Quark-model relations among TMDs in the parton model

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The covariant parton model (CPM) is a consequent application of the parton model concept to the nucleon structure. In this model, there is a choice to put quarks either in a pure-spin state or in a mixed-spin state. We show that the mixed-spin version of the CPM does not support the quark-model relations among transverse momentum dependent parton distributions (TMDs) which were shown to hold in a large class of quark models. One can enforce the quark-model relations to be valid in the CPM by imposing a condition which is equivalent to putting the quarks in a pure-spin state. This gives a complementary perspective on the connection of the pure- and mixed-spin state CPM versions and provides a fresh view on the question whether the quark-model relations could be realized in QCD as "approximate relations" with some useful numerical accuracy.

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I. INTRODUCTION

Transverse momentum dependent parton distributions (TMDs) entail the description of the nucleon structure in deep-inelastic scattering (DIS) processes when in the final state one detects an adequate transverse momentum which is small compared to the hard scale Q of the process [1]. For the understanding of the nonperturbative properties of TMDs, quark models play an important role in two ways. First, undistracted by technical complexities inherent in a full QCD treatment, in models one may investigate in a simpler theoretical framework the significance of a specific physical aspect and gain in this way valuable insights. Second, in situations where some of the TMDs are still not yet well known, results from models may be helpful to interpret first data or give useful estimates for counting rates in future experiments such as the Electron-Ion Collider [2]. In this way, models complement phenomenology and lattice QCD studies.

In this work, we will study the covariant parton model (CPM) which is based on Feynman's parton model concept [3,4]. The latter played a historically important role for the interpretation of DIS processes and establishing QCD and can, in a certain sense, be viewed as a "zeroth-order approximation" to QCD [1,5]. The parton model provides often an effective first step toward an understanding of QCD processes. For instance, the "generalized parton model" of Refs. [6–10] helped to pave the way to modern TMD phenomenology. The exploration of the parton model concept for the sake of studying TMDs and their nonperturbative properties was carried out in Refs. [11–25]. Further applications of the parton model concept can be found in Refs. [26–30].

Because of the absence of interactions, the description of the nucleon structure in the parton model is particularly lucid, and the TMDs are described in terms of covariant functions depending on the variable $P \cdot k$, where P^μ denotes the nucleon momentum and k^μ quark momentum. Despite the simplicity of the model, there was an interesting puzzle. One group claimed that the description of TMDs requires two independent covariant functions [16–23], while the other group claimed that one needs three independent covariant functions [24]. This puzzle was resolved recently by showing that the results of the two groups are equivalent except for the treatment of the quark polarization state [25]. In Refs. [12–23] the quarks were chosen to be in a pure-spin state, while in Ref. [24] they were (implicitly) assumed to be in a mixed-spin state. Other than that, the results of the two groups are equivalent [25].

Here, we will take a different point of view as compared to Ref. [25], where the focus was on technical aspects of the quark correlator. In this work, the starting point is the quark-model aspect of the approach: The CPM is after all a quark model, i.e., a model without gauge field degrees of freedom. In several models of such type, it was observed that certain relations exist between different TMDs to which we shall refer as quark-model relations (QMRs). Not all quark models support the QMRs, but it is worth stressing that a wide class of very different models does.

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The goal of this work is to investigate whether the TMDs in the mixed-spin state version of the CPM [24,25] obey the QMRs. We shall see that this is not the case and show that imposing the validity of the QMRs in this model yields the same condition as when one chooses the quarks to be in a pure-spin state. In other words, if one starts with the mixed-spin state version of the CPM and demands the model to comply with the QMRs supported in other quark models, then one must introduce the pure-spin state model.

Our study is insightful in two ways. First, it gives insights on the CPM and its relation to other quark models. Second, it opens a new perspective on QMRs and may shed light on the question whether they could hold in QCD as approximate relations with a potentially useful numerical accuracy in some range of x and k_T.

The structure of this work is as follows. In Sec. II, we present the QMRs and briefly discuss their understanding within quark models. In Sec. III, we review the CPM and present the results for TMDs in the mixed-spin state version of the CPM. In Sec. IV, we investigate the linear and nonlinear QMRs in the CPM. In Sec. V, we discuss the physical implications of our findings, and in Sec. VI, we draw conclusions and give an outlook for future studies.

II. QUARK-MODEL RELATIONS AMONG TMDs

In contrast to QCD, in quark models, relations among different TMDs can exist due to the simpler model dynamics or due to model symmetries. Some of these relations, such as, e.g., the quark-model Lorentz-invariance relations (qLIRs), are generic in the sense that they hold in quark models which respect Lorentz symmetry and contain no gauge field degrees of freedom [31–33] but are not valid in QCD [34–36]. We quote here only one qLIR, namely,

$$h_T^q(x,k_T) - h_T^{\perp q}(x,k_T) = h_{1L}^{\perp q}(x,k_T), \tag{1}$$

on which it will be instructive to follow up below. A discussion of other qLIRs can be found, for instance, in Ref. [37]. We remark that the notation in Eq. (1) and throughout this work is $k_T = |\vec{k}_T|$, and k_T^2 will always denote $|\vec{k}_T|^2$.

