Local Neumann semitransparent layers: Resummation, pair production, and duality

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We consider local semitransparent Neumann boundary conditions for a quantum scalar field as imposed by a quadratic coupling to a source localized on a flat codimension-one surface. Upon a proper regularization to give meaning to the interaction, we interpret the effective action as a theory in a first-quantized phase space. We compute the relevant heat kernel to all order in a homogeneous background and quadratic order in perturbations, giving a closed expression for the corresponding effective action in $D = 4$. In the dynamical case, we analyze the pair production caused by a harmonic perturbation and a Sauter pulse. Notably, we prove the existence of a strong/weak duality that links this Neumann field theory to the analog Dirichlet one.

I. INTRODUCTION

One of the major successes of quantum field theory (QFT) has been the prediction of the Casimir effect [1], which builds a bridge between the world of macroscopic media and that of quantum effects [2–6]. In a first approximation, the bodies may be modeled as perfect conductors and thus implemented as boundary conditions on the electromagnetic field [1]. Of course, this is not enough to describe the always improving experimental results [7,8]. Moreover, this simplification is thought to be the root of some theoretical issues that include the divergence of the energy density at the boundaries [9,10] and a possible ill definition of the vacuum self-energy [11].

A way to obtain more realistic models is to incorporate information of the bodies' bulk, as in Lifshitz theories [12]. In recent times, this has been done by modeling electromagnetic properties as external (classical) smooth fields [13–23].

Another possibility regards the substitution of perfect Dirichlet or Neumann boundary conditions by more general ones. In the Dirichlet case, one can mention for example semitransparent boundary conditions [24–27] (see also the closely related case of Robin boundary conditions [28]). These can be seen as a special case of the ones described in the previous paragraph, where potentials get localized in a thin shell and allow the recovery of perfect boundary conditions as a special limit of the coupling.

Following these lines one can also consider local semitransparent conditions, in which the coupling (or background field living on the thin shell) becomes space-time dependent [29,30]. This allows the study of interesting phenomena and cases, such as particle creation [31] and smooth nonplanar geometries (say gratings [32]) in the effective-medium approach [33].

A by far less-studied case regards the analog for Neumann boundary conditions. It has been shown in Ref. [34] that Neumann semitransparent (or imperfect) boundary conditions for a scalar field can be modeled by the operator of quantum fluctuations:

$$\Delta_\zeta := -\partial_x^2 - \zeta \partial_x \delta(x) \partial_x, \qquad (1.1)$$

where $\zeta \in \mathbb{R}$ is the coupling constant. From a mathematical point of view, the problem at hand mixes several interesting

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ingredients. Indeed, it is contained in the so-called four-parameter family of point interactions [35], i.e., it arises in the study of self-adjoint extensions of the one-dimensional second-derivative operator acting on functions appropriately defined on $\mathbb{R}/\{0\}$. In this context the interaction was baptized $\delta'(x)$, leading to some misunderstandings in the literature. This point was already explained in the paper by Šeba [36], which showed that the correct interpretation of this self-adjoint extension is in terms of a $\partial\delta\partial$ potential with a renormalized potential, as also discussed in [34,37,38]. In particular, Δ_ζ is not related to the proposal in [39], where a true generalized potential is considered.

One formal way to introduce the operator Δ_ζ is through the implementation of boundary conditions [35,40,41]. In this case, the action of Δ_ζ should be understood as implying the boundary conditions

$$\phi(0^+) - \phi(0^-) = -\zeta\phi'(0), \qquad (1.2)$$

where $0^\pm$ denotes the left/right limits to zero, and the derivative should be regularized for example as $\phi'(0) = \frac{\phi'(0^+)+\phi'(0^-)}{2}$; see [35,38]. A similar interpretation through boundary conditions has been pursued recently in [42–44], where δ-δ' structures have been considered.

In the present paper we will focus on a scalar field living in a D-dimensional flat space. It will satisfy local Neumann semitransparent boundary conditions on a flat thin shell, i.e., we will upgrade ζ to a spacetime-dependent coupling (or field living on the shell). In particular, we will follow the idea that such a field may develop small fluctuations η around a mean (or vacuum expectation) value ζ, so that an exact treatment in ζ but a perturbative one in η should provide rich physical information. We thus begin our exposition in Sec. II with a description of the relevant QFT, consisting of a scalar field interacting with a classical background. In Sec. III we explain how to compute the heat kernel of Δ_ζ to all order in the coupling ζ employing the worldline formalism, i.e., a path-integral approach. In doing so, we devise a tailor-made regularization which allows to perform in Sec. IV a perturbative computation for spacetime-dependent contributions, which is also exact in the constant background.

Then, in Sec. V we analyze the effective action in a four-dimensional setting, showing that the renormalized theory is dual to the problem of local (Dirichlet) semitransparent boundary conditions. Afterwards, we study in Sec. VI the effect of pair production in our system by considering time-dependent background fields, an analog of the widely studied dynamical Casimir effect. In particular, as examples we consider a harmonic perturbation and a Sauter pulse. Finally, in Sec. VII we state our conclusions.

We leave necessary technical information to the appendixes, where we compute the worldline-generating function relevant to our current problem (Appendix A), compute integrals involving chained free heat kernels (Appendix B;

others containing also Hermite polynomials are included in Appendix C) and calculate in closed form a series of Hermite polynomials (Appendix D). We use Planck units so that $\hbar$ and c are taken to be unity. It will prove convenient to split D-dimensional coordinates x into the coordinate perpendicular to the shell, x^D, and those $D-1$ parallel to it, $x^\parallel$; in Minkowski space, the set of spacelike coordinates of the latter is denoted by $\boldsymbol{x}^\parallel$. We define $\tilde{x} := x - L$.

II. MODEL

We begin by considering a real scalar quantum field φ living on a D-dimensional Euclidean flat spacetime; it interacts with an external (classical) real scalar field η that lives on a plate according to the following action:

$$S := \frac{1}{2}\int d^D x [(\partial\varphi)^2 + m^2\varphi^2 - (\eta(x^\parallel) + \zeta)\delta(x^D - L)(\partial^D\varphi)^2].$$
$$(2.1)$$

In this action m is the mass of the field φ and we have split the coordinates into the direction perpendicular to the plate (x^D) and those parallel to it $(x^\parallel)$. Additionally, the plate is placed at $x^D = L$ and ζ may be understood as a mean value of the field η over the plate. From a physical point of view, the classical field describes the properties of the thin plate that are relevant in the interaction with the quantum field. Notice also that η has dimensions of length, independently of the dimension D of the spacetime.

The interpretation of this action in Eq. (2.1) is more evident once we consider a homogeneous configuration for which $\eta(x^\parallel) \equiv 0$ [34]: in the limit $\zeta \to \infty$ one expects to obtain two semispaces, on whose boundaries φ satisfies Neumann boundary conditions, very much akin to the generation of Dirichlet boundary conditions through delta potentials.[1] In this sense, for finite ζ the action (2.1) can be interpreted as imposing Neumann semitransparent boundary conditions on the field φ; allowing for inhomogeneous field configurations $\eta(x^\parallel)$, we will refer to local Neumann semitransparent boundary conditions.

Following the usual way, upon a path-integral quantization one can obtain the generating functional $\mathcal{Z}$ for this system,

$$\mathcal{Z}[J] := \int \mathcal{D}\varphi e^{-S+J\varphi}. \qquad (2.2)$$

Afterwards, integrating out the scalar field φ, we can write the effective action Γ in terms of the operator of quantum fluctuations A; a direct computation gives

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[1]We will see in Sec. III A that the correct limit is indeed $\zeta \to \infty$ and not the naive one $\zeta \to -\infty$.

$$\Gamma_{\text{one-loop}} = \frac{1}{2} \text{Log Det}(A), \qquad (2.3)$$

$$A(-i\partial, x, \eta) := -\partial^2 + m^2 + (\eta(x^{\parallel}) + \zeta)\partial^D[\delta(x^D - L)\partial^D], \qquad (2.4)$$

where $\Gamma_{\text{one-loop}}$ denotes the quantum contributions to Γ. We will assume that $\eta + \zeta \geq 0$, such that A admits only a continuum spectrum. A peculiarity is that the operator A may be interpreted as a Schrödinger operator with a derivative-dependent potential, different from the more diffused case of an only position-dependent potential.[2] As we will see in the next section, this admits an interpretation of boundary conditions in phase space.

On physical grounds, we expect two interesting regimes for our system: one in which η develops small fluctuations around a constant background, admitting thus an expansion in powers of the fluctuations, and one in which topology plays an important role, such that η should be considered to all orders. In the following we will consider the former, leaving the latter for future studies.

III. A PATH INTEGRAL APPROACH: THE WORLDLINE FORMALISM IN PHASE SPACE

One simple way to compute the quantum contributions to the effective action is to employ the well-known equivalence between Log Det and TrLog, as well as Schwinger's propertime trick (or Frullani's representation for the logarithm of a quotient [45]). In this way we may recast

$$\Gamma_{\text{one-loop}} = \frac{1}{2} \text{TrLog} A = -\frac{1}{2} \int_0^{\infty} \frac{dT}{T} \text{Tr} K_A(x, x'; T), \quad (3.1)$$

where the heat kernel $K_A(x, x'; T) := \langle x | e^{-TA} | x' \rangle$ has appeared in a natural way.

From the formal side, the study of the spectral functions of operators with singular operators or generalized boundary conditions has attracted much attention in recent years [46–51]. One efficient way to perform this kind of computation is through the worldline formalism, in which one interprets the heat kernel in terms of path integrals in a first quantization procedure. Indeed, one may notice that the arguments of A are momentum and position operators in a first quantization, realized as $(\hat{p}, \hat{x}) \rightarrow (-i\partial, x)$, so that e^{-TA} can be understood as the evolution operator in imaginary time $t = iT$.

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[2]Another way to look at the problem is to focus on the similarities that the operator A has with Laplace-Beltrami operator with a singular metric. We are not going to pursue this way in this article.

The worldline formalism has been successfully applied to several problems; see the book [52], the reviews [53,54] and references therein. In particular, it has recently proved useful in three situations that are relevant to the present computation: in phase space, where it has been applied to investigate noncommutative quantum field theories [55–57] and Berry phases [58], in the analysis of singular potentials and metrics [31,59,60] and in the study of boundaries [61–64] (see also [65–67] for related path-integral approaches).

As a first step, we will perform an all-order computation with a constant background, setting $\eta \equiv 0$. It should be clear that under such assumption one can disentangle the contribution in the Dth direction, yielding the remaining components just a free path integral. Therefore, for the rest of this section we may simply work in a one-dimensional setup. Additionally, we will consider the massless case, given that the mass may be directly included in the heat kernel at the end of the computations, simply by adding a factor $e^{-m^2 T}$.

