Power-law inflation satisfies Penrose's Weyl curvature hypothesis

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Based on entropy considerations and the arrow of time Penrose argued that the Universe must have started in a special initial singularity with vanishing Weyl curvature. This is often interpreted to be at odds with inflation. Here we argue just the opposite, that Penrose's persuasions are in fact consistent with inflation. Using the example of power law inflation, we show that inflation begins with a past null singularity, where Weyl tensor vanishes when the metric is initially exactly conformally flat. This initial state precisely obeys Penrose's conditions. The initial null singularity breaks T-reversal spontaneously and picks the arrow of time. It can be regulated and interpreted as a creation of a universe from nothing, initially fitting in a bubble of Planckian size when it materializes. Penrose's initial conditions are favored by the initial $O(4)$ symmetry of the bubble, selected by extremality of the regulated Euclidean action. The predicted observables are marginally in tension with the data, but they can fit if small corrections to power law inflation kick in during the last $60e$-folds.

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I. INTRODUCTION

Complex things which break do not reassemble on their own. Putting them together takes a toll and this toll is exacted by the increase of entropy of the system describing the process of breaking and reassembly. This trend can be used to define a global arrow of time in the universe, in contrast to generic microphysical phenomena which typically respect time reversal.

Penrose has taken this observation a step further [1], arguing that this phenomenon implies that the universe originated from a special initial state, characterized by a singularity in whose vicinity the geometry of the universe is very well approximated by conformal flatness, with (almost) vanishing Weyl tensor. This is also often interpreted as a problem for inflation (for a range of viewpoints, see [2–10]). Recall that the idea of inflation [11–13] is to blow up a universe from an initially small region, whose initial contents is much smaller than the vast complexity observed in the universe today. This is regardless of how the contents is inventoried, naively by counting over the initial volume, or more consistently by using the initial apparent horizon size. Either way, the late universe has far more contents than the early one. The difficulty with this obvious fact is that something other than inflation seems to be needed to select this seemingly improbable initial state. In other words, if the entropy count is used as a measure of likelihood, it seems to suggest that inflation presupposes an unlikely initial state.

Curiously, this argument overlooks the simple experiential fact that in many models of inflation the initial state of inflation is both singular and has an almost conformally flat geometry, in full accord with the technical aspects of Penrose's hypothesis. Indeed, the now-classic Borde-Guth-Vilenkin theorem asserts that inflationary spacetimes are past geodesically incomplete [14] (see also [15]), which at least at the semiclassical gravity level implies that inflation starts out of a singularity. Moreover, once inflation sets in,[1] it quickly dilutes initial deviations from homogeneous and isotropic Friedman-Robertson-Walker (FRW) metric [16–25], which being conformally flat has vanishing Weyl tensor, by symmetry. Thus it seems that at least "mechanically," if we accept Penrose's argument that the initial state is singular and Weyl flat, it is completely consistent to get inflation to spring forth from it. In some sense, actually, this state would appear to favor subsequent inflation as the origin of observed structures, since Weyl flatness favors a very smooth initial universe and something is required to break that smoothness spontaneously, instead of explicitly—precisely what inflation is intended to do.

To make our point, we employ the example of power law inflation [26–29]. We explain that the inflationary past ultimately begins with a past null singularity [30,31], for both spatially flat and spatially open FRW metrics. Since

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[1]A careful critic would without doubt express a concern right now that maybe inflation never sets in. We postpone our reply aimed at dispelling this concern for later in this paper.

both of these metrics are initially exactly conformally flat, they have vanishing Weyl tensor. Clearly, the initial null singularity breaks time-reversal and picks the arrow of time. Thus both of these metrics, maximally extended into the past satisfy Penrose's Weyl curvature hypothesis and hence describe universes with an arrow of time which nevertheless inflate. We do not think our examples are unique. Other examples may also exist, which feature spacelike instead of null singularities, such as the closed universe arising from the instantons in the no-boundary proposal. In fact, when we regulate the null singularity examples, which we consider in detail below, the regulators may be spacelike surfaces, which we comment later on. The point we are trying to make, however, is that regardless of the specific nature of the singularity, the selection of the initial state which realizes Penrose's Weyl curvature hypothesis might be a consequence of the quantum completion of inflation, which is anyway necessary, instead of needing a completely separate mechanism.

The question about what specifically selects the initial singularity can be addressed using quantum cosmology and no-boundary proposal. The past null singularity can be understood in terms of the singular Hawking-Turok instantons [32–36],[2] which can be regulated and interpreted as an expanding nonsingular bubble (for various approaches see [38–42]). Using this approach gives the reason for the selection of the initial Weyl-flat state of inflation: it minimizes the Euclidean action thanks to the $O(4)$ symmetry of the configuration and the smallness of the primordial bubble which seeded the universe [43,44].

The model actually yields predictions close to the current BICEP/Keck bounds [45], which can be improved with small corrections[3] to the potential during the last $60e$-folds. Alternatively, if the resolution of the H_0 tension is Early Dark Energy (EDE) [46–48], the CMB fits need a slightly higher scalar spectral index $n_S \sim 0.98$–0.995 [49–51], which is readily retrofitted by power law inflation. Interestingly, for the parameters which are close to the observationally favored values, the regime of universe self-reproduction in power law inflation is relegated to the cutoff physics, and so are superseded by the primordial bubble. This means, once fixed by the birth of the universe, the arrow of time remains unaffected by subsequent dynamics.

