Dynamical system analysis of Myrzakulov gravity

G. Papagiannopoulos,[1] Spyros Basilakos,[2,3,4] and Emmanuel N. Saridakis[2,5,6]

[1]*Department of Physics, National and Kapodistrian University of Athens, Zografou Campus, GR 157 73 Athens, Greece*
[2]*National Observatory of Athens, Lofos Nymfon, 11852 Athens, Greece*
[3]*Academy of Athens, Research Center for Astronomy and Applied Mathematics, Soranou Efesiou 4, 11527 Athens, Greece*
[4]*School of Sciences, European University Cyprus, Diogenes Street, Engomi, 1516 Nicosia, Cyprus*
[5]*CAS Key Laboratory for Researches in Galaxies and Cosmology, Department of Astronomy, University of Science and Technology of China, Hefei, Anhui 230026, People's Rupublic of China*
[6]*Departamento de Matemáticas, Universidad Católica del Norte, Avenida Angamos 0610, Casilla 1280 Antofagasta, Chile*

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We perform a dynamical system analysis of Myrzakulov or $F(R,T)$ gravity, which is a subclass of affinely connected metric theories, where ones uses a specific but nonspecial connection that allows for nonzero curvature and torsion simultaneously. We consider two classes of models, extract the critical points, and examine their stability properties alongside their physical features. In the class 1 models, which possess Λ cold dark matter (CDM) cosmology as a limit, we find the sequence of matter and dark energy eras, and we show that the Universe will result in a dark-energy-dominated critical point for which dark energy behaves like a cosmological constant. Concerning the dark energy equation-of-state parameter, we find that it lies in the quintessence or phantom regime, according to the value of the model parameter. For the class 2 models, we again find the dark-energy-dominated, de Sitter late-time attractor, although the scenario does not possess ΛCDM cosmology as a limit. The cosmological behavior is richer, and the dark energy sector can be quintessencelike, phantomlike, or experience the phantom-divide crossing during the evolution.

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I. INTRODUCTION

An increasing amount of observational data has now led to the establishment of the standard model of cosmology, according to which the Universe has passed through two phases of accelerating expansion, one at early and one at late times. Although the latter can be explained by a cosmological constant, the possibility of a dynamical nature, some possible tensions, as well as the necessity for an additional description of the former phase, may ask for a kind of modification. In general, one has two ways to accomplish this. The first is to maintain general relativity as the underlying gravitational theory but consider extra field contents such as the dark energy sector [1,2] and/or the inflaton field [3]. The second way is to construct modified gravity theories, which in a particular limit tend to general relativity, but which in general exhibit extra degrees of freedom that can drive the nonstandard universe evolution [4–6]. This direction has the additional advantage of bringing gravity closer to a quantum description [7].

Modified gravity can arise from suitable extensions of the Einstein-Hilbert action, such as in $F(R)$ gravity [8], in theories with nonminimal coupling between matter and curvature [9,10], $F(G)$ gravity [11,12], Lovelock gravity [13,14], Horndeski [15] and generalized Galileon [16,17] gravities, etc. A different way to construct gravitational modifications is to use as a base the equivalent, teleparallel formulation of gravity [18,19] and build torsional theories such as $F(T)$ gravity [20–22], theories with nonminimal coupling between matter and torsion [23,24], $F(T,T_G)$ gravity [25], $F(T,B)$ gravity [26], teleparallel Horndeski [27], etc. One can proceed to other geometrical modifications, thus obtaining novel extended gravity theories, such as using nonmetricity [28,29] or constructing more complex structures such as in Finsler geometry [30–32].

An alternative way to construct gravitational modifications is to alter the connection structure of the theory, namely, the extra degrees of freedom will arise from the different connection instead of the different action. This was known in the framework of metric-affine theories [33–36], as well as in Finsler-like theories where the nonlinear connection may bring about extra degrees of freedom [37–41]. In Myrzakulov or $F(R,T)$ gravity [42] [this should not be confused with the $F(R,T)$ gravity where T is the trace of the energy-momentum tensor [43]], one uses a specific but nonspecial connection, which allows for nonzero curvature and torsion simultaneously, which then leads to the appearance of extra degrees of freedom that can make the theory phenomenologically viable [44]. As one can show, it can be expressed as a deformation of both general relativity and its teleparallel equivalent. Hence, this theory lies within the class of Riemann-Cartan family of theories, which in turn belong to the general family of affinely connected metric theories [45]. Nevertheless, the theory at hand maintains zero nonmetricity.

The cosmological applications of Myrzakulov gravity were investigated in [42,44,46–52], while the confrontation with observational data has been performed in [53]. In this work, we are interested in investigating the cosmological behavior by applying the powerful method of dynamical system analysis. Such an approach allows one to extract global information on the cosmological evolution, independent of the specific initial conditions or the intermediate-time behavior [54,55]. In particular, by examining the stable critical points of the autonomously transformed cosmological equations, one can classify the infinite number of possible evolutions into a few different classes obtained asymptotically.

