independent steady-state entanglement can be achieved

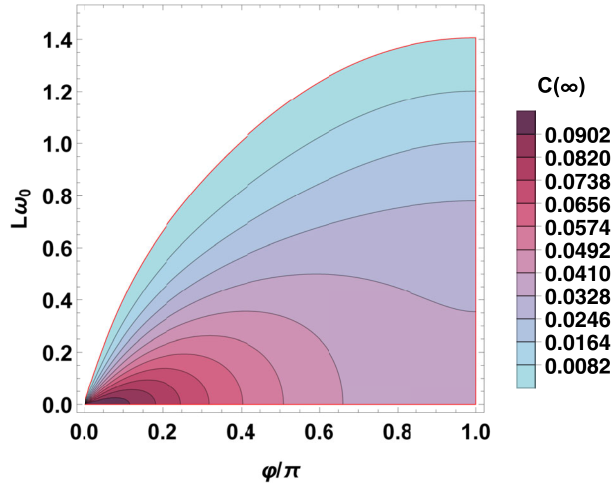

FIG. 3. The contour map of the steady-state concurrence $C(\infty)$ in parameter space $(\varphi/\pi, L\omega_0)$ with $R\omega_0 = 1/10$ and $\Omega/\omega_0 = 3$.

when $\frac{1}{2} < \Omega/\omega_0 < 0.834$. Note that here the angular velocity Ω is smaller than the transition frequency ω_0.

To summarize, in this part, we have investigated analytically the steady state of a pair of rotating Unruh-DeWitt detectors in the ultrarelativistic limit and the nonrelativistic limit, respectively. We have shown that, in the ultrarelativistic limit, initial-state independent steady-state entanglement is always impossible. However, in the nonrelativistic limit, when the angular velocity Ω is appropriate and the orbital separation L and the orbital radius R are sufficiently small, initial-state independent steady-state entanglement can certainly be achieved. Moreover, it is harder to achieve initial-state independent steady-state entanglement when the separation of the detectors is parallel to the rotating axis, in the sense that the rotating angular velocity is required to be larger than the transition frequency of the detectors, while there is no such restriction when the separation of the detectors intersects with the rotating axis.

B. Numerical investigation

In this part, we investigate numerically how the steady-state entanglement of two rotating Unruh-DeWitt detectors is affected by physical variables such as the orbital radius R, the angular velocity Ω, the orbital phase difference φ, and the orbital separation L.

First, we focus on how the initial-state independent steady-state entanglement is affected by the alignment of the detectors with respect to the rotating axis (which is determined by the orbital phase difference φ and the orbital separation L). In Fig. 3, we plot the contour map of the steady-state concurrence $C(\infty)$ in the parameter space $(\varphi/\pi, L\omega_0)$ with $R\omega_0 = 1/10$ and $\Omega/\omega_0 = 3$, from which we conclude as follows.

(1) A sufficiently small orbital separation L (compared with the transition wavelength of the detectors) is a necessary condition for the detectors to obtain steady-state entanglement. Moreover, when the phase angle difference φ of the trajectories of the detectors is fixed at a nonzero value, the smaller the orbital separation L, the larger the steady-state concurrence.

(2) When the orbital separation L is fixed, there always exists an optimal orbital phase difference φ such that the steady-state concurrence the detectors obtained reaches its maximum. Moreover, as the orbital separation L increases, the optimal value of φ changes from 0 (for $L = 0$) to π.

(3) The larger the orbital phase difference φ, the larger the region of the orbital separation L within which the detectors can obtain steady-state entanglement. In this sense, a larger orbital phase difference φ is beneficial to steady-state entanglement.