Taking these considerations into account, we follow the worldline formalism approach to compute the transition amplitudes, so that the relevant heat kernel may be written as

$$K_{\zeta}(x, x'; T)$$
$$:= \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x \mathcal{D}p \, e^{-\int_0^T dt[p^2(t)-ip(t)\dot{x}(t)-\zeta(\delta(x-L)p^2+\frac{1}{4}\delta''(x-L))]}.$$
$$(3.2)$$

In obtaining the master equation (3.2) one needs to Weyl order the operator A, which involves employing the commutation relation between derivatives and coordinates to render the operator symmetric in terms of p and x (see [52,68]); this is the origin of the δ'' contribution and one of the reasons why previous attempts to compute the heat kernel of the operator A may have failed [69]. As a matter of fact, one should keep in mind that this additional δ'' term does not enter in any way in the original operator; instead, it just plays a role in the path-integral interpretation.

One further important point is that the path integral over the momentum variables has no boundary conditions. Since from the QFT point of view we are interested in the trace of the heat kernel, we could have imposed periodic boundary conditions on the phase-space path integral. However, such a choice would not allow the study of the heat kernel out of the diagonal that we are going to undertake in the following section.

A. Heat kernel K_{ζ}

We will compute the heat kernel in a perturbative fashion, expanding the result (3.2) in the coupling parameter ζ. In this way we obtain

$$K_\zeta(x, x'; T) = \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x\mathcal{D}p e^{-\int_0^T dt[p^2(t)-ip(t)\dot{x}(t)]}$$

$$\times \sum_{n=0}^\infty \frac{\zeta^n}{n!} \prod_{j=1}^n \left\{ \int_0^T dt_j \left[\delta(x(t_j) - L)p^2(t_j) \right. \right.$$

$$\left. \left. + \frac{1}{4}\delta''(x(t_j) - L) \right] \right\}. \qquad (3.3)$$

To proceed further we compute the momentum integrals, which are Gaussian upon introducing a source denoted by j:

$$\int \mathcal{D}p e^{-\int_0^T dt(p^2 - i\dot{x}p)} p^2(t_1) = \mathcal{N} \frac{\delta^2}{\delta j(t_1)^2} e^{\frac{1}{4}\int_0^T dt(j+i\dot{x})^2} \bigg|_{j=0}.$$

$$(3.4)$$

The factor $\mathcal{N}$ is a normalization constant whose only role is to be fixed when determining the value of the free path integral and therefore may be safely dismissed. Having recast every momentum as a variation in j, one can simplify the problem noting that $\delta''(x - L) = \partial_L^2 \delta(x - L)$; this enables us to use the Dirac deltas to impose constraints on the paths as following:

$$K_\zeta(x, x'; T)$$

$$= \sum_{n=0}^\infty \left(\frac{\zeta}{4}\right)^n \int_S \prod_{l=1}^n \left[dt_l \left(4\left(\frac{\delta}{\delta j_l}\right)^2 + \partial_{L_l}^2 \right) \right]$$

$$\times \int_{x(0)=x}^{x(t_1)=L_1} \mathcal{D}x e^{\int_0^{t_1} dt \frac{(j+i\dot{x})^2}{4}} \cdots \int_{x(t_n)=L_n}^{x(T)=x'} \mathcal{D}x e^{\int_{t_n}^T dt \frac{(j+i\dot{x})^2}{4}} \bigg|_{j=0},$$

$$(3.5)$$

where for any t-dependent quantity $X(t)$ adding a subindex means $X_l := X(t_l)$ and the integral over the intermediate times has been ordered, such that

$$\int_S \prod_{j=1}^n dt_j := \int_0^T dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1. \qquad (3.6)$$

Additionally, we have introduced one position variable (L_i) for each insertion of the potential, in order to avoid undesired mixing of the derivatives; as we will see shortly, this will bring its own benefits.

In this way the computation is reduced to a chain of partition functions (or generating functionals), which can be readily computed as in Appendix A. Using those results we get

$$K_\zeta(x, x'; T) = \sum_{n=0}^\infty \left(\frac{\zeta}{4}\right)^n \int_S \prod_{l=1}^n \left[dt_l \left(4\left(\frac{\delta}{\delta j_l}\right)^2 + \partial_{L_l}^2 \right) \right]$$

$$\times \prod_{m=0}^n \left[\frac{e^{\frac{1}{4\Delta t_m}(i\Delta x_m + \int_{t_m}^{t_{m+1}} dt j(t))^2}}{\sqrt{4\pi\Delta t_m}} \right] \bigg|_{j=0}, \qquad (3.7)$$

where we have introduced the natural notation for the intermediate displacements $\Delta x_m := x_{m+1} - x_m$ and intermediate periods $\Delta t_m := t_{m+1} - t_m$; correspondingly we define $x_0 := x$, $x_{n+1} := x'$, $t_0 := 0$, $t_{n+1} := T$ and for $i = 1, \cdots n$ we set $x_i := L_i$. Even if at first sight the computation of the variations and derivatives may seem a hard task, the implementation of a Hubbard-Stratonovich transformation to linearize the problem allows a direct computation:

$$K_\zeta(x, x'; T)$$

$$= \sum_{n=0}^\infty \left(\frac{\zeta}{4}\right)^n \int_S \prod_{l=1}^n \left[dt_l \left(4\left(\frac{\delta}{\delta j_l}\right)^2 + \partial_{L_l}^2 \right) \right]$$

$$\times \int_{-\infty}^\infty \prod_{m=0}^n \left[\frac{dk_m}{(2\pi)} e^{-\Delta t_m k_m^2 + k_m(i\Delta x_m + \int_{t_m}^{t_{m+1}} dt j(t))} \right] \bigg|_{j=0}$$

$$= \frac{(-1)}{2\sqrt{\pi}} \sum_{n=0}^\infty \left(\frac{\zeta}{8\sqrt{\pi}}\right)^n \int_S \left[\prod_{l=1}^n dt_l \right] \frac{\Delta x_0 \Delta x_n}{\Delta t_0^{3/2} \Delta t_n^{3/2}}$$

$$\times \left[\prod_{m=1}^{n-1} \frac{(2\Delta t_m - \Delta x_m^2)}{\Delta t_m^{5/2}} \right] \prod_{p=0}^n e^{-\frac{\Delta x_p^2}{4\Delta t_p}}. \qquad (3.8)$$

Notice that had we set $L_i \equiv L$ at this point, the integrals in the intermediate times would have become divergent. Indeed, it has been shown in the past that the problem at hand requires a renormalization of the coupling [34,36] or, alternatively, an extension of the potential to let it act on more general functions [70]. This feature is shared by some related problems, such as point interactions in three dimensions [35,71]. In our case, the regularization is already implemented by the separation of the intermediate points, which in physical terms corresponds to the idea of a plate with finite width suggested in [34]. The limit $L_m \equiv L$ will thus be left to the last stage of the computation.

As explained in Appendix B, one can use the results in [31] to perform the integrals in the intermediate times one by one. Considering the first cases, one realizes that the result for arbitrary n involves Hermite polynomials; using the general result of Appendix C, we obtain

$$K_\zeta(x, x'; T) =: \sum_{n=0}^\infty \zeta^n K_\zeta^{(n)}(x, x'; T), \qquad (3.9)$$

where the nth-order coefficient in this expansion reads

$K_\zeta^{(n)}(x, x'; T)$

$$= -\frac{(-1)^n T^{-\frac{n}{2}-\frac{1}{2}}\text{sign}(\tilde{x})\text{sign}(\tilde{x}')e^{-\frac{(|\tilde{x}|+|\tilde{x}'|)^2}{4T}}H_n\left(\frac{|\tilde{x}|+|\tilde{x}'|}{2\sqrt{T}}\right)}{2\sqrt{\pi}}.$$

$$(3.10)$$

In this expression we have employed the sign function $\text{sign}(\cdot)$ and defined the displaced variables,

$$\tilde{x} := x - L. \qquad (3.11)$$

We can perform a resummation in Eq. (3.9) by using the method described in Appendix D; in this way we get a closed result in terms of the complementary error function $\text{erfc}(\cdot)$:

$$K_\zeta(x, x'; T) = \frac{1}{\sqrt{4\pi T}}e^{-\frac{(\tilde{x}-\tilde{x}')^2}{4T}}$$
$$+ \frac{1}{\sqrt{4\pi T}}\text{sign}(\tilde{x})\text{sign}(\tilde{x}')e^{-\frac{(|\tilde{x}|+|\tilde{x}'|)^2}{4T}}$$
$$- \frac{1}{\zeta}\text{sign}(\tilde{x})\text{sign}(\tilde{x}')e^{\frac{2\zeta(|\tilde{x}|+|\tilde{x}'|)+4T}{\zeta^2}}$$
$$\times \text{erfc}\left(\frac{\zeta(|\tilde{x}|+|\tilde{x}'|)+4T}{2\sqrt{T}\zeta}\right). \qquad (3.12)$$

The expression (3.12) coincides with the one obtained in [69] from a fermionic path integral. One further confirmation of the correctness of our result can be obtained from its trace. Integrating the heat kernel (3.12) over the whole space we get

$$\int dx K_\zeta(x, x; T) = \frac{V}{(4\pi T)^{1/2}} + \frac{1}{2}e^{\frac{4T}{\zeta^2}}\text{erfc}\left(\frac{2\sqrt{T}}{\zeta}\right), \quad (3.13)$$

where V denotes the volume (length) of the whole space. Using this formula as point of departure, we can analyze the small and large coupling regimes by using the following expansions:

$$e^{\frac{4T}{\zeta^2}}\text{erfc}\left(\frac{2\sqrt{T}}{\zeta}\right) = \begin{cases} \frac{1}{2} - \frac{2\sqrt{T}}{\sqrt{\pi}\zeta} + \frac{2T}{\zeta^2} - \frac{16T^{3/2}}{3\sqrt{\pi}\zeta^3} + \mathcal{O}(\zeta^{-4}), \\ \frac{\zeta}{4\sqrt{\pi}\sqrt{T}} - \frac{\zeta^3}{32(\sqrt{\pi}T^{3/2})} + \mathcal{O}(\zeta^4). \end{cases}$$
$$(3.14)$$

In particular, it is immediate to see that we recover the free ($\zeta = 0$) and the Neumann ($\zeta \to \infty$) cases [72].

As a final comment, recall that we are considering a positive ζ. The case with $\zeta < 0$ is subtler, given that a bound state with energy $E_b = -\frac{4}{\zeta^2}$ arises [69]. In order to obtain the heat-kernel trace for negative coupling one should then not only change the sign of ζ in Eq. (3.13) but also add a further contribution coming from the bound state, which reads

$$\int d\Delta K_\zeta(x, x; T) = e^{-TE_b} = e^{\frac{4T}{\zeta^2}}. \qquad (3.15)$$

Interestingly, the appearance of the bound state is already signaled as a nonperturbative factor in Eq. (3.13), which for $\zeta < 0$ generates a strong divergence as $\zeta \to 0$. This reminds us of similar effects in resurgence theory, where information about the nonperturbative sector is stored in the perturbative results; see [73–76] and references therein. A more detailed analysis of this fact will be left to a future publication.