The main focus of this work is another set of relations which have been observed in several very different quark models. These relations, to which we will refer in the following as QMRs, are given by

$$g_{1T}^{\perp q}(x,k_T) = -h_{1L}^{\perp q}(x,k_T), \tag{2a}$$

$$g_T^{\perp q}(x,k_T) = -h_{1T}^{\perp q}(x,k_T), \tag{2b}$$

$$g_L^{\perp q}(x,k_T) = -h_T^{q}(x,k_T), \tag{2c}$$

$$g_1^q(x,k_T) - h_1^q(x,k_T) = h_{1T}^{\perp(1)q}(x,k_T), \tag{2d}$$

$$g_T^q(x,k_T) - h_L^q(x,k_T) = h_{1T}^{\perp(1)q}(x,k_T). \tag{2e}$$

In addition to the linear QMRs (2), also two nonlinear QMRs have been found which are given by

$$\frac{1}{2}[h_{1L}^{\perp q}(x,k_T)]^2 = -h_1^q(x,k_T)h_{1T}^{\perp q}(x,k_T), \tag{3a}$$

$$\frac{1}{2}[g_{1T}^{\perp q}(x,k_T)]^2 = g_{1T}^{\perp q}(x,k_T)g_L^{\perp q}(x,k_T) + g_T^q(x,k_T)g_T^{\perp q}(x,k_T). \tag{3b}$$

The transverse moment of a TMD is defined as

$$h_{1T}^{\perp(1)q}(x,k_T) = \frac{k_T^2}{2M^2}h_{1T}^{\perp q}(x,k_T). \tag{4}$$

The relations (2) and (3) hold in a wide class of quark models which are based on very different model concepts including the spectator model, bag model, or light-front constituent quark model [38–45]. The QMRs (2a), (2d), and (3a) involving twist-2 TMDs were shown to arise from a certain rotational symmetry of the model light cone wave functions [45]. This symmetry is effectively present in many models including, e.g., the pure-spin version of the CPM [16,23].

It is important to remark that not all models support the QMRs. For instance, certain spectator model variants, where (to allow more flexible modeling) a larger number of free model parameters was introduced [46], do not support QMRs. Another example is the quark-target model [47], where the presence of gluon degrees of freedom spoils QMRs.

III. QUARK CORRELATOR AND TMDs IN CPM

In this section, we first review the general structure of the quark correlator in quark models and then discuss the specific results for the correlator and TMDs in the CPM, briefly commenting on the two versions of this model.

A. Quark correlator in a generic quark model

In a theory without explicit gauge degrees of freedom, the quark correlator for the nucleon is defined as follows:

$$\Phi_{ij}^q(k,P,S) = \int \frac{\mathrm{d}^4z}{(2\pi)^4}\mathrm{e}^{ikz}\langle N|\bar{\Psi}_j^q(0)\Psi_i^q(z)|N\rangle, \tag{5}$$

where k^μ is the quark 4-momentum and P^μ and S^μ are the nucleon 4-momentum and polarization vectors satisfying $P^2 = M^2$, $S^2 = -1$, and $P\cdot S = 0$. In quark models, the Lorentz structure of the correlator (5) is described in terms of k^μ, P^μ, and S^μ as follows (we use the convention $\varepsilon^{0123} = 1$ and assume a covariant normalization of nucleon states) [33]:

$$\begin{aligned}\Phi^q(k,P,S) = {} & MA_1^q + \not{P}A_2^q + \not{k}A_3^q + \frac{i}{2M}[\not{P},\not{k}]A_4^q \\ & + i(k\cdot S)\gamma_5 A_5^q + M\not{S}\gamma_5 A_6^q + \frac{(k\cdot S)}{M}\not{P}\gamma_5 A_7^q \\ & + \frac{(k\cdot S)}{M}\not{k}\gamma_5 A_8^q + \frac{[\not{P},\not{S}]}{2}\gamma_5 A_9^q + \frac{[\not{k},\not{S}]}{2}\gamma_5 A_{10}^q \\ & + \frac{(k\cdot S)}{2M^2}[\not{P},\not{k}]\gamma_5 A_{11}^q + \frac{1}{M}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu k_\rho S_\sigma A_{12}^q. \end{aligned} \tag{6}$$

The amplitudes $A_i^q = A_i^q(P\cdot k, k^2)$ in Eq. (6) are real functions of the Lorentz scalars $P\cdot k$ and k^2 [32,33]. The amplitudes A_i^q are chiral even for $i = 2, 3, 6, 7, 8, 12$ and chiral odd for $i = 1, 4, 5, 9, 10, 11$. In QCD and in models with gauge field degrees of freedom, in the definition of the quark correlator (5), Wilson lines must be included which run along a nearly lightlike 4-vector n^μ dictated by hard-momentum flow in the considered process [1]. The presence of the additional vector n^μ allows for 20 further Lorentz structures which are often denoted as B_i amplitudes [34–36]. The T-odd amplitudes A_i^q for $i = 4, 5, 12$ vanish in quark models as do the pertinent T-odd TMDs, because their modeling requires explicit gauge field degrees of freedom [48]. The T-odd amplitudes are included in Eq. (6) merely for completeness. In this work, we will focus on T-even TMDs; see the Appendix for the explicit expressions.

B. Quark correlator in the CPM, and the two model versions

In the CPM, one can explore the equation of motion for the quark fields $(i\not{\partial} - m_q)\Psi^q(z) = 0$ in order to derive the following results for the amplitudes [24,25]:

$$\begin{aligned} & A_1^q = \frac{m_q}{M}A_3^q, \qquad A_2^q = 0, \qquad A_4^q = 0, \qquad A_5^q = 0, \\ & A_6^q = \frac{m_q}{M}A_{10}^q, \qquad A_7^q = -\frac{m_q}{M}A_{11}^q, \qquad A_9^q = 0, \\ & A_{10}^q = \frac{(P\cdot k)}{M^2}A_{11}^q - \frac{m_q}{M}A_8^q, \qquad A_{12}^q = 0. \end{aligned} \tag{7}$$

The T-odd amplitudes A_4^q, A_5^q, and A_{12}^q vanish in the CPM, which is a general quark-model prediction due to the absence of gauge field degrees of freedom [48]. Interestingly, also the T-even amplitudes A_2^q and A_9^q vanish, which is a specific feature of the CPM and is, in general, not the case in other quark models. The amplitudes A_1^q, A_6^q, and A_7^q are proportional to current quark masses and, hence, negligibly small for the light quark flavors. At this stage the relations (7) imply that in the CPM three independent amplitudes exist which can be chosen to be the unpolarized amplitude A_3^q, the chiral-even polarized amplitude A_8^q, and the chiral-odd polarized amplitude A_{11}^q.

