C-P-T fractionalization

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Discrete spacetime symmetries of parity P or reflection R, and time reversal T, act naively as $\mathbb{Z}_2$ involutions in the *passive* transformation on the spacetime coordinates; but together with a charge conjugation C, the total C-P-R-T symmetries have enriched *active* transformations on fields in representations of the spacetime-internal symmetry groups of quantum field theories (QFTs). In this work, we derive that these symmetries can be further fractionalized, especially in the presence of the fermion parity $(-1)^{\mathrm{F}}$. We elaborate on examples including relativistic Lorentz invariant QFTs (e.g., spin-1/2 Dirac or Majorana spinor fermion theories) and nonrelativistic quantum many-body systems (involving Majorana zero modes), and comment on applications to spin-1 Maxwell electromagnetism (QED) or interacting Yang-Mills (QCD) gauge theories. We discover various C-P-R-T-$(-1)^{\mathrm{F}}$ group structures, e.g., Dirac spinor is in a *projective* representation of $\mathbb{Z}_2^C \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$ but in an *(anti)linear* representation of an order-16 non-Abelian finite group, as the central product between an order-8 dihedral (generated by C and P) or quaternion group and an order-4 group generated by T with $T^2 = (-1)^{\mathrm{F}}$. The general theme may be coined as C-P-T or C-R-T fractionalization.

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I. INTRODUCTION AND SUMMARY

Common physics knowledge recites that the time reversal T and parity P are *discrete* spacetime symmetries that cannot be continuously deformed from the identity element —T and P are not part of the proper orthochronous restricted *continuous* Lorentz symmetry group $\mathrm{SO}^+(d,1)$. It is important to distinguish the T and P from the mirror reflection R. As *passive* transformations on the spacetime coordinates $x \equiv (t, \vec{x})$,

$$\begin{aligned} T(t, x_1, \ldots, x_d)T^{-1} &= x'_T \equiv (-t, x_1, \ldots, x_d), \\ P(t, x_1, \ldots, x_d)P^{-1} &= x'_P \equiv (t, -x_1, -\ldots, -x_d), \\ R(t, x_1, \ldots, x_d)R^{-1} &= x'_R \equiv (t, -x_1, +\ldots, +x_d), \end{aligned} \tag{1}$$

where T flips the time coordinate, P flips all $\vec{x}$, but R flips only on one coordinate (here say x_1) with respect to a mirror plane (normal to x_1). We label the spacetime coordinate component x_μ with $\mu = 0, 1, \ldots, d$ for $(d+1)$-spacetime dimensions (denoted as $d+1$d). The transformed coordinates are labeled as x', or x'_μ for each component, with the subscript $T/P/R/etc.$ to indicate which coordinates are transformed. In odd-dimensional spacetime, the P is in fact a subgroup of a continuous spatial rotational symmetry special orthogonal $\mathrm{SO}(d) \subset \mathrm{SO}^+(d,1)$; thus, unluckily, P is not an independent discrete symmetry. We should replace P by the reflection R. For example, the CPT theorem [1–6] should be called the CRT theorem [7,8] in any general dimension of Minkowski spacetime. In this work, we mainly focus on the even-dimensional spacetime, so we can choose either P or R symmetry. We shall mainly use P to match the major literature, but we will comment about R when necessary.

Charge conjugation C, however, cannot manifest itself under a *passive* transformation on the spacetime coordinates but can reveal itself under an *active* transformation on a particle or field, such as a complex-valued spin-0 Lorentz scalar $\phi(x) = \phi(t, \vec{x})$ (which is a function of the spacetime coordinates). The C colloquially flips between particle and antiparticle sectors, or more generally between energetic excitations and antiexcitations,

$$C(\text{excitations})C^{-1} = (\text{antiexcitations}) \tag{2}$$

involving the complex conjugation (denoted *). The *active* transformation acts on this Lorentz scalar ϕ as

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TABLE I. The four-component complex massless Dirac spinor field ψ in $3+1$d contains 8 real degrees of freedom composed from $2\times2\times2$, chiralities (left/right) $\times \hat{S}_z$ spins ($\uparrow/\downarrow$) $\times$ particle/antiparticle. The $+$ or $-$ entry means the quantum number eigenvalue is positive or negative.

Spinor component	$\hat{p}_z$	$\hat{S}_z$	$\hat{h}=\hat{p}\cdot\hat{S}$	Chirality $P_{L/R}$
First	$-$	$+$	$-$	L
Second	$+$	$-$	$-$	L
Third	$+$	$+$	$+$	R
Fourth	$-$	$-$	$+$	R

$$\begin{aligned} C\phi(t,\vec{x})C^{-1} &= \phi'_C(t,\vec{x}) = \phi^*(t,\vec{x}) = \phi^*(x),\\ P\phi(t,\vec{x})P^{-1} &= \phi'_P(t,\vec{x}) = \phi(t,-\vec{x}) = \phi(x'_P),\\ T\phi(t,\vec{x})T^{-1} &= \phi'_T(t,\vec{x}) = \phi(-t,\vec{x}) = \phi(x'_T). \end{aligned} \quad (3)$$

All the above transformations, regardless *passive* or *active*, naively seem to be only $\mathbb{Z}_2$ involutions in mathematics, such that twice transformations are the null (do nothing) transformations.[1] Thus, it reveals a finite group of order 2 structure, namely $\mathbb{Z}_2$.

In this scalar field example, the *C*-*P*-*T* symmetry form a direct product group $\mathbb{Z}_2^C\times\mathbb{Z}_2^P\times\mathbb{Z}_2^T$. One may mistakenly conclude $C^2=P^2=R^2=T^2=+1$ and assume they are all commute in general. The essence of our work is to point out that all these "discrete *C*, *P*, *R*, or *T* symmetries" (which we denote altogether as "*C*-*P*-*R*-*T*" in short) can form a rich non-Abelian finite group structure, in the physical realistic systems pertinent to experiments or theories. We can possibly fractionalize the *C*-*P*-*R*-*T* group structures further, for the state vectors in quantum mechanics or the fields in classical or quantum field theories (QFTs), in various representations (rep) of the spacetime or internal symmetry groups (denoted as $G_{\text{spacetime}}$ and G_{internal}).