A very interesting question is how to interpret the cosmological perturbations, both scalar and tensor, which arise during inflation from the entropic point of view. Scalar perturbations are model dependent, although in all models of inflation they are an intrinsic ingredient of inflationary dynamics. Tensor perturbations are however universal, depending only on the scale of inflation. Both modes however utilize the same "seed," which is the uncertainty principle of quantum fluctuations in the inflationary vacuum. In (quasi)-de Sitter geometries this leads to the spontaneous emergence and growth of anisotropies and inhomogeneities, which may be viewed as an avatar of de Sitter instability [52–57]. This instability, from the entropic point of view, indicates that the pure de Sitter, appearing as the state with vanishing Weyl curvature, is a special state of the theory that dynamically evolves into the more generic states, which include the perturbations. It would be interesting to test this idea in more detail.

[2]A different method to start the universe with a null singularity has been proposed in [37].

[3]We will ignore the specific form of those corrections here, and work with purely exponential potentials because the causal structure analysis is considerably simpler.

II. POWER LAW INFLATION

Power law inflation is driven by a scalar field with an exponential potential, with the field rolling off to $\phi = \infty$. The potential is parametrized by [26–29]

$$V(\phi) = V_0 e^{-c\phi/M_{\rm Pl}}, \tag{1}$$

where ϕ is a canonically normalized scalar field, c is a numerical constant of order unity, and $M_{\rm Pl} \sim 2 \times 10^{18}$ GeV is the Planck scale. Clearly, V_0 is degenerate with the initial value of ϕ. Alternatively, the dynamics can also be parametrized by an equation of state

$$p = w\rho, \tag{2}$$

where p and ρ are pressure and energy density, respectively. In general, for homogeneous solutions the equation of state parameter w is a function of time until the self-similar attractor is reached. When $c \ll \sqrt{2}$ (which is the requirement that the geometry describes an accelerating expansion) a typical configuration will settle into the attractor fixed point within a few Hubble times, and $w \to$ const.

The scalar sources the FRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2\right). \tag{3}$$

Here we will be particularly interested in the $k = 0, -1$ cases, with spatially flat or open hyperbolic slices. Equations of motion are

$$3H^2 + 3\frac{k}{a^2} = \frac{\rho}{M_{\rm Pl}^2}, \qquad \dot\rho + 3H(\rho + P) = 0, \quad \text{with} \qquad \rho = \frac{\dot\phi^2}{2} + V, \qquad p = \frac{\dot\phi^2}{2} - V, \tag{4}$$

where the Hubble parameter is $H = \dot a/a$.

To find the attractor, we substitute ρ and p into (2) and hold w fixed, which gives the first order equation $\dot\phi^2/2 = \frac{1+w}{1-w}V$. This is easy to solve; after straightforward algebra, we find the attractor form of ρ (with $p = w\rho$),

$$\rho = \frac{4}{c^2}\frac{1}{1+w}\frac{M_{\rm Pl}^2}{t^2}. \tag{5}$$

Next, the conservation equation yields $\rho = \rho_0 (a_0/a)^{3(1+w)}$, and so comparing with (5) we find $a \sim t^{\frac{2}{3(1+w)}}$. The Friedmann equation then shows that unless $w = -1/3$, the curvature contribution is subleading relative to the attractor energy density. Neglecting it and substituting ρ of (5) into it yields, using $H = \frac{2}{3(1+w)t}$,

$$1 + w = \frac{c^2}{3}. \tag{6}$$

Clearly, imposing $w \to -1$ requires $|c| \ll 1$. In any case, the attractor is [26–29] (since $\frac{2}{3(1+w)} = \frac{2}{c^2}$, and using $\frac{\dot\phi^2}{2} = \frac{2}{c^2}\frac{M_{\rm Pl}^2}{t^2}$)

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{c^2}}, \qquad \phi = \phi_0 + \frac{2M_{\rm Pl}}{c}\ln\left(\frac{t}{t_0}\right). \tag{7}$$

Here a_0, t_0 and ϕ_0 are integration constants; a_0 is pure gauge, which we can fix to unity choosing $a(t_0) = 1$. The others satisfy $V_0 e^{-c\phi_0/M_{\rm Pl}} t_0^2 = 2M_{\rm Pl}^2 (6 - c^2)/c^4$.

This solution applies at late times. At early times, it may be altered at small t If the universe is spatially curved, specifically open, with $k = -1$, and the curvature initially dominates. In that case, the scale factor changes to $a = t/t_0$, while the scalar field configuration remains largely the same.

In either case, it is evident that $t \to 0$ is an initial singularity. In fact, when $w \le -1/3$, this hypersurface is null [30,31], as we will review below. Here we merely note that the requirement of using Einstein's equations consistently near the singularity imposes a physical cutoff on t_0. Since

$$M_{\rm Pl}^2 R = \frac{48 M_{\rm Pl}^2}{c^4 t_0^2}\left(1 - \frac{c^2}{4}\right), \tag{8}$$

requiring that the effective curvature remains below some cutoff $\mathcal{M}_{\rm UV}^4$ imposes

$$t_0^2 \gtrsim \frac{48 M_{\rm Pl}^2}{c^4 \mathcal{M}_{\rm UV}^4}\left(1 - \frac{c^2}{4}\right). \tag{9}$$

Since $\mathcal{M}_{\rm UV} \lesssim M_{\rm Pl}/\sqrt{N}$ where N is the number of light field theory species [58,59], and $|c| \ll 1$, this implies that $t_0 \gg 1/M_{\rm Pl}$. We will see that this essentially pushes the self-reproduction regime of inflation too close to singularity, and cuts it out of the semiclassical regime.

