Complexity of pure and mixed qubit geodesic paths on curved manifolds

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It is known that mixed quantum states are highly entropic states of imperfect knowledge (i.e., incomplete information) about a quantum system, while pure quantum states are states of perfect knowledge (i.e., complete information) with vanishing von Neumann entropy. In this paper, we propose an information geometric theoretical construct to describe and, to a certain extent, understand the complex behavior of evolutions of quantum systems in pure and mixed states. The comparative analysis is probabilistic in nature, it uses a complexity measure that relies on a temporal averaging procedure along with a long-time limit, and is limited to analyzing expected geodesic evolutions on the underlying manifolds. More specifically, we study the complexity of geodesic paths on the manifolds of single-qubit pure and mixed quantum states equipped with the Fubini-Study metric and the Sjöqvist metric, respectively. We analytically show that the evolution of mixed quantum states in the Bloch ball is more complex than the evolution of pure states on the Bloch sphere. We also verify that the ranking based on our proposed measure of complexity, a quantity that represents the asymptotic temporal behavior of an averaged volume of the region explored on the manifold during the evolution of the systems, agrees with the geodesic length-based ranking. Finally, focusing on geodesic lengths and curvature properties in manifolds of mixed quantum states, we observed a softening of the complexity on the Bures manifold compared to the Sjöqvist manifold.

I. INTRODUCTION

We divide the Introduction in three parts to better motivate the selection of our goals along with their physical relevance. In the first part, we highlight the use of geometric concepts originally introduced in quantum computing and later borrowed by high energy physicists to describe and, to a certain extent, understand the behavior of black holes. In particular, we emphasize the geometric characterization of some complexity notions, including gate complexity and state complexity. In the second part, we outline several distinguishing features that characterize the physics of systems specified by pure and mixed quantum states. Neither geometry nor complexity are mentioned in this second part. In the third part, we finally describe our main objectives.

A. Geometry in quantum computing and high energy physics

Geometry plays a fundamental role in science [1,2], including quantum computing and high energy physics. In Ref. [3], Nielsen and collaborators used methods of Riemannian geometry to propose a way of finding efficient quantum circuits capable of performing certain computational tasks. They proposed a geometric measure of quantum algorithm complexity for quantum circuits constructed with unitary gates. Their formalism led to a geometric continuous-time version of the discrete gate complexity, a measure of complexity quantifying how difficult it is to build a unitary operator [4,5]. In such geometric context, finding optimal quantum circuits is equivalent to finding the shortest path between two points in a certain curved geometry. Essentially, one introduces a Riemannian metric in the space of unitary operators acting on a given number of qubits. The metric quantifies how difficult it is to implement a given quantum computational task. Then, the distance induced by the metric in the space of unitary operators is employed as a measure of the complexity of the quantum operation. In addition to gate complexity, one can also introduce in quantum information science the concept of quantum computational complexity of a state, a measure quantifying how difficult it is to build a unitary transformation that transforms the reference state to the target state [4]. Geometric concepts (including actions, path lengths, volumes, and complexity) play a fundamental role in high energy physics as well. For instance, quantum computational complexity measures of geometric origin appear to play a fundamental role in encoding properties of the interiors of black holes [6].

In Refs. [7,8], it was shown that the quantum computational complexity of the dual quantum state is proportional to the spatial volume of the Einstein-Rosen bridge (i.e., a structure linking two sides of the Penrose diagram of an eternal anti–de Sitter black hole). In Refs. [9,10], it was

argued that the quantum computational complexity of a holographic state is proportional to the action of a certain spacetime region termed Wheeler-DeWitt patch. For very insightful applications of Nielsen's geometric approach to quantum computational complexity of states and gates in the single-qubit and multiqubit scenarios of special relevance in high energy physics, we refer to Refs. [11,12], respectively. The analysis in Ref. [11] is rather illuminating because it clearly shows the effects of replacing a non-deformed Bloch sphere equipped with the usual Fubini-Study metric with a deformed Bloch sphere with a new metric that does not treat all directions in the tangent space in a similar manner. Indeed, in Nielsen's geometric approach, the single-qubit Hilbert space is equipped with a metric that stretches directions that are difficult to move in, assigning them a large distance. Two main consequences of this new metric can be summarized as follows: First, geodesics are no longer generated by time-independent Hamiltonians. Second, suitable choices of the anisotropy penalty factors specifying the new metric can lead to spaces with negative sectional curvature. This, in turn, is responsible for the chaotic growth of perturbations (i.e., exponential maximal complexity). For a work focusing on the connection between a geometric measure of quantum computational complexity and negative curvature, we refer to Ref. [13].

The work in Ref. [12] attracts great interest for several reasons, including the fact that it addresses the issue of ergodicity of geodesics on manifolds of negative curvature. This is especially important in view of a potential application of thermodynamical arguments to complexity evolution. As previously pointed out, Nielsen's approach to quantum computation defines a geometric measure on the space of unitary operators. In Ref. [14], instead, the Fubini-Study metric is used to define a geometry on the space of states to propose a complexity measure assigned to a target state. This complexity is the minimal distance as measured by the Fubini-Study metric among all parametrized curves on the space of states that connect the reference state to the desired target state. Within this approach, the Fubini-Study metric accounts for the complexity by keeping track of the changes of the state (by means of applications of unitary operations) throughout the preparation of the target state. In Ref. [15], a notion of mixed state complexity is extended to impure quantum states by replacing the Fubini-Study metric with the Bures metric (or, alternatively, the quantum Fisher information metric) and, at the same time, extending the nature of quantum transformations acting on the state to nonunitary operations. Finally, following what happens for pure states, the complexity of mixed states is identified with the (Bures) length of the geodesic connecting the reference and target mixed states. To a certain extent and to the best of our knowledge, given the novelty of the introduction of the concept of mixed state complexity, no comparative analysis exists in the literature between complexity behaviors associated with physical systems specified by pure and mixed states. We intend to cover this point in this paper.

B. Pure and mixed quantum states

In quantum information science, when one has complete knowledge about a quantum system, one can use a pure state to describe it. However, complete knowledge is only available in limiting ideal (noiseless) scenarios (i.e., isolated/closed quantum systems). In practice, one only has partial knowledge about a quantum system. Indeed, small errors may happen in the preparation, evolution, or measurement of the system due to imperfect devices or to (external) coupling with other degrees of freedom outside of the system that one is controlling. In these realistic (noisy) cases (i.e., open quantum systems), quantum systems are described by mixed states. These states are specified by classical probability distributions over pure states and are used to represent our probabilistic ignorance of a pure state. The density operator formalism is a very powerful mathematical tool for incorporating a lack of complete knowledge about a quantum system. Within this formalism, the "quantumness" of the system resides in the off-diagonal entries of the density matrix. These are interference terms between the pure states that specify the mixture that defines the mixed state. A particular measure of noisiness of a quantum state is the purity $P(\rho) \overset{\text{def}}{=} \text{Tr}(\rho^2)$ of a density operator ρ. The purity of a pure state is equal to one, and the purity of a mixed state is strictly less than one with $1/N \le P(\rho) \le 1$ for an $N \times N$ density matrix. The departure of a system from a pure state can also be quantified by means of the von Neumann entropy $S_{vN}(\rho) \overset{\text{def}}{=} - \text{Tr}(\rho \log \rho)$. This quantity specifies the degree of mixing of the state describing a given finite-dimensional quantum system. For a pure state, the van Neumann entropy vanishes. Instead, for a maximally mixed state characterized by the complete absence of off-diagonal entries in the density matrix (thus, describing something noninterfering and seemingly classical), the von Neumann is maximal and equals $\log(2)$ for a qubit system.

To the best of our knowledge, there does not exist any comparative geometric analysis of the complexity of pure and mixed states in the literature. From an intuitive standpoint, there are several reasons why one expects mixed states to be more complex than pure states:

(i) Mixed states are generally used to describe highly entropic systems that can exhibit a temperature higher than the one specifying systems in a pure state. In statistical mechanics, for instance, a physical system at thermal equilibrium is described by a thermal (Gibbs) state [16]. The Gibbs state is a mixed state with a well-defined finite temperature value. However, at zero temperature [i.e., $\beta \overset{\text{def}}{=} (k_B T)^{-1} \to \infty$ with k_B denoting the Boltzmann constant), the system is in a pure state. In this limiting case, the

density matrix has every element zero except for a single element on the diagonal. At infinite temperature (i.e., $\beta \to 0$), instead, the system is in a maximally mixed state (i.e., a mixture of pure states with equal statistical weights). For example, consider a spin-1/2 particle in a stationary and uniform magnetic field B_0 along the z direction. The Hamiltonian of the system can be written as $H \stackrel{\text{def}}{=} (\hbar\omega_0/2)\sigma_z$ with $\omega_0 \stackrel{\text{def}}{=} (eB_0)/m$. Clearly, e and m denote the electric charge and the mass of the electron, respectively. Moreover, $\hbar \stackrel{\text{def}}{=} h/(2\pi)$ is the reduced Planck constant and, finally, σ_z is the Pauli phase flip operator. At thermal equilibrium, the density matrix of the system is given by $\rho_{\text{TE}}(\beta) \stackrel{\text{def}}{=} e^{-\beta\text{H}}/\text{Tr}(e^{-\beta\text{H}})$. A simple calculation shows that $\rho_{\text{TE}}(\beta)$ becomes a maximally mixed (or pure) state as T approaches infinity (or zero). For a definition of temperature of arbitrary quantum states, beyond thermal (Gibbs) states used for physical systems at thermal equilibrium, we refer to Ref. [17]. The temperature quantifies the ability of a quantum system to cool down or heat up a thermal environment in Ref. [17]. Finally, for a scheme to measure the temperature of individual pure quantum states by means of quantum interference, we refer to Ref. [18].

(ii) Systems in mixed states are less quantum (or, alternatively, more classical) than systems in pure states. In Ref. [19] it is proven that entanglement, a quintessential quantum property of physical systems, is typical of pure states, while separability is connected with quantum mixtures. For intriguing connections of geometric flavor among purity, separability, and complex behavior in quantum scattering processes, we refer to Refs. [20,21]. Instead, for possible justifications of why chaoticity viewed as temporal complexity is softer in quantum systems compared with classical systems, we hint at Refs. [22–26]. Furthermore, it is known that the existence of speed limits is not something peculiar to quantum systems [27–29]. Indeed, there are speed limits for classical systems as well [30–32]. In Ref. [32], it was shown that the quantum counterpart of the classical speed limits derived in Ref. [30] are obtained by quantum systems specified by density operators describing states that become more and more mixed as $\hbar$ approaches zero.

(iii) Mixed quantum states can undergo a richer variety of transformations compared to pure states [33]. In open system dynamics, one needs to consider general nonunitary quantum evolutions and have the freedom to choose a variety of distance measures between quantum states. Decoherence and measurements are examples of noncontrollable and controllable nonunitary processes, respectively. Quantum channels, for instance, provide us with a formalism for discussing decoherence, the nonunitary evolution of pure states into mixed states [34]. In conventional formulations of quantum mechanics, instead, pure states can only be connected in a unitary fashion. Moreover, the choice of geometric distance measures between pure states is more restrained than that between impure states.

(iv) Mixed states evolutions can exhibit higher speed values than the ones of pure state temporal changes. In Ref. [35], it is shown that the time optimal mixed state evolution can be faster than the time optimal pure state evolution. In Ref. [29], it is demonstrated that non-Markovian (i.e., memory) effects can speed up nonunitary quantum evolutions of arbitrarily driven open quantum systems. In Ref. [36], it is pointed out that finding the optimal unitary for mixed target states is more challenging than for pure target states.

(v) Mixed qubit states have three local degrees of freedom, while pure qubit states only have two local degrees of freedom. From a pure geometric perspective, it is reasonable to expect that mixed states are more complex than pure states [37]. For instance, unlike what happens in optimal-speed unitary evolutions of systems in pure states, tight evolutions of closed quantum systems in mixed states are typically generated by time-varying Hamiltonians [37]. We refer to Table I for a schematic description of physical systems in pure and mixed quantum states in terms of purity, von Neumann entropy, temperature, and entanglement.

Given the lack of a geometric comparative analysis between the complex behaviors exhibited by physical quantum systems specified by pure and mixed quantum states and, in addition, given the variety of distinguishing physical features that characterize the evolution of systems specified by pure and mixed quantum states, we intend to capture here the complexity of these evolutions from a

TABLE I. Schematic description of physical systems in pure and mixed quantum states in terms of purity, von Neumann entropy, temperature, and entanglement.

Knowledge of system	Type of state	Purity	Von Neumann entropy	Temperature	Entanglement
Complete	Pure	Maximal	Minimal	Low	Typical
Partial	Mixed	Not maximal	Not minimal	High	Less typical

geometric standpoint and provide a geometrical picture of these physical differences.

C. Our goals

In this paper, we aim to provide a comparative information geometric analysis of the complexity of geodesic paths of pure and mixed quantum states on the Bloch sphere and in the Bloch ball, respectively. Our investigation is partially inspired by the above mentioned geometrically flavored investigations. Furthermore, it is motivated by our curiosity concerning the possibility of describing and, to a certain extent, understanding from a geometric viewpoint the previously mentioned fingerprints of a greater degree of complexity of mixed quantum states. Finally, it relies on our insights into the concepts of complexity [38], geometric formulations of optimal-speed Hamiltonian evolutions on the Bloch sphere [39,40], and the role played by the thermodynamic length and divergence (or, alternatively, action) in studying the complexity of minimum entropy production probability paths in quantum mechanical evolutions [41,42]. The main questions that we address in this paper can be summarized as follows:

  (i) Can we gain physical insights by identifying the distinguishing features that characterize the geometry along evolution of pure and mixed quantum states?

 (ii) Expressing the concept of complexity in terms of volumes of explored regions on curved manifolds, do geodesic paths on manifolds of mixed quantum states exhibit a higher degree of complexity compared to the complexity of geodesic paths emerging from the geometry along the evolution of pure quantum states?

(iii) Does the choice of the metric on the space of mixed quantum states have crucial observable physical effects on the complexity of the underlying geodesic paths?

The layout of the rest of the paper is as follows. In Sec. II, we introduce our proposed measure of complexity of geodesic paths on curved manifolds. In Sec. III, we introduce the geodesic paths on manifolds of pure and mixed states emerging from the Fubini-Study and the Sjöqvist metrics, respectively. In Sec. IV, we study the complexity of the geodesic paths expressed in terms of temporal averages of volume regions explored by the physical systems during the quantum evolutions. In Sec. V, we include several physics considerations, including comments on the concepts of metric, path length, and curvature employed in our analysis. These comments also help to emphasize the physical significance of our proposed complexity measure. In Sec. VI, we present our final remarks. Finally, several technical details, including a comparative analysis between the Sjöqvist and the Bures metrics for mixed quantum states, appear in Appendixes A, B, C, D, E, and F.

II. INFORMATION GEOMETRIC COMPLEXITY

In this section, we present the notion of information geometric complexity (IGC) along with the concept of information geometric entropy (IGE). These quantities will help quantify how complex are the evolutions of pure and mixed states. Before introducing formal details, let us emphasize at the outset that the IGC is essentially the exponential of the IGE. The latter, in turn, is the logarithm of the volume of the parametric region explored by the system during its evolution from an initial to a final configuration on the underlying manifold. The IGE is an indicator of complexity that was initially proposed in Ref. [43] in the framework of the information geometric approach to chaos (IGAC) [44]. For clarity, we mention in this paper only the necessary information on the IGAC. However, we recommend the interested reader to consider the compact discussions on the IGAC in Refs. [45,46].

In what follows, we begin by presenting the IGE in its original classical setting characterized by probability density functions. Obviously, when transitioning from classical to quantum settings, parametrized families of probability distributions are replaced by families of parametrized density operators.

Assume that N-real valued variables $(\xi^1, \dots, \xi^N)$ parametrize the points $\{p(x;\xi)\}$ of an N-dimensional curved statistical manifold $\mathcal{M}_s$,

$$\mathcal{M}_s \overset{\text{def}}{=} \{p(x;\xi): \xi \overset{\text{def}}{=} (\xi^1, \dots, \xi^N) \in \mathcal{D}_\xi^{\text{tot}}\}. \qquad (1)$$

In addition, assume that the microvariables x specifying the probability distributions $\{p(x;\xi)\}$ are elements of the (continuous) microspace $\mathcal{X}$ while the macrovariables ξ belong to the parameter space $\mathcal{D}_\xi^{\text{tot}}$ defined as

$$\mathcal{D}_\xi^{\text{tot}} \overset{\text{def}}{=} (\mathcal{I}_{\xi^1} \otimes \mathcal{I}_{\xi^2} \dots \otimes \mathcal{I}_{\xi^N}) \subseteq \mathbb{R}^N. \qquad (2)$$

Note that $\mathcal{I}_{\xi^j}$ in Eq. (2) is a subset of $\mathbb{R}^N$ and characterizes the range of acceptable values for the statistical macrovariables ξ^k with $1 \le k \le N$. Within the IGAC framework, it is argued that the IGE is a good measure of temporal complexity of geodesic paths on $\mathcal{M}_s$. The IGE is given by

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \overset{\text{def}}{=} \log \widetilde{\text{vol}}[\mathcal{D}_\xi(\tau)], \qquad (3)$$

with the average dynamical statistical volume $\widetilde{\text{vol}}[\mathcal{D}_\xi(\tau)]$ being defined as

$$\widetilde{\text{vol}}[\mathcal{D}_\xi(\tau)] \overset{\text{def}}{=} \frac{1}{\tau} \int_0^\tau \text{vol}[\mathcal{D}_\xi(\tau')]d\tau'. \qquad (4)$$

Note that $\mathcal{D}_\xi(\tau')$ in Eq. (4) is an N-dimensional subspace of $\mathcal{D}_\xi^{\text{tot}} \subseteq \mathbb{R}^N$ whose elements $\{\xi\}$ with $\xi \overset{\text{def}}{=} (\xi^1, \dots, \xi^N)$ satisfy $\xi^k(\tau_0) \le \xi^k \le \xi^j(\tau_0 + \tau')$ with τ_0 denoting the initial value

taken by the affine parameter τ' that characterizes the geodesics on $\mathcal{M}_s$ as will be described in more detail shortly. In Eq. (4), the temporal average operation is denoted with the tilde symbol. We also emphasize that two sequential integration procedures define $\widetilde{\text{vol}}[\mathcal{D}_{\boldsymbol{\xi}}(\tau)]$ in Eq. (4). The first integration is defined on the explored parameter space $\mathcal{D}_{\boldsymbol{\xi}}(\tau')$ and leads to $\text{vol}[\mathcal{D}_{\boldsymbol{\xi}}(\tau')]$. Then, the second integration describes a temporal averaging procedure, is performed over the duration τ of the evolution on $\mathcal{M}_s$, and finally yields $\widetilde{\text{vol}}[\mathcal{D}_{\boldsymbol{\xi}}(\tau)]$. The volume $\text{vol}[\mathcal{D}_{\boldsymbol{\xi}}(\tau')]$ in the right-hand side of Eq. (4) is the volume of an extended region on $\mathcal{M}_s$ and is given by

$$\text{vol}[\mathcal{D}_{\boldsymbol{\xi}}(\tau')] \stackrel{\text{def}}{=} \int_{\mathcal{D}_{\boldsymbol{\xi}}(\tau')} \rho(\xi^1, ..., \xi^N) d^N\xi. \tag{5}$$

Since we are limiting our present discussion to the IGE in the context of a statistical manifold $\mathcal{M}_s$ of classical probability distributions, $\rho(\xi^1, ..., \xi^N)$ in Eq. (5) is the so-called Fisher density and equals the square root of the determinant $g(\xi)$ of the Fisher-Rao information metric tensor $g_{\mu\nu}^{\text{FR}}(\xi)$, $g^{\text{FR}}(\xi) \stackrel{\text{def}}{=} \det[g_{\mu\nu}^{\text{FR}}(\xi)]$. Therefore, $\rho(\xi^1, ..., \xi^N) \stackrel{\text{def}}{=} \sqrt{g^{\text{FR}}(\xi)}$. Recall that in the continuous microspace setting, $g_{\mu\nu}^{\text{FR}}(\xi)$ is defined as

$$g_{\mu\nu}^{\text{FR}}(\xi) \stackrel{\text{def}}{=} \int p(x|\xi)\partial_\mu \log p(x|\xi)\partial_\nu \log p(x|\xi) dx, \tag{6}$$

with $\partial_\mu \stackrel{\text{def}}{=} \partial/\partial\xi^\mu$. Note that $\text{vol}[\mathcal{D}_{\boldsymbol{\theta}}(\tau')]$ in Eq. (5) assumes a more simple expression for manifolds equipped with metric tensors specified by factorizable determinants,

$$g(\xi) = g(\xi^1, ..., \xi^N) = \prod_{k=1}^{N} g_k(\xi^k). \tag{7}$$

In such a scenario, the IGE in Eq. (3) reduces to

$$\mathcal{S}_{\mathcal{M}_s}(\tau)$$
$$= \log\left\{ \frac{1}{\tau}\int_0^\tau \left[\prod_{k=1}^{N} \left(\int_{\tau_0}^{\tau_0+\tau'} \sqrt{g_k[\xi^k(\eta)]} \frac{d\xi^k}{d\eta} d\eta \right) \right] d\tau' \right\}. \tag{8}$$

We remark the $g(\theta)$ is not factorizable when the microvariables $\{x\}$ are correlated. Therefore, in this case, one is forced to use the general definition of the IGE. We refer to Ref. [47] for a study on the effects of microscopic correlations on the IGE of Gaussian statistical models.

