UV graviton scattering and positivity bounds from IR dispersion relations

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Scattering amplitudes mediated by graviton exchange display IR singularities in the forward limit. This obstructs standard application of positivity bounds based on twice-subtracted dispersion relations. Such divergences can be canceled only if the UV limit of the scattering amplitude behaves in a specific way, which implies a very nontrivial connection between the UV and IR behaviors of the amplitude. We show that this relation can be expressed in terms of an integral transform, obtaining analytic results when $t \log s \to 0$. Carefully applying this limit to dispersion relations, we find that infinite arc integrals, which are usually taken to vanish, can give a nontrivial contribution in the presence of gravity, unlike in the case of finite negative t. This implies that gravitational positivity bounds cannot be trusted unless the size of this contribution is estimated in some way, which implies assumptions on the UV completion of gravitational interactions. We discuss the relevance of these findings in the particular case of QED coupled to gravity.

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I. INTRODUCTION

The analytic properties of the S-matrix are a central element of our understanding of quantum field theory (QFT). Stemming from seminal works on partial-wave unitarity in the 1970s [1–3], there has recently been a modern resurgence of interest in this topic, in connection to the study of effective field theories (EFTs) [4]. S-matrix properties can be used to formulate *positivity bounds* for 2-to-2 scattering amplitudes within the physical region for scattered momenta in the forward limit. These are written in terms of dispersion relations which relate the scattering amplitude at a given kinematical point with the integral of its imaginary part along the whole physical region, which is strictly positive from the optical theorem.

Standard applications then follow a top-down reasoning. One first promotes a given EFT to be the low-energy expansion of an unknown ultraviolet (UV) complete theory satisfying the usual axioms of unitarity, locality, and Lorentz invariance. Positivity bounds are then valid and applicable to the UV-complete theory. However, they can also be evaluated at small center-of-mass energy, in the infrared (IR) region. There, scattering amplitudes are well approximated by those computed in the EFT. As a consequence, positivity implies constraints on the Wilson coefficients accompanying those relevant operators that contribute to the S-matrix elements. This UV-IR connection has been thoroughly used in the literature to constrain, or assess the validity, of many different EFTs; see for example Refs. [5–24].

This approach however, relies on the existence of a mass gap in the spectrum of the EFT, a property needed for the forward limit to be regular. In gapless theories, exchange of massless particles leads to forward limit divergences in the scattering amplitudes, which obstruct a direct application of positivity bounds. This divergence can be relaxed for both scalar and vector degrees of freedom by a proper regularization, but the fundamental problem remains for graviton exchange, which requires an alternative approach. An elegant way out of this is to isolate the divergence on the right-hand side of the dispersion relation—which is exact—so that it can be canceled against the one on the left-hand side [25]. By doing this, one is left with an approximate positivity bound, which can be mildly violated by terms that become important only at very high energies. Nevertheless, these approximate bounds are still very powerful in constraining many IR proposals for gravitational physics [26–28]. This result can also be obtained in a different way, based on the impact parameter formulation [29].

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In order for this cancellation to be possible, an assumption about the high-energy behavior of graviton scattering has to be made, though. In Ref. [25] this was assumed to be of the Regge form, inspired by the Veneziano formula of string scattering [30], but we could of course wonder if this result is unique, or if there are other possible UV behaviors that work. This has been partially answered in Ref. [31], where it was shown that subtraction of the tree-level divergence requires a linear Regge trajectory at leading order in the high-energy region, but nothing has been established so far about subleading corrections. This is an important question because, for example, string scattering is not an exact linear Regge trajectory. Subleading corrections are always present. Thus, it is natural to ask ourselves: are these unique? What kind of subleading terms allow for well-posed dispersion relations and positivity bounds? Is the scattering of strings the only possible UV behavior of graviton scattering that satisfies this condition? And moreover, are positivity bounds insensitive to this choice? In general, we ask ourselves: what is the minimal piece of information about the UV behavior of gravitation needed for dispersion relations to be well posed?

In this work we try to answer this question by reversing the usual direction of thought in the literature on positivity bounds. By looking at the structure of graviton exchange in the IR limit, and exploiting mathematical properties of the dispersion relation, we constrain the UV behavior of graviton-mediated scattering amplitudes. We arrive at an integral formula that relates the IR structure of forward-limit divergences to properties of the UV completion, which is further constrained in the limit $t \log s \to 0$. We also show how this knowledge modifies the standard derivation of positivity bounds, leading even to indeterminate bounds unless extra assumptions about the UV completion are made.

A recent development in a similar direction—reverse bootstrapping UV amplitudes from IR properties—was presented in Ref. [32], where they considered the scattering of photons and gravitons in QED coupled to the Einstein-Hilbert action. Based on one-loop computations done in Ref. [33], they showed that parametrically large negative terms appear in the dispersion relation, naively contradicting positivity bounds unless new physics is introduced at relatively low energies. Instead, the authors of Ref. [32] argued that the presence of these negative large pieces can, and most likely should, have an origin in nontrivial properties of the UV completion, which might in principle be affected by the presence of light particles such as the electron [34]. In this work we show explicitly how this way of thinking, together with our results about the shape of the UV scattering amplitude, can lead to a resolution of the mentioned tension.

This paper is organized as follows. First, we introduce dispersion relations for theories with graviton exchange in Sec. II, and we show how cancellation of IR divergences determines several properties of scattering amplitudes in the UV in Sec. III. Later, in Sec. IV we show how our findings imply a nonvanishing value for the arc integrals contributing to the dispersion relation, and we discuss how this can solve the conundrum that emerges when applying positivity bounds to gravitationally coupled QED in Sec. V. Section VI is devoted to showing how positivity bounds might be rendered useless by our results in the presence of gravitation, while in Sec. VII we explore an explicit example of a UV completion in the form of string amplitudes, finding agreement with our result. Finally, we draw our conclusions in Sec. VIII.

II. DISPERSION RELATIONS WITH GRAVITON EXCHANGE

Let us start by considering the $ab \to ab$ scattering between some identical but otherwise unspecified initial and final states with equal mass m, as described in an EFT with an unknown UV completion, but which we demand to be causal, local, unitary and Lorentz invariant. We also assume that this process includes exchange of a massless graviton. Due to Lorentz invariance, the scattering amplitude $\mathcal{A}(s,t)$ can be uniquely described in terms of the Mandelstam variables s, t, and u, satisfying $s+t+u=4m^2$. The presence of massless gravitons in the scattering channel implies that the amplitude will diverge in the forward limit[1] $t \to 0^-$. The typical expansion of the amplitude in this limit, including tree-level graviton exchange and one graviton loop, has the form

$$\mathcal{A}(s,t) = A_0 \frac{s^2}{M_P^2}\frac{1}{t} + A_1 \frac{s^2}{M_P^4}\log\left(\frac{-t}{\mu_R^2}\right) + (\text{regular terms}) + (\text{higher loops}), \tag{1}$$

where μ_R is the renormalization scale. In the forward limit, this expression has explicit $1/t$ and $\log t$ divergences. The pole is inherited from the one in the graviton propagator, while loop corrections are responsible for generating the logarithmic branch cut, which represents production of soft gravitons of arbitrarily low energy. The values of A_0 and A_1 characterize the particular theory from which this amplitude is obtained. In perturbation theory, they are proportional to the residue in the pole of the propagator of the massless graviton, and to the β function of the a^2b^2 coupling. Higher loops—two and beyond—will produce further logarithmic divergences, but we ignore them hereinafter, since they are further suppressed. From crossing symmetry, the amplitude contains in general equivalent divergences s^{-1}, u^{-1}, $\log(s)$, $\log(u)$. Due to the latter, the

[1]It is important to remark here that the forward limit corresponds to taking $t \to 0$ from the negative side of the real axis, since the physical region corresponds to $t < 0$.