B. Relation with the heat kernel of a Dirac delta potential

One fundamental remark concerns the relation of this heat kernel with a similar problem, viz. that involving a Dirac delta potential. If one defines the operator

$$A^{\text{delta}} := -\frac{d^2}{dx^2} + \lambda\delta(x - L), \qquad (3.16)$$

then its heat kernel has been shown to be [31,77,78]

$$K_{\text{delta}}(x, x'; T; \lambda)$$
$$= \frac{e^{-\frac{(x-x')^2}{4T}}}{\sqrt{4\pi T}} - \frac{\lambda}{4}e^{\frac{1}{4}\lambda[\lambda T + 2(|x-L|+|x'-L|)]}$$
$$\times \text{erfc}\left(\frac{|x-L| + |x'-L| + \lambda T}{2\sqrt{T}}\right). \qquad (3.17)$$

This means that we may recast our result (3.12) as

$$K_\zeta(x, y; T)$$
$$= \frac{(1 + \text{sign}(\tilde{x})\text{sign}(\tilde{x}'))}{2\sqrt{4\pi T}}\left(e^{-\frac{(x-x')^2}{4T}} + e^{-\frac{(\tilde{x}+\tilde{x}')^2}{4T}}\right)$$
$$+ \text{sign}(\tilde{x})\text{sign}(\tilde{x}')\left[K_{\text{delta}}(x, x'; T; 4\zeta^{-1}) - \frac{e^{-\frac{(x-x')^2}{4T}}}{\sqrt{4\pi T}}\right].$$
$$(3.18)$$

This relation is similar in nature to the Fermi-Bose duality introduced by Girardeau [79] and then further developed by Cheon and Shigehara [80] for a gas of interacting particles. However, our case differs from theirs, inasmuch as one can verify that K_ζ possesses no defined symmetry under $\tilde{x} \to -\tilde{x}$ and consequently no statistics can be clearly assigned. Alternatively, projecting our heat kernels K_ζ and K_{delta} respectively to the space of antisymmetric and symmetric functions around L, one can obtain the Fermi-Bose duality at the level of heat kernels.

Importantly, the similarity between the heat kernels involved in (3.18) gets enhanced once we consider their diagonal, since then some partial cancellations occur and the sign functions simplify. As we will see in Secs. IV A and V A, the relation will be upgraded to a map between heat kernels for the case of inhomogeneous backgrounds and

will entail a duality at the level of QFTs, linking Neumann and Dirichlet local semitransparent boundary conditions.

IV. AN EXPANSION FOR LOCAL NEUMANN SEMITRANSPARENT BOUNDARY CONDITIONS

Let us now turn our attention to the operator in Eq. (2.4). As precedently commented, we will split the classical background into a constant ζ plus a small perturbation $\eta(x^{\parallel})$ that may depend on the $(D-1)$ coordinates parallel to the plate. The idea is to perform an expansion in $\eta(x^{\parallel})$, keeping the full dependence on ζ. Following the lines in the preceding section one can obtain a worldline formula for the corresponding heat kernel:

$$K_{\zeta}(x,y;T;\eta) = \int_{x(0)=x}^{x(T)=y} \mathcal{D}x\mathcal{D}p \, e^{-\int_0^T dt[p^2(t)-ip(t)\dot{x}(t)-(\zeta+\eta(x^{\parallel}))(\delta(x^D-L)p^2+\frac{1}{4}\delta''(x^D-L))]}. \tag{4.1}$$

This expression can be readily expanded in powers of η. Moreover, we expect that in a large variety of physical situations η would be such that its average would vanish; as a consequence, we will neglect the first-order contribution, so that the first new contribution will appear at second order in η. In formulas, we obtain

$$K_{\zeta}(x,y;T;\eta) \approx K_{\zeta}(x,y;T) + K_{\zeta}^{(2)}(x,y;T;\eta),$$

$$K_{\zeta}^{(2)}(x,y;T;\eta) := \int_{x(0)=x}^{x(T)=y} \mathcal{D}x\mathcal{D}p \, e^{-\int_0^T dt[p^2(t)-ip(t)\dot{x}(t)-\zeta(\delta(x^D-L)x^D[p]^2+\frac{1}{4}\delta''(x^D-L))]}$$

$$\times \int_0^T \int_0^{s_2} ds_1 ds_2 \prod_{j=1}^2 \left\{ \eta(x^{\parallel}(s_j)) \left(p_D{}^2(s_j) + \frac{\partial_{L_j}^2}{4} \right) \delta(x^D(s_j) - L_j) \right\}, \tag{4.2}$$

in which we have explicitly used the symmetry under exchange of the intermediate times $s_{i=1,2}$. Once more, the computation is more involved than the usual case, given that the potential on one side involves derivatives and on the other should be regularized. For this reason, it proves convenient to employ a series expansion in ζ, instead of trying to make direct use of the heat kernel in Eq. (3.12). Calling $\eta_j := \eta(x^{\parallel}(s_j))$ one obtains

$$K_{\zeta}^{(2)}(x,y;T;\eta) = \int_0^T \int_0^{s_2} ds_1 ds_2 \sum_{n=2}^{\infty} \zeta^n \int_0^T dt_n \cdots \int_0^{t_2} dt_1 \prod_{j=1}^n \left(\delta_{k^D(t_j)}^2 + \frac{\partial_{L_j}^2}{4} \right) \prod_{i=1}^2 \left(\delta_{k^D(s_i)}^2 + \frac{\partial_{L_i}^2}{4} \right)$$

$$\times \int_{x(0)=x}^{x(T)=y} \mathcal{D}x \, e^{\frac{1}{4}\int_0^T dt[i\dot{x}(t)+k(t)]^2} \eta_1 \eta_2 \prod_{j=1}^n \delta(x^D(t_j) - L_j) \prod_{i=1}^2 \delta(x^D(s_i) - L_i). \tag{4.3}$$

If we now want to interpret the Dirac delta functions as fixing the path at given times, then there is an additional difficulty related to the fact that the intermediate times $s_{i=1,2}$ are not ordered with respect to the other times, t_i. However, whatever value $s_{i=1,2}$ may take, they will of course fall into one of the intervals $(0,t_1),(t_1,t_2),\cdots,(t_n,T)$. We can thus define ordered times $\hat{t}_i(lm)$, such that

$$(\hat{t}_0(lm),\hat{t}_1(lm),\cdots) := (0,t_1,\ldots,t_l,s_1,t_{l+1},\ldots,t_m,s_2,t_{m+1},\ldots,T), \tag{4.4}$$

and introduce an analogous definition for the[3] $\hat{L}_i$ and $\hat{x}_i$. Hiding the (lm) dependence for reasons of readability, we obtain

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[3] The hatted coordinates correspond to the Dth component, i.e., $\hat{x}_i = \hat{x}_i^D$; we omit the upper index D to simplify the notation.

$$K_\zeta^{(2)}(x,y;T;\eta] = \frac{(-1)}{2^7\pi^{3/2}}\sum_{n=0}^{\infty}\sum_{l\leq m=0}^{n}\int_{x^\parallel(0)=x^\parallel}^{x^\parallel(T)=y^\parallel}\mathcal{D}x^\parallel\eta_1\eta_2 e^{-\frac{1}{4}\int_0^T dt\dot{x}^\parallel{}^2}$$

$$\times\left\{\left(\frac{\zeta}{8\sqrt{\pi}}\right)^{n-m}\int_0^T ds_2\int_{s_2}^T dt_n\cdots\int_{s_2}^{t_{m+2}}dt_{m+1}\frac{\Delta\hat{x}_{n+2}}{\Delta\hat{t}_{n+2}^{3/2}}\left[\prod_{r=m+2}^{n+1}\frac{(2\Delta\hat{t}_r-\Delta\hat{x}_r^2)}{\Delta\hat{t}_r^{5/2}}\right]\left[\prod_{p=m+2}^{n+2}e^{-\frac{\Delta\hat{x}_p^2}{4\Delta\hat{t}_p}}\right]\right\}$$

$$\times\left\{\left(\frac{\zeta}{8\sqrt{\pi}}\right)^{m-l}\int_0^{s_2}ds_1\int_{s_1}^{s_2}dt_m\cdots\int_{s_1}^{t_{l+2}}dt_{l+1}\left[\prod_{r=l+1}^{m+1}\frac{(2\Delta\hat{t}_r-\Delta\hat{x}_r^2)}{\Delta\hat{t}_r^{5/2}}\right]\left[\prod_{p=l+1}^{m+1}e^{-\frac{\Delta\hat{x}_p^2}{4\Delta\hat{t}_p}}\right]\right\}$$

$$\times\left\{\left(\frac{\zeta}{8\sqrt{\pi}}\right)^{l}\int_0^{s_1}dt_l\cdots\int_0^{t_2}dt_1\frac{\Delta\hat{x}_0}{\Delta\hat{t}_0^{3/2}}\left[\prod_{r=1}^{l}\frac{(2\Delta\hat{t}_r-\Delta\hat{x}_r^2)}{\Delta\hat{t}_r^{5/2}}\right]\left[\prod_{p=0}^{l}e^{-\frac{\Delta\hat{x}_p^2}{4\Delta\hat{t}_p}}\right]\right\}. \tag{4.5}$$

Notice that we may also reorder the series to get the compact expression

$$K_\zeta^{(2)}(x,y;T;\eta] = -\frac{1}{16}\int_0^T ds_2\int_0^{s_2}ds_1\int_{x^\parallel(0)=x^\parallel}^{x^\parallel(T)=y^\parallel}\mathcal{D}x^\parallel\eta_1\eta_2 e^{-\frac{1}{4}\int_0^T dt\dot{x}^\parallel{}^2}S_1(\tilde{y}^D;T-s_2;\zeta)S_2(s_2-s_1;\zeta)S_1(-\tilde{x}^D;s_1;\zeta), \tag{4.6}$$

where the S_1 and S_2 functions are defined as follows:

$$S_1(x;T;\zeta) := \frac{1}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\frac{\zeta}{8\sqrt{\pi}}\right)^n\lim_{\Delta x_0,\cdots\Delta x_{n-1}\to 0}\int_0^T d\alpha_n\cdots\int_0^{\alpha_2}d\alpha_1\frac{xe^{-\frac{x^2}{4\Delta\alpha_n}}}{\Delta\alpha_n^{3/2}}\left[\prod_{r=0}^{n-1}\frac{(2\Delta\alpha_r-\Delta x_r^2)}{\Delta\alpha_r^{5/2}}\right]\left[\prod_{p=0}^{n-1}e^{-\frac{\Delta x_p^2}{4\Delta\alpha_p}}\right],$$

$$S_2(T;\zeta) := \frac{1}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\frac{\zeta}{8\sqrt{\pi}}\right)^n\lim_{\Delta x_0,\cdots\Delta x_n\to 0}\int_0^T d\alpha_n\cdots\int_0^{\alpha_2}d\alpha_1\left[\prod_{p=0}^{n}e^{-\frac{\Delta x_p^2}{4\Delta\alpha_p}}\right]\left[\prod_{r=0}^{n}\frac{(2\Delta\alpha_r-\Delta x_r^2)}{\Delta\alpha_r^{5/2}}\right]. \tag{4.7}$$

In the usual case, i.e., when one considers potentials that do not involve derivatives, the S_i functions would correspond to heat kernels with constant coupling. In the current model, the S_i play instead the role of regularized derivatives of the heat kernel with constant coupling ζ. Employing the results in Appendixes C and D, we can perform a resummation of the power series in ζ and obtain them in closed form:

$$S_1(x;T;\zeta) = \frac{e^{-\frac{x^2}{4T}}}{2\sqrt{\pi}T}\sum_{n=0}^{\infty}\left(-\frac{\zeta\,\mathrm{sign}(x)}{4\sqrt{T}}\right)^n H_{n+1}\left(\frac{x}{2\sqrt{T}}\right)$$

$$= -\frac{4}{\zeta^2}\mathrm{sign}(x)\left[e^{\frac{4T+2\zeta|x|}{\zeta^2}}\mathrm{erfc}\left(\frac{\zeta|x|+4T}{2\zeta\sqrt{T}}\right)-\zeta\frac{e^{-\frac{x^2}{4T}}}{\sqrt{4\pi T}}\right], \tag{4.8}$$

$$S_2(T;\zeta) = -\frac{1}{2\sqrt{\pi}T^{3/2}}\sum_{n=0}^{\infty}\left(-\frac{\zeta}{4\sqrt{T}}\right)^n H_{n+2}(0)$$

$$= -\frac{16}{\zeta^3}\left[e^{\frac{4T}{\zeta^2}}\mathrm{erfc}\left(\frac{4\sqrt{T}}{2\zeta}\right)-\frac{\zeta}{\sqrt{4\pi T}}\right]. \tag{4.9}$$

At this point we can perform the following fast check. If we consider a constant η, then the result in Eq. (4.6) reduces, as expected, to a free heat kernel in the parallel directions, multiplied by $K_{\zeta+\eta}(x,y;T)$ restricted to quadratic order in η.

