How slowly can the early Universe expand?

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When the expansion of the Universe is dominated by a perfect fluid with equation of state parameter w and a sound speed c_s satisfying $w = c_s^2 \leq 1$, the Hubble parameter H and time t satisfy the bound $Ht \geq 1/3$. There has been recent interest in "ultraslow" expansion laws with $Ht < 1/3$ (sometimes described as "fast expanding" models). We examine various models that can produce ultraslow expansion: scalar fields with negative potentials, barotropic fluids, braneworld models, or a loitering phase in the early Universe. Scalar field models and barotropic models for ultraslow expansion are unstable to evolution toward $w = 1$ or $w \to \infty$ in the former case and $w \to \infty$ in the latter case. Braneworld models can yield ultraslow expansion but require an expansion law beyond the standard Friedman equation. Loitering early universe models can produce a quasistatic expansion phase in the early Universe but require an exotic negative-density component. These results suggest that appeals to an ultraslow expansion phase in the early Universe should be approached with some caution, although the loitering early universe may be worthy of further investigation. These results do not apply to ultraslow contracting models.

I. INTRODUCTION

Consider a universe dominated by a perfect fluid with equation of state

$$w = \frac{p}{\rho}, \tag{1}$$

where p and ρ are the pressure and density of the fluid, respectively. The evolution of ρ as a function of the scale factor a is given by

$$a\frac{d\rho}{da} = -3(\rho + p), \tag{2}$$

so that

$$\rho \propto a^{-3(1+w)}. \tag{3}$$

Since the Hubble parameter H corresponds to

$$H \equiv \frac{\dot{a}}{a} = \sqrt{\frac{\rho}{3}} \tag{4}$$

(where we take $\hbar = c = 8\pi G = 1$ throughout), the scale factor a evolves as a power of the time t:

$$a \propto t^{2/3(1+w)}, \tag{5}$$

for $w > -1$. In the standard cosmological model, the universe undergoes a period of radiation domination ($w = 1/3$), followed by an epoch of matter domination ($w = 0$), and it is currently entering a period of dark energy domination with w close to -1.

However, many papers have considered the possibility of more exotic evolution in the early Universe. For instance, the possibility that the Universe underwent a period in which the dominant fluid had a stiff equation of state with $w = 1$ and $a \propto t^{1/3}$ has been investigated in relation to baryogenesis [1], big bang nucleosynthesis (BBN) [2], the relic abundance of dark matter [3–6], and the propagation of gravitational radiation (Ref. [7] and references therein).

More recently, there has been speculation regarding the possibility of ultraslow expansion. For example, if the Universe were dominated by a fluid with $\rho \propto a^{-n}$ and $n > 6$, then $a \propto t^\alpha$ with $\alpha < 1/3$. Since $a(t)$ is strongly constrained from the era of BBN onward, most of the discussion of these models concerns the calculation of relic particle abundances [6,8–10], baryogenesis [11], and leptogenesis [12]. Here, we examine the underlying assumption of these papers: Are there plausible models for which $Ht < 1/3$ over some range in t? Note that some of the terminology in the literature is confusing: An epoch with $\rho \propto a^{-n}$, $n > 6$ is often referred to as "fast expansion." Such an epoch corresponds to fast expansion in the sense that $H(T)$ (where T is the background radiation temperature) is larger than in the standard model, with the ratio between H in this nonstandard model to H in the standard model increasing with T. However, the value of H expressed as a function of time is smaller in the models we consider here than in the standard model, so we describe these models as ultraslow expansion.

Models with ultraslow expansion are constrained by the fact that the speed of sound, $c_s^2 = dp/d\rho$, is required to be subluminal, so $dp/d\rho \leq 1$. For a perfect fluid with

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$w = p/\rho$, we have $w \leq 1$ and $Ht < 1/3$. There are, however, many models for which $dp/d\rho \neq p/\rho$, such as scalar field models, barotropic fluids, and mixtures of perfect fluids, all of which we discuss below to determine if they can provide a period of expansion with $Ht < 1/3$. We examine previously discussed models (scalar fields and braneworld expansion laws) in Secs. II and IV, respectively, and models that have not been previously explored (general barotropic fluids and loitering early universe models) in Secs. III and V. Our results are summarized in Sec. VI.