The plan of the work is the following. In Sec. II, we review Myrzakulov gravity and we present the relevant cosmological equations. In Sec. III, we perform a detailed dynamical analysis of various scenarios in this theory, focusing on the stable late-time solutions, and we discuss the physical behavior. Finally, Sec. IV is devoted to our conclusions.

II. COSMOLOGY IN MYRZAKULOV GRAVITY

In this section, we provide the cosmological equations in a universe governed by Myrzakulov gravity [42,44]. The basic feature of the theory is the modification of the connection, however, maintaining zero nonmetricity. As it is known, choosing a general connection $\omega^a{}_{bc}$ one can construct the curvature and the torsion tensors through the expressions [25]

$$R^a{}_{b\mu\nu} = \omega^a{}_{b\nu,\mu} - \omega^a{}_{b\mu,\nu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}, \quad (1)$$

$$T^a{}_{\mu\nu} = e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu} + \omega^a{}_{b\mu}e^b{}_\nu - \omega^a{}_{b\nu}e^e{}_\mu, \quad (2)$$

with $e_a{}^\mu\partial_\mu$ as the tetrad field satisfying $g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu$, with $g_{\mu\nu}$ as the metric, $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots 1)$, and where greek and latin indices run, respectively, over coordinate and tangent space, with a comma denoting differentiation.

Among the infinite connections, the Levi-Civita Γ_{abc} is the only one that by construction leads to vanishing torsion. For clarity, we will use the superscript "LC" to denote the curvature tensor calculated using Γ_{abc}, i.e., $R^{(LC)a}{}_{b\mu\nu} = \Gamma^a{}_{b\nu,\mu} - \Gamma^a{}_{b\mu,\nu} + \Gamma^a{}_{c\mu}\Gamma^c{}_{b\nu} - \Gamma^a{}_{c\nu}\Gamma^c{}_{b\mu}$. Similarly, imposition of the Weitzenböck connection $W^\lambda{}_{\mu\nu} = e_a{}^\lambda e^a{}_{\mu,\nu}$ leads to zero curvature, and the corresponding torsion tensor becomes $T^{(W)\lambda}{}_{\mu\nu} = W^\lambda{}_{\nu\mu} - W^\lambda{}_{\mu\nu}$, where we use the superscript "W" to denote quantities calculated using $W^\lambda{}_{\mu\nu}$. From contractions of the above tensors, one can find the Ricci scalar corresponding to the Levi-Civita connection,

$$R^{(LC)} = \eta^{ab}e_a{}^\mu e_b{}^\nu[\Gamma^\lambda{}_{\mu\nu,\lambda} - \Gamma^\lambda{}_{\mu\lambda,\nu} + \Gamma^\rho{}_{\mu\nu}\Gamma^\lambda{}_{\lambda\rho} - \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\rho}], \quad (3)$$

as well as the torsion scalar corresponding to the Weitzenböck connection,

$$\begin{aligned} T^{(W)} &= \frac{1}{4}(W^{\mu\lambda\nu} - W^{\mu\nu\lambda})(W_{\mu\lambda\nu} - W_{\mu\nu\lambda}) \\ &\quad + \frac{1}{2}(W^{\mu\lambda\nu} - W^{\mu\nu\lambda})(W_{\lambda\mu\nu} - W_{\lambda\nu\mu}) \\ &\quad - (W_\nu{}^{\mu\nu} - W_\nu{}^{\nu\mu})(W^\lambda{}_{\mu\lambda} - W^\lambda{}_{\lambda\mu}). \end{aligned} \quad (4)$$

In general relativity, one uses $R^{(LC)}$ in the Lagrangian, while in the teleparallel equivalent of general relativity one uses $T^{(W)}$. Both these theories possess two propagating degrees of freedom, describing a massless spin-two field, i.e., the graviton. Thus, in their corresponding modifications, namely, curvature-based modified gravity or torsion-based modified theories, one can acquire extra degrees of freedom by extending the action, and these extra degrees of freedom are the ones that lead to modified cosmological evolution. Nevertheless, after the above discussion we realize that one can introduce extra degrees of freedom through the consideration of nonspecial connections, i.e., going beyond the Levi-Civita and Weitzenböck ones. Hence, if one applies a connection that has both nonzero curvature and torsion, a theory with more degrees of freedom is obtained.

Specifically, as it was presented in [42,44], one can construct a theory that is based on a specific but not special connection that leads to both nonzero curvature and nonzero torsion. The action of such a theory would be

$$S = \int d^4xe\left[\frac{F(R,T)}{2\kappa^2} + L_m\right], \quad (5)$$

with $e = \det(e^a_\mu) = \sqrt{-g}$ and $\kappa^2 = 8\pi G$ the gravitational constant, however, we mention that T and R are the torsion and curvature scalars of the nonspecial connection, namely [25],

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\lambda\nu\mu} - T_\nu{}^{\nu\mu}T^\lambda{}_{\lambda\mu}, \quad (6)$$

$$R = R^{(LC)} + T - 2T_\nu{}^{\nu\mu}{}_{;\mu}, \quad (7)$$

with the semicolon denoting the covariant differentiation with respect to the Levi-Civita connection. Finally, in the above action we have also added the matter Lagrangian L_m.