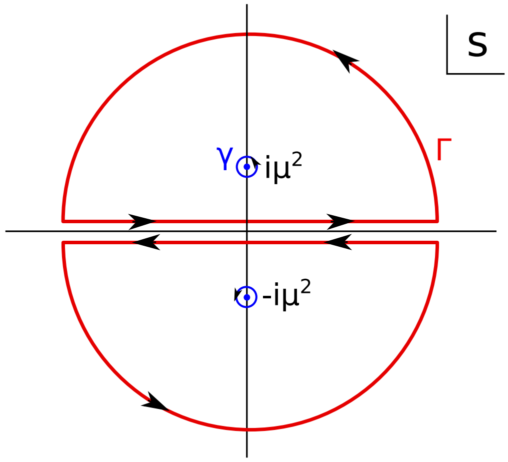

FIG. 1. Integration contours for the dispersion relation in Eqs. (2) and (3).

forward limit amplitude $\mathcal{A}(s, t \to 0^-)$ will also exhibit a branch cut along the whole real axis in the complex plane for s, which obstructs the standard derivation of positivity bounds [4]. This can be avoided, however, by following the derivation in Ref. [31], which we adopt hereinafter. We thus consider the following quantity:

$$\Sigma(\mu, 0^-) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathcal{A}(s, 0^-) s^3 ds}{(s^2 + \mu^4)^3}, \tag{2}$$

where the value 0^- must be understood as $t \to 0^-$ at all times. Thus, we retain only divergent and finite terms, and get rid of those that vanish polynomially in t. Here the integral is taken over the contour γ, as shown in Fig. 1, corresponding to the sum of two small circles surrounding the values $s = \pm i\mu^2$. The choice of $\mu \in \mathbb{R}$ is a matter of convenience, and later we will assume it to be much smaller than the cutoff scale of the low-energy EFT.

Note that the analytic structure of the amplitude is completely determined by the assumptions above. It is fully analytic in the whole s plane, except for the branch cut along the whole real axis. We can thus modify the integration contour to Γ, consisting in two lines $\mathrm{Re}(s) \pm i\epsilon$, with $\epsilon \ll 1$, together with two arcs of infinite radius. Thus

$$\begin{aligned}\Sigma(\mu, 0^-) &= \frac{1}{2\pi i} \oint_\Gamma \frac{\mathcal{A}(s, 0^-) s^3 ds}{(s^2 + \mu^4)^3} \\ &= \int_0^\infty \frac{ds}{\pi} \left(\frac{s^3 \mathrm{Im}\mathcal{A}(s + i\epsilon, 0^-)}{(s^2 + \mu^4)^3} \right. \\ &\quad \left. + \frac{(s - 4m^2)^3 \mathrm{Im}\mathcal{A}^\times (s + i\epsilon, 0^-)}{((s - 4m^2)^2 + \mu^4)^3} \right) + \Sigma_\infty. \end{aligned} \tag{3}$$

Here we have used the Schwarz reflection principle $A^*(s) = A(s^*)$ to relate the integral in the lower part of the complex plane to that in the upper part, and introduced the crossing-symmetric process[2] $\mathcal{A}^\times(s, t)$, obtained by letting $s \to u$, together with a change of variables, to rewrite the whole expression as an integral over positive values of s.

In the previous formula, Σ_∞ stands for the sum of the integrals along the two infinite arcs $\Gamma_C = \Gamma^+ + \Gamma^-$ in the upper and lower parts of the complex plane,

$$\Sigma_\infty = \frac{1}{2\pi i} \oint_{\Gamma_C} ds \frac{\mathcal{A}(s, 0^-)}{s^3}, \tag{4}$$

where μ has been neglected, since for this integral $|s| \to \infty$. The contribution of Σ_∞ is normally ignored by invoking the Froissart-Martin bound [35] in the case of exchange of massive particles. For graviton exchange, it is typically assumed that a certain version of this bound holds in the form

$$\lim_{|s| \to \infty} \left| \frac{\mathcal{A}(s, 0^-)}{s^2} \right| = 0, \tag{5}$$

which seems enough for the arc integrals to vanish. This bound has been rigorously derived for theories in $d > 4$ space-time dimensions, but the realistic case of $d = 4$ remains elusive, so Eq. (5) has to be regarded as an extra assumption at this stage. In this work, we do not want to make such an assumption on the UV behavior of the scattering amplitude, and thus we keep the integral arbitrary hereinafter. Our only starting requirement will be that Σ_∞ does not contain any forward limit singularities stronger than a pole. Namely $(t \cdot \Sigma_\infty)_{t \to 0^-} \sim \text{const}$.

Note that Eq. (3) can be thought of as a formula connecting the IR and UV behaviors of a given theory. While the rhs is an explicit integral along the full range of s, and thus sensitive to the properties of the UV theory, the value of $\Sigma(\mu, 0^-)$ can also be computed in the IR region by using Eq. (2), provided that μ is sufficiently small. In this case, it can even be computed in an EFT approximation of the full theory, as long as $\mu \ll \Lambda$, with Λ being the cutoff of the EFT. This leads to a simple expression in terms of the residues of the integrand in Eq. (2)

$$\Sigma(\mu, 0^-) = \frac{\mathcal{A}_{ss}(i\mu^2, 0^-)}{16} - \frac{3i\mathcal{A}_s(i\mu^2, 0^-)}{16\mu^2}, \tag{6}$$

where $\mathcal{A}_s(x, 0^-) = \partial_s \mathcal{A}(s, 0^-)|_{s=x}$, and equivalently with $\mathcal{A}_{ss}(x, 0^-)$ and the second derivative. For an amplitude of the form (1), which can be obtained from an EFT coupled to general relativity,[3] one obtains

[2]We have also used the fact that $s + t + u = 4m^2$. For a detailed derivation of this expression, cf. Ref. [31].

[3]Or, in general, to any theory whose tree-level gravitational dynamics matches that of the Einstein-Hilbert action.

$$\Sigma(\mu, 0^-) = \frac{1}{2}\left(\frac{A_0}{t} + A_1 \log t\right) + (\text{regular terms}) + (\text{higher loops}). \quad (7)$$

Thus, $\Sigma(\mu, 0^-)$ defined as in Eq. (2) captures exactly the coefficient in front of the s^2 term in the amplitude. Remarkably, this expression still contains singularities in the forward scattering limit $t \to 0^-$, produced by the s^2 dependence in those terms coming from graviton exchange. In the next section we discuss what these singularities tell us about the UV theory.

