

Electron-photon vertex and dynamical chiral symmetry breaking in reduced QED: An advanced study of gauge invariance

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We study the effect of a refined electron-photon vertex on the dynamical breaking of chiral symmetry in reduced quantum electrodynamics. We construct an educated *ansatz* for this vertex that satisfies the required discrete symmetries under parity, time reversal, and charge conjugation operations. Furthermore, it reproduces its asymptotic perturbative limit in the weak coupling regime and ensures the massless electron propagator is multiplicatively renormalizable in its leading logarithmic expansion. Employing this vertex *ansatz*, we solve the gap equation to compute dynamically generated electron mass whose dependence on the electromagnetic coupling is found to satisfy Miransky scaling law. We also investigate the gauge dependence of this dynamical mass as well as that of the critical coupling above which chiral symmetry is dynamically broken. As a litmus test of our vertex construction, both these quantities are rendered virtually gauge independent within a certain interval of values considered for the covariant gauge parameter.

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I. INTRODUCTION

Graphene, the wonder material [1,2], is a physical system with immense potential for technological applications. It has driven a lot of research in both the applied and theoretical physics, not only from the point of view of condensed matter and materials sciences [3,4], but also based on the quantum field theoretic description within the domain of high energy physics [5] and cosmology [6–8]. Its remarkable properties of high electric and thermal conductivity, stiffness, flexibility, and transparency have opened the door to explore a growing family of modern relativistic and relativisticlike materials in one, two, and three spatial dimensions. The underlying honeycomb array of tightly packed carbon atoms and its crystallographic description in terms of two interimposed triangular sublattices provide chiral and valley quantum labels to the charge carrier

electrons. This occurrence is responsible for Klein tunneling [9] as well as other exotic and novel phenomena [10] exhibited by relativistic systems only. That makes graphene an incarnation of quantum electrodynamics (QED) in condensed matter realms. Along with quantum Hall systems [2,11–13] and high- T_c layered cuprate superconductors [14–18], graphene is also a system suitable for its description in terms of relativistic quantum field theoretical considerations, developed and refined in the domain of particle physics [5]. This in turn allows for an exploration of otherwise inaccessible particle physics *phenomenology* in a more controlled and observable ambient of solid state physics. A representative example in this connection is the so-called chiral magnetic effect [19,20], which was first predicted to take place in relativistic heavy ion collisions. It involves chirality flip of quarks, caused by the chiral anomaly. It is a necessary ingredient to produce a non-dissipative current as an observable effect. This effect was proposed to probe the nontrivial vacuum structure of quantum chromodynamics (QCD). However, it has not been observed in isobar collisions in the STAR Collaboration at RHIC [21]. Nevertheless the same basic idea of a physical system in which interactions drive a chirality flip of the fundamental degrees of freedom was found in ZnTe_5 [22], where a neat nondissipative current was observed when this 3D crystal is subject to an array of adequately aligned electric and magnetic fields. After this

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first observation, nondissipative currents driven by the chiral anomaly were also encountered in several other similar materials [23–27].

Some theoretical ideas have also been developed to observe a similar effect in 2D crystals such as graphene [28,29]. More recently, it has been shown that in some 2D materials, one may observe a novel quantum spin Hall phenomenon [30]. A key ingredient for the theoretical realization of these later phenomena is the description of electromagnetic and matter fields living in mixed dimensions. Mixed dimensional theories emerge naturally in the description of 2D materials where experiments are carried out with external electromagnetic fields that permeate the whole space whereas the motion of the charge carriers remains restricted to a plane.

Two independent formulations have been proposed in the literature to describe QED of photons and electrons living in different space dimensions. One vision exploits the equivalence of a theory where electrons live in lower dimensions than photons and a Chern-Simons theory. It has been dubbed as Pseudo-QED [31–34]. Alternatively, a brane-world inspired scenario was developed in [35] to explore the traits of dynamical chiral symmetry breaking (DCSB) in the so-called Reduced QED (RQED), a nomenclature we choose to adopt in this article. The equivalence of these two visions has already been established. It respects causality [33] and unitarity [34]. It exhibits a Coulomb static interaction in the case of graphene [32,36]. It contains an infrared fixed point of the renormalization group as the Fermi velocity tends to the speed of light [36–39]. Two-loop perturbative analysis has been carried out in [40] and later improved with renormalization group arguments [41]. Interestingly for the immediate purpose of our manuscript, DCSB has been explored within the Schwinger-Dyson [42] equations (SDEs) and renormalization group frameworks exploiting the duality between the gap equation in this theory and the corresponding $1/N$ leading truncation in parity preserving ordinary QED₃. In the latter theory, it is known that there exists a critical number of fermion families N_c above which DCSB is restored. In comparison, it is observed that in the quenched version of RQED, where electron-loop contributions to the photon propagator are neglected and the photon dressing function reduces to its tree level expansion, DCSB occurs provided the electromagnetic coupling α exceeds a critical value α_c . The particular values of these critical numbers depend on the gauge parameter and provide a natural motivation for the work we present and the solutions we provide in this article. For the sake of completeness, we would like to mention that DCSB has also been considered in RQED at finite temperature [42–44] and in the presence of a Chern-Simons term [29]. Parity violating solutions to the gap equation have also been explored in [45] in connection with the presence of a Chern-Simons term. This term plays the role of an effective dielectric constant [46], hence having potential experimental realization in graphene related

materials. Effects of strain have also been considered, leading to a lower value of the critical coupling required to break chiral symmetry. Finally, RQED has also been formulated in curved spaces [47]. For a review discussing all this properties and applications of RQED, see [48].

Studying DCSB and its gauge invariance in RQED naturally requires its nonperturbative treatment. Fortunately, extensive amount of analogous research in QED₄ [49–53] and QED₃ [54–57] provides necessary groundwork to carry out similar reliable analysis in RQED. We report the results of this continuum study through state-of-the-art truncation schemes in SDEs. Focusing on the quenched version of the theory, the sole source of our starting *ansatz* is the electron-photon vertex. We construct it by demanding all the key characteristics of RQED to be faithfully respected:

- (i) Ward-Fradkin-Green-Takahashi identity (WFGTI) that connects the electron propagator with the *longitudinal* part of the electron-photon vertex is satisfied nonperturbatively by construction, known as the Ball-Chiu (BC) vertex [58].
- (ii) To expand the transverse part of the vertex, we employ the vector basis and its coefficients in such a manner as to ensure spurious kinematic singularities are absent from our construction [59–61].
- (iii) In the weak coupling regime, the vertex faithfully reproduces its one-loop perturbative expansion for the asymptotic limit of momenta $k^2 \gg p^2$, just it has previously been done in QED₄ [49,50] and QED₃ [62–64].
- (iv) The standard model of particle physics tells us of the intimate connection between its renormalizability and gauge invariance. In the same spirit, we require our vertex *ansatz* to guarantee the multiplicative renormalizability (MR) of the massless electron propagator in its leading logarithmic expansion.
- (v) We also require our vertex to satisfy the discrete symmetries of parity, time reversal, and charge conjugation.

Based upon the above-mentioned constraints, we are able to achieve *nearly* gauge independent Euclidean mass and critical coupling α_c where the DCSB solution bifurcates away from the chirally symmetric one. We believe that obtaining gauge independent DCSB holds the promise to study observable effects in the 2D materials described by RQED in a reliable manner through continuum SDEs.