Considering once more the expression (4.6) for the heat kernel, the integrals in the parallel directions are trivial:

$$\int_{x^\parallel(0)=x^\parallel}^{x^\parallel(T)=y^\parallel}\mathcal{D}x^\parallel\eta_1\eta_2 e^{-\frac{1}{4}\int_0^T dt\dot{x}^\parallel{}^2}$$

$$= \frac{e^{-\frac{(y^\parallel-x^\parallel)^2}{4T}}}{(4\pi T)^{(D-1)/2}}\int\frac{dk_1{}^\parallel dk_2{}^\parallel}{(2\pi)^{2D-2}}\tilde{\eta}_1\tilde{\eta}_2 e^{i(k_1{}^\parallel\cdot x_c{}^\parallel(s_1)+k_2{}^\parallel\cdot x_c{}^\parallel(s_2))}$$

$$\times e^{-\frac{(T-s_1)s_1}{T}\|k_1\|^2-\frac{(T-s_2)s_2}{T}\|k_2\|^2-2\frac{s_1(T-s_2)}{T}k_1{}^\parallel\cdot k_2{}^\parallel}, \tag{4.10}$$

where $x_c{}^\parallel(t) := \frac{(y^\parallel-x^\parallel)}{T}t+x^\parallel$ and the Fourier transforms $\tilde{\eta}_i := \tilde{\eta}(k_i{}^\parallel)$ are defined as

$$\eta(x^\parallel) =: \int\frac{dk^\parallel}{(2\pi)^{D-1}}e^{ik^\parallel\cdot x^\parallel}\tilde{\eta}(k^\parallel). \tag{4.11}$$

Replacing in Eq. (4.6) we obtain our final expression for the contribution to the heat kernel of quadratic order in η:

$$K_\zeta^{(2)}(x,y;T;\eta]$$

$$= -\frac{1}{16}\frac{e^{-\frac{(y^\parallel-x^\parallel)^2}{4T}}}{(4\pi T)^{(D-1)/2}}\int_0^T ds_2\int_0^{s_2}ds_1\int\frac{dk_1{}^\parallel dk_2{}^\parallel}{(2\pi)^{2D-2}}\tilde{\eta}_1\tilde{\eta}_2$$

$$\times e^{i(k_1{}^\parallel\cdot x_c{}^\parallel(s_1)+k_2{}^\parallel\cdot x_c{}^\parallel(s_2))}e^{-\frac{(T-s_1)s_1}{T}\|k_1\|^2-\frac{(T-s_2)s_2}{T}\|k_2\|^2-2\frac{s_1(T-s_2)}{T}k_1{}^\parallel\cdot k_2{}^\parallel}$$

$$\times S_1(\tilde{y}^D;T-s_2;\zeta)S_2(s_2-s_1;\zeta)S_1(\tilde{x}^D;s_1;\zeta). \tag{4.12}$$

A. Mapping the heat kernel: Neumann semitransparent to Dirichlet semitransparent

Looking at the closed expressions for the S_i one notices that they are proportional to the heat kernel for a delta potential, cf. Eq. (3.17). The explicit relations are

$$S_1(x - L; T; \zeta) = \frac{4\text{sign}(x - L)}{\zeta} K_{\text{delta}}(x, L; T; 4\zeta^{-1}), \quad (4.13)$$

$$S_2(T; \zeta) = \frac{16}{\zeta^2} K_{\text{delta}}(L, L; T; 4\zeta^{-1}). \quad (4.14)$$

A more carefully comparison shows that the proportionality extends also to the heat-kernel expression at quadratic order in η, so that, up to a rescaling in the inhomogeneities, the weak-background problem in the delta case (or more precisely, local semitransparent Dirichlet boundary conditions) is mapped to a strong-background regime in our current model and vice versa[4]:

$$K_\zeta^{(2)}(x, y; T; \eta]$$
$$= \text{sign}(\tilde{x}^D)\text{sign}(\tilde{y}^D) K_{\text{delta}}^{(2)}(x, y; T; 4\zeta^{-1}; -4\zeta^{-2}\eta]. \quad (4.15)$$

In particular, this means that we may borrow some results from [31]; as an example, the trace of the heat kernel (which will be employed to compute the effective action) is simply given by

$$\text{Tr} K_\zeta^{(2)}(x, y; T; \eta]$$
$$= \frac{8}{\zeta^4} \frac{T^2}{(4\pi T)^{(D-1)/2}} \int \frac{dk_1^\parallel}{(2\pi)^{D-1}} |\tilde{\eta}_1|^2 \int_0^1 ds_- e^{-T k_1^\parallel{}^2 s_-(1-s_-)}$$
$$\times K_{\text{delta}}(L, L; T(1 - s_-); 4\zeta^{-1}) K_{\text{delta}}(L, L; T s_-; 4\zeta^{-1}). \quad (4.16)$$

An even more detailed inspection shows that this is not an accidental relation valid only for the quadratic expansion in η. Indeed, one can repeat the computations performed in the previous section for an arbitrary order in η, to find that, dismissing the path integrals in the parallel directions, one obtains a chain,

$$K_\zeta^{(n)}(x, y; T; \eta]$$
$$\sim \int ds_1 \cdots ds_n S_1(\tilde{y}^D; T - s_n; \zeta) S_2(s_n - s_{n-1}; \zeta) \cdots$$
$$\times S_2(s_2 - s_1; \zeta) S_1(-\tilde{x}^D; s_1; \zeta), \quad (4.17)$$

which is exactly the one we would have obtained in the delta case. This provides an order by order proof of the

[4]We are defining $K_{\text{delta}}^{(2)}(x, y; T; 4\zeta^{-1}; -4\zeta^{-2}\eta]$ as the contribution to the heat kernel of the operator $A_D^{\text{delta}} := [-\partial^2 + m^2 + (\eta(x^\parallel) + \zeta)\delta(x^D - L)]$ which is quadratic in η.

existence of a map between the heat kernels of these two different problems; more precisely, the map is given by

$$K_\zeta(x, y; T; \eta]$$
$$= \text{sign}(\tilde{x}^D)\text{sign}(\tilde{y}^D) \left[K_{\text{delta}}(x, y; T; 4\zeta^{-1}; 4\zeta^{-2}\eta] - \frac{e^{-\frac{(x-y)^2}{4T}}}{(4\pi T)^{D/2}} \right]$$
$$+ \frac{(1 + \text{sign}(\tilde{x}^D)\text{sign}(\tilde{y}^D))}{2(4\pi T)^{D/2}} \left(e^{-\frac{(x-y)^2}{4T}} + e^{-\frac{(\tilde{x}^D + \tilde{y}^D)^2 + (x^\parallel - y^\parallel)^2}{4T}} \right). \quad (4.18)$$

This map depends strongly on the fact that η lives on the plate and will be broken if one introduces for example an additional potential with support outside from it. As we will see in Sec. V, the mapping that we have discussed will automatically translate into a duality at the level of the renormalized semiclassical field theory.

B. Purely inhomogeneous coupling

One interesting case is that in which the background field is small, such that the fluctuations may become larger than it.[5] In this regime we will be able to obtain a closed expression for the trace of the heat kernel and the effective action in the massive case. Moreover, as we will see this will turn out to be an instructive limiting case.

Let us then begin with expression (4.16). Taking its small-ζ limit involves expanding the heat kernel of the delta potential for large coupling, which gives

$$K_{\text{delta}}(x, x'; T; 4\zeta^{-1})$$
$$= \frac{e^{-\frac{(x-x')^2}{4T}}}{2\sqrt{\pi}\sqrt{T}} - \frac{e^{-\frac{(|\tilde{x}| + |\tilde{x}'|)^2}{4T}}}{2\sqrt{\pi}\sqrt{T}} + \frac{\zeta(|\tilde{x}| + |\tilde{x}'|)e^{-\frac{(|\tilde{x}| + |\tilde{x}'|)^2}{4T}}}{8\sqrt{\pi}T^{3/2}}$$
$$- \frac{\zeta^2 e^{-\frac{(|\tilde{x}| + |\tilde{x}'|)^2}{4T}}((|\tilde{x}| + |\tilde{x}'|)^2 - 2T)}{32\sqrt{\pi}T^{5/2}} + \mathcal{O}(\zeta^3), \quad (4.19)$$

so that after setting $x' \equiv L$ the leading term cancels, rendering the expression more divergent as $T \to 0$:

$$K_{\text{delta}}(x, L; T; 4\zeta^{-1}) = \frac{\zeta e^{-\frac{(L-x)^2}{4T}}(2|L - x| + \zeta)}{16\sqrt{\pi}T^{3/2}}$$
$$- \frac{\zeta^2(L-x)^2 e^{-\frac{(L-x)^2}{4T}}}{32\sqrt{\pi}T^{5/2}} + \mathcal{O}(\zeta^3). \quad (4.20)$$

Setting also $x \equiv L$, the first contribution to K_{delta} is of order ζ^2, which is exactly the power needed to cancel the inverse powers of ζ in Eq. (4.16). In our computation the limit $x, x' \to L$ should be taken at the end, since they act as

[5]But keeping always $\zeta + \eta > 0$ such to avoid instabilities triggered by possible bound states.

regulators; the final expression once the mass contribution is reinstated is given by

$$
\text{Tr}K_0^{(2)}(x,y;T;\eta]
$$
$$
= -\frac{1}{2^5}\frac{e^{-m^2 T}}{T(4\pi T)^{(D-1)/2}}\int\frac{dk^{\|}}{(2\pi)^{D-1}}|\tilde{\eta}|^2 g(Tk^{\|2}), \quad (4.21)
$$

where $g(\cdot)$ is defined in terms of the modified Bessel functions of the first kind $I_\alpha(\cdot)$:

$$
g(b) := be^{-\frac{b}{8}}\left[I_0\left(\frac{b}{8}\right)+I_1\left(\frac{b}{8}\right)\right]. \quad (4.22)
$$

To have an intuition of the result (4.21), one can study the behavior of $g(\cdot)$ for large and small arguments:

$$
g(b) = \begin{cases} \frac{4\sqrt{b}}{\sqrt{\pi}}-\frac{4}{\sqrt{\pi b}}-\frac{6}{\sqrt{\pi b^3}}+\mathcal{O}(b^{-5/2}), \\ b-\frac{b^2}{16}+\frac{b^3}{256}+\mathcal{O}(b^4). \end{cases} \quad (4.23)
$$

Notice that the limit of a homogeneous configuration gives a vanishing contribution, in agreement with the previous results in Sec. III A. Moreover, the first terms in a small-propertime expansion are local. If instead the field η acquires modes with large momenta, then the expansion involves half-integer powers of $k^{\|}$, which is tantamount to saying that nonlocal terms will play an important role.