It is important to remark here that this problem is particular to graviton scattering. If one scatters any other massless species—scalars or vector bosons—which are instead described by a renormalizable theory, the scattering amplitude will still be divergent in the forward limit. However, these divergences come without a quadratic s dependence, which means that they disappear when computing Σ_∞, leading to a regular dispersion relation.

III. GRAVITON SCATTERING AND CANCELLATION OF DIVERGENCES

Let us start by recalling that Eq. (3) is *exact*. On its derivation there is no approximation or expansion whatsoever, besides taking an approximate forward limit. This means that, if the lhs is divergent, the rhs must be so. However, the latter depends only on the imaginary part of the scattering amplitude, which is regular within the physical region by application of the optical theorem

$$\mathrm{Im}\mathcal{A}(s, 0) = s\sqrt{1 - \frac{4m^2}{s}}\sigma(s), \quad (8)$$

and the requirement that the total cross section is finite. As discussed in Ref. [32], this is enough to conclude that the divergence on the rhs of Eq. (3) must come from the failure of the integral to converge when $t \to 0^-$ at some high-energy regime $s \gg M_*^2$, where M_* is thus the scale above which the EFT fails to describe graviton scattering and must be replaced by its UV completion. Assuming the mildest possible analytic behavior of the amplitude leads to a linear Regge trajectory (cf. Ref. [32] and the Appendix in Ref. [31]),

$$\mathrm{Im}\mathcal{A}(s, t)|_{s \gg M_*^2} = r_* s^{2+\alpha' t}. \quad (9)$$

Here $\alpha' \sim M_*^{-2}$ and r_* are provided by the concrete UV completion leading to this form. By assuming this *exact* high-energy behavior, we can cancel the tree-level pole on the lhs of Eq. (3). Such a linear Regge behavior for graviton scattering is typical in string theories. Indeed, it can be obtained from the famous Veneziano amplitude [30], describing the scattering of four closed bosonic strings [36].

However, it is naive to assume that the Regge behavior at high energies is an exact linear trajectory. Indeed, this would lead to two problems. First, it only provides cancellation of the tree-level pole. Moreover, plausible candidates for UV amplitudes, like the scattering of strings, are not Regge exact, as only their leading behavior is of this form. It is then natural to wonder what is the possible allowed form of these subleading corrections. One first mandatory requirement is that they must be able to cancel the logarithmic divergences as well as the pole. As shown in Ref. [31], this requires subleading terms to contain a piece

$$\mathrm{Im}\mathcal{A}(s, t)|_{s \gg M_*^2} = r_* s^{2+\alpha' t}\left(1 + \frac{\zeta}{\log(\alpha' s)}\right), \quad (10)$$

where ζ is a dimensionless constant, but nothing else is known beyond this. We show now however, that we can indeed obtain a good amount of extra information on the UV amplitude by simply requiring the cancellation of divergences, obtaining some results that go beyond the linear Regge trajectory.

In order to proceed, we assume that the high-energy form of the imaginary part of the amplitude reads

$$\mathrm{Im}\mathcal{A}(s, t)|_{s \gg M_*^2} = s^{2+\alpha' t}\phi(s, t), \quad (11)$$

where $\phi(s, t)$ is an arbitrary function. We now take Eq. (3) and split the integration regime on the rhs as

$$\int_0^\infty = \int_0^{M_*^2} + \int_{M_*^2}^\infty. \quad (12)$$

As we previously discussed, all the divergences on this rhs come from the high-energy behavior of the integral, which means that the first piece in the previous expression is regular. Thus, we move it to the lhs and write

$$\Sigma - \Sigma_\infty - \frac{1}{\pi}\int_0^{M_*^2} ds F(s) = \frac{1}{\pi}\int_{M_*^2}^\infty ds F(s), \quad (13)$$

where we have introduced

$$F(s) = \frac{s^3 \mathrm{Im}\mathcal{A}(s + i\epsilon, 0^-)}{(s^2 + \mu^4)^3} + \frac{(s - 4m^2)^3 \mathrm{Im}\mathcal{A}^\times(s + i\epsilon, 0^-)}{((s - 4m^2)^2 + \mu^4)^3}. \quad (14)$$

The lhs here thus contains divergences and regular pieces, which we decide to separate as

$$\lim_{t \to 0^-}\left(\Sigma - \Sigma_\infty - \frac{1}{\pi}\int_0^{M_*^2} ds F(s)\right) = \frac{\beta_0}{t} + \beta_1 \log(-t) + \bar{f}, \quad (15)$$

where $\bar{f}$ is constant. On the other hand, we take the limit $\{m, \mu\} \ll M_*$ on the rhs, as well as the assumption that the external states are bosonic—and thus $\mathcal{A}^\times(s, 0^-) = \mathcal{A}(s, 0^-)$—arriving at

$$\frac{\beta_0}{t} + \beta_1 \log(-t) + \bar{f} = \frac{1}{\pi}\int_{M_*^2}^{\infty} \frac{ds}{s} s^{\alpha' t}\phi(s,t) \\ = \frac{M_*^{2\alpha' t}}{\alpha' \pi}\int_0^\infty d\sigma \phi(\sigma, t) e^{\sigma t}, \quad (16)$$

where in the last step we have performed a change of variables $s = M_*^2 e^{\sigma/\alpha'}$. Finally, taking $x = -t$, recalling that the physical region corresponds to $t < 0$, and thus $x > 0$, and absorbing proportionality coefficients onto the definitions of β_0, β_1 and $\bar{f}$, we arrive at the final expression

$$\frac{\beta_0}{x} + \beta_1 \log(x) + \bar{f} = \int_0^\infty d\sigma \phi(\sigma, x) e^{-\sigma x}, \quad (17)$$

where we can recognize the Laplace measure in the integral on the rhs.

Knowing the UV behavior of $\mathrm{Im}\mathcal{A}(s,t)$, Eq. (17) can easily be used to compute the coefficients β_0, β_1 and $\bar{f}$, in a standard way. However, the inverse problem, obtaining the form of $\phi(\sigma, x)$ from the coefficients of the IR amplitude, is not so simple. Actually, this mathematical problem has, in general, infinitely many possible solutions, but not all of them will satisfy the analyticity properties that we must require for a physical amplitude.[4]

In order to find a proper solution, let us make use here of Watson's lemma [37] for the integral in Eq. (17). We will thus assume that the function $\phi(\sigma, x)$ satisfies[5]

$$\lim_{\sigma\to\infty} \frac{\phi(\sigma, x)}{e^{\gamma\sigma}} = 0, \quad (18)$$

for some $\gamma > 0$, and that it is a meromorphic function[6] around $\sigma = 0$. Thus, it can be expanded in a Laurent series around this point

$$\phi(\sigma, x) = \sum_{n=0}^{\infty}\left(a_n(x)\sigma^n + \frac{b_n(x)}{\sigma^n}\right), \quad (19)$$

with $b_0(x) = 0$. Any individual term of this sum, when plugged into Eq. (17), corresponds to a Laplace transform of σ^a for a certain power a.