The article has been organized as follows: Sec. II begins with a brief introduction to the mathematical foundations of RQED. In Sec. III, we provide necessary preliminaries on the vertex decomposition and its general features. In Sec. IV, we construct a family of *ansätze* for the transverse vertex in perhaps the most economical yet efficient manner by resorting to the constraints of MR and its explicit form in the so-called asymptotic limit at the one-loop level. In Sec. V, we set up the gap equation and engage in a

detailed discussion on the photon propagator in RQED and the appropriate use of the WFGTI in order to ensure the MR of the massless electron propagator. Section VI provides solution of the gap equation, first in the perturbative realm and then the nonperturbative DCSB solution in terms of the critical coupling α_c above which the massive solution bifurcates away from the perturbative massless one. We primarily focus on obtaining gauge independent DCSB. Section VII contains conclusions and offers perspectives for future work.

II. RQED FOUNDATIONS

In order to describe electrons, restricted to move in a dimensionally reduced space-time, coupled to photons free to propagate through the bulk space-time, one initially begins with the well-known QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi + j_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2, \quad (1)$$

where ψ and A^μ are electron and photon fields, respectively, coupled to each other through the electromagnetic current j_μ , where $\mu = 0, 1, 2, 3$. Furthermore, the four-dimensional Dirac matrices γ^μ satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with the commonly used convention $g^{\mu\nu} = (+, -, -, -)$ for the Minkowski space metric tensor. Additionally, m_0 is the bare electron mass, ξ is the covariant gauge parameter, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field tensor. The corresponding action $S_{\text{QED}} = \int d^4x \mathcal{L}_{\text{QED}}$ can be conveniently expressed as

$$S_{\text{QED}} = S_{\bar{\psi}\psi}^{(4)} + \frac{1}{2} \int d^4x [j^\mu \hat{\Delta}_{\mu\nu} j^\nu - A^\mu \hat{\Delta}_{\mu\nu}^{-1} A^\nu], \quad (2)$$

the kinetic and mass terms for electrons, moving in a d -dimensional space-time read

$$S_{\bar{\psi}\psi}^{(d)} = \int d^d x \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi, \quad (3)$$

with $\mu = 0, 1, 2, \dots, d-1$. Moreover, the differential operator $\hat{\Delta}_{\mu\nu}$, namely, the photon propagator in coordinate space, can be cast in terms of its momentum space counterpart by means of a Fourier transformation

$$\hat{\Delta}_{\mu\nu} = - \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} \frac{1}{q^2} \left[g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right], \quad (4)$$

satisfying $\hat{\Delta}_{\mu\alpha}^{-1} \hat{\Delta}^{\alpha\nu} = g_\mu^\nu$ with its corresponding inverse propagator. Such a Green's function accounts for a gauge field propagating through the whole four-dimensional space-time with $\mu, \nu = 0, 1, 2, 3$.

To account for a mixed-dimension system described by RQED with electrons restricted to move on a plane perpendicular to the x_3 axis [31–35,65], the electromagnetic current takes the form

$$j^\mu = \begin{cases} -ie\bar{\psi}\gamma^\mu\psi\delta(x_3) & \text{for } \mu = 0, 1, 2, \\ 0 & \text{for } \mu = 3. \end{cases} \quad (5)$$

Therefore, only the indices $\mu = 0, 1, 2$ contribute to the term $j^\mu \hat{\Delta}_{\mu\nu} j^\nu$ in Eq. (2). The component $\mu = 3$ can thus be integrated out in Eq. (4), leading to [66]

$$\hat{\Delta}_{\mu\nu} = \int \frac{d^3 q}{(2\pi)^3} e^{-iq \cdot x} \frac{1}{2\sqrt{-q^2}} \left[g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{2q^2} \right]. \quad (6)$$

It entails a nonlocal differential operator. This propagator can be obtained from the effective action for RQED (redefining $\xi = 2\zeta - 1$)

$$S_{\text{RQED}} = S_{\bar{\psi}\psi}^{(3)} + \int d^3 x j_\mu A^\mu + \int d^3 x \left[\frac{1}{2} F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F^{\mu\nu} + \frac{1}{\zeta} \partial_\mu A^\mu \frac{1}{\sqrt{-\partial^2}} \partial_\nu A^\nu \right]. \quad (7)$$

From now on, we work in the Euclidean space defined by the metric tensor $\delta_{\mu\nu} = (+, +, +)$ for $\mu, \nu = 4, 1, 2$. In this space, the bare photon propagator takes the form [cf. Eq. (6)]

$$\Delta_{\mu\nu}^{(0)}(q) = \frac{1}{2q} \left[\delta_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{2q^2} \right], \quad (8)$$

where we have defined $q \equiv \sqrt{-q^2}$. Note that this propagator has a softer infrared behavior than the photon propagator in QED₄ and QED₃. Several groups differ in conventions by the global factor of 1/2. When written in terms of the variable ζ , this propagator can be separated into a familiar longitudinal and a transverse component (to q_μ)

$$\Delta_{\mu\nu}^{(0)}(q) = \frac{1}{2q} \left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] + \zeta \frac{q_\mu q_\nu}{2q^3}. \quad (9)$$

However, it must be emphasized that the gauge fixing parameter ζ in RQED is different from that in QED, ξ , due to dimensional reduction. In the above expression for the photon propagator, Eq. (9), the ζ -independent term defines the transverse propagator which is the only component that gets quantum corrections. This is the reason for using such a decomposition in other works. In contrast, the ξ -independent term in Eq. (8) does not define a transverse propagator. Therefore, quantum corrections affect both

ξ -dependent and ξ -independent components. In the present work, we restrict ourselves to work in the quenched approximation. As there are no fermion loops present, there are no quantum corrections to the bare photon propagator. Therefore, both the expressions Eqs. (8), (9) are equally suitable: we choose that of Eq. (8).

Note that the electron-photon vertex plays a vital role in computing the nonperturbative solution of the electron propagator through its gap equation. Therefore, we address this three-point Green's function at length in the following section.

III. THE VERTEX: GENERALITIES

In its general decomposition, the three-point electron-photon vertex can be written in terms of twelve independent spin structures. For the kinematical configuration of Fig. 1, the WFGTI associated with this vertex takes the form

$$iq_\mu \Gamma_\mu(k, p) = S^{-1}(k) - S^{-1}(p), \quad (10)$$

where $q = k - p$. This identity allows us to split the vertex as a sum of *longitudinal* and *transverse* components, as suggested by Ball and Chiu [58]:

$$\Gamma_\mu(k, p) = \Gamma_\mu^L(k, p) + \Gamma_\mu^T(k, p). \quad (11)$$

The longitudinal part $\Gamma_\mu^L(k, p)$ alone satisfies the WFGTI (10), and consumes four of the twelve independent spin structures (one of them is zero in QED), so that, [58]

$$\Gamma_\mu^L(k, p) = \lambda_1(k, p)\gamma_\mu + \lambda_2(k, p)t_\mu\gamma \cdot t - i\lambda_3(k, p)t_\mu, \quad (12)$$

with $t = k + p$, and

$$\begin{aligned} \lambda_1(k, p) &= \frac{1}{2} \left[\frac{1}{F(k^2, \Lambda^2)} + \frac{1}{F(p^2, \Lambda^2)} \right], \\ \lambda_2(k, p) &= \frac{1}{2} \left[\frac{1}{F(k^2, \Lambda^2)} - \frac{1}{F(p^2, \Lambda^2)} \right] \frac{1}{k^2 - p^2}, \\ \lambda_3(k, p) &= \left[\frac{\mathcal{M}(k^2, \Lambda^2)}{F(k^2, \Lambda^2)} - \frac{\mathcal{M}(p^2, \Lambda^2)}{F(p^2, \Lambda^2)} \right] \frac{1}{k^2 - p^2}, \end{aligned} \quad (13)$$

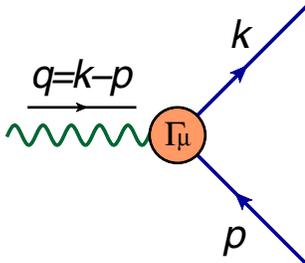


FIG. 1. Diagrammatic representation of the full electron-photon vertex $\Gamma_\mu(k, p)$, with momentum flow indicated.

where Λ is an ultraviolet (UV) cutoff regulator. Note that $\mathcal{M}(k^2, \Lambda^2)$ and $F(k^2, \Lambda^2)$ are the mass function and the wave function renormalization, respectively, related to the electron propagator $S(k)$ through

$$S(k) = \frac{F(k^2, \Lambda^2)}{i\gamma \cdot k + \mathcal{M}(k^2, \Lambda^2)}, \quad (14)$$

with $F(k^2, \Lambda^2) = 1$ and $\mathcal{M}(k^2, \Lambda^2) = m_0$ at the tree level.

