Coefficients at powers of logarithms in the higher-derivatives and minimal-subtractions-of-logarithms renormalization scheme

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For renormalizable theories with a single coupling constant regularized by higher derivatives we investigate the coefficients at powers of logarithms present in the renormalization constants assuming that divergences are removed by minimal subtractions of logarithms. According to this higher-derivatives and minimal-subtractions-of-logarithms (HD + MSL) renormalization prescription, the renormalization constants include only powers of $\ln \Lambda/\mu$, where Λ and μ are the dimensionful regularization parameter and the renormalization point, respectively. We construct general explicit expressions for arbitrary coefficients at powers of this logarithm present in the coupling constant renormalization and in the field renormalization constant which relate them to the β-function and (in the latter case) to the corresponding anomalous dimension. To check the correctness, we compare the results with the explicit four-loop calculation made earlier for $\mathcal{N} = 1$ supersymmetric quantum electrodynamics and (for the supersymmetric case) rederive a relation between the renormalization constants following from the Novikov, Shifman, Vainshtein, and Zakharov equation.

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I. INTRODUCTION

Quantum corrections in most field theory models are divergent in the ultraviolet region. In renormalizable theories these divergences can be eliminated by redefining coupling constants, fields, masses, and gauge parameters [1]. However, the renormalization procedure admits a certain degree of arbitrariness. The renormalization constants are defined ambiguously because they depend on a regularization and a subtraction scheme. For instance, in the case of using dimensional regularization [2–5] ultraviolet divergences have the form of ε poles, and one of the simplest renormalization prescriptions (the minimal subtraction (MS) scheme) is to include only pole terms into the renormalization constants [6]. It is important that various divergent contributions to the renormalization constants are not independent [6]. Namely, the coefficients at higher ε poles in a certain order of the perturbation theory are determined by the coefficients at the lower poles in the previous orders. The corresponding recurrence relations often referred to as the pole equations reveal that all coefficients at the higher order poles can eventually be expressed in terms of the residua of the simple poles. The latter, in turn, are totally determined by the renormalization group functions (RGFs), i.e., by the β-functions and the anomalous dimensions.

The pole equations have also been generalized to different cases including nonrenormalizable theories [7–11]. Similar relations can be written for logarithms in the renormalized Green's functions [12]. In the case of using regularizations formulated in $D = 4$ [13] relations between various divergences analogous to those following from the pole equations should also appear, but, to the best of our knowledge, they have been little studied in literature. However, in certain situations the use of the dimensional technique can be inappropriate. For example, dimensional regularization explicitly breaks supersymmetry [14], while dimensional reduction [15] in its mathematically consistent version causes the loss of supersymmetry in higher orders of perturbation theory [16–19]. On the other hand, an invariant regularization of the supersymmetric theories can be constructed with the help of the higher (covariant) derivative method [20,21]. It can be formulated explicitly in terms of the $\mathcal{N} = 1$ superfields [22,23] and in $\mathcal{N} = 2$ harmonic superspace [24]. For a long time it was believed that the higher derivative regularization is ill-suited for multiloop calculations, mainly due to significant complication of

Feynman rules for vertices. Despite this, a lot of such calculations (see, e.g., [25–33]) for $\mathcal{N}=1$ supersymmetric gauge theories have been made to date. Moreover, the higher covariant derivative regularization appeared to be a very important ingredient of the all-loop perturbative derivation of the Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function [34–37] for $\mathcal{N}=1$ supersymmetric gauge theories made in [38–41]. It was shown [41] that some all-loop NSVZ schemes (i.e., the ones in which the NSVZ β-function is valid in all orders) in non-Abelian cases are given by the higher-derivatives and minimal-subtractions-of-logarithms (HD + MSL) prescription [42]. This means that a theory is regularized by higher covariant derivatives and a renormalization procedure is performed with the help of minimal subtractions of logarithms [43,44]. Thus, the HD + MSL scheme is an obvious analog of the MS scheme in the case of using the dimensional technique, but instead of ε poles the renormalization constants include only powers of $\ln\Lambda/\mu$, where Λ is a dimensionful parameter of the higher derivative regularization and μ is a normalization point. Therefore, it is reasonable to assume that in the HD + MSL scheme it is possible to write the relations between the coefficients at powers of $\ln\Lambda/\mu$ similar to those that follow from the pole equations. Obtaining these relations in the HD + MSL scheme from the renormalization group equations (see, e.g., [1,12]) is the main purpose of this paper. We are going to investigate them in the context of both coupling constant and field renormalizations.

The paper is organized as follows. In Sec. II we describe the renormalization of the coupling constant and introduce the HD + MSL boundary conditions. The explicit expressions for the coefficients at powers of $\ln\Lambda/\mu$ in the coupling constant renormalization (depending on the coefficients in the perturbative expansion of the β-function) are derived in Sec. III. Similar expressions for the coefficients in the field renormalization are obtained in Sec. IV. Some examples are considered in Sec. V. In particular, in Sec. V A we compare the general expressions for the coefficients with the known results for $\mathcal{N}=1$ supersymmetric quantum electrodynamics (SQED) in the four-loop approximation for the coupling constant renormalization and in the three-loop approximation for the matter superfield renormalization. Next, in Sec. V B we derive the equation relating the renormalizations of the coupling constant and of the matter superfields following from the NSVZ equation for $\mathcal{N}=1$ SQED. The similar investigation for non-Abelian $\mathcal{N}=1$ supersymmetric gauge theories with a single coupling is made in Sec. V C. In the Appendix we present expressions for the coefficients at powers of $\ln\Lambda/\mu$ which appear in the functions $\ln\alpha/\alpha_0$ and $(\alpha/\alpha_0)^S$ (for an arbitrary real number S), where α and α_0 are the renormalized and bare coupling constants, respectively.