As one can see from the definitions (1) and (2), T depends on the tetrad, its first derivative, and the connection, and R depends on the tetrad and its first and second derivatives and on the connection and its first derivative. These allow one to introduce the parametrization [44]

$$T = T^{(W)} + v, \tag{8}$$

$$R = R^{(LC)} + u, \tag{9}$$

with u being a scalar quantity depending on the tetrad, its first and second derivatives, and the connection and its first derivative, and v being a scalar depending on the tetrad, its first derivative, and the connection.

The above theory has nontrivial structure and exhibits extra degrees of freedom even in the case where the arbitrary function $F(R,T)$ has a trivial form, since the novel features arise from the nontrivial connection itself, parametrized by the quantities u and v. If this connection becomes the Levi-Civita one, we obtain that $u = 0$ and $v = -T^{(W)}$, and thus we recover the standard $F(R)$ gravity [which for $F(R) = R$ becomes general relativity]. However, if the connection is the Weitzenböck one, then we acquire $v = 0$ and $u = -R^{(LC)}$, and therefore we recover standard $F(T)$ gravity [which for $F(T) = T$ becomes the teleparallel equivalent of general relativity].

In order to proceed to the cosmological applications of the above construction, we follow the minisuperspace procedure [44]. Imposing the flat Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \tag{10}$$

namely, the tetrad $e^a{}_\mu = \mathrm{diag}[1, a(t), a(t), a(t)]$, with $a(t)$ as the scale factor, we find $R^{(LC)} = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$ and $T^{(W)} = -6(\frac{\dot{a}^2}{a^2})$. Taking into account the dependence of u and v on the metric and the connection, we deduce that $u = u(a, \dot{a}, \ddot{a})$ and $v = v(a, \dot{a})$. Furthermore, we take the standard form $L_m = -\rho_m(a)$ [56]. Finally, in order to explore the dynamics of Myrzakulov gravity arising solely from the nonspecial connection itself, we make the simple linear choice $F(R,T) = R + \lambda T$, with λ as the dimensionless coupling parameter.

Inserting the above minisuperspace expressions into (5), we obtain $S = \int L dt$, with

$$L = \frac{3}{\kappa^2}[\lambda + 1]a\dot{a}^2 - \frac{a^3}{2\kappa^2}[u(a, \dot{a}, \ddot{a}) + \lambda v(a, \dot{a})] + a^3\rho_m(a). \tag{11}$$

We can now perform variation and extract the equations of motion for a, and we can moreover consider the Hamiltonian constraint $\mathcal{H} = \dot{a}[\frac{\partial L}{\partial \dot{a}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \ddot{a}}] + \ddot{a}(\frac{\partial L}{\partial \ddot{a}}) - L = 0$; hence resulting in the following Friedmann equations [44]:

$$3H^2 = \kappa^2(\rho_m + \rho_{MG}), \tag{12}$$

$$2\dot{H} + 3H^2 = -\kappa^2(p_m + p_{MG}), \tag{13}$$

where the dark energy sector that arises effectively from the nonspecial connection has energy density and pressure

$$\rho_{MG} = \frac{1}{\kappa^2}\left[\frac{Ha}{2}(u_{\dot{a}} + v_{\dot{a}}\lambda) - \frac{1}{2}(u + \lambda v) + \frac{au_{\ddot{a}}}{2}(\dot{H} - 2H^2) - 3\lambda H^2\right], \tag{14}$$

$$p_{MG} = -\frac{1}{\kappa^2}\left[\frac{Ha}{2}(u_{\dot{a}} + v_{\dot{a}}\lambda) - \frac{1}{2}(u + \lambda v) - \frac{a}{6}(u_a + \lambda v_a - \dot{u}_{\dot{a}} - \lambda\dot{v}_{\dot{a}}) - \frac{a}{2}(\dot{H} + 3H^2)u_{\ddot{a}} - Ha\dot{u}_{\ddot{a}} - \frac{a}{6}\ddot{u}_{\ddot{a}} - \lambda(2\dot{H} + 3H^2)\right], \tag{15}$$

respectively. In the above expressions, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, p_m is the pressure of the matter sector, and the subscripts a, $\dot{a}$, and $\ddot{a}$ mark partial derivatives with respect to these arguments. Note that the effective dark energy sector is conserved, namely, $\dot{\rho}_{MG} + 3H(\rho_{MG} + p_{MG}) = 0$, as it is easily deduced from the above imposing the matter conservation equation $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$ too.

In the following, we focus on two classes of the theory at hand, constructed phenomenologically in order to lead to interesting cosmological evolution.