The *symmetry fractionalization* means the following: the matter field is *not* in the *linear representation* of the original symmetry group G, but in the *projective representation* of G and in the *linear representation* of the extended total group $\tilde{G}$. A typical case is illustrated by a group extension $1\to N\to\tilde{G}\to G\to 1$, where G is the quotient group while the N is the normal subgroup of the total group $\tilde{G}$, so $\tilde{G}/N=G$. A famous example is the gapped $1+1$d isospin-1 Haldane chain with $G=\mathrm{SO}(3)$ symmetry [9], whose $0+1$d boundary can host a twofold degenerated isospin-$1/2$ doublet of $\tilde{G}=\mathrm{SU}(2)$, with $N=\mathbb{Z}_2$. Thus, this doublet is in a projective rep of $G=\mathrm{SO}(3)$, also in a linear rep of $\tilde{G}=\mathrm{SU}(2)$.

In this work, we will find the analogous *C*-*P*-*T* *symmetry fractionalization*. For example, in contrast to a spin-0 scalar field's $G_\phi\equiv\mathbb{Z}_2^C\times\mathbb{Z}_2^P\times\mathbb{Z}_2^T$, we uncover an order-16 non-Abelian $\tilde{G}_\psi\equiv\frac{\mathbb{D}_8^{\mathrm{F},CP}\times\mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$ for a $3+1$d spin-$1/2$ Dirac field [see the later Eq. (6) for explanations]. Remarkably, the fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ generated by $(-1)^{\mathrm{F}}:\psi\mapsto-\psi$ plays a crucial role in the group extension structure $1\to\mathbb{Z}_2^{\mathrm{F}}\to\tilde{G}_\psi\to G_\phi\to 1$. Thus, fermionic systems reveal $\mathbb{Z}_2^{\mathrm{F}}$-enriched structures richer than bosonic systems. This means that Dirac fermion ψ is in a projective rep of G_ϕ, also in an (anti)linear rep of $\tilde{G}_\psi$. (It is antilinear because $\tilde{G}_\psi$ contains the antilinear time-reversal symmetry.)

This beyond-$\mathbb{Z}_2$ group structure for *C*-*P*-*R*-*T* is mostly secretly hidden in the literature and still not yet widely appreciated. [However, a well-known exception is the time-reversal symmetry can be $\mathbb{Z}_4^{T\mathrm{F}}\supset\mathbb{Z}_2^{\mathrm{F}}$ that $T^2=(-1)^{\mathrm{F}}$ in contrast with the usual $\mathbb{Z}_2^T$ with $T^2=+1$, both have applications to the classification of topological superconductors and insulators; see, for instance, [8,10–19]]. Note that Refs. [20,21] discussed the related *C*-*P*-*T* group structure of $3+1$d Dirac field.[2] However, the overall methods and descriptions of the order-16 group between our approach and theirs [20,21] are rather different. Also, the general concept of the symmetry fractionalization structure of the *C*-*P*-*T* group was not obtained nor emphasized in [20,21]. Thus, our work provides its own value, by generalizing the *C*-*P*-*T* symmetry fractionalization structure to other examples. Below, we work through several examples in sections.

II. 3+1D SPIN-1/2 FERMIONIC SPINORS

First, we consider the $3+1$d Dirac theory with a four complex component spinor field ψ. We aim to carry out its *C*-*P*-*R*-*T*-$(-1)^{\mathrm{F}}$ structure acting on ψ in detail. It is convenient to regard the massless Dirac spinor as two complex Weyl spinors $\mathbf{2}_L\oplus\mathbf{2}_R$ (left L and right R) rep in the standard Weyl basis for ψ [22–25]. Each of the four spinor components carries different quantum numbers of momentum ($\hat{p}_z$), Lorentz spin ($\hat{S}_z$), and the chirality (L or R, which is determined by helicity $\hat{h}=\hat{p}\cdot\hat{S}=-$ or $+$, in the massless case), shown in Table I.

We summarize how C, P, and T act on the spinor and its various quantum numbers intuitively in Table II,

[1]Let us clarify the *passive* vs *active* transformations, and their involution. Suppose we take a spatial coordinate x and a scalar function $f(x)$ as an example, the *passive* transformation F_p maps $(x,f(x))$ to $(-x,f(x))$, while the *active* transformation F_a maps $(x,f(x))$ to $(x,f(-x))$. So we see that both $F_p(F_p(x,f(x)))=(x,f(x))$ and $F_a(F_a(x,f(x)))=(x,f(x))$ are $\mathbb{Z}_2$ involutions, such that F_p and F_a are their own inverse functions. The above discussion also follows for the time coordinate t, by replacing x with t. However, we will take the *active* transformation viewpoint on the classical fields or quantum fields. We shall reveal their fractionalization of *C*-*P*-*R*-*T* symmetries, beyond this $\mathbb{Z}_2$-involution structure.

[2]After the journal submission of this work, we thank Physical Review Letters divisional associate editor Daniel N. Kabat for bringing our attention to the early literature [20,21].

TABLE II. Agree with Eq. (5), we show whether each spinor component and its quantum numbers are switched under the *C-P-R-T* transformation. The top horizontal row shows which quantum numbers, and the left vertical column shows how C, P/R, or T acts. The "Yes" entry in the table means the discrete symmetry switches the quantum numbers. The empty entry means the quantum number is preserved.

Discrete symmetry Switch quantum Numbers or not	$p_z > 0 \Updownarrow p_z < 0$	$\hat{S}_z\uparrow \Updownarrow \hat{S}_z\downarrow$	$L \Updownarrow R$	Particle $\Updownarrow$ Antiparticle
C				Yes
P/R	Yes		Yes	
T	Yes	Yes		

(i) The *unitary* C switches between the particle $\Leftrightarrow$ antiparticle, but keeps the momentum p_z, the spins $\hat{S}_z$, and the chirality intact. Note that the antiparticle's first, second, third, fourth components of the four-component spinor have the quantum numbers of the $\hat{S}_z$ and chirality (opposite with respect to those of the particle's): $(-,+,-,+)$ for $\hat{S}_z$, and (R,R,L,L) for chirality. See various clarifications in [26].

(ii) The *unitary* P switches between the momentum $p_z > 0 \Leftrightarrow p_z < 0$, also switches between the chirality $L \Leftrightarrow R$, but keeps the spin $\hat{S}_z$ intact.

(iii) The *antiunitary* T switches between the momentum $p_z > 0 \Leftrightarrow p_z < 0$ and the spin $\hat{S}_z$'s $\uparrow \Leftrightarrow \downarrow$, but keeps the chirality intact.