The result in (4.21) deserves one last comment, which will become important in the analysis of the effective action in the following section. A consequence of the small-ζ limit is that the expression for the heat kernel, in the expansion for small propertime, is more divergent than the finite ζ case by a factor T^{-1}. More generally, every time we increase the order in η by 1, the expansion of the heat kernel for small T will be more divergent by a factor $T^{-1/2}$. This is reminiscent of the findings for a constant coupling, cf. (3.10), signaling that for a small coupling a resummation may be needed in order to see the real behavior for small T.

V. EFFECTIVE ACTION

A. Duality for the field theory

Consider now the mapping in Eq. (4.18) at the level of the effective action adding a mass term. Employing the formula (3.1) for the effective action in terms of the heat kernel's trace, we get the following relation between the quantum contributions to the effective action in our generalized Neumann case, $\Gamma_{1-\text{loop}}$, and those in the generalized Dirichlet case, $\Gamma_{1-\text{loop}}^{\text{delta}}$:

$$
\Gamma_{1-\text{loop}}(\zeta;\eta]
$$
$$
= -\frac{1}{2}\int_0^\infty\frac{dT}{T}e^{-m^2 T}\int d^D x K_\zeta(x,x;T;\eta]
$$
$$
= -\frac{1}{2}\int_0^\infty\frac{dT}{T}e^{-m^2 T}\int d^D x\left[K_{\text{delta}}(x,x;T;4\zeta^{-1};4\zeta^{-2}\eta]\right.
$$
$$
\left.+\frac{e^{-\frac{(x-L)^2}{T}}}{(4\pi T)^{D/2}}\right]
$$
$$
= \Gamma_{1-\text{loop}}^{\text{delta}}(4\zeta^{-1};4\zeta^{-2}\eta]+4^{-D/2}. \quad (5.1)
$$

In as much as we do not consider the interaction with gravity, the constant factor $4^{-D/2}$ is irrelevant in the computation of physical quantities, since it will be absorbed in the renormalization of the cosmological constant. Therefore, we can see that there is a duality between both theories at the quantum level: if one desires to compute the large background expansion in one theory, one may simply study the small background of the other. These assertions are valid independently from the dimensions D of spacetime in which we choose to work.

B. Inhomogeneous and massless case in $D=4$

As an immediate consequence of the duality discussed in the previous paragraphs, we can study the massless case in $D=4$ by borrowing results from the delta-potential case previously obtained by some of the authors of this work [31]. The explicit result for the effective action at quadratic order in η is

$$
\Gamma_{m=0,D=4}^{(2)} := \frac{1}{2}\text{TrLog}A\Big|_{\text{order }\eta^2}
$$
$$
= \int\frac{d^3 k^{\|}}{(2\pi)^3}\tilde{\eta}(k^{\|})\tilde{\eta}(-k^{\|})F(k^{\|},\zeta), \quad (5.2)
$$

where the form factor F has been split into three terms, one divergent as D tends to four, another which is finite and local (F_L), and the remaining which is finite and nonlocal (F_{NL}):

$$
F(k^{\|},\zeta) := -\frac{8}{\zeta^4}\left[\frac{1}{4\pi^2\zeta}\frac{1}{D-4}+F_L(4\zeta^{-1})+F_{NL}(k^{\|},4\zeta^{-1})\right]. \quad (5.3)
$$

The explicit expressions for these contributions, defining $b^2 := 16(\zeta k)^{-2}$ and in terms of Lerch's transcendent function $\Phi(\cdot,\cdot,\cdot)$, read

$$
F_L(4\zeta^{-1}) := \frac{1}{8\pi^2\zeta}\left(\gamma-4+\log\left(\frac{\mu^2\zeta^2}{256\pi^3}\right)\right), \quad (5.4)
$$

$$
F_{NL}(k,4\zeta^{-1}) := \frac{1}{8\pi^2\zeta}\left[-\frac{1}{b}+(2+b)\log\left(1+\frac{2}{b}\right)\right.
$$
$$
\left.+\frac{b}{4(1+b)}\Phi\left(\frac{1}{(1+b)^2},2,\frac{1}{2}\right)\right]. \quad (5.5)
$$

In order to render the nonlocality of F_{NL} more visible, one can perform expansions for large and small b, for which we get either $\log(k^2)$ contributions or half-integer powers of k^2 that preclude a so-called derivative expansion (see [81] for its application to a Casimir configuration which is similar in spirit to ours):

$$F_{NL}(k, 4\zeta^{-1}) = \begin{cases} -\frac{k}{32\pi^2}\left[1 - 4\log\left(\frac{\zeta^2 k^2}{4}\right) - \frac{2(4\log(\zeta^2 k^2/4)+\pi^2+8)}{\zeta k} + \mathcal{O}((k\zeta)^{-2})\right], \\ \frac{3}{8\pi^2\zeta}\left[1 - \frac{\zeta^2 k^2}{216} + \frac{\zeta^4 k^4}{7200} - \frac{\zeta^5(k^2)^{5/2}}{17280} + \frac{11\zeta^6 k^6}{564480} - \frac{\zeta^7(k^2)^{7/2}}{161280} + \mathcal{O}((k\zeta)^8)\right]. \end{cases} \tag{5.6}$$

At this point some comments are in order. First, the leading terms were to be expected from a simple dimensional analysis of the problem. Indeed, this is the reason why corrections proportional to k are so frequently encountered in the bibliography [82].

Second, the vanishing Neumann limit of $\Gamma^{(2)}$ seems to be well motivated: indeed, a small variation around infinity should make no difference, at least as long as η is small, which was one of our hypotheses. The physical mechanism is similar to that in the Dirichlet case, where a larger coupling tends to repel the quantum field from the sheet, correspondingly attenuating the interaction with the background η. The only difference is that the coupling in the current situation involves derivatives of the quantum field.

Third, Eq. (5.3) signals that the theory needs to undergo renormalization. In dimensional regularization, the only term that needs a counterterm is the mass term for η. However, in other schemes one may obtain additional divergent terms, as discussed for general cases in [5,83]. Although this is not the case if $\zeta > 0$, the $\zeta = 0$ case is more subtle and will be discussed in Sec. V C.

Lastly and related to the previous point, in the limit of vanishing coupling the effective action in expression (5.2) is divergent, as can be seen from Eq. (5.3) together with the corresponding definitions and the expansion in Eq. (5.6). Such divergence simply indicates the fact that the expansions for small η and ζ do not commute. To gain insight into this point, consider the constant coupling case. A straightforward computation shows that

$$\frac{\Gamma}{V_\parallel} = \frac{1}{3\pi^3(D-4)\zeta^3} + \frac{-\log(\pi\zeta^2\mu^2) - \psi(\frac{5}{2})}{6\pi^3\zeta^3}$$
$$+ \mathcal{O}(D-4), \tag{5.7}$$

where $\psi(\cdot)$ is the polygamma function and $V_\parallel$ is the volume over the plate. If we now consider homogeneous perturbations by replacing $\zeta \to \zeta + \eta$, the expansion will be in powers of η/ζ, rendering clear our statement. The physical intuition of why the expansion is singular for ζ around zero is related to the instabilities generated by the bound state for $\zeta < 0$. An alternative heuristic way to make sense of those divergences is to interpret them as a need of an additional renormalization. We will analyze this point further in the following section.

C. On the purely inhomogeneous and massive case

To understand better the $\zeta \to 0$ limit of the previous expressions, let us set $\zeta \equiv 0$ right from the beginning; the simpler formulas enable us to include the effects of a nonvanishing mass. One can then compute the effective action employing the heat-kernel's trace in Eq. (4.21); the result for a massive field in such case is[6]

$$\Gamma^{(2)}_{\text{pure}} = \int \frac{d^3 k^\parallel}{(2\pi)^3} \tilde\eta(k^\parallel)\tilde\eta(-k^\parallel) F_{\text{pure}}(k^\parallel, m), \tag{5.8}$$

where the form factor is defined in terms of the hypergeometric function $_2F_1(\cdot, \cdot; \cdot; \cdot)$:

$$F_{\text{pure}}(k, m) := \frac{k^2(m^2)^{3/2}}{384\pi} {}_2F_1\left(-\frac{3}{2}, \frac{1}{2}; 2; -\frac{k^2}{4m^2}\right). \tag{5.9}$$

The asymptotic expressions of this form factor in the limits of large and small mass can be obtained from the corresponding expansions of the hypergeometric function, to read

$${}_2F_1\left(-\frac{3}{2}, \frac{1}{2}; 2; -b\right) = \begin{cases} \frac{8b^{3/2}}{15\pi} + \frac{2\sqrt{b}}{\pi} + \mathcal{O}(b^{-1/2}), \\ 1 + \frac{3b}{8} + \frac{3b^2}{64} - \frac{5b^3}{1024} + \mathcal{O}(b^4). \end{cases} \tag{5.10}$$

In particular, the massless limit corresponds to a nonlocal term proportional to a half-integer power of k^2, namely $(k^2)^{5/2}$.

Notice that the result in Eq. (5.8) is automatically finite in dimensional regularization. However, as mentioned in the previous section the situation may change in other schemes. If instead of dimensional regularization a cutoff is introduced, then also the terms $(k^2)^i|\tilde\eta|^2$, $i = 1, 2$, should be renormalized. Indeed, computing the effective action from (4.21), we introduce a UV cutoff Λ with dimensions of momentum, set $D \equiv 4$ and expand for small T to obtain

---

[6]We are considering $\eta(x^\parallel) > 0$, so that effectively one may extract a mean value and perturbations around it. However, it proves convenient for the following discussion to keep η as one single entity.

$$\Gamma_{\text{pure}}^{(2)} = \int d^3 k^{\parallel} \int_{\Lambda^{-2}}^{1} dt |\tilde{\eta}|^2$$

$$\times \left[\frac{k^{\parallel 2}}{512\pi^{3/2}T^{5/2}} - \frac{k^{\parallel 4} + 16k^{\parallel 2}m^2}{8192\pi^{3/2}T^{3/2}} + \mathcal{O}(T^{-1/2}) \right] + \cdots$$

$$= \frac{1}{768\pi^{3/2}} \int d^3 k^{\parallel} |\tilde{\eta}|^2 \left[k^{\parallel 2}\Lambda(\Lambda^2 - 3m^2) - \frac{k^{\parallel 4}\Lambda}{16} \right] + \cdots,$$

$$(5.11)$$

where the dot points denote finite terms as Λ tends to infinity. As a consequence, the theory lacks predictivity for the terms depicted in (5.11). In particular, one may set all of them to zero, as in the substraction of large mass terms suggested in [84].

