Toward a UV model of kinetic mixing and portal matter. II. Exploring unification in an $SU(N)$ group

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If dark matter interacts with the Standard Model (SM) via the $U(1)_D$ kinetic mixing portal at low energies, it necessitates not only the existence of portal matter particles which carry both dark and SM quantum numbers, but also a possible UV completion into which this $U(1)_D$ and the SM are both embedded. In earlier work, following a bottom-up approach, we attempted to construct a more unified framework of these SM and dark sector interactions. In this paper, we will instead begin to explore, from the top down, the possibility of the unification of these forces via the decomposition of a grand-unified-theory-like group, $G \to G_{\rm SM} \times G_{\rm Dark}$, where $U(1)_D$ is now a low-energy diagonal subgroup of $G_{\rm Dark}$ and where the familiar $G_{\rm SM} = SU(5)$ will play the role of a proxy for the conventional $SU(3)_c \times SU(2)_L \times U(1)_Y$ SM gauge group. In particular, for this study it will be assumed that $G = SU(N)$ with $N = 6$–10. Although not our main goal, models that also unify the three SM generational structure within this same general framework will also be examined. The possibilities are found to be quite highly constrained by our chosen set of model-building requirements, which are likely too strong when they are employed simultaneously to obtain a successful model framework.

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I. INTRODUCTION AND BACKGROUND DISCUSSION

The nature of dark matter (DM) and its possible nongravitational interactions with the Standard Model (SM) remain leading questions in particle physics. Given the measurements of the DM relic density from Planck [1], it is more than likely that interactions of *some* kind must exist coupling the SM to DM and possibly DM to itself. How would these "fit in" with the known forces of the SM in a unified framework, and how are they generated? Although these are not new questions and we may be surprised by the eventual answers, the well-studied DM candidates, such as the QCD axion [2–4] and weakly interacting massive particles [5,6], continue to be hunted for without success over a wide range of fronts and their allowed parameter spaces continue to be eaten into as a result of the null searches by direct or indirect detection experiments as well as those at the LHC [7–10]. The lack of any traditional signatures has inspired a vast effort in examining an ever-growing set of DM candidates spanning wide ranges in both DM particle masses and the strength of their couplings to the SM [11–14]. It has been found that DM may couple to the SM in many various ways, and one very useful tool to classify these possible interactions is via both renormalizable (i.e., dimension ≤ 4) or nonrenormalizable (i.e., dimension > 4) "portals." These portals posit not only the existence of DM itself but also a new set of fields which act as mediators between the SM and the (potentially complex) dark sector of which the DM itself is likely the lightest member. Of the many examples, one that has gotten much attention in the recent literature due to its flexibility is the renormalizable kinetic mixing (KM) or vector portal [15,16] based upon a new gauge interaction. This scenario can allow for the DM to reach its abundance via the familiar thermal mechanism [17,18] albeit for sub-GeV DM masses and employing new non-SM interactions that so far could have evaded detection.

Such a scenario can be realized in many ways enjoying various levels of complexity. The simplest manifestation assumes only the existence of a new $U(1)_D$ gauge group, with a coupling g_D, under which the SM fields are singlets, carrying no dark charges, i.e., $Q_D = 0$ and with the associated new gauge boson termed the "dark photon" (DP) [19,20]. $U(1)_D$ is usually assumed to be broken at or below the $\sim$ few GeV scale so that both the DM and DP have comparable masses. The symmetry breaking in this model usually occurs via the vacuum expectation value(s) [VEV(s)] of at least one dark Higgs field in analogy with the spontaneous symmetry breaking in the SM. Within such a setup, the interaction between the SM and the dark sector

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is generated via KM at the one-loop level between $U(1)_D$ and the SM $U(1)_Y$ gauge fields. Specifically, these gauge bosons experience KM through the action of a set of fields, usually being vectorlike fermions (or complex scalars), here called portal matter (PM) [21–29], that carry both SM and $U(1)_D$ dark charges. After redefinition back to canonically normalized fields removes the effect of the KM and both the SM and $U(1)_D$ symmetries are broken, this KM leads to a coupling of the DP to SM fields of the form $\simeq e\epsilon Q_{em}$, the origin of the DP nomenclature. The strength of the KM generated by these one-loop vacuum polarizationlike graphs is then described by a single dimensionless parameter, ϵ, usually constrained by phenomenology to lie very roughly in the $\sim 10^{-(3\text{–}4)}$ range given the DM and DP sub-GeV mass region that we are assuming. In the conventional normalization [15,16], with $c_w = \cos\theta_w$, ϵ is given by the sum

$$\epsilon = c_w \frac{g_D g_Y}{24\pi^2} \sum_i \eta_i \frac{Y_i}{2} N_{c_i} Q_{D_i} ln \frac{m_i^2}{\mu^2}, \tag{1}$$

with $g_{Y,D}$ being the $U(1)_{Y,D}$ gauge couplings and $m_i(Y_i, Q_{D_i}, N_{c_i})$ are the mass (hypercharge, dark charge, number of colors) of the ith PM field. Here, $\eta_i = 1(1/2)$ if the PM is a chiral fermion (complex scalar) and the hypercharge is normalized so that the electric charge is given by $Q_{em} = T_{3L} + Y/2$. In a more UV-complete theory, such as we are interested in here, this same group theory requires that the sum (for fermions and scalars separately)

$$\sum_i \eta_i \frac{Y_i}{2} N_{c_i} Q_{D_i} = 0, \tag{2}$$

so that ϵ is both finite and, if the PM masses are known, also calculable.

It is important to address the question of how DM, this new $U(1)_D$ gauge interaction, and the various PM fields might fit together with the known SM particles and gauge forces into a more unified structure and, as a result, to also consider the possibility that some further more complex gauge structure(s) might be kinematically accessible to existing and planned colliders in the future [30]. This is a natural extension to the program of grand unification begun long ago [31,32], now augmented by a dark sector with its own matter content and gauge forces. In principle, in addressing these questions, one may want to follow either a bottom-up or a top-down approach, both of which have been previously discussed, with the former method followed in our previous studies [22,25]; in the discussion below, we will make a first attempt at a top-down analysis from our perspective. Of course, one might ask the obvious question if there is any reason to believe that this simple $U(1)_D$ scenario may itself already provide some indirect evidence as to it being part of a larger gauge structure, perhaps even one that is not too far away in energy scale. The following short exercise may be somewhat indicative.

At least for that part of the parameter space when the DM gauge coupling is somewhat large at low energies, it is useful consider the running of the coupling, i.e., $\alpha_D = g_D^2/4\pi$, into the UV. As is very well known, a $U(1)$ gauge theory is not asymptotically free and eventually will become strongly coupled or possibly experience a Landau pole at some point as the energy scale increases. In a (more) UV-complete theory, one would expect new physics of some form to enter before either of these things can happen, so it is possible to roughly estimate by what energy scale this new dynamics must occur. To be specific for demonstration purposes, consider the case of light fermionic DM, having $Q_D = 1$, together with the DP and dark Higgs all lying in a roughly similar mass range $M_L \sim 100$ MeV. More specifically, to escape the direct detection bounds due to elastic DM scattering as well as the rather strong constraints on s-wave annihilation from the cosmic microwave background (CMB) [1,33–35] for fermionic DM in this mass range, we consider the scenario where the DM is pseudo-Dirac with the relevant mass splitting between the two states generated by the same dark Higgs VEV that is responsible for the mass of the DP. Assuming that these fields are the only light degrees of freedom, one can run the value of α_D in a known manner from M_L up to some higher scale, M_U [or until some new physics with $Q_D \neq 0$ enters the renormalization group equations (RGEs)] where one reaches a region of strong coupling and/or encounters a Landau pole.[1] The SM fields do *not* enter into this calculation, as they all have $Q_D = 0$ and so will not couple to the DP to LO in the $\epsilon \to 0$ limit. The result of this simple calculation can be found in Fig. 1. Here, we can see that if $\alpha_D(M_L) \geq 0.175(0.20)$, a not infrequent assumption made in many phenomenological analyses [11–14], its value will become nonperturbative (or even encounter a Landau pole [36]) before $M_U \simeq$ a few TeV when run up from the $M_L = 100$ MeV scale. Even for $\alpha_D(M_L) \simeq 0.12$ (or 0.07), new physics must enter before the $\simeq$1000 TeV (or the traditional grand unification) scale is reached. Semiquantitatively, one finds that this conclusion is not very dependent as to whether these calculations are performed at the one-, two-, or three-loop level as can be gleaned from the figure. Although this simple toy example is only indicative, it gives support to the likelihood that a more complex, probably non-Abelian, broken dark sector gauge structure [22,25,37] is likely to be encountered at higher energies and, perhaps in some cases, may not lie too far above the weak scale, perhaps even being accessible at, e.g., the HL-LHC and at other possible future colliders. Of course, in a top-down analysis, one may speculate that