The two versions of the CPM are best explained by briefly reviewing what the relations (7) imply for the quark correlator. Inserting the results in (7) into Eq. (6) yields [25]

$$\Phi^q(k,P,S) = (\not{k} + m_q)(A_{\rm unp}^q + \gamma_5 \not{w}_q A_{\rm pol}^q), \tag{8}$$

where the unpolarized and polarized amplitudes $A_{\rm unp}^q$ and $A_{\rm pol}^q$, respectively, and w_q^μ are defined as

$$\begin{aligned} & A_{\rm unp}^q = A_3^q, \qquad A_{\rm pol}^q = -\frac{(P\cdot k)A_{11}^q - m_q M A_8^q}{M^2}, \\ & w_q^\mu = S^\mu - P^\mu \frac{(k\cdot S)A_{11}^q}{(P\cdot k)A_{11}^q - m_q M A_8^q} \\ & \qquad + k^\mu \frac{M}{m_q}\frac{(k\cdot S)A_8^q}{(P\cdot k)A_{11}^q - m_q M A_8^q}. \end{aligned} \tag{9}$$

The axial vector w_q^μ has the properties of the quark polarization vector and satisfies $k\cdot w_q = 0$. Notice that we explicitly assume that the quarks have a nonzero mass.[1] At this point, one has two choices in the model related to the treatment of the quark polarization; namely, our on-shell quarks can be in one of the following two states:

$$\begin{aligned} & \text{mixed-spin state:}\ -1 < w_q^2 < 0 \Leftrightarrow \text{three linearly independent amplitudes:}\ A_3^q, \quad A_8^q, \quad A_{11}^q, \\ & \text{pure-spin state:}\ w_q^2 = -1 \Leftrightarrow \text{two linearly independent amplitudes:}\ A_3^q, \quad |A_8^q| = |A_{11}^q|. \end{aligned} \tag{10}$$

In the more general version of the model, one has three independent amplitudes [24] which corresponds to quarks in a mixed-spin state (as long as the inequality $|A_8^q| < |A_{11}^q|$ is valid) [25]. Alternatively, one can put the quarks in a pure-spin state in which case $|A_8^q| = |A_{11}^q|$; i.e., in this version of the model, one has only two independent amplitudes. In this case, the physical-sign solution is

[1] The polarization of massless quarks cannot be described in terms of a polarization vector. But, ultimately in the massless case, one has the same choice of putting a quark in a pure-spin vs mixed-spin state. In this work, it is more insightful to work with the more general case $m_q \neq 0$. But, if desired, the current quark mass effects can be neglected at any stage; cf. [25].

$A_8^q = -A_{11}^q$, which can be determined from a comparison to other models and lattice QCD studies [25].

C. TMDs in the mixed-spin state version of the CPM

The starting point for our investigation is the CPM with quarks in the mixed-spin state. In this section, we quote the results for T-even TMDs starting with the model expressions for unpolarized TMDs [we define $k^\pm = \frac{1}{\sqrt{2}}(k^0 \pm k^3)$]:

$$f_1^q(x, k_T) = 2P^+ \int dk^- [xA_3^q(P\cdot k, k^2)]_{k^+=xP^+}, \tag{11a}$$

$$f^{\perp q}(x, k_T) = 2P^+ \int dk^- [A_3^q(P\cdot k, k^2)]_{k^+=xP^+}, \tag{11b}$$

$$e^q(x, k_T) = 2P^+ \int dk^- \left[\frac{m_q}{M} A_3^q(P\cdot k, k^2)\right]_{k^+=xP^+}. \tag{11c}$$

The expressions for chiral-even polarized TMDs in the mixed-spin state parton model are given by

$$g_1^q(x, k_T) = 2P^+ \int dk^- \left[\frac{x^2M^2 - xP\cdot k + m_q^2}{M^2} A_8^q(P\cdot k, k^2) - \frac{m_q}{M} xA_{11}^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{12a}$$

$$g_{1T}^{\perp q}(x, k_T) = 2P^+ \int dk^- \left[xA_8^q(P\cdot k, k^2) - \frac{m_q}{M} A_{11}^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{12b}$$

$$g_T^q(x, k_T) = 2P^+ \int dk^- \left[\frac{\vec{k}_T^2 + 2m_q^2}{2M^2} A_8^q(P\cdot k, k^2) - \frac{m_q}{M}\frac{P\cdot k}{M^2} A_{11}^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{12c}$$

$$g_L^{\perp q}(x, k_T) = 2P^+ \int dk^- \left[\frac{xM^2 - P\cdot k}{M^2} A_8^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{12d}$$

$$g_T^{\perp q}(x, k_T) = 2P^+ \int dk^- [A_8^q(P\cdot k, k^2)]_{k^+=xP^+}. \tag{12e}$$

Finally, the model expressions for chiral-odd polarized TMDs are given by

$$h_1^q(x, k_T) = 2P^+ \int dk^- \left[\frac{\vec{k}_T^2 - 2xP\cdot k}{2M^2} A_{11}^q(P\cdot k, k^2) + x\frac{m_q}{M} A_8^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{13a}$$

$$h_{1L}^{\perp q}(x, k_T) = 2P^+ \int dk^- \left[xA_{11}^q(P\cdot k, k^2) - \frac{m_q}{M} A_8^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{13b}$$

$$h_{1T}^{\perp q}(x, k_T) = 2P^+ \int dk^- [A_{11}^q(P\cdot k, k^2)]_{k^+=xP^+}, \tag{13c}$$

$$h_L^q(x, k_T) = 2P^+ \int dk^- \left[\frac{x^2M^2 - 2xP\cdot k}{M^2} A_{11}^q(P\cdot k, k^2) + \frac{m_q}{M}\frac{P\cdot k}{M^2} A_8^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{13d}$$

$$h_T^q(x, k_T) = 2P^+ \int dk^- \left[\frac{xM^2 - P\cdot k}{M^2} A_{11}^q(P\cdot k, k^2)\right]_{k^+=xP^+}, \tag{13e}$$

$$h_T^{\perp q}(x, k_T) = 2P^+ \int dk^- \left[-\frac{P\cdot k}{M^2} A_{11}^q(P\cdot k, k^2) + \frac{m_q}{M} A_8^q(P\cdot k, k^2)\right]_{k^+=xP^+}. \tag{13f}$$

For massless quarks in the mixed-spin state version of the CPM, the chiral-even (chiral-odd) polarized TMDs are given entirely in terms of the chiral-even (chiral-odd) amplitude A_8^q (A_{11}^q).