At this point one may understand the $\zeta \to 0$ limit of the results in the previous section as follows. The divergent contributions in Eq. (5.8) as $\zeta \to 0$ should be reabsorbed in a renormalization process; the explicit equivalence between both approaches can be seen by comparing them to Eq. (5.11). Taking this comment into account, one then sees that the $\zeta \to 0$ limit of Eq. (5.2) and the massless limit of expression (5.8) agree at the renormalized level.

One can also envisage what would happen once higher powers in η are considered. At the end of Sec. IV we have mentioned that, in the purely inhomogeneous scenario, the expansion of the heat kernel for small propertimes acquires one additional $T^{-1/2}$ for every extra power of η. This implies that the number of terms to be renormalized will also correspondingly increase. However, this situation is reminiscent of the perturbative expansion of the heat kernel for homogeneous coupling ζ [cf. Eq. (3.10)], where arbitrary large negative powers of the propertime appear. Once the series is resummed, the result (3.12) is seen to have only a $T^{-1/2}$ divergence for small propertime. We expect that a similar mechanism should be behind the need for renormalization of terms like those in (5.11) as long as η is strictly bigger than zero, since we have not seen them in the expansion studied in Sec. V B. In other words, the resummation in ζ performed in Sec. IV, if $\eta/\zeta < 1$, is expected to be enough to avoid the singularities of the effective action's expansion for vanishing total coupling $(\zeta + \eta)$.

VI. DYNAMICAL CASIMIR EFFECT AND PARTICLE CREATION

The term dynamical Casimir effect refers to the study of the electromagnetic field's quantum vacuum properties in time-dependent systems. In particular, it comprises the phenomena of photon generation as a consequence of moving boundaries or the change in the electromagnetic properties of the involved media with time; the interested reader may refer to the reviews [85,86] and references therein.

As is well known, scalar fields have been introduced in the past as simplified models which may help to understand the general behavior of vacuum properties. Following this logic, in the present section we will describe the analog of the dynamical Casimir effect in our scalar model: a time-dependent external field η localized on a plane will play the role of time-varying electromagnetic properties, while instead of photon creation we will observe the production of scalar particles.

The initial obstacle that we encounter is that up to this point we have restricted ourselves to the consideration of the theory in Euclidean space. However, as customarily done, one can appeal to a Wick rotation in order to consider the problem in Minkowski space. This will prove to be enough to mimic a situation of dynamical Casimir effect through time-dependent properties of the wall.

First of all, to examine this dynamical scenario we will assume that the argument of the delta function in Eq. (2.1) corresponds to a spatial coordinate, so that time may only be an argument of the external field η. Calling τ the Euclidean time and x_0 the Minkowski one, the rotation $x_0 =: -i\tau$ (accompanied by analogous rotations for every zeroth component of a tensor) may be performed without encountering singularities in the form factors studied in this manuscript. One obtains then a master formula for the effective action $\Gamma_M^{(2)}$ in Minkowski space at second order in η,

$$\Gamma_M^{(2)} = -\int \frac{d^3 k^{\parallel}}{(2\pi)^3} |\tilde{\eta}_M(k_0, \boldsymbol{k}^{\parallel})|^2 F\left(\sqrt{\boldsymbol{k}^{\parallel 2} - k_0^2 - i\epsilon}, m \right), \quad (6.1)$$

where we have introduced Feynman's prescription through an infinitesimal parameter ϵ, we have taken the branch cut of the square root to be in the negative real axis and the Fourier transform in Minkowski space is defined as[7]

$$\eta(x^{\parallel}) =: \int \frac{dk_0 d\boldsymbol{k}^{\parallel}}{(2\pi)^3} e^{i(-x_0 k_0 + \boldsymbol{x}^{\parallel} \cdot \boldsymbol{k}^{\parallel})} \tilde{\eta}_M(k_0, \boldsymbol{k}^{\parallel}). \quad (6.2)$$

Of course the form factor F, depending on the situation under study, may be chosen among those in Eqs. (5.3) or (5.9).

The expression for the effective action may be used to compute the creation of particles in a dynamical situation. Indeed, in the usual in-out formalism, the vacuum persistence's amplitude is given by the effective action as

$$\langle 0_{\text{out}} | 0_{\text{in}} \rangle = e^{i\Gamma_M}. \quad (6.3)$$

If the effective action develops an imaginary contribution, which may be possible by the appearance of branch cuts in (6.1) after the Wick rotation, then the vacuum becomes

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[7]The set of components that are parallel to the plates and spacelike is denoted by $\boldsymbol{k}^{\parallel}$.

unstable through a process of pair creation, whose probability P is defined as

$$1 - P := e^{-2\mathrm{Im}\Gamma_M}. \tag{6.4}$$

In the most frequently studied situation, i.e., for weak pair-creation processes, we may approximate $P \approx 2\mathrm{Im}\Gamma_M$. Therefore, the quantity in which we are interested is

$$P \approx 2 \int \frac{\mathrm{d}^3 k^{\|}}{(2\pi)^3} |\tilde{\eta}_M(k_0, \boldsymbol{k}^{\|})|^2 \mathrm{Im}\left[F\left(\sqrt{\boldsymbol{k}^{\|2} - k_0^2 - \mathrm{i}\epsilon}, m\right)\right]. \tag{6.5}$$

Turning back to Eq. (5.9), the hypergeometric function $_2F_1(-\frac{3}{2}, \frac{1}{2}; 2; x)$ possesses a branch cut, which in the case of the principal branch runs from 1 to ∞ on the real x axis. We may thus recast the probability of pair creation as

$$P = 2 \int \frac{\mathrm{d}^3 k^{\|}}{(2\pi)^3} |\tilde{\eta}_M(k_0, \boldsymbol{k}^{\|})|^2 \Theta(k_0^2 - \boldsymbol{k}^{\|2} - 4m^2)$$
$$\times \mathrm{Im}\left[F\left(\sqrt{\boldsymbol{k}^{\|2} - k_0^2 - \mathrm{i}\epsilon}, m\right)\right], \tag{6.6}$$

where $\Theta(\cdot)$ is the Heaviside function that signals the threshold of the pair-creation process: the external field η must provide at least the rest energy of two particles for the process to take place. In the case described in Eq. (5.2), such threshold is absent because particles are taken as massless.

A. Harmonic perturbations

A simple model that mimics the dynamical Casimir effect, introduced in [87] for the one-dimensional case and studied also in [31] for an inhomogeneous delta potential, is given by perturbations that are harmonic in time with frequency ω_0; for simplicity we will consider it independent of the spatial coordinates[8]:

$$\eta_H(t) := \eta_0 \cos(\omega_0 t) e^{-\frac{|t|}{T}}, \quad \omega_0, T > 0, \quad \eta_0 \in \mathbb{R}. \tag{6.7}$$

In Eq. (6.7), the exponential factor is employed to impose a boundary in time, since otherwise the number of pairs created becomes infinite. In the limit of large T, a straightforward computation shows that its Fourier transform satisfies

$$|\eta_H(k^0)|^2 = \frac{\pi}{2} \eta_0^2 T [\delta(k^0 - \omega_0) + \delta(k^0 + \omega_0)], \quad \omega_0 T \gg 1, \tag{6.8}$$

which, defining the threshold frequency $\omega_c := \omega_0/(2m)$, thus leads to the following probability of pair-creation rate per unit area of the plate in the purely inhomogeneous case (A denotes the area of the plate):

$$\frac{P_H}{AT} = \eta_0^2 \Theta(\omega_0^2 - 4m^2) \mathrm{Im}[F_{\mathrm{pure}}(-\mathrm{i}\omega_0 + \epsilon, m)]$$
$$= \frac{\eta_0^2}{3072\pi} \Theta(\omega_c^2 - 1)(m^2)^{3/2} \omega_0^2 (\omega_c^2 - 1)^3$$
$$\times {}_2F_1\left(\frac{3}{2}, \frac{7}{2}; 4; 1 - \omega_c^2\right). \tag{6.9}$$

To derive the last line we have employed the result in [88] for the jump across the branch cut of the hypergeometric function, which is proportional to the desired imaginary part.

Analytically, we can compute the expansions for small and large ω_c, which show that ω_0^5 sets the scale of the probability rate:

$$\frac{P_H}{AT} = \frac{\eta_0^2 \omega_0^5}{384\pi} \Theta(\omega_c^2 - 1) \begin{cases} \frac{1}{15\pi} - \frac{1}{4\pi\omega_c^2} + \frac{3(4\log(4\omega_c)-1)}{64\pi\omega_c^4} + \frac{-12\log(4\omega_c)-5}{768\pi\omega_c^6} + \mathcal{O}(\omega_c^{-7}), \\ \frac{1}{8}(\omega_c - 1)^3 - \frac{33}{64}(\omega_c - 1)^4 + \frac{345}{256}(\omega_c - 1)^5 - \frac{2899(\omega_c-1)^6}{1024} + \mathcal{O}((\omega_c - 1)^7). \end{cases} \tag{6.10}$$

On one side, we see that in the massless limit we indeed recover (up to a rescaling) the infinite coupling result for the delta case [31], being the first corrections of order ω_c^{-2}. Once more, on dimensional grounds this situation is reproduced in some analogous setups, such as a moving mirror [89].

On the other side, for large masses the result evidently vanishes as a consequence of the mass threshold, since the energy of the oscillations is not enough to provide the minimum energy of two particles at rest. For values of ω_c slightly larger than the threshold, the particle creation rate behaves as a third power in the difference $(\omega_c - 1)$. These behaviors can be confirmed from the plot in the left panel of Fig. 1, where the pair-production rate per unit area is shown as a function of the frequency; the dashed blue line and the violet dashed-dotted line correspond respectively to $m = 0.1$ and $m = 3$.
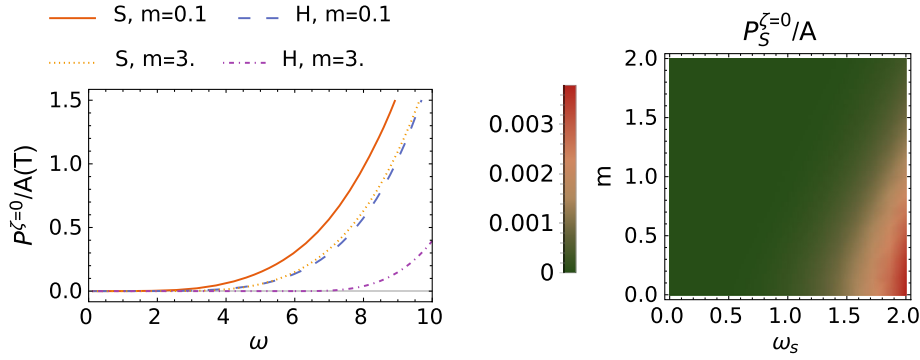
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[8]Following the discussion in Sec. V C, it should be clear that formally it is not enough to consider the amplitude η_0 small; one should also think that there is an additional background ζ (not necessarily much) bigger than η_0 for the expansion in η to be well defined.