Let us focus first on the nonanalytic pieces of the series. By performing the integral—using the analytic continuation of the Γ function—we get

$$b_n(x)\int_\epsilon^\infty d\sigma\, e^{-\sigma x}\,\sigma^{-n} = \frac{(-1)^n}{(n-1)!} b_n(x) x^{n-1}\log(x) \\ + \mathcal{O}(x, \epsilon^{-1}), \quad (20)$$

where $\epsilon \ll 1$ is a regulator. For $n \geq 2$ we get terms that are not present on the lhs of Eq. (17), unless $b_n(x) \sim x^{1-n}$. However, this violates the assumption of analyticity of $\phi(\sigma, x)$ when $x \to 0$. We conclude that all singular terms in the Laurent series for $n \geq 2$ must vanish. We thus have

$$\phi(\sigma, x) = \frac{b_1(x)}{\sigma} + \sum_{n=0}^{\infty} a_n(x)\sigma^n, \quad (21)$$

where $b_1(x) = \beta_1 + \mathcal{O}(x)$, in order to account for the $\log(x)$ forward divergence on the lhs of Eq. (17). This justifies the choice made in Ref. [31].

We shift our focus now to the Taylor series. Since $\phi(\sigma, x)$ is analytic around $x = 0$, we have

$$\lim_{x\to 0} a_n(x) = a_n x^{\eta_n}, \quad (22)$$

where all the η_n are constant and we assume them to be different. Later we will see that this is necessary to avoid double and higher poles in Eq. (17). For now, let us take it as an assumption. We now invoke the partial-wave expansion of the amplitude for a unitary theory, which implies (see Appendix B of Ref. [7] for a proof)

$$\frac{d^k}{dt^k}\mathrm{Im}\mathcal{A}(s,t)|_{t=0} \geq 0, \quad (23)$$

for all k and all values of s within the physical region. Using this fact, we easily conclude by direct computation that

$$a_n \geq 0, \quad \text{for all } n. \quad (24)$$

Knowing this, we now plug the Taylor series in Eq. (21) back into the integral, and by integrating term by term we get

[4]Indeed, a trivial solution is given by $\phi(\sigma, x) = e^{\sigma x}(\frac{\beta_0}{x} + \beta_1 \log(x) + \bar{f})\delta(\sigma - x)$, which does not satisfy analyticity at $x = 0$ for all values of σ.

[5]Exponential boundedness is a softer behavior than the one required by the Froissart-Martin bound. The latter is actually problematic when confronted with the full Veneziano amplitude for string scattering, which does not satisfy it. Instead, Veneziano's amplitude is exponentially bounded, exactly as we require here.

[6]The assumption of meromorphicity is true at one loop, as we show below. However, we might be forced to abandon it in order to account for higher-loop divergences in the IR, such as $\log(\log(x))$. Nevertheless, all these terms will enter with an extra scale suppression and we thus ignore them hereinafter.

$$\sum_{n=0}^{\infty}\int_0^{\infty} ds\, e^{-sx}a_n(x)s^n = \sum_{n=0}^{\infty}\frac{a_n(x)\Gamma(n)}{x^{n+1}}. \tag{25}$$

Since all $a_n > 0$, there cannot be cancellations between different values of n, which implies that all the terms on the rhs must, at most, diverge as a single pole. This implies

$$a_n(x) = a_n x^n + \mathcal{O}(x^{n+1}), \tag{26}$$

for some constant, perhaps vanishing, coefficient a_n. Note however that, in order to cancel the single pole β_0/t in Eq. (17), at least one of the a_n coefficients must be nonvanishing.

At this point one might be worried about the convergence of the sum in Eq. (25), since we are expanding around $\sigma = 0$ and integrating over the whole real line. However, this leads to no problems in the setting discussed here. Let us show this explicitly by cutting the Taylor series at a finite order N

$$\phi(\sigma,x) = \frac{b_1(x)}{\sigma} + \sum_{n=0}^{N} a_n(x)\sigma^n + \mathfrak{R}_{N+1}(\sigma,x). \tag{27}$$

Since this is a Laurent series, there exists a function $K(x)$ such that

$$|\mathfrak{R}_{N+1}(\sigma,x)| < K(x)\sigma^{N+1}, \tag{28}$$

at least in the limit $x \to 0$ of interest. Using this condition we can thus estimate the size of the remainder after cutting the series and exchanging the order of summation and integration. We have

$$\left|\int_0^{\infty} d\sigma\, e^{-\sigma x}\mathfrak{R}_{N+1}(\sigma,x)\right| < \int_0^{\infty} d\sigma\, e^{-\sigma x}|\mathfrak{R}_{N+1}(\sigma,x)| < K(x)\int_0^{\infty} d\sigma\, e^{-\sigma x}\sigma^{N+1}. \tag{29}$$

The last integral is immediate and gives

$$\left|\int_0^{\infty} d\sigma\, e^{-\sigma x}\mathfrak{R}_{N+1}(\sigma,x)\right| < \mathcal{O}\left(\frac{1}{x^{N+2}}\right). \tag{30}$$

Noting that the Nth term in the series contributes at order $a_n(x)x^{-(N+1)}$, this shows that Eq. (25) is thus well behaved as an asymptotic series, which is enough for our purposes here.

Before going forward let us go back to the condition (22). Now it is obvious that all η_n have to be different in the limit $x \to 0$. Unless they satisfy Eq. (26) there would be extra divergences after integration of $\phi(\sigma,x)$. A possible way out would be to have two terms giving rise to the same divergence and canceling each other. However, since all $a_n > 0$, this is not possible. We thus conclude that the form of our asymptotic expansion is indeed unique and reads in the forward limit, once all knowledge is collected

$$\lim_{x\to 0}\phi(\sigma,x) = \frac{\beta_1}{\sigma} + \sum_{n=0}^{\infty} a_n(x\sigma)^n, \qquad \sum_{n=0}^{\infty} a_n\Gamma(n) = \beta_0. \tag{31}$$

Note that the expansion of the function $\phi(\sigma,x)$ in the forward limit, which naively corresponds to $x \to 0$, has become instead an expansion when $x\sigma \to 0$, since it is only under this assumption that Eq. (31) is well behaved as an asymptotic expansion. This suggests that the proper forward limit to account for the presence of graviton exchange, at least at high energies, is actually

$$\tau = \sigma x \sim t\log(s) \to 0, \tag{32}$$

or $t\log|s| \to 0$ in the complex plane for s, which ensures perturbative control of the UV amplitude. As we will see in a moment, this has a strong impact on the derivation of positivity bounds. Since the asymptotic limit $|s| \to \infty$ and $t \to 0$ has to satisfy Eq. (32), the computation of the integral along Γ_C in Eq. (4) has to be taken carefully.