Second, we investigate how the initial-state independent steady-state entanglement is affected by the rotation of the detectors described by the orbital radius R and the rotational angular velocity Ω. To this end, we consider the following two cases as sketched in Fig. 4: *(1)* The parallel case, i.e., the separation of the two Unruh-DeWitt detectors is parallel to the rotating axis. In this case, $\varphi = 0$. *(2)* The intersectant case, i.e., the separation of the two

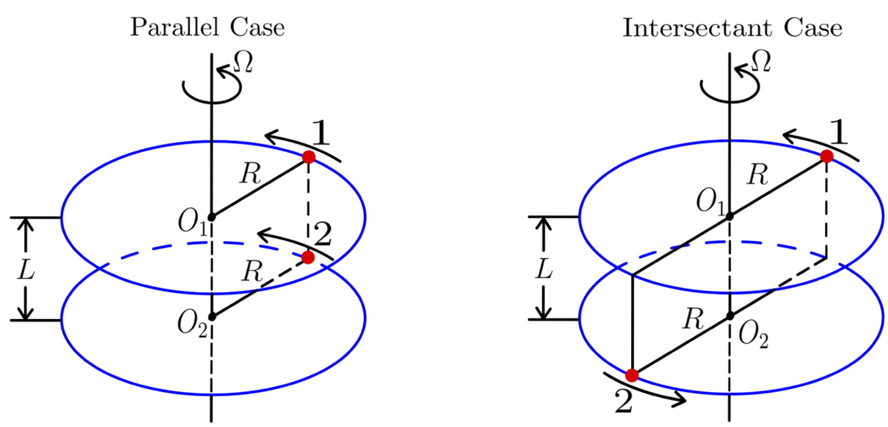

FIG. 4. The parallel case: $\varphi = 0$. The intersectant case: $\varphi = \pi$.

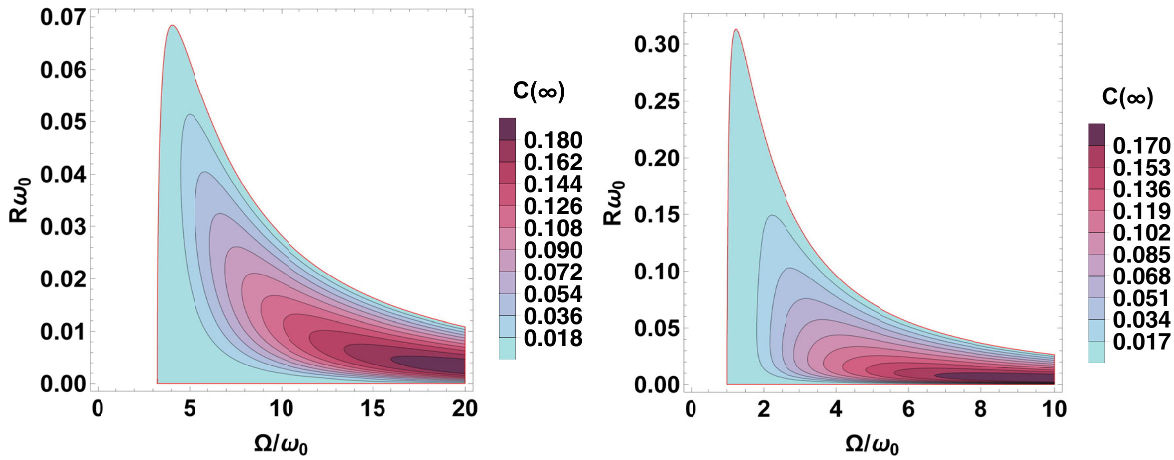

FIG. 5. The contour maps of the steady-state concurrence $C(\infty)$ in parameter space $(\Omega/\omega_0, R\omega_0)$ for the parallel case with $L\omega_0 = 1/10$ (left) and the intersectant case (right). Note that contour lines with $C(\infty) < 10^{-5}$ are not drawn here.

Unruh-DeWitt detectors intersects with the rotating axis. In this case, $\varphi = \pi$.

