Infinite conformal symmetry and emergent chiral fields of topologically nontrivial configurations: From Yang-Mills-Higgs theory to the Skyrme model

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In this manuscript, we discuss a remarkable phenomenon concerning nonlinear and nonintegrable field theories in $(3+1)$ dimensions, living at finite density and possessing nontrivial topological charges and non-Abelian internal symmetries (both local and global). With suitable types of *Ansätze*, one can construct infinite-dimensional families of analytic solutions with nonvanishing topological charges (representing the baryonic number) labeled by both two integer numbers and by free scalar fields in $(1+1)$ dimensions. These exact configurations represent $(3+1)$-dimensional topological solitons hosting $(1+1)$-dimensional chiral modes localized at the energy density peaks. First, we analyze the Yang-Mills-Higgs model, in which the fields depend on all the space-time coordinates (to keep alive the topological Chern-Simons charge), but in such a way to reduce the equations system to the field equations of two-dimensional free massless chiral scalar fields. Then, we move to the nonlinear sigma model, showing that a suitable *Ansatz* reduces the field equations to the one of a two-dimensional free massless scalar field. Then, we discuss the Skyrme model concluding that the inclusion of the Skyrme term gives rise to a chiral two-dimensional free massless scalar field (instead of a free massless field in two dimensions as in the nonlinear sigma model) describing analytically spatially modulated hadronic layers and tubes. The comparison of the present approach both with the instanton-dyon liquid approach and with lattice QCD is shortly outlined.

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I. INTRODUCTION

It is well known that in quantum chromodynamics (QCD) color confinement is closely related to the existence of topologically nontrivial configurations (see [1–14] and references therein), while in the ultraviolet sector quarks and gluons should be liberated [15–17]. The great advances in lattice QCD (LQCD henceforth) [18–25] can only partially compensate the poor analytic control on such nonperturbative issues arising in the phase diagram of non-Abelian gauge theories. Therefore, many open problems would greatly benefit from the presence of explicit solutions relevant to the phase diagram of QCD. In this paper, we will present a concise list of tools, although these are also useful when analyzing different kinds of questions.

An area in which the results and tools of LQCD badly need some further analytic insights is the analysis of the phase diagram of QCD at finite (and low) temperature and with baryon chemical potential; one of the main problematic issues being the infamous sign problem (see [26] for a detailed review). In this case, the methods of AdS/CFT are not especially effective since, only at high enough temperatures, supersymmetric Yang-Mills theory gets very close to Yang-Mills theory (see [27,28] and references therein). Moreover, besides the huge theoretical interest in achieving a deeper understanding of this region of the phase diagram, there are many situations of high phenomenological interest (such as heavy-ion collisions, quark-gluon plasma, neutron stars, and so on) in which novel analytic techniques would be extremely useful to complement the available observations. Among these, one of the most relevant is the appearance of regular-shaped structures at finite density (called nuclear pasta states; see [29–40])

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and the important transport properties within them, whose numerical treatment is quite challenging [41–45].

There are two obvious ways to analyze these issues. One can either begin with the analysis of Yang-Mills theory (which is more fundamental), or one can start directly with the nonlinear sigma model (NLSM) and the Skyrme model (which is the low-energy limit of QCD at leading order in the large N_c 't Hooft expansion [46–51]). These models, at first glance, are very different, as Yang-Mills theory is a gauge theory, while the NLSM and the Skyrme model only possess global symmetries. Thus, one could think that these two possibilities should be treated with different methods. Nevertheless, we will show that it is possible to devise a unified strategy able to identify sectors of the $(3+1)$-dimensional theories that, at the same time, possess arbitrary baryonic charge as well as an infinite-dimensional conformal symmetry. It is quite amusing that the only difference between the infinite-dimensional conformal symmetry appearing in Yang-Mills theory and the NLSM on one side, and Yang-Mills-Higgs and the Skyrme theory, on the other side, is that, in the former cases, one gets an effective two-dimensional conformal field theory (CFT), while in Yang-Mills-Higgs and the Skyrme cases, one gets a two-dimensional chiral CFT. This intriguing result could be related to the fact that the Skyrme theory is the low-energy limit of QCD (and not just of Yang-Mills theory) and knows about chiral symmetry breaking. Needless to say, the possibility to use the tools of two-dimensional CFT in $(3+1)$-dimensional theories (such as Yang-Mills and Skyrme, which are the prototypes of nonlinear and non-integrable field theories) open unexpected and novel perspectives on the analysis of the phase diagram at finite temperature and chemical potential.

A systematic tool to construct a nonspherical hedgehog *Ansatz* suitable to describe finite density effects has been developed in Refs. [52–61] for the Skyrme model and in Refs. [62–64] for the Einstein-Yang-Mills case. In the present article, we will further generalize these results to extend the space of analytical solutions [and the tools that allow obtaining relevant physical information of these systems of topological solitons defined in a $(3+1)$-dimensional finite volume], disclosing the appearance of chiral conformal degrees of freedom representing modulations of hadronic tubes and layers. Although, in the present the paper, we will not discuss the coupling with gravity of the NLSM and Yang-Mills theories, there are already quite a few examples in the literature that show that the current approach is convenient even when the coupling with general relativity is taken into account (see, for instance, [65–71]).

A. About the new analytical solutions

The considerable interest in constructing analytic solutions in theories with non-Abelian internal symmetries (both local and global) and nontrivial topological charges arises from the fact that, in all the theories admitting topological solitons, such charges have a profound physical meaning (such as the baryonic charge, as will be discussed in the following sections). As far as the phase diagram is concerned, it is crucial to analyze what happens when a finite amount of topological charge is forced to live within a limited spatial volume. In this case, practical analytic tools are extremely welcome due, for instance, to the sign problem. On the other hand, the common belief is that it is impossible to develop such tools for (at least) two reasons. First, one necessarily has to abandon spherically symmetric *Ansätze* for the fields. Second, and quite generically, the requirement of a nonvanishing topological charge increases the complexity of the field equations to be solved since a nonvanishing topological density implies that there must be at least three independent degrees of freedom depending nontrivially on three different spatial coordinates in $(3+1)$ dimensions. Resumming,

(1) the departure from spherical symmetry (generated by the presence of "a box" within which the solitons are forced to live), together with
(2) the requirement of a nonvanishing topological charge,

reduce considerably the possibility to derive analytic results on the phase diagram of topologically nontrivial configurations of theories such as Yang-Mills, NLSM, and the Skyrme model. One could reason as follows: the analytic tools of two-dimensional CFT would be handy and welcome in analyzing the phase diagram of $(3+1)$-dimensional Yang-Mills-Higgs theory (or Skyrme model) due to the difficulties analyzing it even with LQCD. Then, why do not we assume that the main fields [either A_μ for Yang-Mills or $U \in SU(2)$ for the NLSM and Skyrme theory] only depend on one spatial coordinate and on time (so that one could hope to use some two-dimensional CFT technologies)?

The answer is that such a naive approach would fail. First of all, the topological charge (to be defined in the following sections) would vanish identically, so that one would gain no information about the phase diagram at finite baryon density. Moreover, already the head-on collision of (topologically trivial) plane waves depending on only two coordinates is intractable from the analytic viewpoint, and numerical methods must be used[1] (see [74–80] and references therein). Hence, at first glance, one might argue that the analytic study of dynamical processes involving solitonic configurations with nonvanishing topological charge in $(3+1)$ dimensions is not feasible.

In fact, here we will show that, from the analytic viewpoint, the above two circumstances (namely, the need to depart from spherical symmetry and the necessity to keep alive the topological charge) are an opportunity rather

[1]For instance, already analysis of head-on collisions of $(1+1)$-dimensional kinks, which is far simpler than Yang-Mills theory in $(3+1)$ dimensions, can only be dealt with numerically [72,73].

than an obstruction. The tools to be developed here give rise, among other things, to genuine $(3+1)$-dimensional nonhomogeneous exact solutions representing spatially modulated hadronic layers and tubes, allowing one to estimate their contributions to the partition function at low temperatures and baryon chemical potential and also to compute relevant quantities.

B. Notation and conventions

In this work, we will use the following convention. Greek indices run over the space-time dimensions with mostly plus signature, and latin indices are reserved for those of the internal space. Also, we work in natural units, such that the Boltzmann's constant k_{B}, the reduced Planck's constant $\hbar$, and the speed of light c are set to one.

As we are interested in studying topological solitons at finite volume, we will use the metric of a box, which in $(3+1)$ space-time dimensions reads

$$ds^2 = -dt^2 + L_r^2 dr^2 + L_\theta^2 d\theta^2 + L_\phi^2 d\phi^2, \tag{1}$$

where $\{r, \theta, \phi\}$ are Cartesian dimensionless coordinates whose ranges will be defined in each case, $\{L_r, L_\theta, L_\phi\}$ are constants with dimension of length that fix the volume of the box in which the solitons are confined, and $g = -L_r^2 L_\theta^2 L_\phi^2$ will denote the metric determinant. Also we denote ∇_μ as the Levi-Civita covariant derivative constructed with the Christoffel symbols, ∂_μ as the partial derivative, and the covariant derivative D_μ acts as

$$D_\mu(\cdot) = \partial_\mu(\cdot) + [A_\mu, \cdot], \tag{2}$$

with A_μ the components of the non-Abelian connection. We will consider as the internal symmetry group the $SU(2)$ Lie group,[2] whose generators are

$$\mathbf{t}_k = i\sigma_k, \tag{3}$$

being σ_k the Pauli matrices. The matrices $\mathbf{t}_i$ satisfy the relation

$$\mathbf{t}_i \mathbf{t}_j = -\delta_{ij}\mathbf{1}_2 - \epsilon_{ijk}\mathbf{t}_k, \tag{4}$$

where $\mathbf{1}_2$ is the 2×2 identity matrix, δ_{ij} is the Kronecker delta, and ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol.

The fundamental field of the Yang-Mills theory, namely, the non-Abelian connection A, splits as

$$A = A_\mu^j \mathbf{t}_j dx^\mu, \tag{5}$$

while the fundamental field of the NLSM and the Skyrme model is the scalar field $U(x) \in SU(2)$, so that

$$R_\mu = U^{-1}\partial_\mu U = R_\mu^j \mathbf{t}_j, \tag{6}$$

is in the $\mathfrak{su}(2)$ algebra.

The relevant topological properties of the solutions that we will construct in this work are encoded in the Chern-Simons (CS) density (for the Yang-Mills theory) and in the baryon charge density (for the NLSMs). These are given, respectively, by

$$\rho_{\mathrm{CS}} = J_0^{\mathrm{CS}}, \quad \text{where}$$
$$J_\mu^{\mathrm{CS}} = \frac{1}{8\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(A^\nu\partial^\rho A^\sigma + \frac{2}{3}A^\nu A^\rho A^\sigma\right), \tag{7}$$

$$\rho_{\mathrm{B}} = \frac{1}{24\pi^2}(U^{-1}\partial U)^3$$
$$\equiv \frac{1}{24\pi^2}\varepsilon_{ijk}\mathrm{Tr}\{(U^{-1}\partial^i U)(U^{-1}\partial^j U)(U^{-1}\partial^k U)\}. \tag{8}$$

The integral of the above densities over a spacelike hypersurface represents the CS charge and the baryonic charge of the corresponding configurations,

$$Q_{\mathrm{CS}} = \int \rho_{\mathrm{CS}}\, dV, \qquad B = \int \rho_{\mathrm{B}}\, dV. \tag{9}$$

The paper is organized as follows: In Sec. II, we study the Yang-Mills theory in $(3+1)$ dimensions showing that, with an appropriate *Ansatz*, the field equations are reduced to that of a two-dimensional free massless scalar field. We also offer that the inclusion of a Higgs field converts the resulting CFT into a chiral theory. In Sec. III, we move to the study of NLSM in $(3+1)$ dimensions, showing that the theory can be reduced to a two-dimensional CFT. In Sec. IV, we show that the inclusion of the Skyrme term in the NLSM defines a chiral CFT for two types of configurations describing nuclear pasta states. In Sec. V, we study the phase diagram and the contribution of the partition functions of the analytic topological solitons. The final section is dedicated to conclusions.