FIG. 1. Pair-production probability/rate per unit area in the massive case. The left panel compares the harmonic and Sauter cases (denoted as H and S respectively) for different values of masses and as a function of the corresponding frequency. Notice that in the harmonic case, the pair-production rate should be understood. The panel on the right is a density plot of the pair-production probability per unit area for the Sauter pulse as a function of the frequency ω_S and the mass m. In all the cases, the amplitudes η_0 are set to unity.

As a generalization of this simple harmonic example, one can also consider a perturbation that resembles a plane wave over the plate,

$$\eta_W(x,t) := \eta_0 \cos(k(x-vt))e^{-\sigma|x-vt|}, \quad k,\sigma>0, \quad \eta_0, \quad v\in\mathbb{R}, \tag{6.11}$$

where k is its wave number, $\sigma \to 0$ is a regulator and v the speed of the wave (recall that $v = 1$ in our units equals c, the speed of light in vacuum). After removing the regulator we get the following pair-production rate per unit area:

$$\frac{P_W}{AT} = \eta_0^2 \Theta((v^2-1)k^2 - 4m^2)$$
$$\times \operatorname{Im}\left[F_{\text{pure}}\left(\sqrt{(1-v^2)k^2 - i\epsilon}, m\right)\right]. \tag{6.12}$$

On one side we see that pair production is possible only if $v \geq 1$. Notice that this situation should be understood not as a traveling wave with speed faster than light, but rather as an active fast modulation of a property over the plate, in a way analogous to that proposed for example in [90].

On the other side, defining an effective frequency $\omega_{\text{eff}}^2 := (v^2-1)k^2$, the result (6.12) can be obtained from (6.9) by simply trading $\omega_0 \to \omega_{\text{eff}}$. In particular, the exclusively time-dependent case η_H can be understood as its infinite speed limit (after an appropriate rescaling of k).

B. Sauter pulse

One widely diffused profile in the literature of QED in external backgrounds is the Sauter pulse [91], defined as

$$\eta_S(t) := \frac{\eta_0}{\cosh^2(\omega_S t)}, \quad \omega_S \in \mathbb{R} - \{0\}, \quad \eta_0 > 0. \tag{6.13}$$

Replacing this profile in our general equation (6.6), we can compute the probability of pair creation P_S.

First, for the massive case (and $\zeta = 0$), we show a density plot of $P_S^{\zeta=0}$ per unit area in the right panel of Fig. 1 as a function of the frequency ω_S and the mass m. On the one hand, as could be expected, as the mass increases the probability of pair creation diminishes, since the cost of creating the pair becomes higher. On the other, if the frequency ω_S becomes larger, the distribution of η in Fourier space is widened, so that creation of pairs is more favored. This is in agreement with the following analytical asymptotics:

$$\frac{P_S^{\zeta=0}}{A} = \eta_0^2 \omega_S^4 \times \begin{cases} \frac{7\zeta_R(7)}{\pi^9} + \cdots, & m/\omega_S \ll 1, \\ \frac{1}{4\pi^4}\frac{m^4}{\omega_S^4}e^{-\frac{2\pi m}{\omega_S}}\left(1 + \frac{23}{4\pi}\frac{\omega_S}{m}\right) + \cdots, & m/\omega_S \gg 1, \end{cases} \tag{6.14}$$

where $\zeta_R(\cdot)$ is Riemann's zeta function. Although the exponential suppression for large masses may remind the one present in the Schwinger pair production for rather general electric fields [92], keep in mind that our process is perturbative in the background field amplitude and therefore intrinsically different in nature.

Additionally, in the left panel of Fig. 1 we show the behavior of the pair-production probability per unit area as a function of the frequency ω_S; the solid red line and the yellow small-dashed line correspond respectively to $m = 0.1$ and $m = 3$. Compared to the harmonic case, if frequencies are small we see that the exponential supression allows for a faster setting in of pair production; in other words, the Sauter pulse always embodies some frequency components above the mass threshold that enable the creation of pairs. On the contrary, for larger frequencies the trend will be reverted and the harmonic pair production will become greater, as dictated by Eqs. (6.10) and (6.14).

We can also consider the massless case together with an arbitrary ζ. An expansion of the pair-production probability for large and small frequencies thus gives
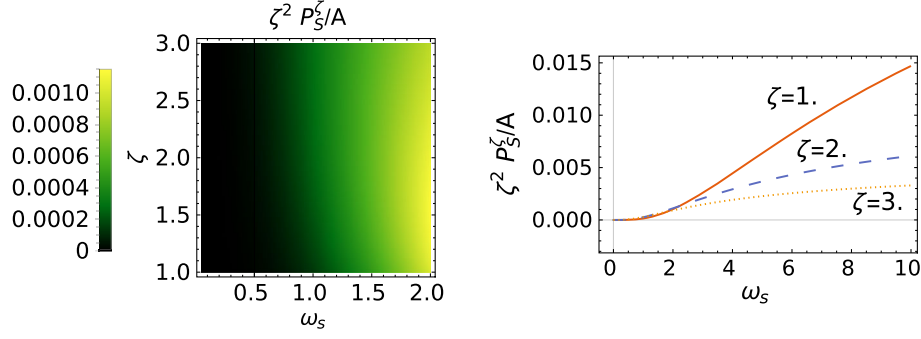
FIG. 2.   Probability of assisted particle production for a Sauter pulse per unit area, $\zeta^2 P_S^\zeta/A$. The left panel shows its density plot as a function of the frequency ω_S and the background ζ. The right panel displays its behavior as the frequency varies, considering different backgrounds. We have set the amplitude of the pulse (η_0) to unity.

$$\frac{P_S^\zeta}{A} = \eta_0^2 \omega_S^4 \times \begin{cases} \frac{7}{\pi^9}\left[\zeta_R(7) - \frac{54}{7\pi^2}\zeta_R(9)(\omega_S\zeta)^2 + \cdots\right], & \zeta\omega_S \ll 1, \\ \frac{1}{(\zeta\omega_S)^4}\left[\frac{12\zeta_R(3)}{\pi^5} - \frac{8}{3\pi\zeta\omega_S} + \cdots\right], & \zeta\omega_S \gg 1, \end{cases}$$

$$(6.15)$$

In particular, for vanishing ζ we recover the expected massless limit of Eq. (6.14). Increasing the value of ζ gives rise to an anti-assisted effect, contrary to the one found for electromagnetic backgrounds for Schwinger pair production [93,94]. Naively, since a constant potential alone does not contribute to pair creation in our scalar setup,[9] one could have expected that its addition would have had no effect on pair creation. Instead, the nonlocal feature of the generated form factor is nontrivial. The fact that it diminishes the pair production can be physically understood from the fact that a larger ζ tends to repel the quantum field, as explained in Sec. V B.

Lastly, taking into account the discourse developed in Sec. V C, we can analyze the case in which we rescale the amplitude of the perturbation, $\eta_0 =: \zeta\tilde{\eta}$, such that $\tilde{\eta}$ is taken to be small. At the practical level, this is analogous to saying that we multiply the pair-production probability by ζ^2. We have plotted in the left panel of Fig. 2 a density plot of the rescaled probability as a function of both ω_S and ζ; one can see that for $\omega_S \lesssim 1.5$, increasing the coupling ζ leads to larger probabilities. The right panel of Fig. 2 provides a clearer picture of this effect, showing the rescaled probability as a function of ω_S for three distinct values of $\zeta \sim \mathcal{O}(1)$. For $\omega_S \sim 1.5$ we see that the hierarchy between the curves is inverted.

VII. CONCLUSIONS

In the present article we have studied the problem of a quantum scalar field theory with local semitransparent Neumann boundary conditions on a plate, which can also

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[9]This is one intrinsic difference with the electromagnetic case, where for example constant electric fields do produce pairs, giving rise to the Sauter-Schwinger effect.

be understood as interactions with an external (classical) field confined to the latter.

First of all, we have shown that the quantum contributions to the effective action can be understood in terms of quantities in the phase space of a first quantization. Worldline techniques are by far much more developed in the case of configuration space [53], rendering the study of such path integrals *per se* an interesting problem.

Second, taking into account the well-known fact that the studied interaction requires regularization, we have devised a regularization appropriate to our path-integral methods. We have shown that in this way we are able to rederive previously obtained results for the homogeneous case. Notice that this regularization provides also a physical interpretation in terms of an effective finite width of the plate, since we have to evaluate the intermediate heat kernels at noncoincident points.

These technical developments allowed the computation of the heat kernel at quadratic level on the perturbations η around a homogeneous background field ζ and the corresponding effective action in the $D \equiv 4$ case. Notably, these results are exact in ζ; they are (generally) given by a nonlocal operator acting on the perturbations η, which can be seen from the appearance of nonanalytic contributions.

Such nonanalytic contributions are responsible for the pair production that we have encountered in Minkowski spacetime, where the perturbations are used to model dynamical properties of the plates, very much akin to the situation in the dynamical Casimir effect. In this scenario we have considered two possibilities, a harmonic excitation and a Sauter pulse. For large masses, the Sauter background displays an exponential cutoff, which is softer than the Heaviside present for the harmonic pulse. In the massless Sauter case, the introduction of a background ζ leads to a scenario of anti-assisted pair production. This effect can be inverted in a region of the parameter space if the amplitude of the pulse scales linearly with ζ, which is possible taking into account the discussion in Sec. V C regarding the smallness of η and ζ.

A special comment deserves the finding of a duality between the field theory obeying local semitransparent Neumann boundary conditions and the equivalent Dirichlet

one. Indeed, we have shown that there exists a mapping between the relevant heat kernels, which becomes an exact strong/weak duality at the level of the corresponding renormalized quantum effective actions. Notice that this is a new duality, different from the Fermi-Bose duality discussed in condensed matter by means of the Girardeau mapping [79,80,95]. To clarify this point, first notice that the latter connects the Lieb-Liniger model [96] and the Cheon-Shigehara one, which are both one-dimensional models, while our results are valid in D dimensions. Second, as discussed in Sec. III B, we impose no type of symmetry under the parity transformation $\tilde{x} \to -\tilde{x}$ and we always work with a scalar bosonic field, contrary to the change in statistics of the Girardeau mapping. Additionally, we do not introduce a self-interaction; instead, our field interacts with a spacetime-dependent background potential η. Nevertheless, taking into account the several experimental accomplishments based on the Fermi-Bose duality [97,98], it will be of interest to explore possible experimental roads of the newly devised duality.

Regarding possible future developments, it will be interesting to try to generalize our results to the problem where nonlocal boundary conditions are imposed. This may provide a way to analyze the appearance of topological effects.

One further peculiarity of the interaction considered in this article is that it can be thought as the first term in an effective field theory expansion, which has undergone a thin-shell limit. As such, it may find applications in trying to understand the nature of dark matter. Indeed, one open possibility is that it may behave as a field with unusual couplings [99,100].

Finally, another useful advance would be the development of a numerical code to tackle the problem discussed here. The situation is more involved than the usual cases, since a naive adaptation of the numerical worldline techniques [101–103] to phase space suffer from the so-called sign problem.