The transverse part $\Gamma_\mu^T(k, p)$ of the vertex decomposition (11), which remains undetermined by the WFGTI, is naturally constrained by

$$q_\mu \Gamma_\mu^T(k, p) = 0. \quad (15)$$

In general, the ultraviolet finite transverse vertex can be expanded out in terms of eight basis vector structures, and their corresponding scalar form factors $\tau_i(k, p)$ [58]:

$$\Gamma_\mu^T(k, p) = \sum_{i=1}^8 \tau_i(k, p) T_\mu^i(k, p). \quad (16)$$

For the kinematical configuration of Fig. 1, we define

$$\begin{aligned} T_\mu^1(k, p) &= i[p_\mu(k \cdot q) - k_\mu(p \cdot q)], \\ T_\mu^2(k, p) &= [p_\mu(k \cdot q) - k_\mu(p \cdot q)](\gamma \cdot t), \\ T_\mu^3(k, p) &= q^2\gamma_\mu - q_\mu(\gamma \cdot q), \\ T_\mu^4(k, p) &= iq^2[\gamma_\mu(\gamma \cdot t) - t_\mu] + 2q_\mu p_\nu k_\rho \sigma_{\nu\rho}, \\ T_\mu^5(k, p) &= \sigma_{\mu\nu} q_\nu, \\ T_\mu^6(k, p) &= -\gamma_\mu(k^2 - p^2) + t_\mu(\gamma \cdot q), \\ T_\mu^7(k, p) &= \frac{i}{2}(k^2 - p^2)[\gamma_\mu(\gamma \cdot t) - t_\mu] + t_\mu p_\nu k_\rho \sigma_{\nu\rho}, \\ T_\mu^8(k, p) &= -i\gamma_\mu p_\nu k_\rho \sigma_{\nu\rho} - p_\mu(\gamma \cdot k) + k_\mu(\gamma \cdot p), \end{aligned} \quad (17)$$

with

$$\sigma_{\nu\rho} = \frac{i}{2}[\gamma_\nu, \gamma_\rho]. \quad (18)$$

This basis is not exactly the one adopted in [58]. We choose to work with a modification of this initial basis which was put forward in [59] and later employed in [60] as well. This modified choice of the basis vectors ensures all transverse form factors of the vertex are independent of any kinematic singularities in one-loop perturbation theory in an arbitrary covariant gauge.

As stated earlier in Sec. I, any *ansatz* for the full vertex must have the same transformation properties as the bare vertex under charge conjugation operation. This requires all the τ_i 's in (16) to be symmetric under the interchange $k \leftrightarrow p$, except τ_4 and τ_6 , which are odd:

$$\tau_i(k, p) = \tau_i(p, k), \quad i = 1, 2, 3, 5, 7, 8, \quad (19)$$

$$\tau_i(k, p) = -\tau_i(p, k), \quad i = 4, 6. \quad (20)$$

From Eq. (13), it is obvious that $\lambda_1(k, p)$, $\lambda_2(k, p)$, and $\lambda_3(k, p)$ are symmetric under $k \leftrightarrow p$, as they should be, in order to preserve the correct transformation properties under charge conjugation operation for the full vertex.

In the following section, we construct the transverse vertex, putting forward an *ansatz* which complies with the key requirements expected of it as detailed in Sec. I. It is also effective, economical, and yields practically gauge invariant DCSB as we see in subsequent sections.

IV. THE TRANSVERSE VERTEX

The transverse vertex is completely determined once the form factors in Eqs. (16), (17) are known. We now proceed to construct an *ansatz* for it.

A. A general *ansatz* for the transverse vertex

We start out by recalling a fairly general *ansatz* which was first proposed in [52], generally referred to as the BB vertex, and later successfully employed in several related works, e.g., [53,61,67]

$$\tau_1(k, p) = \frac{a_1}{(k^2 + p^2)} \lambda_3(k, p), \quad (21)$$

$$\tau_2(k, p) = \frac{2a_2}{(k^2 + p^2)} \lambda_2(k, p), \quad (22)$$

$$\tau_3(k, p) = 2a_3 \lambda_2(k, p), \quad (23)$$

$$\tau_4(k, p) = \frac{a_4(k^2 - p^2)}{4k^2 p^2} \lambda_3(k, p), \quad (24)$$

$$\tau_5(k, p) = -a_5 \lambda_3(k, p), \quad (25)$$

$$\tau_6(k, p) = -\frac{2a_6(k^2 + p^2)}{(k^2 - p^2)} \lambda_2(k, p), \quad (26)$$

$$\tau_7(k, p) = -\left[\frac{a_4 q^2}{2k^2 p^2} + \frac{a_7}{k^2 + p^2} \right] \lambda_3(k, p), \quad (27)$$

$$\tau_8(k, p) = 2a_8 \lambda_2(k, p). \quad (28)$$

Before we proceed any further, we summarize the following important points:

- (i) All eight transverse form factors are taken into consideration, which implies the generality of this *ansatz*.
- (ii) All form factors are proportional to the same structures which appear in the *longitudinal* vertex, namely, $\lambda_2(k, p)$ and $\lambda_3(k, p)$.

- (iii) The form factor $\tau_6(k, p)$ has a kinematic singularity for $k^2 = p^2$. For the QED₄ gap equation, we can employ this form factor with impunity because it cancels with the corresponding term proportional to $\lambda_2(k, p)$ coming from the *longitudinal* vertex. However, for RQED, this cancellation does not take place and we need to modify this form factor to avoid the kinematic singularity. We propose an economical and effective modification:

$$\tau_6(k, p) = a_6 \frac{(k^2 - p^2)}{(k^2 + p^2)} \lambda_2(k, p). \quad (29)$$

We now try to constrain the coefficients a_i through the one-loop expression for the *transverse* vertex. We may call this as the modified BB vertex for RQED.