II. YANG-MILLS-HIGGS THEORY IN (3+1) DIMENSIONS

In this section, before moving to the Yang-Mills-Higgs case (which has not been analyzed previously in the literature), we will study the Yang-Mills theory in $(3+1)$ dimensions, reviewing the results in Ref. [81], showing how the field equations can be reduced to that of a two-dimensional free massless scalar field in $(1+1)$ dimensions keeping alive the topological charge. The concepts introduced here will be helpful also in the

[2]Here we will consider the $SU(2)$ case, but the present results can be extended to the $SU(N)$ case.

following sections, where we will show that a similar construction can also be carried out on NLSMs.

A. Conformal field theory in two dimensions from pure Yang-Mills theory

The Yang-Mills theory in $(3+1)$ dimensions is described by the action

$$I[A] = \frac{1}{2e^2}\int d^4x\sqrt{-g}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{10}$$

where e is the Yang-Mills coupling constant, and the field strength components $F_{\mu\nu}$ are defined in terms of the non-Abelian connection A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The field equations of the theory, obtained varying the action in Eq. (10) with respect to the fundamental field A_μ, are

$$\nabla_\nu F^{\mu\nu} + [A_\nu, F^{\mu\nu}] = 0, \tag{11}$$

while the energy-momentum tensor of the theory turns out to be

$$T_{\mu\nu} = -\frac{2}{e^2}\mathrm{Tr}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{12}$$

One of the main goals of this paper is to construct a formalism able to describe how topologically nontrivial configurations react when they are forced to live within a finite box; this issue must be addressed in the finite density analysis.

The easiest way to take into account finite volume effects is to use the flat metric defined in Eq. (1), with the ranges

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le r \le 4\pi. \tag{13}$$

The above ranges for the coordinates θ, ϕ, and r are related to the Euler angle parametrization for $SU(2)$ valued fields. Let us define the following $U(x) \in SU(2)$ field

$$U = \exp\left(p\theta\frac{\mathbf{t}_3}{2}\right)\exp\left(H(t,\phi)\frac{\mathbf{t}_2}{2}\right)\exp\left(qr\frac{\mathbf{t}_3}{2}\right), \tag{14}$$

where p and q are nonvanishing integers.[3] The theory of Euler angles for $SU(N)$ [82–84] tells us that, when p and q are nonvanishing integers, the range of θ [appearing in the left factor of the decomposition in Eq. (14)] and the range of r [appearing in the right factor of the decomposition in Eq. (14)] must be as in Eq. (13). As far as the central factor $H(t,\phi)$ is concerned, there are two options. If the field $H(t,\phi)$ satisfies periodic boundary conditions,

$$H(t,\phi=0) = H_0 = H(t,\phi=\pi), \tag{15}$$

the CS charge vanishes.[4] The other boundary condition for $H(t,\phi)$ arises naturally, taking into account that $H(t,\phi)$ appears in the central factor of the Euler angles decomposition of an $SU(2)$ element (see [82–84]),

$$H(t,\phi=0) = 0, \qquad H(t,\phi=\pi) = \pi, \tag{16}$$

or

$$H(t,\phi=0) = \pi, \qquad H(t,\phi=\pi) = 0.$$

The option here ensures that the $SU(2)$-valued element U defined in Eqs. (13), (14), and (16) wraps an integer number of times around the group manifold of $SU(2)$; in other words, U has a nonvanishing winding number. In this case, both the CS charge and the CS density in Eq. (7) associated with the gauge field will be nontrivial. It is well known that ρ_{CS} defined in Eq. (7) is the "nonperturbatively induced baryonic charge" of the gauge configuration [85] (see also [86–88] and references therein).

In order to find an *Ansatz* such that ρ_{CS} defined in Eq. (7) will be nonzero and, at the same time, the field equations can be solved analytically, one can follow Refs. [53,54,60–64,81,89], arriving at the following form for the Yang-Mills potential:

$$A_\mu = \sum_{j=1}^{3}\lambda_j\Omega_\mu^j\mathbf{t}_j, \qquad U^{-1}\partial_\mu U = \sum_{j=1}^{3}\Omega_\mu^j\mathbf{t}_j, \tag{17}$$

where $H(t,\phi)$ in Eq. (14) and the λ_i functions in Eq. (17) are explicitly given by

$$H(t,\phi) = \arccos(G), \qquad G = G(t,\phi), \tag{18}$$

$$\lambda_1(t,\phi) = \lambda_2(t,\phi) = \frac{G}{\sqrt{G^2+\exp(2\eta)}} \stackrel{\mathrm{def}}{=} \lambda(t,\phi),$$
$$\lambda_3(t,\phi) = 1, \qquad \eta \in \mathbb{R}, \tag{19}$$

$$G(t,\phi) = \exp(3\eta)\frac{F}{\sqrt{1-\exp(4\eta)\cdot F^2}}, \qquad F = F(t,\phi). \tag{20}$$

The real parameter η will be fixed by requiring that the CS charge is an integer.

The option in Eq. (15) gives rise to the following boundary condition for $F(t,\phi)$:

[3] There will be one more restriction on p and q that will be discussed later on.

[4] Although the CS density can still be nontrivial.

$$F(t,\phi=0)=F_0=F(t,\phi=\pi). \tag{21}$$

In the latter case, the CS charge vanishes. On the other hand, the option in Eq. (16), in terms of $F(t,\phi)$, reads

$$\begin{aligned} F(t,\phi=0)&=-\frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}},\\ F(t,\phi=\pi)&=\frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}}, \end{aligned} \tag{22}$$

in order to have a nonzero CS charge. In this case, both the CS charge and the CS density will be nontrivial. Then, we say that this configuration is topologically nontrivial.

The components of the gauge field can be easily computed taking into account the well-known expression of the Ω^j_μ in the case of the Euler parametrization. Thus, explicitly, A_μ reads

$$\begin{aligned} A_\mu=\lambda(t,\phi)\bigg[&\frac{\mathbf{t}_1}{2}\{-\sin(qr)dH+p\cos(qr)\sin(H)d\theta\}\\ &+\frac{\mathbf{t}_2}{2}\{\cos(qr)dH+p\sin(qr)\sin(H)d\theta\}\bigg]\\ &+\frac{\mathbf{t}_3}{2}[qdr+p\cos(H)d\theta], \end{aligned} \tag{23}$$

where

$$dH=\frac{\partial H}{\partial t}dt+\frac{\partial H}{\partial\phi}d\phi.$$

The fact that $d\lambda\wedge dH=0$, together with the gradients of the coordinates r, θ, and ϕ are mutually orthogonal, simplifies many of the computations. The above *Ansatz* is the key to getting the paper's main results, and the rest is a direct computation.

With the above, the complete set of $(3+1)$-dimensional Yang-Mills field equations with the *Ansatz* in Eqs. (14) and (17)–(20) reduces to

$$\Box F\equiv\left(\frac{\partial^2}{\partial t^2}-\frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)F=0, \tag{24}$$

which corresponds to the field equation of a free massless scalar field in two dimensions.

B. Energy-momentum tensor and topological charge

A direct computation reveals that the topological density for the solution defined above is given by

$$\rho_{\mathrm{CS}}=\frac{pq\exp(3\eta)}{16\pi^2(1-\exp(4\eta)F^2)^{3/2}}\frac{\partial F}{\partial\phi}, \tag{25}$$

which is nonvanishing, as long as $\frac{\partial F}{\partial\phi}\neq0$. On the other hand, the energy density T_{tt} and the on-shell Lagrangian $L_{\text{on-shell}}$ read, respectively,

$$T_{tt}=\frac{p^2}{e^2L_\theta^2}\exp(5\eta)\cosh(\eta)\left[\left(\frac{\partial F}{\partial t}\right)^2+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right], \tag{26}$$

$$L_{\text{on-shell}}=\frac{p^2}{e^2L_\theta^2}\exp(5\eta)\cosh(\eta)\left[\left(\frac{\partial F}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right]. \tag{27}$$

The full energy-momentum tensor reads

$$T_{\mu\nu}=\begin{bmatrix} T_{tt} & 0 & 0 & P_\phi\\ 0 & T_{rr} & 0 & 0\\ 0 & 0 & T_{\theta\theta} & 0\\ P_\phi & 0 & 0 & T_{\phi\phi}\end{bmatrix},$$

where

$$\begin{aligned} T_{rr}&=\frac{p^2L_r^2}{e^2L_\theta^2}\exp(5\eta)\cosh(\eta)\left[\left(\frac{\partial F}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right]\\ &=-\frac{L_r^2}{L_\theta^2}T_{\theta\theta}, \end{aligned} \tag{28}$$

$$T_{\phi\phi}=\frac{p^2L_\phi^2}{e^2L_\theta^2}\exp(5\eta)\cosh(\eta)\left[\left(\frac{\partial F}{\partial t}\right)^2+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right], \tag{29}$$

$$T_{t\phi}=P_\phi=\frac{2p^2\exp(5\eta)\cosh(\eta)}{e^2L_\theta^2}\frac{\partial F}{\partial t}\frac{\partial F}{\partial\phi}. \tag{30}$$

From the above, one can easily verify that the energy-momentum tensor is traceless; $g^{\mu\nu}T_{\mu\nu}=0$, as it should be in Yang-Mills theory in $(3+1)$ dimensions. It is also interesting to note that if one "eliminates" the coordinates r and θ, the resulting two-dimensional energy-momentum tensor in the t and ϕ directions is still traceless (as it happens for a two-dimensional CFT). Explicitly, one can take T_{ab} defined as

$$T_{ab}=\begin{pmatrix} T_{tt} & P_\phi\\ P_\phi & T_{\phi\phi}\end{pmatrix},\qquad a,b=t,\phi,$$

as the effective energy-momentum tensor associated with the massless two-dimensional scalar field F. As it is clear from Eq. (25), the CS density associated with $F_+ + F_-$ (where F_+ and F_- are the left and right movers mode expansion defined explicitly below) is the sum of the topological charge density associated with F_+ plus to one associated with F_- only for small amplitudes, namely, when

$$|\exp(4\eta)F(t,\phi)^2| \ll 1. \tag{31}$$

On the other hand, when the temperature is high enough, it is natural to expect that the thermal fluctuations of $F(t,\phi)$ violate the above condition. That is why the CS density of these configurations (which can be interpreted as baryonic charge density) is only well defined below a specific temperature.

The CS charge reads

$$Q_{\mathrm{CS}} = \frac{pq\exp(3\eta)}{2}\left[\frac{F}{\sqrt{1-\exp(4\eta)F^2}}\right]\Bigg|_{F(t,0)}^{F(t,\pi)}. \tag{32}$$

As it has been already discussed, when $F(t,0) = F(t,\pi)$ the topological charge vanishes. Thus, let us consider the boundary conditions for $F(t,\phi)$ in Eq. (22). The requirement to have an integer topological charge can be expressed as follows. Introducing the useful auxiliary function

$$\Omega(\eta,a,b) \equiv \frac{\exp(3\eta)}{2}\left[\frac{a}{\sqrt{1-\exp(4\eta)a^2}} - \frac{b}{\sqrt{1-\exp(4\eta)b^2}}\right], \tag{33}$$

the topological charge reads

$$Q_{\mathrm{CS}} = pq\cdot\Omega(\eta, a = F(t,\pi), b = F(t,0)).$$

Taking into account the boundary conditions for $F(t,\phi)$ in Eq. (22), the quantity $\Omega(\eta, a = F(t,\pi), b = F(t,0))$ can be further simplified, so that one arrives at the following expression for the topological charge:

$$Q_{\mathrm{CS}} = pq. \tag{34}$$

Consequently, in order to have integer topological charge, the number pq must be an integer. Here it is worth emphasizing that, although the field equations in terms of $F(t,\phi)$ are linear, an important nonlinear effect is manifest in Eqs. (25), (32), and (31). Indeed, in order for the CS density in Eq. (25) to be everywhere well defined, one must require

$$|\exp(4\eta)F(t,\phi)^2| \leq 1. \tag{35}$$

Since the thermal expectation value of $F(t,\phi)^2$ grows with temperature, the condition above implies that the partition function associated with the present family of exact solutions will be well defined only below a certain critical temperature beyond which the CS density is not well defined anymore.