[1]The existence of other light fields with $Q_D \neq 0$ will only strengthen these arguments, since then α_D will run even more quickly.

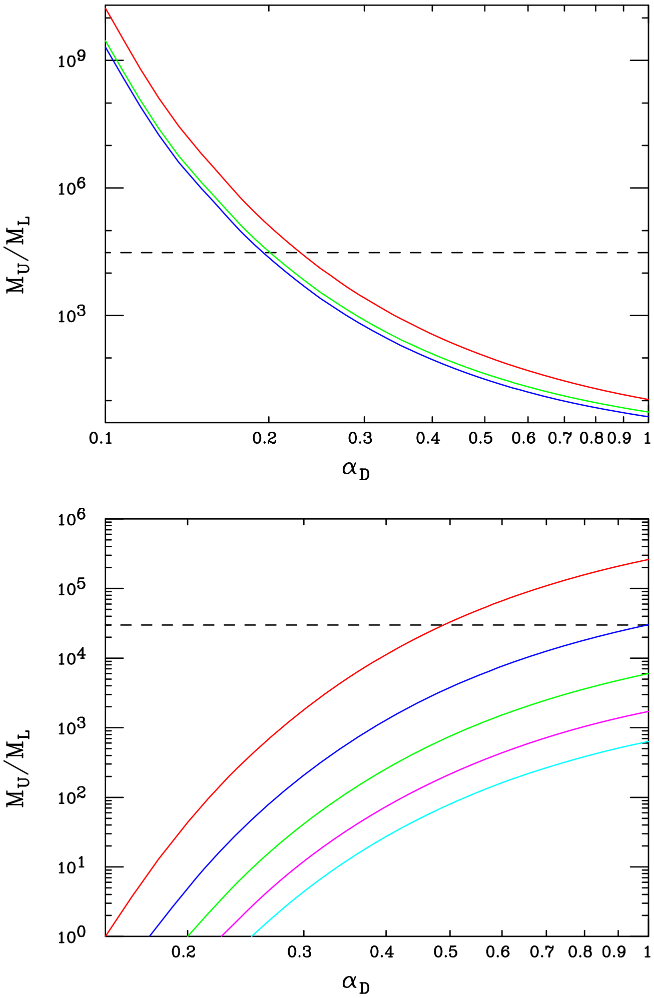

FIG. 1. The running of α_D in the pseudo-Dirac DM example discussed in the text. The top panel shows the location of the Landau pole, M_U, at the one-loop (red curve), two-loop (blue curve), and three-loop (green curve) level in RGE running, employing the $\overline{\text{MS}}$ scheme, as a function of $\alpha_D(M_L)$ where M_L is the low-energy scale associated with the DM, DP, and dark Higgs fields, $\simeq$100 MeV. The dashed line corresponds to $M_U = 3$ TeV when $M_L = 100$ MeV as a guide for the eye. The lower panel shows the three-loop running of α_D (on the x axis) in this same scenario as a function of M_U/M_L. The curves, from top to bottom, are for $\alpha_D(M_L) = 0.15, 0.175, 0.20, 0.225$, and 0.25, respectively. Here, we observe that, e.g., if $\alpha_D(M_L = 100\ \text{MeV}) \geq 0.175$, then $\alpha_D \geq 1$ for $M_U \geq 3$ TeV, again indicated by the dashed line.

this transition is allowed to occur anywhere below the "unification" scale $\sim 10^{16}$ GeV yet still above several TeV or so.

In what follows, we will attempt a preliminary top-down construction wherein the SM gauge interactions and those of an enlarged dark sector are combined into single unification group G, which is broken at some very large scale to the product $G_{\text{SM}} \times G_{\text{Dark}}$ with the $U(1)_D$ totally contained within G_{Dark}. G_{SM} certainly contains (at least) the usual SM $SU(3)_c \times SU(2)_L \times U(1)_Y$ (also known as $3_c 2_L 1_Y$) subgroup. Perhaps the most *minimal* choice of a simple group is the identification $G_{\text{SM}} = SU(5)$ so that, e.g., G itself might naturally be further identified with $SU(N)$, $N \geq 6$. We note that this choice of G_{SM} is far from a unique one, even if we assume it to be simple; i.e., $SO(10)$ and E_6 [38] both certainly come to mind as does the Pati-Salam product group $SU(2)_L \times SU(2)_R \times SU(4)_c$ [39] or even $[SU(3)]^3$ [40] in the case of nonsimple groups.[2] In the setups considered here, we expect that the heavy ($\gtrsim$ a few TeV) fermionic PM fields (which should be vectorlike with respect to the SM) will transform nontrivially under both groups, while dark matter and other purely dark sector fields will transform nontrivially only under G_{Dark}. No matter how G_{Dark} itself gets broken, it will be required that $U(1)_D$ survives intact down to the $\sim$1 GeV scale, and some special "protection" along the way will usually be required to ensure this remains the case. Numerous additional constraints will also need to be imposed in such a setup following from a set of (perhaps too strict) model-building assumptions. Our goal will not be to consider in any detail the phenomenological implications of such models that are judged to be "successful" (if any) in this regard, but we will instead concentrate our efforts on the difficult task of attempting to satisfy all of these basic model-building constraints. As will be fully discussed in the following section, we will examine the corresponding unification group and gauge structures in some detail as well as the various steps of the symmetry-breaking chain leading to the SM and allowing for the possibility of an unbroken $U(1)_D$ down to the $\lesssim$1 GeV scale.

The outline of this paper is as follows: In Sec. II, we provide the details of the basic model framework and the list of assumptions that we will be making in the analysis that follows. Many of these assumptions are fairly "traditional" ones and will be quite familiar from the extensive literature on the subject of grand unification including the SM family structure, some of it dating back well over 40 years. Others, will however, be guided by the additional requirements we need to impose on the dark sector to generate the appropriate PM masses while simultaneously maintaining an unbroken $U(1)_D$ down to the low energies below the electroweak scale. In Sec. III, we will systematically analyze in detail some representative examples of the set of $G = SU(N)$ models, for the $N = 6$–10 series of scenarios with increasing N, from which, as N grows, we will learn valuable lessons that can be employed for even larger values of N. At the end of this section, we also briefly consider values of $N > 10$ as well as some simple alterations in our set of model-building assumptions in light of the previous obtained results. A discussion of our final

[2] One can even imagine that G is itself a product group, e.g., $SU(8)_L \times SU(8)_R$ or even just $SU(5)_A \times SU(5)_B$ [41,42] with G_{Dark} embedded differently.

results, some possible future avenues of further investigation, and our conclusions are then presented in Sec. IV.

