Beyond Schwarzschild–de Sitter spacetimes: A new exhaustive class of metrics inspired by Buchdahl for pure R^2 gravity in a compact form

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Some sixty years ago Buchdahl pioneered a program in search of static spherically symmetric metrics for pure $\mathcal{R}^2$ gravity in vacuo [H. A. Buchdahl, Nuovo Cimento **23**, 141 (1962).]. Surpassing several obstacles, his work culminated in a nonlinear second-order ordinary differential equation (ODE) which required being solved. However, Buchdahl deemed the ODE intractable and abandoned his pursuit for an analytical solution. We have finally managed to overcome this remaining hurdle and bring his program to fruition. Reformulating Buchdahl's ODE, we obtain a *novel* class of metrics (which we shall call the Buchdahl-inspired metrics hereafter) in a compact and transparent expression: $ds^2 = e^{k\int\frac{dr}{rq(r)}}\{p(r)[-\frac{q(r)}{r}dt^2 + \frac{r}{q(r)}dr^2] + r^2 d\Omega^2\}$, in which the pair $\{p, q\}$ are two functions of the radial coordinate r obeying the evolution rules $\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}, \frac{dq}{dr} = (1 - \Lambda r^2)p$, and the Ricci scalar is $\mathcal{R}(r) = 4\Lambda e^{-k\int\frac{dr}{rq(r)}}$. We are able to verify *ex post*, via direct inspection, that the metric given above satisfies the $\mathcal{R}^2$ vacuo field equation $\mathcal{R}(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{R} = 0$, hence establishing its validity. The compact form above casts the Buchdahl-inspired metric in a parallel resemblance with the classic Schwarzschild–de Sitter (SdS) metric, with the case $k = 0$ corresponding to the SdS metric. We show why the Buchdahl-inspired metric, which exhibits nonconstant scalar curvature when $k \neq 0$, defeats a "no-go" theorem proved in Kehagias *et al.* [J. High Energy Phys. 05 (2015) 143.], which posits that pure $\mathcal{R}^2$ gravity vacua are restricted to the Einstein spaces, $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$, and the vanishing Ricci scalar spaces, $\mathcal{R} = 0$. The aforementioned "no-go" theorem assumes a rapid asymptotic falloff for the metric as $r \to \infty$. However, we find that the Buchdahl-inspired metric evades that central assumption, which is overly restrictive. A product of a fourth-derivative gravity, a Buchdahl-inspired metric is specified by four parameters: Λ measuring the scalar curvature at largest distances, k effecting the variation of the curvature on the manifold, and $\{p_0, q_0\}$ initiating the "evolution" of $\{p(r), q(r)\}$ along the radial direction, forming a two-dimensional phase space. The class of Buchdahl-inspired metrics is *exhaustive* as it covers *all* "nontrivial" static spherically symmetric metrics admissible for pure $\mathcal{R}^2$ gravity in vacuo, with the SdS metric being a special case, $k = 0$. Transparently, the quartet $\{\Lambda, k, p_0, q_0\}$ spans a topological space with all members in the class of Buchdahl-inspired metrics being *smoothly connected* to the SdS metrics when k is continuously tuned to 0. In this respect, the Buchdahl-inspired metrics constitute a natural enlargement suitably regarded as a framework "beyond Schwarzschild–de Sitter." Our novel solution thereby completes Buchdahl's six-decades-old program. We also explore the mathematical properties of the Buchdahl-inspired metric in the limit of small k and in the region around the coordinate origin.

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I. MOTIVATION

In a seminal paper entitled "On the Gravitational Field Equations Arising from the Square of the Gaussian Curvature" completed in 1961 [1], Buchdahl pioneered—yet left unfinished—a program to seek static spherically symmetric metrics for *pure* $\mathcal{R}^2$ gravity in vacuo, a theory that excludes the Einstein-Hilbert term at the outset. Back in his time, Buchdahl was motivated to consider the pure $\mathcal{R}^2$ action as an interesting prototype for modified gravity. Recently, the quadratic action has witnessed resurgence [2–9]; one attractive feature of the pure $\mathcal{R}^2$ action is that it is the only theory that is both ghost-free and scale invariant [10].

Despite making significant progress, unfortunately, Buchdahl discontinued his efforts toward the finish line that was within striking distance. The purpose of our current paper is to bridge the final remaining gap in Buchdahl's "abandoned" program. The ultimate outcome is a family of static spherically symmetric vacua, expressible in a compact form, for the pure $\mathcal{R}^2$ action. We shall

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focus on the mathematical aspects of these vacua in this paper, while leaving their potential implications in physics for future research.

As Buchdahl indicated therein [1], if one were to adopt the canonical metric using Schwarzschild coordinates

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

then from the $\mathcal{R}^2$ vacuo field equation one would confront a *coupled* system of two equations for $A(r)$ and $B(r)$, one of fourth- and one of third-differential orders. Eliminating one of the two functions would yield a highly nonlinear *seventh-order* ordinary differential equation (ODE).

Nevertheless, Buchdahl was able to devise a judicious choice for the metric alternative to (1) such that the resulting ODE—albeit nonlinear—is only of *second* differential order *which remained to be solved*. For the reader's convenience, the original Buchdahl equation (as we shall call it as such, hereafter) is

$$2t\frac{d^2q}{dt^2} + \left(\frac{1+t}{1-t} - \frac{3}{4}\frac{k^2}{q^2}\right)\frac{dq}{dt} = 0; \qquad (2)$$

see Eqs. (1.7) and (3.4) in his original paper [1]. The metric he chose is then expressible in terms of the function $q(t)$ [NB: t is not the time coordinate], with the (Buchdahl) parameter k rendering Eq. (2) *nonlinear.*

The Buchdahl equation (2) is very generic; it captures *all* "nontrivial" static spherically symmetric vacua admissible for the pure $\mathcal{R}^2$ action, besides the Einstein spaces (viz. $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$) and the vanishing scalar curvature spaces (viz. $\mathcal{R} = 0$). Accordingly, *if* an *analytical* solution to the Buchdahl equation can be found, then it would yield a powerful tool to tackle the *new physics* inherent in pure $\mathcal{R}^2$ gravity [11–13]. Crucially, as shall be shown in this paper, the new (Buchdahl) parameter k in Eq. (2) would enable the $\mathcal{R}^2$ vacua to develop *nonconstant* scalar curvature.

By and large, the Buchdahl equation was an impressive achievement. Yet Buchdahl abandoned his pursuit for an analytical solution as he judged his ODE intractable.[1] This is an unfortunate twist of events as we find that this is not the case.[2] In this paper, we shall advance a number of mathematical maneuvers to reformulate the Buchdahl equation (2) in a more accessible form. From there, we are able to obtain a compact expression for a new class of metrics, which we shall call *the Buchdahl-inspired metrics*, thereby bringing his six-decades-old endeavor to a successful outcome.