A. Class 1

As a first example, we consider the class where $u = c_1\dot{a} - c_2$ and $v = c_3\dot{a} - c_4$, where c_1,c_2,c_3, and c_4 are constants. For this class, Eqs. (12)–(15) lead to

$$3H^2 = \kappa^2(\rho_m + \rho_{MG}), \tag{16}$$

$$2\dot{H} + 3H^2 = -\kappa^2(p_m + p_{MG}), \tag{17}$$

with

$$\rho_{MG} = \frac{1}{\kappa^2}[c - 3\lambda H^2], \tag{18}$$

$$p_{MG} = -\frac{1}{\kappa^2}[c - \lambda(2\dot{H} + 3H^2)], \tag{19}$$

where we have defined $c \equiv c_2 + c_4$. Thus, the effective dark energy equation-of-state parameter reads

$$w_{DE} = -1 + \frac{2\lambda\dot{H}}{c - 3\lambda H^2}. \tag{20}$$

It is interesting to mention here that the effective dark energy density (18) falls within particular subclasses of the running vacuum cosmology [57–59].

B. Class 2

The second class that we are interested in is the one characterized by $u = c_1 \frac{\dot{a}}{a} \ln \dot{a}$ and $v = s(a)\dot{a}$, where $s(a)$ is an arbitrary function. Hence, expressions (12)–(15) give again the Friedmann equations (16) and (17), but now with

$$\rho_{MG} = \frac{1}{\kappa^2}\left[\frac{c_1}{2}H - 3\lambda H^2\right], \tag{21}$$

$$p_{MG} = -\frac{1}{\kappa^2}\left[\frac{c_1}{2}H + \frac{c_1}{6}\frac{\dot{H}}{H} - \lambda(2\dot{H} + 3H^2)\right], \tag{22}$$

and thus we can find

$$w_{DE} = -1 + \frac{2\lambda\dot{H} - \frac{c_1}{6}\frac{\dot{H}}{H}}{\frac{c_1}{2}H - 3\lambda H^2}. \tag{23}$$

Similar to class 1 above, the effective dark energy density (21) coincides with broader subclasses of the running vacuum cosmology, and as we show below it can lead to very interesting cosmological behavior, despite the fact that it does not have the Λ cold dark matter (CDM) scenario as a particular limit.

III. PHASE-SPACE ANALYSIS

In the previous section, we presented the cosmological equations of the scenario at hand. As we can see, Myrzakulov gravity leads to the appearance of new terms in the Friedmann equations, which are of geometrical origin and, in particular, they arise from the nontrivial connection structure through the parametrization in terms of u and v. In this section, we proceed to the full phase-space analysis of these scenarios, by applying the dynamical system method [54,55]. Hence, we will first introduce suitably the auxiliary variables needed in order to transform the equations into an autonomous dynamical system [54,55,60–70], and then we will extract its critical points. Thus, examining the eigenvalues of the perturbation matrix around each of them, we can conclude their stability properties.

In order to perform the dynamical analysis, we introduce the quantities

$$\begin{aligned} A &= \left[\frac{Ha}{2}(u_{\dot{a}} + v_{\dot{a}}\lambda) - \frac{1}{2}(u + \lambda v) + \frac{a u_{\ddot{a}}}{2}(\dot{H} - 2H^2)\right], \\ B &= [-a(u_a + \lambda v_a - \dot{u}_{\dot{a}} - \lambda\dot{v}_{\dot{a}}) - 3a(\dot{H} + 3H^2)u_{\ddot{a}} \\ &\quad - 6Ha\dot{u}_{\ddot{a}} - a\ddot{u}_{\ddot{a}}]. \end{aligned} \tag{24}$$

Hence, the two Friedman equations can be written as

$$3H^2(1+\lambda) = \kappa^2\rho_m + A, \tag{25}$$

$$(2\dot{H} + 3H^2)(1+\lambda) = \kappa^2\rho_m w_m + A + \frac{B}{6}, \tag{26}$$

where for convenience we have also introduced the matter equation-of-state parameter defined as $w_m = p_m/\rho_m$.

Let us first examine the limit of the scenario at hand to the ΛCDM cosmology. In order to achieve this, we need $\rho_{MG} = -p_{MG}$, which implies that

$$\lambda\dot{H} = -\frac{B}{12}. \tag{27}$$

Although this condition can be satisfied in many ways, the simplest one is to consider the case $\lambda = 0$, namely, to focus on a Lagrangian being just the curvature R corresponding to the nonspecial connection. In this case, if we choose a connection with $u = c_1\dot{a} - c_2$, where c_1 and c_2 are constants, we acquire

$$\rho_{MG} = -p_{MG} = \frac{c_2}{2\kappa^2} \equiv \Lambda. \tag{28}$$

Interestingly enough, we observe that we do obtain ΛCDM cosmology, although in the starting action we had not considered an explicit cosmological constant. Thus, the nontrivial structure of the underlying geometry results in an effective cosmological constant, which reveals the capabilities of the theory. Note that even in this simple case where $\lambda = 0$, and thus T disappears from the action, the nonspecial connection still has a nonzero torsion. In general, such an effective emergence of a cosmological constant due to the richer underlying connection appears in other geometrical modified gravities too [40,71] and reveals the advantages of the theory.