II. BASIC FRAMEWORK AND ANALYSIS ASSUMPTIONS

To begin our search for viable candidate models, we need to set out the underlying assumptions that we will be making in the analysis that follows. Many of these, in one form or another, are very familiar and well known, having been the pillars for many studies in the literature related to extended grand unified theories (GUTS) and the family or generation problem for over four decades. We will make use of these studies here but for a different purpose, i.e., the incorporation of a dark sector with, e.g., PM fields and new interactions. In particular, to see how far we can get this way, we generally will follow the rather conventional, strictly renormalizable, 4-d,[3] nonsupersymmetric approach to these constructions as described quite early on by Georgi [44], but with some additions and modifications introduced almost immediately following this in the work of others authors, e.g., [45–50]. The further model-building requirements and modifications discussed below are specifically inspired to account for the existence of (with respect to SM interactions) vectorlike fermionic PM, which are charged under both the visible and dark sectors and that have masses above the electroweak scale, as well as the survival of an unbroken $U(1)_D$ down to very low energies $\lesssim 1$ GeV.[4] We note, however, the common presence of additional *scalar* PM with both SM and dark quantum numbers throughout these analyses, as such fields are an integral part of the representations necessary for the breaking of the various symmetries that we will encounter.

It is to be noted that some of our assumptions, certainly in combination, may be somewhat overly restrictive, thus making it quite difficult for any model to pass through all the necessary hoops to be successful. The relative importance of the different constraints does change somewhat as N increases as we will see. Be that as it may, our approach can allow us to identify where certain assumptions might be too strictly applied and so point us in future directions for model building.

For our study below, the specific model-building requirements will be taken to be as follows.

(1) We will assume that the unifying group G decomposes as $G_{\rm SM} \times G_{\rm Dark}$ with, for simplicity here, $SU(5)$ acting as a proxy for the SM and playing that role in place of $G_{\rm SM} = 3_c 2_L 1_Y$. Thus, $U(1)_D$ is, trivially, a diagonal subgroup of $G_{\rm Dark}$ that remains unbroken down to the ~ 1 GeV scale so that the dark photon's coupling to the SM results from Abelian KM with the SM $U(1)_Y$. As noted above, the assignment of $SU(5)$ as the SM proxy is certainly far from unique, and other choices, e.g., $SO(10)$, E_6, etc., are clearly possible. It is also possible that $G_{\rm SM}$, though larger than $3_c 2_L 1_Y$, may not be a compact group; this is a strong model-building assumption, i.e., that the "pure SM" physics sector is itself not also extended by, e.g., additional $U(1)$ and/or $SU(2)$ factors. We further note that it is easily possible that $3_c 2_L 1_Y$ may be embedded into G in quite a different manner and not in this simple productlike fashion as is assumed here.

(2) Since the rank of $G \geq 5$ and must have complex irreducible representations, $[\mathbf{R_i}]$, but which are simultaneously required to be real with respect to $SU(3)_c$, it will be assumed that $G = SU(N)$. For simplicity and to obtain representations with (relatively) small dimensionality, it will be further assumed that the various irreducible representations that appear, $[\mathbf{R_i}]$, are solely obtained by taking antisymmetric products of the $SU(N)$ fundamental representation, $\mathbf{N}$.[5] As is well known, this assumption prevents the various resulting fermions from having non-SM-like $SU(3)_c$ and/or weak isospin transformation properties. Note that it will *not* be assumed that the fermion fields must all lie within a single irreducible representation. It is to be noted that this requirement restricts not only the set of SM $SU(5)$ representations that may appear, but simultaneously *also* those of $G_{\rm Dark}$ in a similar fashion and, thus, may be too strong of an assumption.

(3) The combined set of all relevant irreducible fermionic representations $[\mathbf{R_i}]$ of $G = SU(N)$ under consideration for any of the models discussed here, taken together, are assumed to lead to G being anomaly free.

(4) The $SU(N)$ gauge group above the unification scale will be assumed to have the property of asymptotically freedom (AF) at the one-loop level; here, we will include in the relevant β function the gauge and fermion contributions as well as the contributions from the minimal set of Higgs scalars required to break all of the various gauge symmetries and to generate the required particle masses. Specifically, we will demand that

$$\beta_N = -\frac{11}{3}N + \frac{2}{3}\Sigma_f \eta^f T(R_f) + \frac{1}{3}\Sigma_s \eta^s T(R_s) < 0, \tag{3}$$

[3]For example, we do not consider the possibility of using orbifolds or boundary conditions to break gauge symmetries [43] in higher-dimensional setups.

[4]As is usual, all fermions will be are taken to be left-handed in the analysis below.

[5]This need not be the case, but other options have been shown to lead to both representations and sets of such representations of significantly greater dimensionality and, hence, more degrees of freedom; see, for example, Ref. [51].

where the sums extend over the full set of complex (real) chiral fermion representations with $\eta^f = 1$ (1/2) and, correspondingly, complex (real) scalar representations with $\eta^s = 1$ (1/2). As is perhaps obvious, we will see, as N increases, that the overall dimensionality and corresponding β-function contributions of the various fermion and scalar $[\mathbf{R_i}]$ do as well (as does the overall number of degrees of freedom), making this condition ever more difficult to satisfy for large N. Note that we will *not* make the additional, potentially strong, assumption of requiring asymptotic freedom for the usual QCD β function itself near or between the PM and unification scales. In some sense, one may wonder if this AF assumption is really justifiable [52,53]. We can also imagine many different effects, e.g., gravitational influences [54], that may become quite important significantly above the unification scale, especially, as we will see that the role of AF becomes quite important in the discussion below. Note that the introduction of supersymmetry would result only in a strengthening of the AF condition. The runnings of the individual gauge and Yukawa couplings below the unification scale for the models passing all of our requirements (if any) will not be examined here and are left for future work.

(5) We will specifically assume the $SU(N) \to SU(5) \times SU(N-5)' \times U(1)_N$ breaking decomposition via the $SU(N)$ adjoint representation so that we can identify $G_{\rm Dark} = SU(N-5)' \times U(1)_N$.[6] In such a case, the fundamental representation $\mathbf{N}$ decomposes as $\mathbf{N} \to (\mathbf{5}, \mathbf{1})_{\mathbf{N-5}} + (\mathbf{1}, \mathbf{N-5})_{\mathbf{-5}}$. Since the group algebra for the antisymmetric products of the $\mathbf{5}$ of $SU(5)$ closes rapidly, one finds that only fields transforming under the SM $SU(5)$ as $\mathbf{1}, \mathbf{5}, \mathbf{10}$, and/or their complex conjugates will appear in the set $[\mathbf{R_i}]$; i.e., this set of representations can be symbolically decomposed under $SU(5)$ as [44]

$$[\mathbf{R_i}] \to \mathbf{n_1}(\mathbf{1}) + \mathbf{n_5}(\mathbf{5}) + \mathbf{n_{10}}(\mathbf{10}) + \mathbf{n_{\overline{10}}}(\overline{\mathbf{10}}) + \mathbf{n_{\bar{5}}}(\bar{\mathbf{5}}), \tag{4}$$

under which the number of SM generations, n_g, which consists of a single $\bar{\mathbf{5}} + \mathbf{10}$ of $SU(5)$, is given by the difference [44]

$$n_g = n_{\bar{5}} - n_5 = n_{10} - n_{\overline{10}}. \tag{5}$$

Additional $SU(5)$ nonsinglet fields beyond the three sets of $(\bar{\mathbf{5}} + \mathbf{10})$, which are to be identified with the usual SM fermions, might be identifiable as PM if they also satisfy other necessary requirements; e.g., in the case of fermions, they must be vectorlike with respect to the SM and carry $Q_D \neq 0$. Additional pure SM singlet fields are, of course, also allowed and will appear as potentially dark sector fields. Note that, in all generality in this decomposition, Q_D must be given by the sum of generators

$$Q_D = \sum_i a_i \lambda_i^{\rm Diag} + b Q_N, \tag{6}$$

where the a_i, b are constant coefficients, $\lambda_i^{\rm Diag}$ are the well-known set of $N-6$ diagonal generators of $SU(N-5)'$, i.e., $\lambda_{3,8,15,24,\ldots}$, etc., and Q_N is the $U(1)_N$ charge.