For the reader's convenience, we shall briefly present our result in what follows. The Buchdahl-inspired metric is neatly expressible as

$$ds^2 = e^{k\int\frac{dr}{rq(r)}}\left\{p(r)\left[-\frac{q(r)}{r}dt^2 + \frac{r}{q(r)}dr^2\right] + r^2d\Omega^2\right\} \qquad (3)$$

with the Ricci scalar equal to

$$\mathcal{R}(r) = 4\Lambda \exp\left(-k\int\frac{dr}{rq(r)}\right) \qquad (4)$$

and the two auxiliary functions $p(r)$ and $q(r)$ evolving along the radial direction r per

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}, \qquad (5)$$

$$\frac{dq}{dr} = (1 - \Lambda r^2)p. \qquad (6)$$

The deliberate resemblance of Eq. (3) to a Schwarzschild–de Sitter (SdS) metric makes the meaning of terms transparent. The compact form (3)–(6) automatically encompasses the constant-curvature SdS when k equal zero,[3] in which case the nonlinear and singular relation in (5) stays silent. A nonzero k, however, would trigger an interplay between p and q via (5) and (6), in which case the Buchdahl-inspired metric acquires a *nonconstant* scalar curvature per (4), potentially offering a host of intricate phenomenology and *new physics*.

Our paper is organized as follows. In Sec. II we shall rework Buchdahl's original paper in a simplified and straightforward approach. Our twofold aim is to derive the results directly from the $\mathcal{R}^2$ field equation and to arrive at an ODE that is more generic than his original ODE. In Sec. III we shall introduce a shortcut toward the (generalized) Buchdahl equation while circumventing his original Hamiltonian-based procedure. In Sec. IV we shall cast his equation in a more transparent way, and then obtain a compact solution describing the new class of Buchdahl-inspired metrics. In Sec. V we shall outline the verification process that confirms the validity of our Buchdahl-inspired metrics. Between Secs. VI and IX, we shall investigate the Buchdahl-inspired metrics in four situations: (i) recovering the SdS metric at $k = 0$; (ii) deriving a new metric for the small-k limit; (iii) probing the behavior of the metrics

[1] To quote Buchdahl from his original paper (with notes in square brackets ours). On page 4 of [1]: "Unfortunately the simple appearance of [the nonlinear second order ODE] is deceptive. The best I have been able to achieve is to obtain a solution in the form of a sequence of polynomials of ascending powers of t." And on Page 8 of [1]: "[The ODE] does not appear to be soluble in terms of known functions, nor does it appear to be reducible to a simpler form. It therefore seems appropriate to determine a solution in ascending power of t, or in some similar form."

[2] No further attempts either by Buchdahl or by others have been made to solve his ODE since its publication.

[3] A fact to be shown in Sec. VI.

around the coordinate origin; and (iv) uncovering a degeneracy in the overall solution. Section X points out an overly restrictive assumption in a proof proposed in [2] against the existence of nonconstant curvature metrics (and the class of Buchdahl-inspired metrics). The Sec. XI summarizes our work.

II. GENERALIZING THE BUCHDAHL EQUATION: A MORE DIRECT ROUTE

In his original work [1] Buchdahl followed an arduous route. He designed a new Lagrangian, as a "surrogate" to the pure $\mathcal{R}^2$ gravity action, and then applied the variational principle on it. With the benefits of hindsight, we shall rework Buchdahl's formulation in a more straightforward manner. We shall start directly from the $\mathcal{R}^2$ vacuo field equation, conduct the standard calculations, and reach the *generalized* Buchdahl equation. We shall try to retain as much as possible Buchdahl's notation for the reader's convenience.

Following Buchdahl's notation, the metric in spherical coordinate is written in the form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + e^{\mu(r)}d\Omega^2,$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{7}$$

The vacuo field equation in the pure $\mathcal{R}^2$ action is

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{R} = 0, \tag{8}$$

and the "trace" equation in vacuo is

$$\Box\mathcal{R} = 0. \tag{9}$$

Since $\mathcal{R}$ is a function of r only, we have[4]

$$\nabla_\mu\nabla_\nu\mathcal{R} = \partial_\mu\partial_\nu\mathcal{R} - \Gamma^r_{\mu\nu}\partial_r\mathcal{R}. \tag{10}$$

The tt-, $\theta\theta$-, and rr-components of the vacuo field equation (8) read

$$\mathcal{R}_{tt} - \frac{1}{4}g_{tt}\mathcal{R} = -\Gamma^r_{tt}\frac{\mathcal{R}'}{\mathcal{R}}, \tag{11}$$

$$\mathcal{R}_{\theta\theta} - \frac{1}{4}g_{\theta\theta}\mathcal{R} = -\Gamma^r_{\theta\theta}\frac{\mathcal{R}'}{\mathcal{R}}, \tag{12}$$

$$\mathcal{R}_{rr} - \frac{1}{4}g_{rr}\mathcal{R} = -\Gamma^r_{rr}\frac{\mathcal{R}'}{\mathcal{R}} + \frac{\mathcal{R}''}{\mathcal{R}}. \tag{13}$$

The relevant Christoffel symbols and components of the Ricci tensors are

$$\Gamma^r_{tt}e^{\lambda-\nu} = \frac{\nu'}{2}, \tag{14}$$

$$\Gamma^r_{\theta\theta}e^{\lambda-\mu} = -\frac{\mu'}{2}, \tag{15}$$

$$\Gamma^r_{rr} = \frac{\lambda'}{2}, \tag{16}$$

and

$$\mathcal{R}_{tt}e^{\lambda-\nu} = \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu'\mu'}{2}, \tag{17}$$

$$-\mathcal{R}_{\theta\theta}e^{\lambda-\mu} = -e^{\lambda-\mu} + \frac{\mu''}{2} + \frac{\mu'^2}{2} + \frac{\nu'\mu'}{4} - \frac{\lambda'\mu'}{4}, \tag{18}$$

$$-\mathcal{R}_{rr} = \frac{\nu''}{2} + \frac{\nu'^2}{4} + \mu'' + \frac{\mu'^2}{2} - \frac{\nu'\lambda'}{4} - \frac{\lambda'\mu'}{2}. \tag{19}$$

Furthermore, the Jacobian is

$$\sqrt{-g} \triangleq \sqrt{-\det g} = e^{\frac{\nu}{2}+\frac{\lambda}{2}+\mu}\sin\theta, \tag{20}$$

giving

$$\sqrt{-g}g^{rr} = e^{\frac{\nu}{2}-\frac{\lambda}{2}+\mu}\sin\theta. \tag{21}$$

The three functions $\nu(r)$, $\lambda(r)$, $\mu(r)$ are subject to an arbitrary coordinate transform. Buchdahl made a *judicious choice* that

$$\mu(r) \equiv \frac{1}{2}(\lambda(r) - \nu(r)), \tag{22}$$

thus making

$$\sqrt{-g}g^{rr} = \sin\theta. \tag{23}$$

The "trace" equation (9)[5]

$$(\sqrt{-g}g^{rr}\mathcal{R}')' = 0 \tag{24}$$

is vastly simplified to

$$\mathcal{R}'' = 0; \tag{25}$$

hence,

$$\mathcal{R} = \Lambda + kr \tag{26}$$

in which Λ and k are two constants. If $k = 0$, the Ricci scalar is a constant everywhere. For $k \neq 0$ the Ricci scalar deviates from constancy.

[4]Recall that for a scalar field ϕ: $\nabla_\mu\nabla_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi$.

[5]Recall that for a scalar field ϕ: $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$.