II. THE COUPLING CONSTANT RENORMALIZATION AND THE HD + MSL PRESCRIPTION

We consider a $4D$ renormalizable gauge theory with a single dimensionless coupling constant $\alpha\equiv e^2/4\pi$, assuming that all divergences are logarithmical and the higher derivative method [20,21] is used for the regularization. This regularization is introduced by adding a term with higher powers of (covariant) derivatives to the Lagrangian. As a result, higher powers of momenta appear in the denominators of the propagators ensuring the (superficial) ultraviolet convergence of the integrals coming from all diagrams beyond the one-loop approximation. According to [45], the remaining one-loop divergences and subdivergences can be regularized by inserting certain Pauli-Villars determinants into the generating functional. (The form of the Pauli-Villars determinants in the supersymmetric case can be found in [46,47].) We always assume that the masses of the Pauli-Villars fields M_I are proportional to the parameter Λ present in the higher derivative term, and the ratios M_I/Λ do not depend on the coupling constant. This implies that the only independent dimensionful parameter Λ is present in the regularized theory.

Divergences appearing in quantum corrections are absorbed into the renormalization of coupling constant, fields, and other parameters. In particular, the renormalization of the coupling constant is written as

$$\frac{1}{\alpha_0}=\frac{Z_\alpha}{\alpha}, \tag{1}$$

where the bare coupling constant is marked by the subscript 0 and $Z_\alpha=Z_\alpha(\alpha,\ln\Lambda/\mu)$ is the charge renormalization constant. The relation between the renormalized and bare coupling constants can be presented in the form

$$\frac{1}{\alpha}=\frac{1}{\alpha_0}+B_{1,0}+B_{1,1}\ln\frac{\Lambda}{\mu}+\sum_{n=2}^{\infty}\alpha_0^{n-1}\sum_{m=0}^{n-1}B_{n,m}\ln^m\frac{\Lambda}{\mu}, \tag{2}$$

where $B_{n,m}$ are numerical coefficients. Note that these coefficients are not uniquely defined due to their dependence on a subtraction scheme. The HD + MSL renormalization prescription [42] can be defined by the boundary conditions

$$Z_i(\alpha_0,\ln\Lambda/\mu=0)=1, \tag{3}$$

where Z_i denotes all renormalization constants of the theory under consideration. In particular, from the boundary condition for Z_α we obtain

$$\alpha(\alpha_0,\ln\Lambda/\mu=0)=\alpha_0. \tag{4}$$

This implies that in the HD + MSL scheme

$$B_{n,0}=0,\qquad n\geq 1. \tag{5}$$

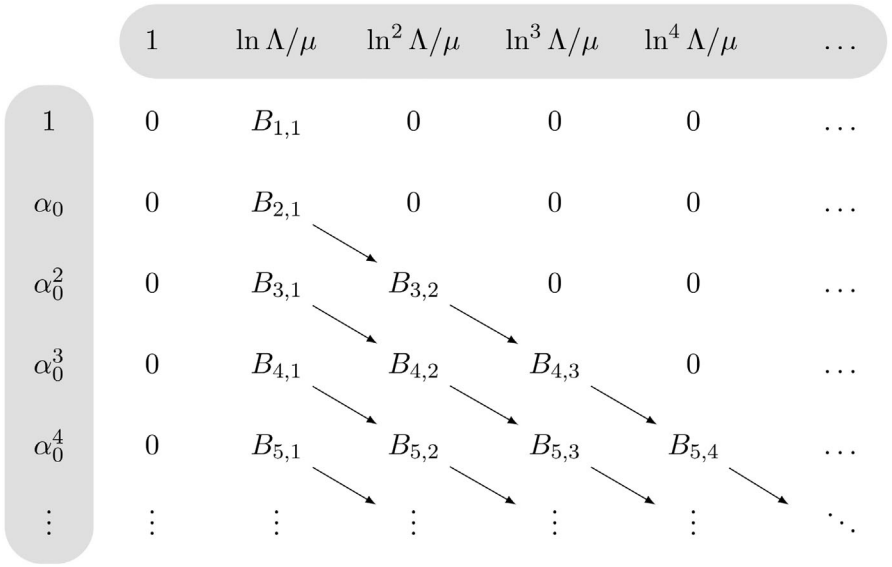

FIG. 1. The coefficients $B_{n,m}$ in the coupling constant renormalization [see Eq. (2)] in the HD + MSL scheme. For $m \geq 2$ they are not independent and can be expressed in terms of the coefficients $B_{n,1}$.

Similar to the pole equations valid in the MS scheme in the case of using the dimensional technique, we expect that the coefficients at higher powers of $\ln\Lambda/\mu$ are determined by the ones at the lower powers in the previous orders of perturbation theory and, eventually, by the coefficients at $\ln^1\Lambda/\mu$ [12]. For the considered HD + MSL scheme this is schematically depicted in Fig. 1. The coefficients $B_{n+1,n}$ with $n \geq 1$ (of the terms containing $\alpha_0^n \ln^n\Lambda/\mu$) correspond to the leading logarithm approximation and form the main diagonal. Based on the analogy with the pole equations we expect that for $n \geq 2$ these coefficients can be expressed in terms of $B_{2,1}$ and $B_{1,1}$. The second diagonal is formed by the coefficients $B_{n+2,n}$ with $n \geq 1$ of the terms containing $\alpha_0^{n+1} \ln^n\Lambda/\mu$. For this diagonal the coefficients at higher powers of $\ln\Lambda/\mu$ appear to be determined by $B_{3,1}$, $B_{2,1}$, and $B_{1,1}$. In the next section we will show how this can be derived for an arbitrary coefficient of any diagonal and present the corresponding explicit expressions.