Let us now turn to observables. The scalar and tensor perturbations spectra evaluated on the attractor are

$$P_{\sf S} = \left(\frac{H^2}{2\pi\dot\phi}\right)^2, \qquad P_{\sf T} = \frac{8H^2}{(2\pi)^2 M_{\rm Pl}^2}. \tag{10}$$

Taking t_* as the instant when the attractor evolution starts to dominate, corresponding to the value ϕ_*, and introducing $\mathcal{N} = \ln(a(t)/a_*)$ as the number of e-folds that transpired until time t, we find that the field variation is $\Delta\phi/M_{\rm Pl} = c\mathcal{N}$, and that the scalar power, tensor power, spectral index n_S, and the tensor-scalar ratio during this epoch are [26–29]

$$P_{\sf S} = P_{\sf S}(t_*)\left(\frac{k}{k_*}\right)^{-2c^2/(2-c^2)}, \qquad P_{\sf T} = rP_{\sf S},$$
$$n_S = 1 - \frac{2c^2}{2-c^2}, \qquad r = 8c^2, \tag{11}$$

where $P_{\sf S}(t_*), k_*$ are the Planck normalization values, $P_{\sf S}(t_*) \simeq 2.1 \times 10^{-9}$ [60]. These formulas are totally independent of $\mathcal{N}$, which is only determined by the variation of ϕ in the field space, $c\mathcal{N} = \Delta\phi/M_{\rm Pl}$. Note that as consequence in these models the spectral running vanishes, $\alpha = \frac{dn_{\sf S}}{d\ln k} = 0$. These examples are a special case of constant roll inflation [61]. If we normalize the parameters by setting $n_{\sf S} \simeq 0.965$ for the CMB anisotropies, we find $c \simeq 0.185$ and $r \simeq 0.274$. As it stands, this is in conflict with bounds on r from BICEP/Keck [45], calibrated to plain vanilla ΛCDM late universe. However, since the exponential potential by itself can't be the whole story, after all needing corrections to accommodate reheating at the very least [39], those deviations could fit [45]. Alternatively, if the resolution of the H_0 tension forces a modification of ΛCDM, by for example inclusion of the EDE [46–48], the primordial scalar spectrum may need to be slightly modified to compensate for the change in the evolution of fluctuations [49–51].

For example, if we pick c such that $r \lesssim 0.036$, to match the bounds of [45], we find

$$c \lesssim 0.067, \qquad n_{\sf S} \gtrsim 0.995. \tag{12}$$

To fit the CMB we may need a slightly higher scalar spectral index $n_{\sf S} \sim 0.98$–0.995 [49–51]. This means that nominally the exponential potentials satisfying (12) might still be in the game. For those values of c, the power controlling the attractor expansion rate is $2/c^2 \gtrsim 444$.

Further, as noted above, the total variation of ϕ for $\mathcal{N}$ e-folds is

$$\Delta\phi = cM_{\rm Pl}\mathcal{N}, \tag{13}$$

which for $\mathcal{N} \sim 60$ yields $\Delta\phi \simeq 4M_{\rm Pl}$, in some tension with the purported swampland bounds [62], but not much. We will not worry too much about this issue here. We do note, however, that for these values of parameters, the bound on the cutoff t_0 of Eq. (9) leads to

$$t_0 \gtrsim \frac{4\sqrt{3}}{c^2}\left(\frac{M_{\rm Pl}}{\mathcal{M}_{\rm UV}}\right)^2 M_{\rm Pl}^{-1}. \tag{14}$$

At earlier times $t < t_0$, quantum gravity is doing most of the driving.

III. CAUSAL STRUCTURE

We now turn to the causal structure of the power law inflation models, following [30,31]. Our particular interest is in the maximally extended past of the solutions with $k = 0, -1$. We already know that the geometries with power law scale factor are singular at $t \to 0$, but the question is, what kind of a singularity is that. For simplicity, we start with $k = 0$, and extend the scale factor

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \tag{15}$$

over the whole real semiaxis $(0, \infty)$. At future infinity, this scale factor is unbounded; however the curvature goes to zero and locally the flat space approximation becomes ever better. To understand the global picture, we look at the Penrose diagram describing such spacetimes. To obtain it, we conformally map the solution on a section of the Einstein static universe, which is a direct product $R \times S^3$ with the metric

$$ds^2 = -d\tau^2 + d\chi^2 + \sin^2(\chi) d\Omega_2. \tag{16}$$

The section of $R \times S^3$ which describes power law inflation is the region bounded by the images of the singularities and/or past and future causal boundaries.

We find the required conformal map as a composition of two maps. First we transition to the conformally flat metric $ds^2 = \omega^2(\bar{x})\eta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$. In the second step, we map this metric to the static Einstein. The first map comprises of changing coordinates by

$$(1+3w)\frac{\bar{t}}{t_0} = 3(1+w)\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}},$$
$$\omega(\bar{t}) = \left(\frac{1+3w}{3(1+w)}\frac{\bar{t}}{t_0}\right)^{\frac{2}{1+3w}}. \tag{17}$$

When $-1 < w < -1/3$, the coordinate $\bar{t}$ is negative and inversely proportional to t, varying from $-\infty$ to 0 as t changes from 0 to ∞: the $\bar{t}$-axis has the same orientation as the t-axis.