(6) While the SM fields must have $Q_D = 0$ by construction, they need *not* be singlets under the full $G_{\rm Dark}$ if the $U(1)_D$ is "properly" embedded within it. Similarly, PM fields must carry $Q_D \neq 0$ and must also transform nontrivially under $G_{\rm SM} = SU(5)$, since they are required to carry SM quantum numbers, in particular, have nonzero hypercharges Y, to induce $U(1)_Y - U(1)_D$ Abelian KM. As noted, the fermionic PM fields must also be vectorlike with respect to the SM to avoid numerous well-known constraints from, e.g., precision electroweak measurements, direct searches, unitarity bounds, and Higgs coupling determinations. The masses of these fermionic PM states must lie above the electroweak scale and also likely $\gtrsim$1–2 TeV [21–29] depending upon their electroweak and color transformation properties. Note that only the $Q_D = 0$ components of the various scalar representations acting as Higgs fields can obtain VEVs that are larger than ~1 GeV to enable the survival of the low-energy KM scenario. Since the fermionic PM fields generally lie in various representations of $G_{\rm Dark}$ and obtain their masses via the Higgs mechanism, they will be chiral with respect to at least some of the $G_{\rm Dark}$ subgroups.

(7) We will assume that the set of representations, $[\mathbf{R_i}]$, can lead to the three SM generations in various ways, the most simple being three copies of a smaller set of representations as is the case in ordinary $SU(5)$; we will refer to models in this class as having $n_g = 1$, and true "family unification" is absent in such scenarios. A more complex and interesting possibility, which we refer to as $n_g = 3$, constructs the three SM generations in a manner that allows any given representation or representations, $[\mathbf{R}]$, to appear more than once but the full set of all representations is *not* a triplification of a smaller subset of fields; this is a clearly a much stronger demand than the previous one and, as we will see, will require representations of larger rank to make remotely

[6]Note that we are not allowing for the possibility that $G_{\rm Dark}$ could just be a smaller subgroup of $SU(N-5)' \times U(1)_N$.

workable. We remind the reader that in Georgi's original work [44], the even stronger requirement was made that *no* representation could appear more that once in the set $[\mathbf{R_i}]$, but this requirement was subsequently relaxed rather soon by other authors, e.g., [45–48,50] with an eye toward reducing the total number of degrees of fermionic freedom and also easing the AF requirement. Here, we will follow these later authors and place some emphasis on models which have a smaller overall number of additional degrees of freedom beyond the usual ones of the SM, although "simple" models of both classes will be investigated. Obviously, fermionic PM will also come in three generations in the $n_g = 1$ models, but this need not be, and will likely not be, the case when $n_g = 3$. Note that family unification itself, as traditionally discussed, is *not* a goal of the current study.

(8) Higgs fields must be present to break $SU(5)$ and also give the SM fermions their masses as usual as well as to break G_{Dark}, possibly in stages, down to $U(1)_D$. The Higgs fields at the penultimate stage of G_{Dark} symmetry breaking, which will also generally lead to the masses of a set of vectorlike (with respect to the SM) fermions which we might directly identify with PM, with masses above the electroweak scale and must allow for the existence of an unbroken $U(1)$ that we can identify with the low-energy $U(1)_D$ under which the SM fields are, by assumption, neutral. As we will see, our assumptions then lead unambiguously to the symmetry-breaking chain

$$G \to G_{\mathrm{SM}} \times G_{\mathrm{Dark}},$$
$$G_{\mathrm{Dark}} \to \cdots \to SU(2)_D \to U(1)_D, \qquad (7)$$

which will be discussed in much detail below where we will consider the algebraically simpler scenario where the breaking of $G_{\mathrm{Dark}} \to SU(2)_D = SU(2)'$ happens in a single step; the possibility of multistep breaking will not alter the results obtained here in any essential manner, although there will undoubtedly be numerical impacts on the RGEs of the various gauge couplings. Note that the $SU(2)_D \to U(1)_D$ breaking cannot occur via the fundamental doublet representation but via, e.g., the adjoint triplet. The usual $U(1)_D$ can then eventually broken at the $\sim$1 GeV scale or below by the VEVs of the many possible $Q_D \neq 0$ neutral Higgs scalars that we encounter which may also be (but need not be) SM singlets. It is also possible that this $U(1)_D$ may be broken by the Stueckelberg mechanism [55], but that will not be helpful in, e.g., generating any needed DM mass terms nor will be it helpful in solving some of the other model-building issues with the symmetry-breaking chains that we will subsequently face, as they involve non-Abelian symmetry breakings.

(9) As noted, we will limit our set of possible particle interactions to those which are renormalizable at each level of symmetry breaking; i.e., we will not consider the contributions of potential higher-dimensional interactions or operators arising from integrating out possible heavy fields appearing in loops.

(10) Although we will not employ it directly as a model-building constraint *per se*, the nature of the DM in this class of models is of some relevance. As noted in the introduction, for thermal DM in the mass range $\lesssim$1 GeV anticipated here, CMB constraints tell us that its annihilation must be substantially suppressed at later times to avoid too much of an injection of electromagnetic energy into the evolving plasma [1,33–35]. One way to do this is to require that this process be p wave so that it is becomes velocity-squared suppressed later on after freeze-out, and such a situation is most easily realized when the DM is a SM singlet, $Q_D \neq 0$, complex scalar that does not obtain a VEV. Although we will not make any specific identification of such a field and some fine-tunings of the scalar potential may be required, as we will see below, the opportunities for the existence of such fields will be quite numerous as they will always occur (at the very least) in both the fundamental and the second-rank antisymmetric Higgs representations of $SU(N)$ of which we will make frequent use. Note that for $N > 6$ such fields will generally transform nontrivially under G_{Dark}. The interplay of this type of DM with the similarly light dark Higgs field(s) may itself be rather complex [27].

These combined requirements, though individually quite reasonable, are together very highly (perhaps overly) constraining, as we will now discover by looking at a broad set of examples. As we will see, in particular, the combined requirements of asymptotic freedom and successful mass generation for all of the fermionic PM fields at or above the TeV scale while simultaneously also requiring electroweak scale masses for the three families of SM fermions and a $U(1)_D$ that survives unbroken down to the $\sim$1 GeV mass range are extremely difficult to satisfy. Clearly, careful but intentional violations of any one or more of these model-building assumptions will lead to broad avenues for possible future investigations.

III. MODEL SURVEY

We now turn to our search for candidate gauge groups with specific fermion representations that satisfy all of the criteria above with ever-increasing values of N—beginning, briefly, with the educational case of $N = 6$.