C. Semiclassical considerations

Let us remember the usual mode expansion of the solutions of Eq. (24). These can be written as

$$F_+ = \phi_0^+ + v_+\left(\frac{t}{L_\phi}+\phi\right) + \sum_{n\neq 0}\left(a_n^+\sin\left[n\left(\frac{t}{L_\phi}+\phi\right)\right] + b_n^+\cos\left[n\left(\frac{t}{L_\phi}+\phi\right)\right]\right), \tag{36}$$

$$F_- = \phi_0^- + v_-\left(\frac{t}{L_\phi}-\phi\right) + \sum_{n\neq 0}\left(a_n^-\sin\left[n\left(\frac{t}{L_\phi}-\phi\right)\right] + b_n^-\cos\left[n\left(\frac{t}{L_\phi}-\phi\right)\right]\right), \tag{37}$$

where, as usual, F_+ refers to the left movers and F_- to the right movers ($v_\pm$ and $\phi_0^\pm$ being integration constants, which must satisfy three constraints that will be discussed below). Hence, the most general topologically nontrivial configuration of the present sector arises, replacing $F = F_+ + F_-$ into Eqs. (14) and (17)–(20). In order to have a clear physical picture of the composition of solutions, it is convenient to choose $a_n^\pm$ and $b_n^\pm$ in such a way that

$$\tilde{F}(t,\phi=0) = \tilde{F}(t,\phi=\pi) = 0,$$

where $\tilde{F}(t,\phi)$ is the part of $F = F_+ + F_-$ coming from the sum over the integers n in Eqs. (36) and (37). Therefore, the topological charge in Eq. (32) is nonzero when

$$v_+ - v_- \neq 0.$$

In particular, $v_\pm$ and $\phi_0^\pm$ in Eqs. (36) and (37) must be chosen as

$$F(t,\phi=0) = \phi_0^+ + \phi_0^- + (v_+ + v_-)\frac{t}{L_\phi} = \frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}} \Rightarrow v_+ + v_- = 0, \quad \phi_0^+ + \phi_0^- = \frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}}, \tag{38}$$

$$F(t,\phi=\pi) = \frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}} + (v_+ - v_-)\pi = -\frac{\exp(-2\eta)}{\sqrt{1+\exp(2\eta)}} \Rightarrow v_- = \frac{\exp(-2\eta)}{\pi\sqrt{1+\exp(2\eta)}}. \tag{39}$$

At a classical level, this is the most straightforward choice of boundary conditions since it identifies which terms are responsible for the topological charge and which are not.

At the semiclassical level, it is very tempting to introduce creation and annihilation operator quantization corresponding to the above mode expansion, as it is usually done in quantizing a free two-dimensional scalar field. However, there are some intriguing differences.

First, in Eqs. (36) and (37), any term in the expansion corresponds to an exact solution of the $(3+1)$-dimensional Yang-Mills equations and not just to a solution of the linearized field equations. Therefore, the Bosonic quantum operators α_n^+, $(\alpha_m^+)^\dagger$ and α_n^-, $(\alpha_m^-)^\dagger$ (which are annihilation and creation operators for the left and right movers, satisfying the obvious commutation relations; see [90]) create exact solutions of the semiclassical Yang-Mills equations. This situation should be compared with the usual case in which, given a particular solution of the $(3+1)$-dimensional Yang-Mills equations, the small fluctuations (both at classical and quantum level) around the given classical configurations are solutions of the linearized field equations (while they are not solutions of the exact field equations, unless, of course, the theory is just a free theory).

Second, the constant terms $\phi_0^\pm$ as well as the linear terms in t and ϕ play an important role. According to Refs. [85–88], the topological charge can be interpreted as the baryonic charge of the configuration. If this interpretation is accepted, when the topological charge is odd, the configuration is a fermion, while when it is even, the configuration is a boson. This observation has no consequences for the operators $(\alpha_n^\pm, (\alpha_{n'}^\pm)^\dagger)$ since these operators are bosonic (because the corresponding classical solutions do not contribute to the topological charge). On the other hand, the creation and annihilation operators associated with the solution's linear part create a boson or a fermion, depending on whether the topological charge is even or odd. Hence, it is tempting to quantize $\phi_0^\pm$ and $v_\pm$ with commutators or anticommutators, depending on the value of the topological charge.

D. Chiral conformal field theory from Yang-Mills-Higgs theory

Now we will show that the construction presented above can be directly generalized to the Yang-Mills-Higgs theory, but with the notable difference that, this time, the theory is reduced to a chiral CFT in $(1+1)$ dimensions instead of just a CFT.

The Yang-Mills-Higgs theory in $(3+1)$ dimensions is defined by the action

$$I[A,\varphi] = \int d^4x\sqrt{-g}\left(\frac{1}{2e^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}\mathrm{Tr}(D_\mu\varphi D^\mu\varphi)\right). \tag{40}$$

Here φ is the Higgs field in the adjoint representation, and the covariant derivative D_μ has been defined in Eq. (2). Varying the action with respect to the fields A_μ and φ, we obtain the field equations of the Yang-Mills-Higgs theory as

$$\nabla_\nu F^{\mu\nu} + [A_\nu, F^{\mu\nu}] + \frac{e^2}{4}[\varphi, D^\mu\varphi] = 0, \tag{41}$$

$$D_\mu D^\mu\varphi = 0 \tag{42}$$

On the other hand, the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{e^2}\mathrm{Tr}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) - \frac{1}{2}\mathrm{Tr}\left(D_\mu\varphi D_\nu\varphi - \frac{1}{2}g_{\mu\nu}D_\alpha\varphi D^\alpha\varphi\right). \tag{43}$$

In order to construct analytical solutions of the Yang-Mills-Higgs theory in $(3+1)$ dimensions, we will use as a starting point the same *Ansatz* for the U field and the connection A_μ introduced for the case without the Higgs contribution, namely, Eqs. (14) and (17). Now, for the Higgs field, we must consider the following general form:

$$\varphi = \sum_{j=1}^{3} f_j(r)h^j(t,\phi)\mathbf{t}_j, \tag{44}$$

where f_j and h_j are functions to be found.

A good choice for the functions introduced above that allows one to reduce significantly the field equations of the Yang-Mills-Higgs system is the following:

$$h_1(t,\phi) = \frac{a}{b}h(t,\phi), \quad h_3(t,\phi) = a\cot(H(t,\phi))\frac{h(t,\phi)}{\lambda(t,\phi)}, \quad \lambda_3 = 1, \tag{45}$$

$$f_1(r) = b\cos(qr)f_3(r), \qquad f_2(r) = a\sin(qr)f_3(r), \qquad f_3(r) = f_0 r, \tag{46}$$

where we have defined

$$h_2(t,\phi) \coloneqq h(t,\phi), \qquad \lambda_1(t,\phi) = \lambda_2(t,\phi) \coloneqq \lambda(t,\phi),$$

with a, b, and f_0 being arbitrary constants.

In fact, it is direct to check that Eqs. (44)–(46), together with Eqs. (14) and (17), reduce the complete set of Yang-Mills-Higgs equations to the following decoupled partial differential equations:

$$\Box H = \left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)H = 0,$$

$$\Box h = \left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)h = 0,$$

$$\Box\lambda = \left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)\lambda = 0,$$

together with

$$\left(\frac{\partial H}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial H}{\partial \phi}\right)^2=\left(\frac{\partial H}{\partial t}-\frac{1}{L_\phi}\frac{\partial H}{\partial \phi}\right)\left(\frac{\partial H}{\partial t}+\frac{1}{L_\phi}\frac{\partial H}{\partial \phi}\right)=0, \tag{47}$$

$$\left(\frac{\partial h}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial h}{\partial \phi}\right)^2=\left(\frac{\partial h}{\partial t}-\frac{1}{L_\phi}\frac{\partial h}{\partial \phi}\right)\left(\frac{\partial h}{\partial t}+\frac{1}{L_\phi}\frac{\partial h}{\partial \phi}\right)=0, \tag{48}$$

$$\left(\frac{\partial \lambda}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial \lambda}{\partial \phi}\right)^2=\left(\frac{\partial \lambda}{\partial t}-\frac{1}{L_\phi}\frac{\partial \lambda}{\partial \phi}\right)\left(\frac{\partial \lambda}{\partial t}+\frac{1}{L_\phi}\frac{\partial \lambda}{\partial \phi}\right)=0. \tag{49}$$

Additionally, from the Yang-Mills equations, the following first-order nonlinear equation emerges:

$$\frac{\partial \lambda}{\partial t}+\tan(H)\lambda(1-\lambda^2)\frac{\partial H}{\partial t}=0\Rightarrow \lambda = \pm\frac{\cos(H)}{\sqrt{\exp(2\lambda_0)+\cos^2(H)}}, \tag{50}$$

which fixes the function λ in terms of H (here, λ_0 is constant). Hence, the constraint above reduces the number of chiral modes to two.

Summarizing, with the *Ansatz* presented above, the complete set of field equations of the Yang-Mills-Higgs theory has been reduced to the field equations of three chiral massless scalar fields in $(1+1)$ dimensions, plus a nonlinear constraint between two of them. Consequently, these families of exact solutions with nonvanishing topological charge are labeled by two integers [p and q, which determine the topological charge in Eq. (34)] and two chiral massless fields in $(1+1)$ dimensions (namely, H and h), since λ depends on H as in Eq. (50). Quite interestingly, the inclusion of the Higgs field leads to two-dimensional chiral massless modes (instead of massless modes).

The energy density $T_{00}^{(1)}$ of the above solutions takes the form

$$\begin{aligned} T_{00}^{(1)} = \frac{(1+e^{2\lambda_0})}{2}\Bigg(&\csc^2(H)\left[a^2f_0^2r^2h^2\cot^2(H) + \frac{e^{4\lambda_0}p^2\sin^4(H)}{e^2L_\theta^2(e^{2\lambda_0}+\cos^2(H))^3}\right]\left((\partial_t H)^2+\frac{1}{L_\phi^2}(\partial_\phi H)^2\right) \\ &+ a^2f_0^2r^2\csc^2(H)\left((\partial_t h)^2+\frac{1}{L_\phi^2}(\partial_\phi h)^2\right) \\ &+\frac{a^2f_0^2}{L_r^2}h\csc^2(H)[h-4L_r^2r^2\cot(H)\partial_t H\partial_t h]\Bigg), \end{aligned} \tag{51}$$

where ∂_t and ∂_ϕ stand for derivative, respectively, with respect to t and ϕ, and the field equations have been used in order to reduce the last term. Here it is worth noting the following fact: at a first glance, because the *Ansatz* reduces the complete set of Yang-Mills-Higgs field equations to a set of linear decoupled equations (one for H and one for h), one could suspect that perhaps the above configurations of Yang-Mils-Higgs theory are, after all, gauge equivalent to Abelian noninteracting configurations. However, if this were the case, then the energy density (which is gauge invariant) should also be the energy density of two decoupled chiral massless modes (which is quadratic in the fields, satisfies linear equations, and only contains kinetic terms of the chiral fields). In the present case, the above expression for the energy density clearly manifests nonlinear interactions between the two main degrees of freedom H and h.

On the other hand, the CS density becomes

$$\rho_{\mathrm{CS}} = -\frac{1}{16\pi^2}pq\sin(H)\frac{\partial H}{\partial \phi}. \tag{52}$$

Integrating in the ranges defined in Eq. (13), the topological charge turns out to be $Q_{\mathrm{CS}}=pq$, where we have used the following boundary conditions:

$$H(t,\phi=\pi)=0, \qquad H(t,\phi=0)=\pi.$$

III. NONLINEAR SIGMA MODEL IN (3+1) DIMENSIONS

Here and in the following sections, we will discuss the NLSM and the Skyrme model in $(3+1)$ dimensions in the $SU(2)$ case, which is more relevant than Yang-Mills-Higgs theory as far as the low-energy phase diagram of QCD. Hence, the primary variable will be an $SU(2)$-valued scalar field U. We will analyze how one can construct in these nonintegrable theories an infinite-dimensional family of exact solutions labeled by two integers, as well as by a free massless scalar field in two dimensions keeping alive the topological charge, which (in this case as well) can be interpreted as the baryonic charge. The key technical point is to find a suitable *Ansatz* that, on the one hand, depends on all the four space-time coordinates (for the topological density to be nonvanishing) and, at the same time, reduces the field equations to the field equations of a free massless scalar field in two dimensions. The high physical interest in the NLSM can be quickly explained, considering its many relevant physical applications. In particular, as far as the present paper is concerned, the model is related to the low-energy limit of QCD and pion's dynamics (see [5,6] and references therein). Thus, the current approach can provide an infinite family of topologically nontrivial solutions, allowing the explicit computation of critical physical quantities (which would be impossible to obtain from perturbation theory). In fact, in many situations of physical interest (especially at finite baryon density), both perturbation theory and even the powerful tools of LQCD may fail (see [91–93] and references therein).