B. One-loop electron-photon vertex

In perturbation theory, the one-loop order expansion of the electron-photon vertex is

$$\Gamma_\mu^{(1)}(k, p) = \gamma_\mu - \frac{\alpha}{2\pi^2} \int d^3\omega \gamma_\alpha S(p - \omega) \gamma_\mu \times S(k - \omega) \gamma_\beta \Delta_{\alpha\beta}(\omega). \quad (30)$$

In analogy with [64,68,69], we have computed the one-loop corrections to the electron-photon vertex in the asymptotic limit, defined as the perturbative expansion with $p^2 \gg k^2 \gg m_0^2$. The leading logarithmic term of the transverse vertex at this level of approximation reads

$$\Gamma_\mu^T(k, p) \stackrel{p^2 \gg k^2}{\cong} -\frac{\alpha}{8\pi p^2} \left(\xi - \frac{1}{3} \right) \log \left(\frac{p^2}{k^2} \right) T_\mu^{asy}, \quad (31)$$

with

$$T_\mu^{asy} = p^2 \gamma_\mu - p_\mu \gamma \cdot p. \quad (32)$$

Hence, in this limit, the transverse vertex can be expressed as [cf. Eq. (58)]

$$\Gamma_\mu^T(k, p) \stackrel{p^2 \gg k^2}{\cong} \frac{1}{2} \left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] \frac{T_\mu^{asy}}{k^2 - p^2}. \quad (33)$$

On the other hand, from Eqs. (16), (17), it is straightforward to see that the leading structure of the transverse vertex in this limit acquires the following form:

$$\Gamma_\mu^T(k, p) \stackrel{p^2 \gg k^2}{\cong} (\tau_3 + \tau_6) T_\mu^{asy}, \quad (34)$$

where $\tau_{3,6} \equiv \tau_{3,6}(k, p)$. Moreover, we have used the fact that, in the asymptotic expansion, the dominant contributions to the transverse vertex come from T_μ^3 and T_μ^6 which simplify to

$$T_\mu^{3asy} = T_\mu^{6asy} \equiv T_\mu^{asy}. \quad (35)$$

As the simplest construction of the transverse vertex, we can single out $\tau_3(k, p)$ and $\tau_6(k, p)$ which correctly provide this desired limit. Writing out these form factors explicitly,

$$\tau_3(k, p) = a_3 \left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] \frac{1}{k^2 - p^2}, \quad (36)$$

$$\tau_6(k, p) = a_6 \left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] \frac{1}{k^2 + p^2}. \quad (37)$$

It is thus straightforward to see from Eqs. (33), (34) that the one-loop behavior of the vertex in the asymptotic limit requires

$$a_3 - a_6 = 1/2. \quad (38)$$

Our aim is to compute nonperturbative solutions of the gap equation using various vertex *ansätze*. Our preferred choice is the electron-photon vertex consonant with the WFGTI and perturbation theory. It consists of the longitudinal BC-vertex, Eq. (12), and the transverse component constructed in terms of the vector structures T_μ^3 and T_μ^6 alone. For the sake of completeness, we also compute and depict some results obtained by employing the bare vertex, the central BC vertex and the full BC vertex.

The starting point for such an endeavor is naturally the gap equation for the electron propagator. This is what we proceed to set up and study in the next section.

V. SETTING UP THE GAP EQUATION

The SDE for the electron propagator, also known as the electron *gap equation*, is diagrammatically depicted in Fig. 2.

Mathematically, the gap equation is written as

$$S^{-1}(k) = S_0^{-1}(k) + \frac{\alpha}{2\pi^2} \int_E d^3p \gamma_\nu S(p) \Gamma_\mu(k, p) \Delta_{\mu\nu}(q), \quad (39)$$

where the subscript E indicates that the integral is performed in the Euclidean space, the subscript “0” denotes the tree level of the corresponding propagator and $\alpha = e^2/4\pi$ is the electromagnetic coupling. We have

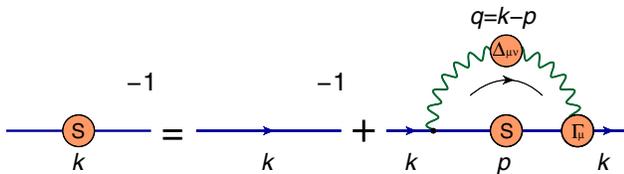


FIG. 2. The gap equation for the electron propagator. The color-filled blobs labeled with S , $\Delta_{\mu\nu}$, and Γ_μ stand for the fully dressed electron and photon propagators, and the three-point vertex, respectively.

already provided a detailed discussion on the full electron-photon vertex $\Gamma_\mu(k, p)$. A similar analysis of the photon propagator $\Delta_{\mu\nu}(q)$ is in place now.

A. The photon propagator

For the sake of an appropriate implementation of the WFGTI in the gap equation, let us split the photon propagator of Eq. (8) in a longitudinal piece

$$\Delta_{\mu\nu}^L(q) = \varrho \kappa \frac{q_\mu q_\nu}{q^3}, \quad (40)$$

and a remaining *nonlongitudinal* component

$$\Delta_{\mu\nu}^{NL}(q) = \frac{\varrho}{q} \left[\delta_{\mu\nu} - \chi \frac{q_\mu q_\nu}{q^2} \right], \quad (41)$$

expressed in Euclidean space such that

$$\Delta_{\mu\nu}(q) = \Delta_{\mu\nu}^L(q) + \Delta_{\mu\nu}^{NL}(q), \quad (42)$$

where $\chi \equiv 1 - \epsilon_e$, $\kappa \equiv \chi \xi$, $\varrho = \Gamma[\chi]/(4\pi)^{\epsilon_e}$. It is important to notice that RQED ($\epsilon_e = 1/2$) yields $\chi = \varrho = 1/2$ and $\kappa = \xi/2$. We have opted for this notation of arbitrary dimensions in alignment with the one adopted in [65].

This separation of the photon propagator, Eqs. (40), (41), plays an important role in QED where it has been demonstrated that it is imperative to apply the WFGTI to the divergence term $q_\mu \Gamma_\mu$ which arises from the contraction of the vertex with the longitudinal (gauge dependent) contribution of the photon propagator in the gap equation. This recipe ensures the absence of spurious terms which are a characteristic of the UV cutoff regularization scheme due to the fact that it breaks translational invariance and can violate gauge invariance if not used carefully [50,70]. Providing a vertex that satisfies the WFGTI and employing this identity in the gap equation only for the appropriate part of the photon propagator are crucial to guarantee the local gauge covariance of the electron propagator. In the quenched approximation, the electron wave function renormalization for QED₄ has been shown to be explicitly renormalizable to all orders in perturbation theory provided an *ansatz* for the vertex in close analogy with full QED₄ is constructed [49]. In the leading logarithmic approximation, the electron wave function renormalization exhibits a power law behavior

$$F(k^2) = 1 + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} \log^n \left(\frac{k^2}{\Lambda^2} \right) = \left(\frac{k^2}{\Lambda^2} \right)^\beta, \quad (43)$$

where $\beta = \beta(\alpha)$ is an unknown coefficient that can be computed order by order in perturbation theory. In the leading logarithmic approximation of QED₄, it is known that $\beta = \alpha \xi / (4\pi)$. For quenched RQED we demonstrate that $\beta = \alpha(\xi - 1/3) / (4\pi)$.

As mentioned earlier, careless implementation of an ultraviolet cutoff regularization breaks gauge invariance and it is manifest in the appearance of spurious terms even in one-loop calculations. An appropriate use and manipulation of the WFGTI in the gap equation is key to getting rid of such spurious terms. For this purpose, it is worth noting that in quenched RQED we can redefine the longitudinal and nonlongitudinal components of the photon propagator by shifting χ and κ with an arbitrary factor η such that the complete photon propagator, Eq. (42), remains the same:

$$\chi \rightarrow \chi - \eta, \quad (44)$$

$$\kappa \rightarrow \kappa - \eta. \quad (45)$$

Bear in mind that the photon propagator and consequently the electron SDE, Eq. (39), remain invariant under the shifts prescribed by Eqs. (44), (45).

