Thermodynamic pressure for massless QCD and the trace anomaly

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(Received 20 September 2022; revised 18 October 2022; accepted 25 October 2022; published 17 November 2022)

From statistical mechanics the trace of the thermal average of any energy-momentum tensor is $\langle T^{\mu}{}_{\mu}\rangle = T\partial P/\partial T - 4P$. The renormalization group formula $\langle T^{\mu}{}_{\mu}\rangle = \beta(g_M)\partial P/\partial g_M$ for QCD with massless fermions requires the pressure to have the structure, $P = T^4 \sum_{n=0}^{\infty} \phi_n(g_M)[\ln(\frac{M}{4\pi T})]^n$, where the factor 4π is for later convenience. The functions $\phi_n(g_M)$ for $n \geq 1$ may be calculated from $\phi_0(g_M)$ using the recursion relation $n\phi_n(g_M) = -\beta(g_M)d\phi_{n-1}/dg_M$. This is checked against known perturbation theory results by using the terms of order $(g_M)^2$, $(g_M)^3$, $(g_M)^4$ in $\phi_0(g_M)$ to obtain the known terms of order $(g_M)^4$, $(g_M)^5$, $(g_M)^6$ in $\phi_1(g_M)$ and the known term of order $(g_M)^6$ in $\phi_2(g_M)$. The above series may be summed and gives the same result as choosing $M = 4\pi T$, viz. $T^4\phi_0(g_{4\pi T})$.

DOI: 10.1103/PhysRevD.106.105013

I. INTRODUCTION

For a symmetric energy-momentum tensor $T^{\mu\nu}$ the dilation current $S^{\mu} = T^{\mu\lambda}x_{\lambda}$ and the four conformal currents $K^{\alpha\mu} = x^2 T^{\alpha\mu} - 2x^{\alpha}T^{\mu\lambda}x_{\lambda}$ are conserved if the energy-momentum tensor is traceless:

$$\partial_{\mu}S^{\mu} = T^{\mu}{}_{\mu},$$
$$\partial_{\mu}K^{\alpha\mu} = -2x^{\alpha}T^{\mu}{}_{\mu}.$$

The classical energy-momentum tensor for QCD with massless fermions is traceless but quantum corrections introduce a renormalization scale that spoils the conservation of scale and conformal currents and renders the trace nonzero [1].

The trace of the thermally averaged energy-momentum tensor is $\langle T^{\mu}{}_{\mu}\rangle = u - 3P$ where $u = \langle T^0{}_0\rangle$ is the energy density and $P = -\sum_{j=1}^{3}\langle T^j{}_j\rangle/3$ is the pressure. The relation

$$\exp(\beta PV) = Z = \mathrm{Tr}\{e^{-\beta H}\}$$

between the pressure and the partition function implies that

$$\frac{\partial}{\partial\beta}(\beta P) = -\frac{\langle H\rangle}{V} = -u,$$

or equivalently

$$T\frac{\partial P}{\partial T} = u + P.$$

The trace of the energy-momentum tensor becomes

$$\langle T^{\mu}{}_{\mu}\rangle = u - 3P = T\frac{\partial P}{\partial T} - 4P. \tag{1}$$

For non-Abelian gauge fields with massless fermions the pressure has the form

$$P = T^4\Phi(g_M, M/T), \tag{2}$$

where M is the renormalization scale. From (1) the trace of the energy-momentum tensor is

$$\langle T^{\mu}{}_{\mu}\rangle = T^5\frac{\partial\Phi}{\partial T}. \tag{3}$$

One would expect the calculation of Φ to be primary and the trace anomaly only an afterthought. However, with the theorem of Drummond *et al.* [2] that

$$\langle T^{\mu}{}_{\mu}\rangle = \beta(g_M)\frac{\partial P}{\partial g_M} \tag{4}$$

the anomaly becomes predictive in that the combination of (3) and (4) gives

$$T\frac{\partial\Phi}{\partial T} = \beta(g_M)\frac{\partial\Phi}{\partial g_M}, \tag{5}$$

which is Eq. (3.11) of Drummond *et al.* [2].