The action of the $SU(2)$ NLSM in $(3+1)$ dimensions is

$$I[U] = \frac{K}{4}\int d^4x\sqrt{-g}\mathrm{Tr}(R^{\mu}R_{\mu}), \tag{53}$$

where K is the coupling constant of the NLSM and R_{μ} has been defined in Eq. (6). It is worth emphasizing that the NLSM only possesses global symmetry and is not classically conformal invariant in $(3+1)$ dimensions (unlike Yang-Mills theory). Nevertheless, despite the enormous differences between these two theories, an approach similar to the one described in the previous section also works in the present case. The field equations obtained varying the action in Eq. (53) with respect to the U field are

$$\nabla_{\mu}R^{\mu} = 0, \tag{54}$$

and the energy-momentum tensor of the model is

$$T_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}\left[R_{\mu}R_{\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha}R_{\alpha}\right]. \tag{55}$$

A. CFT in two dimensions from the NLSM

We will use the metric in Eq. (1) whose ranges for the coordinates can be determined in a similar way as in Eq. (13) (where the theory of Euler angles came into play). Let us define the following $U(x) \in SU(2)$:

$$U = \exp\left(p\theta\frac{\mathbf{t}_3}{2}\right)\exp\left(r\frac{\mathbf{t}_2}{4}\right)\exp\left(F(t,\phi)\frac{\mathbf{t}_3}{2}\right), \tag{56}$$

where p is a nonvanishing integer (there will be one more restriction to be discussed later on). The theory of Euler angles for $SU(N)$ [82–84] tells us that the range of θ [appearing in the left factor of the decomposition in Eq. (56)] and the range of r [appearing in the central factor of the decomposition in Eq. (56)] must be

$$0 \le \theta \le \pi, \qquad 0 \le r \le 2\pi. \tag{57}$$

One can also consider the range of the coordinate ϕ as

$$0 \le \phi \le 2\pi. \tag{58}$$

As far as the exponent in the right factor [namely, $F(t,\phi)$] is concerned, there are again two options. If the field $F(t,\phi)$ satisfies periodic boundary conditions, then the topological charge of the $SU(2)$-valued scalar field U vanishes [although the topological density in Eq. (8) can still be nontrivial]. The other boundary condition for $F(t,\phi)$ arises naturally, taking into account two facts. First of all, one has to require that physical observables [built from traces of product of the $SU(2)$-valued field U and its derivatives] such as the energy-momentum tensor should be periodic in ϕ and this requirement does *not* imply that $F(t,\phi)$ itself is periodic. Second, $F(t,\phi)$ appears in the right factor of the Euler angles decomposition of an $SU(2)$ element (see, for instance, Refs. [82–84]),

$$F(t,\phi=0) - F(t,\phi=2\pi) = \pm 8q\pi, \tag{59}$$

where q is a nonvanishing integer. The option above ensures that the $SU(2)$-valued element U defined in Eqs. (56) and (59) wraps an integer number of times around the group manifold of $SU(2)$ (in other words, U has a nonvanishing winding number). In this case, the topological charge and the topological density associated with U will be nontrivial. Also, in the present section, the term "topologically nontrivial" refers to configurations with $\rho_{\mathrm{B}} \neq 0$: the reason is that configurations with vanishing total baryonic charge but nonvanishing ρ_{B} still describe nontrivial interacting configurations with both regions having positive and negative charge densities.

It is an astounding and powerful result (due to all the analytic nonperturbative tools that will become available) that, despite the nonintegrable character of the NLSM in $(3+1)$ dimensions, the complete set of NLSM field equations in Eq. (54) corresponding to the *Ansatz* in Eq. (56) reduce to the field equation of a free massless scalar field in two dimensions, keeping alive the topological charge density

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_{\phi}^2}\frac{\partial^2}{\partial\phi^2}\right)F(t,\phi) = 0. \tag{60}$$

It is worth emphasizing that F represents a Goldstone mode associated with the phase of the two charged pions. In other words, if one would associate a complex wave function to the two charged pions (keeping out of such wave function the neutral pion), then the scalar field F would be the phase of the wave function.

B. Topological charge and energy density

With the *Ansatz* in Eq. (56), the topological density and topological charge, respectively, read

$$\rho_{\mathrm{B}} = -\frac{p}{32\pi^2}\sin\left(\frac{r}{2}\right)\frac{\partial F}{\partial\phi}, \tag{61}$$

$$B = -\frac{p}{8\pi}[F(t,\phi=2\pi) - F(t,\phi=0)] = \pm pq. \tag{62}$$

It is worth noting that the topological charge density in Eq. (61) has a nontrivial profile depending both on r and on ϕ. The maximum of ρ_{B} are located at $r=\pi$ and at the values of ϕ such that $\partial F/\partial\phi$ is maximum: in three spatial dimensions, these two conditions identify a line. As long as

$\partial F/\partial\phi \neq 0$, the topological density is nonzero. Note that the topological density is a linear function of $F(t,\phi)$, different from the Yang-Mills case presented in the previous section.

The energy density reads

$$T^{\sigma}_{00} = \frac{K}{8}\left[\frac{1}{4}\left(\frac{1}{L_r^2} + \frac{4p^2}{L_\theta^2}\right) + \left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial \phi}\right)^2\right], \quad (63)$$

then the total energy is given by

$$\begin{aligned} E^{\sigma} &= \int \sqrt{-g}\,dr d\theta d\phi T^{\sigma}_{00}, \\ &= \Gamma^{\sigma} + \Psi^{\sigma}\int_0^{2\pi} d\phi\left(\left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial \phi}\right)^2\right), \end{aligned} \quad (64)$$

where

$$\Gamma^{\sigma} = \frac{K\pi^3 L_\phi}{8L_r L_\theta}(L_\theta^2 + 4p^2L_r^2), \quad \Psi^{\sigma} = \frac{K\pi^2 L_r L_\theta L_\phi}{4}. \quad (65)$$

On the other hand, the on-shell action becomes

$$\begin{aligned} I^{\sigma}_{\text{on-shell}}[F] = -\frac{K}{8}\int \sqrt{-g}\,dr d\theta d\phi &\left[\frac{1}{4}\left(\frac{1}{L_r^2} + \frac{4p^2}{L_\theta^2}\right)\right. \\ &\left. -\left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial \phi}\right)^2\right]. \end{aligned} \quad (66)$$

It is important to note that the energy does not grow linearly with the topological charge, as can be seen from Eqs. (62) and (63). This fact indicates that these solutions describe interacting systems (as otherwise, the energy would be linear in the topological charge).

C. Conformal field theory and some semiclassical considerations

The usual mode expansion of the solutions of Eq. (60) is, of course, the same as in the previous section in Eqs. (36) and (37), where F_+ refers to the left movers and F_- to the right movers ($v_\pm$ and $\phi_0^\pm$ being integration constants that must satisfy three constraints, which will be discussed below). Hence, the most general topologically nontrivial configuration of the present sector arises replacing $F = F_+ + F_-$ in Eqs. (36) and (37) into Eq. (56).

Also, in the present case, the most natural choice corresponds to take $a_n^\pm$ and $b_n^\pm$ in such a way that

$$\tilde{F}(t,\phi=0) = \tilde{F}(t,\phi=2\pi) = 0,$$

where $\tilde{F}(t,\phi)$ is the part of $F = F_+ + F_-$ coming from the sum over the integers n in Eqs. (36) and (37). Therefore, B in Eq. (62) is nonzero when $v_+ - v_- \neq 0$. Also, $v_\pm$ in Eqs. (36) and (37) must be chosen as

$$F(t,\phi=0) = \phi_0^+ + \phi_0^- + (v_+ + v_-)\frac{t}{L_\phi} \Rightarrow v_+ + v_- = 0, \quad (67)$$

$$F(t,\phi=2\pi) = \phi_0^+ + \phi_0^- + (v_+ - v_-)2\pi \Rightarrow v_+ - v_- = 4q. \quad (68)$$

Unlike what happens in the Yang-Mills case, here there is no constraint on $\phi_0^+ + \phi_0^-$. Hence, the topological charge is

$$B = pq.$$

At the classical level, this is the most straightforward possible choice of boundary conditions, since it allows us to identify the terms in the expansion modes responsible for the topological charge and which are not. However, plenty of different options will be discussed in forthcoming papers.

Also, in the present case, the semiclassical quantization of these configurations corresponds to the quantization of the free massless scalar field $F(t,\phi)$, with the boundary conditions described above having a nonvanishing topological charge. However, as discussed in the previous sections, some interesting differences exist.

First, in Eqs. (36) and (37) any term in the expansion corresponds to an exact solution of the $(3+1)$-dimensional NLSM field equations and not just to a solution of the linearized field equations. Therefore, the bosonic quantum operators α_n^+, $(\alpha_m^+)^\dagger$, and α_n^-, $(\alpha_m^-)^\dagger$ (which are annihilation and creation operators for the left and right movers, satisfying the obvious commutation relations, see [90]) are quantum operators that create exact solutions of the semiclassical NLSM field equations.

Second, the constant terms $\phi_0^\pm$ as well as the linear terms in t and ϕ play an important role as these are associated with classical solutions that carry the topological charge (while the modes satisfying periodic boundary conditions do not contribute to the topological charge). Thus, depending on whether B is odd or even, one should quantize the modes associated with the linear terms in the expansion of F as fermionic or bosonic. Hence, when B is odd, F has a component that should be considered as an emergent fermionic field.

IV. THE SKYRME MODEL IN (3+1) DIMENSIONS

A very natural question is this: does the Skyrme term spoil the remarkable relation discussed in the previous section between the simplest two-dimensional CFT and a nonintegrable theory in $(3+1)$ dimensions at finite baryon density in topologically nontrivial sectors? The importance of the Skyrme model lies in the fact that the NLSM in flat space-time does not admit static topologically nontrivial soliton solutions with finite energy, known as Derrick's scale argument [94]. The Skyrme term is introduced to get around this problem and stabilize the soliton (skyrmion).

The obvious physical relevance of finite density effects arises from the difficulties in providing cold and dense nuclear matter as a function of baryon number density with a good analytic understanding. The nonperturbative nature of low-energy QCD prevents (the very complex and interesting structure of) its phase diagram from being described in detail (see [95–100] and references therein): this is the reason why researchers in this area mainly use numerical and lattice approaches. In particular, a very intriguing part in the QCD phase diagram, which appears at finite baryon density,[5] is related to the appearance of ordered structures (similar to the Larkin-Ovchinnikov-Fulde-Ferrell phase [109]). These ordered structures at finite density are, by now, a well-established feature (see, for instance, [110–112] and references therein). These are just some of the reasons why it is mandatory to shed more light on these issues with theoretical tools, as often even the numerical approaches are not effective with high topological charges.

Here we will show that the Skyrme term discloses a remarkable phenomenon: namely, the present construction still works (with precisely the same *Ansatz*) but now, when the Skyrme coupling is nonzero, instead of a two-dimensional CFT, one gets a two-dimensional chiral CFT; namely, either left or right movers must be eliminated. This new result is likely to be related to the fact that the Skyrme model includes the effects of the low-energy limit of QCD so that the Skyrme model knows, somehow, about chiral symmetry breaking.