We apply the WFGTI, Eq. (10), on the longitudinal photon propagator term in the gap equation without loss of generality, leading to

$$\begin{aligned} & \int_E d^3 p \gamma_\nu S(p) \Gamma_\mu(k, p) \Delta_{\mu\nu}^L(q) \\ &= q\kappa \int_E \frac{d^3 q}{q^3} \gamma \cdot q - iq\kappa \int_E \frac{d^3 p}{q^3} \gamma \cdot q S(p) S^{-1}(k), \end{aligned} \quad (46)$$

where we have shifted $d^3 p \rightarrow d^3 q$ in the first term on the right-hand side of the above equation in order to show that such a term vanishes in a translationally invariant theory as it is an odd integral. Moreover, for the term $\Gamma_\mu(k, p) \Delta_{\mu\nu}^{NL}(q)$

in Eq. (39) we use the explicit form of the vertex defined through Eqs. (11)–(13), (16), (17). After bringing out the subtle role played by the photon propagator in RQED, we can now focus on the mathematical and technical details of the gap equation itself.

B. The gap equation

We can project out two coupled, integral equations for \mathcal{M} and F from the matrix gap equation. In an arbitrary gauge, these equations can, respectively, be written as

$$\begin{aligned} \frac{\mathcal{M}(k^2)}{F(k^2)} &= m_0 + \frac{\alpha q \kappa}{2\pi^2} \int_E \frac{d^3 p}{q^3} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} \frac{1}{F(k^2)} \\ &\quad \times \{ \mathcal{M}(p^2) q \cdot k - \mathcal{M}(k^2) q \cdot p \} \\ &\quad + \frac{\alpha q}{2\pi^2} \int_E d^3 p \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} G_{\mathcal{M}}(k, p), \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{1}{F(k^2)} &= 1 - \frac{\alpha q \kappa}{2\pi^2} \int_E \frac{d^3 p}{q^3} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} \frac{1}{F(k^2)} \\ &\quad \times \left\{ q \cdot p + \mathcal{M}(k^2) \mathcal{M}(p^2) \frac{q \cdot k}{k^2} \right\} \\ &\quad + \frac{\alpha q}{2\pi^2} \int_E \frac{d^3 p}{k^2} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} G_F(k, p), \end{aligned} \quad (48)$$

where we have adopted the notation $F(k^2) \equiv F(k^2, \Lambda^2)$ and the same for \mathcal{M} . Moreover, the electron-photon vertex form factors contribute to the gap equation via the scalar functions $G_{\mathcal{M}}$ and G_F :

$$\begin{aligned} qG_{\mathcal{M}}(k, p) &= (3 - \chi) \mathcal{M}(p^2) \lambda_1 + \left[t^2 - \frac{(k^2 - p^2)^2}{q^2} \chi \right] \mathcal{M}(p^2) \lambda_2 - \left[t \cdot p - \frac{(q \cdot p)(k^2 - p^2)}{q^2} \chi \right] \lambda_3 \\ &\quad + \nabla(k, p) \tau_1 + 2\nabla(k, p) \mathcal{M}(p^2) \tau_2 + 2q^2 \mathcal{M}(p^2) \tau_3 - 2[(k^2 - p^2)(q \cdot p) + \nabla(k, p)] \tau_4 \\ &\quad - 2(q \cdot p) \tau_5 - 2(k^2 - p^2) \mathcal{M}(p^2) \tau_6 - [(k^2 - p^2)(t \cdot p) - \nabla(k, p)] \tau_7, \end{aligned} \quad (49)$$

$$\begin{aligned} qG_F(k, p) &= [(1 - 3\chi)k \cdot p + 2\chi u(k, p)] \lambda_1 - \left[(k \cdot p) t^2 + 2\nabla(k, p) - \frac{(k \cdot p)(k^2 - p^2)^2}{q^2} \chi \right] \lambda_2 \\ &\quad - \left[t \cdot k - \frac{(q \cdot k)(k^2 - p^2)}{q^2} \chi \right] \mathcal{M}(p^2) \lambda_3 + \nabla(k, p) \mathcal{M}(p^2) \tau_1 - (k^2 + p^2) \nabla(k, p) \tau_2 \\ &\quad + 2(q \cdot k)(q \cdot p) \tau_3 + 2[q^2(t \cdot k) - \nabla(k, p)] \mathcal{M}(p^2) \tau_4 + 2(q \cdot k) \mathcal{M}(p^2) \tau_5 - 2(k^2 - p^2)(k \cdot p) \tau_6 \\ &\quad + [(k^2 - p^2)(t \cdot k) + \nabla(k, p)] \mathcal{M}(p^2) \tau_7 + \nabla(k, p) \tau_8, \end{aligned} \quad (50)$$

where we have used the simplifying notation $\lambda_i \equiv \lambda_i(k, p)$ and $\tau_i \equiv \tau_i(k, p)$. We have also defined

$$\nabla(k, p) = k^2 p^2 - (k \cdot p)^2, \quad (51)$$

$$u(k, p) = 2k \cdot p - \frac{\nabla(k, p)}{q^2}. \quad (52)$$

We are now in a position to go ahead and solve these coupled equations for any given vertex *ansatz*. We take up this task in perturbative and nonperturbative realms in the next section.

VI. SOLVING THE GAP EQUATION

In order to have a better intuitive grasp over the solutions of the gap equation, we start from perturbation theory. It is only natural to demand that any physically acceptable nonperturbative solution must reduce to its perturbative counterpart in the weak coupling regime.

A. Perturbative solution

Note that in the weak coupling regime, the gap equation is expected to reproduce the perturbative result for the electron propagator. In the chiral limit ($m_0 = 0$), the leading logarithmic expansion for the wave function renormalization in quenched RQED at one-loop order reads

$$F(k^2) = 1 + \frac{\alpha Q}{\pi} (\kappa - \phi) \log\left(\frac{k^2}{\Lambda^2}\right) + \frac{2}{3} \frac{\alpha Q}{\pi} \phi, \quad (53)$$

where the last term in the above Eq. (53), parametrized by the constant $\phi = \chi - 1/3$, spoils the power law behaviour of $F(k^2)$, Eq. (43). This term is spurious. It does not appear if we work in the dimensional regularization scheme. Just like in QED₄, its presence owes itself to the use of the cutoff regulator which we cannot avoid for the nonperturbative treatment presented in the next section. The good news is that just like in QED₄, we can cure the inadequacy of the cutoff regulator by an appropriate use of the WFGTI which ensures this spurious term is formally removed. It is achieved by redefining χ and κ as prescribed through Eqs. (44), (45). To clarify how to achieve this, we focus on the survey of two special cases:

Case 1: Let us choose not to apply the WFGTI in any term of the kernel of the gap equation. It is equivalent to taking $\eta = \kappa$ in Eqs. (44), (45). It implies making the following shift:

$$\chi \rightarrow \chi' = (1 - \epsilon_e)(1 - \xi), \quad (54)$$

$$\kappa \rightarrow \kappa' = 0, \quad (55)$$

yielding $\phi = -(\xi - 1/3)/2$.

Case 2: Like the case of QED₄, we can be selective in applying the WFGTI on a redefined longitudinal part of the photon propagator such that $\phi = 0$. According to Eqs. (44), (45), it entails making the shift

$$\chi \rightarrow \tilde{\chi} = 1/3, \quad (56)$$

$$\kappa \rightarrow \tilde{\kappa} = \epsilon_e - 2/3 + \xi(1 - \epsilon_e). \quad (57)$$

In this second case, the wave function renormalization exhibits the power law behavior of Eq. (43) in the leading logarithmic approximation with $\beta = \alpha(\xi - 1/3)/(4\pi)$ as stated before.