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Note that (4) is similar to the zero temperature operator identity $T^{\mu}{}_{\mu} = \beta(g_M)\partial\mathcal{L}/\partial g_M$.

Section II shows how Eq. (5) ensures that P is independent of the renormalization scale M and requires P to have the structure shown in the abstract. Section III tests the recursion relation using known results for $\phi_0(g_M)$ from perturbation theory to calculate the three known terms in $\phi_1(g_M)$ and the only known term of $\phi_2(g_M)$ and illustrates how to improve perturbation theory.

II. STRUCTURE OF P

A. Independence of the renormalization scale M

As indicated in Eq. (2) the renormalization scale appears in Φ through g_M and through $r = M/T$. The full M derivative of Φ is

$$M\frac{d\Phi}{dM} = M\frac{dg_M}{dM}\frac{\partial\Phi}{\partial g_M}\bigg|_r + M\frac{dr}{dM}\frac{\partial\Phi}{\partial r}\bigg|_{g_M}. \tag{6}$$

In the first term use $Mdg_M/dM = \beta(g_M)$; in the second, use $Mdr/dM = r$ and $r\partial\Phi/\partial r = -T\partial\Phi/\partial T$ so that

$$M\frac{d\Phi}{dM} = \beta(g_M)\frac{\partial\Phi}{\partial g_M}\bigg|_r - T\frac{\partial\Phi}{\partial T}\bigg|_{g_M} = 0 \tag{7}$$

after using Eq. (5).

Comment: One can reverse the argument and derive the anomaly relation (4) of Drummond *et al.* [2] by starting with the assertion that P is a physical quantity and must therefore be independent of the renormalization scale.

B. Origin of $[\ln(M/T)]^n$

Since $\Phi(g_M, M/T)$ is independent of M, it must be only a function of $T/\Lambda_{\rm QCD}$. It is convenient to consider Φ as a function ϕ_0 of $\ln(\xi T/\Lambda_{\rm QCD})$, where ξ is some constant

$$\Phi(g_M, T/M) = \phi_0(\ln(\xi T/\Lambda_{\rm QCD})), \tag{8}$$

and to introduce variables

$$u = \ln(M/\Lambda_{\rm QCD}),$$
$$v = \ln(M/\xi T). \tag{9}$$

The running coupling is a function of u determined by $\beta(g_M) = dg_M/du$; Φ is a function of $u - v$:

$$\Phi(g_M, M/T) = \phi_0(u - v) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{d^n\phi_0(u)}{du^n}v^n, \tag{10}$$

after a Taylor series expansion. The definition

$$\phi_n(g_M) = \frac{(-1)^n}{n!}\frac{d^n\phi_0(g_M)}{du^n} \tag{11}$$

allows the series to be written as

$$\Phi(g_M, M/T) = \sum_{n=0}^{\infty}\phi_n(g_M)\left[\ln\left(\frac{M}{\xi T}\right)\right]^n. \tag{12}$$

The recursion relation $n\phi_n(g_M) = -d\phi_{n-1}/du$, which follows from (11), may be expressed as

$$\phi_n(g_M) = -\frac{1}{n}\beta(g_M)\frac{d\phi_{n-1}}{dg_M} \quad (n \geq 1). \tag{13}$$

One can confirm directly that the series (12) satisfies $d\Phi/dM = 0$.

Comment: If ξ is changed to ξ', then

$$\ln\left(\frac{M}{\xi T}\right) = \ln\left(\frac{M}{\xi' T}\right) + \ln\left(\frac{\xi'}{\xi}\right). \tag{14}$$

The binomial theorem allows the series (12) to be expressed in terms of powers of $\ln(M/\xi' T)$ with modified functions $\phi_n'(g_M)$.