II. SCALAR FIELDS WITH NEGATIVE POTENTIALS

Scalar fields providing a significant component of the energy density of the Universe have frequently been invoked in cosmology. They were first introduced as the main component of models for inflation (see, e.g., Refs. [13,14] for reviews). Later, under the name "quintessence," scalar fields were investigated as an alternative to the cosmological constant as a mechanism to drive the observed accelerated expansion of the Universe [15–23] (see Refs. [24,25] for reviews).

The equation governing the evolution of a scalar field ϕ in a potential $V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{6}$$

where the dot denotes the time derivative. The pressure and density of the scalar field are given by

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \tag{7}$$

and

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{8}$$

respectively, so the equation of state parameter w from Eq. (1) is

$$w = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{9}$$

Note that the sound speed in these models is $c_s^2 = 1$, independent of the value of w.

From Eq. (9), it is clear that $-1 \leq w \leq 1$ as long as $V(\phi) \geq 0$. However, negative potentials can give rise to $w > 1$ and ultraslow expansion or contraction. Scalar field models with negative potentials have long been of interest [26–30]. In particular, it has been noted that a negative exponential potential of the form

$$V(\phi) = -V_0 e^{-\lambda\phi}, \tag{10}$$

where $V_0 > 0$ and $\lambda > 0$, can lead to a constant value of w with $w \gg 1$ during the contracting phase of the ekpyrotic model [31–33]. [Note that the evolution of all of the physically relevant parameters is invariant if we take instead $V(\phi) = -V_0 e^{\lambda\phi}$ and invert the sign of $\dot{\phi}$.] It is then reasonable to assume that this negative exponential potential can also yield ultraslow expansion. Indeed, D'Eramo *et al.* [6] provided a solution to the equation of motion with the potential of Eq. (10) that yields just such an expansion:

$$\phi = \phi_i + \frac{2}{\lambda}\ln\left(\frac{t}{t_i}\right), \tag{11}$$

which gives an equation of state parameter

$$w = \frac{\lambda^2}{3} - 1. \tag{12}$$

Thus, for $\lambda > \sqrt{6}$, we have $w > 1$ and ultraslow expansion.

The first thing to note about this solution is that it represents a scalar field rolling uphill in the potential, which corresponds to very unnatural initial conditions. Second, the solution is unstable. This was first noted by Heard and Wands [27], who investigated the evolution of scalar fields with both positive and negative exponential potentials (see also the later discussion in Ref. [30]). The corresponding solution for a contracting universe, on the other hand, is stable.

To illustrate the nature of this instability, and to investigate the rate at which small perturbations to the fixed-point solution grow, consider the expression for the evolution of w for a scalar field in an exponential potential of the form given by Eq. (10) in the limit where the scalar field dominates the expansion [34,35]:

$$a\frac{dw}{da} = (w - 1)\sqrt{3(1 + w)}[\sqrt{3(1 + w)} - \lambda]. \tag{13}$$

The derivation of Eq. (13) assumes that $\dot{\phi} > 0$, but it is trivial to generalize it to the opposite case. As expected, this equation has a solution of the form $dw/da = 0$ and w given by Eq. (12), corresponding to the last factor on the right-hand side of Eq. (13) equal to zero. Now, suppose we perturb this solution with a small change to w. For $w > 1$, a positive change in w gives $dw/da > 0$ and $w \to \infty$, while a negative change yields $dw/da < 0$ and $w \to 1$. Thus, the solution given by Eqs. (11) and (12) is unstable. Note that the opposite is true for $w < 1$; in this case, the solution is stable.