It is also worth noticing that for both the cases considered, we have

$$\frac{1}{F(k^2)} - \frac{1}{F(p^2)} = -\frac{\alpha}{4\pi} \left(\xi - \frac{1}{3}\right) \log\left(\frac{k^2}{p^2}\right), \quad (58)$$

which proves to be a useful guide to express the one-loop corrections to the electron-photon vertex in terms of the electron dressing functions, as we study in Sec. IV, thus helping us construct a nonperturbative *ansatz* for the electron-photon vertex.

So far, we have shown that ensuring the MR for the massless electron propagator, i.e., the power law behavior of Eq. (43), at the one-loop level only requires a specific shifting of χ and κ , cf. Eqs. (56), (57). However, two-loop and higher order perturbative expansions of the electron propagator and its gauge covariance will be sensitive to the choice of the vertex *ansatz*. For the sake of comparison, we compute the two-loop contributions to the electron dressing functions for the bare vertex ($\Gamma_\mu = \gamma_\mu$) and the BC-vertex, Eqs. (12), (13). For this purpose, we first compute the one-loop corrections to F and \mathcal{M} from Eqs. (47), (48) for the case of a small mass (i.e., we neglect squared mass terms). The result is used recursively in the gap equation in order to compute two-loop contributions. Up to leading logarithmic terms, we arrive at the following expressions:

$$F(k^2) = 1 + \beta \log\left(\frac{k^2}{\Lambda^2}\right) + \frac{1}{2} \Phi_F \beta^2 \log^2\left(\frac{k^2}{\Lambda^2}\right), \quad (59)$$

$$\frac{\mathcal{M}(k^2)}{m_0} = 1 + \lambda \log\left(\frac{\Lambda^2}{k^2}\right) + \frac{1}{2} \Phi_{\mathcal{M}} \lambda^2 \log^2\left(\frac{\Lambda^2}{k^2}\right), \quad (60)$$

where $\beta = \alpha Q(\kappa - \chi + 1/3)/\pi$ in consonance with the one-loop expansion for F , Eq. (53), and $\lambda = 8\alpha Q/(3\pi)$. Moreover, for the bare vertex we have

$$\Phi_F = (\kappa + \chi - 1/3)/(\kappa - \chi + 1/3),$$

$$\Phi_{\mathcal{M}} = \{23 + 6(\kappa + \chi) - 3(\kappa - \chi)[8 + 3(\kappa - \chi)]\}/32, \quad (61)$$

and for the BC vertex

$$\Phi_F = (\kappa - \chi + 4/3)/(\kappa - \chi + 1/3),$$

$$\Phi_{\mathcal{M}} = \{40 - 3(\kappa - \chi)[1 + 3(\kappa - \chi)]\}/32. \quad (62)$$

It is important to notice from Eqs. (59), (60) that a power law solution for F and \mathcal{M} , cf. Eq. (53), demands $\Phi_F = \Phi_{\mathcal{M}} = 1$. However, there is no shift in χ and κ that ensures this requirement. Moreover, for both bare and BC vertices, the mass function picks up a gauge dependence at

second order in perturbation theory via $\kappa \sim \xi$ in $\Phi_{\mathcal{M}}$. These results show that even the full longitudinal vertex fails to ensure the local gauge covariance of the electron propagator and highlights the necessity of incorporating a transverse component for the electron-photon vertex.

In order to elucidate the analytical structure of the transverse vertex and its impact on the gauge covariance of the electron propagator, it is worth realizing that the leading logarithmic contributions to F and \mathcal{M} at two loops, Eqs. (59), (60), arise from integrals of the form

$$\int_k^\Lambda \frac{dp}{p} = \frac{1}{2} \log\left(\frac{\Lambda^2}{k^2}\right),$$

$$\int_k^\Lambda \frac{dp}{p} \log\left(\frac{p^2}{\Lambda^2}\right) = \frac{1}{4} \log^2\left(\frac{\Lambda^2}{k^2}\right),$$

which in turn correspond to evaluating the integrals in the so-called asymptotic regime where $p^2 \gg k^2 \gg m_0^2$. Since the two-loop contributions to the electron propagator are determined by the one-loop corrections of the electron-photon vertex, in particular its transverse pieces, we reaffirm that not only a proper treatment of the photon propagator and an adequate use of the WFGTI are essential to study the gap equation but an inclusion of a refined transverse vertex is also required to incorporate the constraints of gauge covariance and perturbation theory as advocated before as well.

B. Nonperturbative solution and the DCSB

In order to compute non perturbative solutions of the electron SDE or the gap equation, Eq. (39), it is necessary to choose an *ansatz* for the electron-photon vertex defined through Eqs. (11), (12), (16), (17). We compute dynamically generated massive solutions for the electron propagator in the chiral limit, i.e., when the bare electron mass $m_0 = 0$. In this case, the electron mass generated is entirely dynamical in nature. In particular, we compute $M_E = \mathcal{M}(k^2)$ which can be interpreted as the Euclidean mass. Note that the only mass scale available in quenched RQED is the ultraviolet regulator Λ . The dynamically generated mass M_E is naturally proportional to it. Therefore, we choose to plot M_E/Λ as a function of the electromagnetic coupling α . At $\alpha = \alpha_c$, a DCSB solution emerges, bifurcating away from the perturbative solution $M_E = 0$ which corresponds to the fact that no electron mass is generated at any order in perturbation theory if we start from the bare mass $m_0 = 0$. As α_c separates the phase of massless electrons from that of massive electrons, we expect it to be gauge invariant just as the Euclidean mass itself.¹ Note

¹It is the Minkowski pole mass which should strictly be gauge invariant. However, as M_E lies close to $M(0)$, we would expect it to be at least nearly gauge invariant.

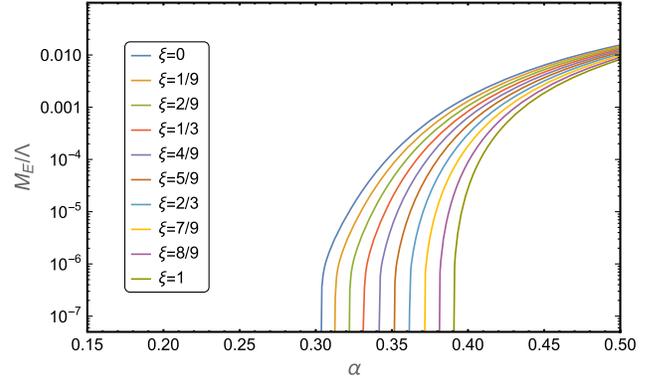


FIG. 3. Dynamically generated dimensionless Euclidean mass M_E/Λ as a function of the electromagnetic coupling α for the bare vertex *ansatz* for different values of the covariant gauge parameter ξ . The gauge dependence of M_E/Λ and α_c is noticeably sizable even in a small interval of ξ , i.e., (0,1).

that we only depict results after the appropriate use of the WFGTI in the kernel of the gap equation. To start with, we compare the results of the following two vertices:

- (i) *Bare*, with $\lambda_1 = 1$ and $\lambda_{2,3} = \tau_{1,2,\dots,8} = 0$.
- (ii) *Our ansatz*, with $\lambda_{1,2,3}$ defined in Eq. (13) and all τ 's set to zero except τ_6 defined in Eq. (37) with $a_6 = -1/2$.

Figure 3 clearly depicts the fact that the bare vertex results in a considerable amount of undesirable gauge dependence in the Euclidean mass M_E and α_c . We now repeat the exercise for our proposed vertex *ansatz* which not only agrees with the asymptotic one-loop result for the full vertex but also guarantees the wave-function renormalization to be MR in the leading logarithmic approximation, see Fig. 4.

The gauge independence is almost magically instated. We find that our choice of the vertex yields $\alpha_c(\xi) \approx 0.212$.