III. COEFFICIENTS IN THE COUPLING CONSTANT RENORMALIZATION

In this paper we will (standardly) define RGFs in terms of the renormalized coupling constant. In particular, the β-function is given by the expression

$$\beta(\alpha) \equiv \left.\frac{d\alpha(\alpha_0, \ln\Lambda/\mu)}{d\ln\mu}\right|_{\alpha_0=\text{const}}, \tag{6}$$

where the derivative is taken at a fixed value of the bare coupling constant. Note that in the HD + MSL scheme RGFs defined in terms of the renormalized couplings coincide with the ones defined in terms of the bare couplings after a formal replacement $\alpha \to \alpha_0$, see [42] for details.

Differentiating Eq. (2) with respect to $\ln\mu$ and using the definition (6) we obtain the equation

$$\frac{\beta(\alpha)}{\alpha^2} = -\left.\frac{d}{d\ln\mu}\left(\frac{1}{\alpha}\right)\right|_{\alpha_0=\text{const}} = B_{1,1} + \sum_{n=2}^{\infty}\alpha_0^{n-1}\sum_{m=1}^{n-1} m B_{n,m}\ln^{m-1}\frac{\Lambda}{\mu}. \tag{7}$$

Next, we substitute into this equation the standard perturbative expansion for the β-function

$$\beta(\alpha) = \sum_{n=1}^{\infty}\beta_n\alpha^{n+1} \tag{8}$$

and formally set $\Lambda = \mu$. Then, taking into account the boundary condition (4) it is possible to relate the coefficients of the β-function to the coefficients of the series (2) in the HD + MSL scheme,

$$\beta_n = B_{n,1}, \qquad n \geq 1. \tag{9}$$

To express the remaining coefficients $B_{n,m}$ with $n > m \geq 2$ in terms of β_n, we substitute the expansion (8) into the left-hand side of Eq. (7) and differentiate the resulting equation $(p-1)$ times $(p \geq 2)$ with respect to $\ln\mu$ at a fixed value of the bare coupling constant,

$$\sum_{k=2}^{\infty}\beta_k\frac{d^{p-1}\alpha^{k-1}}{d\ln\mu^{p-1}}=(-1)^{p-1}\sum_{n=2}^{\infty}\alpha_0^{n-1}\sum_{m=1}^{n-1}m(m-1)\ldots(m-p+1)\theta[m-p]B_{n,m}\ln^{m-p}\frac{\Lambda}{\mu}. \tag{10}$$

Here

$$\theta[m-p]\equiv\begin{cases}0, & m<p,\\ 1, & p\geq m\end{cases} \tag{11}$$

is the Heaviside step function, and we took into account that the coefficients β_k do not depend on $\ln\Lambda/\mu$. It is convenient to represent the derivatives in the left-hand side of Eq. (10) using the chain rule,

$$\left.\frac{d}{d\ln\mu}\right|_{\alpha_0=\text{const}}=\frac{d\alpha}{d\ln\mu}\cdot\frac{d}{d\alpha}=\beta(\alpha)\frac{d}{d\alpha}=\sum_{n=1}^{\infty}\beta_n\alpha^{n+1}\frac{d}{d\alpha}. \tag{12}$$

With the help of this equation we obtain

$$\frac{d^{p-1}\alpha^{k-1}}{d\ln\mu^{p-1}}=\left(\sum_{k_{p-1}=1}^{\infty}\beta_{k_{p-1}}\alpha^{k_{p-1}+1}\frac{d}{d\alpha}\right)\left(\sum_{k_{p-2}=1}^{\infty}\beta_{k_{p-2}}\alpha^{k_{p-2}+1}\frac{d}{d\alpha}\right)\ldots\left(\sum_{k_1=1}^{\infty}\beta_{k_1}\alpha^{k_1+1}\frac{d}{d\alpha}\alpha^{k-1}\right), \tag{13}$$

where the derivatives with respect to α act on everything that stands to the right of them. After calculating these derivatives the considered expression can be written as

$$\begin{aligned}\frac{d^{p-1}\alpha^{k-1}}{d\ln\mu^{p-1}}=&\sum_{k_1=1}^{\infty}(k-1)\beta_{k_1}\sum_{k_2=1}^{\infty}(k-1+k_1)\beta_{k_2}\sum_{k_3=1}^{\infty}(k-1+k_1+k_2)\beta_{k_3}\ldots\\&\times\sum_{k_{p-1}=1}^{\infty}(k-1+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\alpha^{k-1+k_1+k_2+\cdots+k_{p-1}}.\end{aligned} \tag{14}$$

Next, we set $\Lambda=\mu$ in Eq. (10) and use the HD + MSL boundary condition (4). Then only terms with $m=p$ in the right-hand side of Eq. (10) survive, so that

$$\begin{aligned}&\sum_{k=2}^{\infty}\beta_k\sum_{k_1=1}^{\infty}(k-1)\beta_{k_1}\sum_{k_2=1}^{\infty}(k-1+k_1)\beta_{k_2}\sum_{k_3=1}^{\infty}(k-1+k_1+k_2)\beta_{k_3}\ldots\sum_{k_{p-1}=1}^{\infty}(k-1+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\alpha_0^{k-1+k_1+k_2+\cdots+k_{p-1}}\\&=(-1)^{p-1}p!\sum_{n=1}^{\infty}\alpha_0^{n+p-1}B_{n+p,p}.\end{aligned} \tag{15}$$

Equating the coefficients at the same powers of α_0 in Eq. (15) we obtain the explicit expression for $B_{n+p,p}$ with $n\geq 1$ and $p\geq 2$,

$$\begin{aligned}B_{n+p,p}=&(-1)^{p-1}\frac{1}{p!}\sum_{k=2}^{n+1}\beta_k\sum_{k_1}(k-1)\beta_{k_1}\sum_{k_2}(k-1+k_1)\beta_{k_2}\sum_{k_3}(k-1+k_1+k_2)\beta_{k_3}\ldots\\&\times\sum_{k_{p-1}}(k-1+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\Big|_{k_1+k_2+\cdots+k_{p-1}=n+p-k},\end{aligned} \tag{16}$$

where the sums over the indices k_i are taken only over the positive integers satisfying the condition $k_1+k_2+\cdots+k_{p-1}=n+p-k$. Besides, we set the upper limit of summation over k equal to $n+1$ since the higher values obviously do not meet this condition.