The Skyrme action is given by

$$I[U] = \frac{K}{4}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left(R_\mu R^\mu + \frac{\lambda}{8}[R_\mu, R_\nu][R^\mu, R^\nu]\right),$$

where K and λ are positive coupling constants.[6] The field equations of the model are obtained varying the last action with respect to the U field; we get

$$\nabla^\mu\left(R_\mu + \frac{\lambda}{4}[R^\nu, [R_\mu, R_\nu]]\right) = 0, \qquad (69)$$

being these three nonlinear coupled second-order partial differential equations.

The energy-momentum tensor reads

$$T_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}\left(R_\mu R_\nu - \frac{1}{2}g_{\mu\nu}R_\alpha R^\alpha + \frac{\lambda}{4}\left(g^{\alpha\beta}[R_\mu, R_\alpha][R_\nu, R_\beta] - \frac{1}{4}g_{\mu\nu}[R_\alpha, R_\beta][R^\alpha, R^\beta]\right)\right). \qquad (70)$$

The topological density and charge are defined in Eqs. (8) and (9). Now, we will study two types of analytical configurations that will lead to a chiral CFT. The description of the box is based on the metric given in Eqs. (1), (57), and (58).

A. Chiral conformal field theory from the Skyrme model. Type-I: Euler *Ansatz* for the lasagna phase

We will consider, once again, the matter field *Ansatz* in Eq. (56). When one plugs Eq. (56) into the Skyrme equations in Eq. (69), the field equations reduce to

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)F = 0, \qquad (71)$$

$$\left(\frac{\partial F}{\partial t}\right)^2 - \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2 = \left(\frac{\partial F}{\partial t} - \frac{1}{L_\phi}\frac{\partial F}{\partial\phi}\right)\left(\frac{\partial F}{\partial t} + \frac{1}{L_\phi}\frac{\partial F}{\partial\phi}\right) = 0. \qquad (72)$$

As in the NLSM and Yang-Mills cases, the first equation describes the simplest bosonic CFT in two dimensions. Thus, from Eq. (71), $F = F_+ + F_-$ (where $F_\pm$ represent the contributions of the left and right movers). However, Eq. (72) can be satisfied only by killing either F_+ or F_-. Hence, we still get a two-dimensional CFT, but this time it is a chiral CFT. Once again, this result is a huge analytic achievement as the field equations have been reduced exactly, keeping alive the topological density, to the field equations of a free massless chiral scalar field in $(1+1)$ dimensions. Also, the topological charge is the same as in the NLSM case defined in Eq. (62).

In this case, the energy density is given by

$$T_{00}^{(2)} = \frac{K}{8}\left[\frac{1}{4}\left(\frac{1}{L_r^2} + \frac{4p^2}{L_\theta^2}\right) + \left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right] + \frac{K\lambda}{32L_r^2L_\theta^2}\left[\frac{p^2}{4} + \left(\frac{1}{4}L_\theta^2 + p^2L_r^2\sin^2\left(\frac{r}{2}\right)\right) \times\left(\left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right)\right], \qquad (73)$$

so that the expression for the energy becomes

[5] See [101–108] and references therein, for the construction of nonhomogeneous condensates at finite density in chiral perturbation theory.

[6] The parameters K and λ are related to the meson decay coupling constant F_π and the Skyrme coupling e via $F_\pi = 2\sqrt{K}$ and $K\lambda e^2 = 1$, where $F_\pi = 141$ MeV and $e = 5.45$.

$$\begin{aligned} E^{(2)} &= \int \sqrt{-g}drd\theta d\phi T^{(2)}_{00} \\ &= \Gamma^{(2)} + \Psi^{(2)} \int_0^{2\pi} d\phi \left(\left(\frac{\partial F}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial \phi}\right)^2 \right), \end{aligned} \tag{74}$$

where

$$\begin{aligned} \Gamma^{(2)} &= \frac{K\pi^3 L_\phi}{32 L_r L_\theta}(4L_\theta^2 + p^2(\lambda + 16L_r^2)), \\ \Psi^{(2)} &= \frac{K\pi L_\phi}{64 L_r L_\theta}(\pi L_\theta^2(\lambda + 16L_r^2) + 8p^2\lambda L_r^2). \end{aligned} \tag{75}$$

As in the NLSM, the energy does not grow linearly with the topological charge, implying the presence of interactions between particles. Also, the topological charge density is linear in $F(t,\phi)$ instead of nonlinear, as in the Yang-Mills case presented in the previous sections. These configurations describe modulated nuclear lasagna layers in which the periodic part in the mode expansion of the field $F(t,\phi)$ (which does not carry topological charge) represents the modulations in the ϕ direction, while the linear part is responsible for the "bare lasagna" (namely, the lasagna without modulations that have been analyzed in [113,114]). Figure 1 shows the energy density of two lasagna-type configurations, one with modulation and the other without modulation. It is worth emphasizing that it is also necessary to introduce a cutoff in the computation of the (semi) classical partition function because the Skyrme theory is an effective low-energy model. We will detail this point in the next section.

B. Chiral conformal field theory from the Skyrme model. Type-II: Exponential *Ansatz* for the spaghetti phase

This time, for the U field, we adopt the standard (exponential) parametrization of an element of $SU(2)$, that is,

$$U^{\pm 1}(x^\mu) = \cos(\alpha)\mathbf{1}_2 \pm \sin(\alpha) n^i \mathbf{t}_i, \tag{76}$$

where

$$\begin{aligned} &n^1 = \sin\Theta\cos\Phi, \qquad n^2 = \sin\Theta\sin\Phi, \qquad n^3 = \cos\Theta, \\ &\alpha = \alpha(x^\mu), \qquad \Theta = \Theta(x^\mu), \qquad \Phi = \Phi(x^\mu), \qquad n^i n_i = 1. \end{aligned} \tag{77}$$

From Eq. (8) it follows that the topological charge density takes the following general form:

$$\rho_{\mathrm{B}} = -\frac{1}{2\pi^2}\sin^2\alpha \sin\Theta d\alpha \wedge d\Theta \wedge d\Phi. \tag{78}$$

Hence, in order to have topologically nontrivial configurations, we must demand that $d\alpha \wedge d\Theta \wedge d\Phi \neq 0$. On the other hand, as we want to construct analytical solutions, it is necessary to have a good *Ansatz* that significantly reduces the Skyrme field equations. Considering the approach developed in [58,59] leads to the following:

$$\begin{aligned} &\alpha = \alpha(r), \qquad \Theta = Q\theta, \qquad \Phi = F(t,\phi), \\ &Q = 2v + 1, \qquad v \in \mathbb{N}. \end{aligned} \tag{79}$$

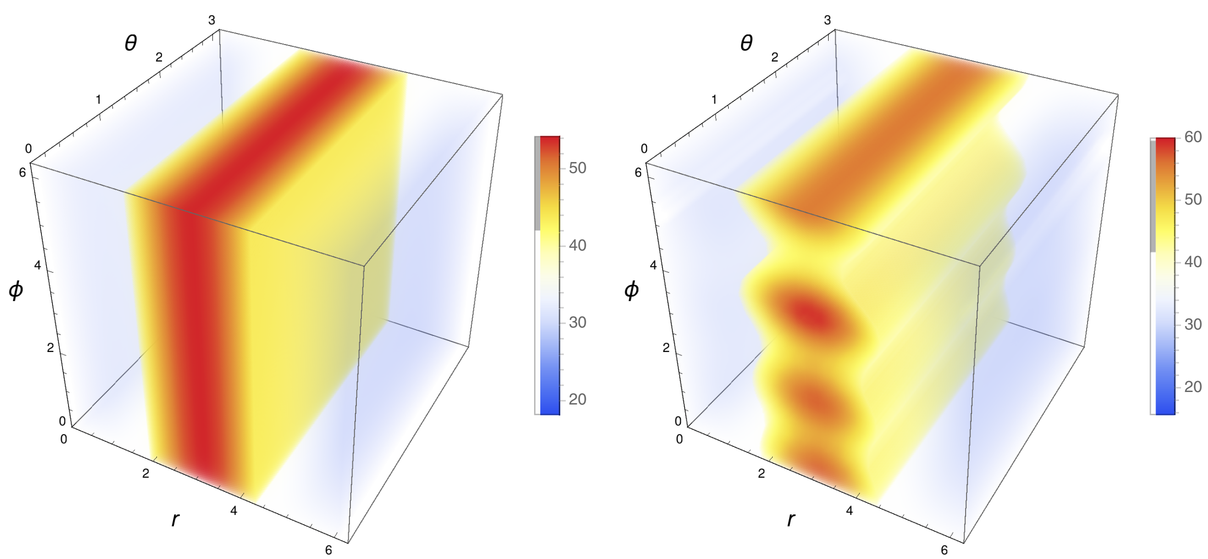

FIG. 1. Energy density with and without modulation of nuclear lasagna configurations with baryonic charge $B = 6$. For both cases, we have set $K = \lambda = L_r = L_\theta = L_\phi = 1$, $p = 3$, $q = 2$, and $\phi_0 = 0$. Left: nuclear lasagna without modulation where $a_i = b_i = 0$. Right: snapshot at $t = 0$ of nuclear lasagna with a modulation in the ϕ direction where the non-null modulation coefficients were set as $a_1 = -a_3 = b_1 = b_2 = 0.1$.

It is a direct computation to verify that, by replacing the *Ansatz* defined in Eqs. (77) and (79) into the Skyrme field equations, one gets the following system of equations:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{L_\phi^2}\frac{\partial^2}{\partial\phi^2}\right)F=0, \tag{80}$$

$$\left(\frac{\partial F}{\partial t}\right)^2-\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2=\left(\frac{\partial F}{\partial t}-\frac{1}{L_\phi}\frac{\partial F}{\partial\phi}\right)\left(\frac{\partial F}{\partial t}+\frac{1}{L_\phi}\frac{\partial F}{\partial\phi}\right)=0, \tag{81}$$

$$\alpha''-\frac{Q^2}{2}\frac{(L_r^2-\lambda\alpha'^2)\sin(2\alpha)}{L_\theta^2+Q^2\lambda\sin^2(\alpha)}=0. \tag{82}$$

Once again, the *Ansatz* in Eqs. (77) and (79) discloses many remarkable simplifications. Not only does the equation for α decouple from F [when F satisfies Eqs. (80) and (81)], but it can be also reduced to a simple quadrature,

$$\frac{d\alpha}{dr}=\pm\eta(\alpha,E_0),$$
$$\eta(\alpha,E_0)=\left[\frac{L_\theta^2}{L_\theta^2+Q^2\lambda\sin^2(\alpha)}\left(E_0-\frac{Q^2}{2}\frac{L_r^2}{L_\theta^2}\cos(2\alpha)\right)\right]^{\frac{1}{2}}, \tag{83}$$

where E_0 is an integration constant to be fixed by analyzing the boundary conditions,

$$F(t,\phi=0)-F(t,\phi=2\pi)=2p\pi, \tag{84}$$

and

$$\alpha(2\pi)-\alpha(0)=m\pi,\qquad m\in\mathbb{Z}.$$

In fact, by integrating Eq. (83) and considering the above boundary conditions, we get the following equation for E_0:

$$\pm m\int_0^\pi\frac{1}{\eta(\alpha,E_0)}d\alpha=2\pi.$$

From the above condition, it is clear that, for large m, the integration constant E_0 scales as m^2,

$$E_0=m^2\xi_0,\qquad \xi_0>0,$$

where ξ_0 (which can also be interpreted as an integration constant) does not depend on m for large m.

Moreover, in this case, Eq. (80) describes the simplest bosonic CFT in two dimensions. Thus, from Eq. (80), $F=F_++F_-$ but, once again, Eq. (81) can be satisfied only by killing either F_+ or F_-. Thus, as in the last case, we still get a chiral massless scalar field in $(1+1)$ dimensions. We stress the very intriguing phenomenon of the appearance of chiral modes without the presence of any actual edge. These chiral modes are "hosted" by the hadronic tubes[7]: hence these configurations describe modulated nuclear spaghetti configurations. Indeed, the linear part in the mode expansion of the field $F(t,\phi)$ is responsible for the "bare spaghetti," namely, the nuclear spaghetti without modulations along the axis that have been analyzed in [113,114]. On the other hand, the periodic part in the mode expansion of the field $F(t,\phi)$ (which does not carry topological charge) represents the modulations of the tubes in the ϕ direction.