IV. ARC INTEGRALS IN THE FORWARD LIMIT

By means of analyticity and crossing symmetry, one can actually go beyond our results in the previous section and recover the asymptotics of the whole amplitude from its imaginary part. Indeed, for $|s| \gg M_*^2$ we can write a totally standard dispersion relation for the scattering amplitude by using Cauchy's integral theorem. We have [31,38]

$$\mathcal{A}(s,t) = \frac{s^2}{2\pi i}\oint_{\gamma_s}\frac{\mathcal{A}(z,t)dz}{z^2(z-s)} = F(s,t) + F(-s-t,t). \tag{33}$$

Here γ_s is a small circle around $z = s$ and we have exploited crossing symmetry to obtain an explicit expression in terms of s and $u = -s-t$. The function $F(s,t)$ reads[7]

$$F(s,t) = \frac{s^2}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\mathcal{A}(z,t)dz}{z^2(z-s)} = \frac{s^2}{\pi}\int_0^{M_*^2}\frac{\mathrm{Im}\mathcal{A}(z,t)dz}{z^2(z-s)} + \frac{s^2}{\pi}\int_{M_*^2}^{\infty}\frac{a_0 z^{\alpha' t}dz}{z-s} + \mathcal{O}(t\log(s)), \tag{34}$$

where we have taken into account only the leading term in the expansion (31).

[7] See Ref. [38] for a detailed derivation. Here we are taking the subtraction point $\mu_p^2 \ll |s|$, and taking into account that all pathologies of the scattering amplitude, such as the pole, are contained in the IR.

In the region of validity $|s| \gg M_*^2$, and the first integral becomes proportional to s/M_*^2, so that it can be neglected. The asymptotics of the last one for large $|s|$ leads instead to

$$F(s,t) = -\frac{a_0 e^{-i\pi\alpha' t}}{\sin(\pi\alpha' t)} s^{2+\alpha' t}. \tag{35}$$

Thus, we see that the total leading part of the amplitude, and not only its imaginary part, is actually completely determined. It reads

$$A(s,t) = -\frac{a_0 e^{-i\pi\alpha' t}}{\sin(\pi\alpha' t)} (s^{2+\alpha' t} + (-s-t)^{2+\alpha' t}) + \mathcal{O}(t\log(s)). \tag{36}$$

This result of course reproduces the imaginary part $a_0 s^{2+\alpha' t}$, while being at the same time its analytic and crossing-symmetric continuation.[8]

After obtaining this asymptotic form for the UV scattering amplitude including exchange of gravitons, we can now focus on understanding whether we can really set the contribution of the infinite arcs Σ_∞ to the dispersion relation (3) to vanish or not. In the case of gapped theories, the Froissart-Martin bound $\mathcal{A}(s,t) < s\log^2 s$ guarantees that $\Sigma_\infty = 0$. For Eq. (36), we instead find a different result.

Let us then compute the integral in Eq. (4) explicitly. Taking into account that the arc Γ_R is described by $s = Re^{i\theta}$ with $R \to \infty$, and that

$$\frac{1}{2\pi i}\oint_{\Gamma_R} ds \frac{s^{2+\alpha' t}}{s^3} = \frac{R^{\alpha' t}}{2\pi}\int_0^{2\pi} d\theta e^{i\alpha' t\theta} = \frac{R^{\alpha' t}}{2\pi}\frac{e^{2\pi i\alpha' t} - 1}{i\alpha' t}, \tag{37}$$

we get

$$\Sigma_\infty = -\frac{2a_0 e^{-i\pi\alpha' t}}{\sin(\pi\alpha' t)}\frac{R^{\alpha' t}}{2\pi}\frac{e^{2\pi i\alpha' t} - 1}{i\alpha' t} = -\frac{2a_0}{\pi\alpha' t}R^{\alpha' t} = -\frac{2a_0}{\pi\alpha' t}, \tag{38}$$

where we have taken into account that the asymptotic expansion is well controlled only when $t\log R \to 0$. Since in this limit $R^{\alpha' t} \to 1$, we get a nonvanishing contribution, unlike in a case where one first takes the limit $R \to \infty$ with small but finite t [39].

For the sake of completeness, let us notice that if one takes the integral over the branch cut in Eq. (3) not from M_*^2 to infinity, but from M_*^2 to R, one obtains instead

$$\Sigma_{\rm UV} = \frac{2}{\pi}\int_{M_*^2}^\infty \frac{ds \mathrm{Im}\mathcal{A}(s,t)}{s^3} = \frac{2}{\pi}\int_{M_*^2}^R \frac{ds}{s} a_0 s^{\alpha' t} = \frac{2a_0}{\pi\alpha' t}(R^{\alpha' t} - (M_*^2)^{\alpha' t}). \tag{39}$$

In the limit $t\log R \to 0$ both terms go to unity and we get

$$\Sigma_{\rm UV} = O(t) + O(M_*^4) \tag{40}$$

without the $1/t$ divergence, which appears instead to be captured in the arc contribution Σ_R. If instead we carefully take $R \to \infty$ first at finite negative t, the familiar result with $\Sigma_\infty = 0$ is reproduced [39]. Thus, it seems that the combination $\Sigma_{\rm UV} + \Sigma_\infty$ does not depend on the way we take the limits in R and t, although individual terms do. Hereinafter we use the limit $\tau \to 0$ motivated by our findings in Sec. III. This simplifies the computation by allowing us to use only the leading term in the τ expansion for the imaginary part of the amplitude.

With this choice, and for the simple amplitude (36), the infinite arc brings a $1/t$ term. However, subleading terms might also lead to finite contributions. In particular, by noting that the new pole contribution comes from the real part of the amplitude, which is not constrained by unitarity or any of the other arguments here, it seems that any finite value can be obtained from the UV, simply by modifying $\mathrm{Re}\mathcal{A}(s,t)$. For example, we can take the equally valid amplitude

$$A(s,t) = -\frac{a_0 e^{-i\pi\alpha' t}}{\sin(\pi\alpha' t)}(s^{2+\alpha' t} + (-s-t)^{2+\alpha' t}) + \beta e^{-i\pi\alpha' t}(s^{2+\alpha' t} + (-s-t)^{2+\alpha' t}), \tag{41}$$

which leads to

$$\Sigma_\infty = -\frac{2a_0}{\pi\alpha' t} + \beta + O(t). \tag{42}$$

This indeed contains a finite piece that must be matched by the infrared terms on the lhs of Eq. (3) in order for the dispersion relation to be valid. Notice however, that β is not constrained at all, and in particular it is not forced to be positive. It can be large ($|\beta| \gg M_*^{-4}$) and negative. Thus, its value will influence the applicability of positivity bounds based on the relation (3). We will discuss this point later.

Here we did not include the $\log(-t)$ divergences in the dispersion relation. Given that they can be canceled by the subleading contributions to $\mathrm{Im}\mathcal{A}(s,t)$ at large s, their impact on Σ_R should also be subleading. An accurate computation shows that the term $s^{2+\alpha' t}/(\log s)$ appearing in the imaginary part would always lead to a vanishing contribution. Thus, only those terms needed to cancel the leading IR divergences induce a nontrivial value of the infinite arc integrals. However, let us also note that

[8]Notice that for large $s > 0$ and small $t < 0$ the second term does not contribute to the discontinuity.

although subleading IR divergences are not sensitive to the real part of the amplitude in the UV, they are still relevant to recover its imaginary part, as previously discussed in this work.