Of course, even a transient solution with $w > 1$ could be sufficient to produce interesting changes to the evolution of relic particle densities, as discussed in Refs. [6,8–12], so it is important to examine the rate at which w evolves away from its unstable fixed-point value. If we write $w = w_0 + \Delta$, with $w_0 = \lambda^2/3 - 1$, and expand Eq. (13) to linear order in Δ, we obtain

$$a\frac{d\Delta}{da} = \frac{3}{2}(w_0 - 1)\Delta, \qquad (14)$$

so the evolution of w near w_0 is

$$w = w_0 + \Delta_0 a^{(3/2)(w_0-1)}, \qquad (15)$$

where we define $a = 1$ to be the scale factor at which $w = w_0 + \Delta_0$. Thus, w diverges from its fixed-point value w_0 as a power of the scale factor, with this power increasing for larger values of w_0. As an example, the largest value of w considered in Ref. [6] is $w = 5/3$, for which the density of the scalar field scales as $\rho_\phi \propto a^{-8}$. In this case, we have $w = w_0 + \Delta_0 a$. The effects examined in Ref. [6] require this value of w to be maintained over a factor ~ 100 in scale factor. In order to achieve this, w needs to be initially tuned to within less than 1% of its fixed-point value.

III. BAROTROPIC MODELS

As noted in the Introduction, a perfect fluid with $w = p/\rho$ has a sound speed $c_s^2 = w$, so $c_s^2 \leq 1$ forces $w \leq 1$. However, we can break the equivalence between w and c_s^2 by going to a more complex relation between p and ρ. In barotropic models, the pressure is a fixed function of the density:

$$p = f(\rho). \qquad (16)$$

(Perfect fluids are the special case for which $p = w\rho$.) Barotropic models have been studied extensively as possible models for dark energy. Particular models of this form include the Chaplygin gas [36,37] and the generalized Chaplygin gas [38], the linear/affine equation of state [39–43], the quadratic equation of state [42], the van der Waals equation of state [44,45], and more complicated equations of state [46,47]. A general study of the properties of barotropic models for dark energy was undertaken in Refs. [48,49]. Note that there is a simple mapping between the barotropic models discussed here and purely kinetic k-essence models [48], so the results presented here can be extended in a straightforward way to the latter set of models.

These models seem plausible as a source for ultraslow expansion because we now have $dp/d\rho \neq p/\rho$, so one can construct models for which $c_s^2 = dp/d\rho \leq 1$ but $p/\rho > 1$. As an example, consider one of the simplest barotropic models, in which p is a linear function of ρ [39–43]:

$$p = p_0 + \alpha\rho, \qquad (17)$$

where p_0 and α are constants. The requirement that $c_s^2 \leq 1$ gives $\alpha \leq 1$. Using Eq. (2), we find the following relation between the density and scale factor:

$$\rho = Ca^{-3(\alpha+1)} - \frac{p_0}{\alpha + 1}, \qquad (18)$$

where C is a constant. In the limit of early times (small a), the first term dominates and the density simply scales as a perfect fluid with $w = \alpha$. When the two terms on the right-hand side of Eq. (18) become of roughly equal magnitude, there is a transient period for which $w > 1$ and the expansion becomes ultraslow. However, we rapidly have $w \to \infty$ and $\rho \to 0$.

This fate for barotropic fluids with $w > 1$ is not peculiar to this particular choice of model; it is generic to all such models. To see why this is the case, we use the expression for the evolution of w in barotropic models given in Ref. [48]:

$$a\frac{dw}{da} = 3(1 + w)(w - c_s^2). \qquad (19)$$

If $c_s^2 \leq 1$ (as required) and $w > 1$, then the evolution is manifestly unstable; $dw/da > 0$, and w rapidly evolves to ∞. Hence, it does not appear that any barotropic model can provide sustained ultraslow expansion.