The second map is defined by

$$\frac{r}{t_0} = \frac{1}{2}\left(\tan\left(\frac{\chi+\tau}{2}\right) + \tan\left(\frac{\chi-\tau}{2}\right)\right),$$
$$\frac{\bar{t}}{t_0} = \frac{1}{2}\left(\tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right)\right). \tag{18}$$

Since $r \in [0, \infty)$, $\bar{t} \in (-\infty, 0)$, and $\chi \in [0, \pi]$, it follows that $\tau \in [-\pi, 0]$. Putting together these formulas, the flat power law inflation metric is

$$ds^2 = \mathcal{C}^2 t_0^2 \frac{[\cos(\frac{\chi-\tau}{2})\cos(\frac{\chi+\tau}{2})]^{\frac{4}{|1+3w|}-2}}{4\sin^{\frac{4}{|1+3w|}}(|\tau|)} \times (-d\tau^2 + d\chi^2 + \sin^2(\chi) d\Omega_2), \tag{19}$$

where $\mathcal{C}$ is an $\mathcal{O}(1)$ constant, and

$$\frac{|1+3w|}{6(1+w)}\left(\frac{t}{t_0}\right)^{\frac{|1+3w|}{3(1+w)}} = \left(\tan\left(\frac{\chi-\tau}{2}\right) - \tan\left(\frac{\chi+\tau}{2}\right)\right)^{-1}. \tag{20}$$

Using this, we see that the ultimate future of power law inflation, $t \to \infty$, for any fixed value of r, maps onto $\tan(\frac{\chi-\tau}{2}) = \tan(\frac{\chi+\tau}{2})$: i.e., precisely the latitude circle $\tau = 0$ on the cylinder. Because the spacetime ends there, we cut out the portion of the cylinder $R \times S^2$ above it. On the other hand, the singularity corresponds to the limit $t \to 0$ for any fixed r. By (18), (20), we see that it maps onto the curve $\tan(\frac{\chi-\tau}{2}) \to \infty$, which corresponds to $\tau = \chi - \pi$. It is clear that this is the null semicircle connecting the points $(-\pi, 0)$ and $(0, \pi)$ on the cylinder. Since this hypersurface is the ultimate singular past of power law inflation, we must throw out the portion of the cylinder beneath it. Then we unwrap what remains, and find the causal structure of Fig. 1.

Each point in Fig. 1 corresponds to an angular S^2. The ultimate past, which realizes the outcome of the Borde-Guth-Vilenkin theorem [14], is a *null singularity*. If power law inflation never ends, the future is a spacelike infinity. Any observer must have a future horizon, which in their rest frame is the null inward line ending in the upper left corner

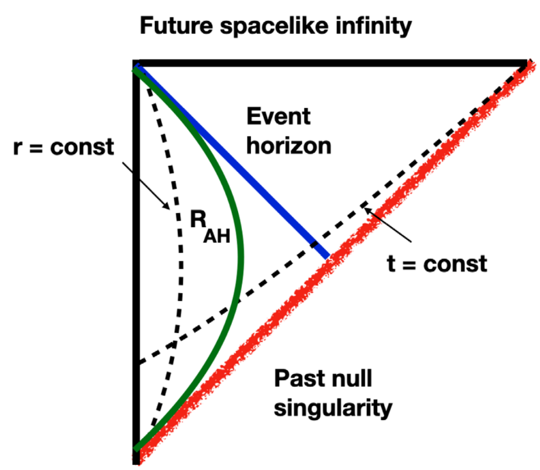

FIG. 1. Causal structure of a spatially flat endless power law inflation. Depicted are the event horizon, the apparent horizon $\mathcal{R}_{AH}$ and $r =$ const. and $t =$ const. hypersurfaces.

of the diagram. Any observer would find the universe at any given finite time to be of finite size, being able to causally explore only the interior of the diamond bounded by the horizon and the singularity.

The causal structure analysis so far concerns spatially flat geometry $k = 0$. What if the universe is open, $k = -1$? As we noted above, in this case, the expansion rate is set by a competition between spatial curvature and the exponential potential. At late times, the potential wins because of the attractor behavior. However, early on the curvature can be dominant. When that happens, the scale factor is a linear function of the comoving time, $a = t/t_0$. In this limit the metric is

$$ds^2 = -dt^2 + \left(\frac{t}{t_0}\right)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_2\right). \tag{21}$$

At first glance one might think the metric is locally just a Milne wedge of the flat Minkowski in an accelerated reference frame. However, thanks to t_0 this is not so: there is a real curvature singularity at $t \to 0$. The singularity is again null, as we can see by mapping the slice of the spacetime near $t = 0$ onto the static Einstein universe. In this case the analog of Eq. (17) is

$$\bar{t} = t_0 \ln(t/t_0) + \ldots, \qquad \omega(\bar{t}) = e^{\frac{\bar{t}}{t_0}} + \ldots. \tag{22}$$

and so

$$\ln(t/t_0) + \ldots = \frac{1}{2}\left(\tan\left(\frac{\chi+\tau}{2}\right) - \tan\left(\frac{\chi-\tau}{2}\right)\right). \tag{23}$$

The ellipses denote the subleading terms when $t \to 0$. Hence the singularity again maps on the past null semicircle $\tau = \chi - \pi$.

At larger values of t this geometry changes into the attractor-controlled section, where the curvature is locally negligible. If power law inflation lasts forever, the Penrose diagram is very similar to Fig 1, except for the local differences near the null singularity, as depicted in Fig 2.