V. QED WITH GRAVITY

Following the derivation of positivity bounds from twice-subtracted dispersion relations, several works examined their consequences for different physical theories of interest for model building. In particular, bounds in the presence of graviton exchange were closely studied in recent works [32,33]. There, and by looking at photon scattering, the authors showed that positivity bounds are easily violated by gravitational contributions. In this section we examine how our findings and, in particular, the contribution of the infinite arcs, relax this issue.

Let us then consider QED coupled to gravitation, with the action

$$S = \int \sqrt{-g}\, d^4x \left(-\frac{M_P^2}{2} R - \frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(D_\mu \gamma^\mu - m)\psi \right), \tag{43}$$

where $F_{\mu\nu}$ is the photon field strength, and ψ is a fermion field with charge e. Following Ref. [33] we look at $2 \to 2$ photon scattering, including tree-level graviton exchange and one-loop fermion corrections, thus retaining contributions up to $\mathcal{O}(e^6)$ and $\mathcal{O}(M_P^{-4})$. In the forward limit, and by taking $s \gg m^2$, this reads

$$A(s,t) = -\frac{s^2}{M_P^2 t} + \frac{1}{M_P^2}\left(-\frac{11 e^2 s^2}{360\pi^2 m^2} + \frac{e^2 s}{12\pi^2} \right) + \frac{11 e^4 s^2}{720\pi^2 m^4}. \tag{44}$$

The different topologies contributing to this scattering are shown in Fig. 2.

The rhs of Eq. (3) contains four different pieces when computed for this amplitude. The first one is the integral of the imaginary part of the amplitude for those contributions that survive in the limit $M_P \to \infty$. These can be computed analytically, since the action (43) corresponds to a renormalizable theory in this limit, and were obtained in Ref. [33]. We will borrow their result here. The second piece is the contribution given by pure gravitational terms in the IR. Again, these can be computed explicitly by using the amplitude above, as the authors of Ref. [33] did. We also have those coming from the UV part of the integral, when $s \gg M_*^2$, and which depend on the UV completion, as previously discussed. We name them $\Sigma_{\rm UV}$. Finally, we have the contribution of the infinite arcs, which we have learned cannot be taken to vanish *a priori*. The total result for the right-hand side—in the limit of small μ—is then

$$\Sigma(\mu^2, 0^-) = \frac{11 e^4}{360\pi^2 m^4} - \frac{11 e^2}{360\pi^2 m^2 M_P^2} + \Sigma_{\rm UV} + \Sigma_\infty. \tag{45}$$

On the other hand, the lhs of Eq. (3) can be directly computed and reads in this case

$$\Sigma(\mu^2, 0^-) = A_{ss}(s) = -\frac{2}{M_P^2 t} - \frac{11 e^2}{180\pi^2 m^2 M_P^2} + \frac{11 e^4}{360\pi^2 m^4}. \tag{46}$$

As we can see, both results agree in the decoupling limit of gravitational interactions. This is not a surprise because, as we have already pointed out, the action (43) is renormalizable in this limit. This means that Eq. (3) becomes trivial. The divergence $1/t$ however, has a pure gravitational origin and, as we have discussed, will be canceled by the interplay between $\Sigma_{\rm UV}$ and Σ_∞. However, the contributions of order $(mM_P)^{-2}$, showing up on both sides, do not cancel each other. If we were being naive, assuming that $\Sigma_\infty = 0$ and simply canceling the pole with $\Sigma_{\rm UV}$, then we would find a clash between the two approaches to derive $\Sigma(\mu^2, 0^-)$. The only way out is to assume the existence of new physics turning on before gravitational interactions, such that the amplitude gets modified and leads to a cancellation of the undesired piece

$$-\frac{11 e^2}{360\pi^2 m^2 M_P^2} + \frac{\Theta}{\Lambda^2} = 0, \tag{47}$$

where Θ is a constant. From simple dimensional analysis we can see that this implies that the cutoff of the Standard Model (SM)—in other words, the scale of introduction of new physics—would be $\Lambda \sim \sqrt{mM_P}$, which in QED leads to $\Lambda \sim 10^8$ GeV, significantly lower than the Planck scale. This option was studied in Ref. [32].

However, there is another natural solution to the problem: the possibility of having a nonzero infinite arc contribution Σ_∞ with negative sign. As we have shown in previous sections, the value of Σ_∞ contains contributions from the real part of the gravitational amplitude at high

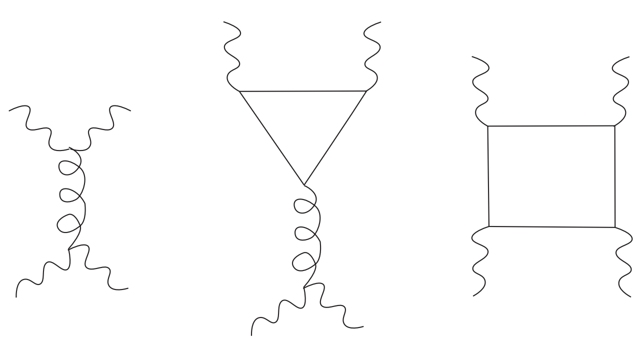

FIG. 2. Topologies providing the leading contributions to the photon scattering amplitude at large s in the forward limit.

energies—the term β in Eq. (41)—which are not constrained at all. Thus, they can potentially cancel the remaining negative contribution of order $(mM_P)^{-2}$. Moreover, by considering this possibility, we can obtain nontrivial information about the UV completion of gravitational interactions. In particular, in the case of QED, we can conclude that the UV amplitude in the Regge limit should "know" about the presence of the light particle (electron), since it needs to contain a large negative contribution related to it. For instance, by borrowing Eq. (41), a possible consistent UV amplitude in the forward limit is

$$A(s,t) = e^{-i\pi\alpha' t}\left(-\frac{11e^2}{360\pi^2 m^2 M_P^2} - \frac{\pi\alpha'}{M_P^2 \sin(\pi\alpha' t)}\right) \times (s^{2+\alpha' t} + (-s-t)^{2+\alpha' t}), \tag{48}$$

but this option is, of course, not unique and other possible amplitudes could lead to similar physical results, cf. Ref. [34]. This is simply an example of a situation in which IR-UV decoupling is not present. There are contributions—the electron loop here—that contribute to the amplitude at all energies. Let us stress that the Regge slope α' cannot be fixed from IR considerations, and it is instead connected to properties of the UV completion.

By considering this solution to the conundrum unveiled in Refs. [32,33], QED can be "rescued" and trusted as a good EFT when coupled to gravitation up to the Planck scale. Of course, a realistic model of QED breaks down before M_P, since it needs to be embedded in the electroweak model, but a similar reasoning can be made even in the general case of the SM, where the problem is even worse due to neutrino loops, which bring the cutoff down to values just slightly beyond the LHC's reach.