D. On-shellness and a useful identity

In the CPM the quarks are on shell; i.e., the amplitudes $A_i^q(P\cdot k, k^2)$ are actually functions of the type

$$A_i^q(P\cdot k, k^2) = F_i^q(P\cdot k)\delta(k^2 - m_q^2). \tag{14}$$

The explicit expressions for the functions $F_i^q(P\cdot k)$ can be found in Refs. [23–25] and will not be needed in this work. However, we will need an identity among the kinematic variables which holds under the k^- integration and can be derived as follows. Obviously, due to Eq. (14) we have

$$\begin{aligned} 0 &= \int dk^- (k^2 - m_q^2) A_i^q(P\cdot k, k^2)|_{k^+=xP^+} \\ &= \int dk^- (2k^+k^- - \vec{k}_T^2 - m_q^2) A_i^q(P\cdot k, k^2)|_{k^+=xP^+}. \end{aligned}$$

Next, we notice that $2k^+k^- = 2xP^+k^- = 2xP\cdot k - 2xP^-k^+ = 2xP\cdot k - 2x^2P^+P^- = 2xP\cdot k - x^2M^2$ holds under the integral where $k^+ = xP^+$. Inserting this in the above intermediate step, dividing by $2x$, and rearranging, we obtain

$$\int dk^-(P\cdot k)A_i^q(P\cdot k,k^2)|_{k^+=xP^+}$$
$$=\int dk^-\left(\frac{x^2M^2+\vec{k}_T^2+m_q^2}{2x}\right)A_i^q(P\cdot k,k^2)|_{k^+=xP^+}. \quad (15)$$

Thus, we see that under the k^- integral due to the mass-shell condition implicit in the amplitudes [cf. Eq. (14)] we can replace the variable $P\cdot k$ by an expression determined in terms of x, k_T, and the nucleon and quark masses. This identity will be helpful in the following.

IV. CONSEQUENCES OF IMPOSING QMRs IN MIXED-SPIN STATE CPM

Before we investigate the QMRs in the CPM, it is instructive to discuss first the example of the qLIR in Eq. (1). Here and in the following, it is convenient to reformulate the relations such that all TMDs appear on one side of the equation. Inserting the model expressions (13b), (13e), and (13f) for h_T^q, $h_T^{\perp q}$, and $h_{1L}^{\perp q}$, we obtain

$$h_T^q(x,k_T)-h_T^{\perp q}(x,k_T)-h_{1L}^{\perp q}(x,k_T)$$
$$=2P^+\int dk^-\Bigg[\frac{xM^2-P\cdot k}{M^2}A_{11}^q(P\cdot k,k^2)$$
$$-\left(-\frac{P\cdot k}{M^2}A_{11}^q(P\cdot k,k^2)+\frac{m_q}{M}A_8^q(P\cdot k,k^2)\right)$$
$$-\left(xA_{11}^q(P\cdot k,k^2)-\frac{m_q}{M}A_8^q(P\cdot k,k^2)\right)\Bigg]_{k^+=xP^+}=0.$$

We see that the qLIR (1) is valid for any $A_8^q(P\cdot k,k^2)$ and $A_{11}^q(P\cdot k,k^2)$. This was to be expected. The qLIRs require only the absence of gauge field degrees of freedom and, thus, must be valid in every quark model respecting Lorentz invariance. The investigation of this and other qLIRs is a useful cross-check for the theoretical consistency of a model but does not yield new insights. In this respect, the QMRs are more insightful, as we shall discuss next.

A. QMR between gear-worm functions, Eq. (2a)

The TMDs $g_{1T}^{\perp q}(x,k_T)$ and $h_{1L}^{\perp q}(x,k_T)$ are sometimes called gear-worm functions. In the spectator model study of Ref. [38], the QMR (2a) between these TMDs was derived which was later confirmed in several other quark models. Inserting the CPM expressions (12b) for $g_{1T}^{\perp q}(x,k_T)$ and (13b) for $h_{1L}^{\perp q}(x,k_T)$, the relation (2a) can be expressed as

$$g_{1T}^{\perp q}(x,k_T)+h_{1L}^{\perp q}(x,k_T)$$
$$=2P^+\int dk^-\left(x-\frac{m_q}{M}\right)[A_8^q(P\cdot k,k^2)$$
$$+A_{11}^q(P\cdot k,k^2)]_{k^+=xP^+}. \quad (16)$$

Clearly, in the mixed-spin state version of the CPM, where the amplitudes A_8^q and A_{11}^q are unrelated, the relation (2a) is not valid. If we would like the CPM to comply with this QMR, then this is possible if and only if we impose the condition $A_8^q=-A_{11}^q$ which corresponds to the pure-spin state version of the CPM; cf. Eq. (10).