With Buchdahl's choice (22), the relevant Ricci components become

$$\mathcal{R}_{tt} = \frac{\nu''}{2} e^{\nu-\lambda}, \tag{27}$$

$$\mathcal{R}_{\theta\theta} = 1 + e^{-\frac{\nu}{2}-\frac{\lambda}{2}}\left(\frac{\nu''}{4} - \frac{\lambda''}{4}\right), \tag{28}$$

$$\mathcal{R}_{rr} = -\frac{\lambda''}{2} + \frac{\lambda'^2}{8} - \frac{3\nu'^2}{8} + \frac{\nu'\lambda'}{4}. \tag{29}$$

From (14), (26), (27) the tt-equation (11) reads

$$\frac{\nu''}{2} e^{\nu-\lambda} + \frac{1}{4} e^{\nu}(\Lambda + kr) = -\frac{\nu'}{2} e^{\nu-\lambda}\frac{k}{\Lambda + kr}, \tag{30}$$

leading to

$$\nu'' + \frac{k}{\Lambda + kr}\nu' + \frac{1}{2}(\Lambda + kr)e^{\lambda} = 0. \tag{31}$$

From (15), (26), (28) the $\theta\theta$-equation (12) reads

$$1 + e^{-\frac{\nu}{2}-\frac{\lambda}{2}}\left(\frac{\nu''}{4} - \frac{\lambda''}{4}\right) - \frac{1}{4} e^{\frac{\lambda}{2}-\frac{\nu}{2}}(\Lambda + kr) = \left(\frac{\lambda'}{4} - \frac{\nu'}{4}\right) e^{-\frac{\nu}{2}-\frac{\lambda}{2}}\frac{k}{\Lambda + kr}, \tag{32}$$

leading to

$$\lambda'' - \nu'' + \frac{k}{\Lambda + kr}(\lambda' - \nu') + (\Lambda + kr)e^{\lambda} = 4e^{\frac{\nu}{2}+\frac{\lambda}{2}}, \tag{33}$$

which, combined with (31), becomes

$$\lambda'' + \frac{k}{\Lambda + kr}\lambda' + \frac{3}{2}(\Lambda + kr)e^{\lambda} = 4e^{\frac{\nu}{2}+\frac{\lambda}{2}}. \tag{34}$$

From (16), (26), (29) the rr-equation (13) reads

$$-\frac{\lambda''}{2} + \frac{\lambda'^2}{8} - \frac{3\nu'^2}{8} + \frac{\nu'\lambda'}{4} - \frac{1}{4} e^{\lambda}(\Lambda + kr) = -\frac{\lambda'}{2}\frac{k}{\Lambda + kr}, \tag{35}$$

leading to

$$\lambda'' - \frac{k}{\Lambda + kr}\lambda' + \frac{\Lambda + kr}{2} e^{\lambda} - \frac{\lambda'^2}{4} + \frac{3\nu'^2}{4} - \frac{\nu'\lambda'}{2} = 0. \tag{36}$$

Now, eliminating λ'' from Eqs. (34) and (36), we get

$$2e^{\frac{\nu}{2}+\frac{\lambda}{2}} - \frac{k}{\Lambda + kr}\lambda' - \frac{\Lambda + kr}{2} e^{\lambda} - \frac{\lambda'^2}{8} + \frac{3\nu'^2}{8} - \frac{\nu'\lambda'}{4} = 0. \tag{37}$$

Next, we make the following coordinate change, which is slightly *different* from Buchdahl in his original paper:

$$\Lambda + kr = \Lambda e^{kz}. \tag{38}$$

The first and second derivatives acting on r become

$$\frac{d}{dr} = \frac{dz}{dr}\frac{d}{dz} = \frac{e^{-kz}}{\Lambda}\frac{d}{dz}, \tag{39}$$

$$\frac{d^2}{dr^2} = \frac{dz}{dr}\frac{d}{dz}\left(\frac{e^{-kz}}{\Lambda}\frac{d}{dz}\right) \tag{40}$$

$$= \frac{e^{-kz}}{\Lambda}\left(-\frac{ke^{-kz}}{\Lambda}\frac{d}{dz} + \frac{e^{-kz}}{\Lambda}\frac{d^2}{dz^2}\right) \tag{41}$$

$$= \frac{e^{-2kz}}{\Lambda^2}\left(\frac{d^2}{dz^2} - k\frac{d}{dz}\right), \tag{42}$$

upon which Eqs. (31), (34), (37), respectively, become

$$\frac{e^{-2kz}}{\Lambda^2}(\nu_{zz} - k\nu_z) + \frac{ke^{-2kz}}{\Lambda^2}\nu_z + \frac{\Lambda}{2} e^{kz+\lambda} = 0, \tag{43}$$

$$\frac{e^{-2kz}}{\Lambda^2}(\lambda_{zz} - k\lambda_z) + \frac{ke^{-2kz}}{\Lambda^2}\lambda_z + \frac{3\Lambda}{2} e^{kz+\lambda} = 4e^{\frac{\nu}{2}+\frac{\lambda}{2}}, \tag{44}$$

$$\frac{ke^{-2kz}}{\Lambda^2}\lambda_z + \frac{\Lambda}{2}e^{kz+\lambda} + \frac{e^{-2kz}}{8\Lambda^2}\lambda_z^2 - \frac{3e^{-2kz}}{8\Lambda^2}\nu_z^2 + \frac{e^{-2kz}}{4\Lambda^2}\nu_z\lambda_z = 2e^{\frac{\nu}{2}+\frac{\lambda}{2}}, \tag{45}$$

hence giving

$$\nu_{zz} + \frac{\Lambda^3}{2} e^{3kz+\lambda} = 0, \tag{46}$$

$$\lambda_{zz} + \frac{3\Lambda^3}{2} e^{3kz+\lambda} = 4\Lambda^2 e^{2kz+\frac{\nu}{2}+\frac{\lambda}{2}}, \tag{47}$$

$$\lambda_z^2 - 3\nu_z^2 + 2\nu_z\lambda_z + 8k\lambda_z + 4\Lambda^3 e^{3kz+\lambda} = 16\Lambda^2 e^{2kz+\frac{\nu}{2}+\frac{\lambda}{2}}. \tag{48}$$

Further define

$$\nu = -u + v - kz + \ln 4, \tag{49}$$

$$\lambda = 3u + v - 3kz + 3\ln 4, \tag{50}$$

$$\mu = \frac{\lambda}{2} - \frac{\nu}{2} = 2u - kz + \ln 4, \tag{51}$$

from which, together with (46)–(48), we obtain

$$u_{zz} = 16\Lambda^2 e^{u}(1 - \Lambda e^{2u})e^{v}, \tag{52}$$

$$v_{zz} = 16\Lambda^2 e^u (1 - 3\Lambda e^{2u}) e^v, \tag{53}$$

$$u_z v_z = 16\Lambda^2 e^u (1 - \Lambda e^{2u}) e^v + \frac{3k^2}{4}. \tag{54}$$

If $\Lambda = 1$, these equations would be equivalent to Eqs. (3.1), (3.3), and (3.4) in Buchdahl's original paper [1].

Let us recap: So far, we have obtained the three equations (52)–(54) for two unknown functions $u(z)$ and $v(z)$. However, the three equations are *not* independent. Upon taking derivative with respect to z, Eq. (54) yields

$$u_{zz} v_z + u_z v_{zz} = 16(e^u - 3\Lambda e^{3u}) e^v u_z + 16(e^u - \Lambda e^{3u}) e^v v_z, \tag{55}$$

which is *trivially* satisfied by Eqs. (52) and (53). Therefore, the system is *not* overdetermined. We shall discard Eq. (53) while keeping Eqs. (52) and (54) from now on.