Below, we manifest the *C-P-T* transformation of Table II explicitly in a set of gamma matrices acting on the spinor ψ. We adopt the standard Pauli matrix convention,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for the gamma matrices of Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with the metric signature $(+,-,-,-)$ in the chiral Weyl basis,

$$\gamma^0 = \sigma^1 \otimes \sigma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}.$$
$$\gamma^j = \mathrm{i}\sigma^2 \otimes \sigma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad \text{for } j = 1,2,3.$$
$$\gamma^5 = -\sigma^3 \otimes \sigma^0 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}. \tag{4}$$

The *active C-P-T* transformation on the fields changes ψ to ψ' (instead of the passive transformation on coordinates), but we adopt the primed coordinate notations, x'_P and x'_T, introduced earlier in Eq. (1),

$$C\psi(x)C^{-1} = \psi'_C(x) = -\mathrm{i}\gamma^2\psi^*(x) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\psi^*(x).$$
$$P\psi(x)P^{-1} = \psi'_P(x) = \gamma^0\psi(x'_P) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\psi(x'_P).$$
$$T\psi(x)T^{-1} = \psi'_T(x) = -\gamma^1\gamma^3\psi(x'_T) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\psi(x'_T).$$
$$(CPT)\psi(x)(CPT)^{-1} = \psi'_{CPT}(x) = \gamma^5\psi^*(-x).$$
$$T^2 = (CP)^2 = (-1)^{\mathrm{F}}. \quad C^2 = P^2 = (CPT)^2 = +1. \tag{5}$$

The *unitary* C says $C(z\psi(x))C^{-1} = z(-\mathrm{i}\gamma^2\psi^*(x))$ with a *linear* map on a complex number $z \in \mathbb{C}$. The C in Eq. (5) indeed agrees with Table II, by taking into account that the spin ($\hat{S}_z$) and chirality (L/R) quantum numbers of antiparticle ψ^* are opposite to that of particle ψ in Table I, namely $(-,+,-,+)$ and (R,R,L,L) for each of four components of spinor ψ^*.

The *antiunitary* T actually requires a complex conjugation K to do the *antilinear* map $T(z\psi(x))T^{-1} = -z^*\gamma^1\gamma^3\psi(x'_T)$. The complex conjugation K maps $z \in \mathbb{C} \mapsto \mathrm{K}z\mathrm{K} = z^* \in \mathbb{C}$ with a state-vector-basis-dependence on the Hilbert space. But luckily these specific Weyl basis gamma matrices in Eq. (5) make this K not manifest because all the *linear* maps (i.e., $-\mathrm{i}\gamma^2, \gamma^0, -\gamma^1\gamma^3$, and γ^5) in Eq. (5) contain only the *real* coefficient matrices.

Clearly the Dirac spinor theory (here, $d+1 = 3+1$) action $\int \mathrm{d}^{d+1}x\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$ preserves the discrete symmetry transformations in Eq. (5). Lo and behold, based on a chain of remarks listed below Eq. (6), we discover the total discrete non-Abelian finite group structure, of $C/P/T$ and $(-1)^{\mathrm{F}}$, summarized as $\tilde{G}_\psi \equiv \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$,

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & \mathbb{Z}_2^{\mathrm{F}} & \longrightarrow & \mathbb{D}_8^{\mathrm{F},CP} & \longrightarrow & \mathbb{Z}_2^C \times \mathbb{Z}_2^P & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{Z}_4^{T\mathrm{F}} & \hookrightarrow & \tilde{G}_\psi \equiv \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}} \text{ central product} & & & & \\ & & \downarrow & & \Vert & & & & \\ & & \mathbb{Z}_2^T & & \tilde{G}_\psi \equiv \frac{\mathbb{Q}_8^{\mathrm{F},CP,PT} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}} \text{ central product} & & & & \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array} \quad (6)$$

Let us now elaborate on Eq. (6) in detail step-by-step,

(1) We have $T^2 = (-1)^{\mathrm{F}}$ so the time reversal $\mathbb{Z}_2^T$ and fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ combines to be an order-4 Abelian group $\mathbb{Z}_4^{T\mathrm{F}} \supset \mathbb{Z}_2^{\mathrm{F}}$, such that the total group $\mathbb{Z}_4^{T\mathrm{F}}$ sits in the group extension of the quotient $\mathbb{Z}_2^{\mathrm{F}}$ extended by the normal subgroup $\mathbb{Z}_2^{\mathrm{F}}$, written as a short exact sequence,

$$1 \to \mathbb{Z}_2^{\mathrm{F}} \to \mathbb{Z}_4^{T\mathrm{F}} \to \mathbb{Z}_2^T \to 1. \quad (7)$$

(2) Remarkably, $CP = (-1)^{\mathrm{F}}PC$ here, while we can show $CP\psi P^{-1}C^{-1} = -\mathrm{i}\gamma^2\gamma^0\psi^*(x'_P)$ and $PC\psi C^{-1}P^{-1} = +\mathrm{i}\gamma^2\gamma^0\psi^*(x'_P)$ in this particular basis. This means the C and P do not commute in the fermion parity odd $(-1)^{\mathrm{F}} = -1$ sector (illustrated in Fig. 1), but they commute in the bosonic $(-1)^{\mathrm{F}} = +1$ sector. The C and P form a non-Abelian finite group of order-8, a dihedral group $\mathbb{D}_8$, denoted by a standard group theory notation via enlisting its generators (on the left) and their multiplicative properties (on the right),

$$\mathbb{D}_8^{\mathrm{F},CP} \equiv \langle CP, C | (CP)^4 = C^2 = +1, C(CP)C = (CP)^{-1} \rangle. \quad (8)$$

Note that we can either understand the $\mathbb{D}_8^{\mathrm{F},CP} = \mathbb{Z}_4^{CP} \rtimes \mathbb{Z}_2^C$ via the group extension $1 \to \mathbb{Z}_4^{CP} \to \mathbb{D}_8^{\mathrm{F},CP} \to \mathbb{Z}_2^C \to 1$ with the order-4 $\mathbb{Z}_4^{CP}$ sits at the normal subgroup and the $\mathbb{Z}_2^C$ (or $\mathbb{Z}_2^P$) sits at the quotient; or we can understand the $\mathbb{D}_8^{\mathrm{F},CP}$ as the quotient $\mathbb{Z}_2^C \times \mathbb{Z}_2^P$ extended by the fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ as another group extension,

$$1 \to \mathbb{Z}_2^{\mathrm{F}} \to \mathbb{D}_8^{\mathrm{F},CP} \to \mathbb{Z}_2^C \times \mathbb{Z}_2^P \to 1. \quad (9)$$

Note that $(CP)^2 = T^2 = (-1)^{\mathrm{F}}$.