For the lowest values of n (corresponding to the main and two subleading diagonals) the coefficients (16) can be presented in a different form,

$$B_{1+p,p} = (-1)^{p-1}\frac{1}{p}\beta_1^{p-1}\beta_2; \tag{17}$$

$$B_{2+p,p} = (-1)^{p-1}(\beta_1^{p-1}\beta_3 + (H_p - 1)\theta[p-2]\beta_1^{p-2}\beta_2^2); \tag{18}$$

$$B_{3+p,p} = (-1)^{p-1}\frac{(p+1)}{2}\left\{\beta_1^{p-1}\beta_4 + 2(H_p - 1)\theta[p-2]\beta_1^{p-2}\beta_2\beta_3 + \left(\frac{p+3}{p+1} + H_{p+1}(H_{p+1} - 2) - H_{p+1,2}\right)\theta[p-3]\beta_1^{p-3}\beta_2^3\right\}, \tag{19}$$

where the Heaviside step function is given by Eq. (11), and the generalized harmonic numbers are defined as

$$H_{p,s} \equiv \sum_{k=1}^{p}\frac{1}{k^s}. \tag{20}$$

For $s = 1$ they correspond to the usual harmonic numbers $H_p \equiv H_{p,1}$. Some particular values of the generalized harmonic numbers are presented in Table I.

Then, for the lowest coefficients in the second and third subdiagonals in the coupling constant renormalization we get

$$\begin{aligned}
B_{4,2} &= -\beta_1\beta_3 - \frac{1}{2}\beta_2^2;\\
B_{5,2} &= -\frac{3}{2}\beta_1\beta_4 - \frac{3}{2}\beta_2\beta_3;\\
B_{5,3} &= \beta_1^2\beta_3 + \frac{5}{6}\beta_1\beta_2^2;\\
B_{6,3} &= 2\beta_1^2\beta_4 + \frac{10}{3}\beta_1\beta_2\beta_3 + \frac{1}{2}\beta_2^3;\\
B_{6,4} &= -\beta_1^3\beta_3 - \frac{13}{12}\beta_1^2\beta_2^2;\\
B_{7,4} &= -\frac{5}{2}\beta_1^3\beta_4 - \frac{65}{12}\beta_1^2\beta_2\beta_3 - \frac{35}{24}\beta_1\beta_2^3.
\end{aligned} \tag{21}$$

In the Appendix using the method described in this section we find the coefficients at powers of $\ln\Lambda/\mu$ in the expressions $\ln\alpha/\alpha_0$ and $(\alpha/\alpha_0)^S$ (for an arbitrary real number S). The results are given by Eqs. (A10) and (A11), respectively.

IV. COEFFICIENTS IN THE FIELD RENORMALIZATION

Similarly, it is possible to find explicit expressions for the coefficients in the field renormalization constants. Let us denote such a constant by Z. Then we can present $\ln Z$ as the series

$$\ln Z\left(\alpha\left(\alpha_0, \ln\frac{\Lambda}{\mu}\right), \ln\frac{\Lambda}{\mu}\right) = \sum_{n=1}^{\infty}\alpha_0^n\sum_{m=0}^{n}C_{n,m}\ln^m\frac{\Lambda}{\mu}. \tag{22}$$

The corresponding anomalous dimension is defined in terms of the renormalized coupling constant by the equation

$$\gamma(\alpha) \equiv \left.\frac{d\ln Z}{d\ln\mu}\right|_{\alpha_0=\text{const}} \tag{23}$$

and has the perturbative expansion

$$\gamma(\alpha) = \sum_{n=1}^{\infty}\gamma_n\alpha^n, \tag{24}$$

where the coefficients γ_n do not contain $\ln\Lambda/\mu$. From the other side, substituting the expression (22) into Eq. (23) we obtain the anomalous dimension as a function of the bare coupling constant,

$$\gamma\left(\alpha\left(\alpha_0, \ln\frac{\Lambda}{\mu}\right)\right) = -\sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}mC_{n,m}\ln^{m-1}\frac{\Lambda}{\mu}. \tag{25}$$

Next, we equate Eqs. (24) and (25) and repeat the reasoning of Sec. III. Namely, it is necessary to differentiate the resulting equation $(p-1)$ times with respect to $\ln\mu$, set $\mu = \Lambda$, and apply the HD + MSL boundary conditions (3) and (4). Equating the coefficients of the terms with the same powers of α_0 we obtain explicit expressions for the coefficients $C_{n,m}$ in the HD + MSL scheme,

$$C_{n,0} = 0; \qquad C_{n,1} = -\gamma_n, \qquad n \geq 1; \tag{26}$$

TABLE I. Some lower values of the generalized harmonic numbers $H_{p,s}$.

	p							
s	1	2	3	4	5	6	7	…
1	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	…
2	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	…
3	1	$\frac{9}{8}$	$\frac{251}{216}$	$\frac{2035}{1728}$	$\frac{256103}{216000}$	$\frac{28567}{24000}$	$\frac{9822481}{8232000}$	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$$C_{n+p,p} = (-1)^p \frac{1}{p!} \sum_{k=1}^{n+1} \gamma_k \sum_{k_1} k\beta_{k_1} \sum_{k_2} (k+k_1)\beta_{k_2} \sum_{k_3} (k+k_1+k_2)\beta_{k_3} \ldots \times \sum_{k_{p-1}} (k+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}} \Big|_{k_1+k_2+\cdots+k_{p-1}=n+p-k}, \qquad n \geq 0, \quad p \geq 2. \tag{27}$$

Their structure is very similar to Eq. (16) except for the presence of the coefficients γ_k instead of β_k in the first sum and slightly different numerical factors. Actually, making the formal replacement

$$\gamma_k \to -\beta_{k+1} \tag{28}$$

in the first sum of Eq. (27) one can obtain the coefficient $B_{n+1+p,p}$ from $C_{n+p,p}$.