III. OUR SHORTCUT LEADING TO THE *GENERALIZED* BUCHDAHL EQUATION

Note that Eq. (52) is of second differential order and Eq. (54) is of first differential order. Eliminating one of the functions u or v would *in principle* produce a *third* differential order ODE.

To proceed, Buchdahl next exploited some clever analogy of Eqs. (52)–(54) with a Hamiltonian dynamics. However, with the benefit of hindsight, we have found a *shortcut* to be presented in what follows.

Define q as a function of u:

$$q \coloneqq u_z, \tag{56}$$

giving

$$u_{zz} = q_z = q_u u_z = q_u q. \tag{57}$$

Also, by viewing v as a function of u, we have

$$v_z = v_u u_z = v_u q. \tag{58}$$

Combining (52) and (57), we get

$$q q_u = 16\Lambda^2 e^u (1 - \Lambda e^{2u}) e^v. \tag{59}$$

Combining (54), (56), and (58), we get

$$q^2 v_u = 16\Lambda^2 e^u (1 - \Lambda e^{2u}) e^v + \frac{3k^2}{4}. \tag{60}$$

Now, make a substitution

$$u = \ln x, \tag{61}$$

which leads to

$$q_u = \frac{q_x}{u_x} = x q_x, \tag{62}$$

$$v_u = \frac{v_x}{u_x} = x v_x. \tag{63}$$

From Eqs. (59) and (60) we thus get

$$q q_x = 16\Lambda^2 (1 - \Lambda x^2) e^v, \tag{64}$$

$$q^2 v_x = 16\Lambda^2 (1 - \Lambda x^2) e^v + \frac{3k^2}{4x} = q q_x + \frac{3k^2}{4x}. \tag{65}$$

Differentiating Eq. (64) with respect to x,

$$q_x^2 + q q_{xx} = 16\Lambda^2 (1 - \Lambda x^2) e^v v_x - 32\Lambda^3 x e^v, \tag{66}$$

and rewriting it as

$$q_x^2 + q q_{xx} = q q_x v_x - \frac{2\Lambda x q q_x}{1 - \Lambda x^2}. \tag{67}$$

Substituting Eq. (65) into the right-hand side (RHS) of Eq. (67),

$$q_x^2 + q q_{xx} = q_x^2 + \frac{3k^2 q_x}{4xq} - \frac{2\Lambda x q q_x}{1 - \Lambda x^2}, \tag{68}$$

which leads to

$$q_{xx} + \frac{2\Lambda x}{1 - \Lambda x^2} q_x = \frac{3k^2}{4xq^2} q_x. \tag{69}$$

At $\Lambda = 1$, it duly recovers

$$x q_{xx} + \left(\frac{2x^2}{1 - x^2} - \frac{3k^2}{4q^2}\right) q_x = 0, \tag{70}$$

which is precisely Eqs. (4.8) and (3.4) in Buchdahl's 1962 *Nuovo Cimento* paper [1].

Remarkably, the resulting ODE is of *second* (instead of third) differential order. Finally, upon substituting $x \coloneqq \sqrt{t}$, Eq. (69) becomes

$$2t q_{tt} + \left(\frac{1 + \Lambda t}{1 - \Lambda t} - \frac{3k^2}{4q^2}\right) q_t = 0, \tag{71}$$

which, at $\Lambda = 1$, recovers Eqs. (4.10) and (3.4) in Buchdahl's paper [1].

We shall call Eq. (70) the *generalized* Buchdahl equation hereafter. Our next task is to make further progress with this equation.

IV. A NEW CLASS OF BUCHDAHL-INSPIRED METRICS

As we alluded to in the Motivation, Buchdahl deemed that his nonlinear ODE (71)—although "deceptively simple"—was insoluble and irreducible to simpler forms. He discontinued his pursuit for an analytical solution and instead sought a power-expansion solution; see Footnote 1 in our current paper for his reasoning.

We find that this is not the case. The task of this section is to reformulate the generalized Buchdahl equation in a more transparent way, via which the final metric can be attained. We shall consider $\Lambda \in \mathbb{R}$ in general. It turns out that the generalized Buchdahl ODE (69) can be cast in a more convenient form as

$$\frac{d}{dx}\left(\frac{q_x}{1-\Lambda x^2}\right) = \frac{3k^2}{4xq^2}\left(\frac{q_x}{1-\Lambda x^2}\right). \tag{72}$$

Next, let us define a new function $p(x)$ per

$$p(x) \coloneqq \frac{q_x}{1-\Lambda x^2}, \tag{73}$$

which, upon combining with (72), produces a set of two coupled nonlinear first-order ODEs:

$$p_x = \frac{3k^2}{4x}\frac{p}{q^2}, \tag{74}$$

$$q_x = (1-\Lambda x^2)p. \tag{75}$$

In terms of x, the functions u and v are, using Eqs. (61) and (64),

$$e^u = x, \tag{76}$$

$$e^v = \frac{qq_x}{16\Lambda^2(1-\Lambda x^2)} = \frac{qp}{16\Lambda^2}, \tag{77}$$

and the functions ν, λ, and μ are, using Eqs. (49)–(51),

$$e^\nu = e^{-u+v-kz+\ln 4} = \frac{4}{\Lambda^2 e^{kz}}\frac{qp}{16x}, \tag{78}$$

$$e^\lambda = e^{3u+v-3kz+3\ln 4} = \frac{64}{\Lambda^2 e^{3kz}}\frac{x^3qp}{16}, \tag{79}$$

$$e^\mu = e^{2u-kz+\ln 4} = \frac{4}{e^{kz}}x^2. \tag{80}$$

From (38) we have

$$dr = \Lambda e^{kz}dz, \tag{81}$$

and since we also know from (56) and (61) that

$$q = u_z = \frac{du}{dx}\frac{dx}{dz} = \frac{1}{x}\frac{dx}{dz}, \tag{82}$$

which leads to

$$dz = \frac{dx}{xq}, \tag{83}$$

we thus have

$$dr = \Lambda e^{kz}\frac{1}{xq}dx. \tag{84}$$

The metric initially expressed in (7) becomes

$$\begin{aligned} ds^2 &= -e^\nu dt^2 + e^\lambda dr^2 + e^\mu d\Omega^2 \\ &= -\frac{pq}{4\Lambda^2 e^{kz}x}dt^2 + \frac{4pqx^3}{\Lambda^2 e^{3kz}}\left(\frac{\Lambda e^{kz}}{xq}dx\right)^2 + \frac{4x^2}{e^{kz}}d\Omega^2 \end{aligned} \tag{85}$$

$$= \frac{4}{e^{kz}}\left\{\frac{p}{4}\left[-\frac{q}{4x}\frac{dt^2}{\Lambda^2} + \frac{4x}{q}dx^2\right] + x^2 d\Omega^2\right\}. \tag{86}$$

Finally, using the notation of r in place of x, and making the following replacements:

$$\begin{cases} p \to 4p \\ q \to 4q \\ k \to -4k \\ kz \to -kz + \ln 4 \\ t \to \Lambda t \end{cases}, \tag{87}$$

we arrive at the family of Buchdahl-inspired metrics presented below.