IV. BRANEWORLD MODELS

In braneworld cosmologies, the observable universe is a brane embedded in a higher-dimensional bulk. The standard-model fields are confined to our 3-brane, while gravity alone propagates in the bulk. The most widely investigated model of this type is the type II Randall-Sundrum model, in which the brane has positive tension and the bulk contains a positive cosmological constant [50,51]. In such models, in flat spacetime, we can write the Hubble parameter as

$$H^2 = \frac{\rho}{3}\left(1 + \frac{\rho}{\rho_0}\right) + \frac{C}{a^4} + \frac{\Lambda}{3}, \qquad (20)$$

where C and ρ_0 are constants, with the latter depending on the 5-dimensional Planck mass, and Λ is the cosmological constant. At early times, if we neglect C, the Hubble parameter evolves as $H \propto \rho$ instead of $H \propto \sqrt{\rho}$. The effects of this altered evolution on the relic dark matter abundance were investigated in Refs. [52–55]. Reference [53], in particular, noted that this modification to the Friedman equation could lead to ultraslow expansion. When $H \propto \rho$, we have, instead of Eq. (5),

$$a \propto t^{1/3(1+w)}. \qquad (21)$$

Thus, even a radiation-dominated universe leads to ultra-slow expansion at early times, with $a \propto t^{1/4}$. For an arbitrary perfect fluid, with $w \leq 1$, the slowest possible expansion is given by $a \propto t^{1/6}$.

It is clear that braneworld models can produce evolution slower than the bound set by a stiff perfect fluid. However, such behavior comes with an expansion law that does not obey the standard Friedman equation, so standard

calculations of, e.g., relic particle evolution must incorporate this modified expansion law, as was done in Refs. [52–55].

V. LOITERING EARLY UNIVERSE

The loitering universe is an idea that goes back to Lemaitre [56]. This model requires a spatially closed universe containing matter and a cosmological constant, so H is given by

$$H^2 = \frac{\rho_{M0}}{3}\left(\frac{a}{a_0}\right)^{-3} + \frac{\Lambda}{3} - \frac{\kappa}{a^2}, \qquad (22)$$

where ρ_{M0} and a_0 are the present-day matter density and scale factor, respectively; Λ is the (positive) cosmological constant; and $\kappa = 1$ is the curvature. The last term in Eq. (22) must be large enough to alter the evolution of H at late times, but not so large that H ever reaches 0, which corresponds to recollapse. There has been persistent interest in this model [57], but it is clear that it does not correspond to our best current observations; in particular, the curvature of the universe is now known to be very small [58]. To remedy this defect, Sahni and Shtanov [59] proposed a loitering model in the context of braneworld models. This model allows loitering to occur in a flat universe and at somewhat higher redshifts than in the original model described by Eq. (22).

However, neither model can accommodate loitering in the early Universe. The reason is that both of these models evolve asymptotically to a cosmological-constant-dominated evolution. This is desirable at present but cannot be incorporated into an acceptable evolution in the early Universe. Instead, we require a loitering solution that evolves asymptotically to a radiation-dominated early universe consistent with observations. (Note also that Ref. [60] proposed a loitering phase in the context of brane gas cosmology; this phase takes place in the very early Universe, before the epoch of interest here.)

In analogy to Eq. (22), we seek a universe containing multiple perfect fluids but which is dominated by radiation (with density $\rho_\gamma \propto a^{-4}$) at late times. Therefore, any additional fluids we add to the radiation must have a density that decreases faster than a^{-4}. The fastest possible decay occurs for a stiff fluid with $c_s^2 = 1$ and $\rho_S \propto a^{-6}$. Finally, to allow for a loitering phase, we add a component with an intermediate equation of state and negative energy density, $\rho_5 \propto a^{-5}$ and $\rho_5 < 0$. Then, for this mixture of fluids, the total density is given by

$$\rho = \rho_{\gamma i}\left(\frac{a}{a_i}\right)^{-4} - \rho_{5i}\left(\frac{a}{a_i}\right)^{-5} + \rho_{Si}\left(\frac{a}{a_i}\right)^{-6}, \quad (23)$$