In the IGAC theoretical setting, the leading asymptotic behavior of $\mathcal{S}_{\mathcal{M}_s}(\tau)$ in Eq. (8) characterizes the complexity of the statistical models being analyzed. To this end, we consider the leading asymptotic term in the equation for the IGE,

$$\mathcal{S}_{\mathcal{M}_s}^{\text{asymptotic}}(\tau) \sim \lim_{\tau\to\infty}[\mathcal{S}_{\mathcal{M}_s}(\tau)]. \tag{9}$$

Observe that $\mathcal{D}_{\boldsymbol{\xi}}(\tau')$ specifies the domain of integration that appears in the expression of $\text{vol}[\mathcal{D}_{\boldsymbol{\xi}}(\tau')]$ in Eq. (5), and is defined as

$$\mathcal{D}_{\boldsymbol{\xi}}(\tau') \stackrel{\text{def}}{=} \{\xi : \xi^k(\tau_0) \le \xi^k \le \xi^k(\tau_0 + \tau')\}, \tag{10}$$

where $\tau_0 \le \eta \le \tau_0 + \tau'$ and τ_0 is the initial value of the affine parameter η. In Eq. (10), $\xi^k = \xi^k(\eta)$ satisfies the geodesic equations

$$\frac{d^2\xi^k}{d\eta^2} + \Gamma_{ij}^k \frac{d\xi^i}{d\eta}\frac{d\xi^j}{d\eta} = 0, \tag{11}$$

with Γ_{ik}^j in Eq. (11) being the usual Christoffel connection coefficients,

$$\Gamma_{ij}^k \stackrel{\text{def}}{=} \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}). \tag{12}$$

Note that the elements of $\mathcal{D}_{\boldsymbol{\xi}}(\tau')$ in Eq. (10), an N-dimensional subspace of $\mathcal{D}_{\boldsymbol{\xi}}^{\text{tot}}$, are N-dimensional macrovariables $\{\xi\}$ with components ξ^j bounded by fixed integration limits $\xi^j(\tau_0)$ and $\xi^j(\tau_0 + \tau')$. The temporal functional form of such limits can be determined by integrating the N-coupled nonlinear second order (Ordinary Differential Equations) ODEs in Eq. (11). Having introduced the IGE, we term information geometric complexity the quantity $\mathcal{C}_{\mathcal{M}_s}(\tau)$ given by

$$\mathcal{C}_{\mathcal{M}_s}(\tau) \stackrel{\text{def}}{=} \widetilde{\text{vol}}[\mathcal{D}_{\boldsymbol{\xi}}(\tau)] = e^{\mathcal{S}_{\mathcal{M}_s}(\tau)}. \tag{13}$$

As mentioned earlier, we shall focus on the asymptotic temporal behavior of the IGC as specified by $\mathcal{C}_{\mathcal{M}_s}^{\text{asymptotic}}(\tau) \stackrel{\tau\to\infty}{\sim} e^{\mathcal{S}_{\mathcal{M}_s}(\tau)}$.

The IGC $\mathcal{C}_{\mathcal{M}_s}(\tau)$ can be interpreted by explaining the meaning of the IGE $\mathcal{S}_{\mathcal{M}_s}(\tau)$ in Eq. (3). The IGE is an affine temporal average of the N-fold integral of the Fisher density over geodesics regarded as maximum probability trajectories and, in addition, measures the number of the explored macrostates in $\mathcal{M}_s$. In particular, the IGE at a given instant is the logarithm of the volume of the effective parameter space navigated by the system at that specific instant. The temporal averaging procedure in Eq. (4) is introduced to average out the conceivably very complicated fine details of the probabilistic dynamical description of the system on $\mathcal{M}_s$. Furthermore, the long-time limit in Eq. (9) is used to properly specify the selected dynamical indicators of complexity by neglecting the transient effects which enter the calculation of the expected value of the volume of the effective parameter space. In summary, the IGE provides an asymptotic coarse-grained inferential

characterization of the complex dynamics of a system in the presence of partial knowledge. For further details on the IGC and IGE, we refer to Refs. [38,48,49].

Our discussion has followed the original IGAC setting where we assumed to deal with an underlying continuous microspace yielding a macrospace equipped with a classical Fisher-Rao information metric in its integral form. However, shifting to a discrete microspace leading to a macrospace with a Fisher-Rao information metric expressed in terms of a summation is straightforward,

$$g_{\mu\nu}^{\mathrm{FR}}(\xi) = \sum_{k=1}^{N} \frac{1}{p_k(\xi)} \frac{\partial p_k(\xi)}{\partial \xi^\mu} \frac{\partial p_k(\xi)}{\partial \xi^\nu}. \qquad (14)$$

From Eq. (14), the Fisher-Rao infinitesimal line element ds_{FR}^2 becomes

$$ds_{\mathrm{FR}}^2 = g_{\mu\nu}^{\mathrm{FR}}(\xi) d\xi^\mu d\xi^\nu = \sum_{k=1}^{N} \frac{dp_k^2}{p_k}, \qquad (15)$$

where $dp_k \overset{\text{def}}{=} (\partial_\mu p_k) d\xi^\mu$. Moreover, the parameters $\{\xi^k\}_{1 \le k \le N}$ were originally viewed in the IGAC context as statistical macrovariables emerging, for instance, as suitable expectation values of the microvariables of the physical system in the presence of partial knowledge. However, due to the fact that in principle the IGE can be fully constructed from a geometric standpoint once the infinitesimal line element ds^2 is known, its extension to quantum manifolds of density matrices $\{\rho_\xi(x)\}$ specified by a set of parameters $\{\xi\}$ with $\xi \in \mathcal{D}_\xi^{\mathrm{tot}} \subseteq \mathbb{R}^N$, including experimentally controllable parameters such as temperature and magnetic field intensity, is simple as well. For clarity, note that $\rho_\xi(x) \in \mathcal{M}_s^{(\mathrm{quantum})}$ replaces $p_\xi(x) \overset{\text{def}}{=} p(x;\xi) \in \mathcal{M}_s^{(\mathrm{classical})}$ with $\mathcal{M}_s^{(\mathrm{classical})}$ equal to $\mathcal{M}_s$ in Eq. (1).

Clearly, to provide estimates of the IGE and of the IGC in Eqs. (3) and (13), respectively, we need to first find the geodesic paths on the manifolds. Therefore, in the next section, we present the geodesic paths on the manifolds of pure and mixed states equipped with the Fubini-Study and Sjöqvist metrics, respectively.

III. GEODESIC PATHS

We introduce here the geodesic paths on manifolds of pure and mixed states equipped with the Fubini-Study and the Sjöqvist metrics, respectively.

A. Geodesic paths on the Bloch sphere: The Fubini-Study metric

We begin by discussing geodesic paths on manifolds of pure states equipped with the Fubini-Study metric. In quantum mechanics, it is known that the only Riemannian

metric on the set of rays, up to a constant factor, which is invariant under all unitary transformations is the angle in Hilbert space (also known as the Wootters angle),

$$\theta_{\mathrm{Wootters}}(|\psi_i\rangle, |\psi_f\rangle) \overset{\text{def}}{=} \arccos[|\langle \psi_i|\psi_f\rangle|], \qquad (16)$$

with $|\psi_i\rangle$ and $|\psi_f\rangle$ being two pure states. It is also known that a concept of statistical distance can be defined between different preparations of the same quantum system, or to put it another way, between different rays in the same Hilbert space [50]. This notion of statistical distance is specified completely by the size of statistical fluctuations taking place in measurements prepared to discriminate one state from another. A major finding obtained by Wootters in Ref. [50] was showing that such statistical distance coincides with the usual distance (i.e., angle) between rays. The infinitesimal line element that corresponds to the Hilbert space angle is the so-called Fubini-Study metric $g_{\mu\nu}^{\mathrm{FS}}(\xi)$, the natural metric on the manifold of Hilbert space rays. The physical interpretation of this metric in terms of statistical fluctuations in the outcomes of intrinsically probabilistic quantum measurements that aim at distinguishing one pure state from another is a major result obtained in Ref. [50]. Before introducing the Fubini-Study metric $g_{\mu\nu}^{\mathrm{FS}}(\xi)$ in an explicit manner, we remark that the extension of Wootters's reasoning to the problem of distinguishing mixed quantum states was carried out by Braunstein and Caves in Ref. [51]. In the case of mixed states, the Bures angle θ_{Bures} and the Bures metric $g_{\mu\nu}^{\mathrm{Bures}}(\xi)$ replace the Hilbert space angle $\theta_{\mathrm{Wootters}}$ and the Fubini-Study metric $g_{\mu\nu}^{\mathrm{FS}}$, respectively. The Bures angle represents the length of a geodesic joining two density operators ρ_i and ρ_f and is given by

$$\theta_{\mathrm{Bures}}(\rho_i, \rho_f) \overset{\text{def}}{=} \arccos[\mathrm{F}_B(\rho_i, \rho_f)]. \qquad (17)$$

In Eq. (17), $\mathrm{F}_B(\rho_i, \rho_f)$ is the Bures fidelity defined as

$$\mathrm{F}_B(\rho_i, \rho_f) \overset{\text{def}}{=} \left[\mathrm{Tr}\left(\sqrt{\rho_i^{1/2} \rho_f \rho_i^{1/2}} \right) \right]^2. \qquad (18)$$

For clarity, we point out that using Eqs. (17) and (18), $\theta_{\mathrm{Bures}}(\rho_i, \rho_f)$ equals $\arccos(\sqrt{\langle \psi_i|\rho_f|\psi_i\rangle})$ when $\rho_i \overset{\text{def}}{=} |\psi_i\rangle\langle\psi_i|$. Furthermore, when both ρ_i and ρ_f are pure states, $\theta_{\mathrm{Bures}}(\rho_i, \rho_f)$ reduces to $\theta_{\mathrm{Wootters}}(|\psi_i\rangle, |\psi_f\rangle)$ in Eq. (16). Finally, for completeness, we remark here that the Bures distance $d_{\mathrm{Bures}}(\rho_i, \rho_f)$ is different from the Bures angle in Eq. (17) and is formally defined as

$$d_{\mathrm{Bures}}(\rho_i, \rho_f) \overset{\text{def}}{=} \sqrt{2\left[1 - \sqrt{\mathrm{F}_B(\rho_i, \rho_f)} \right]}. \qquad (19)$$

Returning to the formal introduction of $g_{\mu\nu}^{\mathrm{FS}}(\xi)$, consider two neighboring single qubit pure states $|\psi\rangle$ and $|\bar\psi\rangle$ defined as

$$|\psi\rangle \overset{\text{def}}{=} \sum_{k=0}^{1} \sqrt{p_k}\, e^{i\phi_k} |e_k\rangle,$$

$$|\bar{\psi}\rangle \overset{\text{def}}{=} \sum_{k=0}^{1} \sqrt{p_k + dp_k}\, e^{i(\phi_k + d\phi_k)} |e_k\rangle, \qquad (20)$$

respectively, with $\{|e_k\rangle\}$ being an orthonormal basis of the Hilbert space of single qubit state vectors. The infinitesimal line element between $|\psi\rangle$ and $|\bar{\psi}\rangle$ in Eq. (20) is given by the Fubini-Study metric ds_{FS}^2 [51],

$$ds_{\text{FS}}^2 \overset{\text{def}}{=} 1 - |\langle \bar{\psi} | \psi \rangle|^2$$
$$= \frac{1}{4} \sum_{k=0}^{1} \frac{dp_k^2}{p_k} + \left[\sum_{k=0}^{1} p_k d\phi_k^2 - \left(\sum_{k=0}^{1} p_k d\phi_k \right)^2 \right]. \quad (21)$$

Using the Bloch sphere parametrization of single qubit states,

$$|\psi\rangle = |\psi(\theta, \varphi)\rangle \overset{\text{def}}{=} \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) |1\rangle, \quad (22)$$

where $0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$, we get by comparing Eqs. (20) and (22) that

$$p_0(\theta, \varphi) = \cos^2\left(\frac{\theta}{2}\right), \qquad p_1(\theta, \varphi) = \sin^2\left(\frac{\theta}{2}\right),$$
$$\phi_0(\theta, \varphi) = 0, \qquad \phi_1(\theta, \varphi) = \varphi. \qquad (23)$$

Therefore, substituting Eq. (23) into Eq. (21), the Fubini-Study metric ds_{FS}^2 reduces to

$$ds_{\text{FS}}^2 = g_{\mu\nu}^{\text{FS}}(\xi) d\xi^\mu d\xi^\nu = \frac{1}{4}[d\theta^2 + \sin^2(\theta) d\varphi^2]. \quad (24)$$

In Eq. (24), $g_{\mu\nu}^{\text{FS}}(\xi)$ is the Fubini-Study metric tensor, $1 \le \mu$, $\nu \le 2$, and $\xi = (\xi^1, \xi^2) \overset{\text{def}}{=} (\theta, \varphi)$. For completeness, we point out that the Fubini-Study distance between two antipodal (i.e., orthogonal) states on the Bloch sphere is $\pi/2$. Instead, the geodesic distance between two antipodal states is π. Indeed, $ds_{\text{FS}}^2 = (1/4) ds_{\text{BSM}}^2$ where ds_{BSM}^2 denotes the Bloch sphere metric (BSM) defined as [52]

$$ds_{\text{BSM}}^2 \overset{\text{def}}{=} d\hat{n} \cdot d\hat{n}. \qquad (25)$$

In Eq. (25), $\hat{n}$ is the unit vector in $\mathbb{R}^3$ given by

$$\hat{n} \overset{\text{def}}{=} \frac{\langle \psi(\theta, \varphi) | \vec{\sigma} | \psi(\theta, \varphi) \rangle}{\langle \psi(\theta, \varphi) | \psi(\theta, \varphi) \rangle}$$
$$= (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)), \quad (26)$$

with $\vec{\sigma} \overset{\text{def}}{=} (\sigma_x, \sigma_y, \sigma_z)$ being the Pauli vector operator and $|\psi(\theta, \varphi)\rangle$ given in Eq. (22). From Eq. (24), the only nonvanishing Christoffel connection coefficients are

$$\Gamma_{22}^1 = -\sin(\theta)\cos(\theta), \qquad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\cos(\theta)}{\sin(\theta)}. \quad (27)$$

Therefore, geodesic paths satisfy the geodesic equations in Eq. (11) being specified by the following system of two coupled second order nonlinear ODEs:

$$\ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}^2 = 0, \qquad \ddot{\varphi} + 2\frac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\,\dot{\varphi} = 0, \quad (28)$$

where $\dot{\theta} \overset{\text{def}}{=} d\theta/d\eta$ with η being an affine parameter. Integration of Eq. (28) under suitable working conditions yields geodesic paths given by

$$\theta(\eta) = \cos^{-1}[a_{\text{FS}}\sin(\eta)],$$
$$\varphi(\eta) = \varphi_i + \tan^{-1}[c_{\text{FS}}\tan(\eta)], \qquad (29)$$

where $a_{\text{FS}}^2 \overset{\text{def}}{=} 1 - c_{\text{FS}}^2$ and $c_{\text{FS}} = c_{\text{FS}}(\theta_i, \dot{\varphi}_i) \overset{\text{def}}{=} \dot{\varphi}_i \sin^2(\theta_i) = $ const. Note that both $\theta(\eta)$ and $\varphi(\eta)$ in Eq. (29) are bounded functions for any $\eta \ge 0$. We remark that the speed of evolution along these paths is constant and equals $v_{\text{FS}} \overset{\text{def}}{=} (1/2)[\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2]^{1/2}$. For a detailed derivation of the relations in Eq. (29) along with their extension to arbitrary working conditions, we refer to Appendix A. Having found the geodesic paths in Eq. (29), we focus now on geodesics on manifolds of mixed quantum states equipped with the Sjöqvist metric.

B. Geodesic paths in the Bloch ball: The Sjöqvist metric

We begin by mentioning the motivation underlying the introduction of the Sjöqvist metric from a practical standpoint in science. From a physical standpoint, the Sjöqvist metric can be related to measurable quantities in suitably prepared interferometric measurements. For this reason, it is sometimes called "interferometric" metric. The metric can be regarded as the infinitesimal distance $\delta s^2(\rho, \rho + \delta\rho) \approx g(\dot{\rho}, \dot{\rho})\delta t^2$ between two neighboring mixed states ρ and $\rho + \delta\rho$ with $\delta\rho = \dot{\rho}\delta t$. The mixed state ρ encodes the internal degree of freedom of a particle entering a Mach-Zehnder interferometer with two beam splitters. The mixed state $\rho' \overset{\text{def}}{=} \rho + \delta\rho$ equals $U\rho U^\dagger$ with U being a unitary applied to the particle for a small but finite time δt. From an experimental standpoint, the line element δs^2 is related to the probability P_0 of finding the particle in the 0-beam (that is, the beam where the unitary transformation U was applied) after passing the second beam splitter. In particular, up to the leading nontrivial order in δt, one finds that $P_0 = 1 - (1/4)\delta s^2$. For more details on a

direct experimental access to the Sjöqvist line element, we refer to Refs. [53,54]. In Ref. [54], the Sjöqvist metric is generalized by extending its applicability to degenerate density matrices as well. Interestingly, studying finite-temperature equilibrium phase transitions, dramatically different behaviors between the Sjöqvist and the Bures metrics are noticed in Ref. [54]. Specifically, the Sjöqvist metric appears to be more sensitive to the change in parameters than the Bures one. Indeed, unlike what happens for the Bures metric, the Sjöqvist metric infers both zero-temperature and finite-temperature phase transitions. We will return to this point on the difference between the Sjöqvist and Bures metrics later in our paper.