A. $SU(6)$

We start our analysis by quickly considering the $SU(6)$ unification scenario, as it provides a very simple toy example of where things go badly wrong almost from the start. We begin by employing the familiar $SU(6) \to SU(5) \times U(1)_6$ breaking pattern wherein a single SM generation is embedded in the anomaly free set of representations $2(\bar{\mathbf{6}}) + \mathbf{15}$ (which is just the $\mathbf{27}$ of E_6 [56]), which under $SU(5) \times U(1)_6$ is simply $2[\bar{\mathbf{5}}_{-1} + \mathbf{1}_5] + [\mathbf{10}_2 + \mathbf{5}_{-4}]$ in obvious notation. This example is educational, because it contains both an additional set of vector-like (with respect to the SM) $\mathbf{5} + \bar{\mathbf{5}}$ fermion fields which *could* play the role of PM as well as an additional Abelian gauge group, $U(1)_6$, which *could* be just $U(1)_D$, two of the necessary ingredients we require for a successful model. However, we see immediately that this is not the case, as this possibility fails in the most trivial way: All of the fermions representations are seen to carry a nonzero $U(1)_6$ charge, while we have demanded that all of the ordinary SM chiral fields have $Q_D = 0$ so that we *cannot* identify $U(1)_D$ with $U(1)_6$. Since $U(1)_6$ is the only new gauge group factor beyond $SU(5)$, this simple possibility is clearly excluded. Furthermore, in parallel with this, we note that requiring the SM fermions to have $Q_D = 0$ would also force the potential candidate PM fields to also have $Q_D = 0$ in this framework. Specifically, in $n_g = 1(3)$ type scenarios, we require that we can identify one (three) $\bar{\mathbf{5}} + \mathbf{10}$ representations with the SM fermions and necessarily having $Q_D = 0$.

This $SU(6)$ discussion teaches us a valuable, if perhaps obvious, lesson when considering the more general $SU(N) \to SU(5) \times SU(N-5)' \times U(1)_N$ decomposition as we will see below. The set of relevant $SU(N)$ chiral fermions will very commonly include at least one $\mathbf{N}$ (or its conjugate), which subsequently decomposes as noted above as $\mathbf{N} \to (\mathbf{5}, \mathbf{1})_{\mathbf{N-5}} + (\mathbf{1}, \mathbf{N-5})_{\mathbf{-5}}$ whose first contributor we will commonly want to identify as "the SM $\mathbf{5}$" of $SU(5)$. Since this field necessarily has a nonzero $U(1)_N$ charge and is also a singlet of $SU(N-5)'$ so that no other diagonal generators are relevant, we must conclude that $U(1)_D$ has *no* contribution from $U(1)_N$ in this type of construction. Similarly, except for accidental cases, this also implies that all of the representations obtained via antisymmetric products of $\mathbf{N}$ with itself will also carry nonzero values of $U(1)_N$. Together, this directly implies that in such setup we must have

$$Q_D = \sum_i a_i \lambda_i^{\mathrm{Diag}}, \qquad (8)$$

where the $\lambda_i^{\mathrm{Diag}}$ are defined above and any potential Q_N contribution to Q_D must now be *absent*, i.e., $b = 0$. The same argument applies in the presence of any of the Higgs representations that produce SM fermion masses. If these are singlets of $SU(N-5)'$, then either their values of Q_N must all be zero or Q_N cannot be allowed to contribute to Q_D in such a setup; otherwise, $U(1)_D$ would be broken at the electroweak scale. This simple result has nontrivial implications, and we will see how it will play out more clearly in the subsequent examples we analyze more fully below. An important exemption to this conclusion *may* occur in scenarios where the spectrum of states is sufficiently rich that we can try to identify all of the SM $\bar{\mathbf{5}}$'s and $\mathbf{10}$'s with fields which are *not* also $SU(N-5)'$ singlets and, simultaneously, all carry nonzero values of Q_N. Such very rare cases, however, will encounter other problems such as, e.g., running afoul of the $SU(N)$ β-function constraint above or having Higgs fields which are $SU(N-5)'$ singlets carrying a nonzero Q_N charge.

Note that, since almost all the Higgs fields that we will encounter below will carry a nonzero value for the $U(1)_N$ charge, this symmetry will generally be broken at the same mass scale where the $SU(N-5)'$ group itself first breaks.

B. $SU(7)$

$SU(7)$ offers another opportunity to see where our requirements will cause models to fail and the general setup again simply "goes wrong" although in ways which are a bit more subtle than in the $SU(6)$ example above. Note that in this case, since $G_{\mathrm{Dark}} = SU(2)' \times U(1)_7$ [and some of the SM fields are always in $SU(2)'$ singlets], Q_D must be proportional to the diagonal λ_3 generator of $SU(2)'$ by the arguments made above.

$SU(7)$ has been considered as a potential GUT or family group since the earliest days, and we can take advantage of that huge body of work here. There are many sets of $SU(7)$ representations which satisfy (most of) our basic requirements that have been previously examined in other contexts, e.g., [44,45,48,57–60]. These models differ mainly in how they address the family or generation problem; i.e., in the notation introduced above, are they of the $n_g = 1$ or $n_g = 3$ variety? Certainly, the former are somewhat simpler, but both types of setups will lead to similar problems in the present context. The following are a nonexhaustive but fairly representative set of asymptotically free scenarios of both model classes appearing in the literature [44,45,48,57–60]:

$$\begin{aligned} &(a) \quad 3[3(\bar{\mathbf{7}}) + \mathbf{21}], \\ &(b) \quad 3[2(\bar{\mathbf{7}}) + \mathbf{35}], \\ &(c) \quad 8(\bar{\mathbf{7}}) + \mathbf{2}(\mathbf{21}) + \mathbf{35}, \\ &(d) \quad 7(\bar{\mathbf{7}}) + \mathbf{2}(\mathbf{35}) + \mathbf{21}, \end{aligned} \qquad (9)$$

with the first two of these being examples of scenarios with $n_g = 1$ while the last two are examples of $n_g = 3$.[7] Note that

[7] Note that a pair of singlet fields should also appear to remove Witten anomalies [59] but are not directly relevant to our present discussion, so they are omitted for simplicity.

other scenarios can be easily constructed [48] by "adding or subtracting" multiples of the combination of representations $\bar{\mathbf{7}}+\mathbf{21}+\overline{\mathbf{35}}$ [61,62] which forms an anomaly free set with $n_g=0$. Under the assumed breaking $SU(7)\to SU(5)\times SU(2)'\times U(1)_7$, we find that these representations, as well as the $SU(7)$ adjoint $\mathbf{48}$, will decompose as [63,64]

$$\begin{aligned}
\bar{\mathbf{7}} &\to (\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{2}}+(\mathbf{1},\mathbf{2})_{\mathbf{5}},\\
\mathbf{21} &\to (\mathbf{5},\mathbf{2})_{-\mathbf{3}}+(\mathbf{10},\mathbf{1})_{\mathbf{4}}+(\mathbf{1},\mathbf{1})_{-\mathbf{10}},\\
\mathbf{35} &\to (\mathbf{5},\mathbf{1})_{-\mathbf{8}}+(\mathbf{10},\mathbf{2})_{-\mathbf{1}}+(\overline{10},1)_{\mathbf{6}},\\
\mathbf{48} &\to (\mathbf{24},\mathbf{1})_{\mathbf{0}}+(\mathbf{1},\mathbf{3})_{\mathbf{0}}+(\mathbf{1},\mathbf{1})_{\mathbf{0}}+(\mathbf{5},\mathbf{2})_{\mathbf{7}}+(\bar{\mathbf{5}},\mathbf{2})_{-\mathbf{7}}. \qquad (10)
\end{aligned}$$

We observe that, as usual, the adjoint of Higgs in the $\mathbf{48}$ is responsible for both the initial $SU(7)$ breaking as well as that of the standard $SU(5)$ via the $(\mathbf{24},\mathbf{1})_{\mathbf{0}}$ component which we can imagine takes place at a comparable scale. We also find as expected that the potential fermionic PM fields are necessarily *chiral* with respect to both $SU(2)'$ and $U(1)_7$ gauge groups.