Having the above discussion in mind, we can deduce that class 1 defined in Sec. II A corresponds to a deviation from ΛCDM cosmology, accepting it as a particular limit and thus satisfying the basic requirements to be a viable theory, while still maintaining the possibility to improve ΛCDM behavior. On the other hand, class 2 defined in Sec. II B does not have ΛCDM cosmology as a limit, nevertheless, and interestingly enough, as we will later show it can lead to a cosmological behavior in agreement with observations.

We can now proceed to the dynamical analysis of the above specific classes, keeping a general $\lambda \neq 0$.

A. Class 1

We start with class 1 of Sec. II A. In this case, definitions (24) lead to

$$\begin{aligned} A &= \frac{1}{2}(c_2 + \lambda c_4) \equiv C, \\ B &= 0. \end{aligned} \tag{29}$$

In order to transform the cosmological equations into an autonomous form, we introduce the dark matter and dark energy density parameters as our dimensionless variables, namely,

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2(1+\lambda)}, \tag{30}$$

$$\Omega_{MG} \equiv \frac{C}{3H^2(1+\lambda)}, \tag{31}$$

and therefore the first Friedmann equation (16) becomes $1 = \Omega_m + \Omega_{MG}$ [note that the case $\lambda = -1$ is not physically interesting, since according to (25) it leads to $\rho_m = -C/\kappa^2 = \text{const.}$; hence, in the following, we focus on the case $\lambda \neq -1$]. Additionally, the second Friedmann equation (17) becomes

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + \Omega_m w_m - \Omega_{MG}). \tag{32}$$

Using this expression, as well as (30) and (31), Eq. (20) can be rewritten as follows:

$$w_{DE} = -1 - \frac{3\lambda(1 + \Omega_m w_m - \Omega_{MG})}{\Omega_{MG}(1+\lambda) - 3\lambda}. \tag{33}$$

In summary, the dynamical system can be straightforwardly written as

$$\frac{d\Omega_m}{d\ln a} = -3\Omega_m[w_m - \Omega_m w_m + \Omega_{MG}], \tag{34}$$

$$\frac{d\Omega_{MG}}{d\ln a} = 3\Omega_{MG}[\Omega_m w_m + 1 - \Omega_{MG}]. \tag{35}$$

Since the first Friedman equation acts as a constraint, we finally remain with one-dimensional phase space. The corresponding critical points $P(\Omega_m, \Omega_{MG})$ are summarized in Table I, alongside their features and stability conditions. Note that in this case (33) provides the useful expression

$$w_{DE} = -1 - \frac{3\lambda\Omega_m(1+w_m)}{(1-\Omega_m)(1+\lambda) - 3\lambda}. \tag{36}$$

As we observe, point P_1 corresponds to a dark-energy-dominated Universe, in which dark energy behaves like a cosmological constant, and the fact that in the usual case of dust matter it is stable implies that it will be the late-time state of the Universe independent of the initial conditions. On the other hand, point P_2 is a matter-dominated, non-accelerating solution, and the fact that for dust matter equation of state it is a saddle point implies that this point can describe the necessary intermediate era of the Universe, in which matter structure is formed [69,70].

In order to show the above feature in a more transparent way, in Fig. 1 we present the behavior of the system in the (w_{DE}, Ω_m) space, in the case of dust matter, for various values of λ. As we can see, the system passes through the saddle point P_2 before it results in the stable late-time attractor P_1. Additionally, in order to examine the system at both intermediate and late times, in Fig. 2 we present Ω_m as a function of the redshift $z = -1 + a_0/a$ (setting the current scale factor $a_0 = 1$), since $\dot{H} = -(1+z)H(z)H'(z)$, with primes denoting derivatives with respect to z. We

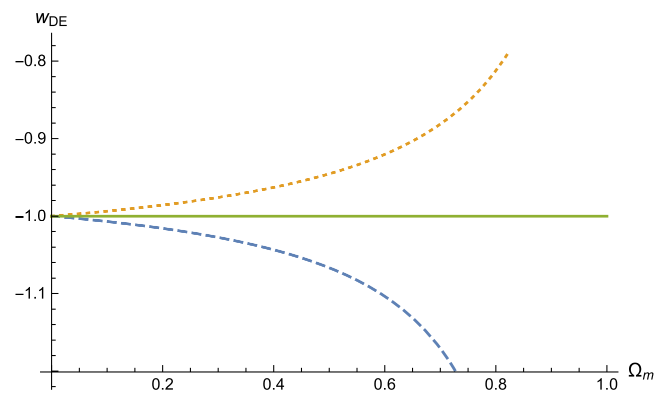

FIG. 1. The behavior of the system in the (w_{DE}, Ω_m) space for class 1 models of (16) and (17) with (18) and (19), for $w_m = 0$ and with $\lambda = 0.02$ (blue dashed), $\lambda = 0$ (green solid), and $\lambda = -0.02$ (orange dotted).