B. QMR between $g_T^{\perp q}$ and $h_{1T}^{\perp q}$, Eq. (2b)

The QMR (2b) connecting the TMDs $g_T^{\perp q}$ and $h_{1T}^{\perp q}$ was, to the best of our knowledge, first discussed in Ref. [40]. Inserting CPM expressions (12e) and (13c) for $g_T^{\perp q}(x,k_T)$ and $h_{1T}^{\perp q}(x,k_T)$, respectively, into Eq. (2b) yields

$$g_T^{\perp q}(x,k_T)+h_{1T}^{\perp q}(x,k_T)$$
$$=2P^+\int dk^-[A_8^q(P\cdot k,k^2)+A_{11}^q(P\cdot k,k^2)]_{k^+=xP^+}. \quad (17)$$

Again we see that if the amplitudes A_8^q and A_{11}^q are unrelated, then the QMR (2b) is not valid which is the case in the mixed-spin state version of the model. For the CPM to comply with this QMR, it is necessary to introduce the condition $A_8^q=-A_{11}^q$ which brings us to the pure-spin state version of the CPM; cf. Eq. (10).

C. QMR between $g_L^{\perp q}$ and h_T^q, Eq. (2c)

The QMR (2c) connecting the twist-3 TMDs $g_L^{\perp q}$ and h_T^q was derived for the first time in Ref. [38] and later confirmed in other models. Inserting respectively the model expressions (12d) and (13f) for the TMDs $g_L^{\perp q}(x,k_T)$ and $h_T^q(x,k_T)$ into Eq. (2c) leads immediately to

$$g_L^{\perp q}(x,k_T)+h_T^q(x,k_T)$$
$$=2P^+\int dk^-\left(\frac{xM^2-P\cdot k}{M^2}\right)[A_8^q(P\cdot k,k^2)$$
$$+A_{11}^q(P\cdot k,k^2)]_{k^+=xP^+}. \quad (18)$$

Also in this case we see that in the mixed-spin state version of the CPM the relation (2c) is not valid, unless we demand that $A_8^q=-A_{11}^q$ which is equivalent to introducing the pure-spin state version of the CPM; cf. Eq. (10).

D. QMR of helicity, transversity, and pretzelosity, Eq. (2d)

This QMR was, to the best of our knowledge, first discussed in Ref. [41]. The difference of g_1^q and h_1^q was known to be related in models to quark orbital angular momentum [49,50], implying that pretzelosity is related to quark orbital angular momentum [43]. Although only a model relation, this is the only connection of quark orbital angular momentum to TMDs known so far and attracted a lot of interest. The QMR (2d) and its connection to quark orbital angular momentum have been confirmed in several other model studies. Inserting the model expressions (12a), (13a), and (13c) into Eq. (2d), we obtain the lengthy expression

$$g_1^q(x,k_T)-h_1^q(x,k_T)-h_{1T}^{\perp(1)q}(x,k_T) = 2P^+\int dk^-\left[\left(\frac{x^2M^2-xP\cdot k+m_q^2-xm_qM}{M^2}\right)A_8^q(P\cdot k,k^2) + \left(\frac{xP\cdot k-xm_qM-k_T^2}{M^2}\right)A_{11}^q(P\cdot k,k^2)\right]_{k^+=xP^+}.$$

In order to proceed, we eliminate $P\cdot k$ under the integral by means of the identity (15). After rearranging, the result can be expressed as

$$g_1^q(x,k_T)-h_1^q(x,k_T)-h_{1T}^{\perp(1)q}(x,k_T) = 2P^+\int dk^-\frac{(xM-m_q)^2-k_T^2}{2M^2}[A_8^q(P\cdot k,k^2) + A_{11}^q(P\cdot k,k^2)]_{k^+=xP^+}. \tag{19}$$

As in the previous cases, we see that in the mixed-spin state version of the CPM the relation (2d) is not supported. For this QMR to be valid in the CPM, we must introduce the condition $A_8^q=-A_{11}^q$ which is equivalent to introducing the pure-spin state version of the CPM; cf. Eq. (10).

E. QMR of twist-3 TMDs g_T^q and h_L^q to pretzelosity, Eq. (2e)

We now turn our attention to the last linear QMR which connects $g_T^q(x,k_T)$, $h_L^q(x,k_T)$, and the transverse moment of pretzelosity. Inserting in Eq. (2e) the model expressions (12c), (13c), and (13d) for $g_T^q(x,k_T)$, $h_L^q(x,k_T)$, and $h_{1T}^{\perp q}(x,k_T)$ gives

$$g_T^q(x,k_T)-h_L^q(x,k_T)-h_{1T}^{\perp(1)q}(x,k_T) = 2P^+\int dk^-\left(\frac{2m_q^2+k_T^2}{2M^2}-\frac{m_q}{M}\frac{P\cdot k}{M^2}\right) \times [A_8^q(P\cdot k,k^2)+A_{11}^q(P\cdot k,k^2)]_{k^+=xP^+}. \tag{20}$$

As in the previous cases, we observe that the QMR (2e) is not valid in the mixed-spin state version of the CPM and can be satisfied only when one introduces the condition $A_8^q=-A_{11}^q$, i.e., the pure-spin state version of the model.

F. Nonlinear QMR between h_1^q, $h_{1L}^{\perp q}$, and $h_{1T}^{\perp q}$ in Eq. (3a)

The nonlinear QMR (3a) was derived in Ref. [44]. Inserting the model expressions (13a)–(13c) into the nonlinear relation (3a), we obtain

$$2h_1^q(x,k_T)h_{1T}^{\perp q}(x,k_T)+h_{1L}^{\perp q}(x,k_T)^2 = (2P^+)^2\iint dk^-dk'^-\left[x\frac{m_q}{M}(A_8^q(P\cdot k,k^2)A_{11}^q(P\cdot k',k'^2)-A_8^q(P\cdot k',k'^2)A_{11}^q(P\cdot k,k^2)) + \frac{m_q^2}{M^2}(A_8^q(P\cdot k,k^2)A_8^q(P\cdot k',k'^2)-A_{11}^q(P\cdot k,k^2)A_{11}^q(P\cdot k',k'^2))\right]_{k^+=xP^+}, \tag{21}$$