Using Eq. (27) it is possible to derive explicit expressions for all coefficients in certain diagonals. In particular, the coefficients corresponding to the main and two subleading diagonals can be presented in the form

$$C_{p,p} = \frac{(-1)^p}{p} \beta_1^{p-1} \gamma_1; \tag{29}$$

$$C_{1+p,p} = (-1)^p (\beta_1^{p-1}\gamma_2 + (H_p - 1)\theta[p-2]\beta_1^{p-2}\beta_2\gamma_1); \tag{30}$$

$$C_{2+p,p} = (-1)^p \frac{(p+1)}{2} \left\{ \beta_1^{p-1}\gamma_3 + \theta[p-2]\beta_1^{p-2} \left((2H_{p+1} - 3)\beta_2\gamma_2 + \frac{(p-1)}{(p+1)}\beta_3\gamma_1 \right) + \left(\frac{p+3}{p+1} + H_{p+1}(H_{p+1} - 2) - H_{p+1,2} \right) \theta[p-3]\beta_1^{p-3}\beta_2^2\gamma_1 \right\}, \tag{31}$$

where the generalized harmonic numbers $H_{p,s}$ are defined by Eq. (20). In particular, from Eqs. (30) and (31) we obtain that the lowest coefficients in the second and third subdiagonals are written as

$$C_{3,2} = \beta_1\gamma_2 + \frac{1}{2}\beta_2\gamma_1; \qquad C_{4,2} = \frac{3}{2}\beta_1\gamma_3 + \beta_2\gamma_2 + \frac{1}{2}\beta_3\gamma_1;$$
$$C_{4,3} = -\beta_1^2\gamma_2 - \frac{5}{6}\beta_1\beta_2\gamma_1; \qquad C_{5,3} = -2\beta_1^2\gamma_3 - \frac{7}{3}\beta_1\beta_2\gamma_2 - \beta_1\beta_3\gamma_1 - \frac{1}{2}\beta_2^2\gamma_1;$$
$$C_{5,4} = \beta_1^3\gamma_2 + \frac{13}{12}\beta_1^2\beta_2\gamma_1; \qquad C_{6,4} = \frac{5}{2}\beta_1^3\gamma_3 + \frac{47}{12}\beta_1^2\beta_2\gamma_2 + \frac{3}{2}\beta_1^2\beta_3\gamma_1 + \frac{35}{24}\beta_1\beta_2^2\gamma_1. \tag{32}$$

V. EXAMPLES

A. Renormalization of $\mathcal{N} = 1$ SQED with N_f flavors

Let us compare the general expressions obtained in the previous sections with the results of explicit multiloop calculations made for $\mathcal{N} = 1$ SQED with N_f flavors regularized by higher derivatives. The three-loop anomalous dimension of the matter superfields for this theory was found in [25]. After that, the four-loop β-function was constructed with the help of the NSVZ equation. The results in the HD + MSL scheme are given by the equations

$$\frac{\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2}{\pi^3}\left[\frac{N_f}{2} + (N_f)^2\left(\ln a + 1 + \frac{A_1}{2}\right)\right] + \frac{\alpha^3}{\pi^4}\left[\frac{N_f}{2} + (N_f)^2\left(\ln a + \frac{3}{4} + C\right) + (N_f)^3\left((\ln a + 1)^2 - \frac{A_2}{4} + D_1 \ln a + D_2\right)\right] + \mathcal{O}(\alpha^4); \tag{33}$$

$$\gamma(\alpha)=-\frac{\alpha}{\pi}+\frac{\alpha^2}{\pi^2}\left[\frac{1}{2}+N_f\left(\ln a+1+\frac{A_1}{2}\right)\right]-\frac{\alpha^3}{\pi^3}\left[\frac{1}{2}+N_f\left(\ln a+\frac{3}{4}+C\right)+(N_f)^2\left((\ln a+1)^2-\frac{A_2}{4}+D_1\ln a+D_2\right)\right]+\mathcal{O}(\alpha^4), \tag{34}$$

where a, A_1, A_2, C, D_1, and D_2 are some parameters which depend on a particular version of the higher covariant derivative regularization. Explicit expressions for them can be found in [25]. For example, $a=M/\Lambda$, where M is the mass of the Pauli–Villars superfield (needed for removing residual one-loop divergences).

The equations describing the coupling constant and matter superfield renormalizations in the HD + MSL scheme in the considered case are written as

$$\frac{1}{\alpha}=\frac{1}{\alpha_0}+\frac{N_f}{\pi}\ln\frac{\Lambda}{\mu}+\frac{\alpha_0 N_f}{\pi^2}\ln\frac{\Lambda}{\mu}-\frac{\alpha_0^2 N_f}{\pi^3}\left[\frac{1}{2}\ln\frac{\Lambda}{\mu}+N_f\left(\ln a+1+\frac{A_1}{2}\right)\ln\frac{\Lambda}{\mu}+\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu}\right]+\frac{\alpha_0^3 N_f}{\pi^4}\left[\frac{1}{2}\ln\frac{\Lambda}{\mu}+N_f\left(\ln a+\frac{3}{4}+C\right)\ln\frac{\Lambda}{\mu}+(N_f)^2\left((\ln a+1)^2-\frac{A_2}{4}+D_1\ln a+D_2\right)\ln\frac{\Lambda}{\mu}+(N_f)^2\left(\ln a+1+\frac{A_1}{2}\right)\ln^2\frac{\Lambda}{\mu}+\frac{(N_f)^2}{3}\ln^3\frac{\Lambda}{\mu}\right]+\mathcal{O}(\alpha_0^4); \tag{35}$$