A. The Buchdahl-inspired metrics

$$ds^2 = e^{k\int\frac{dr}{rq(r)}}\left\{p(r)\left[-\frac{q(r)}{r}dt^2 + \frac{r}{q(r)}dr^2\right] + r^2 d\Omega^2\right\} \tag{88}$$

in which the evolution rules are

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}, \tag{89}$$

$$\frac{dq}{dr} = (1-\Lambda r^2)p, \tag{90}$$

and, using (26), (38), (83), and (87), the Ricci scalar equals

$$\mathcal{R}(r) = 4\Lambda e^{-k\int\frac{dr}{rq(r)}}. \tag{91}$$

There are two *separate* sets of metrics depending on the sign of Λ:

(i) Asymptotically de Sitter: $\Lambda > 0$ and $r \in [0, \Lambda^{-\frac{1}{2}}]$,

$$\mathcal{R}(r) = 4\Lambda \exp\left[k\int_r^{\Lambda^{-\frac{1}{2}}} \frac{dr'}{r'q(r')}\right]. \tag{92}$$

(ii) Asymptotically anti–de Sitter: $\Lambda < 0$ and $r \in [0, \infty)$,

$$\mathcal{R}(r) = 4\Lambda \exp\left[k\int_r^{\infty} \frac{dr'}{r'q(r')}\right]. \tag{93}$$

Note that, in the two expressions above, r is used as the lower bound in the integrals, hence the flip in the sign of k. In either case, the upper bound for the integral in $\mathcal{R}(r)$ is chosen such that, at the largest distance allowable, the Ricci scalar converges to 4Λ.

Compatible with a fourth-derivative action, each metric is specified by four parameters: Λ (the large-scale curvature), k (the deviation from constant curvature), $p(r_0)$ and $q(r_0)$ at a reference distance r_0.

We shall tentatively call the class of metrics represented in (88)–(93) the Buchdahl-inspired metrics and the coordinate system (t, r, θ, ϕ) used therein the Buchdahl coordinates. The Buchdahl-inspired metrics are complete and *exhaustive*. *All* "nontrivial" static spherically symmetric vacuo metrics in pure $\mathcal{R}^2$ gravity fall under the umbrella of the Buchdahl-inspired metrics.

V. VERIFYING OUR SOLUTION VIA DIRECT INSPECTION

It is desirable to confirm *ex post* that our solution expressed in (88)–(91) obeys the $\mathcal{R}^2$ vacuo field equation. We shall carry out this due diligence exercise via direct inspection. The task is nontrivial because of the cross dependence between $p(r)$ and $q(r)$. Below is our maneuver.

First, we consider the line element

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + e^{\mu(r)}d\Omega^2, \tag{94}$$

in which, by virtue of (88),

$$\nu(r) := \ln\left(f(r)\frac{p(r)q(r)}{r}\right), \tag{95}$$

$$\lambda(r) := \ln\left(f(r)\frac{p(r)r}{q(r)}\right), \tag{96}$$

$$\mu(r) := \ln(f(r)r^2). \tag{97}$$

We further equate

$$f(r) := \exp\left(k\int\frac{dr}{rq(r)}\right), \tag{98}$$

while leaving $p(r)$ and $q(r)$ *unspecified* at the moment.

The relevant Christoffel symbols and Ricci tensor components are given in (14)–(19). We use the symbolic manipulator MAXIMA ONLINE interface to compute these six components and the Ricci scalar $\mathcal{R}$. They are found to contain $p(r)$ and $q(r)$ and their higher-differential order terms up to the fourth order.

Next, we specify

$$p'(x) = \frac{3k^2}{4r}\frac{p(r)}{q^2(r)}, \tag{99}$$

$$q'(x) = (1 - \Lambda r^2)p(r), \tag{100}$$

and then use MAXIMA ONLINE to compute $p''(r)$, $q''(r)$, $p'''(r)$, $q'''(r)$, $p''''(r)$, $q''''(r)$ and express each of them solely in terms of $p(r)$ and $q(r)$. We then substitute these quantities into the Christoffel symbols, the Ricci tensor components, and the Ricci scalar obtained above. Despite their cumbersome appearances, after all the dust settles, MAXIMA ONLINE determines that

$$\mathcal{R}\left(\mathcal{R}_{tt} - \frac{1}{4}g_{tt}\mathcal{R}\right) + \Gamma^r_{tt}\mathcal{R}' \equiv 0, \tag{101}$$

$$\mathcal{R}\left(\mathcal{R}_{\theta\theta} - \frac{1}{4}g_{\theta\theta}\mathcal{R}\right) + \Gamma^r_{\theta\theta}\mathcal{R}' \equiv 0, \tag{102}$$

$$\mathcal{R}\left(\mathcal{R}_{rr} - \frac{1}{4}g_{rr}\mathcal{R}\right) + \Gamma^r_{rr}\mathcal{R}' - \mathcal{R}'' \equiv 0 \tag{103}$$

identically. In addition, it produces

$$\mathcal{R}(r) \equiv \frac{4\Lambda}{f(r)} \quad \forall\ r. \tag{104}$$

These outcomes solidly validate that our solution given in (88)–(91) *satisfies* the $\mathcal{R}^2$ vacuo field equation.

Our MAXIMA codes used for this section are available in [14]. We must note that another researcher independently and successfully verified our solution using *Mathematica*; his working notebook is accessible in the public domain [15].

VI. RECOVERING SCHWARZSCHILD–DE SITTER METRIC AS SPECIAL CASE AT $k = 0$

Consider a metric with constant curvature, $\mathcal{R} \equiv 4\Lambda\ \forall\ r$. This requires $k = 0$ and, from (89),

$$\frac{dp}{dr} = 0 \tag{105}$$

or $p = p_0 \equiv 1$ without loss of generality. Then, from (90), we subsequently have

$$\frac{dq}{dr} = 1 - \Lambda r^2, \tag{106}$$

$$q = r - \frac{\Lambda}{3} r^3 - r_s, \tag{107}$$

$$\frac{q}{r} = 1 - \frac{\Lambda}{3} r^2 - \frac{r_s}{r}, \tag{108}$$

with r_s being a constant of integration. The metric in (88) becomes

$$ds^2 = -\left(1 - \frac{\Lambda}{3} r^2 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3} r^2 - \frac{r_s}{r}} + r^2 d\Omega^2, \tag{109}$$

which is nothing but the classic SdS metric. This result also means that the SdS metric is the *only* vacuo metric with *constant* curvature available in pure $\mathcal{R}^2$ gravity.

A Buchdahl-inspired metric can be made *arbitrarily* close to the SdS metric by tuning the parameter k to zero. Hence, the quartet $\{\Lambda, k, p_0, q_0\}$ spans a topological space where all members in the space are smoothly connected to the $k = 0$ member (namely, the set of SdS metrics).