In Fig. 5, we plot the contour maps of the steady-state concurrence $C(\infty)$ in the parameter space $(\Omega/\omega_0, R\omega_0)$ for the parallel case and the intersectant case, respectively. From Fig. 5, we conclude as follows.

(1) In either the parallel case or the intersectant case, in order to obtain initial-state independent steady-state entanglement, a sufficiently small orbital radius and a sufficiently large rotational angular velocity are required. Also, the rotational angular velocity cannot be too large [otherwise the necessary condition *(2)* in Eq. (41) cannot be satisfied]. This means that there always exists an optimal rotational angular velocity such that the steady-state concurrence the detectors obtained reaches its maximum.

(2) Compared with the parallel case, the detectors in the intersectant case have more spacious rooms for orbital radius and rotational angular velocity to obtain initial-state independent steady-state entanglement, i.e., the detectors can obtain steady-state entanglement at a larger orbital radius or a smaller rotational angular velocity. In this sense, the detectors in the intersectant case are more easily able to obtain initial-state independent steady-state entanglement than the ones in the parallel case. Let us note that, as discussed analytically [See Eqs. (54) and (55) and the discussions below], steady-state entanglement is also possible in the intersectant case ($\varphi = \pi$) when the angular velocity is smaller than the transition frequency, i.e., $\Omega < \omega_0$. However, the steady-state concurrence obtained is extremely small (less than 10^{-5}), so it is not shown in Fig. 5.

To summarize, in this part, we have investigated numerically how the initial-state independent steady-state entanglement of a pair of rotating Unruh-DeWitt detectors is affected by the configuration and the rotation of the detectors. First, by comparing the steady-state concurrence obtained by detectors with different configurations, we find that the detectors in the intersectant case (i.e., when the separation of the detectors intersects with the rotating axis) have more spacious rooms for orbital radius and rotational angular velocity to obtain steady-state entanglement. In this sense, steady-state entanglement independent of the initial state is more likely to be obtained in this case. Second, in any configuration, a sufficiently small separation between the detectors compared with the transition wavelength and an angular velocity larger than the transition frequency are required to obtain a significant amount of steady-state entanglement.

V. SUMMARY

In this paper, we have studied the steady-state entanglement of two rotating Unruh-DeWitt detectors coupled with a fluctuating massless scalar field in vacuum. A necessary and sufficient condition for the detectors to achieve steady-state entanglement is derived, which can be classified according to the dependence on the initial state. With the help of the condition obtained, the requirements on the physical parameters such as the angular velocity Ω, the orbital radius R, the orbital phase difference φ, and the orbital separation L to obtain steady-state entanglement for two Unruh-DeWitt detectors rotating in coaxial orbits with the same orbital radius and angular velocity is systematically investigated, both analytically and numerically. It is found that, when the separation between the two detectors is vanishing, the detectors can obtain steady-state entanglement dependent on the initial state of the two-detector system, which is the same as what happens in the uniform acceleration case and the case of static detectors in a thermal bath. When the separation between the detectors is nonvanishing but small compared with the transition wavelength of detectors, the detectors can obtain steady-state entanglement whatever the initial state is in the nonrelativistic limit. This is a unique phenomenon caused by the centripetal acceleration, which can never happen in the uniform acceleration case and the case of static detectors in a thermal bath. In contrast, in the ultrarelativistic limit, the detectors can never obtain steady-state entanglement independent of the initial state. In addition, the detectors have more spacious rooms for orbital radius and rotational angular velocity to obtain initial-state independent steady-state entanglement when their spacial separation intersects with the rotation axis compared with detectors aligned parallel to the rotating axis. In this sense, the steady-state entanglement independent of the initial state is more likely to be achieved in this case.

ACKNOWLEDGMENTS

This work was supported in part by the NSFC under Grants No. 11690034, No. 11805063, and No. 12075084, and the Hunan Provincial Natural Science Foundation of China under Grant No. 2020JJ3026.