The energy density is given by

$$\begin{aligned}T_{00}^{(3)}=&\frac{K}{2}\left\{\frac{\alpha'^2}{L_r^2}+\left[\frac{Q^2}{L_\theta^2}+\left(\left(\frac{\partial F}{\partial t}\right)^2\right.\right.\right.\\&\left.\left.\left.+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right)\sin^2(Q\theta)\right]\sin^2(\alpha)\right\}\\&+\frac{K\lambda}{2}\left\{\frac{Q^2}{L_\theta^2}\sin^2(Q\theta)\sin^2(\alpha)\left(\left(\frac{\partial F}{\partial t}\right)^2\right.\right.\\&\left.+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right)+\frac{\alpha'^2}{L_r^2}\left[\frac{Q^2}{L_\theta^2}+\sin^2(Q\theta)\left(\left(\frac{\partial F}{\partial t}\right)^2\right.\right.\\&\left.\left.\left.+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right)\right]\right\}\sin^2(\alpha),\end{aligned} \tag{85}$$

then the total energy is given by

$$\begin{aligned}E^{(3)}&=\int\sqrt{-g}drd\theta d\phi T_{00}^{(3)},\\&=\Gamma^{(3)}+\Psi^{(3)}\int_0^{2\pi}d\phi\left[\left(\frac{\partial F}{\partial t}\right)^2+\frac{1}{L_\phi^2}\left(\frac{\partial F}{\partial\phi}\right)^2\right],\end{aligned} \tag{86}$$

where

$$\Gamma^{(3)}=\frac{mK\pi^2L_\phi}{L_rL_\theta}\int_0^\pi d\alpha\Omega(\alpha,m,Q),$$
$$\Psi^{(3)}=\frac{mK\pi L_\phi}{4L_rL_\theta}\int_0^\pi d\alpha\tilde{\Omega}(\alpha,m,Q),$$

and

$$\Omega(\alpha,m,Q)=\eta(\alpha,E_0)(L_\theta^2+\lambda Q^2\sin^2(\alpha))+\frac{L_r^2Q^2}{\eta(\alpha,E_0)}\sin^2(\alpha), \tag{87}$$

[7]This can be seen as follows: the local maxima of the energy density [see Eq. (85)], which coincides with the maximum of the topological density, is found in the center of the tubes, where $\sin^2(\alpha)\sin^2(Q\theta)=1$. The chiral massless modes have their support around these points. On the other hand, when $\sin^2(\alpha)\sin^2(Q\theta)=0$, the contribution of the chiral modes to the energy density vanishes.

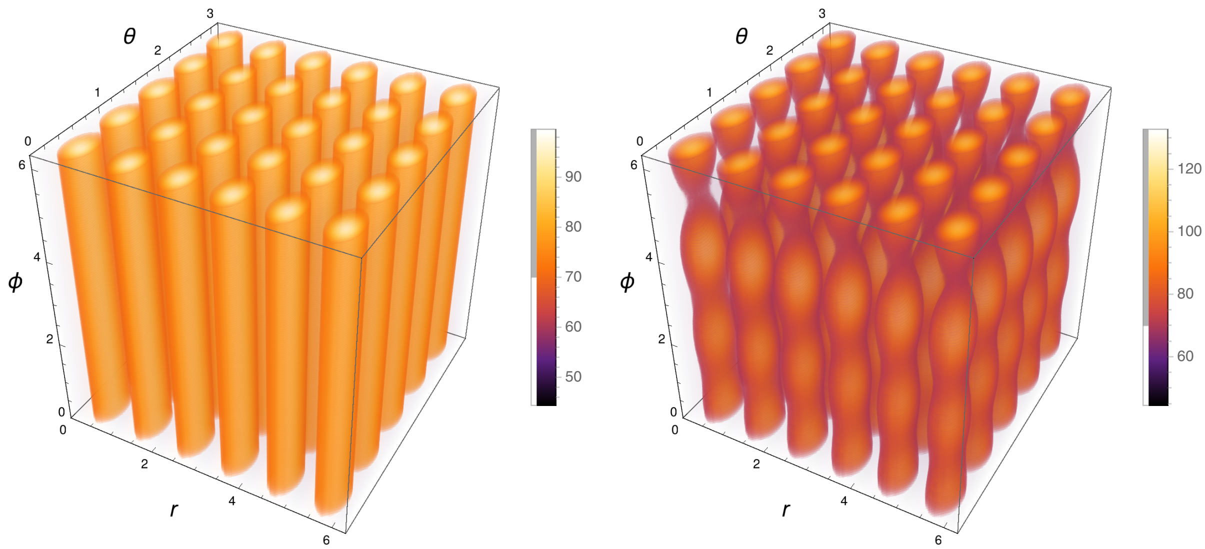

FIG. 2. Energy density with and without modulation of nuclear spaghetti configurations with baryonic charge $B = 6$. For both cases, we have set $K = \lambda = L_r = L_\theta = L_\phi = 1$, $p = 1$, $m = 6$, $q = 5$, and $\phi_0 = 0$. Left: nuclear spaghetti without modulation, where $a_i = b_i = 0$. Right: snapshot at $t = 0$ of nuclear spaghetti with a modulation in the ϕ direction, where the non-null modulation coefficients were set as $a_1 = -a_3 = b_1 = b_2 = 0.1$.

$$\tilde{\Omega}(\alpha, m, Q) = \eta(\alpha, E_0)\lambda L_\theta^2 \sin^2(\alpha) + \frac{\sin^2(\alpha)}{\eta(\alpha, E_0)} L_r^2 (L_\theta^2 + \lambda Q^2 \sin^2(\alpha)), \quad (88)$$

while the topological charge density reads

$$\rho_B = \frac{1}{2\pi^2} (\sin^2(\alpha)\alpha')(\sin(Q\theta))(\partial_\phi F) dr \wedge d(Q\theta) \wedge d\phi. \quad (89)$$

Note that the positions of the maximum of ρ_B are located at

$$Q\theta = \frac{\pi}{2} + N\pi, \qquad \sin^2(\alpha) = 1,$$

(N being an integer) and at the values of r and ϕ such that both $\sin^2(\alpha)\alpha'$ and $\partial_\phi F$ have maximum. In three spatial dimensions, these three conditions identify isolated points, and the same happens for the energy density of these configurations (as we mentioned above). Taking into account the boundary conditions satisfied by α and F, one arrives at the following value of the baryonic charge:

$$B = mp.$$

It is important to emphasize that both the energy density and the topological charge density depend on all three spatial coordinates: to the best of the authors' knowledge, these are the first analytic examples of soliton crystals in which both the energy density and the baryon density manifest a genuine three-dimensional behavior. Figure 2 shows the energy density of two spaghetti-type configurations, one with modulation and the other without modulation.

V. PARTITION FUNCTIONS

This section will discuss the semiclassical partition function associated with some of the families of topologically nontrivial configurations constructed in the previous sections. The wording "semiclassical partition functions" in this section refers to the following: all the exact solutions described previously are characterized both by some discrete labels (which determine the baryonic charge) and (for any possible choice of the discrete labels) by a massless chiral field F in $(1+1)$ dimensions (or two chiral massless fields in the Yang-Mills-Higgs case). The classical partition functions will include a sum of

$$e^{-\beta(E-\mu_B B)}$$

(where E is the energy of the solution and B is the baryonic charge) over all the possible discrete labels and (for any choice of the discrete labels) over the chiral massless field F, satisfying the boundary conditions defined in the previous sections, corresponding to the given choice of discrete labels.[8] On the other hand, we can take advantage of the fact that the massless chiral field F satisfies a linear equation. This allows us to "promote" the classical partition function over F to a semiclassical partition function by

[8]The main difference between the Yang-Mills-Higgs and Skyrme cases is that, in the former, two chiral modes contribute to the total energy, while in the latter, only one.

quantizing the massless chiral degree of freedom F in the obvious way.[9]

We will focus mainly on the Skyrme theory since it is more directly relevant as far as the low-temperature phase diagram is concerned (being the Skyrme theory, the low-energy limit of QCD at leading order in the ‘t Hooft expansion). The relations with the instanton-dyon liquid approach [115–118] will be shortly analyzed. A complete treatment of the quantum partition functions associated with these families should include (for any member of these families) the other possible fluctuations (such as small perturbations of the other 2 degrees of freedom of the Skyrme model and not just of F). Unfortunately, this task would involve the computation of functional determinants in $(3+1)$-dimensional backgrounds, which depend explicitly on time and spatial coordinates: such a computation can be done neither analytically nor numerically. However, it is worth emphasizing that it is already a quite remarkable fact that one of the modes (namely, F) can be quantized exactly. Moreover, the comparison with [115–118] below clearly shows that the partition function to be defined in the following sections captures much relevant information.

Schematically, the contribution of the current families of exact solutions to the partition function Z is

$$Z \approx \sum_{\substack{\text{over all the}\\ \text{solutions}\\ \text{of the family}}} \exp[-\beta(E_{\mathrm{Cl\,Sol}} - \mu_B B_{\mathrm{Cl\,Sol}})],$$

where the sum is over all the solutions of the given family.[10] Here $E_{\mathrm{Cl\,Sol}}$ is the total energy of a classical solution, $B_{\mathrm{Cl\,Sol}}$ is the baryonic charge of the configuration, β is the inverse of the temperature T, and μ_B is the baryon chemical potential.

In particular, for the lasagna phase constructed from the Euler representation and for the spaghetti phase constructed from the exponential representation in the Skyrme model, the expressions for $E_{\mathrm{Cl\,Sol}}$ in Eqs. (74) and (86) can be written, respectively, as

$$E^{(2)}_{\mathrm{Cl\,Sol}} \coloneqq E^{(2)} = \tilde{\Gamma}^{(2)} + \Psi^{(2)} \int_0^{2\pi} \left[\left(\frac{\partial \tilde{F}}{\partial t} \right)^2 + \frac{1}{L_\phi^2} \left(\frac{\partial \tilde{F}}{\partial \phi} \right)^2 \right] d\phi, \tag{90}$$

where

$$\tilde{\Gamma}^{(2)} = \Gamma^{(2)} + \frac{64\pi q^2}{L_\phi^2} \Psi^{(2)},$$

and

$$E^{(3)}_{\mathrm{Cl\,Sol}} \coloneqq E^{(3)} = \tilde{\Gamma}^{(3)} + \Psi^{(3)} \int_0^{2\pi} \left[\left(\frac{\partial \tilde{F}}{\partial t} \right)^2 + \frac{1}{L_\phi^2} \left(\frac{\partial \tilde{F}}{\partial \phi} \right)^2 \right] d\phi, \tag{91}$$

where

$$\tilde{\Gamma}^{(3)} = \Gamma^{(3)} + \frac{4\pi p^2}{L_\phi^2} \Psi^{(3)}.$$

Here $\{\Gamma^{(2)}, \Psi^{(2)}\}$ and $\{\Gamma^{(3)}, \Psi^{(3)}\}$ have been defined, respectively, in Eqs. (74) and (86), and $\tilde{F}$ is the part of F coming from the sum over the integers n in Eqs. (36) and (37). It is important to remember that the linear terms in Eqs. (90) and (91) that come from the modes expansion of the function F must be nonzero in order to have a nonvanishing topological charge. Also, it is worth emphasizing that the novel solutions presented in the manuscript with the arbitrary dependence of F on u (which is the lightlike coordinate orthogonal to the spacelike coordinates which enter explicitly in the *Ansatz*) can be considered as saddle points: from the intuitive viewpoint, the periodic part of $F(u)$ represents the lowest energy normal modes of the hadronic tubes and layers in very much the same way as a vibrating string encodes the lowest energy normal modes of static strings.

A. Partition function for fixed value of the baryonic charge

In the following, we will focus on the nuclear lasagna phase, which is slightly simpler to analyze than the spaghetti phase, using as starting point Eq. (90). The reason is that the total and free energies associated with hadronic tubes depend on inverse elliptic functions, while the ones of hadronic layers depend on functions that are polynomial in the physically relevant variables (so that these are easier to handle). On the other hand, the qualitative low-temperature behavior for large baryonic charges is similar in both cases.

Let us consider a fixed value of the baryonic charge B in Eq. (62) and let us turn off, momentarily, the baryon chemical potential μ_B.