Here, we see that it is easy to identify one linear combination of the $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{2}}$ fields appearing in the $\bar{\mathbf{7}}$'s with the usual $\bar{\mathbf{5}}$ of $SU(5)$ that is an $SU(2)'$ singlet and, thus, automatically will have $Q_D=0$, since Q_7 does not contribute to this quantity as discussed above. In case (b), we need to identify the usual $\mathbf{10}$ containing SM fields with $(\mathbf{10},\mathbf{2})_{-\mathbf{1}}$ in the $\mathbf{35}$, but this representation is an $SU(2)'$ doublet, both of whose members must carry a value of $Q_D\sim\lambda_3\neq 0$; this excludes case (b) as a realistic possibility. In case (a), the corresponding identification of the $(\mathbf{10},\mathbf{1})_{\mathbf{4}}$ in the $\mathbf{21}$ with the $\mathbf{10}$ of the usual $SU(5)$ avoids this particular issue, since it automatically has $Q_D=0$ and these same types of choices would need to also be made elsewhere. We will return to this issue below.

The next, somewhat correlated, pair of obstacles we face are the generation of the various fermion masses as well as the breaking of $G_{\rm Dark}$ (hopefully) down to $U(1)_D$. The required Higgs fields to do these two jobs can be found from among the same set of representations given above for the fermions but with the particular choices dependent upon which scenario (a), (c), or (d) is being considered. Symbolically, in all three of these cases, such Higgs-Yukawa terms take the generic form (or some subset thereof) of the products of couplings

$$\begin{aligned}
&\sim \bar{\mathbf{7}}\cdot\mathbf{35}\cdot\mathbf{21_H}+\bar{\mathbf{7}}\cdot\mathbf{21}\cdot\bar{\mathbf{7}}_{\mathbf{H}}+\mathbf{21}\cdot\mathbf{35}\cdot\mathbf{21_H}\\
&\quad+\mathbf{21}\cdot\mathbf{21}\cdot\mathbf{35_H}+\mathbf{35}\cdot\mathbf{35}\cdot\mathbf{7_H}+\text{H.c.}, \qquad (11)
\end{aligned}$$

where the subscript H labels a Higgs representation. As promised, here we see the first representative examples of *scalar* PM fields, carrying both dark and SM charges, as necessary ingredients to the overall gauge symmetry breaking and mass generation process. For all of these cases, however, one will generally be attempting to pair up (at least some of), e.g., the additional $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{2}}$'s in the $\bar{\mathbf{7}}$'s with the $(\mathbf{5},\mathbf{2})_{-\mathbf{3}}$'s in the $\mathbf{21}$ via a $(\mathbf{1},\mathbf{2})_{\mathbf{5}}$ from a Higgs in a $\mathbf{7_H}$ to generate vectorlike mass terms for all of the PM fermions. Since these representations contain fields which are SM color triplets, we know from previous analyses [21–29] recasting LHC searches that such states must have masses which are in excess of $\sim$1–2 TeV so that the single $SU(2)'$-breaking VEV of $\mathbf{7_H}$'s must be at least several TeV. Now this $SU(2)'$ doublet VEV will also break both $SU(2)'$ as well as $U(1)_7$ but, as is well known [65], will *not* leave any remaining unbroken subgroup of $G_{\rm Dark}$ that can be identifiable as $U(1)_D$, since $U(1)_D$ must be a subgroup of $SU(2)_D$ by the discussion above.[8] However, if two different $SU(2)'$ doublets obtain VEVs which are not "aligned" (i.e., both $T_3'=\pm 1/2$ members obtaining VEVs even if this is in different doublets) in $SU(2)'$ space, then even this remaining $U(1)$ will also be broken. Since both members of the $(\mathbf{1},\mathbf{2})_{\mathbf{5}}$ $SU(2)'$ doublet necessarily carry $Q_D\neq 0$, even this single VEV will break $U(1)_D$ at a high scale, violating our requirements above.

These initial considerations will then exclude case (a) immediately without any further analysis, but some more straightforward checking is required to see what happens in cases (c) and (d). It does not take long, however, to convince oneself that in both of these cases a $(\mathbf{1},\mathbf{2})_{\mathbf{5}}$ from a Higgs in a $\mathbf{7_H}$ will still be required to give masses to some set of non-SM, vectorlike fermions at or above the TeV mass scale. For example, the $(\mathbf{10},\mathbf{2})_{-\mathbf{1}}+(\overline{\mathbf{10}},\mathbf{1})_{\mathbf{6}}$ representations in the $\mathbf{35}$ will pair up to form such vectorlike states via the VEV of the $(\mathbf{1},\mathbf{2})_{-\mathbf{5}}$, $SU(2)'$ isodoublet in the 7_H in both scenarios (c) and (d). Since these fields also contain VL-color-triplet fermions that need large masses, by our previous discussion this excludes these cases as well since this VEV necessarily breaks $U(1)_D$. Extending these arguments to their logical conclusion, we find that the identification of $G=SU(7)$ is a failure, since $SU(2)'$ doublet VEVs are always required, leaving $U(1)_D$ broken at a large scale. It is interesting (and unfortunate) to note that if we had *not* needed these $SU(2)'$ doublets to generate the PM vectorlike fermion masses, we could have simply employed the $(\mathbf{1},\mathbf{3})_{\mathbf{0}}$ in the $\mathbf{48}$ to break the $SU(2)'$ gauge symmetry and this would have left us with an unbroken $U(1)_D$ having the desired charge assignments.

C. $SU(8)$

As in the case of $SU(7)$, $G=SU(8)$ offers many model-building opportunities that have been discussed from time to time over the past few decades but which we can still divide into $n_g=1$ and $n_g=3$ subsets. Note that, in this scenario, we recall that $Q_D=a_1\lambda_3+a_2\lambda_8$, since now $G_{\rm Dark}=SU(3)'\times U(1)_8$ with $SU(3)'$ being rank 2, allowing for the possibility of a single $Q_D=0$ field within an $SU(3)'$ $\mathbf{3}/\bar{\mathbf{3}}$

[8]Remember that Q_D does not have a contribution from the $U(1)_7$ charge as discussed above.

representation. Here, we will see another example of how things can go wrong which will also be common for the larger unification groups we encounter below.

Some of the most common but yet not exhaustive set of example $SU(8)$ models appearing in the literature, e.g., [44,45,48,66–70], are given in the following list:

$$
\begin{aligned}
&\text{(a)} \quad 3[4(\bar{\mathbf{8}}) + \mathbf{28}],\\
&\text{(b)} \quad 3[\bar{\mathbf{8}} + \overline{\mathbf{28}} + \mathbf{56}],\\
&\text{(c)} \quad 3[2(\mathbf{8}) + \mathbf{3}(\overline{\mathbf{28}}) + \mathbf{2}(\mathbf{56})],\\
&\text{(d)} \quad 5(\overline{\mathbf{28}}) + \mathbf{4}(\mathbf{56}),\\
&\text{(e)} \quad 9(\bar{\mathbf{8}}) + \mathbf{28} + \mathbf{56}, \qquad (12)
\end{aligned}
$$

where the first three obviously have $n_g = 1$ while the last two have $n_g = 3$. As before, other possibilities can be obtained by adding or subtracting "multiples" of the combination of representations $3(\bar{\mathbf{8}}) + \mathbf{2}(\mathbf{28}) + \overline{\mathbf{56}}$ [62], which itself forms an anomaly free set with $n_g = 0$. Under the $SU(8) \to SU(5) \times SU(3)' \times U(1)_8$ decomposition one finds for the relevant representations that