$$\ln Z=\frac{\alpha_0}{\pi}\ln\frac{\Lambda}{\mu}-\frac{\alpha_0^2}{\pi^2}\left[\frac{1}{2}\ln\frac{\Lambda}{\mu}+N_f\left(\ln a+1+\frac{A_1}{2}\right)\ln\frac{\Lambda}{\mu}+\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu}\right]+\frac{\alpha_0^3}{\pi^3}\left[\frac{1}{2}\ln\frac{\Lambda}{\mu}+N_f\left(\ln a+\frac{3}{4}+C\right)\ln\frac{\Lambda}{\mu}+(N_f)^2\left((\ln a+1)^2-\frac{A_2}{4}+D_1\ln a+D_2\right)\ln\frac{\Lambda}{\mu}+(N_f)^2\left(\ln a+1+\frac{A_1}{2}\right)\ln^2\frac{\Lambda}{\mu}+\frac{(N_f)^2}{3}\ln^3\frac{\Lambda}{\mu}\right]+\mathcal{O}(\alpha_0^4). \tag{36}$$

From Eqs. (33)–(36) we obtain that in the HD + MSL scheme the coefficients of the β-function and of the anomalous dimension are given by the expressions

$$\begin{aligned}
&\beta_1=B_{1,1}=\frac{N_f}{\pi};\\
&\beta_2=B_{2,1}=\frac{N_f}{\pi^2};\\
&\beta_3=B_{3,1}=-\frac{N_f}{\pi^3}\left[\frac{1}{2}+N_f\left(\ln a+1+\frac{A_1}{2}\right)\right];\\
&\beta_4=B_{4,1}=\frac{N_f}{\pi^4}\left[\frac{1}{2}+N_f\left(\ln a+\frac{3}{4}+C\right)+(N_f)^2\left((\ln a+1)^2-\frac{A_2}{4}+D_1\ln a+D_2\right)\right];
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\gamma_1=-C_{1,1}=-\frac{1}{\pi};\\
&\gamma_2=-C_{2,1}=\frac{1}{\pi^2}\left[\frac{1}{2}+N_f\left(\ln a+1+\frac{A_1}{2}\right)\right];\\
&\gamma_3=-C_{3,1}=-\frac{1}{\pi^3}\left[\frac{1}{2}+N_f\left(\ln a+\frac{3}{4}+C\right)+(N_f)^2\left((\ln a+1)^2-\frac{A_2}{4}+D_1\ln a+D_2\right)\right].
\end{aligned} \tag{38}$$

Substituting them into Eq. (16) [or, equivalently, into the relevant equations in (21)] we obtain the expressions for the coefficients at higher powers of $\ln\Lambda/\mu$ in the coupling constant renormalization,

$$B_{3,2}=-\frac{1}{2}\beta_1\beta_2=-\frac{(N_f)^2}{2\pi^3}; \tag{39}$$

$$B_{4,2} = -\beta_1\beta_3 - \frac{1}{2}\beta_2^2 = \frac{(N_f)^3}{\pi^4}\left(\ln a + 1 + \frac{A_1}{2}\right); \quad (40)$$

$$B_{4,3} = \frac{1}{3}\beta_1^2\beta_2 = \frac{(N_f)^3}{3\pi^4}. \quad (41)$$

The coincidence of these values with the coefficients in Eq. (35) confirms the correctness of the general equations presented above.

Similarly, substituting Eqs. (37) and (38) into Eq. (27) or Eq. (32) we find the coefficients at higher powers of $\ln\Lambda/\mu$ in $\ln Z$,

$$C_{2,2} = \frac{1}{2}\beta_1\gamma_1 = -\frac{N_f}{2\pi^2}; \quad (42)$$

$$C_{3,2} = \beta_1\gamma_2 + \frac{1}{2}\beta_2\gamma_1 = \frac{(N_f)^2}{\pi^3}\left(\ln a + 1 + \frac{A_1}{2}\right); \quad (43)$$

$$C_{3,3} = -\frac{1}{3}\beta_1^2\gamma_1 = \frac{(N_f)^2}{3\pi^3}. \quad (44)$$

Again, these values are exactly the same as the coefficients in Eq. (36), so that the general equations derived earlier really produce the correct result for the theory under consideration.

B. Constraints on the renormalization constants following from the NSVZ β-function for $\mathcal{N}=1$ SQED with N_f flavors

As another example, we consider $\mathcal{N}=1$ SQED with N_f flavors and check a relation between the renormalizations of the coupling constant and of the matter superfields following from the exact NSVZ β-function [48,49]

$$\frac{\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi}(1-\gamma(\alpha)). \quad (45)$$

According to [42], for the considered theory in the HD + MSL scheme the NSVZ equation is valid in all orders, although for an arbitrary renormalization prescription this is not in general satisfied. Using Eqs. (8) and (24) we see that in the Abelian case due to the NSVZ equation the β-function in a certain loop is related to the anomalous dimension in the previous loop,

$$\beta_{n+1} = -\frac{N_f}{\pi}\gamma_n. \quad (46)$$

Substituting the expression for γ_n following from this equation into Eq. (27) and comparing the result with Eq. (16) we see that the coefficients at higher powers of $\ln\Lambda/\mu$ satisfy the constraints

$$B_{n+p+1,p} = \frac{N_f}{\pi}C_{n+p,p}, \quad n \geq 0, \quad p \geq 2. \quad (47)$$

Taking into account that $B_{1,1} = N_f/\pi$ and using the HD + MSL boundary conditions (3) and (4) we obtain the relation

$$\begin{aligned} 0 &= \left(B_{1,1} - \frac{N_f}{\pi}\right)\ln\frac{\Lambda}{\mu} \\ &\quad + \sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}\left(B_{n+1,m} - \frac{N_f}{\pi}C_{n,m}\right)\ln^m\frac{\Lambda}{\mu} \\ &= \frac{1}{\alpha} - \frac{1}{\alpha_0} - \frac{N_f}{\pi}\ln\frac{\Lambda}{\mu} - \frac{N_f}{\pi}\ln Z. \end{aligned} \quad (48)$$

Exactly this equation is obtained after integrating the NSVZ β-function (45) with respect to $\ln\mu$, again, taking into account Eqs. (3) and (4). Therefore, Eqs. (16) and (27) have been verified by a nontrivial all-order calculation.