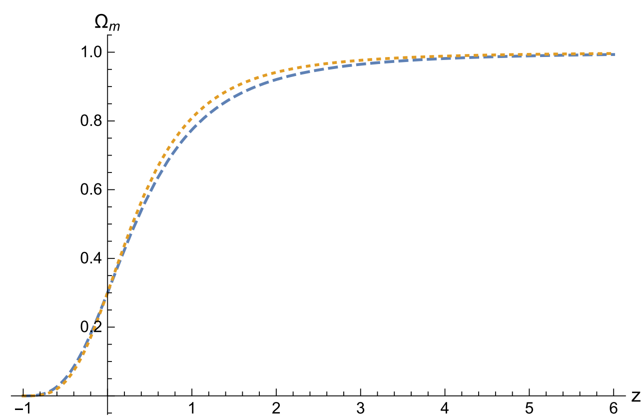

FIG. 2. The evolution of the matter density parameter $\Omega_m(z)$ as a function of the redshift, for class 1 models of (16) and (17) with (18) and (19), for $w_m = 0$ (blue dashed) and $w_m = 0.1$ (orange dotted).

TABLE I. The physically interesting critical points of class 1, namely, of (16) and (17) with (18) and (19), their features, and their stability conditions.

Point	(Ω_m, Ω_{MG})	Existence	w_{DE}	Acceleration	Stability
P_1	(0,1)	Always	-1	Yes	$w_m > -1$
P_2	(1,0)	Always	w_m	$w_m < -\frac{1}{3}$	$w_m < -1$

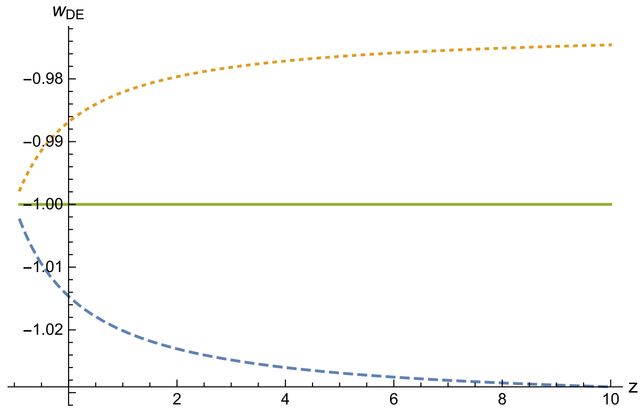

FIG. 3. The evolution of the dark energy equation-of-state parameter $w_{DE}(z)$ as a function of the redshift, for class 1 models of (16) and (17) with (18) and (19), for $w_m = 0$ and with $\lambda = 0.02$ (blue dashed), $\lambda = 0$ (green solid), and $\lambda = -0.02$ (orange dotted).

choose different values of w_m, and we fix C in order to have $\Omega_m(z = 0) \equiv \Omega_{m0} \approx 0.31$ as required by observations [72]. As we observe, the Universe follows the required evolution, with the sequence of matter and dark energy epochs.

Moreover, in Fig. 3 we depict the corresponding behavior of the dark energy equation-of-state parameter w_{DE} for various values of λ. As we see, although for every λ at asymptotic late times (i.e., for $z \to -1$) w_{DE} is stabilized at the cosmological constant value -1, as it was found in Table I, the behavior at intermediate redshifts and at the present Universe is different. In particular, for $\lambda < 0$ the dark energy sector behaves as quintessence, while for $\lambda > 0$ the w_{DE} lies in the phantom regime. This was expected from the form of (36) and reveals that class 1 offers a unified description of both quintessence and phantom regimes, without pathologies. Finally, as we see, in the case $\lambda = 0$ the scenario at hand recovers ΛCDM cosmology.

B. Class 2

Let us now proceed to the investigation of class 2 of Sec. II B. In this case, definitions (24) lead to

$$A = \frac{c_1}{2} H \equiv DH. \tag{37}$$

Similar to the class 1 case, for $\lambda \neq -1$ we can introduce the dimensionless auxiliary variables

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2(1+\lambda)}, \tag{38}$$

$$\Omega_{MG} \equiv \frac{D}{3H(1+\lambda)}, \tag{39}$$

and thus the first Friedmann equation becomes the constraint $1 = \Omega_m + \Omega_{MG}$. Additionally, for $\lambda \neq -1$ the second Friedmann equation becomes

$$\frac{\dot{H}}{H^2} = -\frac{3(1 + \Omega_m w_m - \Omega_{MG})}{2 - \Omega_{MG}}. \tag{40}$$

Hence, using this expression and (38) and (39), we can rewrite expression (23) as

$$w_{DE} = -1 - \frac{(1 + \Omega_m w_m - \Omega_{MG})[\lambda(2 - \Omega_{MG}) - \Omega_{MG}]}{(2 - \Omega_{MG})[(1+\lambda)\Omega_{MG} - \lambda]}. \tag{41}$$