(3) The Eq. (6)'s vertical and horizontal group extensions are already explained in Eqs. (7) and (9) as two short exact sequences. The standard notation of the inclusion "$\hookrightarrow$" in $G_{\mathrm{sub}} \hookrightarrow G$ means that G contains G_{sub} as a subgroup. This order-16 non-Abelian finite group $\tilde{G}_\psi \equiv \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$ contains both $\mathbb{D}_8^{\mathrm{F},CP}$ and $\mathbb{Z}_4^{T\mathrm{F}}$ subgroups, as their inclusion notations ($\hookrightarrow$) suggest. The $\tilde{G}_\psi$ is the *central product* between $\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}$ mod out their common $\mathbb{Z}_2^{\mathrm{F}}$ center subgroup, as their $\mathbb{Z}_2^{\mathrm{F}}$ is identical. Amusingly this $\tilde{G}_\psi$ is isomorphic to the 16-element rank-2 matrix group known as Pauli group $\equiv \langle \sigma^1, \sigma^2, \sigma^3 \rangle$ generated by Pauli matrices that act on the two-dimensional Hilbert space of 1 qubit.

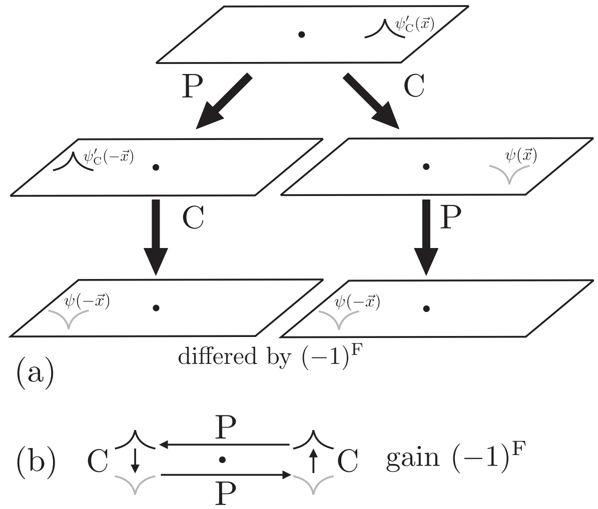

FIG. 1. Schematic illustrations (a) CP and PC act on a local Dirac fermionic excitation, two final configurations differed by $(-1)^{\mathrm{F}}$ due to $CP = (-1)^{\mathrm{F}}PC$. Namely, the following two procedures differed by a (-1) sign for a Dirac fermion: (i) Apply P to map the particle to its mirror partner, then apply C to map the particle to its antiparticle. (ii) Apply C to map the particle to its antiparticle, then apply P to map the antiparticle to its mirror partner. More generally, the parity P here (in even spacetime dimensions) can be replaced by the reflection R. The P or R transformation is with respect to the origin (the black dot). The white planes indicate the spatial dimensions. The ψ'_C and ψ are fermionic particle and antiparticle excitation creation operators, respectively. The convex or concave cusps represent the particle or hole excitations. (b) A consecutive procedure $CPCP = (-1)^{\mathrm{F}}$ gives a minus sign to a fermionic excitation.

(4) Now we show $\tilde{G}_\psi \equiv \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}} = \frac{\mathbb{Q}_8^{\mathrm{F},CP,PT} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$ group isomorphism, which basically says two facts: (1) the first group generated by C, P, T, and the second group generated by CP, PT, CT, and T, are exactly the same order-16 non-Abelian group, (2) an order-8 quaternion group,

$$\mathbb{Q}_8^{\mathrm{F},CP,PT} = \langle CP, PT, CT | (CP)^2 = (PT)^2 = (CT)^2 = (-1)^{\mathrm{F}} \rangle \tag{10}$$

is generated by $\mathbf{i} = CP$, $\mathbf{j} = PT$, and $\mathbf{k} = CT$ via a standard notation $\mathbb{Q}_8 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1 \rangle$.

(5) Because the Dirac spinor ψ sits in the complex $\mathbf{2}_L \oplus \mathbf{2}_R$ rep of spacetime symmetry Spin(3,1), we can ask: How does the order-16 non-Abelian finite group fit into the Dirac theory's spacetime-internal symmetry group,

$$G_{\substack{\text{spacetime}\\ -\text{internal}}} = G_{\text{spacetime}} \ltimes_N G_{\text{internal}} \tag{11}$$

(the semidirect product mod out the common normal subgroup N is denoted as "$\ltimes_N$")? In Minkowski signature flat spacetime, we have $G_{\text{spacetime}} = \mathrm{Pin}(d,1)$, which not only is a double cover of $\mathrm{O}(d,1)$, but also contains a normal subgroup $\mathrm{Spin}(d,1)$. All these $\mathrm{Pin}(d,1)$, $\mathrm{O}(d,1)$, and $\mathrm{Spin}(d,1)$ sit inside the group extension,

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{Z}_2^{\mathrm{F}} & & \mathbb{Z}_2^{\mathrm{F}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathrm{Spin}(d,1) & \longrightarrow & \mathrm{Pin}(d,1) & \longrightarrow & \mathbb{Z}_2^T & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathrm{SO}(d,1) & \longrightarrow & \mathrm{O}(d,1) & \xrightarrow{\det=\pm1} & \mathbb{Z}_2^T & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{12}$$

Note that a special orthogonal $\mathrm{SO}(d,1)$ contains two components $[\pi_0(\mathrm{SO}(d,1)) = \mathbb{Z}_2]$, the proper orthochronous Lorentz group $\mathrm{SO}^+(d,1)$ and another component that can be switched via the simultaneous R and T (say $\mathbb{Z}_2^{RT}$). Thus,

$$\begin{aligned} &1 \to \mathrm{SO}^+(d,1) \to \mathrm{SO}(d,1) \to \mathbb{Z}_2^{RT} \to 1, \\ &1 \to \mathrm{SO}^+(d,1) \to \mathrm{O}(d,1) \to \mathbb{Z}_2^R \times \mathbb{Z}_2^T \to 1. \end{aligned} \tag{13}$$

Note that here we choose the $\mathrm{Pin}(d,1)$ instead of $\mathrm{Pin}(1,d)$ because a generic nonisomorphism $\mathrm{Pin}(d,1) \ncong \mathrm{Pin}(1,d)$, while the former has their T^2 and Clifford algebra as [8]

$$\begin{aligned} &T^2 = (-1)^{\mathrm{F}}, \qquad \mathrm{Cliff}_{d,1} \colon e_0^2 = -1, \\ &e_j^2 = 1, \quad \text{with} \quad j = 1, \ldots, d, \end{aligned}$$

the later has a different property, note we required here,

$$\begin{aligned} &T^2 = +1, \qquad \mathrm{Cliff}_{1,d} \colon e_0^2 = 1, \\ &e_j^2 = -1, \quad \text{with} \quad j = 1, \ldots, d. \end{aligned}$$

In short, $\mathrm{Pin}(d,1)$ not only contains the $\mathbb{Z}_2^{\mathrm{F}}$ center, but also contains four connected components, i.e., $\pi_0(\mathrm{Pin}(d,1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, same as $\pi_0(\mathrm{O}(d,1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, disconnected from each other flipped by $\mathbb{Z}_2^R$ and $\mathbb{Z}_2^T$.