VII. THE SMALL k LIMIT

For $k = 0$ we already have the solution considered in the preceding section:

$$p(r) \equiv 1 \quad \text{(without loss of generality)}, \tag{110}$$

$$q(r) = r - \frac{\Lambda}{3} r^3 - r_s. \tag{111}$$

Let us consider up to $\mathcal{O}(k)$,

$$p(r) = 1 + \mathcal{O}(k), \tag{112}$$

$$q(r) = \left(r - \frac{\Lambda}{3} r^3 - r_s\right) + \mathcal{O}(k). \tag{113}$$

Plugging them into (89) leads to

$$\frac{dp}{dr} = \mathcal{O}(k^2), \tag{114}$$

which then means

$$p(r) = 1 + \mathcal{O}(k^2). \tag{115}$$

Note that this expression is valid up to $\mathcal{O}(k^2)$ instead of merely $\mathcal{O}(k)$ as in (112). Plugging (115) into (90) yields

$$\frac{dq}{dr} = (1 - \Lambda r^2) + \mathcal{O}(k^2), \tag{116}$$

and then

$$q = \left(r - \frac{\Lambda}{3} r^3 - r_s\right) + \mathcal{O}(k^2). \tag{117}$$

Once again, this expression is valid up to $\mathcal{O}(k^2)$ instead of merely $\mathcal{O}(k)$ as in (113). The conformal factor in the metric is thus

$$e^{k \int \frac{dr}{r q(r)}} = e^{k \int \frac{dr}{r^2 (1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2)} + \mathcal{O}(k^3)}. \tag{118}$$

The metric in (88) becomes

$$ds^2 = e^{k \int \frac{dr}{r^2 (1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2)}} \left\{ -\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega^2 \right\} + \mathcal{O}(k^2), \tag{119}$$

and the Ricci scalar is

$$\mathcal{R} = 4\Lambda \left[1 - k \int \frac{dr}{r^2 (1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2)}\right] + \mathcal{O}(k^2). \tag{120}$$

This new metric is valid up to $\mathcal{O}(k^2)$ and would be useful for physical situations with small k, i.e., with a weak deviation from constant scalar curvature. The metric is determined by *three* parameters Λ, r_s, and k, each representing a length scale.

At $\mathcal{O}(k^2)$, the new metric (119) only differs from the SdS metric (109) by the conformal factor $e^{k \int \frac{dr}{r^2 (1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2)}}$. Note that the pure $\mathcal{R}^2$ action is *not* subject to the conformal symmetry. As a result, the conformal factor is a physical quantity; it explicitly participates in the Ricci scalar rendering the latter *nonconstant* as is evident in (120).

VIII. BEHAVIOR OF BUCHDAHL-INSPIRED METRIC AROUND THE COORDINATE ORIGIN

For any metric, the most interesting behavior should be around the origin where singularities might occur. In the limit of $r \to 0$, the "evolution" rules (89) and (90) become

$$\frac{dp}{dr} = \frac{3k^2}{4r} \frac{p}{q^2}, \tag{121}$$

$$\frac{dq}{dr} \approx p. \tag{122}$$

The sign of p solely determines the direction of flows for both $p(r)$ and $q(r)$. Figure 1 shows the phase space spanned by $\{p, q\}$ with q the horizontal axis and p the

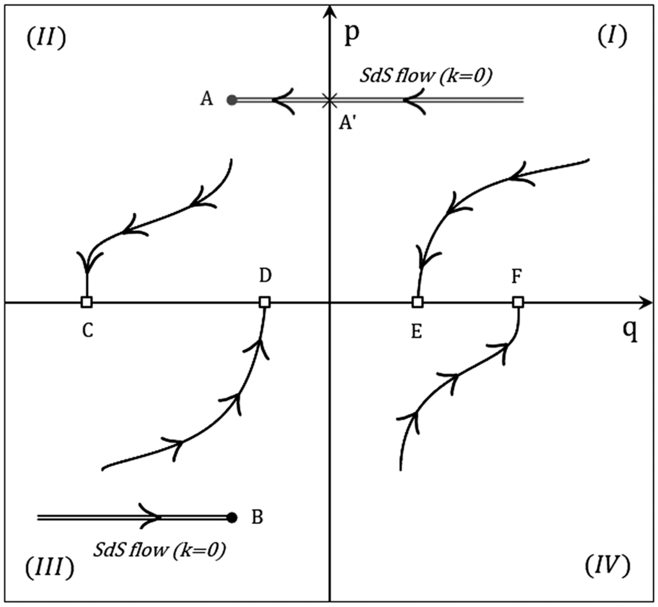

FIG. 1. Evolution of $\{p(r), q(r)\}$ as r approaches 0. Points A and B are the end points of SdS flows (i.e., $k = 0$). Points C, D, E, and F are the end points of Buchdahl flows ($k \neq 0$), each starting from one of the four quadrants.

vertical axis. As r moves *toward* the coordinate origin, Quadrants (I) and (II) correspond to monotonic decreasing p and q; Quadrants (III) and (IV) to monotonic increasing p and q. Figure 1 shows the direction of the flow if we start from a reference distance $r_0 > 0$ and move *toward* the origin. The SdS flows correspond to $k = 0$ [thus, $p \equiv 1$ and $q(r) = r - r_s + \frac{\Lambda}{3} r^3$ making $\lim_{r\to 0} q(r) = -r_s$]; thus their end points belong to Quadrants (II) or (III).

The horizontal axis is an attractor for all quadrants (note: we let r move *toward* the coordinate origin). This can be shown below.

We shall let p and q converge to p_* and q_* when $r \to 0$ in the following manner:

$$p \approx p_* + \bar{p} r^\eta, \tag{123}$$

$$q \approx q_* + \bar{q} r^\zeta, \tag{124}$$

with $\eta > 0$ and $\zeta > 0$. First, let us assume $p_* \neq 0$; from (121)

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2} \approx \frac{3k^2 p_*}{4q_*^2}\frac{1}{r}, \tag{125}$$

making

$$p \approx -\frac{3k^2 p_*}{4q_*^2}\frac{1}{r^2} + \text{const}, \tag{126}$$

which would diverge as $r \to 0$ in contradiction with the requirement (123). Hence, p_* must equal 0. This means that *every* trajectory must hit the horizontal axis as $r \to 0$ from above. We shall only consider $q_* \neq 0$ to this end. Since $p_* = 0$, the evolution rules (121) and (122) become

$$\eta \bar{p} r^{\eta-1} \approx \frac{3k^2}{4}\frac{\bar{p}}{q_*^2} r^{\eta-1}, \tag{127}$$

$$\zeta \bar{q} r^{\zeta-1} \approx \bar{p} r^\eta, \tag{128}$$

giving

$$\eta = \frac{3k^2}{4q_*^2} > 0, \tag{129}$$

$$\zeta = \eta + 1 > 0, \tag{130}$$

$$\bar{p} = \zeta \bar{q}. \tag{131}$$

Close to the origin, the functions thus are

$$p(r) \approx (\eta + 1)\bar{q} r^\eta, \tag{132}$$

$$q(r) \approx q_* + \bar{q} r^{\eta+1}. \tag{133}$$

The scalar curvature close to the origin behaves as

$$\mathcal{R}(r) \approx 4\Lambda \exp\left[-k\int \frac{dr}{rq_*}\right] = 4\Lambda r^{-\frac{k}{q_*}}. \tag{134}$$

As $r \to 0^+$, the Ricci scalar vanishes or diverges depending on the sign of k/q_*.