$$
\begin{aligned}
\bar{\mathbf{8}} &\to (\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}} + (\mathbf{1},\bar{\mathbf{3}})_{\mathbf{5}},\\
\mathbf{28} &\to (\mathbf{5},\mathbf{3})_{-\mathbf{2}} + (\mathbf{10},\mathbf{1})_{\mathbf{6}} + (\mathbf{1},\bar{\mathbf{3}})_{-\mathbf{10}},\\
\mathbf{56} &\to (\mathbf{5},\bar{\mathbf{3}})_{-\mathbf{7}} + (\mathbf{10},\mathbf{3})_{\mathbf{1}} + (\overline{\mathbf{10}},\mathbf{1})_{\mathbf{9}} + (\mathbf{1},\mathbf{1})_{-\mathbf{15}},\\
\mathbf{70} &\to (\mathbf{5},\mathbf{1})_{-\mathbf{12}} + (\mathbf{10},\bar{\mathbf{3}})_{-\mathbf{4}} + (\overline{\mathbf{10}},\mathbf{3})_{\mathbf{4}} + (\bar{\mathbf{5}},\mathbf{1})_{\mathbf{12}},\\
\mathbf{63} &\to (\mathbf{24},\mathbf{1})_{\mathbf{0}} + (\mathbf{1},\mathbf{8})_{\mathbf{0}} + (\mathbf{1},\mathbf{1})_{\mathbf{0}} + (\mathbf{5},\bar{\mathbf{3}})_{\mathbf{8}} + (\bar{\mathbf{5}},\mathbf{3})_{-\mathbf{8}}. \qquad (13)
\end{aligned}
$$

Note that the **70** is a real representation as, of course, is the **63** adjoint. For $N = 8$, as noted above, the asymptotic freedom requirement now starts to be felt in a nontrivial way, since several of these possibilities might fail immediately without even considering the potential scalar contributions to the β function which will only make matters worse: We find, however, that only case (c) fails this requirement when we take three multiples of a single set of representations to obtain the three SM generations, and so it no longer needs to be realistically considered in the discussions that follow.

The gauge symmetry breaking in the $SU(8)$ scenario is rather familiar with the $SU(5)$ **24** performing its usual role. A single fundamental $\mathbf{3}/\bar{\mathbf{3}}$ will break $SU(3)'$ down to $SU(2)'$ [65], while the VEV of the $T'_3 = 0$ member of the $SU(2)'$ triplet within the adjoint, $(\mathbf{1},\mathbf{8})_{\mathbf{0}}$, will then break $SU(2)'$ down to $U(1)_D$ as desired. We note, however, that multiple fundamental $\mathbf{3},\bar{\mathbf{3}}$'s whose VEVs are not aligned, i.e., not all having $Q_D = 0$, will result in $SU(3)'$ breaking completely *without* any surviving $U(1)$'s [65] below the electroweak and TeV scales that we can identify with $U(1)_D$ as is required by our model-building constraints above. Note that *only* scalar fields in the $\mathbf{3},\bar{\mathbf{3}}$ representations appear that can break $SU(3)'$ here. Further, since several distinct (yet VEV-aligned) fundamentals appear with different values of the $U(1)_8$ charge, this symmetry will break at the same mass scale as does $SU(3)'$; this will be a common feature for all the models below with $G = SU(N), N \geq 8$. We again note that the PM fields are chiral under the unbroken $SU(2)'$ group.

Turning to the fermion mass terms, as we might expect, the required Higgs fields (apart from the usual adjoint) are essentially also members of this same set of representations as are the fermions. Similarly to the $SU(7)$ case above, we can again symbolically write the Higgs-Yukawa interaction terms for the fermion masses in a generic form (or as some subset thereof depending upon the case) of the products

$$
\begin{aligned}
&\sim \bar{\mathbf{8}}\cdot\mathbf{28}\cdot\bar{\mathbf{8}}_{\mathbf{H}} + \bar{\mathbf{8}}\cdot\mathbf{56}\cdot\overline{\mathbf{28}}_{\mathbf{H}} + \mathbf{28}\cdot\mathbf{56}\cdot\mathbf{56}_{\mathbf{H}} + \mathbf{56}\cdot\mathbf{56}\cdot\mathbf{28}_{\mathbf{H}}\\
&\quad + \mathbf{28}\cdot\mathbf{28}\cdot\mathbf{70}_{\mathbf{H}} + \text{permutations} + \text{H.c.}, \qquad (14)
\end{aligned}
$$

so that the number of possible PM and SM fermion mass generation terms are each somewhat restricted in all cases.

Note that, in case (a), we can select one linear combination the four $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}}$ fields contained in the four $\bar{\mathbf{8}}$'s to be the "conventional" SM $\bar{\mathbf{5}}$ of $SU(5)$ while also choosing the $(\mathbf{10},\mathbf{1})_{\mathbf{6}}$ from the 28 as the usual **10**. Since both of these fields are already $SU(3)'$ singlets, the first issue we had to deal with in the $SU(7)$ model above is trivially bypassed and the SM fields will have $Q_D = 0$ automatically as required. Simultaneously, the Higgs fields needed to supply vectorlike masses to the non-SM fermions are now either in singlets or in $\mathbf{3}/\bar{\mathbf{3}}$'s of $SU(3)'$ as we can tell from the representation decompositions above.

Case (a) has a somewhat simple symmetry-breaking sector as, apart from the adjoint **63**, it requires only the first and fifth terms in Eq. (10), i.e., Higgs fields in both the $\bar{\mathbf{8}}_{\mathbf{H}}$ and $\mathbf{70}_{\mathbf{H}}$ representations, and, since the three families are a simple replication of one subset, we can consider for simplicity fermions in a single combination of $4(\bar{\mathbf{8}}) + \mathbf{28}$ fields. Thinking at the $SU(5)$ level, we can identify one linear combination of the four $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}}$'s as the "SM field," while the other three $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}}$'s must then match up with the $(\mathbf{5},\mathbf{3})_{-\mathbf{2}}$ to form the vectorlike PM fields. From this, we learn two things: (i) The SM fermion masses are necessarily generated by two distinct SM $SU(2)_L$ Higgs isodoublets, one from the $\bar{\mathbf{8}}_{\mathbf{H}}$ and the other from the $\mathbf{70}_{\mathbf{H}}$ with the same type of coupling structure that occurs in the type-II two Higgs doublet model [71]. (ii) Even more importantly, the PM mass term in this model must necessarily be of the general form (in obvious $53'1_8$ language)

$$
y_{ia}[(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}}]_{\mathbf{i}}(\mathbf{5},\mathbf{3})_{-\mathbf{2}}[(\mathbf{1},\bar{\mathbf{3}})_{\mathbf{5}}]^{\mathbf{a}}_{\mathbf{H}} + \text{H.c.}, \qquad (15)
$$

where the y's are Yukawa couplings, the index $i = 1$–3 labels the three remaining $\bar{\mathbf{5}}$'s, and here we will allow for the possibility of more than one relevant Higgs antitriplet, labeled by the index $a = 1, \ldots$.