(6) The three discrete subgroups, $\mathbb{Z}_2^R$, $\mathbb{Z}_2^T$, and $\mathbb{Z}_2^{\mathrm{F}}$ are found as some normal subgroup or quotient group in Eq. (12). But where is the missing charge conjugation $\mathbb{Z}_2^C$?

(i) In general, the charge conjugation is better defined mathematically [8] as a new element of the extended group in the *CRT* theorem, acting by *conjugate linear* (*antilinear*) maps on the Hilbert space of statevectors. This follows Wigner's theorem on symmetries of a quantum system [27]: any transformation of projective Hilbert space that preserves the absolute value of the inner products can be represented by a *linear* or *antilinear* transformation of Hilbert space, which is unique up to a phase factor.

(ii) In a particularly narrow-minded purpose here, we can include naturally the internal symmetry $G_{\text{internal}} = \mathrm{U}(1)$ into the full spacetime-internal symmetry of Dirac theory's $G_{\substack{\text{spacetime}\\ -\text{internal}}} = \mathrm{Pin}(d,1) \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)$ in Eq. (11), such that the charge conjugation C is the complex conjugation of the U(1), which maps $g = \mathrm{e}^{\mathrm{i}q\theta} \in \mathrm{U}(1)$ to $g^* = \mathrm{e}^{-\mathrm{i}q\theta} \in \mathrm{U}(1)$. Thus, the charge conjugation generates the outer automorphism of the U(1): $\mathrm{Out}(\mathrm{U}(1)) = \mathbb{Z}_2^C$.

In $3+1$d, the outer automorphism of $G_{\substack{\text{spacetime}\\ -\text{internal}}}$ still is $\mathrm{Out}(\mathrm{Pin}(3,1) \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)) = \mathbb{Z}_2$, the only natural charge conjugation available.

The benefit of this viewpoint is that $G_{\substack{\text{spacetime}\\ -\text{internal}}} = \mathrm{Pin}(d,1) \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)$ relates to the so-called AII class topological insulator's symmetry group in the Wigner-Dyson-Altland-Zirnbauer symmetry classification [28–30].

(iii) In summary of the above, we put four $\mathbb{Z}_2$ groups together: $\mathbb{Z}_2^P$, $\mathbb{Z}_2^R$, $\mathbb{Z}_2^T$ into disconnected components of Eq. (12), and the $\mathbb{Z}_2^C$ can be introduced either (1) generally by a conjugate linear map on the Hilbert space of state vectors, or (2) narrowly by an outer automorphism of G_{internal} or $G_{\substack{\text{spacetime}\\ -\text{internal}}}$. Then, the order-16 group can be fitted into both Eq. (6) and Eq. (12)'s framework.

(iv) We can also view the $\tilde{G}_\psi \equiv \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$ extended from the bosonic $G_\phi \equiv \mathbb{Z}_2^C \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$ via a $\mathbb{Z}_2^{\mathrm{F}}$ extension:

$$1 \to \mathbb{Z}_2^{\mathrm{F}} \to \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}} \to \mathbb{Z}_2^C \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T \to 1. \tag{14}$$

Then the spin-0 boson ϕ sits at an (*anti*)*linear representation* of G_ϕ, but the spin-1/2 Dirac fermion ψ sits at a *projective representation* of G_ϕ. The ψ carries fractional quantum numbers of G_ϕ is in fact in an (*anti*)*linear representation* of $\tilde{G}_\psi$. The spinor ψ is thus a fractionalization of a scalar ϕ. The *symmetry extension* [31] as $1 \to \mathbb{Z}_2^{\mathrm{F}} \to \tilde{G}_\psi \to G_\phi \to 1$ implies that whether ψ may or may not have 't Hooft anomaly in G_ϕ, but ψ can become anomaly free via the pullback to $\tilde{G}_\psi$.

(7) In addition, we can study other similar spacetime-internal symmetry, compatible with $G_{\text{spacetime}}$ contains Lorentz (boost and rotation) symmetry and $G_{\text{internal}} = \mathrm{U}(1)$ while they both share $\mathbb{Z}_2^{\mathrm{F}}$. This can be done, by solving the group extension [8,32]: $1 \to \mathrm{O}(d,1) \to G_{\substack{\text{spacetime}\\ -\text{internal}}} \to \mathrm{U}(1) \to 1$, and enumerating the solutions of $G_{\substack{\text{spacetime}\\ -\text{internal}}}$, based on Minkowski or Euclidean notations,

$$\begin{aligned} &\mathrm{Pin}(d,1) \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1) \ \text{ or } \ \mathrm{Pin}^{\tilde{c}+} \equiv \mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)\colon \mathrm{AII},\\ &\mathrm{Pin}(1,d) \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1) \ \text{ or } \ \mathrm{Pin}^{\tilde{c}-} \equiv \mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)\colon \mathrm{AI},\\ &\mathrm{Pin}(d,1) \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1) \ \text{ or } \ \mathrm{Pin}^{c} \equiv \mathrm{Pin}^\pm \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)\colon \mathrm{AIII}. \end{aligned} \tag{15}$$

These groups are known to be compatible with AII, AI, and AIII symmetry classifications of quantum (e.g., condensed or nuclear) matters [28–30]. The AI and AII have $T^2 = +1$ and $T^2 = (-1)^{\mathrm{F}}$, respectively, the antiunitary T does *not* commute with a chargelike (operator $\hat{q}$) U(1),

$$TU_{\mathrm{U}(1)} = U_{\mathrm{U}(1)}^{-1} T, \qquad \text{namely } T\mathrm{e}^{\mathrm{i}\hat{q}\theta} = \mathrm{e}^{-\mathrm{i}\hat{q}\theta} T, \tag{16}$$

known also as the symmetry of topological insulators.