where $k=(k^+,k^-,\vec{k}_T)$ and $k'=(k^+,k'^-,\vec{k}_T)$ and we used the identity (15) to eliminate the variable $P\cdot k$ under the integral. In order to proceed, we repeat the calculation leading to Eq. (21) with the dummy integration variables k^- and k'^- interchanged and take the average of the two results. In this way, the "mixed terms" in the first term on the right-hand side of Eq. (21) with $A_8^qA_{11}^q$ cancel out, and we obtain

$$2h_1^q(x,k_T)h_{1T}^{\perp q}(x,k_T)+h_{1L}^{\perp q}(x,k_T)^2 = (2P^+)^2\iint dk^-dk'^-\frac{m_q^2}{M^2}[A_8^q(P\cdot k,k^2)A_8^q(P\cdot k',k'^2) - A_{11}^q(P\cdot k,k^2)A_{11}^q(P\cdot k',k'^2)]_{k^+=xP^+}. \tag{22}$$

It is convenient to rewrite this result in the following equivalent way:

$$2h_1^q(x,k_T)h_{1T}^{\perp q}(x,k_T)+h_{1L}^{\perp q}(x,k_T)^2 = (2P^+)^2\iint dk^-dk'^-\frac{m_q^2}{M^2}[(A_8^q(P\cdot k,k^2) + A_{11}^q(P\cdot k,k^2))(A_8^q(P\cdot k',k'^2)-A_{11}^q(P\cdot k',k'^2))]_{k^+=xP^+}. \tag{23}$$

In order to show that Eq. (23) is equivalent to Eq. (22), one can apply the trick with repeating the calculation with the dummy integration variables k^- and k'^- interchanged and taking the average.

As in the case of linear QMRs, the nonlinear relation (3a) is, in general, not valid in the CPM version with quarks in a mixed-spin state. Interestingly and in contrast to the linear case, the violation of the nonlinear QMR (3a) is, however, a small effect proportional to m_q^2/M^2 which is numerically of the order $\mathcal{O}(10^{-6})$ for the light u and d flavors. This observation may have interesting consequences on which we shall comment in Sec. V.

If we insist on the nonlinear QMR (3a) to be exactly valid for $m_q\neq 0$, then we see from the final expression (23) that there are two solutions: $A_8^q = \pm A_{11}^q$. It is not surprising to find two solutions, as we deal with a quadratic equation. Both solutions were encountered in Ref. [25], and $A_8^q = +A_{11}^q$ was recognized to be an unphysical solution as it would imply opposite signs for quark helicity and transversity TMDs in contradiction to results from other models and lattice QCD. The solution $A_8^q = -A_{11}^q$ leads to like signs for quark helicity and transversity TMDs in agreement with other models and lattice QCD and constitutes, therefore, the physical solution [25]. Thus, the CPM with massive quarks complies exactly with the nonlinear QMR (3a) if and only if we use the pure-spin version of the model.

G. Nonlinear QMR between $g_{1T}^{\perp q}$, $g_L^{\perp q}$, g_T^q, and $g_T^{\perp q}$ in Eq. (3b)

The nonlinear QMR (3b) was also derived in Ref. [44]. Inserting the model expressions (12b)–(12e) into the nonlinear relation (3b), we obtain

$$
\begin{aligned}
&2g_{1T}^{\perp q}(x,k_T)g_L^{\perp q}(x,k_T) + 2g_T^q(x,k_T)g_T^{\perp q}(x,k_T) - g_{1T}^{\perp q}(x,k_T)^2 \\
&\quad = (2P^+)^2 \iint dk^- dk'^- \bigg[x\frac{m_q}{M}(A_8^q(P\cdot k,k^2)A_{11}^q(P\cdot k',k'^2) - A_8^q(P\cdot k',k'^2)A_{11}^q(P\cdot k,k^2)) \\
&\qquad + \frac{m_q^2}{M^2}(A_8^q(P\cdot k,k^2)A_8^q(P\cdot k',k'^2) - A_{11}^q(P\cdot k,k^2)A_{11}^q(P\cdot k',k'^2))\bigg]_{k^+=xP^+},
\end{aligned}
\tag{24}
$$

where we used the identity (15) to eliminate the variable $P\cdot k$ in the coefficient of the $A_8^q(k\cdot P,k^2)A_8^q(P\cdot k',k'^2)$ term (in other cases, $P\cdot k$ cancels out). The expression under the integral of (24) coincides with the expression in (21), and the further steps continue from here in the same way as in Sec. IV F including all considerations and conclusions.

V. DISCUSSION OF THE RESULTS

In order to better understand the physical implications of our results, it is instructive to briefly review the relation between the two versions of the CPM [25]. As mentioned in Sec. III B, in Ref. [25] it was recognized that for massive quarks, $m_q \neq 0$, the quark correlator can be expressed compactly by introducing an axial vector w_q^μ which has the properties of a quark polarization vector and satisfies $k\cdot w_q = 0$. One then has a choice: A quark can be in a pure-spin state with $w_q^2 = -1$ or mixed-spin state with $-1 < w_q^2 < 0$. These two choices lead to the two versions of the model; cf. Eq. (10).

It is an interesting question which of the two CPM versions might be more realistic from a phenomenological point of view. At first glance, one could suspect the mixed-spin state version of the CPM to be phenomenologically more realistic due to a larger flexibility with three independent covariant functions which can be determined from parametrizations of unpolarized, helicity, and transversity parton distribution functions [51–63]. (The scale at which the covariant functions are determined from parametrizations is part of the modeling. We follow up on this important point in more detail in Sec. VI.) This question can be answered by future studies, when more of the TMDs become better known and constrained by data.