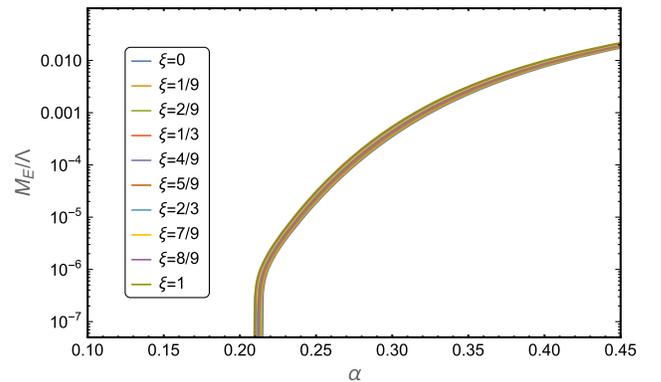


FIG. 4. Dynamically generated dimensionless Euclidean electron mass M_E/Λ as a function of the electromagnetic coupling α for our proposed vertex construction for different values of the covariant gauge parameter ξ . Both M_E/Λ and α_c are now practically gauge independent.

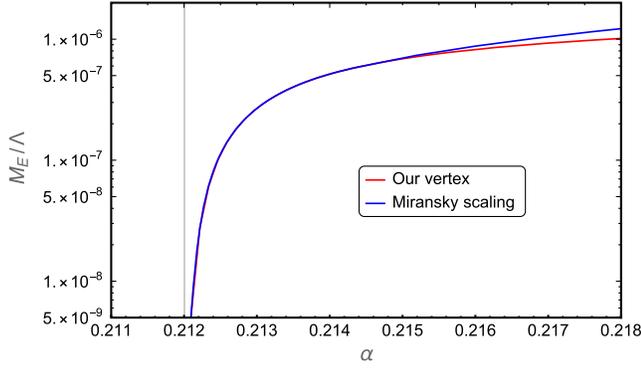


FIG. 5. Miransky scaling using our vertex for $\xi = 1/3$.

We confirm that the α dependence of M_E/Λ in the close vicinity of α_c satisfies Miransky scaling law, see Fig. 5:

$$\frac{M_E}{\Lambda} = \text{Exp} \left[-\frac{\pi\kappa_1}{\sqrt{\alpha/\alpha_c - 1}} + \kappa_2 \right], \quad (63)$$

where $\kappa_1 = 0.051$ and $\kappa_2 = -12.857$.

At this point, it appears worth its while to plot the dynamical mass M_E/Λ as a function of the covariant gauge parameter ξ , Fig. 6. We choose $\alpha = 0.35$ merely as a representative value to bring out the stark difference between the two truncation schemes as regards the gauge dependence of this physical observable. It is remarkable to note the potency of our proposal in this connection.

We also plot $\alpha_c(\xi)$ to have a more quantitative insight into its gauge (in)dependence, Fig. 7. Moreover, one might consider it illustrative to also compare the results with the following two vertices occasionally adopted in literature:

- (i) *Central Ball-Chiu* (CBC), with λ_1 defined in Eq. (13) and $\lambda_{2,3} = \tau_{1,2,\dots,8} = 0$.

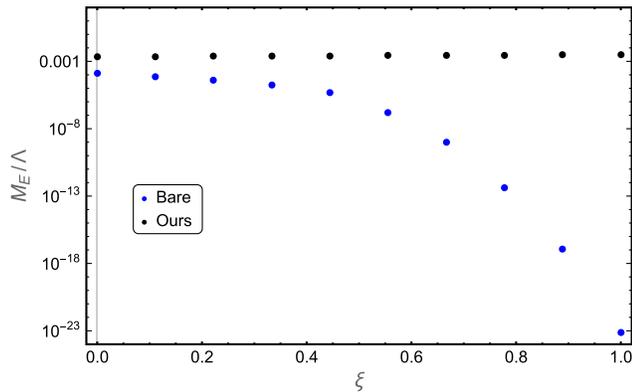


FIG. 6. Dynamically generated dimensional Euclidean mass as a function of the covariant gauge parameter ξ for the bare vertex and our proposed vertex *ansatz*. We choose $\alpha = 0.35$ to draw the plot.

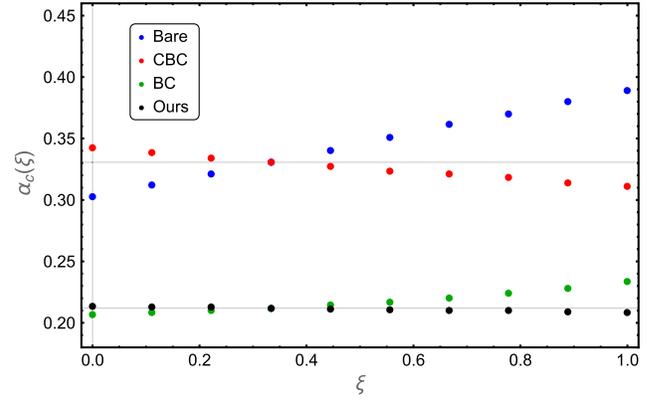


FIG. 7. The critical coupling α_c as a function of the covariant gauge parameter ξ for four different choices of the vertex *ansatz*, mentioned in the text.

- (ii) *Ball-Chiu* (BC), with $\lambda_{1,2,3}$ defined in Eq. (13) and $\tau_{1,2,\dots,8} = 0$.

Notably the bare vertex and the CBC-vertex perform almost equally badly. The BC vertex reduces the gauge dependence. However, our proposal renders α_c practically gauge independent. These observations naturally lead us to conclude the article on a positive note.

VII. CONCLUSIONS AND PERSPECTIVES

It is quite satisfactory to observe that a truncation of the infinite tower of SDEs which respects key features of the underlying quantum field theory of RQED, namely, its discrete symmetries, matching with perturbation theory in the domain of weak coupling and the MR of the massless electron propagator expectedly leads to practically gauge invariant results, within the interval (0,1) of the gauge parameter ξ for the observables under study, i.e., the dynamically generated Euclidean mass and the critical coupling which marks the onslaught of DCSB. What is remarkable is that only one transverse form factor $\tau_6(k, p)$ suffices to construct our vertex *ansatz* and to achieve the goal we set ourselves. Having constructed a reliable truncation of the infinite tower of the SDEs, we believe we are now in a pole position to study the physical properties of graphene and other relevant materials of interest within this formalism. This is for the future.