APPENDIX: THE CALCULATION DETAILS OF $\mathcal{P}(\omega, \varphi, L)$

According to the definition Eq. (13), the function $P(\Delta t, \varphi, L)$ can be expressed as

$$\begin{aligned}
P(\Delta t,\varphi,L) &= -\frac{1}{4\pi^2}\frac{1}{(\Delta t - i\varepsilon)^2 - 4R^2\sin^2(\Omega\Delta t/2+\varphi/2) - L^2} \\
&= \int \frac{e^{i(Lk_z-\omega_k\Delta t)}e^{i2k_\perp R\sin(\frac{\Omega\Delta t+\varphi}{2})\sin(\theta-\frac{\Omega(t'+t)+\varphi}{2})}}{(2\pi)^3 2\omega_k}d^3k \\
&= \int_0^{+\infty}\frac{k_\perp dk_\perp}{4\pi^2}\int_{-\infty}^{+\infty}\frac{e^{i(Lk_z-\omega_k\Delta t)}dk_z}{2\omega_k}\int_{-\pi}^{+\pi}\frac{e^{i2k_\perp R\sin(\frac{\Omega\Delta t+\varphi}{2})\sin(\theta-\frac{\Omega(t'+t)+\varphi}{2})}}{2\pi}d\theta \\
&= \int_0^{+\infty}\frac{k_\perp dk_\perp}{4\pi^2}\int_{-\infty}^{+\infty}\frac{e^{i(Lk_z-\omega_k\Delta t)}}{2\omega_k}J_0\left(2k_\perp R\sin\left(\frac{\varphi+\Omega\Delta t}{2}\right)\right)dk_z \qquad \text{(A1)}\\
&= \sum_{n=-\infty}^{+\infty}e^{in\varphi}\int_0^{+\infty}dk_\perp\frac{k_\perp J_n^2(k_\perp R)}{4\pi^2}\int_{-\infty}^{+\infty}\frac{e^{iLk_z}}{2\omega_k}e^{-i\Delta t(\omega_k-n\Omega)}dk_z, \qquad \text{(A2)}
\end{aligned}$$

where $\omega_k = \sqrt{k_x^2+k_y^2+k_z^2}$, with $k_x = k_\perp\cos\theta$, $k_y = k_\perp\sin\theta$. To obtain Eq. (A2) from Eq. (A1), we have used

$$J_0\left(2k_\perp R\sin\left(\frac{\varphi+\Omega\Delta t}{2}\right)\right) = \sum_{n=-\infty}^{+\infty}e^{in(\Omega\Delta t+\varphi)}J_n^2(k_\perp R). \qquad \text{(A3)}$$

Then, the Fourier transform of the function $P(\Delta t, \varphi, L)$ can be calculated as

$$\begin{aligned}
\mathcal{P}(\omega,\varphi,L) &= \int_{-\infty}^{+\infty}P(\Delta t,\varphi,L)e^{i\omega\Delta t}d\Delta t \\
&= \sum_{n=-\infty}^{+\infty}e^{in\varphi}\int_0^{+\infty}dk_\perp\frac{k_\perp J_n^2(k_\perp R)}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iLk_z}}{2\omega_k}\delta(\omega_k-n\Omega-\omega)dk_z, \\
&= \sum_{n>\frac{-\omega}{\Omega}}^{+\infty}\frac{e^{in\varphi}}{2\pi}\int_0^{n\Omega+\omega}J_n^2\left(\sqrt{(n\Omega+\omega)^2-x^2}R\right)\cos(Lx)dx \\
&= \sum_{n>\frac{-\omega}{\Omega}}^{+\infty}\frac{e^{in\varphi}R^{2|n|}(n\Omega+\omega)^{2|n|+1}}{2\pi(2|n|+1)!}F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix}-R^2(n\Omega+\omega)^2 \\ -L^2(n\Omega+\omega)^2/4\end{pmatrix}\right], \qquad \text{(A4)}
\end{aligned}$$

where the function $F^{0:1;0}_{1:1;0}\left[\begin{smallmatrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{smallmatrix}\begin{pmatrix}x \\ y\end{pmatrix}\right]$, which is a generalized Kampé De Fériet's double hypergeometric series (function) [36–38], can be expressed as