Meanwhile, one could also try to address this question based on what is known about TMDs from other quark models. A striking observation is that a large class of quark models supports the QMRs. Thus, one could wonder whether, based on a comparison to other models, for instance, the linear QMRs (2) should also hold in the CPM. If one would like the CPM to comply with the QMRs, then one must introduce a condition between the polarized amplitudes, namely, $A_8^q = -A_{11}^q$ as shown in Sec. IV, which leads at once to the pure-spin state version of the CPM.

To be more precise, when one approaches the issue from the point of view of a quark polarization vector w_q^μ, the pure-spin condition $w_q^2 = -1$ tells us only that $|A_8^q| = |A_{11}^q|$, and the CPM *per se* is not able to predict the sign of the chiral-odd TMDs. It is necessary to resort to results from other models and lattice QCD to determine the physical solution [25]. Here, the situation is different. The linear QMRs already "encode" the information from other models about the relative signs of the polarized chiral-even and chiral-odd TMDs. By imposing the linear QMRs in the mixed-spin version of the CPM, one is unambiguously led to the condition $A_8^q = -A_{11}^q$ without encountering any spurious unphysical solution.

Thus, there are two ways to introduce the CPM with quarks in a pure-spin state: (a) by demanding that $w_q^2 = -1$ and determining the physical solution or (b) by demanding that the model be compliant with the QMRs observed in other quark models. The two procedures are conceptually quite different but nevertheless equivalent. This is an interesting observation in itself and gives new insights on the CPM. Notice that this observation is independent of whether one considers massive quarks or neglects quark mass effects; cf. footnote 1.

These considerations are of interest beyond the CPM and give rise to a question regarding the spin state of quarks in other models which, to the best of our knowledge, has not been addressed in literature. Considering that the TMDs in the CPM comply with QMRs if and only if the quarks are in a pure-spin state, one may wonder whether the reverse is true: If a quark model supports the QMRs, are the quarks in this model necessarily in a pure-spin state? We do not know the answer. It will be interesting to address this question in other models.

The above remarks about the QMRs leading to the condition $A_8^q = -A_{11}^q$ refer to the linear case (2). For the nonlinear QMRs (3), the situation is different. These relations are quadratic in TMDs, and hence it is not surprising to encounter two solutions $A_8^q = \pm A_{11}^q$, one of which is physical and the other unphysical. As a consequence, with nonlinear QMRs (3) alone, we would need to use additional constraints to determine the physical and eliminate the unphysical solution—analogously as it was done with the two solutions of the condition $w_q^2 = -1$ in Ref. [25].

However, there is an interesting difference between the ways the CPM can comply with linear and nonlinear QMRs which bears an unexpected observation. In the more general mixed-spin state version of the CPM, the violation of the nonlinear QMRs (3) is proportional to the square of the current quark masses. In other words, already in the mixed-spin version of the CPM the nonlinear QMRs (3) are supported modulo current quark mass effects proportional to m_q^2/M^2 which is numerically an effect of the order of 10^{-6} for the up and down flavors.

This is an interesting observation for the following reason. The description of TMDs in QCD becomes equivalent to that in the parton model in the Wandzura-Wilczek-type (WW-type) approximation [39]. This approximation consists in exploring the QCD equations of motion for twist-3 TMDs to relate them to the better known twist-2 TMDs and the so-called tilde terms which are contributions due to quark-gluon-quark matrix elements and current quark mass terms. Neglecting the tilde and current quark mass terms constitutes the WW-type approximation. [The attribute "type" is added to distinguish the more complex TMD case from the original WW-approximation for the collinear function $g_T^q(x)$ [64].] The exploration of the free equation of motion in the parton model generates exactly the same mass terms as in QCD but, of course, no tilde terms. In this sense, the predictions of the parton model are equivalent to the description of TMDs in QCD in the WW-type approximation.

The linear QMRs (2) hold in the CPM only if one introduces an *additional* constraint, which is equivalent to putting the quarks in a pure-spin state. It remains to be seen whether this leads to a realistic modeling of the nucleon structure from a phenomenological point of view. However, the nonlinear QMRs (3) do not require such an additional condition and are valid also for (light) quarks in a mixed-spin state. This could imply that the nonlinear QMRs (3) are more likely to be supported in QCD because no additional (pure-spin state) condition is required for their validity.

The observation that the nonlinear QMRs (3) could be valid in the WW-type approximation is interesting. The WW-type approximation has been explored for phenomenological applications, for instance, in Ref. [65]. The quality of this approximation cannot be determined *a priori*, and it needs to be investigated on a case by case basis because different quark-gluon-quark matrix elements are neglected in each case. In some cases, the WW-type approximation was shown to work with a phenomenologically useful approximation [65,66], and in one case there is support from lattice QCD [67]. It will be very interesting to investigate whether the nonlinear QMRs (3) could be valid in QCD with a similarly useful approximation.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have investigated the quark-model relations in the mixed-spin version of the covariant-parton model. The equations of motion in the CPM imply some conditions among the amplitudes in the quark correlator but leave the amplitudes A_8^q and A_{11}^q unrelated. We have shown that the linear QMRs are not valid, unless one introduces the condition $A_8^q = -A_{11}^q$. This condition is equivalent to putting the quarks in a pure-spin state (more precisely, the pure-spin state condition only implies $|A_8^q| = |A_{11}^q|$ and does not determine the relative sign).

Our results are of interest, because they give insights on the CPM and raise interesting questions about quark models and QMRs. The observation that imposing linear QMRs is equivalent to putting the quarks in a pure-spin state is primarily an insight about the CPM but may be of interest also beyond this model for the following reason. In the CPM, the two statements (i) quarks are in the pure-spin state and (ii) model complies with QMRs are equivalent. It will be interesting to investigate whether this is the case also in other models: If a quark model obeys the QMRs, are then the quarks in this model in a pure-spin state? This aspect deserves further investigations.