Clearly, both of these cases are reminiscent of the spatially flat charts of de Sitter, with the exception that the past horizon is replaced by a null singularity. Nevertheless as long as the metrics are purely FRW—isotropic and homogeneous—Weyl tensor vanishes there. Unlike in de Sitter the future horizon is not at constant spatial separation from the observer, but grows according to ($w < -1/3$)

$$L_H = a(t) \int_t^\infty \frac{dt'}{a(t')} = \frac{3(1+w)}{|1+3w|} t, \tag{24}$$

which shows that the volume of any spacelike hypersurface inside the causal diamond grows extremely large. Yet the volume outside grows even larger [30,31].

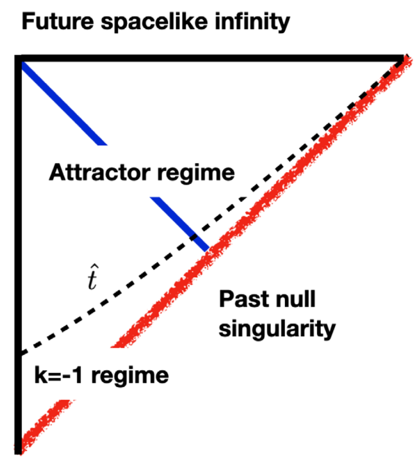

FIG. 2. Causal structure of a spatially open power law inflation. It is an amalgam of the past $k = -1$ regime and a future power law attractor, matched together at a time $\sim\hat{t}$ (which in reality is a slab of world volume few Hubble times thick).

The cosmic inventory, as tallied by a single observer who receives the signals from their past, can be accounted for by the capacity of the holographic screen, which is bounded by the area of the apparent horizon [63–65]. The apparent horizon $\mathcal{R}_{AH}$ is a boundary of the normal region of space, which colloquially we may think of the largest region that behaves as a locally Minkowski space. Specifically, it is the largest region inside which the beams of all outward geodesics, future or past oriented, spread out. On the apparent horizon, at least one class refocuses. This means, the apparent horizon behaves like a lens. In our case, the exterior of the apparent horizon in all our examples is an antitrapped region, meaning that all past oriented null geodesics outside of the apparent horizon, inward or outward bound, are converging. This is because of the null singularity in the past.

To find the location of the apparent horizon, recall that it is the hypersurface where at least one family of null lines has vanishing expansion. If we consider a sphere of radius ar with area $A \sim a^2(t)r^2$, along the radial null geodesics $dt = \pm a(t)dr$, the gradient of A is $A' \sim a'r + ar'$ where the prime denotes the derivative with respect to the affine parameter of the null line. The extremum yields the comoving size of the apparent horizon to be $r = 1/\dot{a}(t)$, and so the proper apparent horizon size is[4]

$$\mathcal{R}_{AH} = \frac{1}{H} = \frac{3(1+w)}{2} t. \tag{25}$$

Clearly, since $\mathcal{R}_{AH}/L_H = |1+3w|/2 < 1$ for $-1 < w < -1/3$, $\mathcal{R}_{AH}$ is always inside the future horizon.

[4]In truth, $\mathcal{R}_{AH} = 1/\sqrt{H^2 + k/a^2}$, but we neglect the curvature term assuming the attractor to be a long stage. In the regime where the curvature term dominates over the scalar the variation of $\mathcal{R}_{AH}$ is slower than linear, but it still goes to zero on the singularity.

On the diagram of Fig. 1, it is the arc $\mathcal{R}_{AH}$ between the lower left corner and the upper left corner.

Given the discussion above, it should be obvious that to get a realistic cosmology out of power law inflation, we need to end inflation and reheat the universe. We also need to perturb the reheating surface with the scalar and, unavoidably, tensor fluctuations which we discussed in the previous section. Under those conditions, it is easy to see that the causal structure of such a universe is represented by the Penrose diagram of Fig. 3. There we allow for the possibility that very early on the universe is open and curvature dominated, then transitions to the attractor regime, which ends globally with reheating. The reheating surface will be smooth only down to one part in 10 000, due to the inflationary fluctuations. As we discussed in the preceding section, the dynamics can match the observations, with some tweaks.

It is interesting to get an idea of the complexity of this universe at its various stages. We pick an observer and let them count what they see, from the comfort of their rest frame. They do so by collecting the photons arriving from afar, and originating from as early as near the null singularity (or gravitons instead of photons, since the universe is far more transparent for those). As those null rays approach the observer—they are future oriented inward geodesics—they cross the apparent horizon and focus to the origin in the normal region of spacetime surrounding the observer. The total amount of information coming in must satisfy the horizon area bound [63–65],

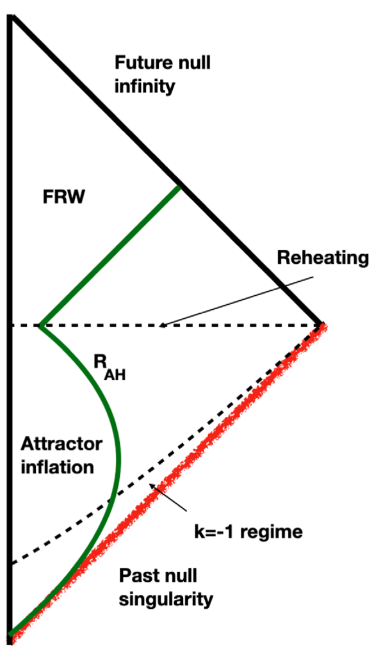

FIG. 3. Causal structure of a spatially open power law inflation which exits to radiation and matter dominated FRW. It is an amalgam of the past $k = -1$ regime, a future power law attractor, and the postinflationary decelerating FRW.