Comment: From $u - 3P = T^5\partial\Phi/\partial T$ it follows that the energy density and entropy density are

$$u = T^4\left[3\Phi + T\frac{\partial\Phi}{\partial T}\right], \tag{15}$$

$$s = T^3\left[4\Phi + T\frac{\partial\Phi}{\partial T}\right]. \tag{16}$$

III. RESULTS FROM PERTURBATION THEORY

The $\mathcal{O}(g_M^2)$ term in P was calculated by Shuryak [3]; the $\mathcal{O}(g_M^3)$ term by Kapusta [4]; to this order there was no $\ln(M/T)$. The $\mathcal{O}(g_M^4)$ term was calculated by Arnold and Zhai [5]; the $\mathcal{O}(g_M^5)$ by Zhai and Kastening [6]; in both cases $\ln(M/T)$ appeared. The same result was obtained by Braaten and Nieto [7] using hard thermal loop resummation.

At $\mathcal{O}(g_M^6)$ nonperturbative magnetic screening effects arise [8–10]. Kajantie *et al.* [11] were able to calculate the $\mathcal{O}(g_M^6)$ perturbative terms and found both $\ln(M/T)$ and $\ln^2(M/T)$. A convenient reference that discusses all the results is Sec. 8.4 of Kapusta and Gale [12].

A. Checks against known results

For comparison with the published results from perturbation theory it is convenient to insert a prefactor in the series expression for the pressure and choose $\xi = 4\pi$:

$$P = \frac{\pi^2 d_A}{9} T^4 \sum_{n=0}^{\infty} \phi_n(g_M)\left[\ln\left(\frac{M}{4\pi T}\right)\right]^n, \qquad (17)$$

where d_A is the dimension of the adjoint representation.

With the order $(g_M)^2$, $(g_M)^3$, and $(g_M)^4$ terms of $\phi_0(g_M)$, the recursion relation (13) gives the first three terms of $\phi_1(g_M)$ and the first term of $\phi_2(g_M)$. Using the notation $\phi_n^{(k)}(g_M)$ for the $\mathcal{O}(g_M)^k$ term in $\phi_n(g_M)$ the necessary inputs are

$$\phi_0^{(2)}(g_M) = -\left(\frac{g_M}{4\pi}\right)^2\left(C_A + \frac{5}{2}S_F\right),$$
$$\phi_0^{(3)}(g_M) = \left(\frac{g_M}{4\pi}\right)^3 (C_A + S_F)^{3/2} 16/\sqrt{3},$$
$$\phi_0^{(4)}(g_M) = \left(\frac{g_M}{4\pi}\right)^4 \{48 C_A (C_A + S_F)\ln W + R\},$$

where $W = (g_M/2\pi)\sqrt{(C_A + S_F)/3}$ and

$$R = C_A^2 R_1 + C_A S_F R_2 + S_F^2 R_3 + S_{2F} R_4. \qquad (18)$$

The coefficients R_j are given in [5,12] in terms of Riemann zeta functions and the Euler constant. For later comparison with [11] it is convenient to employ the approximate numerical values

$$R_1 = 79.2626, \qquad R_2 = 18.9212,$$
$$R_3 = -0.6914, \qquad R_4 = 9.6145. \qquad (19)$$

The standard notation [12] for SU(N) with n_f fermions in the fundamental representation is $d_A = N^2 - 1$, $C_A = N$, $d_F = N n_f$, $S_F = n_f/2$, $S_{2F} = (N^2 - 1) n_f / 4N$. The first two terms in the beta function are

$$\beta(g_M) = -\beta_0 g_M^3 - \beta_1 g_M^5 + \cdots,$$
$$\beta_0 = \left(\frac{11}{3}C_A - \frac{4}{3}S_F\right)/(4\pi)^2,$$
$$\beta_1 = \left(\frac{34}{3}C_A^2 - \frac{20}{3}C_A S_F - 4 S_{2F}\right)/(4\pi)^4. \qquad (20)$$

The predictions of the recursion relation (13) are

$$A. \quad \phi_1^{(4)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M}\phi_0^{(2)}(g_M),$$
$$B. \quad \phi_1^{(5)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M}\phi_0^{(3)}(g_M),$$
$$C. \quad \phi_1^{(6)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M}\phi_0^{(4)}(g_M) + \beta_1 g_M^5 \frac{d}{dg_M}\phi_0^{(2)}(g_M),$$
$$D. \quad \phi_2^{(6)}(g_M) = \frac{1}{2}\beta_0 g_M^3 \frac{d}{dg_M}\phi_1^{(4)}(g_M). \qquad (21)$$

The result for A,

$$\phi_1^{(4)}(g_M) = \left(\frac{g_M}{4\pi}\right)^4\left\{-C_A^2\frac{22}{3} - C_A S_F\frac{47}{3} + S_F^2\frac{20}{3}\right\}, \qquad (22)$$

agrees with [5–7,11].