As $r \to 0^+$, the metric is approximately

$$ds^2 \approx r^{\frac{k}{q_*}}\left\{(\eta + 1)\frac{\bar{q}}{k} r^\eta \left[-\frac{q_*}{kr} d\tilde{t}^2 + \frac{kr}{q_*} dr^2\right] + r^2 d\Omega^2\right\}, \tag{135}$$

which is specified by exactly *three* parameters $\{\Lambda, \frac{q_*}{k}, \frac{\bar{q}}{k}\}$ with $\eta = \frac{3}{4}(\frac{k}{q_*})^2$ and $\tilde{t} := kt$.

IX. A DEGENERACY IN PARAMETER SPACE OF BUCHDAHL-INSPIRED METRIC

As the limit $k \to 0$ corresponds to the SdS metric, we shall consider only $k \neq 0$ herein. If we make the following substitutions:

$$q := k\tilde{q}, \tag{136}$$

$$p := k\tilde{p}, \tag{137}$$

$$t := k^{-1}\tilde{t}, \tag{138}$$

then the metric in (88) becomes

$$ds^2 = e^{\int \frac{dr}{r\tilde{q}(r)}} \left\{ \tilde{p}(r) \left[-\frac{\tilde{q}(r)}{r} d\tilde{t}^2 + \frac{r}{\tilde{q}(r)} dr^2 \right] + r^2 d\Omega^2 \right\}, \tag{139}$$

in which

$$\mathcal{R}(r) = 4\Lambda \exp\left[-\int \frac{dr}{r\tilde{q}(r)} \right], \tag{140}$$

$$\frac{d\tilde{p}}{dr} = \frac{3}{4r} \frac{\tilde{p}}{\tilde{q}^2}, \tag{141}$$

$$\frac{d\tilde{q}}{dr} = (1 - \Lambda r^2)\tilde{p}. \tag{142}$$

Accordingly, despite being a product of a fourth-derivative action, a Buchdahl-inspired metric is effectively characterized by only three parameters. This degeneracy helps simplify the classification of Buchdahl-inspired metrics. We shall carry out this task in a companion paper [16].

Note that in Sec. VII when treating the weak nonconstancy for the Ricci scalar, we made k explicit. Nevertheless, the metric obtained therein was specified by *three* length scales $\{|\Lambda|^{-\frac{1}{2}}, r_s, k\}$ in perfect agreement with the number of degrees of freedom allowable by the degeneracy uncovered in this section.

X. HOW DOES BUCHDAHL-INSPIRED METRIC CIRCUMVENT A "PROOF" OF NONEXISTENCE?

In [2] Kehagias *et al.* sought black hole solutions for the pure quadratic action. Curiously, they omitted the Buchdahl equation and consequently overlooked the new class of Buchdahl-inspired metrics uncovered in our current paper. They considered only the two "automatic" vacuo configurations: (i) the zero-Ricci-scalar spaces, $\mathcal{R} = 0$, and (ii) the Einstein spaces, $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$. Therein, they offered a neat proof that apparently rules out the existence of nonconstant curvature metrics (to which Buchdahl-inspired metrics belong). However, the class of Buchdahl-inspired metrics *defeat* their proof by evading its central assumption. Below is how it happens.

Let us first recap the essence of the proof of Kehagias *et al.* Their proof is a type of no-go, stating that all admissible $\mathcal{R}^2$ vacua must have constant scalar curvature. The authors in [2] started with the trace equation of the pure $\mathcal{R}^2$ action *in vacuo*

$$\Box \mathcal{R} = 0. \tag{143}$$

For the following metric:

$$ds^2 = -\mu(r)dt^2 + \frac{dr^2}{\nu(r)} + r^2 d\Omega^2, \tag{144}$$

the trace equation takes the form[6]

$$(r^2 \sqrt{\mu\nu} \mathcal{R}')' = 0. \tag{145}$$

This leads to

$$(r^2\sqrt{\mu\nu}\mathcal{R}'\mathcal{R})' = \cancel{(r^2\sqrt{\mu\nu}\mathcal{R}')'}\mathcal{R} + r^2\sqrt{\mu\nu}(\mathcal{R}')^2 \tag{146}$$

from which one obtains the following identity:

$$\int_0^\infty dr\, r^2\sqrt{\mu\nu}(\mathcal{R}')^2 = \int_0^\infty dr (r^2\sqrt{\mu\nu}\mathcal{R}'\mathcal{R})'. \tag{147}$$

The RHS of (147) can be cast into a three-volume integral, which then turns into a two-dimensional (2D) surface integral at infinity by virtue of the Gauss-Ostrogradsky divergence theorem[7]:

$$\int_0^\infty dr (r^2\sqrt{\mu\nu}\mathcal{R}'\mathcal{R})'$$

$$= \frac{1}{4\pi} \int d\Omega \int_0^\infty dr\, r^2 \vec{\nabla}(\sqrt{\mu\nu}\mathcal{R}(\vec{\nabla}\mathcal{R})) \tag{148}$$

$$= \frac{1}{4\pi} \int d^3V \vec{\nabla}(\sqrt{\mu\nu}\mathcal{R}(\vec{\nabla}\mathcal{R})) \tag{149}$$

$$= \frac{1}{4\pi} \oint_S d\vec{S}\, \sqrt{\mu\nu}\, \mathcal{R}(\vec{\nabla}\mathcal{R}) \tag{150}$$

$$= \lim_{r\to\infty} r^2 \sqrt{\mu\nu}\mathcal{R}\mathcal{R}'. \tag{151}$$

Now, the authors of [2] posited that *if* $\mathcal{R}'$ *falls to zero rapidly enough at large distances*, then the limit in (151) vanishes, making

$$\int_0^\infty dr\, r^2\sqrt{\mu\nu}(\mathcal{R}')^2 = 0. \tag{152}$$

Because of the non-negativity of the left-hand side (LHS) of (152), this would force $\mathcal{R}' = 0$ *everywhere*. QED.

However, Buchdahl-inspired metrics invalidate this very assumption: their Ricci scalar decays *not* as rapidly to warrant (152). As a counterexample, in Sec. VII we obtained a metric with the Ricci scalar behaving at large distances as, per Eq. (120),

$$\mathcal{R} \approx 4\Lambda - \frac{4k}{r^3}, \tag{153}$$

making

[6]Recall that for a scalar field ϕ: $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$.

[7]Recall that in spherical coordinates, for $\phi(r)$ and $\vec{A} = A(r)\hat{r}$: $\vec{\nabla}\phi = \partial_r\phi(r)\hat{r}$ and $\vec{\nabla}.\vec{A} = \frac{1}{r^2}\partial_r(r^2 A(r))$. The 3D divergence theorem for a generic vector field $\vec{A}$: $\int_V d^3V\vec{\nabla}.\vec{A} = \oint_S d\vec{S}.\vec{A}$.

$$\mathcal{R}' \approx \frac{12k}{r^4}; \tag{154}$$

thence

$$\lim_{r\to\infty} |r^2\sqrt{\mu\nu}\mathcal{R}\mathcal{R}'| = \lim_{r\to\infty}\left|\frac{48\Lambda k}{r^2}\sqrt{\mu\nu}\right| = 16\Lambda^2|k| \neq 0, \tag{155}$$

given that $\mu \simeq \nu \simeq 1 - \frac{\Lambda}{3}r^2$ as large distances. In general, the growth in μ and ν balances out the decay in $\mathcal{R}'$; the proof in [2] overlooked this compensation effect.