$S \lesssim \mathcal{A}_{AH}/4G_N$. Since the apparent horizon expands, the information contents grows, but during the accelerated epoch the variation is very slow. The apparent horizon area evolves according to $\dot{\mathcal{A}}_{AH}/\mathcal{A}_{AH} = c^2 H$, which by using $\mathcal{N} = \ln(a/a_*)$ we can express as variation per e-fold,

$$\frac{d\mathcal{A}_{AH}}{\mathcal{A}_{AH}} = c^2 d\mathcal{N}, \tag{26}$$

and so during the attractor stage since $c \ll 1$, the maximal entropy is growing very slowly. This slow increase[5] continues until the end of inflation, after which the growth rate changes to $d\mathcal{A}_{AH}/\mathcal{A}_{AH} = \mathcal{O}(1)d\mathcal{N}$, with the precise details being controlled by the postinflationary cosmic inventory.

The evolution in the semiclassical regime being adiabatic, with a globally fixed arrow of time as selected by the null singularity, means the "entropy" is crossing the apparent horizon during inflation very slowly, and after inflation much more rapidly. Still, close to the singularity the geometry may still undergo a phase of self-reproduction. If so then different segments of the attractor regime of inflation could be subject to different perturbations, that can trigger the onset of exit at different times, or perhaps even prevent it altogether. If so those phenomena could alter the arrow of time in some parts of the spacetime. However, if we demand that the attractor dynamics yields observables close to the current limits, the self-reproduction regime is excised out of the semiclassical limit. We can verify this as follows. The boundary of self-reproduction is approximately given by the field values where $P_{\mathsf{S}} \simeq 1$, or more accurately the equality of the classical field variation integrated over a Hubble time and the quantum fluctuation induced by cosmic acceleration,

$$\int_{\text{Hubble time}} d\phi \simeq \frac{H}{2\pi}. \tag{27}$$

In other words, where the field variation is slow enough, the quantum Brownian drift can compensate it, and "reboot" inflation. Using $\dot{\phi} = \frac{2M_{\rm Pl}}{ct}$ and $H = \frac{2}{c^2 t}$ yields for $c \ll 1$

$$t_{\mathsf{boundary}} = \frac{1}{\pi c^3} M_{\rm Pl}^{-1}. \tag{28}$$

Self-reproduction could only occur for $t < t_{\mathsf{boundary}}$, and the slow roll regime of inflation for $t > t_{\mathsf{boundary}}$ (we could have phrased this condition in terms of the gauge invariant variable ϕ instead, but since we gauge fixed the solution that is not necessary). However: our result for the cutoff t_0 of Eq. (14) severely obstructs the self-reproduction regime. Namely, comparing (14) and (28),

[5]Which could be associated with the horizon crossing of the perturbations [66,67].

$$\frac{t_0}{t_{\rm boundary}} \gtrsim 4\sqrt{3}\pi c\left(\frac{M_{\rm Pl}}{\mathcal{M}_{\rm UV}}\right)^2 \simeq 4\sqrt{3}\pi cN, \quad (29)$$

where as we noted above N is the number of light species in the theory, below the cutoff. If we take those to only count the Standard Model degrees of freedom, $N \sim 120$, and so the right-hand side is $\sim 2612c$. If we further require that n_S is not greater than 0.998, we find $c \gtrsim 0.044$. This means that for the values of c closest to fitting the data, the ratio of Eq. (29) is much greater than unity, $\frac{t_0}{t_{\rm boundary}} \gg 1$. Since only the time interval $t > t_0$ is allowed in the effective theory, it means that the regime of self-reproduction is basically confined to the spacetime sliver right next to the null singularity in Fig. 3 that it is pointless to think about it. In other words, the self-reproduction regime is behind the Planckian cutoff surface above the null singularity, and it makes no sense physically in the solutions depicted by Fig. 3. As a result, the arrow of time, once set, remains preserved in those solutions. Taking the solution to start from the null singularity as a homogeneous and isotropic FRW implies the vanishing of its Weyl tensor in the far past. This will be violated later, by evolution, since quantum fluctuations of the scalar will perturb the geometry, and this will contribute to the entropy production in the late universe. This is all fully consistent with Penrose's Weyl curvature hypothesis. The question is, what selects this initial condition.

IV. COSMIC BUBBLES

A rationale for selecting the initial condition which approximates really well the null singularity with vanishing Weyl tensor could be provided using the framework of no boundary proposal for quantum cosmology [68] and weighing the probabilities by the tunneling wave function prescription for the initial conditions [69,70]. We will argue below that the process which mediates the creation of the universe depicted by the causal structure of Fig. 3 is closely related to the Hawking-Turok instanton [32–36]. We start by first briefly reviewing the Hawking-Turok instanton.

The idea is to imagine a theory of open inflating universe which tunnels from nothing, with a generic potential that can support 60e-folds of inflation. This universe originates by a formation of a bubble of spacetime, and the universe is its dynamical interior. The pre-genesis stage is described by an Euclidean geometry which resembles a squashed sphere [32]. The scalar gradients will get large in some region of the Euclidean space, and produce a singularity which lies on the hypersurface of vanishing extrinsic curvature along which the analytical continuation is carried out [32–36]. In this regime, the simple limit of relevant equations is

$$ds^2 = d\sigma^2 + b^2(\sigma)(d\psi^2 + \sin^2\psi d\Omega_2), \quad (30)$$

for the Euclidean metric and

$$\phi'' + 3\frac{b'}{b}\phi' = \partial_\phi V, \qquad \frac{b''}{b} = -\frac{1}{3M_{\rm Pl}^2}\left(\frac{\phi'^2}{2} + V\right),$$
$$\frac{b'^2}{b^2} = \frac{1}{b^2} + \frac{1}{3M_{\rm Pl}^2}\left(\frac{\phi'^2}{2} - V\right), \quad (31)$$