C. Constraints on the renormalization constants following from the NSVZ β-function for non-Abelian $\mathcal{N}=1$ supersymmetric theories

In the non-Abelian case the NSVZ β-function [34–37] for $\mathcal{N}=1$ supersymmetric theories with a simple gauge group G and the chiral matter superfields in the representation R is given by the expression[1]

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha,\lambda)/r}{2\pi(1 - C_2\alpha/2\pi)}, \quad (49)$$

where

$$\begin{aligned} &\mathrm{tr}(T^AT^B) \equiv T(R)\delta^{AB}; \quad (T^AT^A)_i{}^j \equiv C(R)_i{}^j; \\ &f^{ACD}f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \dim G = \delta^{AA}. \end{aligned} \quad (50)$$

According to [41], it is also valid in all-loops in the HD + MSL scheme. Note that, in general, for non-Abelian theories RGFs can also depend on Yukawa couplings λ^{ijk} abbreviated with λ. However, in this paper we consider theories with a single coupling constant, so that below we will omit the dependence on λ. Then we rewrite the NSVZ equation in the form

$$\frac{\beta(\alpha)}{\alpha^2} + \frac{1}{2\pi}(3C_2 - T(R)) + \frac{1}{2\pi r}C(R)_i{}^j\gamma_j{}^i(\alpha) = \frac{C_2}{2\pi}\frac{\beta(\alpha)}{\alpha} \quad (51)$$

and substitute the expansions (8) and (24) into it. Equating the coefficients at the same powers of the coupling constant α gives the equations

[1]The NSVZ equations for theories with multiple gauge couplings can be found in [50].

$$\beta_1 = -\frac{1}{2\pi}(3C_2 - T(R));$$
$$\beta_{n+1} = \frac{C_2}{2\pi}\beta_n - \frac{1}{2\pi r}C(R)_i{}^j(\gamma_n)_j{}^i, \quad n \geq 1. \tag{52}$$

The solution of the recurrence relations (52) is written as

$$\beta_{n+1} = \left(\frac{C_2}{2\pi}\right)^n \beta_1 - \frac{1}{2\pi r}C(R)_i{}^j \sum_{k=0}^{n-1}\left(\frac{C_2}{2\pi}\right)^k (\gamma_{n-k})_j{}^i. \tag{53}$$

With the help of Eq. (52) and the expressions (16), (27), and (A10) we obtain the equation relating coefficients in the expansions of $1/\alpha$, $\ln\alpha/\alpha_0$, and $(\ln Z)_j{}^i$,

$$B_{n+1+p,p} = -\frac{C_2}{2\pi}\tilde{B}_{n+p,p} + \frac{1}{2\pi r}C(R)_i{}^j(C_{n+p,p})_j{}^i, \quad n \geq 0, \quad p \geq 1. \tag{54}$$

Taking into account that $B_{1,1} = -(3C_2 - T(R))/2\pi$ and using Eqs. (2), (22), and (A1) this relation can be presented as

$$\frac{1}{\alpha} - \frac{1}{\alpha_0} + \frac{1}{2\pi}(3C_2 - T(R))\ln\frac{\Lambda}{\mu} - \frac{1}{2\pi r}C(R)_i{}^j(\ln Z)_j{}^i + \frac{C_2}{2\pi}\ln\frac{\alpha}{\alpha_0} = 0. \tag{55}$$

It exactly coincides with the equation obtained after integrating Eq. (51) with respect to $\ln\mu$ and taking into account the HD + MSL boundary conditions (3) and (4). This fact can be considered as a very nontrivial correctness test of the calculations made in this paper.

VI. CONCLUSION

In this paper we present explicit expressions for the coefficients at powers of $\ln\Lambda/\mu$ in the renormalization of the coupling constant and fields for renormalizable theories with a single coupling in the HD + MSL scheme (assuming that the corresponding divergences are logarithmical). In the HD + MSL scheme a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms, when only $(\ln\Lambda/\mu)^k$ with $k \geq 1$ are included into the renormalization constants. This subtraction scheme is similar to minimal subtraction in the case of using the dimensional technique, and the expressions obtained in this paper [Eqs. (16) and (27)] are analogous to the solutions of the pole equations. They express coefficients at various powers of $\ln\Lambda/\mu$ in terms of the coefficients of the β-function and (in the case of the field renormalization) of the anomalous dimension. As a test, we have verified that these expressions correctly reproduce the coefficients in the renormalization of $\mathcal{N} = 1$ SQED with N_f flavors in the four-loop approximation for the β-function and in the three-loop approximation for the anomalous dimension of the matter superfields. Note that in the HD + MSL scheme RGFs of $\mathcal{N} = 1$ supersymmetric theories satisfy the NSVZ relation in all orders [41]. As was demonstrated in this paper, the relation between the renormalizations of the coupling constant and of the chiral matter superfields following from the NSVZ equation can be constructed with the help of the equations derived in this paper. Moreover, for various theories these equations can be used for multiloop calculations as a nontrivial test of the correctness. Also in the case of using the dimensional technique with $\Lambda \neq \mu$, where Λ is the dimensionful regularization parameter, (see, e.g., [51–53]) it would be interesting to compare the coefficients at powers of $1/\varepsilon$ and $\ln\Lambda/\mu$. Possibly, the equations obtained in this paper can be useful for this purpose.