Resuming the formal introduction of the Sjöqvist metric, consider two rank-two neighboring nondegenerate density operators $\rho(t)$ and $\rho(t + dt)$ connected via a smooth path $t \mapsto \rho(t)$ characterizing the evolution of a quantum system. The nondegeneracy requirement assures that the gauge freedom in the spectral decomposition of the density operators is represented by the phase of the eigenvectors. This, in turn, implies there is a one-to-one correspondence between a rank-two nondegenerate density operator $\rho(t)$ and the set of two orthogonal rays $\{e^{i\phi_k(t)}|e_k(t)\rangle : 0 \leq \phi_k(t) < 2\pi\}$ that specify the spectral decomposition along the path $t \mapsto \rho(t)$. Clearly, if some nonzero eigenvalue of $\rho(t)$ was degenerate, the above mentioned correspondence would not be valid any longer. The infinitesimal line element between $\rho(t)$ and $\rho(t + dt)$ in the working assumption that $\{\sqrt{p_k(t)}e^{i\phi_k(t)}|e_k(t)\rangle\}_{k=0,1}$ represents the spectral decompositions along the path $t \mapsto \rho(t)$ is given by the Sjöqvist metric $ds^2_{\text{Sjöqvist}}$ [53],

$$ds^2_{\text{Sjöqvist}} \stackrel{\text{def}}{=} \min\left[d^2(t, t+dt)\right], \qquad (30)$$

with $d^2(t, t + dt)$ defined as

$$d^2(t, t+dt) \stackrel{\text{def}}{=} \sum_{k=0}^{1} \| \sqrt{p_k(t)}e^{i\phi_k(t)}|e_k(t)\rangle$$
$$- \sqrt{p_k(t+dt)}e^{i\phi_k(t+dt)}|e_k(t+dt)\rangle \|^2. \quad (31)$$

Following the line of reasoning in Ref. [53], $ds^2_{\text{Sjöqvist}}$ can be recast as

$$ds^2_{\text{Sjöqvist}} = \frac{1}{4}\sum_{k=0}^{1}\frac{dp_k^2}{p_k} + \sum_{k=0}^{1}p_k ds_k^2, \qquad (32)$$

where $dp_k = \dot{p}_k dt$ and, recalling Ref. [55], ds_k^2 in Eq. (32) is the Fubini-Study metric along the pure state $|e_k\rangle$

$$ds_k^2 \stackrel{\text{def}}{=} \langle \dot{e}_k|(\hat{1} - |e_k\rangle\langle e_k|)|\dot{e}_k\rangle$$
$$= \langle de_k|de_k\rangle - |\langle e_k|de_k\rangle|^2, \qquad (33)$$

with $\hat{1}$ being the identity operator on the Hilbert space of single qubit quantum states. Using the Bloch sphere parametrization of single qubit mixed states in the Bloch ball, we have

$$\rho = \frac{\hat{1} + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1 + r\cos(\theta) & r\sin(\theta)e^{-i\varphi} \\ r\sin(\theta)e^{i\varphi} & 1 - r\cos(\theta) \end{pmatrix}, \qquad (34)$$

where $\vec{r}$ is the polarization vector given by $\vec{r} \stackrel{\text{def}}{=} r\hat{n}$ with $\hat{n}$ defined in Eq. (26). Note that for mixed quantum states, $0 \leq r < 1$ and $\det(\rho) = (1/2)(1 - \vec{r}^2) \geq 0$ because of the positiveness of ρ. For pure quantum states, instead, $r = 1$ and $\det(\rho) = 0$. From Eq. (34), we observe that the spectral decomposition of ρ is given by

$$\rho = \sum_{k=0}^{1} p_k|e_k\rangle\langle e_k|. \qquad (35)$$

The two distinct eigenvalues $\{p_k\}_{k=0,1}$ are given by

$$p_0 = p_0(r, \theta, \varphi) \stackrel{\text{def}}{=} \frac{1+r}{2},$$
$$p_1 = p_1(r, \theta, \varphi) \stackrel{\text{def}}{=} \frac{1-r}{2}, \qquad (36)$$

respectively. The orthonormal eigenvectors corresponding to p_0 and p_1 in Eq. (36) are

$$|e_0\rangle = |e_0(r, \theta, \varphi)\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi}\sqrt{1 + \cos(\theta)} \\ \frac{\sin(\theta)}{\sqrt{1+\cos(\theta)}} \end{pmatrix}$$
$$= \begin{pmatrix} e^{-i\varphi}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}, \qquad (37)$$

and

$$|e_1\rangle = |e_1(r, \theta, \varphi)\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi}\sqrt{1 - \cos(\theta)} \\ \frac{\sin(\theta)}{\sqrt{1-\cos(\theta)}} \end{pmatrix}$$
$$= \begin{pmatrix} -e^{-i\varphi}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}, \qquad (38)$$

respectively. Finally, using Eqs. (36), (37), and (38), $ds^2_{\text{Sjöqvist}}$ in Eq. (32) becomes

$$ds^2_{\text{Sjöqvist}} = g^{\text{Sjöqvist}}_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu = \frac{1}{4}\left[\frac{dr^2}{1-r^2} + d\Omega^2\right], \qquad (39)$$

with $d\Omega^2 \overset{\text{def}}{=} d\theta^2 + \sin^2(\theta)d\varphi^2$. In Eq. (39), $g_{\mu\nu}^{\text{Sjöqvist}}(\xi)$ is the Sjöqvist metric tensor, $1 \leq \mu, \nu \leq 3$, and $\xi = (\xi^1, \xi^2, \xi^3) \overset{\text{def}}{=} (r, \theta, \varphi)$. Note when r is constant and equals one, $ds_{\text{Sjöqvist}}^2$ in Eq. (39) reduces to ds_{FS}^2 in Eq. (24). For completeness, we recall that the Bures metric extends to mixed quantum states and reduces to the Fubini-Study metric on pure states [56–58]. Furthermore, as shown in Ref. [51], it is equivalent, up to a proportionality factor of four, to the quantum Fisher information metric. Interestingly, we remark that the Bures infinitesimal line element ds_{Bures}^2 between ρ and $\rho + d\rho$ with ρ given in Eq. (34) is given by

$$ds_{\text{Bures}}^2 = g_{\mu\nu}^{\text{Bures}}(\xi)d\xi^\mu d\xi^\nu = \frac{1}{4}\left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right]. \quad (40)$$

For an explicit derivation of Eq. (40), we refer to Appendix B. From Eqs. (39) and (40), we notice that the angular part of $ds_{\text{Sjöqvist}}^2$ does not exhibit the r^2-factor which, instead, appears in ds_{Bures}^2. The lack of this factor implies that the Sjöqvist metric is singular at the origin of the Bloch ball where $r = 0$ and, unlike the Bures metric, is not defined for degenerate density operators. Finally, we refer to Ref. [54] for a recent extension of the Sjöqvist metric for the space of nondegenerate density matrices, to the degenerate case, i.e., the case in which the eigenspaces have dimension greater than or equal to one. We will go back to this point on the difference between the Sjöqvist and Bures metrics later in our paper (see also Appendix C and Appendix D).

Returning to the Sjöqvist metric analysis, we see from Eq. (39) that the only nonvanishing Christoffel connection coefficients are

$$\Gamma_{11}^1 = \frac{r}{1-r^2}, \quad \Gamma_{33}^2 = -\sin(\theta)\cos(\theta), \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{\cos(\theta)}{\sin(\theta)}. \quad (41)$$

Therefore, geodesics satisfy the geodesic equations in Eq. (11) described in terms of a system of three coupled second order nonlinear ODEs,

$$\ddot{r} + \frac{r}{1-r^2}\dot{r}^2 = 0, \qquad \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}^2 = 0,$$
$$\ddot{\varphi} + 2\frac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\dot{\varphi} = 0, \quad (42)$$

where $\dot{r} \overset{\text{def}}{=} dr/d\eta$ with η being an affine parameter. Interestingly, observe that although the ODE satisfied by the radial parameter r in Eq. (42) is nonlinear, it is not coupled to the ODEs describing the evolution of the angular parameters θ and φ. Furthermore, the angular motion is identical to the one that emerges when employing the Fubini-Study metric. Therefore, we refer to Eq. (29) and

to Appendix A for details on the angular motion. Instead, integration of the radial equation of motion in Eq. (42) yields,

$$r_{\text{Sjöqvist}}(\eta) = \sin\left[\sin^{-1}(r_i) + \frac{\dot{r}_i}{\sqrt{1-r_i^2}}\eta\right], \quad (43)$$

where $r_i \overset{\text{def}}{=} r(\eta_i)$, $\dot{r}_i \overset{\text{def}}{=} \dot{r}(\eta_i)$, and η_i is set equal to zero. We emphasize that the speed of evolution along geodesic paths is constant and equals $v_{\text{Sjöqvist}} \overset{\text{def}}{=} (1/2)[(1-r^2)^{-1}\dot{r}^2 + \dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2]^{1/2}$. For an explicit derivation of Eq. (43) along with a discussion on alternative geodesic parametrizations like the one used in Ref. [53], we refer to Appendix C. Finally, for a discussion on the integration of the geodesic equations in a Bloch ball equipped with the Bures metric $g_{\mu\nu}^{\text{Bures}}(\xi)$, we refer to Appendix D. In Appendix E, instead, we present a summary of curvature properties of the manifold of pure states equipped with $g_{\mu\nu}^{\text{FS}}(\xi)$ along with those of a manifold of mixed quantum states endowed with $g_{\mu\nu}^{\text{Sjöqvist}}(\xi)$ and $g_{\mu\nu}^{\text{Bures}}(\xi)$. More specifically, for each scenario, we find the expressions of the tensor metric components, infinitesimal line elements, Christoffel connection coefficients, Ricci tensor components, Riemann curvature tensor components, scalar curvatures and, finally, sectional curvatures.

At this point, having found the geodesic paths on curved manifolds equipped with the Fubini-Study and Sjöqvist metrics, we are ready to use our complexity quantifiers in Eqs. (3) and (13) to determine how complex evolutions on pure and mixed states are.

IV. COMPLEXITY OF QUANTUM EVOLUTION

We study here the complexity of the geodesic paths expressed in terms of temporal averages of volume regions explored by the physical systems during the quantum evolutions.

A. Actions, lengths, and accessible volumes

Before studying the complexity, let us first comment on the relevance of the concepts of length and action in the geometric formulation of physical theories. In the Introduction, we mentioned these concepts play a key role in the understanding of the physics of black holes [7–10]. However, lengths and actions also play a very important role in the geometric formulation of thermodynamics [59]. In this case, these two quantities are generally termed thermodynamic length and thermodynamic divergence, respectively. Indeed, including the theory of fluctuations into the axioms of equilibrium thermodynamics [60], thermodynamic systems can be characterized by Riemannian manifolds furnished of a thermodynamic metric tensor that is identical to the Fisher-Rao information metric [61]. Within this geometric setting for thermodynamics, the above mentioned

Riemannian structure allows one to introduce a notion of length for fluctuations about equilibrium states as well as for thermodynamic processes proceeding via equilibrium states. In analogy to Wootters's statistical distance between probability distributions as presented in Ref. [50], the thermodynamic length of a path connecting two points on a manifold of thermal states can be viewed as a measure of the maximal number of statistically distinguishable thermodynamic states along the path [62]. In particular, the larger the fluctuations, the closer the points are together. The thermodynamic divergence of a path, instead, is a measure of the losses in the process quantified by the total entropy produced along the path. For more details, we refer to Refs. [60,63,64]. Having in mind Wootters's approach, note that the concepts of action and length are formally different when studying the geometry along the evolution of states. For the sake of reasoning, assume that the line element of the Riemannian space is given by $ds^2 \stackrel{\text{def}}{=} g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu$. Then, the action A is given by

$$\text{A} \stackrel{\text{def}}{=} \frac{1}{2}m\int_0^\tau g_{\mu\nu}(\xi)\dot{\xi}^\mu\dot{\xi}^\nu d\eta, \qquad (44)$$

with $\dot{\xi} \stackrel{\text{def}}{=} d\xi/d\eta$. The length $\mathcal{L}$ of a path $\xi^\mu(\eta)$ with $0 \leq \eta \leq \tau$, instead, is defined as

$$\mathcal{L} \stackrel{\text{def}}{=} \int_0^\tau \sqrt{g_{\mu\nu}(\xi)\dot{\xi}^\mu\dot{\xi}^\nu}\, d\eta. \qquad (45)$$

However, for particles of mass m moving along geodesics with constant velocity, both velocity and energy are conserved. In this case, the path $\mathcal{L}$ and the action A are linearly related. Indeed, one has

$$\text{A} = \sqrt{\frac{mE}{2}}\mathcal{L}, \qquad (46)$$

where $E \stackrel{\text{def}}{=} (1/2)mv^2$ and $v^2 \stackrel{\text{def}}{=} g_{\mu\nu}(\xi)\dot{\xi}^\mu\dot{\xi}^\nu$ are both constant. Interestingly, we remark that while the length $\mathcal{L}$ is invariant under reparametrization of the affine parameter η, the action A is not. Before proceeding with the calculation of lengths in Eq. (45) and volumes of explored regions in Eq. (4) yielding the complexity of geodesic paths on manifolds of quantum states, we make a couple of remarks that can help our intuition when considering the Sjöqvist and Bures cases with pure calculations. First, considering Eqs. (39) and (40) while performing a change of variables defined by $r \stackrel{\text{def}}{=} \sin(\alpha_r)$ with $0 \leq \alpha_r \leq \pi/2$, we find that $4ds_{\text{Sjöqvist}}^2 = d\alpha_r^2 + d\Omega_{\text{sphere}}^2$ and $4ds_{\text{Bures}}^2 = d\alpha_r^2 + \sin^2(\alpha_r)d\Omega_{\text{sphere}}^2$ with $d\Omega_{\text{sphere}}^2 \stackrel{\text{def}}{=} d\theta^2 + \sin^2(\theta)d\varphi^2$. The structure of the Sjöqvist line element recast in this new form is reminiscent of the structure of a line element in the usual cylindrical coordinates (ρ, φ, z), $ds_{\text{cylinder}}^2 = dz^2 + d\Omega_{\text{cylinder}}^2$ with

$d\Omega_{\text{cylinder}}^2 \stackrel{\text{def}}{=} d\rho^2 + \rho^2 d\varphi^2$, once one identifies the pair $(\alpha_r, d\Omega_{\text{sphere}})$ with the pair $(\rho, d\Omega_{\text{cylinder}})$. Therefore, one can imagine associating a cylinder with a constant (varying) radius to the Sjöqvist (Bures) geometry, respectively. Note that the varying radius in the Bures case is upper bounded by the constant value that specifies the radius in the Sjöqvist geometry. Second, the volumes of the accessible regions of the manifolds in the Sjöqvist and Bures scenarios are given by

$$V_{\text{Sjöqvist}}^{(\text{accessible})} \stackrel{\text{def}}{=} \frac{1}{8}\int_0^1\int_0^\pi\int_0^{2\pi}\frac{\sin(\theta)}{\sqrt{1-r^2}}drd\theta d\varphi = \frac{\pi^2}{4}, \quad (47)$$

and

$$V_{\text{Bures}}^{(\text{accessible})} \stackrel{\text{def}}{=} \frac{1}{8}\int_0^1\int_0^\pi\int_0^{2\pi}\frac{r^2\sin(\theta)}{\sqrt{1-r^2}}drd\theta d\varphi = \frac{\pi^2}{8}, \quad (48)$$

respectively. Clearly, from Eqs. (47) and (48), we note that $V_{\text{Bures}}^{(\text{accessible})} \leq V_{\text{Sjöqvist}}^{(\text{accessible})}$. This fact, in turn, is compatible with our intuitive picture proposed in our first remark. We remark that it is possible that only parts of the accessible geometric regions are indeed explored during the evolution. In what follows, we shall finally calculate the lengths and the volumes of the effectively explored regions of the manifolds in the (pure) Fubini-Study and (mixed) Sjöqvist cases.

B. Evolution on the Bloch sphere

1. Length

In what follows, we focus on calculating the length of geodesics in the unit Bloch ball that lay int the xz-plane specified by the condition $\varphi = 0$. In the Fubini-Study metric case, we have

$$\mathcal{L}_{\text{FS}}(\eta_f) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^{\eta_f}\frac{d\theta}{d\eta}d\eta = \frac{1}{2}\dot{\theta}_i\eta_f, \qquad (49)$$

or, alternatively, in terms of the angular variable θ,

$$\mathcal{L}_{\text{FS}}(\theta_f) \stackrel{\text{def}}{=} \frac{\theta_f}{2}. \qquad (50)$$

Note that $\mathcal{L}_{\text{FS}}$ in Eq. (50) denotes the Fubini-Study distance $(\theta_f/2)$ and is half the geodesic distance (θ_f) on the Bloch sphere. For completeness, we emphasize that in obtaining Eq. (49) we exploited the relation $\ddot{\theta} = 0$. This relation can be obtained from Eq. (28) once one imposes the constraint of constant φ. Moreover, in getting Eq. (50), we assumed $\theta_i \stackrel{\text{def}}{=} \theta(\eta_i) = 0$, with $\eta_i = 0$.

2. Complexity

From the Fubini-Study metric ds_{FS}^2 in Eq. (24), we note that $g_{\text{FS}}(\theta, \varphi) = \sin^2(\theta)/16$ with g_{FS} denoting the determinant of the metric tensor $g_{\mu\nu}^{\text{FS}}$. Therefore, using Eq. (29), the instantaneous explored volume region $V_{\text{FS}}(\eta)$ as defined in Eq. (5) becomes

$$V_{\text{FS}}(\eta) = \frac{1}{4} a_{\text{FS}} \sin(\eta) \arctan [c_{\text{FS}} \tan(\eta)]. \quad (51)$$

Recall that $a_{\text{FS}}^2 \stackrel{\text{def}}{=} 1 - c_{\text{FS}}^2$ with $c_{\text{FS}} = c_{\text{FS}}(\theta_i, \dot{\varphi}_i) \stackrel{\text{def}}{=} \dot{\varphi}_i \sin^2(\theta_i) = \text{const}$. Then, setting $\theta_i = \pi/2$ for illustrative purposes, we note that the modulus of the volume of the explored region of the manifold of pure states in Eq. (51) is upper bounded by $\pi/8$, $|V_{\text{FS}}(\eta)| \leq \pi/8$. Therefore, at a time η, less than one-eighth of the accessible region of the manifold is actually explored since $V_{\text{FS}}^{(\text{accessible})} = \pi$. From Eq. (51), the average explored region as given in Eq. (4) represents the IGC in Eq. (13) and turns into

$$C_{\text{FS}}(\tau) = \frac{1}{4} a_{\text{FS}} \frac{I_{V_{\text{FS}}}(\tau)}{\tau}. \quad (52)$$

The function $I_{V_{\text{FS}}}(\tau)$ in Eq. (52) is defined as the integral of $V_{\text{FS}}(\eta)$ with $0 \leq \eta \leq \tau$ and is given by

$$I_{V_{\text{FS}}}(\tau) \stackrel{\text{def}}{=} \frac{c_{\text{FS}}}{\sqrt{c_{\text{FS}}^2 - 1}} \arctan \left[\sqrt{c_{\text{FS}}^2 - 1} \sin(\tau) \right]$$
$$- \cos(\tau) \arctan [c_{\text{FS}} \tan(\tau)]. \quad (53)$$

The above mentioned integral was performed with the help of the *Mathematica* software. Furthermore, we remark that $I_{V_{\text{FS}}}(\tau)$ is a bounded function for any $\tau \geq 0$. The asymptotic temporal expression of the IGC $C_{\text{FS}}(\tau)$ in Eq. (52) will be compared with the one that we obtain in the case of mixed state evolutions with the distinguishability metric provided by $g_{\mu\nu}^{\text{Sjöqvist}}(\xi)$.

C. Evolution in the Bloch ball

1. Length

In what follows, we focus on calculating the length of geodesics in the unit Bloch ball that lay int the xz-plane specified by the condition $\varphi = 0$. In the Sjöqvist metric case, recall that $r'^2(1 - r^2)^{-1} = \text{const}$ and $\ddot{\theta} = 0$. Therefore, the length $\mathcal{L}_{\text{Sjöqvist}}(\eta_f)$ is given by

$$\mathcal{L}_{\text{Sjöqvist}}(\eta_f) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^{\eta_f} \sqrt{1 + \frac{r'^2}{1 - r^2} \frac{d\theta}{d\eta}} \, d\eta$$
$$= \frac{1}{2} \sqrt{1 + \frac{r_i'^2}{1 - r_i^2}} \dot{\theta}_i \eta_f, \quad (54)$$

or, alternatively,

$$\mathcal{L}_{\text{Sjöqvist}}(\theta_f) \stackrel{\text{def}}{=} \frac{1}{2} \sqrt{\theta_f^2 + [\sin^{-1}(r_f) - \sin^{-1}(r_i)]^2}. \quad (55)$$

Comparing Eqs. (55) and (50), we note that $\mathcal{L}_{\text{Sjöqvist}}$ reduces to $\mathcal{L}_{\text{FS}}$ when $r_f = r_i = 1$. For $r_f \neq r_i$, we generally have

$$\mathcal{L}_{\text{Sjöqvist}}(\theta_f) \geq \mathcal{L}_{\text{FS}}(\theta_f). \quad (56)$$

Equation (56) implies that the length of a geodesic path connecting two arbitrary points $P_i \stackrel{\text{def}}{=} (r_i, \theta_i, 0)$ and $P_f \stackrel{\text{def}}{=} (r_f, \theta_f, 0)$ with $r_i \neq r_f$ in the Bloch ball laying in the xz-plane (i.e., $\varphi = 0$) is longer than the length of two arbitrary points $\tilde{P}_i \stackrel{\text{def}}{=} (\theta_i, 0)$ and $\tilde{P}_f \stackrel{\text{def}}{=} (\theta_f, 0)$ laying on the Bloch sphere with $r_i = r_f = 1$ that intercepts the xz-plane (i.e., $\varphi = 0$). Equation (56) hints to what might happen when comparing the information geometric complexity of the evolutions of pure and quantum states as we shall see shortly. Complexities are expressed in terms of volumes. In parametric spaces of dimension higher than one, relations between length and volumes are not straightforward. Therefore, we could not easily take the hint as a "proof" of the higher complexity of the evolution of mixed quantum states. For this reason, we actually estimate this quantity in an explicit manner in the following subsection.