For this class of scenarios, the dynamical system can be straightforwardly written as

$$\frac{d\Omega_m}{d\ln a} = \frac{3\Omega_m[\Omega_{MG} - w_m(-2 + \Omega_{MG} + 2\Omega_m)]}{\Omega_{MG} - 2}, \tag{42}$$

$$\frac{d\Omega_{MG}}{d\ln a} = \frac{6\Omega_{MG}(\Omega_{MG} - w_m\Omega_m - 1)}{\Omega_{MG} - 2}. \tag{43}$$

Because of the constraint first Friedman equation, we result to a one-dimensional phase space. Hence, in this case (41) gives the useful expression

$$w_{DE} = -1 - \frac{\Omega_m(1 + w_m)[\lambda - 1 + \Omega_m(\lambda + 1)]}{(1 + \Omega_m)[(1+\lambda)(1 - \Omega_m) - \lambda]}. \tag{44}$$

The critical points are summarized in Table II. In the same table, we provide their features and their stability conditions. Interestingly enough, class 2 exhibits the same critical points as class 1, namely, the dark-energy-dominated, de Sitter universe P_1, which is stable for dust matter, and the matter-dominated, nonaccelerating universe P_2, which is a saddle point for dust matter equation of state. The importance of the current behavior is that it is obtained not only without the consideration of an explicit cosmological constant, but also through the quite rich and complicated dark energy density (21), which does not accept the ΛCDM model as a particular limit.

Nevertheless, although class 2 has the same critical points as class 1, the behavior of the system at intermediate times is radically different. In Fig. 4 we show the (w_{DE}, Ω_m) diagram, in the case of dust matter. Although the system passes through the saddle point P_2 before it results in the stable late-time attractor P_1, the corresponding curves are different from those of Fig 1. Furthermore, in Fig. 5 we present Ω_m as a function of the redshift z and

TABLE II. The physically interesting critical points of class 2, namely, of (16) and (17) with (21) and (22), their features, and their stability conditions.

Point	(Ω_m, Ω_{MG})	Existence	w_{DE}	Acceleration	Stability
P_1	(0,1)	Always	-1	Yes	$w_m > -1$
P_2	(1,0)	Always	w_m	$w_m < -\frac{1}{3}$	$w_m < -1$

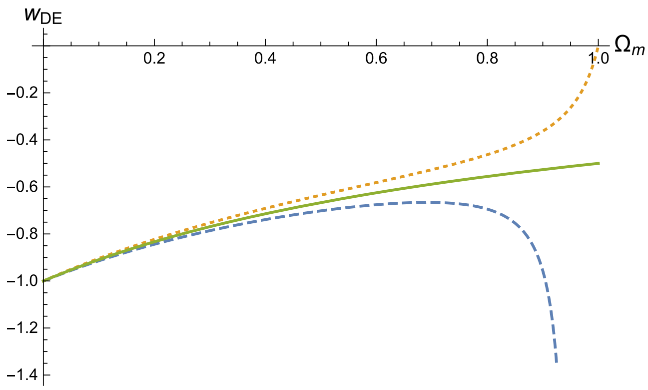

FIG. 4. The behavior of the system in the (w_{DE}, Ω_m) space for class 2 models of (16) and (17) with (21) and (22), for $w_m = 0$ and with $\lambda = 0.02$ (blue dashed), $\lambda = 0$ (green solid), and $\lambda = -0.02$ (orange dotted).

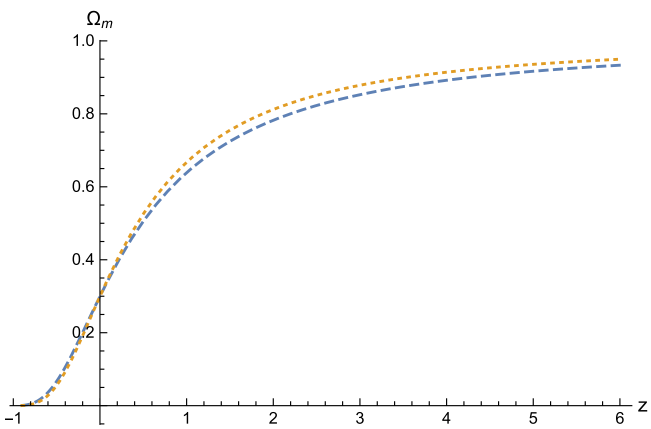

FIG. 5. The evolution of the matter density parameter $\Omega_m(z)$ as a function of the redshift, for class 2 models of (16) and (17) with (21) and (22), for $w_m = 0$ (blue dashed) and $w_m = 0.1$ (orange dotted).