For AIII, regardless $T^2 = +1$ or $(-1)^{\mathrm{F}}$, the antiunitary T commutes with an isospinlike (operator $\hat{s}$) U(1),

$$TU_{\mathrm{U}(1)} = U_{\mathrm{U}(1)} T, \qquad \text{namely } T\mathrm{e}^{\mathrm{i}\hat{s}\theta} = \mathrm{e}^{\mathrm{i}\hat{s}\theta} T, \tag{17}$$

known also as the symmetry of topological superconductors. Note that $T\mathrm{i}T^{-1} = -\mathrm{i}$, $T\hat{q}T^{-1} = \hat{q}$, and $T\hat{s}T^{-1} = -\hat{s}$.

(i) The AII case has a total $\tilde{G}_\psi = \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$ in Eq. (6).

(ii) The AI case has $T^2 = +1$, so we replace Eq. (6)'s $\mathbb{Z}_4^{T\mathrm{F}}$ by another subgroup $\mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^T$ instead. Then Eq. (6) reduces to a different order-16 non-Abelian $\tilde{G}_\psi = \mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_2^T$.

(iii) The AIII case has a subtle U(1) and T relation given by Eq. (17), e.g., one can realize this new T' as the combined $T' = CT$ [17,18] of Eq. (5). We leave this and other symmetry realizations in upcoming works [33].

(8) *Majorana fermion*: Other than the Dirac spinor ψ discussed above, we can ask what happens to Majorana spinor? Once we impose the Majorana condition,

$$C\psi(x)C^{-1} = \psi_C(x) = -\mathrm{i}\gamma^2\psi^*(x) = \psi(x),$$

the $\mathbb{Z}_2^C$ acts trivially as an identity on Majorana spinor. Therefore, we shall reduce the total group structure to P-R-T-$(-1)^{\mathrm{F}}$ without C. Then Eq. (6)'s total group $\tilde{G}_\psi$ reduces to an order-8 abelian group, $\mathbb{Z}_2^P \times \mathbb{Z}_4^{T\mathrm{F}}$ for the AII case, and $\mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$ for the AI case.

III. 1+1D SPIN-1/2 FERMIONIC SPINORS

Now we move on to the C-P-R-T fractionalization structure for $1+1$d relativistic fermions.

Dirac fermion: We can regard a $1+1$d massless Dirac spinor ψ as two complex Weyl spinors in $\mathbf{1}_L \oplus \mathbf{1}_R$ (left L + right R) rep, easily seen in the Weyl basis gamma matrices,

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The *active* C-P-T transformation on ψ gives

$$C\psi(x)C^{-1} = \psi'_C(x) = \gamma^5\psi^*(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\psi^*(x).$$
$$P\psi(x)P^{-1} = \psi'_P(x) = \gamma^0\psi(x'_P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi(x'_P).$$
$$T\psi(x)T^{-1} = \psi'_T(x) = \gamma^0\psi(x'_T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi(x'_T).$$
$$(CPT)\psi(x)(CPT)^{-1} = \psi'_{CPT}(x) = \gamma^5\psi^*(-x).$$
$$C^2 = P^2 = T^2 = (CPT)^2 = +1. \quad (CP)^2 = (-1)^{\mathrm{F}}. \tag{18}$$

(i) Remarkably, $CP = (-1)^{\mathrm{F}}PC$, so we still have Eq. (9)'s $\mathbb{D}_8^{\mathrm{F},CP}$.
(ii) Again, T is antiunitary, so precisely $T(z\psi(x))T^{-1} = z^*\gamma^0\psi(x'_T)$, but luckily the complex conjugation K is not manifest in this gamma matrix basis. Since $T^2 = +1$, the $\mathbb{Z}_4^{T\mathrm{F}}$ in Eq. (6) is replaced by the $\mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^T$.
(iii) PT commutes with every group element, so we derive that the order-16 total group is $\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_2^{PT}$. This particular case is within AI case in Eq. (15), we leave other spacetime-internal symmetry realizations (e.g., AII, AIII) in upcoming works [33].

Majorana fermion: A $1+1$d Majorana spinor imposes the condition,

$$C\psi(x)C^{-1} = \psi_C(x) = \gamma^5\psi^*(x) = \psi(x),$$

the $\mathbb{Z}_2^C$ acts trivially as an identity on the *real* Majorana spinor. Then we reduce the Eq. (6)'s total group to an order-8 group $\mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$.

IV. 0 + 1D MAJORANA FERMION ZERO MODES

Kitaev's fermionic chain [34] is a $1+1$d nonrelativistic quantum system, hosting a Majorana zero mode on each open end of a $0+1$d boundary. The $0+1$d low energy effective boundary action is $\int \mathrm{d}t\chi \mathrm{i}\partial_t\chi$ for each $0+1$d real Majorana fermion χ. There is no parity P in $0+1$d, and no C for the real Majorana. When the bulk of k fermionic chains with $k \bmod 8 \neq 0$ are protected by $G = \mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^T$ symmetry, the k-boundary's zero modes are not gappable (with the dimension of Hilbert space as $2^{\frac{k}{2}}$) as long as G is preserved due to the 't Hooft anomaly in G is classified by $k \in \mathbb{Z}_8$ [35,36]. References [37–43] suggest that at $k = 2$ (or $k = 2 \bmod 4$, in general) has various supersymmetric quantum mechanical interpretations. Concretely, we follow Ref. [41], which shows this boundary can realize an extended symmetry $\tilde{G} = \mathbb{D}_8^{\mathrm{F},T} = \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^{\mathrm{F}}$. The two-dimensional Hilbert space $\mathcal{H} = \{|\mathrm{B}\rangle, |\mathrm{F}\rangle\} = \mathcal{H}_{\mathrm{B}} \oplus \mathcal{H}_{\mathrm{F}}$ has a bosonic and a fermionic ground state, say $|\mathrm{B}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\mathrm{F}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The fermion parity $(-1)^{\mathrm{F}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3$ and the time reversal $T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ K $= \sigma^2$ K do not commute, i.e., $(-1)^{\mathrm{F}}T(-1)^{\mathrm{F}} = T^{-1} = -T$. Also, $T^2 = -\sigma^0 = -1$ and $T^4 = +1$. This example can be interpreted as a generalization of *symmetry extension* [31] (in contrast to *symmetry breaking*) to cancel (or trivialize) the $k = 2$ anomaly in G by a *supersymmetry extension* pullback to $\tilde{G}$ [41]. *Supersymmetry extension* means that there exists some symmetry generator (here T) such that this generator switches between bosonic $|\mathrm{B}\rangle$ and fermionic $|\mathrm{F}\rangle$ sectors; thus, this generator does not commute with the fermion parity $(-1)^{\mathrm{F}}$. It can be also understood as a T fractionalization from an order-4 Abelian $G = \mathbb{Z}_2^{\mathrm{F}} \times \mathbb{Z}_2^T$ (with $T^2 = +1$) to an order-8 non-Abelian $\tilde{G} = \mathbb{D}_8^{\mathrm{F},T} = \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^{\mathrm{F}}$ (with $T^2 = -1$ and $T^4 = +1$).