2. Complexity

From the Sjöqvist metric $ds_{\text{Sjöqvist}}^2$ in Eq. (39), we observe that the determinant of the metric tensor $g_{\mu\nu}^{\text{Sjöqvist}}$ satisfies the relation

$$g_{\text{Sjöqvist}}(r, \theta, \varphi) = \frac{1}{64} \frac{\sin^2(\theta)}{1 - r^2}. \quad (57)$$

Therefore, making use of Eqs. (43) and (29), the instantaneous explored volume region $V_{\text{Sjöqvist}}(\eta)$ as given in Eq. (5) turns into

$$V_{\text{Sjöqvist}}(\eta) = \frac{1}{8} a_{\text{FS}} \frac{\dot{r}_i}{\sqrt{1 - r_i^2}} \eta \sin(\eta) \arctan [c_{\text{FS}} \tan(\eta)]. \quad (58)$$

Recall that $a_{\text{FS}}^2 \stackrel{\text{def}}{=} 1 - c_{\text{FS}}^2$ with $c_{\text{FS}} = c_{\text{FS}}(\theta_i, \dot{\varphi}_i) \stackrel{\text{def}}{=} \dot{\varphi}_i \sin^2(\theta_i) = \text{const}$. Then, putting $\theta_i = \pi/2$ for simplicity, we observe that the modulus $|V_{\text{Sjöqvist}}(\eta)|$ of the volume of the explored region of the manifold of mixed states in Eq. (58) is upper bounded by a function that grows linearly with η with proportionality coefficient given by $\dot{r}_i / \sqrt{1 - r_i^2}$ and not by a constant function as in the case of the evolution of pure quantum states. Moreover, recalling that $V_{\text{Sjöqvist}}^{(\text{accessible})} = \pi^2/4 \leq \pi = V_{\text{FS}}^{(\text{accessible})}$, we clearly expect a more complex behavior for sufficiently large values of η in

the case of the geometry along evolution of mixed states since

$$\frac{V_{\text{Sjöqvist}}^{(\text{explored})}(\eta)}{V_{\text{Sjöqvist}}^{(\text{accessible})}} \geq \frac{V_{\text{FS}}^{(\text{explored})}(\eta)}{V_{\text{FS}}^{(\text{accessible})}}. \qquad (59)$$

From Eq. (58), the average explored region as given in Eq. (4) denotes the IGC in Eq. (13) and becomes

$$\mathcal{C}_{\text{Sjöqvist}}(\tau) = \frac{1}{\tau} \int_0^\tau V_{\text{Sjöqvist}}(\eta) d\eta, \qquad (60)$$

that is,

$$\mathcal{C}_{\text{Sjöqvist}}(\tau) = \frac{1}{8} a_{\text{FS}} \frac{\dot{r}_i}{\sqrt{1 - r_i^2}}$$
$$\times \frac{1}{\tau} \int_0^\tau \eta \sin(\eta) \arctan\left[c_{\text{FS}} \tan(\eta)\right] d\eta. \qquad (61)$$

Denoting $f(\eta) \stackrel{\text{def}}{=} \eta$ and $\dot{g}(\eta) \stackrel{\text{def}}{=} \sin(\eta) \arctan\left[c_{\text{FS}} \tan(\eta)\right]$, we integrate by parts the integral in Eq. (61) and get

$$\int_0^\tau \eta \sin(\eta) \arctan\left[c_{\text{FS}} \tan(\eta)\right] d\eta$$
$$= \left[\eta I_{V_{\text{FS}}}(\eta)\right]_{\eta=0}^{\eta=\tau} - \int_0^\tau I_{V_{\text{FS}}}(\eta) d\eta. \qquad (62)$$

Note that $g(\eta) = I_{V_{\text{FS}}}(\eta)$ with $I_{V_{\text{FS}}}(\eta)$ given in Eq. (53). Finally, substituting Eq. (62) into Eq. (61) and considering the asymptotic temporal behavior of $\mathcal{C}_{\text{Sjöqvist}}(\tau)$, we obtain

$$\mathcal{C}_{\text{Sjöqvist}}^{\text{asymptotic}}(\tau) = \frac{1}{8} a_{\text{FS}} \frac{\dot{r}_i}{\sqrt{1 - r_i^2}} I_{V_{\text{FS}}}^{\text{asymptotic}}(\tau). \qquad (63)$$

At this point, considering the ratio between $\mathcal{C}_{\text{Sjöqvist}}^{\text{asymptotic}}(\tau)$ in Eq. (63) and the asymptotic temporal behavior of $\mathcal{C}_{\text{FS}}(\tau)$ in Eq. (52), we get that the relative asymptotic complexity growth in terms of a ratio exhibits a linear behavior given by

$$\frac{\mathcal{C}_{\text{Sjöqvist}}^{\text{asymptotic}}(\tau)}{\mathcal{C}_{\text{FS}}^{\text{asymptotic}}(\tau)} \sim \tau. \qquad (64)$$

In the long-time limit, Eq. (64) expresses the fact that the evolution of mixed states in the Bloch ball equipped with the Sjöqvist metric explores averaged volumes of regions larger that the ones inspected during the evolution of pure states on the Bloch sphere supplied with the Fubini-Study metric. In particular, there appears to be an asymptotic linear growth of the ratio between the two IGCs. Finally, in terms of the IGE defined in Eq. (3), we obtain an asymptotic entropy growth of the relative difference between the two IGEs given by

$$\mathcal{S}_{\text{Sjöqvist}}^{\text{asymptotic}}(\tau) - \mathcal{S}_{\text{FS}}^{\text{asymptotic}}(\tau) \sim \log(\tau). \qquad (65)$$

Equation (65) displays the asymptotic logarithmic discrepancy between the IGE in the mixed and pure quantum state scenarios. Since the IGC is simply the exponential of the IGE, we can interpret this entropic deviation as follows. To a larger IGE there corresponds a larger IGC. Larger IGCs are larger asymptotic averaged explored volumes. Larger volumes encode, via the metric, larger fluctuations. The larger the fluctuations, the closer the points (i.e., the states) are together. The closer points are together, the greater is the likelihood of incorrectly distinguishing quantum states during the evolutions of the quantum system. This, in turn, leads to higher entropic configurations which are typical of quantum systems in a mixed quantum state. Note that Eqs. (56), (59), (64), and (65) are nonconflicting and consistent relations in support of arguments yielding to a higher degree of complexity of evolutions of mixed states compared to pure quantum states from a geometric perspective. Indeed, Eq. (56) is an inequality in terms of lengths. Equation (59) is an inequality between ratios expressed by means of accessible and instantaneous explored volumes. Finally, Eqs. (64) and (65) are complexity and entropic relations that are expressed by means of long-time limits of averaged explored volumes of regions on and inside the Bloch ball. Interestingly, note that the asymptotic temporal rates of change of the two IGCs in Eq. (64) scale in a similar fashion, $d\mathcal{C}_{\text{Sjöqvist}}^{\text{asymptotic}}/d\tau \sim d\mathcal{C}_{\text{FS}}^{\text{asymptotic}}/d\tau$. This is a consequence of a balancing effect that occurs between the asymptotic averaged explored volumes, $\bar{V}_{\text{Sjöqvist}}^{\text{asymptotic}} \sim \tau \bar{V}_{\text{FS}}^{\text{asymptotic}}$, and between the asymptotic temporal rates of change of the averaged explored volumes, $d\bar{V}_{FS}^{\text{asymptotic}}/d\tau \sim (1/\tau) d\bar{V}_{\text{Sjöqvist}}^{\text{asymptotic}}/d\tau$. From a curvature analysis perspective, the manifold of pure states equipped with the Fubini-Study metric is an isotropic two-dimensional manifold of constant positive sectional curvature $\mathcal{K}_{\text{FS}} = 4$ and constant scalar curvature $\mathcal{R}_{\text{FS}} = 8$. Instead, the manifold of mixed states equipped with the Sjöqvist metric is an anisotropic three-dimensional manifold of nonconstant but positive sectional curvature and constant scalar curvature $\mathcal{R}_{\text{Sjöqvist}} = 8$. The positivity of sectional curvatures in both scenarios leads to the presence of convergence in the geodesic spread analysis on both manifolds. However, given the anisotropic nature in the mixed quantum states manifold with the Sjöqvist metric, the study of the geodesic spread equation would be more complicated in the scenario of mixed states distinguished via the Sjöqvist metric. Our remarks concerning the asymptotic rates of change of volumes and IGEs are an indication of this distinct convergent behavior in the pure and mixed quantum states scenarios. For further details on curvature properties along with a comparison between the Sjöqvist and Bures manifolds, we refer to Appendix E.

V. PHYSICAL CONSIDERATIONS

In this section, we present physical comments on the concepts of metric, path length, and curvature employed in our investigation. Moreover, we clarify the physics behind the evolution of quantum states on curved manifolds in terms of Bloch coordinates. These comments will help highlight even further the physical significance of our proposed complexity measure in Eq. (13).

A. Metric, path length, and curvature

For completeness, we begin by recalling that in quantum mechanics a physical state is not represented by a normalized state vector $|\psi(t)\rangle \in \mathcal{H} \backslash \{0\}$ but by a ray. A ray is the one-dimensional subspace to which this vector belongs. Two normalized vectors are equivalent, $|\psi'\rangle \sim |\psi\rangle$, if they belong to the same ray, i.e., if $|\psi'\rangle = e^{i\phi}|\psi\rangle$ with $\phi \in U(1)$. This equivalence relation specifies equivalence classes on the sphere $\mathcal{S}^{2N_{\mathcal{H}}+1}$, with $\dim_{\mathbb{C}} \mathcal{H} = N_{\mathcal{H}} + 1$. Finally, the set of equivalence classes $\mathcal{S}^{2N_{\mathcal{H}}+1}/U(1)$ forms the space of physical states (rays) which is denoted here by $\mathcal{P}$. The space of rays is the projective Hilbert space $\mathcal{P}$ which, in turn, is isomorphic to the complex projective space $\mathbb{C}P^N$.

1. Metric

The metric [Eqs. (24), (39), and (40)] on the manifold of quantum states is fixed once the quantum mechanical fluctuation in energy is specified [55]. Focusing on pure states, when the uncertainty $\Delta A(\eta)$ in the generator of motion $A(\eta)$ with respect to the parameter η in the projective Hilbert space $\mathcal{P}$ is provided, the metric

$$ds_{\mathrm{FS}}^2 = g_{\mu\nu}(\xi) \frac{d\xi^\mu}{d\eta} \frac{d\xi^\nu}{d\eta} d\eta^2 = \Delta A^2(\eta) d\eta^2 \qquad (66)$$

is fixed. In particular, assuming $\eta = t$ and $A(\eta) = \mathrm{H}(t)/\hbar$, we have that $ds_{\mathrm{FS}} = [\Delta E(t)/\hbar]dt$ and the dispersion $\Delta E(t)$ of the generator of motion can originate from a variety of Hamiltonians $\mathrm{H}(t)$. Therefore, the geometry of the projective Hilbert space, specified by the metric on it, cannot be modified by the dynamics of the system governed by the Hamiltonian $\mathrm{H}(t)$ [65]. This was a major result obtained by Anandan and Aharonov in Ref. [66]. For pure states, the distance function in the projective Hilbert space $\mathcal{P}$ is the distance between two quantum states along a given curve in $\mathcal{P}$ as measured by the Fubini-Study metric defined from the inner product of the representative states in the $(N_{\mathcal{H}} + 1)$-dimensional Hilbert space $\mathcal{H}$. Anandan and Aharonov showed that this distance equals the time integral of the uncertainty of the energy, and does not depend on the particular Hamiltonian used to move the quantum system along a given curve in $\mathcal{P}$. It is dependent only on the points in $\mathcal{P}$ to which the quantum states project. In summary, from a physics standpoint, the metric tensor and its components on the projective Hilbert space $\mathcal{P}$ are linked to the

dispersion of suitable quantum-mechanical operators (for instance, the Hamiltonian operator) acting on the underling Hilbert space $\mathcal{H}$. This connection between metrics and quantum fluctuations is an important physical consideration to keep in mind throughout our work. This connection extends to the geometric analysis of quantum mixed states as well [53,67,68].

2. Path length

To explain the physical meaning of the Riemannian distance [Eqs. (49) and (54)] between two arbitrarily chosen pure quantum states, we follow Wootters [50]. For mixed states, we hint to the work by Braunstein and Caves in Ref. [51]. Two infinitesimally close points ξ and $\xi + d\xi$ along a path $\xi(\eta)$ with $\eta_1 \le \eta \le \eta_2$ are statistically distinguishable if $d\xi$ is at least equal to the standard fluctuation of ξ [62]. The line element along the path is ds_{FS}^2 with $ds_{\mathrm{FS}}^2 = g_{\mu\nu}^{\mathrm{FS}}(\xi)d\xi^\mu d\xi^\nu$. The length of the path $\xi(\eta)$ with $\eta_1 \le \eta \le \eta_2$ between $\xi_1 \stackrel{\mathrm{def}}{=} \xi(\eta_1)$ and $\xi_2 \stackrel{\mathrm{def}}{=} \xi(\eta_2)$ is defined as

$$\mathcal{L} \stackrel{\mathrm{def}}{=} \int_{\xi_1}^{\xi_2} \sqrt{ds_{\mathrm{FS}}^2} = \int_{\eta_1}^{\eta_2} \sqrt{\frac{ds_{\mathrm{FS}}^2}{d\eta^2}} d\eta, \qquad (67)$$

and represents the maximal number $\tilde{N}$ of statistically distinguishable states along the path. In particular, the geodesic distance between ξ_1 and ξ_2 is the path of shortest distance between ξ_1 and ξ_2 and is the minimum of $\tilde{N}$. This connection between path length and number of statistically distinguishable states along the path is a relevant physical remark to consider throughout our investigation. This viewpoint extends naturally to the geometric analysis of quantum mixed states as well [51].

3. Curvature

What is the physical significance of curvature (Appendix E) in our investigation? We recall that in the Riemannian geometrization of classical Newtonian mechanics [1], the curvature $\mathcal{R}$ of the manifold corresponds, roughly speaking, to the curvature of the potential V expressed by means of the second derivative of V, $\mathcal{R} \sim \partial^2 V$, with the Hamiltonian of the system given by $\mathrm{H}(p, q) = p^2/(2m) + V(q)$. More generally, in arbitrary differential geometric settings, the curvature of the manifold determines the stability (or, alternatively, the instability) of the geodesics via the Jacobi equation of geodesic spread. This latter curvature interpretation remains valid in our work. However, we can provide a more specific interpretation for the concept of curvature in our analysis. As pointed out by Braunstein and Caves in Ref. [69], unlike what happens in general relativity, the geometry on the space of quantum states does not describe the dynamical evolution of the physical system. Rather, it places limits on

our ability to discriminate one state from another via measurements. In a sense, the geometry of quantum states puts the emphasis on the fact that quantum mechanics is rooted in making statistical inferences based on observed experimental data. Quantum measurement theory, in turn, is statistical inference in its essence [70]. Therefore, given the fact that the problem of distinguishing neighboring quantum states can be formulated as a parameter estimation problem [51], given that quantum mechanics can be regarded as a theory for making statistical inferences based on observed experimental data [70], and, finally, since the curvature of a manifold is a measure of how difficult it is to do estimations at a given point in statistical science [71], it is reasonable to interpret the curvature of a manifold of quantum states equipped with suitably defined metric structures as an indicator of how difficult it is to distinguish quantum states by means of parameter estimation at a given point of the state space. In particular, the higher the curvature, the more difficult it is to do estimation at that point. This is the point of view that we adopt in this paper. For remarks on a physical interpretation of curvature of manifolds underlying the information geometry of non-interacting gases satisfying the Fermi-Dirac and Bose-Einstein statistics, we refer to Ref. [72]. Finally, for a work on the estimation of the curvature of a quantum manifold via measurement on a quantum particle constrained to propagate on the manifold itself, we refer the reader to Ref. [73].

B. Evolution of Bloch coordinates

Having clarified the meaning of metric, path length, and curvature employed in our work we devote some time explaining the relation between Bloch coordinates and quantum states on the Bloch sphere and inside the Bloch ball (Appendixes A, C, and D). Let us point out from the start that in our work $\xi^\mu(\eta) \stackrel{\text{def}}{=} (\xi^1(\eta), \xi^2(\eta), \xi^3(\eta))$ in Eq. (66) is specified by the Bloch coordinates, that is, $\xi^\mu(\eta) = (r(\eta), \theta(\eta), \varphi(\eta))$. In this paper, we focused on the integration of evolution equations for Bloch coordinates used to parametrize quantum states, either pure or mixed. This choice was not dictated by mathematical convenience only. Indeed, there is a clear physical path connecting Bloch coordinates, Bloch vectors, and, finally, pure and mixed quantum states. For simplicity, we set $\eta = t$ in this discussion. We observe that from the time evolution of the Bloch coordinates, both radial and angular, one can generally recover the time evolution of the density operators for arbitrary quantum states via the time evolution of the Bloch vector. Conversely, the opposite is also possible. For a detailed study concerning the time evolution of the Bloch vector of a single two-level atom that interacts with a single quantized electromagnetic field mode according to the Jaynes-Cummings model, we refer to Ref. [74]. To be explicit here, we note that the Bloch vector $\vec{p}(t)$ is defined as

$$\vec{p}(t) \stackrel{\text{def}}{=} \text{tr}[\rho(t)\vec{\sigma}]$$
$$= (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \quad (68)$$

with $r = r(t)$, $\theta = \theta(t)$, and $\varphi = \varphi(t)$. Focusing for simplicity on the case of unitary quantum evolution, the density operator $\rho(t) = (1/2)[\text{I} + \vec{p}(t) \cdot \vec{\sigma}]$ in Eq. (68) satisfies the von Neumann equation $i\hbar\dot{\rho} = [\text{H}(t), \rho(t)]$ with $\dot{\rho} \stackrel{\text{def}}{=} d\rho/dt$ and I denoting the identity operator on the single-qubit quantum state space. Moreover, for a system in a pure state that evolves under a time independent Hamiltonian H, $\rho(t) = |\psi(t)\rangle\langle\psi(t)| = U(t)\rho(0)U^\dagger(t)$ with $U(t)$ being the unitary time evolution operator given by $\exp[-(i/\hbar)\text{H}t]$ and $|\psi(t)\rangle = \cos[\theta(t)/2]|0\rangle + e^{i\varphi(t)}\sin[\theta(t)/2]|1\rangle$ in the Bloch sphere parametrization. To make very clear this link among Bloch coordinates, Bloch vectors, pure states, and mixed states, we consider a simple illustrative example for pure states evolution. Assume the system, a spin-1/2 particle, is initially in the state $|\psi(0)\rangle$ parametrized in terms of $(\theta(0), \varphi(0)) = (\pi/2, 0)$. Solving the Schrödinger's evolution equation $i\hbar\partial_t|\psi(t)\rangle = \text{H}(t)|\psi(t)\rangle$ with $\text{H} \stackrel{\text{def}}{=} \hbar\omega_0\sigma_z$, we get

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \frac{1}{2}\begin{pmatrix} 1 & e^{-2i\omega_o t} \\ e^{2i\omega_o t} & 1 \end{pmatrix}$$
$$= \frac{1}{2}[\text{I} + \vec{p}(t) \cdot \vec{\sigma}], \quad (69)$$

with $\vec{p}(t) = (\cos(2\omega_0 t), \sin(2\omega_0 t), 0)$. Furthermore, the Bloch angles at time t become $(\theta(t), \varphi(t)) = (\pi/2, 2\omega_0 t)$. In general, the state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ can be parametrized as $|\psi(t)\rangle = a(t)|0\rangle + b(t)|1\rangle$ with $\mathbb{C} \ni a(t) = |a(t)|e^{i\varphi_a(t)}$, $\mathbb{C} \ni b(t) = |b(t)|e^{i\varphi_b(t)}$, $\varphi_{a,b}(t) \in \mathbb{R}$, and $|a(t)|^2 + |b(t)|^2 = 1$. With this pure state parametrization, the Bloch angles $\theta(t)$ and $\varphi(t)$ satisfy the relations, $\tan[\theta(t)] = |b(t)|/|a(t)|$ and $\varphi(t) = \varphi_b(t) - \varphi_a(t)$, respectively.