for the scalar and gravitational equations. The prime is a derivative with respect to σ. In this regime, the field ϕ is rolling in the upside-down potential $-V$. Let us initially consider a point where the geometry is regular, and hence sufficiently close to it must be locally R^4. If we place the coordinate origin at that point, near it we must have $b \to \sigma + \cdots$, and by symmetry $\phi' \to 0, \phi \to$ const. (otherwise we would encounter a singularity in ϕ'', and consequently in ϕ too). Moving away from this point, b grows, but at a rate which is decreasing due to the b'' equation. So b reaches a maximum, and turns around. Past it, the scalar derivatives grow fast for generic potentials, and take over, forcing ϕ to diverge at some $\sigma = \sigma_*$. In this limit $b \to (\frac{3}{2}\frac{\mathcal{C}^2}{M_{\rm Pl}^2})^{1/6}(\sigma_* - \sigma)^{1/3}$ and $\phi \to \text{const.} - \sqrt{\frac{2}{3}}M_{\rm Pl}\ln(\sigma_* - \sigma)$. Note that this behavior generalizes the spherical limit $b = \sin\sigma$ which describes $\phi = $ const., with a constant potential.

The metric (30), with these properties of b, can now be analytically continued in two steps. First, changing the latitude coordinate ψ to $\psi = \pi/2 + i\tau$ at the equatorial hypersphere gives

$$ds^2 = d\sigma^2 + b^2(\sigma)(-d\tau^2 + \cosh^2\tau d\Omega_2), \quad (32)$$

which describes an anisotropic cosmology just "north" of the equator [32], which has a timelike singularity at $\sigma = \sigma_*$. This geometry also has a horizon at $\sigma = 0$, where its Euclidean counterpart had a regular point. We can analytically continue across $\sigma = 0$, therefore, by using $\tau = i\pi/2 + \chi$ and $\sigma = it$, while defining $a(t) = -ib(it)$ [32]. Since b has no singular points along the imaginary axis, $a(t)$ is well defined. The metric in this latter region is

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sinh^2\chi d\Omega_2), \quad (33)$$

i.e. precisely an open universe. This is precisely the same metric as our metric of Eq. (3), with one exception: here, $t \to 0$ is a regular null hypersurface, a horizon rather than a null singularity. The singularity is now resolved, and hides in the past of the horizon, as depicted in Fig. 4 (see [32–36]). Notice that since the metrics are analytic continuations of each other, and, e.g., (30) is conformally flat, Weyl tensor remains zero everywhere. This shows that the replacement of the null past singularity by a timelike regulator which asymptotes to null does not affect the interpretation of the solutions.

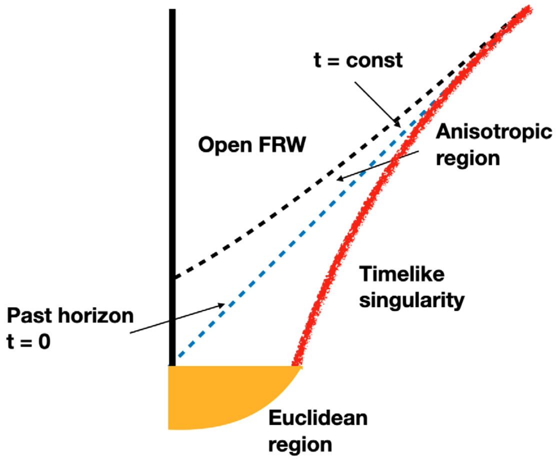

FIG. 4. Resolving the singularity: on top, a spatially open power law inflation which can exit to radiation and matter dominated FRW; $t = 0$ null surface is now a horizon. There is a timelike singularity behind it. This singularity can be excised by cutting out the region of space around it and replacing it with a bubble of flat space surrounded by a tensional domain wall [38], whose world volume lies between the singularity and the $t = 0$ horizon, or its generalizations [39–42].

In the final step, as in [38–42] we excise the region around the singularity and replace it with a bubble, surrounded by a domain wall with some tension. We will not repeat all the technical procedure here, instead referring the reader to the various options in [38–42]. The important point is that the world volume of the spherical domain wall asymptotes the past horizon from below. The surface energy density of the bubble is controlled by its initial size, and so the smaller it starts, the larger the density will be. In turn, this controls the scale of the Euclidean action of the resolved configuration. For example in perhaps the simplest regularized case, where the bubble’s interior is a ball of flat space, replacing the singular region.

Note that the metric surgery with cutting and pasting various pieces together across a domain wall will not change the Weyl tensor of the configuration for the metrics which are $O(4)$ symmetric. The reason is that the symmetry conditions are very restrictive for the metric, and only allow a single “free” function to appear in the metric—the scale factor, which is also the conformal factor. It is the only term in the metric which picks up the boundary conditions. Thus the Weyl tensor remains insensitive to the singularity regulator. If Weyl is zero without the regulator, it remains zero with it.