If we change the bulk symmetry to be protected by a $G = \mathbb{Z}_4^{T\mathrm{F}}$, then Ref. [41] finds that the $k = 2$ Majorana zero mode anomaly can be canceled (or trivialized) by a *supersymmetry extension* pullback to an order-16 non-Abelian group $\tilde{G} = \mathbb{M}_{16}$ [41]. It can be also understood as a T fractionalization from an order-4 Abelian G [with $T^2 = (-1)^{\mathrm{F}}$ and $T^4 = +1$] to $\mathbb{M}_{16}$ (with $T^4 = -1$ and $T^8 = +1$ [37,41]).

V. 3 + 1D SPIN-1 MAXWELL OR YANG-MILLS GAUGE THEORY

We briefly analyze *C-P-R-T* group structure for the spin-1 gauge theories, pure U(1) Maxwell or SU(N) Yang-Mills (YM) theories of $3+1$d actions $\int \mathrm{Tr}(F \wedge \star F) - \frac{\theta}{8\pi^2} g^2 \mathrm{Tr}(F \wedge F)$ of a 2-form field strength $F = \mathrm{d}a - \mathrm{i}ga \wedge a$ with a θ term. We will see that generalized global symmetries [44] (i.e., 1-form symmetries $G_{[1]}$ that act on 1d Wilson or 't Hooft line operators in contrast to 0d point particle operators) can enrich the group structure. Follow the notations of [45], the *active C-P-T* transformations act on the spin-1 gauge bosons in terms of 1-form gauge field, $a = a_\mu \mathrm{d}x^\mu = a_0\mathrm{d}t + a_j\mathrm{d}x^j = (a_0^\alpha \mathrm{d}t + a_j^\alpha \mathrm{d}x^j)T^\alpha$ with the real-valued four-vector component (namely $a_\mu^\alpha \in \mathbb{R}$) and the Hermitian Lie algebra generator (namely the Hermitian conjugate $T^{\alpha\dagger} = T^\alpha$ and a real Lie structure constant $f^{\alpha\beta\gamma} \in \mathbb{R}$ in the commutator $[T^\alpha, T^\beta] = \mathrm{i} f^{\alpha\beta\gamma} T^\gamma$), as

$$\begin{aligned}
C a_\mu^\alpha(x) C^{-1} &= \mp\, (a_0^\alpha(x), a_j^\alpha(x)), \qquad C T^\alpha C^{-1} = T^\alpha. \\
P a_\mu^\alpha(x) P^{-1} &= (a_0^\alpha(x'_P), -a_j^\alpha(x'_P)), \qquad P T^\alpha P^{-1} = T^\alpha. \\
T a_\mu^\alpha(x) T^{-1} &= (\pm a_0^\alpha(x'_T), \mp\, a_j^\alpha(x'_T)), \qquad T T^\alpha T^{-1} = T^{\alpha *}. \\
CT a_\mu^\alpha(x) (CT)^{-1} &= (-a_0^\alpha(x'_T), +a_j^\alpha(x'_T)). \\
CPT a_\mu^\alpha(x) (CPT)^{-1} &= -(a_0^\alpha(-x), a_j^\alpha(-x)).
\end{aligned} \tag{19}$$

The gauge field associated with a real symmetric Lie algebra generator (namely the complex conjugate $T^{\alpha *} = T^\alpha$) has the upper version of the sign choices. The gauge field associated with an imaginary antisymmetric Lie algebra generator (namely $T^{\alpha *} = -T^\alpha$) has the lower version of the sign choices. However, overall, we can *rewrite* the *C*-*P*-*T* symmetries on the combined $a_\mu = a_\mu^\alpha T^\alpha$ from Eq. (19) equivalently as

$$\begin{aligned}
C a_\mu(x) C^{-1} &= (a_0^\alpha(x), a_j^\alpha(x))(-T^\alpha) = -a_\mu^*(x). \\
P a_\mu(x) P^{-1} &= (a_0(x'_P), -a_j(x'_P)). \\
T a_\mu(x) T^{-1} &= (a_0(x'_T), -a_j(x'_T)). \\
CT a_\mu(x) (CT)^{-1} &= (-a_0^*(x'_T), +a_j^*(x'_T)). \\
CPT a_\mu(x) (CPT)^{-1} &= -(a_0^*(-x), a_j^*(-x)) = -a_\mu^*(-x).
\end{aligned} \tag{20}$$

Other than *C*-*P*-*R*-*T* symmetries (manifest at $\theta = 0, \pi$), the pure U(1) gauge theory has 1-form electric and magnetic symmetries, denoted as $\mathrm{U}(1)^e_{[1]} \times \mathrm{U}(1)^m_{[1]}$, while the pure SU(2) YM has a 1-form electric symmetry $\mathbb{Z}^e_{2,[1]}$ [44]. It can be shown that *kinematically*, the U(1) gauge theory has

$$(\mathrm{U}(1)^e_{[1]} \times \mathrm{U}(1)^m_{[1]}) \rtimes \mathbb{Z}_2^C,$$

and where $\mathbb{Z}_2^P \times \mathbb{Z}_2^T$ are contained in the Lie group $\mathrm{O}(d,1)$; the SU(2) YM has instead $\mathbb{Z}_2^P \times \mathbb{Z}_2^T \times \mathbb{Z}^e_{2,[1]} \subset \mathrm{O}(d,1) \times \mathbb{Z}^e_{2,[1]}$ [no $\mathbb{Z}_2^C$ due to no SU(2) outer automorphism] which fermionic/bosonic extension is studied carefully in [45] also in [32]. These global symmetries *C*-*P*-*R*-*T*-$G_{[1]}$ are preserved *kinematically* at $\theta = 0$ and π, but the gauge *dynamical* fates (spontaneously symmetry breaking or not) are highly constrained by their 't Hooft anomalies of higher symmetries. (These 't Hooft anomalies are firstly discovered in [44,46], later found to be captured by precise invertible topological QFTs via cobordism invariants by [45,47]. Dynamical constraints of these anomalies are explored in particular by [45,48].)