Having clarified the physical meaning of geometrical concepts employed in our analysis, the main take-home message is the following. We have estimated in this paper the complexity of geodesic paths of both pure and mixed quantum states by means of a complexity measure [Eq. (13)] expressed in terms of explored volumes of the suitably metricized curved manifolds that underlay the dynamics (i.e., the change in Bloch parameters, with changes specified by the parametric evolution operator). The metric structure on the curved manifolds of quantum states is fixed by quantum-mechanical fluctuations. Moreover, just as path lengths can be interpreted in terms of the maximal number of distinguishable states traversed during the evolution along the path, the volumes of the parametric space explored in a fixed temporal interval can be regarded as representing the maximal number of different states visited during the regional exploration. Clearly, the role played by the infinitesimal increment $d\xi$ in the

path exploration is replaced by the infinitesimal volume element $dV \overset{\text{def}}{=} \sqrt{g(\xi)}\, d^N \xi$ in the regional travel, with $g(\xi) \overset{\text{def}}{=} \det[g_{\mu\nu}(\xi)]$ and N being the dimensionality of the curved manifold. Essentially, the Riemannian volume element dV helps gauging the number of distinct states explored within an infinitesimal volume of a region of the manifold [75]. We are ready now for our conclusions.

VI. CONCLUDING REMARKS

We present here a summary of our main findings along with limitations and possible future directions.

A. Summary of results

In this paper, we provided a comparative information geometric analysis of the complexity of geodesic paths of pure and mixed quantum states on the Bloch sphere and inside the Bloch ball, respectively. In this geometric setting, pure and mixed states were chosen to be distinguished by means of the Fubini-Study [Eq. (24)] and the Sjöqvist metric [Eq. (39)], respectively. After finding the geodesic paths connecting arbitrary points on (see Appendix A) and inside the Bloch ball (see Appendix C), we analytically estimated the IGE [Eq. (3)] and the IGC [Eq. (13)] in both scenarios. The long-time limit of this pair of entropic measures of complexity of evolution of system in pure [see Eq. (52)] and mixed [see Eq. (63)] states were compared. We observed a degree of complexity for the evolution of mixed states with the Sjöqvist geometry higher than the one specifying the complexity for the evolution of pure states with the Fubini-Study geometry [see Eqs. (64) and (65)].

The metric structure on the manifold of quantum states is specified by quantum fluctuations. Path lengths and volumes can be physically interpreted as indicators of the maximal number of distinguishable states crossed along trajectories and in volumes of regions of the manifold, respectively. To a higher count of distinct states passed over in a fixed time interval, there corresponds a higher degree of complexity of the evolution on the underlying manifold. Within this physically meaningful geometric description, mixed state (geodesic) evolutions appear to be generally more complex than pure state evolutions.

Our main findings can be outlined as follows:

(1) We proposed a different information geometric way [Eqs. (3) and (13)] to describe and, to a certain

extent, understand the complex behavior of evolutions of quantum systems in pure and mixed states. The ranking is probabilistic in nature; it requires a temporal averaging procedure along with a long-time limit, and is limited to comparing expected geodesic evolutions on the underlying manifolds.

(2) We showed [Eqs. (64) and (65)] that the complexity of geodesic paths (Appendix C) corresponding to the evolution of mixed quantum states in the Bloch ball equipped with the Sjöqvist metric is higher than the complexity of geodesic paths (Appendix A) arising from the evolution of pure states on the Bloch sphere furnished with the Fubini-Study metric.

(3) We found that the ranking in terms of the information geometric complexity (64), a quantity that represents the asymptotic temporal behavior of an averaged volume of the region explored on the manifold during the evolution, is in agreement with the ranking in terms of lengths [Eq. (56)] and, in addition, volume ratios in terms of accessible and instantaneous explored volumes (59). For a schematic summary, we refer to Table II.

(4) We confirmed that the choice of the metric on the space of mixed states matters. Specifically, we observed fingerprints of a softening of the complexity on the Bures manifold [Eq. (D20)] compared to the Sjöqvist manifold. This is in agreement with the presence of longer lengths of geodesic paths on the Sjöqvist manifold [Eq. (D19)]. Furthermore, the two manifolds exhibit different curvature properties. The Bures manifold is isotropic, while the Sjöqvist manifold is anisotropic (Appendix E). For a schematic outline, we refer the reader to Table III.

B. Outlook

Like most scientific studies, our investigation suffers a few limitations. From a computational standpoint, our proposed complexity measure requires volume calculations which are much more difficult than action or path length calculations since there are differential equations to be solved. This particular point is in agreement with what was stated in Ref. [10]. Therefore, exact analytical solutions are rare and approximate numerical solutions are unavoidable in more realistic physical scenarios. From a conceptual

TABLE II. Schematic description of geometric properties (i.e., curvature, length, and complexity) along evolutions of pure and mixed quantum states on manifolds equipped with the Fubini-Study and Sjöqvist metrics, respectively.

Type of state	Metric	Sectional curvature	Path length	Information geometric complexity
Pure	Fubini-Study	Constant	Shorter	Lower
Mixed	Sjöqvist	Nonconstant	Longer	Higher

TABLE III. Schematic description of distinct features of the Sjöqvist and Bures metrics in terms of sectional curvatures, path lengths, and information geometric complexities.

Metric	Sectional curvature	Path length	Information geometric complexity
Sjöqvist	Nonconstant	Longer	Stronger
Bures	Constant	Shorter	Weaker

perspective, there are at least two weaknesses. First, there is a freedom in the choice of the metric for mixed quantum states. For further details on the Sjöqvist metric that concern its extension to the degenerate case along with its relation to the Bures metric, we refer to Appendix F. Second, we only limited our work to the study of two-level quantum systems. The ambiguity in the metric affects the notion of speed which, in turn, is related to the concepts of length, action, and complexity. The dependence of the ratio of distance and time on the choice of the metric on the space of mixed quantum states is in agreement with the considerations carried out in Ref. [76]. The restriction to a single qubit, while simple and insightful, cannot be expected to cover the full richness of a higher-dimensional quantum dynamics occurring in an exponentially larger Hilbert space. In particular, scaling laws with respect to the dimensionality of the Hilbert space of suitable physical quantities cannot be addressed in this limiting scenario. Moreover, one fundamental quantum phenomenon that escapes a single qubit treatment is entanglement. This second limitation is similar to the one presented in Ref. [11].

Despite these limitations, we believe our work is relevant also in view of the fact that it paves the way to further lines of inquiry. For instance, it naturally triggers the following questions:

(i) Using our findings along with the ones presented in Refs. [3,11], can one compare the complexity of geodesic motion on differently deformed Bloch spheres by adding anisotropic penalty factors to the Fubini-Study metric?

(ii) What happens to the evolution of mixed states inside a deformed Bloch ball when introducing anisotropic penalty factors? Is the relative ranking in terms of complexity between pure and mixed states preserved under any arbitrary deformations of the Bloch sphere?

(iii) Can one compare the complexity of geodesic paths on a deformed Bloch sphere with the complexity of geodesic paths in a nondeformed Bloch ball?

(iv) Is there a minimal complexity metric for mixed quantum states?

(v) Using our curvature calculations and the analysis presented in Ref. [12], how much does one need to deform the Bloch sphere (that is, introducing

anisotropic penalty factors to get negative sectional curvature) and how high should the dimensionality of a quantum system be in order to address the issue of ergodicity and properly apply thermodynamic arguments to complexity evolution?

The introduction of anisotropic penalty factors can be motivated by experimental considerations. For example, considering a spin-$1/2$ particle in an external magnetic field, it may be the case that is is easier to apply the field in some direction rather that in another. In this case, the penalty factor would be larger where it is more complicated to apply the field. Incorporating these factors in our analysis would open up the lines of investigation of relevance in the context of finding optimal Hamiltonian evolutions, both in terms of efficiency [39,40,77–81] and complexity [41,42], in the presence of physical constraints dictated by experimental limitations.

We hope our work will inspire other scientists and pave the way toward further investigations in this fascinating research direction. For the time being, we leave a more in-depth quantitative discussion on these potential extensions and applications of our theoretical findings to future scientific efforts.

APPENDIX A: GEODESIC PATHS ON THE BLOCH SPERE

In this appendix, we derive the geodesic paths on the two-sphere. In the first derivation, we use simple geometric arguments to obtain the equation of a great circle in spherical coordinates. In the second derivation, we integrate the geodesic equations to get explicit expressions of geodesic paths $\theta = \theta(\eta)$ and $\varphi = \varphi(\eta)$. Then, combining the geodesic paths equations, we show that we also get the equation of a great circle in spherical coordinates that matches our first derivation.

1. Geometric derivation

Geodesics on the two-sphere lie on great circles. Great circles, in turn, can be obtained by intersecting a plane passing through the origin in $\mathbb{R}^3$ with the two-sphere. Assume that the equations of the plane and the two-sphere of unit radius are given by

$$\alpha x + \beta y + \gamma z = 0, \quad \text{and} \quad x^2 + y^2 + z^2 = 1, \quad \text{(A1)}$$

respectively. In Eq. (A1), α, β, and γ belong to $\mathbb{R}$. Using spherical coordinates, we set $x \stackrel{\text{def}}{=} \sin(\theta)\cos(\varphi)$, $y \stackrel{\text{def}}{=} \sin(\theta)\sin(\varphi)$, and $z \stackrel{\text{def}}{=} \cos(\theta)$. Finally, combining the two relations in Eq. (A1), we get the equation of a great circle in spherical coordinates

$$\cot(\theta) = \pm a \cos(\varphi - \bar{\varphi}). \tag{A2}$$

The constants a and $\bar{\varphi}$ in Eq. (A2) are such that $a^2 \stackrel{\text{def}}{=} (\alpha^2 + \beta^2)/\gamma^2$ and $\tan(\bar{\varphi}) \stackrel{\text{def}}{=} \beta/\alpha$, respectively.

2. Dynamics derivation

Consider the system of two coupled second order non-linear ODEs,

$$\begin{cases} \frac{d^2\theta}{d\eta^2} - \sin(\theta)\cos(\theta)\left(\frac{d\varphi}{d\eta}\right)^2 = 0 \\ \frac{d^2\varphi}{d\eta^2} + 2\frac{\cos(\theta)}{\sin(\theta)}\frac{d\theta}{d\eta}\frac{d\varphi}{d\eta} = 0 \end{cases}. \tag{A3}$$

Note that the second relation in Eq. (A3) is equivalent to

$$\frac{d}{d\eta}\left(\frac{d\varphi}{d\eta}\sin^2(\theta)\right) = 0, \tag{A4}$$

that is,

$$\left(\frac{d\varphi}{d\eta}\right)^2 = \frac{c_{\text{FS}}^2}{\sin^4(\theta)}, \tag{A5}$$

with $c_{\text{FS}} = c_{\text{FS}}(\theta_i, \dot{\varphi}_i) \stackrel{\text{def}}{=} \dot{\varphi}_i \sin^2(\theta_i)$ being a real constant where $\theta_i \stackrel{\text{def}}{=} \theta(\eta_i)$ and $\dot{\varphi}_i \stackrel{\text{def}}{=} (d\varphi/d\eta)_{\eta=\eta_i}$ with η_i set equal to zero. Using Eq. (A5), the first relation in Eq. (A3) yields

$$\frac{d^2\theta}{d\eta^2} = c_{\text{FS}}^2 \frac{\cos(\theta)}{\sin^3(\theta)}. \tag{A6}$$

From Eqs. (A5) and (A6), we note that the original system of coupled ODEs in Eq. (A3) can be uncoupled. However, Eq. (A6) is nonlinear and we shall use some tricks to find a way of integrating it. Imposing the unit-speed condition, $\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta) = \text{const}$ with $\text{const} = 1$, we employ Eqs. (A5) and (A6) to get

$$\int d\eta = \pm \int \frac{\sin(\theta)}{\sqrt{\sin^2(\theta) - c_{\text{FS}}^2}} d\theta. \tag{A7}$$

Let us perform a change of variables and put $\epsilon \stackrel{\text{def}}{=} \cos(\theta)$. Then, integration of Eq. (A7) yields

$$\eta = \eta_i \pm \left[\arctan\left(\frac{\epsilon_i}{\sqrt{a_{\text{FS}}^2 - \epsilon_i^2}}\right) - \arctan\left(\frac{\epsilon}{\sqrt{a_{\text{FS}}^2 - \epsilon^2}}\right)\right], \tag{A8}$$

where $a_{\text{FS}}^2 \stackrel{\text{def}}{=} 1 - c_{\text{FS}}^2$. For simplicity, assume $\theta_i \stackrel{\text{def}}{=} \theta(\eta_i) = \cos^{-1}(\epsilon_i) = \pi/2$ with $\eta_i = 0$. Then, $\epsilon_i = 0$ and manipulation of Eq. (A8) yields

$$\cos(\theta) = \pm a_{\text{FS}}\sin(\eta). \tag{A9}$$

We remark that for an arbitrary θ_i, the analog of Eq. (A9) squared is simply given by

$$\cos^2(\theta) = a_{\text{FS}}^2 \sin^2\left[\eta + \arctan\left(\frac{\epsilon_i}{\sqrt{a_{\text{FS}}^2 - \epsilon_i^2}}\right)\right]. \tag{A10}$$

As a consistency check, we note that for $\eta_i = 0$, Eq. (A10) correctly yields

$$\cos^2(\theta_i) = a_{\text{FS}}^2 \sin^2\left[\arctan\left(\frac{\epsilon_i}{\sqrt{a_{\text{FS}}^2 - \epsilon_i^2}}\right)\right] = \epsilon_i^2. \tag{A11}$$

We also point out here that we could have set $\tilde{v}_{\text{FS}}^2 \stackrel{\text{def}}{=} \dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta) = \text{const}$ with $\text{const} \neq 1$. Then, Eq. (A9) would simply become $\cos(\theta) = \pm a_{\text{FS}}\sin(\tilde{v}_{\text{FS}}\eta)$ with $a_{\text{FS}}^2 \stackrel{\text{def}}{=} 1 - (c_{\text{FS}}/\tilde{v}_{\text{FS}})^2$. Equation (A9) allows us to express $\theta = \theta(\eta)$. Next, we need to find the relation $\varphi = \varphi(\eta)$. Using Eqs. (A5) and (A9), we get

$$\frac{d\varphi}{d\eta} = \frac{c_{\text{FS}}}{1 - a_{\text{FS}}^2 \sin^2(\eta)}. \tag{A12}$$

Integration of Eq. (A12) leads to

$$\varphi(\eta) = \varphi_i + \frac{c_{\text{FS}}}{\sqrt{-1 + a_{\text{FS}}^2}}\tanh^{-1}\left[\sqrt{-1 + a_{\text{FS}}^2}\tan(\eta)\right]. \tag{A13}$$

Note that $\sqrt{-1 + a_{\text{FS}}^2} = ic_{\text{FS}}$, with $i \in \mathbb{C}$ denoting the imaginary unit. Furthermore, recalling from Ref. [82] that the inverse hyperbolic tangent function of a complex variable satisfies the relation $\tanh^{-1}(z) = -i\tan^{-1}(iz)$, setting $z \stackrel{\text{def}}{=} ix$ with $x \in \mathbb{R}$ and $z \in \mathbb{C}$, we obtain

$$-i\tanh^{-1}(ix) = \arctan(x). \tag{A14}$$

Finally, using Eqs. (A13) and (A14), we get

$$\varphi(\eta) = \varphi_i + \tan^{-1}[c_{\text{FS}}\tan(\eta)], \tag{A15}$$

that is,

$$\tan[\varphi(\eta) - \varphi_i] = c_{\text{FS}}\tan(\eta). \tag{A16}$$

For completeness, we point out that using Eqs. (A9) and (A16) along with the following two trigonometric identities,

$$\cos^2(\theta) = \frac{\cot^2(\theta)}{1 + \cot^2(\theta)}, \quad \sin^2(\eta) = \frac{\tan^2(\eta)}{1 + \tan^2(\eta)}, \quad \text{(A17)}$$

we get after some straightforward algebraic manipulations the equation of a great circle,

$$\cot(\theta) = \pm\sqrt{\frac{1 - c_{\text{FS}}^2}{c_{\text{FS}}^2}} \sin(\varphi - \varphi_i). \quad \text{(A18)}$$

Equation (A18) is identical to Eq. (A2) once we identify a and $\bar{\varphi}$ with $\sqrt{\frac{1 - c_{\text{FS}}^2}{c_{\text{FS}}^2}}$ and $\varphi_i + \pi/2$, respectively.

$$d\rho = \frac{1}{2} \begin{pmatrix} \cos(\theta)dr - r\sin(\theta)d\theta & e^{-i\varphi}[\sin(\theta)dr + r\cos(\theta) - ir\sin(\theta)d\varphi] \\ e^{i\varphi}[\sin(\theta)dr + r\cos(\theta) + ir\sin(\theta)d\varphi] & -\cos(\theta)dr + r\sin(\theta)d\theta \end{pmatrix}. \quad \text{(B3)}$$

Finally, using Eqs. (36), (37), (38), and (B3), ds_{Bures}^2 in Eq. (B1) reduces to ds_{Bures}^2 in Eq. (40). For further details on the Bures metric for high-dimensional quantum systems, we refer to Refs. [84,85].

APPENDIX C: GEODESIC PATHS IN THE BLOCH BALL WITH THE SJÖQVIST METRIC

In this appendix, we study the geodesic paths in the Bloch ball using the Sjöqvist metric. Recall that the form of the geodesic equation remains unchanged under affine transformations. An affine transformation of a parameter η is the change of variable $\eta \to \eta' = \tilde{a}\,\eta + \tilde{b}$, with $\tilde{a}$ and $\tilde{b}$ in $\mathbb{R}$. Any other transformation will generate extra terms in the geodesic equation [86]. The affine parameter for a geodesic is unique up to an affine change of variables. Therefore, it is important to keep in mind that geodesics are curves with a preferred parametrization. In what follows, we explicitly analyze geodesic paths parametrized in terms of two distinct affine parametrizations.