VI. FATE OF GRAVITATIONAL POSITIVITY BOUNDS

The prototypical application of Eq. (3) is to derive positivity bounds, constraints on the values of Wilson coefficients of EFTs, by explicitly computing the value of $\Sigma(\mu, 0^-)$ in the IR. These are obtained by simply considering the following expression:

$$\Sigma = \int_{s_{\rm th}}^{\infty} \frac{ds}{\pi}\left(\frac{s^3 \mathrm{Im}\mathcal{A}(s,0)}{(s^2+\mu^4)^3} + \frac{(s-4m^2)^3 \mathrm{Im}\mathcal{A}^\times(s,0)}{((s-4m^2)^2+\mu^4)^3}\right), \tag{49}$$

where we have assumed that $\Sigma_\infty = 0$ for the moment. Here $s_{\rm th}$ stands for the threshold of particle production where the branch cut starts on the real axis. For scattering processes without massless particles in the exchange channel, this corresponds to $s_{\rm th} = 4m_l^2$, where m_l is the mass of the lightest exchanged state, and the integral runs along the physical regime for the Mandelstam variable s [4]. In the case of a massless exchange, we have $s_{\rm th} = 0$. Taking into account that the optical theorem (8) implies that the integrand on the rhs is always positive, from unitarity requirements of the UV completion, we can conclude that

$$\Sigma > 0, \tag{50}$$

which in turn will imply conditions on the Wilson coefficients contributing to the scattering amplitudes and ultimately to Σ.

The bounds (50) can be improved by noting that part of the rhs can actually be computed within an EFT. Splitting the integral on the rhs as $\int_{s_{\rm th}}^{\infty} = \int_{s_{\rm th}}^{\Lambda^2} + \int_{\Lambda^2}^{\infty}$, we can move the first piece to the left and conclude in the same fashion that

$$\Sigma - \int_{s_{\rm th}}^{\Lambda^2} \frac{ds}{\pi}\left(\frac{s^3 \mathrm{Im}\mathcal{A}(s,0)}{(s^2+\mu^4)^3} + \frac{(s-4m^2)^3 \mathrm{Im}\mathcal{A}^\times(s,0)}{((s-4m^2)^2+\mu^4)^3}\right) > 0. \tag{51}$$

These *improved* positivity bounds have also been referred to in the literature as *beyond* positivity bounds [40].

If $s_{\rm th} \neq 0$ there is a simpler way to bound the coefficient in front of s^2 in the amplitude. One can equivalently derive a bound for

$$\bar{\Sigma} = \frac{1}{2\pi i}\oint \frac{\mathcal{A}(s)ds}{(s-\mu^2)^3} = \frac{1}{2}\mathcal{A}_{ss}(s) > 0, \tag{52}$$

which is applicable for $\mu^2 < s_{\rm th}$ [4,8,40]. This approach, however, cannot be directly applied for scattering of massless particles. The presence of a branch cut with $s_{\rm th} = 0$ requires using the more complicated dispersion relation (3).

These types of bounds can be systematically obtained from many different dispersion relations, by simply taking more subtractions; see Refs. [38,41]. Even amplitudes containing graviton exchange can provide rigorous bounds for the coefficients in front of higher powers of s (s^4 and beyond) as well as for their t derivatives [24], which are regular in the forward limit. This happens because the $1/t$ pole (as well as the loop IR singularities) is accompanied by a s^2 power at most.[9] For this reason, only the application of positivity bounds for the s^2 term gets obstructed in the presence of graviton exchange and graviton loops.

Of course, one can proceed naively by regularizing the divergence in the same way as we have done here, by keeping $t < 0$, and by simply bounding the coefficient accompanying the divergence to be negative, since it dominates the bound. However, this is just the residue in the pole of the graviton propagator, whose sign is already

[9]This may not be true in theories with higher-spin states, which we are not considering here.

constrained to satisfy trivial requirements of perturbative unitarity. Thus, no new information is obtained from positivity bounds in this case, unless one resolves the singularity. This can be done provided that the contribution of the infinite arc Σ_∞ vanishes. In this case the divergence can be simply canceled by the proper Regge behaviour in the UV. However, there are finite remainders whose sign cannot be determined *a priori*, and thus one arrives at an approximate positivity bound

$$\Sigma > -O(M_*^{-4}), \tag{53}$$

which allows for small negativity [25,31].

As we have mentioned, this is only true under the extra assumption that the infinite arc contribution is either zero in the limit $t \to 0$ or shown to be parametrically smaller than $O(M_*^{-4})$. However, as we have discussed in Sec. V with the example of QED coupled to gravity, the contribution of the infinite arc can actually be negative and parametrically large, violating this assumption. This reflects the fact that the loops of light particles can affect the amplitude in the forward limit even in the UV region of large s. If this happens, then no bounds can be set for the finite part of the s^2 term in the amplitude, since the former are modified to

$$\Sigma > \Sigma_\infty - \mathcal{O}(M_*^{-4}),$$

which is meaningless without a systematic way to determine the size of Σ_∞. Any amount of negativity can always be explained by contributions to the UV amplitude which, to the best of our knowledge, do not contradict any of the basic principles of QFT.

Although the identification of the undetermined term as part of the infinite arc integral is related to our choice of kinematics in the UV, controlled by $\tau \to 0$, let us stress that the previous conclusion is not tied to it. For other choices, Σ_∞ might vanish, but a similar contribution would arise from the branch cut, leading to the same physical conclusion [39].

As a final note, let us note that a possible way out of this conundrum is the case when the IR amplitudes are parametrically larger than the UV contribution to the arcs. This requires the existence of a cutoff scale in the IR Λ such that the amplitude in this region can be organized as an EFT

$$\mathcal{A}(s,t) = \sum_{n=0}^{\infty} \mathcal{A}_n(s,t)\Lambda^{-n}, \tag{54}$$

while gravitational dynamics will contribute with terms ordered in inverse powers of M_P. In the case in which $\Lambda \ll M_P$, the contribution of Σ_∞ can thus be safely neglected, so that we recover an approximate positivity bound

$$\Sigma > \mathcal{O}(M_P^{-2}). \tag{55}$$

This justifies the application of positivity bounds to the case of gapped theories much below the gravitational scale—and the neglect of gravity even though everything universally couples to gravitation—but the problem survives if one wants to account for graviton exchange. Information about the UV completion is needed.

VII. DO STRING THEORIES PROVIDE BONA FIDE POSITIVITY BOUNDS?

We cannot say that we are currently in a position to provide a number of known nonperturbative amplitudes for the test of our findings above for the UV behavior of gravitational scattering. However, string theory gives us some hints on how some examples are constructed. In this section we use string amplitudes as a test ground to see if one indeed recognizes $\tau = t\log(s)$ as the expansion parameter of the forward limit, and to assess what happens to the large arc integral, i.e., whether they can provide a constant contribution, similar to the one that solves the QED conundrum in Sec. V, upon some conditions or not.