The result for B,

$$\phi_1^{(5)}(g_M) = \left(\frac{g_M}{4\pi}\right)^5\left(\frac{C_A + S_F}{3}\right)^{1/2} \times (C_A^2 176 + C_A S_F 112 - S_F^2 64), \qquad (23)$$

agrees with [6,7,11].

The result for C is

$$\phi_1^{(6)}(g_M) = 4\left(\frac{g_M}{4\pi}\right)^6\left\{\left(\frac{11}{3}C_A - \frac{4}{3}S_F\right)R + \left(C_A + \frac{5}{2}S_F\right)\left(-\frac{17}{3}C_A^2 + \frac{10}{3}C_A S_F + 2S_{2F}\right) + \left(\frac{11}{3}C_A - \frac{4}{3}S_F\right)C_A(C_A + S_F)(12 + 48\ln W)\right\}. \qquad (24)$$

To compare this with [11] it is necessary to evaluate (24) for SU(3):

$$\phi_1^{(6)}(g_M) = 4\left(\frac{g_M}{4\pi}\right)^6\left\{432\left(11 - \frac{2}{3}n_f\right)\left(1 + \frac{1}{6}n_f\right)\ln W + 1035 + \frac{325}{4}n_f - \frac{49}{12}n_f^2 + \left(11 - \frac{2}{3}n_f\right)R\right\}. \qquad (25)$$

Substituting the numerical values of R gives the final result

$$\phi_1^{(6)}(g_M) = 4\left(\frac{g_M}{4\pi}\right)^6\left\{432\left(11 - \frac{2}{3}n_f\right)\left(1 + \frac{1}{6}n_f\right)\ln W + 8882 - 11.6186 n_f - 29.1767 n_f^2 + 0.1152 n_f^3\right\}. \qquad (26)$$

In [11] the $\mathcal{O}(g_M^6)$ results are expressed in terms of $(\alpha_M/\pi)^3$ and $\ln(M/2\pi T)$. When [11] is reexpressed in terms of $(g_M/4\pi)^6$ and $\ln(M/4\pi T)$, it agrees completely with Eq. (26).

The final calculation D gives

$$\phi_2^{(6)}(g_M) = -\left(\frac{g_M}{4\pi}\right)^6 4\left(C_A + \frac{5}{2}S_F\right)\left(\frac{11}{3}C_A - \frac{4}{3}S_F\right)^2. \quad (27)$$

For SU(3) with n_f multiplets of fermions

$$\phi_2^{(6)}(g_M) = -\left(\frac{g_M}{4\pi}\right)^6 1452\left(1 + \frac{5}{12}n_f\right)\left(1 - \frac{2}{33}n_f\right)^2, \quad (28)$$

which is exactly the same as [11].

B. Improving perturbation theory

At order $(g_M)^6$ nonperturbative effects appear in $\phi_0^{(6)}(g_M)$ but not in $\phi_1^{(6)}(g_M)$ or $\phi_2^{(6)}(g_M)$ calculated above. The argument of Linde [8,9,12] shows that certain diagrams that appear to be of order $(g_M)^k$ with $k > 6$ are so infrared sensitive that nonperturbative magnetic shielding will render them of order $(g_M)^6$. Thus $\phi_0^{(6)}(g_M)$ receives contributions from diagrams with infinitely many loops. Nevertheless $\phi_0(g_M)$ is still a series of the form

$$\phi_0(g_M) = \sum_{k=0}^{\infty} \phi_0^{(k)}(g_M). \quad (29)$$

The $k = 1$ term vanishes; the $k = 2$ term is the first to depend on g_M. Because the beta function begins with $(g_M)^3$, the recursion relation (13) implies that $\phi_0^{(k)}(g_M)$ will generate terms of order $(g_M)^{2n+k}[\ln(M/4\pi T)]^n$. The series (17) for P may be considered a double series:

$$P = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \phi_n^{(2n+k)}(g_M)\left[\ln\left(\frac{M}{4\pi T}\right)\right]^n. \quad (30)$$