We leave additional analysis and other general gauge groups of gauge theories (see examples in Ref. [49] for SU(N) YM with $N > 2$, and Refs. [50,51] for 2 + 1d) for future works [33].

VI. APPLICATIONS

As applications, we briefly apply the above results to physical pertinent systems.

(1) For any proposed duality between two seemingly different QFTs, their global symmetries must be matched. So the *C*-*P*-*T* fractionalization provides a constraint to verify the duality.

(2) Quantum electro/chromodynamics ($\mathrm{QED}_4/\mathrm{QCD}_4$):

(i) For Dirac fermions coupled to U(1) background fields [which $\mathrm{U}(1) \supset \mathbb{Z}_2^{\mathrm{F}}$, the full spacetime-internal symmetry contains $\mathrm{Pin}^{\tilde{c}+}$ in Eq. (15) and $\tilde{G}_\psi = \frac{\mathbb{D}_8^{\mathrm{F},CP} \times \mathbb{Z}_4^{T\mathrm{F}}}{\mathbb{Z}_2^{\mathrm{F}}}$]. By dynamically gauging the U(1), the outcome QED_4 reduces the $\mathrm{Pin}^{\tilde{c}+}$ to O(3,1) while reduces the $\tilde{G}_\psi$ to $\mathbb{Z}_2^C \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$. However, if the Dirac fermion has a large mass at ultraviolet (UV), at infrared (IR) there could be new emergent 1-form symmetries [44] (whose charged objects are one-dimensional Wilson or 't Hooft lines), which do not commute with the $\mathbb{Z}_2^C$.

(ii) Dirac fermions can be in the fundamental or adjoint reps of SU(2) when coupling to SU(2) gauge fields. In the case of the fundamental rep, $\mathrm{SU}(2) \supset \mathbb{Z}_2^{\mathrm{F}}$, so the fundamental QCD_4 obtained by gauging SU(2) reduces $\tilde{G}_\psi$ to $\mathbb{Z}_2^C \times \mathbb{Z}_2^P \times \mathbb{Z}_2^T$. However, for the adjoint rep, $\mathrm{SU}(2) \not\supset \mathbb{Z}_2^{\mathrm{F}}$, the resulting adjoint QCD_4 keeps the same order-16 $\tilde{G}_\psi$. In fact, this C-P-T fractionalization $\tilde{G}_\psi$ can provide a constraint to verify the UV-IR duality between the UV adjoint QCD_4 theory and the IR Dirac fermion theory previously studied in [52–55].

(iii) For Dirac fermions coupled to SU(3) in the fundamental rep [which $\mathrm{SU}(3) \not\supset \mathbb{Z}_2^{\mathrm{F}}$], the resulting real-world SU(3) QCD_4 indeed can keep this C-P-T fractionalization order-16 $\tilde{G}_\psi$. Moreover, the CPT theorem and Vafa-Witten theorem [56] say that CPT and P cannot be spontaneously broken in a vectorlike QCD theory. If the strong CP problem further indicates that the CP (thus T) is not violated in the real-world QCD_4 [namely, say $\theta = 0$ for the θ term $\frac{\theta}{8\pi^2} g^2 \mathrm{Tr}(F \wedge F)$], then all discrete C-P-T are preserved which implies that the order-16 $\tilde{G}_\psi$ can be preserved in the vacuum of the real-world QCD_4, at least within the strong force sector.

Of course, the weak force sector breaks P and CP, so $\tilde{G}_\psi$ is still violated within the full Standard Model.

VII. FRACTIONAL SPIN-STATISTICS AND *CPT*

Since the early studies by Pauli [57], and by Schwinger-Pauli-Lüder [1–6], physicists are intrigued by the subtle relation between the spin-statistics theorem and the CPT theorem. Some observations and comments are in order:

(i) We were well-informed that quantum excitations in $2+1$d, called anyons, can have the fractional spin s (self-statistics gives a Berry phase $\mathrm{e}^{\mathrm{i}2\pi s}$) and also Abelian or non-Abelian statistics (mutual statistics); see the reviews [58,59].

(ii) In higher dimensions ($3+1$d or above), there are no 0d particlelike anyons (of 1d worldline) with fractional statistics; but there are extended objects (1d loop-like anyonic strings on 2d world sheets, or nd branes on $(n+1)$d world volumes) that can also have fractional statistics, either Abelian or non-Abelian statistics [60–62]—when those world trajectories of these objects forming nontrivial mathematical *link invariants* in the spacetime [63–65].

(iii) Fractional C-P-T symmetry does not necessarily imply fractional spin statistics of anyons beyond fermions. For example, the $3+1$d Dirac spinor of Eq. (6) and Eq. (14) shows that the fermion ψ sits in the projective rep of G_ϕ and carries fractionalized C-P-T quantum numbers of G_ϕ, but ψ sits in the (anti)linear rep of $\tilde{G}_\psi$. The ψ does not have anyonic statistics, but only has fermionic statistics (spin $s = 1/2$, but still fractionalized with respect to a bosonic integer spin).

(iv) Vice versa, fractional spin-statistics of anyons do not imply a fractional C-P-T symmetry, because intrinsic topological orders (that give rise to anyons) do not necessarily require any global symmetry.

(v) The *spin-statistics theorem* colloquially says the *self-braiding statistics* of an excitation can be deformed to the *mutual-braiding statistics* between two (or more) excitations, illustrated by Dirac belt and Feynman plate tricks [66]. Thus, this theorem reveals the *topological properties of matter*: the topological links of world trajectories of (semi-classical or entangled quantum) matter excitations inside the spacetime manifold.

(vi) The *CPT or CRT theorem* colloquially says that our physical laws are also obeyed by a CRT image of our universe. Thus, this theorem reveals the *topological properties of spacetime*, the disconnected components of the spacetime symmetry groups, and how the matter-antimatter are transformed under those discrete symmetries.

(vii) We propose that the relation between the spin-statistics theorem and the CPT theorem may also shed light on the relation between the *fractional spin statistics* and the *fractionalized* C-P-R-T structure. Follow the promise of the fractional spin-statistics studies in the past decades [58,59], we anticipate that the fractional C-P-R-T topic presented here will also offer various future applications, both relativistic or nonrelativistic, in high-energy physics or quantum material systems.

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