The nonzero value in (155) renders the no-go proof in [2] inapplicable for the Buchdahl-inspired metric.[8]

Before closing this section, we must make two additional comments:

First, the no-go proof provided in [2] was previously offered by Nelson for the $\mathcal{R} + \mathcal{R}^2 + \mathcal{C}_{\mu\nu\rho\sigma}\mathcal{C}^{\mu\nu\rho\sigma}$ action [3]. Nelson's proof similarly relied on an overly restrictive assumption on the asymptotic falloff for $\mathcal{R}'$ as $r \to \infty$.

Second, in a 2015 paper [4], Lü *et al.* reported the existence of further black hole solutions (above the Schwarzschild solution) for the *Einstein-Weyl* gravity, $\mathcal{R} + \mathcal{C}_{\mu\nu\rho\sigma}\mathcal{C}^{\mu\nu\rho\sigma}$, viz. with the $\mathcal{R}^2$ term being suppressed. These solutions—albeit *not* in an analytical form—would be in defiance of Nelson's no-go proof [3]. The authors therein [4] identified a (sign) error in Nelson's proof rendering it inapplicable for the Einstein-Weyl gravity. However, these authors did not refute Nelson's proof for the pure $\mathcal{R}^2$ gravity; they did not point out the problem with the asymptotic falloff assumed in Nelson's no-go proof, which would have precluded the existence of Buchdahl-inspired metrics, as we have shown in this section.

XI. SUMMARY

In this paper, we show that pure $\mathcal{R}^2$ gravity admits nontrivial vacuo configurations beyond the vanishing Ricci scalar spaces ($\mathcal{R} = 0$) and the Einstein space ($\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$).

The new solutions are inherent in a program which Buchdahl originated circa 1962. In a seminal—yet obscure—*Nuovo Cimento* paper [1], Buchdahl set forth to seek static spherically symmetric solutions for the pure $\mathcal{R}^2$ action. His work culminated in a nonlinear second-order ODE that *remained to be solved*. If a solution to his ODE can be found, then a complete set of vacua for pure $\mathcal{R}^2$ gravity would be readily obtained.

Despite its importance and potential, the Buchdahl equation has largely escaped the attention of the gravitation research community since its inception. Among the mere 40+ publications that cited Buchdahl's original *Nuovo Cimento* work, none have attempted to solve his ODE.[9] In this paper, we have finally obtained a *novel* set of compact solutions to the Buchdahl equation, thereby accomplishing his six-decades-old goal seeking nontrivial vacuo metrics for pure $\mathcal{R}^2$ gravity.

Our main result: We reformulated Buchdahl's original work via a more straightforward route starting directly from the $\mathcal{R}^2$ vacuo field equation; we thus departed from Buchdahl's arduous route that used the variational principle on a "surrogate" Lagrangian. Along the way, we introduced a few shortcuts. We are able to arrive at a *generalized* Buchdahl equation in the form of a nonlinear second-order ODE:

$$\frac{d^2q}{dr^2} + \frac{2\Lambda r}{1-\Lambda r^2}\frac{dq}{dr} = \frac{3k^2}{4rq^2}\frac{dq}{dr}. \tag{156}$$

This ODE embodies the four parameters, $\{\Lambda, k, q(r_0), \frac{dq}{dr}|_{r=r_0}\}$, of the *fourth-order* $\mathcal{R}^2$ theory.

Next, in place of the second-order ODE (156), we are able to recast it in terms of two coupled nonlinear first-order ODEs:

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}, \tag{157}$$

$$\frac{dq}{dr} = (1-\Lambda r^2)p. \tag{158}$$

From here, we are able to express the final solution in a neat resemblance to the SdS metric to make the terms transparent and self-explanatory. The Buchdahl-inspired metrics are in a compact representation:

$$ds^2 = e^{k\int\frac{dr}{rq(r)}}\left\{p(r)\left[-\frac{q(r)}{r}dt^2 + \frac{r}{q(r)}dr^2\right] + r^2 d\Omega^2\right\} \tag{159}$$

with the Ricci scalar equal

$$\mathcal{R}(r) = 4\Lambda\exp\left(-k\int\frac{dr}{rq(r)}\right). \tag{160}$$

As is generally expected from a fourth-order theory, a Buchdahl-inspired metric is specified by four parameters: Λ as the large-distance scalar curvature, the (Buchdahl) parameter k controlling the deviation of the Ricci scalar from constancy, $\{p_0, q_0\}$ initiating the "evolution" flow.

Validity of our solution: To allay any doubts, in Sec. V, we verified by *direct inspection* that the metric given in (157)–(160) obeys the $\mathcal{R}^2$ vacuo field equation

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{R} = 0, \tag{161}$$

[8] As an aside comment, the proof in [2] was not watertight. It should also have handled the intricacy introduced into the 3D divergence theorem by way of the curved *space* (which in general is not 3D Euclidean).

[9] Based on NASA ADS and InspireHEP citation trackers.

hence establishing the validity of our solution. The verification process will be detailed in [14]. Note that another researcher also successfully carried out his own verification of our results, with his *Mathematica* notebook accessible in the public domain [15].

Circumventing a no-go theorem: In [2] it was proved that pure $\mathcal{R}^2$ vacua were restricted to the vanishing Ricci scalar spaces, $\mathcal{R} = 0$, and the Einstein spaces, $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$. This no-go proof, if it were correct, would rule out the existence of vacua with nonconstant scalar curvature. Since Buchdahl-inspired metrics project nonconstant scalar curvature, as is evident per (160) for $k \neq 0$, we must identify the cause of the conflict. In Sec. X we found that the no-go proof in [2] imposed a rapid asymptotic falloff for the metric at largest distances. Buchdahl-inspired metrics, however, evade this overly restrictive assumption, thereby being able to circumvent the proof.

Recovering the SdS metric at $k = 0$: The case of $k = 0$ corresponds to the SdS metric in which $p(r)$ can be set identically equal to 1 and $q(r)$ contains a Schwarzschild radius; see Sec. VI.

Properties of the Buchdahl-inspired metrics: We examined the metrics in three situations: (i) the small k limit; (ii) the region around the coordinate origin; and (iii) a degeneracy in the parameter space of the metrics. These results are shown in Secs. VII, VIII, and IX, respectively. A thorough systematic study of the metrics shall be provided in [16].

A framework "beyond Schwarzschild–de Sitter": The family of Buchdahl-inspired metrics (157)–(160) is *exhaustive*: it covers all nontrivial static spherically symmetric vacuo configurations admissible in pure $\mathcal{R}^2$ gravity. Its parameters $\{\Lambda, k, p_0, q_0\}$ form a topological space that encloses the constant-curvature SdS metrics ($k = 0$) and smoothly connects each nonconstant curvature member to an SdS metric when k is tuned to 0.

The Buchdahl-inspired metrics thus constitute a bona fide enlargement of the SdS metric. It offers a nontrivial example in the context of $3 + 1$ higher-order gravity that encompasses the SdS metric yet—at the same time—*transcends* it. Hence the Buchdahl-inspired metrics embody a framework "beyond Schwarzschild–de Sitter."

In closing, the compact representation (157)–(160) of the Buchdahl-inspired metrics should equip future researchers with a powerful tool to explore *new physics* in pure $\mathcal{R}^2$ gravity with relative ease.

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