Perturbative calculations through order $(g_M)^5$ determine $\phi_n^{(2n+k)}(g_M)$ for $2n + k \leq 5$:

$$P_{[n]}^{(k\leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5}\sum_{n=0}^{\frac{1}{2}(5-k)} \phi_n^{(2n+k)}(g_M)\left[\ln\left(\frac{M}{4\pi T}\right)\right]^n. \quad (31)$$

The difference between $P_{[n]}^{(k\leq 5)}$ and $P_{[n]}^{k\leq 4}$ is not small [6,7,13].

There is no need to terminate the sum over n; one can easily compute the full sum

$$P^{(k\leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5}\sum_{n=0}^{\infty} \phi_n^{(2n+k)}(g_M)\left[\ln\left(\frac{M}{4\pi T}\right)\right]^n. \quad (32)$$

The input is of the form

$$\phi_0^{(k)}(g_M) = \left(\frac{g_M}{4\pi}\right)^k \left\{A_k + B_k \ln\left[\frac{g_M}{2\pi}\sqrt{(C_A + S_F)/3}\right]\right\}, \quad (33)$$

where $A_1 = 0$ and B_4 is the only nonzero B_k for $k \leq 5$. As before, define $u = \ln(M/\Lambda_{\mathrm{QCD}})$. At large M, one can use $(g_M)^2 = [\beta_0 u]^{-1}$ and the parametrization

$$\phi_0^{(k)}(g_M) = \frac{1}{u^{k/2}}(a_k + b_k \ln u). \quad (34)$$

The nth order derivatives of $\phi_0(g_M)$ required by Eq. (11) give

$$\phi_n^{(2n+k)}(g_M) = \frac{1}{u^{k/2+n}}\left[a_k S_n - 2\frac{dS_n}{dk}b_k + S_n b_k \ln u\right],$$
$$S_n = \frac{\Gamma(n + k/2)}{n!\Gamma(k/2)}. \quad (35)$$

With $v = \ln(M/4\pi T)$ Eq. (10) requires the sum

$$\sum_{n=0}^{\infty} S_n \left(\frac{v}{u}\right)^n. \quad (36)$$

By the ratio test this sum converges for $|v/u| < 1$, which is satisfied provided $M > \sqrt{4\pi T \Lambda_{\mathrm{QCD}}}$ and $4\pi T > \Lambda_{\mathrm{QCD}}$. The result is

$$\sum_{n=0}^{\infty} S_n \left(\frac{v}{u}\right)^n = \left[1 - \frac{v}{u}\right]^{-k/2}. \quad (37)$$

Applying d/dk as required in (35) gives

$$P^{(k\leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5} \frac{1}{(u-v)^{k/2}}[a_k + b_k \ln(u - v)]. \quad (38)$$

The dependence on the renormalization scale M disappears since $u - v = \ln(4\pi T/\Lambda_{\mathrm{QCD}})$. When a_k and b_k are expressed in terms of A_k, B_k, and $u - v = (\beta_0 g_{4\pi T})^{-1}$, the result is

$$P^{(k\leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5}\left(\frac{g_{4\pi T}}{4\pi}\right)^k \times \left\{A_k + B_k \ln\left[\frac{g_{4\pi T}}{2\pi}\sqrt{(C_A + S_F)/3}\right]\right\}, \quad (39)$$

or more concisely

$$P^{(k\leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5} \phi_0^{(k)}(g_\mu)\Big|_{\mu=4\pi T}. \quad (40)$$

In short, convergence of the infinite sum on n in (30) is automatic; whether a finite number of $\phi_0^{(k)}(g_M)$ in the series for (29) for $\phi_0(g_M)$ is a good approximation, i.e., whether perturbation theory is reliable, is an open question [13].

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