1. The Sjöqvist θ-affine parametrization

In Ref. [53], Sjöqvist focused on finding geodesic paths connecting points in the Bloch ball laying in a plane that contains the origin. In other words, using spherical coordinates (r, θ, φ) and keeping $\varphi = \text{const}$, geodesics were obtained by minimizing $\int ds_{\text{Sjöqvist}}$ over all curves connecting points (r_i, θ_i) and (r_f, θ_f). More specifically, one

APPENDIX B: THE BURES INFINITESIMAL ELEMENT

In this appendix, we derive Eq. (40). Recall that for the single qubit case the Bures distance between two infinitesimally close density matrices ρ and $\rho + d\rho$ is given by [83]

$$ds_{\text{Bures}}^2 \stackrel{\text{def}}{=} \frac{1}{2} \sum_{n,m=0}^{1} \frac{|\langle e_m | d\rho | e_n \rangle|^2}{p_m + p_n}, \quad \text{(B1)}$$

where $\{|e_n\rangle\}_{n=0,1}$ is an orthonormal basis of eigenvectors of ρ with eigenvalues $\{p_n\}_{n=0,1}$. Therefore, $\rho = p_0|e_0\rangle\langle e_0| + p_1|e_1\rangle\langle e_1|$. From the expression of ρ in Eq. (34), we get

$$d\rho = \frac{\partial\rho}{\partial r}dr + \frac{\partial\rho}{\partial\theta}d\theta + \frac{\partial\rho}{\partial\varphi}d\varphi, \quad \text{(B2)}$$

that is,

obtains the curve $[\theta_i, \theta_f] \ni \theta \mapsto r_{\text{Sjöqvist}}(\theta) \in (0, 1]$ that minimizes the length $\mathcal{L}_{\text{Sjöqvist}}(\theta_i, \theta_f)$ defined as

$$\mathcal{L}_{\text{Sjöqvist}}(\theta_i, \theta_f) \stackrel{\text{def}}{=} \int_{s_i}^{s_f} \sqrt{ds_{\text{Sjöqvist}}^2} = \frac{1}{2}\int_{\theta_i}^{\theta_f} \text{L}(r', r, \theta)d\theta. \quad \text{(C1)}$$

In Eq. (C1), $r' \stackrel{\text{def}}{=} dr/d\theta$ and $\text{L}(r', r, \theta)$ is the Lagrangian-like function given by

$$\text{L}(r', r, \theta) \stackrel{\text{def}}{=} \sqrt{1 + \frac{r'^2}{1 - r^2}}. \quad \text{(C2)}$$

From Eq. (C2), note that $\text{L} = \text{L}(r', r)$ does not depend explicitly on θ. Therefore, $\partial\text{L}/\partial\theta = 0$. In this case, it happens that the Euler-Lagrange equation

$$\frac{d}{d\theta}\frac{\partial\text{L}(r', r)}{\partial r'} - \frac{\partial\text{L}(r', r)}{\partial r} = 0, \quad \text{(C3)}$$

reduces to the so-called Beltrami identity,

$$\text{L}(r', r) - r'\frac{\partial\text{L}(r', r)}{\partial r'} = \text{const}. \quad \text{(C4)}$$

Indeed, combining Eq. (C3) with the identity

$$\frac{d\mathrm{L}(r', r)}{d\theta} = \frac{\partial \mathrm{L}(r', r)}{\partial r'} r'' + \frac{\partial \mathrm{L}(r', r)}{\partial r} r', \qquad \text{(C5)}$$

we get

$$\frac{d\mathrm{L}(r', r)}{d\theta} = \frac{d}{d\theta}\left(r' \frac{\partial \mathrm{L}(r', r)}{\partial r'}\right). \qquad \text{(C6)}$$

Equation (C6) finally leads to the Beltrami identity in Eq. (C4). Using Eqs. (C2) and (C4), we get

$$\frac{1}{\sqrt{1 + \frac{r'^2}{1 - r^2}}} = \text{const} \equiv c_{\mathrm{S}}, \qquad \text{(C7)}$$

that is,

$$\frac{r'^2}{1 - r^2} = \text{const} \equiv \mathrm{k} \stackrel{\text{def}}{=} \frac{1 - c_{\mathrm{S}}^2}{c_{\mathrm{S}}^2}, \qquad \text{(C8)}$$

Integrating Eq. (C8) and imposing the boundary conditions $r(\theta_i) = r_i$ and $r(\theta_f) = r_f$ with $\theta_i = 0$, we finally get the geodesic path

$$r_{\mathrm{Sjöqvist}}(\theta) = \sin\left[\sin^{-1}(r_i) + \frac{\sin^{-1}(r_f) - \sin^{-1}(r_i)}{\theta_f}\theta\right]. \quad \text{(C9)}$$

Evaluating Eq. (C7) at $\theta_i = 0$ and using Eq. (C9), we note that the constant c_{S} can be expressed in terms of r_i, r_f, and θ_f as

$$
\begin{aligned}
c_{\mathrm{S}} &= c_{\mathrm{S}}(r_i, r_f, \theta_f) \\
&\stackrel{\text{def}}{=} \frac{\theta_f}{\sqrt{\theta_f^2 + [\sin^{-1}(r_f) - \sin^{-1}(r_i)]^2}},
\end{aligned}
\qquad \text{(C10)}
$$

with $0 \leq c_{\mathrm{S}} \leq 1$, as correctly expected. For completeness, we point out that in terms of initial conditions $r(\theta_i) = r_i$ and $r'(\theta_i) = r'_i$ with $\theta_i = 0$, $r_{\mathrm{Sjöqvist}}(\theta)$ in Eq. (C9) can be recast as

$$r_{\mathrm{Sjöqvist}}(\theta) = \sin\left[\sin^{-1}(r_i) + \frac{r'_i}{\sqrt{1 - r_i^2}}\theta\right], \qquad \text{(C11)}$$

where we used $\cos\left[\sin^{-1}(r_i)\right] = \sqrt{1 - r_i^2}$ in the raw solution of the form $r(\theta) = \sin(c_1\theta + c_2)$ with real integrations constants c_1 and c_2. As a final remark, observe from Eq. (C9) that θ plays the role of the affine parameter that characterizes the points on the curve of minimal length connecting the initial and final points in the Bloch ball.

2. The canonical η-affine parametrization

To find the geodesic paths in the Bloch ball suing the Sjöqvist metric and parametrized in terms of the "canonical" affine parameter η, we need to integrate Eq. (42). We note from Eq. (42) that the differential equations for the angular and radial motion are not coupled. In particular, the angular motion is identical to the one that emerges when using the Fubini-Study metric. Therefore, we refer to Appendix A for the characterization of the angular motion. Let us focus here on the radial motion specified by the relation

$$\ddot{r} + \frac{r}{1 - r^2}\dot{r}^2 = 0, \qquad \text{(C12)}$$

with $\dot{r} \stackrel{\text{def}}{=} dr/d\eta$. Assuming initial conditions given by $r(\eta_i) = r_i$ and $\dot{r}(\eta_i) = \dot{r}_i$ with $\eta_i = 0$, integration of Eq. (C12) yields

$$r_{\mathrm{Sjöqvist}}(\eta) = \sin\left[\sin^{-1}(r_i) + \frac{\dot{r}_i}{\sqrt{1 - r_i^2}}\eta\right]. \qquad \text{(C13)}$$

Equation (C13) represents the radial geodesic path parametrized in terms of the affine parameter η, the time coordinate, related to the proper length $ds_{\mathrm{Sjöqvist}}$. Note that the speed of evolution along geodesic paths is constant and equals,

$$v_{\mathrm{Sjöqvist}} \stackrel{\text{def}}{=} (1/2)[(1 - r^2)^{-1}\dot{r}^2 + \dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2]^{1/2}. \quad \text{(C14)}$$

The constancy of $v_{\mathrm{Sjöqvist}}$ in Eq. (C14) can be verified by exploiting the constancy of v_{FS} along with the constancy of $(1 - r^2)^{-1}\dot{r}^2$ by means of Eq. (C13). Interestingly, assuming $\varphi = \text{const}$, the geodesic equation $\theta(\eta)$ becomes $\ddot{\theta} = 0$. Therefore, assuming $\theta(\eta_i) = 0$ and $\theta(\eta_f) = \theta_f$, we get $\theta(\eta) = (\theta_f/\eta_f)\eta$. Then, considering an affine change of variables defined by $\eta \to \theta = \theta(\eta) \stackrel{\text{def}}{=} (\theta_f/\eta_f)\eta$, Eq. (C12) becomes

$$r'' + \frac{r}{1 - r^2}r'^2 = 0, \qquad \text{(C15)}$$

with $r' \stackrel{\text{def}}{=} dr/d\eta$. Finally, assuming $r(\theta_i) = r_i$ and $r(\theta_f) = r_f$, integration of Eq. (C15) yields exactly $r_{\mathrm{Sjöqvist}}(\theta)$ in Eq. (C9).

APPENDIX D: GEODESIC PATHS IN THE BLOCH BALL WITH THE BURES METRIC

In this appendix, we study the geodesic paths in the Bloch ball using the Bures metric.

1. The Sjöqvist-like θ-affine parametrization

Following the line of reasoning used in the Sjöqvist affine parametrization case in Appendix C, we use spherical coordinates (r, θ, φ) and keep $\varphi = \text{const}$. Then, geodesics are obtained by minimizing $\int ds_{\mathrm{Bures}}$ over all curves

connecting points (r_i, θ_i) and (r_f, θ_f). More specifically, one arrives at the curve $[\theta_i, \theta_f] \ni \theta \mapsto r_{\text{Bures}}(\theta) \in [0, 1]$ that minimizes the length $\mathcal{L}_{\text{Bures}}(\theta_i, \theta_f)$ defined as

$$\mathcal{L}_{\text{Bures}}(\theta_i, \theta_f) \overset{\text{def}}{=} \int_{s_i}^{s_f} \sqrt{ds_{\text{Bures}}^2} = \frac{1}{2} \int_{\theta_i}^{\theta_f} \text{L}(r', r, \theta) d\theta. \quad \text{(D1)}$$

In Eq. (D1), $r' \overset{\text{def}}{=} dr/d\theta$ and $\text{L}(r', r, \theta)$ is the Lagrangian-like function given by

$$\text{L}(r', r, \theta) \overset{\text{def}}{=} \sqrt{r^2 + \frac{r'^2}{1 - r^2}}. \quad \text{(D2)}$$

From Eq. (D2), we follow step-by-step the analysis carried out in the Sjöqvist metric case and arrive at the following analog of Eq. (C7):

$$\frac{r^2}{\sqrt{r^2 + \frac{r'^2}{1-r^2}}} = \text{const.} \equiv c_{\text{B}}. \quad \text{(D3)}$$

Manipulating Eq. (D3) and imposing the boundary conditions $r(\theta_i) = r_i$ and $r(\theta_f) = r_f$ with $\theta_i = 0$, we obtain

$$\int_{r_i}^{r_f} \frac{dr}{\sqrt{r^2(\frac{r^2}{c_{\text{B}}^2} - 1)(1 - r^2)}} = \int_{\theta_i}^{\theta_f} d\theta, \quad \text{(D4)}$$

with $0 < c_{\text{B}} \le r \le 1$. For notational simplicity, let us put $a_{\text{B}} \overset{\text{def}}{=} 1/c_{\text{B}}^2 > 1$. Then, integration of Eq. (D4) by use of *Mathematica* gives

$$\text{I}(r) \overset{\text{def}}{=} \int \frac{dr}{\sqrt{r^2(\frac{r^2}{c_{\text{B}}^2} - 1)(1 - r^2)}}$$

$$= \frac{r\sqrt{r^2 - 1}\sqrt{a_{\text{B}}r^2 - 1}}{\sqrt{-r^2(r^2 - 1)(a_{\text{B}}r^2 - 1)}} \tanh^{-1}\left(\frac{\sqrt{r^2 - 1}}{\sqrt{a_{\text{B}}r^2 - 1}}\right)$$

$$+ \text{const.} \quad \text{(D5)}$$

Manipulation of Eq. (D5) yields

$$\text{I}(r) = i \tanh^{-1}\left(i\frac{\sqrt{1 - r^2}}{\sqrt{a_{\text{B}}r^2 - 1}}\right) + \text{const.}, \quad \text{(D6)}$$

that is,

$$\text{I}(r) = -\arctan\left(\frac{\sqrt{1 - r^2}}{\sqrt{a_{\text{B}}r^2 - 1}}\right) + \text{const.} \quad \text{(D7)}$$

Substituting Eq. (D7) into Eq. (D5), we finally get that the radial geodesic path in the Bures case is given by

$$r_{\text{Bures}}(\theta) = \left\{\frac{1 + \tan^2[\mathcal{A} - (\theta - \theta_i)]}{1 + a_{\text{B}}\tan^2[\mathcal{A} - (\theta - \theta_i)]}\right\}^{1/2}, \quad \text{(D8)}$$

where the constant $\mathcal{A}$ in Eq. (D8) is defined as

$$\mathcal{A} = \mathcal{A}(r_i, a_{\text{B}}) \overset{\text{def}}{=} \arctan\left(\frac{\sqrt{1 - r_i^2}}{\sqrt{a_{\text{B}}r_i^2 - 1}}\right). \quad \text{(D9)}$$

For consistency's check, we note that we correctly obtain $r_{\text{Bures}}(\theta_i) = r_i$ with $r_{\text{Bures}}(\theta)$ in Eq. (D8). Moreover, $0 \le r_{\text{Bures}}(\theta) \le 1$ since $a_{\text{B}} > 1$ in Eq. (D8). Clearly, Eq. (D8) with θ_i set equal to zero should be compared, for a given pair of initial conditions r_i and r_i', with its corresponding analog in the framework of Sjöqvist metric [that is, Eq. (C11)]. Such comparison can be carried out once we express a_{B} and $\mathcal{A}$ in Eq. (D8) in terms of r_i and r_i'. After some algebra, it happens that

$$a_{\text{B}} = a_{\text{B}}(r_i, r_i') \overset{\text{def}}{=} \frac{1}{r_i^2} + \frac{r_i'^2}{r_i^4(1 - r_i^2)}, \quad \text{and}$$

$$\mathcal{A} = \mathcal{A}(r_i, r_i') \overset{\text{def}}{=} \tan^{-1}\left[\frac{r_i}{r_i'}(1 - r_i^2)\right]. \quad \text{(D10)}$$

For completeness, we point out that a_{B} and $\mathcal{A}$ can only be implicitly expressed in terms of the two boundary conditions $r_i = r(\theta_i)$ and $r_f = r(\theta_f)$ via the two relations

$$r_i^2 = \frac{1 + \tan^2 \mathcal{A}}{1 + a_{\text{B}}\tan^2 \mathcal{A}}, \quad r_f^2 = \frac{1 + \tan^2(\mathcal{A} - \theta_f)}{1 + a_{\text{B}}\tan^2(\mathcal{A} - \theta_f)}. \quad \text{(D11)}$$

Finally, setting $\theta_i = 0$ and using Eq. (D10), $r_{\text{Bures}}(\theta)$ in Eq. (D8) can be formally compared with $r_{\text{Sjöqvist}}(\theta)$ in Eq. (C11).

2. The canonical η-affine parametrization

From Eq. (40), the only nonvanishing Christoffel connection coefficients are

$$\Gamma_{11}^1 = \frac{r}{1 - r^2}, \quad \Gamma_{22}^1 = -(1 - r^2)r,$$

$$\Gamma_{33}^1 = -(1 - r^2)r\sin^2(\theta),$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{33}^2 = -\sin(\theta)\cos(\theta),$$

$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{\cos(\theta)}{\sin(\theta)}. \quad \text{(D12)}$$

Therefore, geodesics satisfy the geodesic equations in Eq. (11) described in terms of a system of three coupled second order nonlinear ODEs,

$$\begin{cases} \ddot{r} + \frac{r}{1-r^2}\dot{r}^2 - (1-r^2)r[\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2] = 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}^2 = 0 \\ \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\frac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\dot{\varphi} = 0 \end{cases} , \quad \text{(D13)}$$

where $\dot{r} \overset{\text{def}}{=} dr/d\eta$ with η being an affine parameter. We note from Eqs. (42) and (D13) that, unlike what happens with the Sjöqvist metric, when using the Bures metric in the Bloch ball, the radial and angular evolutions are coupled. Integration of the system in Eq. (D13) is rather complicated and, in what follows, we limit our attention to geodesic paths with constant φ. In this case, Eq. (D13) reduces to

$$\begin{cases} \ddot{r} + \frac{r}{1-r^2}\dot{r}^2 - (1-r^2)r\dot{\theta}^2 = 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0 \end{cases} . \quad \text{(D14)}$$

Manipulating the second relation in Eq. (D14), we note that $r^2\dot{\theta} = \text{const}$. Then, given our knowledge of $r = r(\theta)$ in Eq. (D8), we get

$$\int_{\theta_i}^{\theta} r^2(\theta)d\theta = r_i^2\dot{\theta}_i\eta, \quad \text{(D15)}$$

where η_i is assumed to be equal to zero. Using Eq. (D8), integration of Eq. (D15) with *Mathematica* along with some algebraic manipulations yield $\theta = \theta(\eta)$ as

$$\theta_{\text{Bures}}(\eta)$$
$$= \theta_i + \mathcal{A}$$
$$- \tan^{-1}\left\{ \frac{1}{\sqrt{a_B}}\tan\left[\tan^{-1}(\sqrt{a_B}\tan\mathcal{A}) - \sqrt{a_B}r_i^2\dot{\theta}_i\eta\right] \right\}. \quad \text{(D16)}$$

Observe that $\theta_{\text{Bures}}(\eta)$ is a bounded function for any $\eta \geq 0$. For consistency, note that we correctly obtain from Eq. (D16) that $\theta_{\text{Bures}}(\eta_i) = \theta_i$ and $\dot{\theta}_{\text{Bures}}(\eta_i) = \dot{\theta}_i$. Finally, using Eq. (D16) and recalling that $r^2\dot{\theta} = r_i^2\dot{\theta}_i$, we get $r = r(\eta)$ as

$$r_{\text{Bures}}(\eta)$$
$$= \left(\frac{a_B + \tan^2\left[\tan^{-1}\left(\sqrt{a_B}\tan\mathcal{A}\right) - \sqrt{a_B}r_i^2\dot{\theta}_i\eta\right]}{a_B + a_B\tan^2\left[\tan^{-1}\left(\sqrt{a_B}\tan\mathcal{A}\right) - \sqrt{a_B}r_i^2\dot{\theta}_i\eta\right]} \right)^{1/2}. \quad \text{(D17)}$$

Note that, $0 \leq r_{\text{Bures}}(\eta) \leq 1$ for any $\eta \geq 0$ since $a_B > 1$ in Eq. (D17). As a side remark, note that the speed of evolution along these geodesic paths with φ-fixed is constant and equals,

$$v_{\text{Bures}} \overset{\text{def}}{=} (1/2)[(1-r^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2]^{1/2}. \quad \text{(D18)}$$

The constancy of v_{Bures} in Eq. (D18) is a consequence of Eq. (D3) along with the constancy of $r^2\dot{\theta}$. As a consistency check, observe that Eq. (D17) correctly leads to $r_{\text{Bures}}(\eta_i) = r_i$ and $r_{\text{Bures}}^2(\eta)\dot{\theta}_{\text{Bures}}(\eta) = r_i^2\dot{\theta}_i = \text{const}$. Observe that $r_{\text{Bures}}(\eta)$ in Eq. (D17) is the analog of $r_{\text{Sjöqvist}}(\eta)$ in Eq. (C13). It is evident from Eq. (D17) that the radial variable $r_{\text{Bures}}(\eta)$ depends on the angular motion via $\dot{\theta}_i$ and $r_i' = (dr/d\theta)_{\theta=\theta_i}$. This latter quantity enters the expressions of a_B and $\mathcal{A}$ as presented in Eq. (D10).

In what follows, we briefly compare features that appear in the Bures and Sjöqvist cases. Using Eq. (D3) and exploiting the constancy of $r^2\dot{\theta}$, we have

$$\mathcal{L}_{\text{Bures}}(\theta_f) = \frac{1}{2}\sqrt{r_i^2 + \frac{r_i'^2}{1-r_i^2}\dot{\theta}_i\eta_f} \leq \mathcal{L}_{\text{Sjöqvist}}(\theta_f), \quad \text{(D19)}$$

with $\mathcal{L}_{\text{Sjöqvist}}(\theta_f)$ in Eq. (54). Moreover, using Eqs. (A9), (C13), (D16), and (D17) and noting that $V_{\text{Sjöqvist}}^{(\text{accessible})} = \pi^2/4 = 2V_{\text{Bures}}^{(\text{accessible})}$, we get after some algebra that for sufficiently large values of η and fixing φ,

$$\frac{V_{\text{Bures},\varphi=\text{const}}^{(\text{explored})}(\eta)}{V_{\text{Bures}}^{(\text{accessible})}} \leq \frac{V_{\text{Sjöqvist},\varphi=\text{const}}^{(\text{explored})}(\eta)}{V_{\text{Sjöqvist}}^{(\text{accessible})}}. \quad \text{(D20)}$$

In particular, the qualitative behavior of the explored volumes in Eq. (D20) is given by

$$V_{\text{Sjöqvist},\varphi=\text{const}}(\eta) \sim \eta\cos^{-1}[\sin(\eta)], \quad \text{and}$$

$$V_{\text{Bures},\varphi=\text{const}}(\eta) \sim \sqrt{\frac{\tan^2(\eta)}{1+\tan^2(\eta)}}\tan^{-1}(\eta). \quad \text{(D21)}$$

Since the sequential application of the averaging and asymptotic limit procedures preserve the ranking in Eq. (D20), we expect that the complexity of the evolution on the Bures manifold is softer than that on the Sjöqvist manifold. This is not completely unexpected given that $\mathcal{L}_{\text{Bures}}(\theta_f) \leq \mathcal{L}_{\text{Sjöqvist}}(\theta_f)$ and, above all, the presence of a correlational structure in the equations of motion between the radial and angular directions. Such a structure is absent in the Sjöqvist case. Correlational structures do tend to shrink the explored volumes of regions on the manifold underlying the dynamics and, therefore, tend to weaken the complexity of the evolution [87,88]. In summary, although Eq. (D20) assumes $\varphi = \text{const}$, the information geometric complexity of the evolution of quantum systems in mixed quantum states seems to depend on the choice of the metric selected on the underlying manifold.

APPENDIX E: CURVATURE OF QUANTUM STATE MANIFOLDS

In this appendix, we outline some curvature properties of the manifold of pure states equipped with the Fubini-Study metric along with those of a manifold of mixed quantum states endowed with the Sjöqvist and Bures metrics. In particular, for each case, we report expressions of the tensor metric components, infinitesimal line elements, Christoffel connection coefficients, Ricci tensor components, Riemann curvature tensor components, scalar curvatures and, finally, sectional curvatures.