where a_i is an arbitrary fiducial value of the scale factor at which the densities of the three components are given by $\rho_{\gamma i}$, ρ_{5i}, and ρ_{Si}. The first and second derivatives of the scale
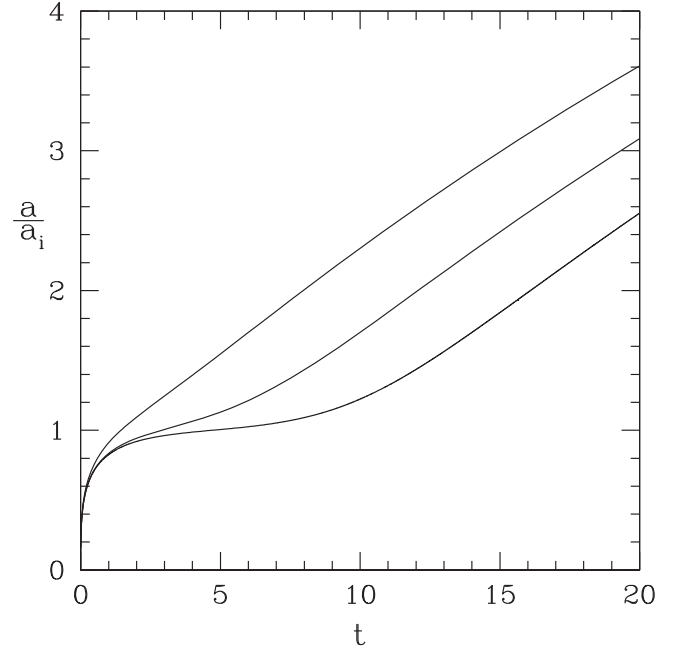

FIG. 1. Evolution of the scale factor a as a function of time t for the loitering early universe produced by the density given in Eq. (23) with $\rho_{5i} = 2\rho_{\gamma i}$ and $\rho_{Si}/\rho_{\gamma i} = 1.1, 1.01, 1.001$ (top to bottom).

factor are given by $(\dot{a}/a)^2 = (1/3)\rho$ and $\ddot{a}/a = -(1/6)(\rho + 3p)$. Setting $\dot{a} = \ddot{a} = 0$ at $a = a_i$ gives $\rho_{5i} = 2\rho_{\gamma i}$ and $\rho_{Si} = \rho_{\gamma i}$. Choosing ρ_{Si}, ρ_{5i}, and $\rho_{\gamma i}$ near these values will then give loitering behavior for a near a_i. This is illustrated in Fig. 1, where we set $\rho_{5i} = 2\rho_{\gamma i}$ and choose ρ_{Si} close to $\rho_{\gamma i}$ to produce varying degrees of loitering.

It is clear from Fig. 1 that the density given by Eq. (23) can lead to a quasistatic expansion or simply a significant slowing of the expansion rate, depending on the relative densities of the components. However, this model has several problems. It requires a fine-tuning of the ratios of the densities in Eq. (23), although this is also true of the original late-time loitering universe. It also introduces a poorly motivated component with negative energy density, while in the standard loitering model, the curvature automatically behaves as an effectively negative energy density with the desired evolution. Models with pairs of perfect fluids having, respectively, positive and negative densities were examined in Ref. [61], which also gives some motivation for the latter. Despite these problems, the evolution produced by this model is rather intriguing. It is capable of producing an expansion rate slower than any of the other models considered here; for an appropriate choice of parameters, the universe can approach a nearly static state for a significant period of time.

VI. CONCLUSIONS

We find that there are no compelling models for ultra-slow expansion within the framework of the standard

Friedman equation. Scalar field models and barotropic models are both unstable. Braneworld models allow for ultraslow expansion but only in the context of a modified expansion law. The loitering early universe introduced in the previous section is perhaps the most interesting, as it allows for a nearly static phase in the early Universe.

It would be interesting to explore such a phase in connection with models for relic particle evolution and baryogenesis. However, this model requires a negative energy component with an unusual equation of state, and it is not clear that there is a plausible motivation for such a component.

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