We also learn an interesting lesson about QMRs. In QCD, each TMD is an independent function describing a different aspect of the nucleon structure, and no relations among TMDs exist. In quark models, the situation can be simpler and relations among TMDs may exist. Such relations become particularly interesting if they are supported by a wide class of different quark models as is the case with the QMRs which arise from a certain symmetry of the nucleon wave function which is present in many (though not all) quark models.

QMRs become even more interesting if they require only general model assumptions. In the CPM, the linear QMRs require a strong model assumption; namely, the quarks must be in pure spin state. The situation is different for the nonlinear QMRs. These relations become exact in the CPM for quarks in a pure-spin state and/or for massless quarks. However, even in the most general case in the CPM, i.e., for massive quarks in a mixed-spin state, the nonlinear QMRs are still valid to a very good approximation, namely, up to negligibly small quadratic quark mass effects $\propto m_q^2/M^2$.

Thus, the nonlinear QMRs are practically supported in the CPM independently of the quark spin state. In other words, the nonlinear QMRs require no strong model assumption (like the pure-spin condition). From the point of view of the CPM, all that is required for the nonlinear QMRs is the absence of interactions. From the point of view of QCD, this, in turn, means that the nonlinear QMRs could be valid in the WW-type approximation.

It is important to remark that even if the QMRs were valid at one scale, due to the different evolution equations of the different TMDs, they would not be valid at other scales. However, considering their crude nature, the "accuracy" of quark models can be expected to be around $\mathcal{O}(30\%\text{–}40\%)$ [68,69], and the TMD evolution effects are not a dominant uncertainty. It will be interesting to see whether phenomenological extractions or lattice QCD results will support, at some scale, predictions from quark models like the CPM within such model accuracy.

The spin state of a quark in QCD is not easy to determine [70–72]. The comparison of the CPM predictions to phenomenological results for TMD extractions will constitute one way to infer to which extent the quarks in QCD can be viewed as being in a pure- or mixed-spin state. It will be interesting to shed more light on the polarization state of the quarks in the nucleon based on dedicated phenomenological, model, and lattice QCD studies.

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APPENDIX: QUARK-MODEL EXPRESSIONS FOR T-EVEN TMDs

In this Appendix, to make this work self-contained, we list the quark-model expressions for T-even TMDs in terms of the amplitudes defined in Eq. (6). These expressions are valid in all models without gauge field degrees of freedom. In QCD, the TMDs depend on the renormalization scale μ^2 and the scale ζ at which light cone divergences are regulated. In this work, we do not indicate the scales for brevity. The determination of these scales in a model calculation is an important part of the modeling. In previous works in the CPM, the scales were assumed to be $\mu^2 = \zeta \simeq (3-4)\ \mathrm{GeV}^2$.

In the twist-2 case, the expressions for TMDs read

$$f_1^q(x,k_T) = 2P^+ \int dk^- [A_2^q + xA_3^q]_{k^+=xP^+}, \tag{A1}$$

$$g_1^q(x,k_T) = 2P^+ \int dk^- \left[-A_6^q - \frac{P\cdot k - M^2 x}{M^2}(A_7^q + xA_8^q)\right]_{k^+=xP^+}, \tag{A2}$$

$$g_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- [A_7^q + xA_8^q]_{k^+=xP^+}, \tag{A3}$$

$$h_1^q(x,k_T) = 2P^+ \int dk^- \left[-A_9^q - xA_{10}^q + \frac{\vec{k}_T^2}{2M^2}A_{11}^q\right]_{k^+=xP^+}, \tag{A4}$$

$$h_{1L}^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[A_{10}^q - \frac{P\cdot k - M^2 x}{M^2}A_{11}^q\right]_{k^+=xP^+}, \tag{A5}$$

$$h_{1T}^{\perp q}(x,k_T) = 2P^+ \int dk^- [A_{11}^q]_{k^+=xP^+}. \tag{A6}$$

In the twist-3 case, the expressions are given by

$$e^q(x,k_T) = 2P^+ \int dk^- [A_1^q]_{k^+=xP^+}, \tag{A7}$$

$$f^{\perp q}(x,k_T) = 2P^+ \int dk^- [A_3^q]_{k^+=xP^+}, \tag{A8}$$

$$g_T^q(x,k_T) = 2P^+ \int dk^- \left[-A_6^q + \frac{\vec{k}_T^2}{2M^2}A_8^q\right]_{k^+=xP^+}, \tag{A9}$$

$$g_L^{\perp q}(x,k_T) = 2P^+ \int dk^- \left[-\frac{P\cdot k - M^2 x}{M^2}A_8^q\right]_{k^+=xP^+}, \tag{A10}$$

$$g_T^{\perp q}(x,k_T) = 2P^+ \int dk^- [A_8^q]_{k^+=xP^+}, \tag{A11}$$

$$h_L^q(x,k_T) = 2P^+ \int dk^- \left[-A_9^q - \frac{P\cdot k}{M^2}A_{10}^q + \frac{(P\cdot k - M^2 x)^2}{M^4}A_{11}^q\right]_{k^+=xP^+}, \tag{A12}$$

$$h_T^q(x,k_T) = 2P^+ \int dk^- \left[-\frac{P\cdot k - M^2 x}{M^2}A_{11}^q\right]_{k^+=xP^+}, \tag{A13}$$

$$h_T^{\perp q}(x,k_T) = 2P^+ \int dk^- [-A_{10}^q]_{k^+=xP^+}. \tag{A14}$$

In QCD also B_i^q amplitudes enter; see, e.g., [37] for the full expressions. But in quark models the 14 T-even TMDs are expressed in terms of nine T-even A_i^q amplitudes. This implies five relations, namely, the qLIRs mentioned in Sec. II.

We also remark that, in contrast to QCD, in the CPM no UV or rapidity divergences appear. This allows one to relate TMDs and collinear parton distribution functions simply as $f_1^q(x) = \int d^2k_T f_1^q(x, k_T)$ with a finite k_T integration, which in QCD [1] as well as in some models [73] is spoiled by the appearance of divergences.

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