In order to avoid very long algebraic expressions, we will consider $q = p$ (since this choice keeps the essential features of the problem). As for fixed values of the discrete label p, these configurations are characterized by a massless chiral field F in two dimensions; the contribution Z_p of the present family to the Skyrme partition function is

$$Z_p(\beta) = \int DF \mathcal{Z}_F = \int DF \exp\{-\beta E^{(2)}_{\mathrm{Cl\,Sol}}\},$$
$$\mathcal{Z}_F = \exp\{-\beta E^{(2)}_{\mathrm{Cl\,Sol}}\}, \tag{92}$$

where $E^{(2)}_{\mathrm{Cl\,Sol}}$ have been defined in Eq. (90) [the case of hadronic tubes, defined by the energy in Eq. (91), has a

[9] A more detailed treatment of the partition functions associated with these families will appear in future publications.

[10] Here the sum over all the solutions of the family means a sum over p, q, and $\tilde{F}$.

similar qualitative behavior]. The path integral over the massless chiral field can be done in the usual way, taking into account the obvious quantization (see, for instance, [90]) of the mode expansion for F (here, we will consider the total baryonic charge to be even to avoid complications with Grassmann variables associated with ϕ_0^- and v_-),

$$F_- = \phi_0^- + v_-\left(\frac{t}{L_\phi} - \phi\right) + \sum_{n\neq 0}\left(a_n^- \sin\left[n\left(\frac{t}{L_\phi} - \phi\right)\right] + b_n^- \cos\left[n\left(\frac{t}{L_\phi} - \phi\right)\right]\right). \tag{93}$$

The corresponding semiclassical partition function for the hadronic layers (with fixed discrete labels) reads

$$Z_p(\beta) = \sum_{n=1}^{+\infty} \delta(n) \exp\left[-\beta(\tilde{\Gamma}^{(2)} + \Psi^{(2)} n)\right], \tag{94}$$

where the integer n comes from the quantization of the Hamiltonian, $\int_0^{2\pi}\left[\left(\frac{\partial \tilde{F}}{\partial t}\right)^2 + \frac{1}{L_\phi^2}\left(\frac{\partial \tilde{F}}{\partial \phi}\right)^2\right] d\phi$, of the massless chiral mode and $\delta(n)$ is the corresponding degeneracy (related to the number partition). Taking into account Eq. (75), one can rewrite $\tilde{\Gamma}^{(2)}$ and $\Psi^{(2)}$ as follows:

$$\begin{aligned}
\tilde{\Gamma}^{(2)} &= \Sigma_1 + \Sigma_2 p^2 + \Sigma_3 p^4,\\
\Psi^{(2)} &= \Sigma_4 + \Sigma_5 p^2, \qquad \Sigma_1 = \frac{K\pi^3 L_\theta L_\phi}{8L_r},\\
\Sigma_2 &= \frac{K\pi^3}{32 L_r L_\theta L_\phi}(\lambda + 16L_r^2)(L_\phi^2 + 32L_\theta^2),\\
\Sigma_3 &= \frac{8K\pi^2 L_r \lambda}{L_\theta L_\phi}, \qquad \Sigma_4 = \frac{K\pi^2 L_\theta L_\phi}{64 L_r}(\lambda + 16L_r^2),\\
\Sigma_5 &= \frac{K\lambda\pi L_r L_\phi}{8L_\theta}.
\end{aligned} \tag{95}$$

The above is useful to separate the terms that depend on the discrete labels (which are proportional to Σ_2, Σ_3, and Σ_5) from the terms that do not depend on any discrete label of the family (which are proportional to Σ_1 and Σ_4). Note that the partition function in Eq. (94), when $p \neq q$, will be given by

$$Z_{p,q}(\beta) = \exp\left[-\beta\left(\Sigma_1 + \Sigma_2 \frac{(L_\phi^2 p^2 + 32L_\theta^2 q^2)}{(L_\phi^2 + 32L_\theta^2)} + \Sigma_3 p^2 q^2\right)\right] \times \sum_{n=1}^{+\infty} \delta(n) \exp\left[-\beta((\Sigma_4 + \Sigma_5 p^2) n)\right].$$

Now, in our case (with $p = q$), Eq. (94) can be written as

$$Z_p(\beta) = \exp[-\beta(\Sigma_1 + \Sigma_2 p^2 + \Sigma_3 p^4)] \times \sum_{n=1}^{+\infty} \delta(n) \exp[-\beta((\Sigma_4 + \Sigma_5 p^2) n)]. \tag{96}$$

These results are similar to the usual two-dimensional chiral CFT with the difference that, in the sum over n, the inverse temperature β has been rescaled by $(\Sigma_4 + \Sigma_5 p^2)$. If p is fixed, then the result is the usual chiral massless bosons partition function with rescaled temperature $\beta_r \coloneqq (\Sigma_4 + \Sigma_5 p^2)\beta$. However, be aware that we also have to sum over the label p. Such a partition function can also be written as

$$\zeta(z) \sim \exp\left(-\beta(\Sigma_1 + \Sigma_2 p^2 + \Sigma_3 p^4)\right) \sum_{n=1}^{+\infty} \delta(n) \exp(-\beta_r n), \tag{97}$$

where $\delta(n)$ is the degeneracy of the energy level n, which can be easily obtained (for large n) using the Hardy-Ramanujan-Cardy formula. The fundamental formula for the asymptotic growth of the partitions $\delta(n)$ was found a long time ago by Hardy and Ramanujan [119]: for $n \gg 1$, we get

$$\delta(n) \sim \frac{1}{4\sqrt{3}} \frac{\exp\left(\pi\sqrt{\frac{2n}{3}}\right)}{n}. \tag{98}$$

Such discrete label n represents exact excitation energies[11] (related to the "quanta" of modulations either of the hadronic layers or of the hadronic tubes) over bare lasagna or spaghetti configurations. Looking at Eq. (93), the "bare" Euler or Exponential configurations (which have been discussed previously in the literature, see [58–60,113,114] and references therein) possess $a_n = 0 = b_n$, while $v_- \neq 0$ (and fixed by the boundary condition to have integer baryonic charge). Since Skyrme theory is the low-energy limit of QCD, it is natural to introduce a cutoff Δ on the sum over n: such a Δ can be interpreted as the scale beyond which the Skyrme model is not a good description anymore. Therefore, instead of Eq. (97), we will consider the following expression:

$$\zeta_\Delta(z) \sim \exp\left(-\beta(\Sigma_1 + \Sigma_2 p^2 + \Sigma_3 p^4)\right) \sum_{n=1}^{\Delta} \delta(n) \exp(-\beta_r n). \tag{99}$$

[11] We use the expression exact excitation energies, since these chiral modes are not only small excitations on top of hadronic tubes or layers but, in fact, these configurations are exact solutions of the full Skyrme field equations. On the other hand, in the usual cases, one can only study small fluctuations around topological solitons as solutions of the linearized field equations around those solitons.

The Δ depends, in general, both on the temperature and the chemical potential: $\Delta = \Delta(T, \mu_B)$. Although we have been unable to find in the literature a widely accepted expression for the cutoff $\Delta = \Delta(T, \mu_B)$ as a function of T and μ_B, in the following subsection, we will show that reasonable choices of Δ provide us with analytic results in qualitative agreement, both with the available lattice data [120], as well as with a different analytical approach [115].

B. Partition function at finite baryon chemical potential

In order to get the full contribution of the present family to the semiclassical Skyrme partition function with nonvanishing baryon chemical potential μ_B, one also has to sum over p, since p determines the baryon charge $B = p^2$ (remember that we have considered for simplicity $q = p$). In this way, we get

$$
\begin{aligned}
Z^* &= \sum_{p=-\infty}^{+\infty} \exp(-\beta(\Sigma_1 + (\Sigma_2 - \mu_B)p^2 + \Sigma_3 p^4)) \\
&\quad \times \sum_{n=1}^{\Delta} \delta(n) \exp(-\beta_r n) \\
&\Leftrightarrow Z^* = \sum_{n=1}^{n_{\max}} \sum_{p=-\infty}^{+\infty} \delta(n) \exp(-\beta(\Sigma_1 + (\Sigma_2 - \mu_B)p^2 + \Sigma_3 p^4)) \\
&\quad \times \exp(-\beta(\Sigma_4 + \Sigma_5 p^2)n), \quad n_{\max} = \Delta(T, \mu_B). \qquad (100)
\end{aligned}
$$

The above double sum is clearly convergent since it is possible to exchange the order of the sums. As we will see below, the cutoff $n_{\max} = \Delta(T, \mu_B)$ can be fixed in such a way to achieve a qualitative agreement with the description of LQCD for the phase diagram. Note that, unlike what happens in LQCD, in the present approach the inclusion of the baryon chemical potential is not harmful.

It is worth emphasizing the intriguing similarities of the present partition function with the semiclassical partition functions computed using the Poisson duality and the instanton-dyon liquid approach in supersymmetry (SUSY) Yang-Mills theory (see [115–118] and references therein). Let us consider first the case in which $\frac{L_r}{L_\phi}$ is very small ($\frac{L_r}{L_\phi} \ll 1$) so that Σ_3 can be neglected [see Eq. (95)]. In this case (which corresponds to a box that is much longer in the ϕ direction than in the r direction), if one analyzes Eq. (11) at page 5 of Ref. [115], one can see that the label k (and the corresponding sum) of $Z_{\rm inst}$ in that reference is analogous to the sum over n in Eq. (100) in the present approach, as k appears linearly in the exponent of $Z_{\rm inst}$ as n in Eq. (100). On the other hand, the label n of $Z_{\rm inst}$ (and the corresponding sum) in Eq. (11) of Ref. [115] is analogous to our topological sums over p since the label n appears quadratically in the exponent of $Z_{\rm inst}$ of Ref. [115], as p in our case. The only two relevant differences between the present expressions and $Z_{\rm inst}$ are the following. First, in the sum in Eq. (11) of Ref. [115], there is the factor $(\beta/g^2)(k^3/\beta M)^3$, where M is defined below Eq. (11) of Ref. [115], while in our case we have the degeneracy factor $\delta(n)$. The factor arises from the one-loop effects around the instantons and can be computed explicitly thanks to the powerful results made available by SUSY, which are basic building blocks in the approach introduced in [116,117]. However, in the low-energy/temperature limit of QCD, there is no SUSY, so the computations of one-loop effects are far more complicated. It is worth reminding the reader that each term in the expansion in Eq. (93) corresponds to an exact solution of the Skyrme field equations (with energy and baryon densities depending on all three spatial coordinates in a nontrivial way), so that, for any fixed n in Eq. (100), one should compute the corresponding one-loop determinant around this nontrivial nonsupersymmetric background. This fact, together with the lack of SUSY, makes the computation of this one-loop determinant unfeasible in our case. Second, in the present approach, we have introduced a cutoff on the sum over n, as the Skyrme model is not valid anymore at very high temperature/energies, while SUSY Yang-Mills theory is well behaved in the UV. Despite these differences, we find the similarities between the two approaches quite striking. On the other hand, if $\frac{L_r}{L_\phi}$ is not very small, then the thermodynamical behavior of the present families of topologically nontrivial configurations will deviate from the predictions of the instanton-dyon liquid approach (when the adimensional parameter $\frac{L_r}{L_\phi}$ plays a key role). The very rich but quite complicated phase diagram associated with these families will be analyzed in a future publication.