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix}x \\ y\end{pmatrix}\right] = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\frac{(|n|+\frac{1}{2})_k}{(|n|+\frac{3}{2})_{k+l}(2|n|+1)_k}\frac{x^k y^l}{k!l!}, \qquad \text{(A5)}$$

with $(a)_b \equiv \Gamma(a+b)/\Gamma(a)$ being the Pochhammer sign. Here we have used the following formula:

$$J_n^2(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k [2(k+|n|)]!}{k![(k+|n|)!]^2 (k+2|n|)!} \left(\frac{x}{2}\right)^{2(k+|n|)} \tag{A6}$$

in the last step in Eq. (A4). When $y=0$ or $x=0$, the double-variable function Eq. (A5) reduces to single-variable generalized hypergeometric functions as

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix} x \\ 0 \end{pmatrix}\right] = {}_1F_2\left[|n|+\frac{1}{2}; |n|+\frac{3}{2}, 2|n|+1; x\right],$$

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix} 0 \\ y \end{pmatrix}\right] = {}_0F_1\left[|n|+\frac{3}{2}; y\right]. \tag{A7}$$

The generalized Kampé De Fériet's double hypergeometric function can be written as a series of the single-variable generalized hypergeometric function ${}_0F_1$ or ${}_1F_2$, i.e.,

$$F^{0:1;0}_{1:1;0}\left[\begin{matrix} : & |n|+\frac{1}{2}; & ; \\ |n|+\frac{3}{2}: & 2|n|+1; & ; \end{matrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \sum_{k=0}^{+\infty} \frac{(|n|+\frac{1}{2})_k x^k}{(|n|+\frac{3}{2})_k (2|n|+1)_k k!} {}_0F_1\left[|n|+k+\frac{3}{2}; y\right] \tag{A8}$$

$$= \sum_{l=0}^{+\infty} \frac{y^l}{(|n|+\frac{3}{2})_l l!} {}_1F_2\left[|n|+\frac{1}{2}; |n|+l+\frac{3}{2}, 2|n|+1; x\right]. \tag{A9}$$

Therefore, Eq. (A4) can be further expressed as

$$\mathcal{P}(\omega,\varphi,L) = \sum_{n>\frac{-\omega}{\Omega}}^{+\infty} \sum_{k=0}^{+\infty} \frac{e^{i(n\varphi+k\pi)} R^{2|n|+2k} (n\Omega+\omega)^{2|n|+2k+1}}{2\pi(2|n|+2k+1)(2|n|+k)!k!} {}_0F_1\left[|n|+k+\frac{3}{2}; -L^2(n\Omega+\omega)^2/4\right] \tag{A10}$$

$$= \sum_{n>\frac{-\omega}{\Omega}}^{+\infty} \sum_{l=0}^{+\infty} \frac{e^{i(n\varphi+l\pi)} R^{2|n|} L^{2l} (n\Omega+\omega)^{2|n|+2l+1}}{2\pi|n|!l!(|n|+l+1)_{|n|+l+1}} {}_1F_2\left[|n|+\frac{1}{2}; |n|+l+\frac{3}{2}, 2|n|+1; -R^2(n\Omega+\omega)^2\right], \tag{A11}$$

which is more convenient for numerical calculation. Then, with the help of the built-in generalized hypergeometric function ${}_0F_1$ or ${}_1F_2$ in Mathematica, the function $\mathcal{P}(\omega,\varphi,L)$ can be calculated numerically to an arbitrary precision. In our numerical results (Figs. 3 and 5 in the main text), the relative truncation errors are estimated to be less than 10^{-6}.

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