Garriga found that the matching conditions $b'/b|_{\mathrm{out}} = -\kappa|\mathcal{C}|/3b^3$ and $b'/b|_{\mathrm{in}} = 1/b$ with the bubble wall tension modeled by $\mu = \mu_0 - \alpha e^{\kappa\phi}$ yield the regulator contribution to the Euclidean action which is

$$S_{\mathrm{sing}} = \frac{1}{3} S_{\mathrm{GH}} = \frac{\pi^2|\mathcal{C}|}{\kappa}. \tag{34}$$

With the inflationary potential also included, one will find additional contributions. A very thorough survey of possible instantons and the actions which govern their nucleation rate is given in [40]. In the case when the Hawking-Turok instanton is regulated by a tensional domain, the full $O(4)$ Euclidean action is given by [40]

$$S_{HT/D} = -\frac{24\pi^2}{3M_{\mathrm{Pl}}^2 H^2}(1 - \cos(H\sigma_m)). \tag{35}$$

where $3M_{\mathrm{Pl}}^2 H^2 = U(\phi_{\mathrm{initial}})$ and σ_m is the location of the domain wall which serves as a seam between two geometries. The trick used by [40] to construct the regular solution is to orbifold around the wall[6] instead to think of it as a boundary between the Hawking-Turok solution in the bulk and a ball of flat space excising the singularity. This can be interpreted as a creation of two jointed open universes, or by identifying the two, a single Hawking-Turok geometry with a singularity excised by a wall at the end of the world. This actually may assist with obstructing the interpretation of the regulated solution as coming from a bubble of nothing in $6D$, which may be problematic for its use as a regulator of Hawking-Turok processes [41,42]. We will not delve into this very interesting issue any further here. Instead we will treat the action of (35) as an estimate of the Hawking-Turok nucleation rate, even thought it is probably sensitive to the precise details of the UV completion of the configuration.

Note, that the wall will get closer—i.e., approach its asymptotically null world volume—the faster the smaller it starts, because it starts closer to the horizon initially. The regulated geometry is depicted in Fig. 5. It should be clear that for all practical intents and purposes, if the universe arises as the interior of an initially small bubble, whose energy density is at or above the cutoff, this wall will behave practically as an almost null singularity: its world volume will be approximately null, and its energy density at the cutoff. Near the wall Weyl tensor will still be zero, and the geometry will behave to leading order just like the solution depicted in Fig. 3.

Using the tunneling from nothing probability prescription to estimate the likelihood of such a universe, [69,70],

$$P \sim e^{S_{\mathrm{euclidean}}} \simeq e^{-\frac{24\pi^2}{3M_{\mathrm{Pl}}^2 H^2}(1-\cos(H\sigma_m))}, \tag{36}$$

explains the selection of the initial conditions. First off, the $O(4)$ symmetry is favored over more complicated initial configurations by minimizing the action. Second, the initial values of ϕ which maximize the initial value of the potential are preferred over those which make it small. Both of these conditions select inflationary history, and the

[6] This is very similar to warped braneworld constructions of, e.g., [71–74].

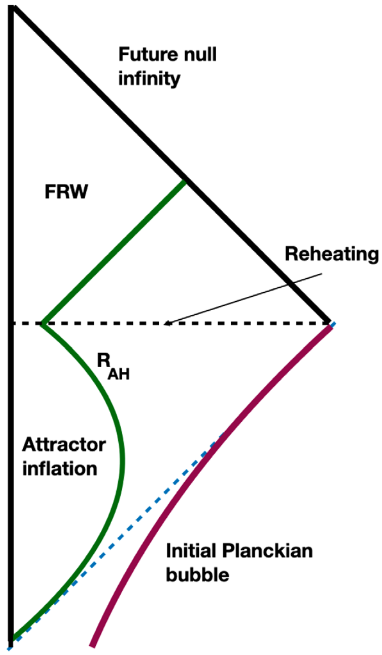

FIG. 5. Resolving the singularity: the beginning a spatially open power law inflation which exits to radiation and matter dominated FRW. The past null singularity is interpreted as a domain wall of a bubble which initially nucleated at Planckian density.

exponential potential cuts off the possible attainable number of e-folds—by not plateauing in the UV. This explains, at least in this context, how inflation starts.[7] Likewise, these conditions are also compatible with Penrose's conjecture, since the initially $O(4)$ invariant geometry gives a vanishing Weyl tensor, and the (almost) null (regulated) singularity picks the global time direction. One can then study entropy production, initially by studying metric perturbations, as in, e.g., [75,76], and later with the contributions from reheating and postinflationary evolution. The evolution of the apparent horizon area, Eq. (26), is consistent with postnucleation entropy growth. Since many of the specific details can be found in the literature, we will not delve into the details here.

[7] And addresses the issue of footnote 1.

V. SUMMARY

In this article, we have presented an argument that Penrose's vanishing Weyl curvature hypothesis, along with the initial singularity in the universe, motivated by the entropy considerations and the observed global arrow of time, is actually consistent with the inflationary paradigm. As an example, we used power law inflation which initially starts with a Hawking-Turok nucleation process, with likelihood described by the tunneling from nothing probability. Note that here we demonstrated the compatibility of Penrose's Weyl curvature hypothesis and inflation—where by inflation we mean the (semi)classical evolution of the background *augmented* with the selection of tunneling from nothing probability as a theory of initial conditions—without explicitly showing a more microscopic origin of either of these premises. That suffices for our purposes here. Going beyond this goal requires a more precise exploration of the realms of quantum gravity, not easily accessible by present means.

Curiously, the resulting dynamics could even be in marginal agreement with the current data. We note however that similar conclusions should hold for other models of inflation which start with the universe in a small bubble. The presence of the past (null) singularity will be generic for flat or open FRW universes in the extreme past whenever the field value and the potential in that regime are not exactly constant. The gradients near the past horizon will induce a large backreaction, and require regularization. Thus the general conclusions presented here may hold even for potentials which fit the data better.

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