The idea of the present section is to provide sound pieces of evidence that the families of topologically nontrivial configurations constructed in the previous sections have a reasonable thermodynamical behavior. In order to get an idea of the thermodynamical behavior of these modulated topological solitons, we can approximate the sums in Eq. (100) by integrals [in the limit in which $\frac{L_r}{L_\phi} \ll 1$, so that Σ_3 can be neglected; see Eq. (95)], arriving at the following formula:

$$
\begin{aligned}
Z_{GP}(\tilde{\mu}_B, T) &= \exp(-\beta\Sigma_1) \int_1^{\Delta(T,\mu_B)+1} dn\, \delta(n) \exp(-\beta\Sigma_4 n) \int_{-\infty}^{+\infty} dp\, \exp(-p^2 \beta(\Sigma_5 n - \tilde{\mu}_B) - \beta\Sigma_3 p^4) \\
&= \frac{\exp(-\beta\Sigma_1)}{2\sqrt{\Sigma_3}} \int_1^{\Delta(T,\mu_B)+1} dn\, \delta(n) \exp\left(\beta\left[\frac{(n\Sigma_5 - \tilde{\mu}_B)^2}{8\Sigma_3} - \Sigma_4 n\right]\right) \sqrt{n\Sigma_5 - \tilde{\mu}_B}\, K_{1/4}\left(\frac{\beta(n\Sigma_5 - \tilde{\mu}_B)^2}{8\Sigma_3}\right), \qquad (101)
\end{aligned}
$$

where $\tilde{\mu}_B := \mu_B - \Sigma_2$, $K_n(z)$ denotes the modified Bessel function of the second kind, and $\delta(n)dn$ gives the number of states with energies between n and $n + dn$. Note that the condition $\Sigma_2 + \Sigma_5 > \mu_B$ must be fulfilled. Also, we have introduced a $+1$ in the upper integration limit of n for numerical analysis reasons.

The integral in Eq. (101) cannot be computed exactly. Since we have to evaluate it numerically, we can consider the following generalized form of $\delta(n)$. For this section, let us consider a modified expression of Eq. (98) as follows:

$$\delta(n) \sim \frac{1}{4\sqrt{3}} \frac{\exp\left(\sqrt{\frac{2n}{3}}\pi\right)}{(n^a + b^2)}, \qquad \text{as } n \to \infty, \qquad a, b \in \mathbb{R}_{>0}. \tag{102}$$

The original formula in Eq. (98) recovers by setting $a = 1$ and $b = 0$. Substituting Eq. (102) into Eq. (101), we get

$$Z_{GP}(\tilde{\mu}_B, T) \approx \frac{\exp\left(-\beta\Sigma_1 + \sqrt{\frac{2n}{3}}\pi\right)}{8\sqrt{3}} \int_1^{\Delta(T,\mu_B)+1} dn\, f(n, \mu_B), \tag{103}$$

where

$$f(n, \mu_B) = \frac{\sqrt{n\Sigma_5 - \tilde{\mu}_B}}{\sqrt{\Sigma_3}(n^a + b^2)} \exp\left(\beta\left[\frac{(n\Sigma_5 - \tilde{\mu}_B)^2}{8\Sigma_3} - \Sigma_4 n\right]\right) \times K_{1/4}\left(\frac{\beta(\tilde{\mu}_B - n\Sigma_5)^2}{8\Sigma_3}\right). \tag{104}$$

By considering the expansion $L_r/L_\phi \ll 1$, the last function reduces to

$$f(n, \mu_B) \approx \frac{2\sqrt{\pi}e^{-n\beta\Sigma_4}}{(n^a + b^2)\sqrt{\beta(\tilde{\mu}_B - n\Sigma_5)}}, \qquad \text{as } \Sigma_3 \ll 1. \tag{105}$$

The partition function in Eq. (103) with the expansion $\Sigma_3 \ll 1$ allows one to extract different thermodynamical properties of the present families of "dressed" topological solitons: we will compare the results obtained from the above partition function with the available numerical results from LQCD. Before doing that, we should note that the explicit dependency on the temperature in the limit of integration through the cutoff $\Delta(T, \mu_B)$ can also be considered as a modification in the Hamiltonian with additional explicitly T-dependent terms. As it was shown by Gorenstein and Yang [121], this kind of modification produces specific changes in some thermodynamics functions. At finite chemical potential, a generalization of the solution of Gorenstein and Yang modifies the entropy S and the internal energy U as [122]

$$S'(V, T, \Delta(T, \mu_B)) \equiv S(V, T, \Delta(T, \mu_B)) - \frac{\partial\Delta}{\partial T}\left(\frac{\partial A}{\partial\Delta}\right)_{V,T}, \tag{106}$$

$$U'(V, T, \Delta(T, \mu_B)) \equiv U(V, T, \Delta(T, \mu_B)) - T\frac{\partial\Delta}{\partial T}\left(\frac{\partial A}{\partial\Delta}\right)_{V,T}, \tag{107}$$

where $A(V, T)$ is identified with the free energy obtained from the standard formula

$$A(\mu_B, T, V) = -T \log Z_{GP}(\mu_B, T, V). \tag{108}$$

All the other thermodynamics functions are unchanged and can be found from $A(\mu_B, T, V)$ using the standard thermodynamics relations. In order to compare our simulations with LQCD, we are particularly interested in computing the pressure

$$P = -\left(\frac{\partial A}{\partial V}\right)_T, \tag{109}$$

with $V = 8\pi^3 L_r L_\theta L_\phi$ being the finite volume. In the limit of a T-independent Hamiltonian, the last term in Eq. (106) has to be zero, so that the standard expressions of the statistical mechanics are recovered.

As is well known, perturbative QCD calculations should describe, at extremely high temperatures and chemical potential, the quarks and gluons degrees of freedom: the quark-gluon plasma. To make our results comparable with perturbative QCD computations, at those energies, we will also add to the pressure and entropy the perturbative terms, to order g^2, computed in the perturbative QCD approach [123,124], given by

$$\begin{aligned} S'(V, T, \Delta(T, \mu_B)) &= S(V, T, \Delta(T, \mu_B)) - \frac{\partial\Delta}{\partial T}\left(\frac{\partial A}{\partial\Delta}\right)_{V,T} \\ &\quad + \left(\frac{\pi^2(7N_cN_f + 4N_g)}{45} - \frac{N_g(4N_c + 5N_f)}{144}g^2\right)T^3, \end{aligned} \tag{110}$$

$$P = -\left(\frac{\partial A}{\partial V}\right)_T + \left(\frac{\pi^2(7N_cN_g + 4N_g)}{180} - \frac{N_g(4N_c + 5N_f)}{576}g^2\right)T^4, \tag{111}$$

where g is the coupling constant of QCD, $N_g = (N_c^2 - 1)$, N_c is the number of colors, and N_f is the number of flavors.

A possible choice of the cutoff $\Delta(T, \mu_B)$ at a fixed chemical potential that allows one to compare closely our results to LQCD data is

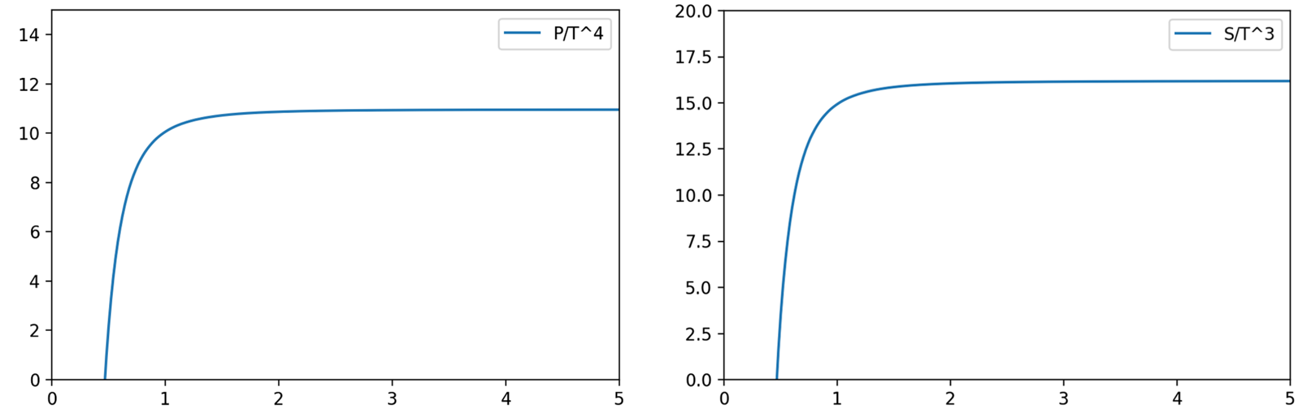

FIG. 3. Plots of the pressure normalized by T^4 and the entropy normalized by T^3 as functions of the temperature T. We run our simulations with the values $\mu_B = 0.1$, $V = 1$, $N_g = N_c^2 - 1$, $N_c = 3, N_f = 2, g = 0.1, a = 1, b = 0$.

$$\Delta(T, \mu_B) = 0.050 \times \log(0.712 - 2.148T + 0.294T^2 + 79.971T^{-1} + 14.254T^{-2} + 25.413e^{-1.605T}), \tag{112}$$

where we have fixed $\mu_B = 0.1$. In the present case, the temperature has energy units (our energy unit is 100 MeV, as is common in QCD), so that the first term inside the logarithmic is dimensionless, the second coefficient has inverse energy units $[2.148 \times (100 \text{ MeV})^{-1}]$, and so on. On the other hand, it would be nice to determine the precise functional form of $\Delta(T, \mu_B)$ from first principles: we hope to come back to this interesting issue in a future publication. In the meantime, our aim is only to show that the partition function associated with the configurations described in the previous sections can be relevant as simple choices of $\Delta(T, \mu_B)$ and give rise to good qualitative agreement with LQCD.

Now, one of the primary thermodynamic observables that we compute is the pressure P according to the formula in Eq. (110). With the expression of $\Delta(T, \mu_B)$ above, the starting value for P/T^4 at $T_{\text{in}} \equiv 0.480$ is given by $P_{\text{in}}/T_{\text{in}}^4 = 0.942$. The results at different values of T are shown in Fig. 3. The comparison of this plot can be made with those from Refs. [125–127]. Another crucial thermodynamic observable of our interest is the entropy S related to the pressure by basic thermodynamics relations. The starting point of the entropy at $T_{\text{in}} = 0.480$ is $S_{\text{in}}/T_{\text{in}}^3 = 1.679$. The entropy per unit of T^3 is shown in Fig. 3.

Clearly, these plots exhibit good qualitative agreement with the results of LQCD of those references. We will come back to a more detailed analysis of the low-temperature behavior of the present topologically nontrivial configurations in a future publication.

VI. CONCLUSIONS AND PERSPECTIVES

In the present work, we constructed exact and topologically nontrivial solutions of the Skyrme and Yang-Mills-Higgs theory at finite baryon density in $(3 + 1)$ dimensions. These analytic configurations are characterized both by two discrete labels (determining the baryonic charge) and by a massless chiral field F in $(1 + 1)$ dimensions (in the Yang-Mills-Higgs case, there are two chiral massless modes). Physically, the chiral massless modes characterize exact excitations on top of hadronic layers and tubes. Thus, nontrivial modes of F represent either hadronic tubes, which are not homogeneous along the axis of the tubes, or hadronic layers, which are not homogeneous in the directions tangent to the layers themselves. In other words, the chiral massless modes hosted in the topologically nontrivial configurations constructed in the previous sections represent "exact" excitations, since these chiral modes are not only small excitations on top of hadronic tubes or layers, but these configurations are exact solutions of the full Skyrme field equations with nontrivial topological density (the same is true in the Yang-Mills-Higgs case). This situation should be compared with the usual circumstances when one can only study small fluctuations around topological solitons as solutions of the linearized field equations around those solitons. Hence, these are the first exact analytic examples describing ordered arrays of $(3 + 1)$-dimensional topological solitons with nontrivial inhomogeneities. In the case of the Skyrme model, the plots of the energy and baryon densities of the two types of solutions show that these configurations are appropriate to describe inhomogeneous nuclear pasta states, where chiral modes modulate the tubes and layers.

From the technical viewpoint, the fact that the present approach can reduce the complete set of field equations of the Skyrme model in $(3 + 1)$ dimensions to the equation of a massless chiral field in $(1 + 1)$ dimensions (keeping alive the baryonic charge) opens the remarkable possibility to use tools from CFT in $(1 + 1)$ dimensions to analyze the low-temperature behavior of QCD. We have discussed the semiclassical grand canonical partition function associated with one of the present families. We have calculated (by approximating the partition function of the hadronic layer with a suitable one-dimensional integral) the pressure and the entropy, obtaining an excellent qualitative agreement with results from LQCD. Our results also allow discussing out-of-equilibrium features in the low-energy limit of QCD in $(3+1)$ dimensions using the well-established tools of two-dimensional CFT (see [128] and references therein). We will analyze these issues in a future publication.

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