Let us try the four-graviton scattering amplitude derived in type II superstring theory. It can be written in the following form [25,42]:

$$\mathcal{A}_{\rm string}(s,t) = -A(s^2t^2 + s^2u^2 + t^2u^2) \times \left.\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}\right|_{u=-s-t}, \tag{56}$$

where A is positive and the string constant is set to $\alpha' = 4$ for simplicity. This amplitude represents the tree-level four-graviton scattering in the closed superstring NS-NS sector. The polynomial factor accounts for the polarization states of the spin-2 particle. This result has some limitations, though. Neither NS-R nor open-closed string interactions are included here. Moreover, it takes into account only scattering of string states without addressing the question of how branes enter the game. Note that D-branes, which superficially accommodate the previously successful Chan-Paton factors, are responsible for the inclusion of SM particles in string phenomenology scenarios [43,44]. All this means that the amplitude above is far from describing our real world, but it is still of great interest for our purposes here, because it is an example of an extremely successful theoretically justified nonperturbative scattering description.

The expansion of the amplitude (56) for $s \to \infty$ and $t \to 0$ can be performed straightforwardly. Arranging the terms in powers of t and keeping the leading s contribution we get

$$\mathcal{A}_{\text{string}}(s,t) = A\frac{s^2}{t}\left(1 + 2t\log(s) + 2t^2\log^2(s) + \frac{4}{3}t^3\log^3(s) + \frac{2}{3}t^4\log^4(s) + \ldots\right), \tag{57}$$

where we readily reveal the canonical s^2/t pole for the spin-2 massless particle, and recognize the presence of $\tau = t\log(s)$ as the expansion parameter. We note that the appearance of τ is a very nontrivial property. It was not granted *a priori* to observe it here. However, our considerations in previous sections suggested its presence as a necessary condition for a healthy amplitude. Its emergence here thus serves as a very nontrivial sanity check of our results.

Computing $\Sigma(\mu, 0^-)$ out of $\mathcal{A}_{\text{string}}(s,t)$ one gets

$$\Sigma(\mu, 0^-) = -\frac{A}{2t} + 4At + O(t^2), \tag{58}$$

where we notice that the first subleading contribution is linear in t, with no constant $\mathcal{O}(1)$ term. This result can be obtained by following the procedure outlined in Ref. [25]. It requires summing over residues of poles arising at the integer negative points of the Γ function. Remarkably, note that since the constant term is absent, the amplitude (56) does not provide the large negative contribution that saves the day in the case of QED, cf. Sec. V.

Therefore, we conclude that pure NS-NS superstring amplitudes cannot heal the curious contribution observed in Ref. [33]. However, there is hope for this to happen once SM particles are included in the amplitude, through coupling to D-branes. This is definitely an ambitious open question to be understood in the string framework, as one needs generalizations of SM amplitudes computed from the string perspective. Alternatively, there may still be a window in the string framework on its own if one includes other contributions arising from NS-R interactions or from open-closed string interactions. This analysis is however clearly beyond the scope of the present work.

VIII. CONCLUSIONS

In this paper we have shown that the requirement of cancellation of IR forward divergences appearing in graviton-mediated scattering is enough to constrain the form of the imaginary part of the scattering amplitude at very high energies, above a scale M_*. In particular, we have proven that whatever the form of the UV completion of gravitation is, $\mathrm{Im}\mathcal{A}(s,t)$ must admit an asymptotic expansion of the form (31) in the limit $\tau \propto t\log s \to 0$. The appearance of the parameter τ is a highly nontrivial feature that is however reproduced in the known case of the Veneziano amplitude of string theory [30], as we discussed in Sec. VII.

The determination of the form of $\mathrm{Im}\mathcal{A}(s,t)$ has an immediate impact on the construction of positivity bounds, which are widely used to constrain EFTs of matter coupled to gravitation. In their derivation, there appear integrals along arcs with radius $|s| \to \infty$, which are typically taken to vanish, either by invoking the Froissart-Martin bound for gapped theories, or with other arguments in the gapless case. By using our expansion we showed however that this cancellation is not guaranteed and instead depends on the form of the *real* part of the scattering amplitude, which is not constrained at all, to the best of our knowledge. In the case that this real contribution exists, the predictability of gravitational positivity bounds is doomed, since the previous simple expression $\Sigma > 0$, which is computable within an EFT, gets modified as

$$\Sigma - \Sigma_\infty > 0, \tag{59}$$

which is meaningless unless some input about the contribution of the UV completion Σ_∞ is given case by case.

Although this situation is overall negative for the applicability of positivity bounds, it can also have a bright side. The undetermined contribution from the UV completion could compensate the presence of anomalously large negative terms appearing in Σ in the case of QED coupled to Einstein-Hilbert gravity, which we have studied in Sec. V, and in the general case of the SM. A naive solution to both preserving the fate of positivity bounds, and accepting these terms, is to assume the existence of new physics above a relatively low-energy scale $\Lambda \sim \sqrt{m_l M_P}$, where m_l is the mass of the lightest fermion. In the case of the SM this can be within the LHC scale and thus puts in tension the validity of the SM itself. In contrast to this solution, the existence of a nonvanishing contribution Σ_∞, coming from the real part of the scattering amplitude of the UV completion, can solve the issue and preserve the validity of the SM up to the Planck scale. However, this would imply that positivity bounds cannot give us any new information in these situations.

Alternatively, we can look at this as an opportunity to realize a reverse bootstrapping. We can use the theories in the IR to compute the value of Σ, and use it to determine contributions to $\mathrm{Re}\mathcal{A}(s,t)$ at high energies through Σ_∞. This could give important insight on how light particles contribute to graviton scattering even beyond the Planck scale.

Finally, we have tested our results by looking at the nonperturbative amplitude for graviton scattering obtained from the scattering of NS-NS closed superstrings. This amplitude is indeed organized as an asymptotic expansion in $\tau \to 0$ in the double limit $s \to \infty$ and $t \to 0$, confirming our results. However, it does not provide the negative large term Σ_∞ required to save QED and the SM from a low cutoff. Although this could be interpreted as a hint of the true existence of this cutoff, we believe instead that it points to the necessity of a better understanding of scattering amplitudes in the string framework. In particular, the problem at hand seems to require going beyond the simple Veneziano amplitude and also accounting for interactions with SM particles through attaching the strings to D-branes, or perhaps by also including NS-R sectors, and open-closed string interactions. Additionally, it is also interesting to question if there exist other UV completions besides string scattering that satisfy the requirements discussed in this work. Even more, we wonder if it is possible to construct model-independent amplitudes that not only cancel the IR forward divergences in graviton scattering, but also render the SM safe until the Planck scale, and what we can say about these amplitudes.

One possible direction of research along these lines could be to consider triple-product amplitudes. Indeed, by looking at Eq. (56) we see that, apart from the polarization factor, the expression is a triple product of $\mathcal{B}(z) = \Gamma(-z)/\Gamma(1+z)$ such that $\mathcal{A}(s,t) \sim \mathcal{B}(s)\mathcal{B}(t)\mathcal{B}(u)$. Recently a new set of amplitudes with triple-product structure was considered in Ref. [45], where it was claimed that a wide class of functions $\mathcal{B}(z)$ leads to a unitary construction. It would be interesting to see whether those new amplitudes obey the constraints obtained in the present paper using a general model-independent approach.

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