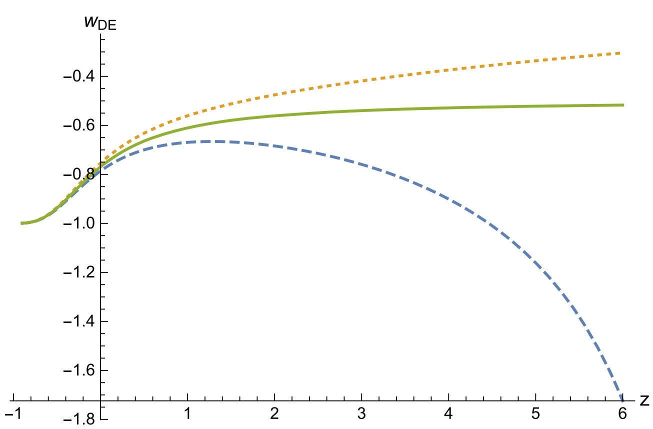

FIG. 6. The evolution of the dark energy equation-of-state parameter $w_{DE}(z)$ as a function of the redshift, for class 2 models of (16) and (17) with (21) and (22), for $w_m = 0$ and with $\lambda = 0.02$ (blue dashed), $\lambda = 0$ (green solid), and $\lambda = -0.02$ (orange dotted).

fixing D in order to have $\Omega_m(z=0) \equiv \Omega_{m0} \approx 0.31$, where we can see the sequence of matter and dark energy eras.

Finally, in Fig. 6 we present the evolution of the dark energy equation-of-state parameter $w_{DE}(z)$ for various values of λ. Similar to the previous class of models, at asymptotically late times w_{DE} tends to the cosmological constant value -1, as it was found in Table II; however, in the present case this is not trivial since the scenario at hand does not possess ΛCDM cosmology as a limit. Additionally, the behavior at intermediate redshifts is even more different, and the dark energy sector can lie in the quintessence regime, in the phantom regime, or experience the phantom-divide crossing during the evolution. This was expected from the form of (44) and reveals the capabilities of this class of models.

IV. DISCUSSION

We performed a dynamical system analysis of Myrzakulov gravity. The latter is a subclass of affinely connected metric theories, in which one uses a specific but nonspecial connection, which allows for nonzero curvature and torsion simultaneously. Thus, one obtains extra degrees of freedom, which in turn lead to extra terms in the Friedman equations that can lead to interesting phenomenology. Hence, by applying the dynamical system approach and performing a phase-space analysis, one is able to bypass the nonlinearities of the equations and investigate the global behavior of the system, independent of the specific initial conditions or the intermediate-time behavior evolution.

We considered two classes of models and for each case we transformed the equations into an autonomous dynamical system. We extracted the critical points, and we examined their stability properties alongside their physical features. In the class 1 models, which possess ΛCDM cosmology as a limit, we found that independent of the initial conditions the Universe will result in a dark-energy-dominated critical point in which dark energy behaves like a cosmological constant. Moreover, we found a matter-dominated, nonaccelerating solution, which is a saddle point and thus it can describe the necessary corresponding intermediate matter era of the Universe. Hence, the Universe follows the required evolution, with the sequence of matter and dark energy eras. Concerning the dark energy equation-of-state parameter w_{DE}, we showed that, although at asymptotic late times it is stabilized at the cosmological constant value -1 for every value of the model parameter λ, the behavior at intermediate redshifts and at the present Universe is different, since for $\lambda < 0$ the dark energy sector behaves as quintessence, while for $\lambda > 0$ the w_{DE} lies in the phantom regime.

For the class 2 models, we again found the dark-energy-dominated, de Sitter late-time attractor and the saddle critical point corresponding to a matter-dominated, nonaccelerating universe. Furthermore, at asymptotically late times, w_{DE} tends to the cosmological constant value -1. However, the interesting feature is that this was obtained without the scenario possessing ΛCDM cosmology as a particular limit. This class can also describe the sequence of matter and dark energy epochs; nevertheless, at intermediate times the behavior is radically different than the previous class, since the dark energy sector can lie in the quintessence regime, in the phantom regime, or experience the phantom-divide crossing during the evolution.

Let us stress here that, as we mentioned above, the two examined classes of theories, at a cosmological framework, fall within the class of generalized running vacuum theories [57–59]. Hence, one can perform the big bang nucleosynthesis analysis in the same way [73] and deduce that the early-universe evolution is not spoiled in the present models, too.

In summary, the phase-space analysis revealed the interesting features of Myrzakulov gravity and, in particular, the ability to possess a stable de Sitter solution as a late-time attractor even without the explicit consideration of a cosmological constant. It would be interesting to apply the Noether symmetry approach [74] in order to extract exact analytic solutions at intermediate times too. Furthermore, since the resulting cosmological equations are similar to subclasses of the running vacuum cosmology, it is necessary to further investigate their possible connection and examine whether the current framework offers the way to provide a Lagrangian for running vacuum models, a well-known open issue in the corresponding literature. Finally, it would be interesting to investigate the relation and differences of the present theory with theories with Weyl connection (not to be confused with Weyl gravity, which uses the standard Levi-Civita connection), which have an altered connection but nonzero nonmetricity [75]. These studies will be performed in separate works.

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