ACKNOWLEDGMENTS

APPENDIX A: PARTITION FUNCTION IN THE WORLDLINE

In this appendix we will recall how to compute the partition function for an open scalar line in the worldline formalism, considering a source k for $\dot{x}$:

$$Z_{\dot{x}}[k] := \frac{\int_{x(t_1)=x}^{x(t_2)=x'} \mathcal{D}x\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{x}^2(t)}{4} + \int_{t_1}^{t_2} dt\, k(t)\dot{x}(t)}}{\int_{x(t_1)=x}^{x(t_2)=x'} \mathcal{D}x\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{x}^2(t)}{4}}}. \quad (A1)$$

First of all, we may perform a change of variable in the path, so that we get a path integral over trajectories that obey initial and final Dirichlet boundary conditions. In order to do so, we split the path into a classical and a quantum contribution; substituting $x(t) \to y(t) + (x' - x)(t - t_1)/(t_2 - t_1) + x$ we get

$$Z_{\dot{x}}[k] = \frac{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y\, e^{-\int_{t_1}^{t_2} dt \frac{\left(\dot{y}(t) + \frac{x'-x}{t_2-t_1}\right)^2}{4} + \int_{t_1}^{t_2} dt\, k(t)\left(\dot{y}(t) + \frac{x'-x}{t_2-t_1}\right)}}{\int_{x(t_1)=x}^{x(t_2)=x'} \mathcal{D}x\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{x}^2(t)}{4}}}$$

$$= e^{\frac{x'-x}{t_2-t_1}\int_{t_1}^{t_2} dt\, k(t)} \frac{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4} - \int_{t_1}^{t_2} dt\, \dot{k}(t) y(t)}}{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4}}}, \quad (A2)$$

where in the last line we have performed an integration by parts to get rid of the derivatives acting on $y(t)$ in the source term.

We have thus reduced our problem to the computation of the partition function with Dirichlet boundary conditions, which can be readily solved by inverting the kinetic term. Indeed, a straightforward computation gives

$$\frac{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4} - \int_{t_1}^{t_2} dt\, \dot{k}(t) y(t)}}{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y\, e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4}}} = e^{\frac{1}{4}\int_{t_1}^{t_2} dt \int_{t_1}^{t_2} d[s] \dot{k}(s) G(s,t) \dot{k}(t)}, \quad (A3)$$

where the required symmetric Green function is defined as

$$G(s,t) := \frac{4}{(t_2 - t_1)}(s - t_1)(t_2 - t), \quad t_1 < s < t < t_2. \quad (A4)$$

This Green function satisfies as customarily the differential equation

$$-\frac{1}{4}\partial_s^2 G(s,t) = \delta(s - t), \quad (A5)$$

as well as the boundary conditions $G(t_1, t) = G(t_2, t) = 0 = G(s, t_1) = G(s, t_2)$.

One can further simplify the expression in the present case integrating by parts in the exponent; explicitly employing the boundary conditions satisfied by the Green function, we get

$$\frac{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4} - \int_{t_1}^{t_2} dt k(t)y(t)}}{\int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y e^{-\int_{t_1}^{t_2} dt \frac{\dot{y}^2(t)}{4}}}$$
$$= e^{\frac{1}{4}\int_{t_1}^{t_2} dt \int_{t_1}^{t_2} ds k(s)\partial_s\partial_t G(s,t)k(t)}, \qquad (A6)$$

A direct computation shows that the second partial derivative involved in the computation is given by

$$\partial_s\partial_t G(s,t) = -\frac{4}{(t_2-t_1)}[1 - \delta(t-s)(t_2-t_1)]. \quad (A7)$$

Adding all these results together we are led to our final expression:

$$Z_{\dot{x}}[k] = e^{\frac{x'-x}{t_2-t_1}\int_{t_1}^{t_2} dt k(t) - \frac{1}{(t_2-t_1)}(\int_{t_1}^{t_2} dt k(t))^2 + \int_{t_1}^{t_2} dt k^2(t)}. \quad (A8)$$

APPENDIX B: INTERMEDIATE-TIME INTEGRALS OF CHAINED HEAT KERNELS

In Sec. III A, we are led to expressions that involve integrals of chains of free heat kernels in the intermediate time t, combined with increasing negative powers of that time and its distance to the end time T. One can compute the first of them by considering the first order in ζ of the result [[31], Eq. (16)]:

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{\sqrt{t}\sqrt{T-t}} = \pi \text{erfc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{T}}\right). \qquad (B1)$$

The algorithm to obtain integrals with higher powers in the intermediate time involves differentiating this expression in terms of a or b. As a matter of completion, we mention the first of them:

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{\sqrt{t}(T-t)^{3/2}} = \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}}{\sqrt{b}\sqrt{T}}, \qquad (B2)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{\sqrt{t}(T-t)^{5/2}} = \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}(2\sqrt{a}\sqrt{b}+2b+T)}{2b^{3/2}T^{3/2}}, \qquad (B3)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{3/2}\sqrt{T-t}} = \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}}{\sqrt{a}\sqrt{T}}, \qquad (B4)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{3/2}(T-t)^{3/2}} = \frac{\sqrt{\pi}(\sqrt{a}+\sqrt{b})e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}}{\sqrt{a}\sqrt{b}T^{3/2}}, \quad (B5)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{3/2}(T-t)^{5/2}}$$
$$= \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}(\sqrt{a}(4b+T)+2a\sqrt{b}+2b^{3/2})}{2\sqrt{a}b^{3/2}T^{5/2}}, \quad (B6)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{5/2}\sqrt{T-t}} = \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}(2\sqrt{a}\sqrt{b}+2a+T)}{2a^{3/2}T^{3/2}}, \quad (B7)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{5/2}(T-t)^{3/2}}$$
$$= \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}(2\sqrt{a}(\sqrt{a}+\sqrt{b})^2+\sqrt{b}T)}{2a^{3/2}\sqrt{b}T^{5/2}}, \quad (B8)$$

$$\int_0^T dt \frac{e^{-\frac{a}{t}-\frac{b}{T-t}}}{t^{5/2}(T-t)^{5/2}}$$
$$= \frac{\sqrt{\pi}e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{T}}(T(a^{3/2}+b^{3/2})+2\sqrt{a}\sqrt{b}(\sqrt{a}+\sqrt{b})^3)}{2a^{3/2}b^{3/2}T^{7/2}}. \quad (B9)$$

APPENDIX C: INTEGRALS INVOLVING HERMITES, GAUSSIANS AND INVERSE POWERS OF THE INTERMEDIATE TIME

A crucial step in obtaining an all-order expression for the heat kernel in Sec. III A is the derivation of a closed expression for the integral

$$A_{n,m}(x,y) := \int_0^1 dt \frac{e^{-\frac{x^2}{4t}}}{t^{(1+n)/2}} \frac{e^{-\frac{y^2}{4(1-t)}}}{(1-t)^{(1+m)/2}} H_n\left(\frac{|x|}{2\sqrt{t}}\right)$$
$$\times H_m\left(\frac{|y|}{2\sqrt{1-t}}\right). \qquad (C1)$$

Employing the well-known formula to generate the Hermite polynomials as derivatives of a Gaussian,

$$H_n(x) = (-1)^n e^{x^2}\partial_x^n e^{-x^2}, \qquad (C2)$$

we can recast Eq. (C1) as

$$A_{n,m}(x,y) = (-2)^{n+m}\text{sign}^n(x)\text{sign}^m(y)\partial_x^n\partial_y^m$$
$$\times \int_0^1 dt \frac{1}{t^{1/2}}\frac{1}{(1-t)^{1/2}} e^{-\frac{x^2}{4t}}e^{-\frac{y^2}{4(1-t)}}$$
$$= (-2)^{n+m}\pi sign^n(x)sign^m(y)$$
$$\times \partial_x^n\partial_y^m\text{erfc}\left(\frac{\sqrt{x^2}+\sqrt{y^2}}{2}\right), \qquad (C3)$$

where in the last line we have employed our result (B1). If $x > 0$, then we may set $\sqrt{x^2} = x$ in Eq. (C3), since the derivatives act by definition only locally. If also $y > 0$, employing once more the generating formula (C2) we find

$$A_{n,m}(x,y) = 2\sqrt{\pi}e^{-\frac{(x+y)^2}{4}}H_{n+m-1}\left(\frac{x+y}{2}\right). \quad (C4)$$

Extending this analysis to any sign of x and y, and allowing also for a final time T different from 1, we prove the following relation:

$$\int_0^T dt \frac{e^{-\frac{x^2}{4t}}}{t^{(1+n)/2}} \frac{e^{-\frac{y^2}{4(T-t)}}}{(1-t)^{(1+m)/2}} H_n\left(\frac{|x|}{2\sqrt{t}}\right) H_m\left(\frac{|y|}{2\sqrt{T-t}}\right)$$
$$= 2\sqrt{\pi}\frac{e^{-\frac{(x+y)^2}{4T}}}{T^{(m+n)/2}} H_{n+m-1}\left(\frac{|x|+|y|}{2\sqrt{T}}\right), \quad (C5)$$

valid for[10] $x, y \in \mathbb{R} - \{0\}$ and $n, m = 0, 1, \ldots$, if Hermite's polynomials with negative index are understood in terms of parabolic cylinder functions.

APPENDIX D: SERIES OF HERMITE POLYNOMIALS

In Sec. III A, the resummation of the heat kernel's ζ expansion involves the computation of a series of Hermite functions,

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[10]Notice that if $x = 0$ or $y = 0$, then the departing integral in Eq. (C3) will be in general ill defined. This is not an obstacle to the computations in the body of this article, given that we always work in a regularized framework, in which we have to consider the limiting cases $x, y \to 0$.

$$T(x,\beta) := \sum_{n=0}^{\infty} H_n(x)\beta^n. \quad (D1)$$

A way to obtain a closed expression for this series is to notice that, by using (C2), we have

$$T(x,\beta) = \sum_{n=0}^{\infty} e^{x^2}(-\beta)^n \frac{d^n}{dx^n} e^{-x^2} = e^{x^2} \frac{1}{1+\beta\frac{d}{dx}} e^{-x^2}, \quad (D2)$$

where we have employed the formal expression of the geometric series. Solving it for T we arrive at the following differential equation:

$$\left(1 + \beta\frac{d}{dx}\right)e^{-x^2}T(x,\beta) = e^{-x^2}, \quad (D3)$$

whose more general solution is given by

$$\tilde{T}(x,\beta) = c_1(\beta)e^{x^2-\frac{x}{\beta}} + \frac{\sqrt{\pi}}{2\beta}e^{\frac{1}{4\beta^2}+x^2-\frac{x}{\beta}}\mathrm{erf}\left(x - \frac{1}{2\beta}\right). \quad (D4)$$

The "constant of integration" c_1 may be fixed by analyzing the behavior of T for small β, from which we get

$$c_1(\beta) = -\frac{\sqrt{\pi}e^{\frac{1}{4\beta^2}}}{2\beta}. \quad (D5)$$

This implies that the desired function is given by

$$T(x,\beta) = -\frac{\sqrt{\pi}e^{\frac{(1-2\beta x)^2}{4\beta^2}}\mathrm{erfc}(x - \frac{1}{2\beta})}{2\beta}. \quad (D6)$$

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