1. Preliminaries

Given a metric tensor $g_{\mu\nu}(\xi)$ with corresponding line element $ds^2 \overset{\text{def}}{=} g_{\mu\nu} d\xi^\mu d\xi^\nu$, the Christoffel connection coefficients are defined as [89]

$$\Gamma^\rho_{\mu\nu} \overset{\text{def}}{=} \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \qquad (\text{E1})$$

where $\partial_\mu \overset{\text{def}}{=} \partial/\partial\xi^\mu$ and $g^{\rho\sigma}$ are the coefficients of the inverse metric tensor such that $g^{\rho\sigma} g_{\sigma\beta} \overset{\text{def}}{=} \delta^\rho_\beta$ with δ denoting the Kronecker delta symbol. From the expression of the Christoffel connection coefficients in Eq. (E1), the Ricci tensor and Riemann curvature tensor components can be defined as [89]

$$\mathcal{R}_{\mu\nu} \overset{\text{def}}{=} \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} - \Gamma^\gamma_{\mu\alpha} \Gamma^\alpha_{\nu\gamma}, \qquad (\text{E2})$$

and

$$\mathcal{R}^\alpha_{\mu\nu\rho} \overset{\text{def}}{=} \partial_\nu \Gamma^\alpha_{\mu\rho} - \partial_\rho \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\rho} - \Gamma^\alpha_{\beta\rho} \Gamma^\beta_{\mu\nu}, \qquad (\text{E3})$$

respectively. In terms of the quantities in Eqs. (E2) and (E3), the scalar curvature $\mathcal{R}$ is given by

$$\mathcal{R} \overset{\text{def}}{=} \mathcal{R}_{\mu\nu} g^{\mu\nu} = \mathcal{R}_{\alpha\beta\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}. \qquad (\text{E4})$$

We remark that the sign of the scalar curvature of a curved manifold is subject to convention. For instance, following Weinberg's sign convention [89], the scalar curvature of a two-sphere of unit radius equals -2. Here, however, we are using the opposite sign convention. In Weinberg's book, $(\mathcal{R}_{\mu\nu})_{\text{Weinberg}} \overset{\text{def}}{=} -\mathcal{R}_{\mu\nu}$ with $\mathcal{R}_{\mu\nu}$ in Eq. (E2). Adopting our sign convention, the scalar curvature of a two-sphere of unit radius equals $+2$. The scalar curvature $\mathcal{R}$ of a manifold $\mathcal{M}$ in Eq. (E4) can also be recast as the sum of all sectional curvatures $\mathcal{K}(\hat{e}_i, \hat{e}_j)$ of planes spanned by pairs $\{\hat{e}_i, \hat{e}_j\}$ of orthonormal basis elements $\{\hat{e}_k\}$ with $1 \le k \le |T_p\mathcal{M}|$ [90],

$$\mathcal{R} = \sum_{i \neq j} \mathcal{K}(\hat{e}_i, \hat{e}_j). \qquad (\text{E5})$$

The pair $\{\hat{e}_i, \hat{e}_j\}$ is a basis for a 2-plane $\Pi \subset T_P\mathcal{M}$, a two-dimensional subspace of the tangent space to $\mathcal{M}$ at P. The sectional curvature $\mathcal{K}(\hat{e}_i, \hat{e}_j)$ is defined as [89]

$$\mathcal{K}(\hat{e}_i, \hat{e}_j) \overset{\text{def}}{=} \frac{\text{Riemann}(\hat{e}_i, \hat{e}_j, \hat{e}_j, \hat{e}_i)}{\langle \hat{e}_i, \hat{e}_i \rangle \langle \hat{e}_j, \hat{e}_j \rangle - \langle \hat{e}_i, \hat{e}_j \rangle^2} \qquad (\text{E6})$$

where Riemann$(a, b, b, a) \overset{\text{def}}{=} \mathcal{R}_{\alpha\beta\gamma\delta} a^\alpha b^\beta b^\gamma a^\delta$ with a, b being two arbitrary vectors on the 2-plane Π spanned by $\{\hat{e}_i, \hat{e}_j\}$ and, finally, $\langle a, b \rangle \overset{\text{def}}{=} g_{\mu\nu} a^\mu b^\nu$. The constancy of the sectional curvatures is related to the concept of maximally symmetric manifold. Specifically, an isotropic n-dimensional manifold $\mathcal{M}$ is a maximally symmetric manifold with $n(n + 1)/2$ independent Killing vectors where the geometry does not depend on directions. For a maximally symmetric manifold, the following simplifying relations hold true among the scalar curvature $\mathcal{R}$, the constant sectional curvature $\mathcal{K}$, the Ricci tensor components $\mathcal{R}_{\alpha\beta}$, and the Riemann curvature tensor components $\mathcal{R}_{\alpha\beta\gamma\delta}$ [91],

$$\mathcal{R} = n(n-1)\mathcal{K}, \qquad \mathcal{R}_{\alpha\beta} = (n-1)\mathcal{K} g_{\alpha\beta},$$
$$\mathcal{R}_{\alpha\beta\gamma\delta} = \frac{\mathcal{R}}{n(n-1)} (g_{\beta\delta} g_{\alpha\gamma} - g_{\beta\gamma} g_{\alpha\delta}). \qquad (\text{E7})$$

Isometries play a key role in the characterization of maximally symmetric manifolds. Recall that an isometry of the metric $g_{\mu\nu}(\xi)$ is a distance-preserving transformation $\xi \to \xi'$ such that [89]

$$g_{\mu\nu}(\xi) = \frac{\partial \xi'^\rho}{\partial \xi^\mu} \frac{\partial \xi'^\sigma}{\partial \xi^\nu} g_{\rho\sigma}(\xi'). \qquad (\text{E8})$$

All the infinitesimal isometries of the metric $g_{\mu\nu}(\xi)$ are determined by the Killing vectors of the metric. Consider an infinitesimal coordinate transformation $\xi^\mu \to \xi'^\mu \overset{\text{def}}{=} \xi^\mu + \epsilon k^\mu(\xi)$, with $|\epsilon| \ll 1$. A vector field $k^\mu(\xi)$ is a Killing vector for the metric $g_{\mu\nu}(\xi)$ if it satisfies the so-called Killing condition,

$$D_\rho k_\sigma + D_\sigma k_\rho = 0, \qquad (\text{E9})$$

where $D_\rho k_\sigma \overset{\text{def}}{=} \partial_\rho k_\sigma - \Gamma^\lambda_{\sigma\rho} k_\lambda$ and $\partial_\rho \overset{\text{def}}{=} \partial/\partial\xi^\rho$. For completeness, we point out that what really determines the infinitesimal isometries of a metric $g_{\mu\nu}(\xi)$ is the space of vector fields spanned by the Killing vectors since any linear combination of Killing vectors with constant coefficients is a Killing vector. In general, it is highly nontrivial solving the Killing conditions in Eq. (E9).

2. Type of manifold

a. Manifold equipped with the Fubini-Study metric

In the case of the two-dimensional manifold of pure states equipped with the Fubini-Study metric $g_{\mu\nu}^{\text{FS}}(\xi)$ with $\xi \overset{\text{def}}{=} (\xi^1, \xi^2) = (\theta, \varphi)$, the infinitesimal line element is given by $ds_{\text{FS}}^2 \overset{\text{def}}{=} (1/4)[d\theta^2 + \sin^2(\theta)d\varphi^2]$. In this case, the nonzero Christoffel connection coefficients in Eq. (E1) are given by

$$\Gamma_{22}^1 = -\sin(\theta)\cos(\theta), \qquad \Gamma_{12}^2 = \frac{\cos(\theta)}{\sin(\theta)}. \qquad (E10)$$

Furthermore, using Eq. (E10), the nonzero Ricci and Riemann curvature tensor components in Eqs. (E2) and (E3) are

$$\mathcal{R}_{11} = 1, \qquad \mathcal{R}_{22} = \sin^2(\theta), \qquad (E11)$$

and

$$\mathcal{R}_{1212} = \frac{1}{4}\sin^2(\theta), \qquad (E12)$$

respectively. For completeness, we point out that exploiting the symmetry properties of the Riemann curvature tensor, we also have $\mathcal{R}_{1221} = \mathcal{R}_{2112} = -\mathcal{R}_{1212}$ and $\mathcal{R}_{2121} = \mathcal{R}_{1212}$. To calculate the sectional curvatures $\mathcal{K}(\hat{e}_i, \hat{e}_j)$ in Eq. (E6), we note that the unit tangent vectors $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$ in spherical coordinates are given by

$$\hat{e}_r \overset{\text{def}}{=} \frac{\partial_r \vec{r}}{\|\partial_r \vec{r}\|} = \sin(\theta)\cos(\varphi)\hat{x} + \sin(\theta)\sin(\varphi)\hat{y} + \cos(\theta)\hat{z},$$

$$\hat{e}_\theta \overset{\text{def}}{=} \frac{\partial_\theta \vec{r}}{\|\partial_\theta \vec{r}\|} = \cos(\theta)\cos(\varphi)\hat{x} + \cos(\theta)\sin(\varphi)\hat{y} - \sin(\theta)\hat{z},$$

$$\hat{e}_\varphi \overset{\text{def}}{=} \frac{\partial_\varphi \vec{r}}{\|\partial_\varphi \vec{r}\|} = -\sin(\varphi)\hat{x} + \cos(\varphi)\hat{y}, \qquad (E13)$$

where $\vec{r} \overset{\text{def}}{=} r\sin(\theta)\cos(\varphi)\hat{x} + r\sin(\theta)\sin(\varphi)\hat{y} + r\cos(\theta)\hat{z}$, and $\|\cdot\|$ denotes the usual Euclidean norm. In the Fubini-Study case, we have that $ds_{\text{FS}}^2 = \mathbf{ds}_{\text{FS}} \cdot \mathbf{ds}_{\text{FS}}$ with the infinitesimal vector element $\mathbf{ds}_{\text{FS}}$ given by

$$\mathbf{ds}_{\text{FS}} \overset{\text{def}}{=} \frac{1}{2}\hat{e}_\theta d\theta + \frac{1}{2}\sin(\theta)\hat{e}_\varphi d\varphi, \qquad (E14)$$

where $1/2$ and $(1/2)\sin(\theta)$ in Eq. (E14) denote the so-called scale factors of the metric [92]. Then, inserting Eqs. (E13) and (E12) into Eq. (E6), we find

$$\mathcal{K}_{\text{FS}}(\hat{e}_\theta, \hat{e}_\varphi) = \mathcal{K}_{\text{FS}}(\hat{e}_\varphi, \hat{e}_\theta) = 4. \qquad (E15)$$

Thus, from Eq. (E15), we conclude that the manifold of pure states equipped with the Fubini-Study metric is an isotropic manifold of constant (positive) sectional curvature with (positive) constant Ricci curvature $\mathcal{R}_{\text{FS}} = 8$. As a final remark, we emphasize that for a two-sphere with metric $4ds_{\text{FS}}^2 \overset{\text{def}}{=} d\theta^2 + \sin^2(\theta)d\varphi^2$, Killing vectors can be found [93]

$$k_1 \overset{\text{def}}{=} L_x/i\hbar = \sin(\varphi)\partial_\theta + \cot(\theta)\cos(\varphi)\partial_\varphi,$$

$$k_2 \overset{\text{def}}{=} L_y/i\hbar = -\cos(\varphi)\partial_\theta + \cot(\theta)\sin(\varphi)\partial_\varphi,$$

$$k_3 \overset{\text{def}}{=} L_z/i\hbar = -\partial_\varphi. \qquad (E16)$$

Then, the most general Killing vector k is a linear combination of these three independent Killing vectors $\{k_1, k_2, k_3\}$. The three vectors describe rotations and are just the angular momentum operators $\{L_x, L_y, L_z\}$, the generators of the three-dimensional rotation group $SO(3; \mathbb{R})$, expressed in spherical coordinates [94].

b. Manifold equipped with the Sjöqvist metric

For the three-dimensional manifold of mixed states equipped with the Sjöqvist metric $g_{\mu\nu}^{\text{Sjöqvist}}(\xi)$ with $\xi \overset{\text{def}}{=} (\xi^1, \xi^2, \xi^3) = (r, \theta, \varphi)$, the infinitesimal line element is $ds_{\text{Sjöqvist}}^2 \overset{\text{def}}{=} (1/4)[(1-r^2)^{-1}dr^2 + d\theta^2 + \sin^2(\theta)d\varphi^2]$. The nonzero Christoffel connection coefficients in Eq. (E1) are

$$\Gamma_{11}^1 = \frac{r}{1-r^2}, \quad \Gamma_{33}^2 = -\sin(\theta)\cos(\theta), \quad \text{and} \quad \Gamma_{23}^3 = \frac{\cos(\theta)}{\sin(\theta)}. \qquad (E17)$$

Moreover, exploiting Eq. (E17), the nonvanishing Ricci and Riemann curvature tensor components in Eqs. (E2) and (E3) become

$$\mathcal{R}_{22} = 1, \qquad \mathcal{R}_{33} = \sin^2(\theta), \qquad (E18)$$

and

$$\mathcal{R}_{2323} = \frac{1}{4}\sin^2(\theta), \qquad (E19)$$

respectively. For completeness, we emphasize that exploiting the symmetry properties of the Riemann curvature tensor, we also have $\mathcal{R}_{2332} = \mathcal{R}_{3223} = -\mathcal{R}_{2323}$ and $\mathcal{R}_{3232} = \mathcal{R}_{2323}$. In the Sjöqvist metric, we have that $ds_{\text{Sjöqvist}}^2 = \mathbf{ds}_{\text{Sjöqvist}} \cdot \mathbf{ds}_{\text{Sjöqvist}}$ with the infinitesimal vector element $\mathbf{ds}_{\text{Sjöqvist}}$ defined as

$$\mathbf{ds}_{\text{Sjöqvist}} \overset{\text{def}}{=} \frac{1}{2}\frac{1}{\sqrt{1-r^2}}\hat{e}_r dr + \frac{1}{2}\hat{e}_\theta d\theta + \frac{1}{2}\sin(\theta)\hat{e}_\varphi d\varphi, \qquad (E20)$$

where$(1/2)(1 - r^2)^{-1/2}$, $1/2$ and $(1/2)\sin(\theta)$ in Eq. (E20) denote the so-called scale factors of the metric. Then, inserting Eqs. (E13) and (E19) into Eq. (E6), we find

$$\mathcal{K}_{\text{Sjöqvist}}(\hat{e}_\theta, \hat{e}_\varphi) = \mathcal{K}_{\text{Sjöqvist}}(\hat{e}_\varphi, \hat{e}_\theta) = 4, \quad (E21)$$

and

$$\mathcal{K}_{\text{Sjöqvist}}(\hat{e}_r, \hat{e}_\theta) = \mathcal{K}_{\text{Sjöqvist}}(\hat{e}_\theta, \hat{e}_r) = \mathcal{K}_{\text{Sjöqvist}}(\hat{e}_r, \hat{e}_\varphi)$$
$$= \mathcal{K}_{\text{Sjöqvist}}(\hat{e}_\varphi, \hat{e}_r) = 0. \quad (E22)$$

Thus, from Eqs. (E21) and (E22), we conclude that the manifold of mixed states equipped with the Sjöqvist metric is an anisotropic manifold of nonconstant (positive) sectional curvature with an overall (positive) constant Ricci curvature $\mathcal{R}_{\text{Sjöqvist}} = 8$.

c. Manifold equipped with the Bures metric

In the case of the three-dimensional manifold of mixed states equipped with the Bures metric $g_{\mu\nu}^{\text{Bures}}(\xi)$ with $\xi \overset{\text{def}}{=} (\xi^1, \xi^2, \xi^3) = (r, \theta, \varphi)$, the infinitesimal line element is given by $ds_{\text{Bures}}^2 \overset{\text{def}}{=} (1/4)[(1 - r^2)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)d\varphi^2]$. In this case, the nonzero Christoffel connection coefficients in Eq. (E1) are given by

$$\Gamma_{11}^1 = \frac{r}{1 - r^2}, \quad \Gamma_{22}^1 = -r(1 - r^2),$$

$$\Gamma_{33}^1 = -r(1 - r^2)\sin^2(\theta), \quad \Gamma_{12}^2 = \frac{1}{r},$$

$$\Gamma_{33}^2 = -\sin(\theta)\cos(\theta), \quad \Gamma_{13}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \frac{\cos(\theta)}{\sin(\theta)}. \quad (E23)$$

Furthermore, employing Eq. (E23), the nonzero Ricci and Riemann curvature tensor components in Eqs. (E2) and (E3) are

$$\mathcal{R}_{11} = \frac{2}{1 - r^2}, \quad \mathcal{R}_{22} = 2r^2, \quad \mathcal{R}_{33} = 2r^2 \sin^2(\theta), \quad (E24)$$

and, modulo symmetries of the Riemann curvature tensor,

$$\mathcal{R}_{1212} = \frac{1}{4}\frac{r^2}{1 - r^2}, \quad \mathcal{R}_{1313} = \frac{1}{4}\frac{r^2}{1 - r^2}\sin^2(\theta),$$

$$\mathcal{R}_{2323} = \frac{1}{4}r^4 \sin^2(\theta), \quad (E25)$$

respectively. In the Bures case, we have that $ds_{\text{Bures}}^2 = \mathbf{ds}_{\text{Bures}} \cdot \mathbf{ds}_{\text{Bures}}$ with the infinitesimal vector element $\mathbf{ds}_{\text{Bures}}$ given by

$$\mathbf{ds}_{\text{Bures}} \overset{\text{def}}{=} \frac{1}{2}\frac{1}{\sqrt{1 - r^2}}\hat{e}_r dr + \frac{r}{2}\hat{e}_\theta d\theta + \frac{r}{2}\sin(\theta)\hat{e}_\varphi d\varphi, \quad (E26)$$

where $(1/2)(1 - r^2)^{-1/2}$, $r/2$ and $(r/2)\sin(\theta)$ in Eq. (E26) are the scale factors of the metric. Then, inserting Eqs. (E13) and (E25) into Eq. (E6), we find

$$\mathcal{K}_{\text{Bures}}(\hat{e}_r, \hat{e}_\theta) = \mathcal{K}_{\text{Bures}}(\hat{e}_\theta, \hat{e}_r) = \mathcal{K}_{\text{Bures}}(\hat{e}_r, \hat{e}_\varphi)$$
$$= \mathcal{K}_{\text{Bures}}(\hat{e}_\varphi, \hat{e}_r) = \mathcal{K}_{\text{Bures}}(\hat{e}_\theta, \hat{e}_\varphi)$$
$$= \mathcal{K}_{\text{Bures}}(\hat{e}_\varphi, \hat{e}_\theta) = 4. \quad (E27)$$

Thus, from Eq. (E27), we conclude that the manifold of mixed states equipped with the Bures metric is an isotropic manifold of constant (positive) sectional curvature with (positive) constant Ricci curvature $\mathcal{R}_{\text{Bures}} = 24$.

APPENDIX F: FURTHER DETAILS ON THE SJÖQVIST METRIC

In this appendix, we provide some comparative statements between the Bures and the Sjöqvist metrics. Furthermore, we briefly present the extension of the original Sjöqvist metric to degenerate mixed quantum states.