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- [1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, *Nature (London)* **438**, 197 (2005).
- [2] Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, *Nature (London)* **438**, 201 (2005).
- [3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, *Rev. Mod. Phys.* **81**, 109 (2009).
- [4] A. K. Geim and K. S. Novoselov, *Nat. Mater.* **6**, 183 (2007).
- [5] V. P. Gusynin, S. G. Sharapov, and J. P. Carbotte, *Int. J. Mod. Phys. B* **21**, 4611 (2007).
- [6] A. Cortijo and M. A. H. Vozmediano, *Eur. Phys. J. Special Topics* **148**, 83 (2007).
- [7] M. A. H. Vozmediano, F. de Juan, and A. Cortijo, *J. Phys. Conf. Ser.* **129**, 012001 (2008).
- [8] M. Vozmediano, M. Katsnelson, and F. Guinea, *Phys. Rep.* **496**, 109 (2010).
- [9] M. I. Katsnelson, K. S. Novoselov, and A. K. Geim, *Nat. Phys.* **2**, 620 (2006).
- [10] M. I. Katsnelson, *Eur. Phys. J. B* **51**, 157 (2006).
- [11] G. W. Semenoff, *Phys. Rev. Lett.* **53**, 2449 (1984).
- [12] F. D. M. Haldane, *Phys. Rev. Lett.* **61**, 2015 (1988).
- [13] V. P. Gusynin and S. G. Sharapov, *Phys. Rev. Lett.* **95**, 146801 (2005).
- [14] N. Dorey and N. Mavromatos, *Nucl. Phys.* **B386**, 614 (1992).
- [15] M. Franz and Z. Tešanović, *Phys. Rev. Lett.* **87**, 257003 (2001).
- [16] I. F. Herbut, *Phys. Rev. B* **66**, 094504 (2002).
- [17] M. Franz, Z. Tešanović, and O. Vafek, *Phys. Rev. B* **66**, 054535 (2002).
- [18] K. Farakos and N. E. Mavromatos, *Mod. Phys. Lett. A* **13**, 1019 (1998).
- [19] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, *Nucl. Phys.* **A803**, 227 (2008).
- [20] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, *Phys. Rev. D* **78**, 074033 (2008).
- [21] M. Abdallah *et al.* (STAR Collaboration), *Phys. Rev. C* **105**, 014901 (2022).
- [22] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikoscic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, *Nat. Phys.* **12**, 550 (2016).
- [23] C.-Z. Li, L.-X. Wang, H. Liu, J. Wang, Z.-M. Liao, and D.-P. Yu, *Nat. Commun.* **6**, 10137 (2015).
- [24] X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang, X. Dai, and G. Chen, *Phys. Rev. X* **5**, 031023 (2015).
- [25] Z. Wang, Y. Zheng, Z. Shen, Y. Lu, H. Fang, F. Sheng, Y. Zhou, X. Yang, Y. Li, C. Feng, and Z.-A. Xu, *Phys. Rev. B* **93**, 121112 (2016).
- [26] F. Arnold *et al.*, *Nat. Commun.* **7**, 11615 (2016).
- [27] C. e. a. Zhang, *Nat. Commun.* **8**, 13741 (2017).
- [28] A. J. Mizher, S. Hernandez-Ortiz, A. Raya, and C. Villavicencio, *Eur. Phys. J. C* **78**, 912 (2018).
- [29] J. A. C. Olivares, L. Albino, A. J. Mizher, and A. Raya, *Phys. Rev. D* **102**, 096023 (2020).
- [30] D. Dudal, F. Matusalem, A. J. Mizher, A. R. Rocha, and C. Villavicencio, *Sci. Rep.* **12**, 5439 (2022).
- [31] E. Marino, *Nucl. Phys.* **B408**, 551 (1993).
- [32] E. C. Marino, *Quantum Field Theory Approach to Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2017).
- [33] R. L. P. G. do Amaral and E. C. Marino, *J. Phys. A* **25**, 5183 (1992).
- [34] E. C. Marino, L. O. Nascimento, V. S. Alves, and C. Morais Smith, *Phys. Rev. D* **90**, 105003 (2014).
- [35] E. V. Gorbar, V. P. Gusynin, and V. A. Miransky, *Phys. Rev. D* **64**, 105028 (2001).
- [36] J. González, F. Guinea, and M. Vozmediano, *Nucl. Phys.* **B424**, 595 (1994).
- [37] J. González, F. Guinea, and M. A. H. Vozmediano, *Phys. Rev. Lett.* **77**, 3589 (1996).
- [38] J. González, F. Guinea, and M. A. H. Vozmediano, *Phys. Rev. B* **59**, R2474 (1999).
- [39] J. González, F. Guinea, and M. A. H. Vozmediano, *Phys. Rev. B* **63**, 134421 (2001).
- [40] A. V. Kotikov and S. Teber, *Phys. Rev. D* **89**, 065038 (2014).
- [41] D. Dudal, A. J. Mizher, and P. Pais, *Phys. Rev. D* **99**, 045017 (2019).
- [42] V. S. Alves, W. S. Elias, L. O. Nascimento, V. Juričić, and F. Peña, *Phys. Rev. D* **87**, 125002 (2013).
- [43] L. O. Nascimento, V. S. Alves, F. Peña, C. M. Smith, and E. Marino, *Phys. Rev. D* **92**, 025018 (2015).
- [44] Jean Báez Cuevas, A. Raya, and J. C. Rojas, *Phys. Rev. D* **102**, 056020 (2020).
- [45] M. E. Carrington, *Phys. Rev. B* **99**, 115432 (2019).
- [46] G. C. Magalhães, V. S. Alves, E. C. Marino, and L. O. Nascimento, *Phys. Rev. D* **101**, 116005 (2020).
- [47] P. I. C. Caneda and G. Menezes, *Phys. Rev. D* **103**, 065010 (2021).
- [48] J. A. C. Olivares, A. J. Mizher, and A. Raya, *Rev. Mex. Fis.* **68**, 040101 (2022).
- [49] D. C. Curtis and M. R. Pennington, *Phys. Rev. D* **42**, 4165 (1990).
- [50] A. Bashir and M. R. Pennington, *Phys. Rev. D* **50**, 7679 (1994).
- [51] A. Bashir and M. R. Pennington, *Phys. Rev. D* **53**, 4694 (1996).
- [52] A. Bashir, R. Bermudez, L. Chang, and C. D. Roberts, *Phys. Rev. C* **85**, 045205 (2012).
- [53] L. Albino, A. Bashir, L. X. G. Guerrero, B. E. Bennich, and E. Rojas, *Phys. Rev. D* **100**, 054028 (2019).
- [54] A. Bashir, A. Huet, and A. Raya, *Phys. Rev. D* **66**, 025029 (2002).
- [55] A. Bashir and A. Raya, *Nucl. Phys.* **B709**, 307 (2005).
- [56] A. Bashir and A. Raya, *Few Body Syst.* **41**, 185 (2007).
- [57] A. Bashir, A. Raya, S. Sanchez-Madrigal, and C. D. Roberts, *Few Body Syst.* **46**, 229 (2009).
- [58] J. S. Ball and T.-W. Chiu, *Phys. Rev. D* **22**, 2542 (1980).
- [59] A. Kizilersu, M. Reenders, and M. R. Pennington, *Phys. Rev. D* **52**, 1242 (1995).
- [60] A. I. Davydychev, P. Osland, and L. Saks, *Phys. Rev. D* **63**, 014022 (2000).
- [61] R. Bermudez, L. Albino, L. X. Gutiérrez-Guerrero, M. E. Tejada-Yeomans, and A. Bashir, *Phys. Rev. D* **95**, 034041 (2017).
- [62] A. Bashir, A. Kizilersu, and M. R. Pennington, arXiv: hep-ph/9907418.

- [63] A. Bashir, A. Kizilersu, and M. R. Pennington, *Phys. Rev. D* **62**, 085002 (2000).
- [64] A. Bashir, A. Raya, and S. Sanchez-Madrigal, *Phys. Rev. D* **84**, 036013 (2011).
- [65] S. Teber, *Phys. Rev. D* **86**, 025005 (2012).
- [66] D. Dudal, A. J. Mizher, and P. Pais, *Phys. Rev. D* **98**, 065008 (2018).
- [67] L. Albino, A. Bashir, B. El-Bennich, E. Rojas, F. E. Serna, and R. C. da Silveira, *J. High Energy Phys.* **11** (2021) 196.
- [68] D. Curtis and M. Pennington, *Phys. Rev. D* **44**, 536 (1991).
- [69] A. Bashir, A. Kizilersu, and M. R. Pennington, *Phys. Rev. D* **57**, 1242 (1998).
- [70] Z.-h. Dong, H. J. Munczek, and C. D. Roberts, *Phys. Lett. B* **333**, 536 (1994).