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APPENDIX: THE COEFFICIENTS IN THE EXPANSIONS OF $\ln\alpha/\alpha_0$ AND $(\alpha/\alpha_0)^S$

Repeating the reasoning of Sec. III in the HD + MSL scheme it is possible to write the coefficients in the expressions

$$\ln\frac{\alpha}{\alpha_0} \equiv \sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}\tilde{B}_{n,m}\ln^m\frac{\Lambda}{\mu} \quad \text{and} \tag{A1}$$

$$\left(\frac{\alpha}{\alpha_0}\right)^S \equiv 1 + \sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}B^{(S)}_{n,m}\ln^m\frac{\Lambda}{\mu} \tag{A2}$$

in terms of β_n. Note that the case considered in Sec. III corresponds to $S = -1$,

$$B^{(-1)}_{n,m} = B_{n,m}; \qquad B^{(-1)}_{p,p} = 0, \qquad p \geq 2. \tag{A3}$$

First, we differentiate the expressions (A1) and (A2) with respect to $\ln\mu$ at a fixed value of the bare coupling constant α_0 and substitute the perturbative series for the β-function into the resulting expressions,

$$\frac{d}{d\ln\mu}\ln\frac{\alpha}{\alpha_0} = \frac{\beta(\alpha)}{\alpha} = \sum_{k=1}^{\infty}\beta_k\alpha^k; \tag{A4}$$

$$\frac{d}{d\ln\mu}\left(\frac{\alpha}{\alpha_0}\right)^S = (\alpha_0)^{-S}S\alpha^{S-1}\beta(\alpha) = (\alpha_0)^{-S}S\sum_{k=1}^{\infty}\beta_k\alpha^{k+S}. \tag{A5}$$

Then, we differentiate these equations $(p-1)$ times with respect to $\ln\mu$ and, after that, set $\mu=\Lambda$,

$$\sum_{k=1}^{\infty}\beta_k\frac{d^{p-1}\alpha^k}{d\ln\mu^{p-1}}\bigg|_{\mu=\Lambda}=\frac{d^p}{d\ln\mu^p}\sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}\tilde{B}_{n,m}\ln^m\frac{\Lambda}{\mu}\bigg|_{\mu=\Lambda};\tag{A6}$$

$$(\alpha_0)^{-S}S\sum_{k=1}^{\infty}\beta_k\frac{d^{p-1}\alpha^{k+S}}{d\ln\mu^{p-1}}\bigg|_{\mu=\Lambda}=\frac{d^p}{d\ln\mu^p}\sum_{n=1}^{\infty}\alpha_0^n\sum_{m=1}^{n}B^{(S)}_{n,m}\ln^m\frac{\Lambda}{\mu}\bigg|_{\mu=\Lambda}.\tag{A7}$$

Calculating the derivatives [in the left-hand side for this purpose we use Eq. (14)] gives the equations

$$\sum_{k=1}^{\infty}\beta_k\sum_{k_1=1}^{\infty}k\beta_{k_1}\sum_{k_2=1}^{\infty}(k+k_1)\beta_{k_2}\sum_{k_3=1}^{\infty}(k+k_1+k_2)\beta_{k_3}\ldots\sum_{k_{p-1}=1}^{\infty}(k+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\alpha_0^{k+k_1+k_2+\cdots+k_{p-1}}$$
$$=(-1)^p p!\sum_{n=0}^{\infty}\alpha_0^{n+p}\tilde{B}_{n+p,p};\tag{A8}$$

$$S\sum_{k=1}^{\infty}\beta_k\sum_{k_1=1}^{\infty}(k+S)\beta_{k_1}\sum_{k_2=1}^{\infty}(k+S+k_1)\beta_{k_2}\sum_{k_3=1}^{\infty}(k+S+k_1+k_2)\beta_{k_3}\ldots\sum_{k_{p-1}=1}^{\infty}(k+S+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\alpha_0^{k+k_1+k_2+\cdots+k_{p-1}}$$
$$=(-1)^p p!\sum_{n=0}^{\infty}\alpha_0^{n+p}B^{(S)}_{n+p,p}.\tag{A9}$$

Equating the coefficients at the same powers of the coupling constant α_0 we obtain the required expressions for the coefficients in Eqs. (A1) and (A2),

$$\tilde{B}_{n+p,p}=(-1)^p\frac{1}{p!}\sum_{k=1}^{n+1}\beta_k\sum_{k_1}k\beta_{k_1}\sum_{k_2}(k+k_1)\beta_{k_2}\sum_{k_3}(k+k_1+k_2)\beta_{k_3}\ldots$$
$$\times\sum_{k_{p-1}}(k+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\bigg|_{k_1+k_2+\cdots+k_{p-1}=n+p-k};\tag{A10}$$

$$B^{(S)}_{n+p,p}=S(-1)^p\frac{1}{p!}\sum_{k=1}^{n+1}\beta_k\sum_{k_1}(k+S)\beta_{k_1}\sum_{k_2}(k+S+k_1)\beta_{k_2}\sum_{k_3}(k+S+k_1+k_2)\beta_{k_3}\ldots$$
$$\times\sum_{k_{p-1}}(k+S+k_1+k_2+\cdots+k_{p-2})\beta_{k_{p-1}}\bigg|_{k_1+k_2+\cdots+k_{p-1}=n+p-k},\tag{A11}$$

where $n\geq 0$ and $p\geq 1$. Certainly, for $S=-1$ Eq. (A11) gives the expressions (9) for $p=1$ and (16) for $p\geq 2$. Note that from Eq. (A11) we obtain

$$B^{(S)}_{p,p}=\frac{(-1)^p}{p!}S(S+1)\ldots(S+p-1)\beta_1^p,\tag{A12}$$

so that $B^{(-1)}_{p,p}=0$ for all $p\geq 2$.

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