1. Comparison with the Bures metric

Following the Morozova-Cencov-Petz theorem as reported in Ref. [85], every (Riemannian and monotone) metric in the Bloch ball at a point where the density matrix is diagonal, $\rho = (1/2)\text{diag}(1 + r, 1 - r)$, can be expressed as

$$ds^2 = \frac{1}{4}\left[\frac{dr^2}{1 - r^2} + \frac{1}{f(\frac{1-r}{1+r})}\frac{r^2}{1 + r}d\Omega^2\right], \quad (F1)$$

with $0 < r < 1$. In Eq. (F1), $d\Omega^2 \overset{\text{def}}{=} d\theta^2 + \sin^2(\theta)d\varphi^2$ is the metric on the unit 2-sphere while $f: \mathbb{R}_+ \to \mathbb{R}_+$ is the so-called Morozova-Cencov function $f = f(t)$. This function satisfies three conditions: (i) f is operator monotone, (ii) f is self-inversive with $f(1/t) = f(t)/t$, (iii) $f(1) = 1$. From Eq. (F1), we emphasize that condition (iii), $f(1) = 1 \neq 0$, serves to avoid a conical singularity in the metric at the maximally mixed state where $r = 0$ [that is, $t = t(r) \overset{\text{def}}{=} (1 - r)/(1 + r) = 1$]. For details on the Morozova-Cencov-Petz theorem and further discussion on the meaning of conditions (i), (ii), and (iii), we refer to Refs. [85,95]. For details on the monotonicity of operator functions, we refer to Refs. [96–100]. In the case of the Bures metric,

$$ds_{\text{Bures}}^2 = \frac{1}{4}\left[\frac{dr^2}{1 - r^2} + r^2 d\Omega^2\right]. \quad (F2)$$

From Eqs. (F1) and (F2), we find that $f_{\text{Bures}}(t) \overset{\text{def}}{=} (1 + t)/2$. Clearly, $f_{\text{Bures}}(t)$ satisfies conditions (i), (ii), and (iii). In the case of the Sjöqvist metric,

$$ds^2_{\text{Sjöqvist}} = \frac{1}{4}\left[\frac{dr^2}{1-r^2} + d\Omega^2\right]. \tag{F3}$$

From Eqs. (F1) and (F3), we find that $f_{\text{Sjöqvist}}(t) \stackrel{\text{def}}{=} (1/2)[(1-t)^2/(1+t)]$. We observe that although $f_{\text{Sjöqvist}}(t)$ is self-inversive, $f_{\text{Sjöqvist}}(1) = 0$. Therefore, as pointed out in Ref. [53], the Sjöqvist metric in Eq. (F3) is singular at the origin of the Bloch ball where $r = 0$ (i.e., $t \equiv t(0) = 1$) and the angular components of the metric tensor diverge because $f_{\text{Sjöqvist}}(1) = 0$. For this reason, the original Sjöqvist metric is limited to nondegenerate mixed quantum states. When considering degenerate quantum states ρ, the Sjöqvist metric must be generalized as discussed in Refs. [54,101]. We briefly address this point in the next subsection.

2. Extension to the degenerate case

From Ref. [53] and the main text of this paper [see Eq. (32)], we recall that in the n-dimensional case $ds^2_{\text{Sjöqvist}}$ can be recast as

$$(ds^2_{\text{Sjöqvist}})^{(\text{non-degenerate})} = \frac{1}{4}\sum_{k=1}^{n}\frac{dp_k^2}{p_k} + \sum_{k=1}^{n} p_k ds_k^2, \tag{F4}$$

where $dp_k = \dot{p}_k dt$, $ds_k^2 \stackrel{\text{def}}{=} \langle de_k|de_k\rangle - |\langle e_k|de_k\rangle|^2$ is the Fubini-Study metric along the pure state $|e_k\rangle$, and $\hat{I}$ being the identity operator on the n-dimensional Hilbert space. Equation (F4) is valid in the nondegenerate case. Before introducing its extension to the degenerate case, we make a remark. Both the Bures and the Sjöqvist metrics can be viewed as the sum of a classical and a quantum contribution. In both cases, the classical contribution is the Fisher-Rao metric between two probability distributions.

The quantum contributions, however, differ in general. In the Bures case [see Eq. (B1)], the quantum contribution emerges from the noncommutativity of the density matrices ρ and $\rho + d\rho$. When $[\rho + d\rho, \rho] = 0$, the problem becomes classical and the Bures metric reduces to the classical Fisher-Rao metric. In the Sjöqvist case [see Eq. (F4)], the quantum contribution is the sum of the pure state Fubini-Study metrics ds_k^2 along the state vectors $\{|e_k\rangle\}$ weighted with their corresponding probability $\{p_k\}$, $\sum_k p_k ds_k^2$. Returning to the extension of $ds^2_{\text{Sjöqvist}}$ in Eq. (F4) to the case of degenerate mixed quantum states, following Refs. [54,101], we have

$$(ds^2_{\text{Sjöqvist}})^{(\text{degenerate})} = \frac{1}{4}\sum_{k=1}^{m} r_k \frac{dp_k^2}{p_k} + \sum_{k=1}^{m} p_k \text{tr}(P_k dP_k dP_k), \tag{F5}$$

where $\rho \stackrel{\text{def}}{=} \sum_{k=1}^{m} p_k P_k$, $r_k \stackrel{\text{def}}{=} \text{tr}(P_k)$ is the rank of the orthogonal projector P_k, and $r \stackrel{\text{def}}{=} \sum_{k=1}^{m} r_k$ is the rank of the state ρ. Obviously, unlike what happens in Eq. (F4), in Eq. (F5) not all projectors P_k are rank-one operators because of the possible presence of degenerate eigenvalues of ρ. Note that $m \leq l$ with l denoting the cardinality of the set of pure states that specify the probabilistic mixture (i.e., quantum ensemble) that defines $\rho \in \mathbb{C}^{n \times n}$. Observe also that l can be greater than n when ρ is nondegenerate. For a complete classification of quantum ensembles yielding a given density matrix, we refer to [102]. For completeness, we note that when $P_k = |e_k\rangle\langle e_k|$ and $r_k = 1$ for any $1 \leq k \leq m$, $\text{tr}(P_k dP_k dP_k)$ equals $\langle de_k|de_k\rangle - |\langle e_k|de_k\rangle|^2$ and, in addition, Eq. (F5) reduces to Eq. (F4). For details on the derivation of Eq. (F5), we refer the reader to Refs. [54,101].

[1] L. Casetti, M. Pettini, and E. G. D. Cohen, Geometric approach to Hamiltonian dynamics and statistical mechanics, Phys. Rep. **337**, 237 (2000).

[2] M. Pettini, *Geometry and Topology in Hamiltonian Dynamics and Statistical Mechanics* (Springer-Verlag, New York, 2007).

[3] M. A. Nielsen, M. R. Dowling, and A. C. Doherty, Quantum computation as geometry, Science **311**, 1133 (2006).

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).

[5] A. Bhattacharyya, P. Nandy, and A. Sinha, Renormalized Circuit Complexity, Phys. Rev. Lett. **124**, 101602 (2020).

[6] L. Susskind, *Three Lectures on Complexity and Black Holes* (Springer, Cham, 2020).

[7] D. Stanford and L. Susskind, Complexity and shock wave geometries, Phys. Rev. D **90**, 126007 (2014).

[8] D. A. Roberts, D. Stanford, and L. Susskind, Localized shocks, J. High Energy Phys. 03 (2015) 051.

[9] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Holographic Complexity Equals Bulk Action?, Phys. Rev. Lett. **116**, 191301 (2016).

[10] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Complexity, action, and black hole, Phys. Rev. D **93**, 086006 (2016).

[11] A. R. Brown and L. Susskind, Complexity geometry of a single qubit, Phys. Rev. D **100**, 046020 (2019).

[12] R. Auzzi, S. Baiguera, G. B. De Luca, A. Legramandi, G. Nardelli, and N. Zenoni, Geometry of quantum complexity, Phys. Rev. D **103**, 106021 (2021).

[13] A. R. Brown, L. Susskind, and Y. Zhao, Quantum complexity and negative curvature, Phys. Rev. D **95,** 045010 (2017).

[14] S. Chapman, M. P. Heller, H. Marrachio, and F. Pastawski, Toward a Definition of Complexity for Quantum Field Theory States, Phys. Rev. Lett. **120,** 121602 (2018).

[15] S.-M. Ruan, Circuit complexity of mixed states, Ph.D. in Physics, University of Waterloo, 2021.

[16] K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1987).

[17] P. Lipka-Bartosik, M. Perarnau-Llobet, and N. Brunner, What is the temperature of a quantum state?, arXiv: 2205.0001.

[18] M. T. Mitchinson, A. Purkayastha, M. Brenes, A. Silva, and J. Goold, Taking the temperature of a pure quantum state, Phys. Rev. A **105,** L030201 (2022).

[19] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58,** 883 (1998).

[20] D.-H. Kim, S. A. Ali, C. Cafaro, and S. Mancini, Information geometric modeling of scattering induced quantum entanglement, Phys. Lett. A **375,** 2868 (2011).

[21] D.-H. Kim, S. A. Ali, C. Cafaro, and S. Mancini, Information geometry of quantum entangled wave-packets, Physica (Amsterdam) **391A,** 4517 (2012).

[22] L. A. Caron H. Jirari, H. Kröger, X. Q. Luo, G. Melkonyan, and K. J. M. Moriarty, Quantum chaos at finite temperature, Phys. Lett. A **288,** 145 (2001).

[23] L. A. Caron, D. Huard, H. Kroger, G. Melkonyan, K. J. M. Moriarty, and L. P. Nadeau, Is quantum chaos weaker than classical chaos?, Phys. Lett. A **322,** 60 (2004).

[24] H. Kroger, J.-F. Laprise, G. Melkonyan, and R. Zomorrodi, Quantum chaos versus classical chaos: Why is quantum chaos weaker?, in *The Logistic Map to the Route to Chaos*, edited by M. Ausloos and M. Dirickx (Springer-Verlag, Berlin, Heidelberg, 2006), pp. 355–367.

[25] C. Cafaro, A. Giffin, C. Lupo, and S. Mancini, Insights into the softening of chaotic statistical models by quantum considerations, AIP Conf. Proc. **1443,** 366 (2012).

[26] C. Cafaro, A. Giffin, C. Lupo, and S. Mancini, Softening the complexity of entropic motion on curved statistical manifolds, Open Syst. Inf. Dyn. **19,** 1250001 (2012).

[27] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum Speed Limits for Physical Processes, Phys. Rev. Lett. **110,** 050402 (2013).

[28] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. Huelga, Quantum Speed Limits in Open Quantum Systems Dynamics, Phys. Rev. Lett. **110,** 050403 (2013).

[29] S. Deffner and E. Lutz, Quantum Speed Limit for Non-Markovian Dynamics, Phys. Rev. Lett. **111,** 010402 (2013).

[30] M. Okuyama and M. Ohzehi, Quantum Speed Limit is Not Quantum, Phys. Rev. Lett. **120,** 070402 (2018).

[31] B. Shanahan, A. Chenu, N. Margolus, and A. del Campo, Quantum Speed Limits across the Quantum-to-Classical Transition, Phys. Rev. Lett. **120,** 070401 (2018).

[32] K. Bolonek-Lason, J. Gonera, and P. Kosinki, Classical and quantum speed limits, Quantum **5,** 482 (2021).

[33] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[34] C. Cafaro and S. Mancini, Characterizing the depolarizing quantum channel in terms of Riemannian geometry, Int. J. Geom. Methods Mod. Phys. **09,** 1260020 (2012).

[35] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Time optimal quantum evolution of mixed states, J. Phys. A **41,** 045303 (2008).

[36] D. Girolami, How Difficult Is It to Prepare a Quantum State?, Phys. Rev. Lett. **122,** 010505 (2019).

[37] N. Hornedal, D. Allan, and O. Sonnerborn, Extensions of the Mandelstam-Tamm quantum speed limit to systems in mixed states, New J. Phys. **24,** 055004 (2022).

[38] D. Felice, C. Cafaro, and S. Mancini, Information geometric methods for complexity, Chaos **28,** 032101 (2018).

[39] C. Cafaro, S. Ray, and P. M. Alsing, Geometric aspects of analog quantum search evolutions, Phys. Rev. A **102,** 052607 (2020).

[40] C. Cafaro, S. Ray, and P. M. Alsing, Optimal-speed unitary quantum time evolutions and propagation of light with maximal degree of coherence, Phys. Rev. A **105,** 052425 (2022).

[41] C. Cafaro and P. M. Alsing, Information geometry aspects of minimum entropy production paths from quantum mechanical evolutions, Phys. Rev. E **101,** 022110 (2020).

[42] C. Cafaro, S. Ray, and P. M. Alsing, Complexity and efficiency of minimum entropy production probability paths from quantum dynamical evolutions, Phys. Rev. E **105,** 034143 (2022).

[43] C. Cafaro and S. A. Ali, Jacobi fields on statistical manifolds of negative curvature, Physica (Amsterdam) **234D,** 70 (2007).

[44] C. Cafaro, The information geometry of chaos, Ph.D. Thesis, State University of New York at Albany, Albany, NY, 2008; Information-geometric indicators of chaos in Gaussian models on statistical manifolds of negative Ricci curvature, Int. J. Theor. Phys. **47,** 2924 (2008); Information geometry, inference methods and chaotic energy levels statistics, Mod. Phys. Lett. B **22,** 1879 (2008); Works on an information geometrodynamical approach to chaos, Chaos Solit. Fractals **41,** 886 (2009).

[45] S. A. Ali, C. Cafaro, S. Gassner, and A. Giffin, An information geometric perspective on the complexity of macroscopic predictions arising from incomplete information, Adv. Theor. Math. Phys. **2018,** 2048521 (2018).

[46] C. Cafaro and S. A. Ali, Information geometric measures of complexity with applications to classical and quantum physical settings, Foundations **1,** 45 (2021).

[47] S. A. Ali, C. Cafaro, D.-H. Kim, and S. Mancini, The effect of microscopic correlations on the information geometric complexity of Gaussian statistical models, Physica (Amsterdam) **389A,** 3117 (2010).

[48] S. A. Ali and C. Cafaro, Theoretical investigations of an information geometric approach to complexity, Rev. Math. Phys. **29,** 1730002 (2017).

[49] C. Cafaro, A. Giffin, S. A. Ali, and D.-H. Kim, Reexamination of an information geometric construction of entropic indicators of complexity, Appl. Math. Comput. **217,** 2944 (2010).

[50] W. K. Wootters, Statistical distance and Hilbert space, Phys. Rev. D **23,** 357 (1981).

[51] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72,** 3439 (1994).

[52] R. Uzdin, U. Guenther, S. Rahav, and N. Moiseyev, Time-dependent Hamiltonians with 100% evolution speed efficiency, J. Phys. A **45,** 415304 (2012).

[53] E. Sjöqvist, Geometry along evolution of mixed quantum states, Phys. Rev. Res. **2,** 013344 (2020).

[54] H. Silva, B. Mera, and N. Paunkovic, Interferometric geometry from symmetry-broken Uhlmann gauge group with applications to topological phase transitions, Phys. Rev. B **103,** 085127 (2021).

[55] J. P. Provost and G. Vallee, Riemannian structure on manifolds of quantum states, Commun. Math. Phys. **76,** 289 (1980).

[56] D. Bures, An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite ω^* -algebras, Trans. Am. Math. Soc. **135,** 199 (1969).

[57] A. Uhlmann, The "transition probability" in the state space of a *-algebra, Rep. Math. Phys. **9,** 273 (1976).

[58] M. Hübner, Explicit computation of the Bures distance for density matrices, Phys. Lett. A **163,** 239 (1992).

[59] C. Cafaro, O. Luongo, S. Mancini, and H. Quevedo, Thermodynamic length, geometric efficiency, and Legendre invariance, Physica (Amsterdam) **590A,** 126740 (2022).

[60] G. Ruppeiner, Thermodynamics: A Riemannian geometric model, Phys. Rev. A **20,** 1608 (1979).

[61] G. E. Crooks, Measuring Thermodynamic Length, Phys. Rev. Lett. **99,** 100602 (2007).

[62] L. Diosi, G. Forgacs, B. Lukacs, and H. L. Frisch, Metricization of thermodynamic-state space and the renormalization group, Phys. Rev. A **29,** 3343 (1984).

[63] F. Weinhold, Metric geometry of equilibrium thermodynamics, J. Chem. Phys. **63,** 2479 (1975).

[64] P. Salamon and R. S. Berry, Thermodynamic Length and Dissipated Availability, Phys. Rev. Lett. **51,** 1127 (1983).

[65] A. K. Pati, Relation between "phases" and "distance" in quantum evolution, Phys. Lett. A **159,** 105 (1991).

[66] J. Anandan and Y. Aharonov, Geometry of Quantum Evolution, Phys. Rev. Lett. **65,** 1697 (1990).

[67] P. Zanardi, P. Giorda, and M. Cozzini, Information-Theoretic Differential Geometry of Quantum Phase Transitions, Phys. Rev. Lett. **99,** 100603 (2007).

[68] P. Zanardi, L. C. Venuti, and P. Giorda, Bures metric over thermal state manifolds and quantum criticality, Phys. Rev. **76A,** 062318 (2007).

[69] S. L. Braunstein and C. M. Caves, Geometry of quantum states, in *Quantum Communications and Measurement*, edited by B. V. Belavkin, O. Hirota, and R. L. Hudson (Springer, Boston, MA, 1995), pp. 21–30.

[70] S. Luo, Wigner-Yanase Skew Information and Uncertainty Relations, Phys. Rev. Lett. **91,** 180403 (2003).

[71] B. Efron, Defining the curvature of a statistical problem (with applications to second order efficiency), Ann. Stat. **3,** 1189 (1975).

[72] P. Pessoa and C. Cafaro, Information geometry for Fermi-Dirac and Bose-Einstein quantum statistics, Physica (Amsterdam) **576A,** 126061 (2021).

[73] D. Bonalda, L. Seveso, and M. G. A. Paris, Quantum sensing of curvature, Int. J. Theor. Phys. **58,** 2914 (2019).

[74] H. Azuma, Dynamics of the Bloch vector in the thermal Jaynes-Cummings model, Phys. Rev. A **77,** 063820 (2008).

[75] J. Myung, V. Balasubramanian, and M. A. Pitt, Counting probability distributions: Differential geometry and model selection, Proc. Natl. Acad. Sci. U.S.A. **97,** 11170 (2000).

[76] D. C. Brody and B. Longstaff, Evolution speed of open quantum dynamics, Phys. Rev. Res. **1,** 033127 (2019).

[77] C. Cafaro and P. M. Alsing, Continuous-time quantum search and time-dependent two-level quantum systems, Int. J. Quantum. Inform. **17,** 1950025 (2019).

[78] C. Cafaro and P. M. Alsing, Theoretical analysis of a nearly optimal analog quantum search, Phys. Scr. **94,** 085103 (2019).

[79] C. Cafaro, S. Gassner, and P. M. Alsing, Information geometric perspective on off-resonance effects in driven two-level quantum systems, Quantum Rep. **2,** 166 (2020).

[80] F. Campaioli, W. Sloan, K. Modi, and F. A. Pollock, Algorithm for solving unconstrained unitary quantum brachistochrone problems, Phys. Rev. A **100,** 062328 (2019).

[81] E. O'Connor, G. Guarnieri, and S. Campbell, Action quantum speed limits, Phys. Rev. A **103,** 022210 (2021).

[82] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972).

[83] H.-J. Sommers and K. Zyczkowski, Bures volume of the set of mixed states, J. Phys. A **36,** 10083 (2003).

[84] P. B. Slater, Bures metric for certain high-dimensional quantum systems, Phys. Lett. A **244,** 35 (1998).

[85] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, England, 2006).

[86] C. Cafaro, D. Felice, and P. M. Alsing, Quantum Groverian geodesic paths with gravitational and thermal analogies, Eur. Phys. J. Plus **135,** 900 (2020).

[87] C. Cafaro and S. Mancini, On the complexity of statistical models admitting correlations, Phys. Scr. **82,** 035007 (2010).

[88] C. Cafaro and S. Mancini, Quantifying the complexity of geodesic paths on curved statistical manifolds through information geometric entropies and Jacobi fields, Physica Amsterdam) **240D,** 607 (2011).

[89] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, Inc., New York, 1972).

[90] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature* (Springer-Verlag, New York, Inc., 1997).

[91] F. De Felice and C. J. S. Clarke, *Relativity on Curved Manifolds* (Cambridge University Press, Cambridge, England, 1990).

[92] M. L. Boas, *Mathematical Methods in the Physical Sciences* (John Wiley & Sons, Inc., New York, 2006).

[93] A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton, NJ, 1975).

[94] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics* (Cambridge University Press, Cambridge, England, 2017).

[95] D. Petz and Cs. Sudar, Extending the Fisher metric to density matrices, in *Geometry in Present Days Science*, edited by O. E. Barndorff-Nielsen and E. B. Vendel (World Scientific, Singapore, 1999), pp. 21–34.

[96] R. Bhatia, *Matrix Analysis* (Springer-Verlag, Berlin, 1997).

[97] M. K. Kwong, Some results on matrix monotone functions, Linear Algebra Appl. **118,** 129 (1989).

[98] D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl. **244,** 81 (1996).

[99] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, Linear Algebra Appl. **429,** 972 (2008).

[100] P. Gibilisco, F. Hansen, and T. Isola, On a correspondence between regular and nonregular operator monotone functions, Linear Algebra Appl. **430,** 2225 (2009).

[101] H. V. da Silva, Quantum information geometry and applications, M.S. thesis in engineering physics, IT Lisboa, 2021.

[102] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, Phys. Lett. A **183,** 14 (1993).