First, consider the simplest case when only a single Higgs field ($a = 1$) is present; since only one element of the $[(\mathbf{1}, \bar{\mathbf{3}})_\mathbf{5}]_\mathbf{H}$ is allowed to have a ($Q_D = 0$) VEV, this projects out a single corresponding element in the $(\mathbf{5}, \mathbf{3})_{-\mathbf{2}}$, implying only a single fermion bilinear can be constructed in the $SU(3)'$ subspace so that in the full $53'1_8$ space only a single set of five fermion bilinears can obtain masses; i.e., one of the $\bar{\mathbf{5}} \cdot \mathbf{5}$'s obtains a mass term and only five (degenerate) fermion masses are the result. If we increase the number of Higgs fields and allow for *arbitrary* alignment of their VEVs, then three Higgs fields will generate all the desired mass terms. However, as is well known, in such a situation the $SU(3)'$ group breaks completely [65], leaving us without a low-energy $U(1)_D$ gauge group. If, instead, we add extra Higgs fields where all the VEVs occur in the same element of the representation so that they are aligned, as is required so that $U(1)_D$ remains unbroken, this will not alter the result obtained with only a single Higgs field with only one $\bar{\mathbf{5}} \cdot \mathbf{5}$ mass term resulting. Thus, it is impossible to generate tree-level masses for these remaining two candidate PM fields at the $SU(3)'$-breaking scale. This is a disaster for this case, because at the subsequently $SU(2)'$-breaking scale (which must lie above a few TeV), to avoid breaking $U(1)_D$ while also generating the required gauge boson masses, i.e., those apart from that of the dark photon, the $T'_3 = 0$ member of the real $SU(2)'$ triplet in the adjoint is employed. Giving a $T'_3 \neq 0$ member of any scalar representation a VEV will automatically break $U(1)_D$ at or above the few TeV scale, and, thus, at least at tree level, the remaining $\bar{\mathbf{5}} \cdot \mathbf{5}$ terms must be absent. Since not all the needed masses can be generated at the $SU(3)'$-breaking scale, case (a) is excluded; this will be a very common feature of the many scenarios we will encounter below.

In case (e), since $n_g = 3$, three linear combinations of the nine $\bar{\mathbf{8}}$'s contain the three $\bar{\mathbf{5}}$'s which are also $SU(3)'$ singlets that we must identify as $Q_D = 0$ SM fields, while the **28** contains a $(\mathbf{10}, \mathbf{1})_\mathbf{6}$ which we can also identify with a SM $Q_D = 0$ field. However, we still need two more $SU(5)$ **10**'s with $Q_D = 0$ to identify with the remaining SM fields, and the only possible source for these lies within the $(\mathbf{10}, \mathbf{3})_\mathbf{1}$ in the **56** which is an $SU(3)'$ triplet. This requires that when the operator $Q_D = a_1\lambda_3 + a_2\lambda_8$ acts on this triplet it produces two zero eigenvalues by a suitable choice of $a_{1,2}$. But, of course as we know, this cannot happen, as at most one zero eigenvalue can be obtained. This implies that one of the SM generations necessarily carries $Q_D \neq 0$, which is not phenomenologically acceptable and violates our model-building assumptions above, thus excluding case (e).

The situation is found to be quite different in case (d), where we see immediately that *none* of the usual $SU(5)$ $\bar{\mathbf{5}}$'s or **10**'s are $SU(3)'$ singlets. However, we are able to freely choose one component of these triplet and antitriplet representations to have $Q_D = 0$; we can, without loss of generality, further take this to always be, e.g., the lowermost component in such triplet fields for purposes of this discussion. Then we see that, while a consistent pair of mass terms may be obtainable for the choice of SM fields, we cannot simultaneously generate ∼ few TeV-scale $SU(3)'$-breaking masses required for *all* of the potential vectorlike PM fermions in the remaining set of $\mathbf{5} + \bar{\mathbf{5}}$ and $\mathbf{10} + \overline{\mathbf{10}}$ representations without also breaking the $U(1)_D$ gauge symmetry, as was seen in case (a), as the required Yukawa mass terms will take the symbolic form

$$\sim (\mathbf{10}, \mathbf{3})_\mathbf{1} \cdot (\overline{\mathbf{10}}, \mathbf{1})_\mathbf{9} \cdot (\mathbf{1}, \bar{\mathbf{3}})^\mathbf{H}_{-\mathbf{10}} + (\bar{\mathbf{5}}, \bar{\mathbf{3}})_\mathbf{2} \cdot (\mathbf{5}, \bar{\mathbf{3}})_{-\mathbf{7}} \cdot (\mathbf{1}, \bar{\mathbf{3}})^\mathbf{H}_\mathbf{5} + \mathrm{H.c.} \tag{16}$$

Although there are two different species of antitriplet Higgs fields appearing here, their VEVs must still be aligned along the $Q_D = 0$ in direction, as we recall that only a single component of any of the triplet or antitriplet Higgs fields can obtain a VEV if we want to obtain an unbroken $U(1)_D$. This apparently excludes case (d).

However, maybe we can obtain some additional freedom in this particular case by recalling the caveat we noted above about the requirement that Q_D can be only some linear combination of the diagonal $SU(N-5)'$ generators, omitting any possible contribution for $U(1)_N$, i.e., above being just $Q_D = a_1\lambda_3 + a_2\lambda_8$, but now allowing for an additional Q_8 contribution. Perhaps we can apply this to case (d), as *neither* of the fields that we identify with the SM ($\bar{\mathbf{5}} + \mathbf{10}$) are $SU(3)'$ singlets but are instead triplets or antitriplets, i.e., by taking the SM fermions to be just the $Q_D = 0$ members of three copies of $(\bar{\mathbf{5}}, \bar{\mathbf{3}})_\mathbf{2} + (\mathbf{10}, \mathbf{3})_\mathbf{1}$. But then we immediately see that the Higgs responsible for generating the masses of the d, e-type SM fermions must be the $SU(3)'$ singlet field $(\bar{\mathbf{5}}, \mathbf{1})_{-\mathbf{3}}$ which is required to have $Q_D = 0$ so that $U(1)_D$ survives unbroken below the electroweak scale—yet this Higgs scalar carries a nonzero value of $Q_8 = -3$. Thus, it remains true that Q_D cannot have a contribution from the $U(1)_8$ generator Q_8, even in this case. Furthermore, one finds that, even if we allow for the possibility that the $Q_D = 0$ element is in different locations within the (anti)fundamental representation depending upon the value of Q_8, this result persists. At the very least we see that, in general, when $G = SU(N)$, if a Higgs representation which is needed to generate SM particle masses is an $SU(5-N)'$ singlet, it is not allowed to carry a nonzero value of Q_N if Q_N contributes to Q_D. It is interesting to note that in the already excluded case (c), something very similar happens where the $(\bar{\mathbf{5}}, \mathbf{1})^\mathbf{H}_{-\mathbf{3}}$ must generate the SM d, e-type mass terms so that Q_D must again be independent of Q_N. Furthermore, we see that a further necessary, but not sufficient, condition to allow for a Q_N contribution to Q_D is to make sure that the $\bar{\mathbf{5}}$ and **10** SM fields do not transform as conjugate representations under the $SU(N-5)'$ group so that the Higgs field(s) needed to generate mass terms are not $SU((N-5)'$ singlets. Similar arguments can be applied to the other fields as well.

Now having ruled out this case, as well as (e) above, we see that all of the $n_g = 3$ scenarios are excluded. It is also clear from this discussion that the survival threshold is set higher for these model varieties than for those with $n_g = 1$; this will continue to be the case as we increase N as will be seen below.

In the remaining case (b), the symmetry-breaking requirements can be somewhat more complex since, while the usual SM $SU(5)$ **10** must be identified with the lower member of the $(\mathbf{10},\mathbf{3})_\mathbf{1}$ in the $SU(8)$ $\overline{\mathbf{28}}$, the SM $\bar{\mathbf{5}}$ can *either* be the $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}}$ in the $\bar{\mathbf{8}}$, as in case (a), or the lower member of the $(\bar{\mathbf{5}},\bar{\mathbf{3}})_\mathbf{2}$ in the $\overline{\mathbf{28}}$. Both of these assignments are made possible by assuming, e.g., that the lower members of all triplets and antitriplets have $Q_D = 0$ as we did in case (d). Both of these choices allow for the generation of the SM d-quark and charged lepton masses via either $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}} \cdot (\mathbf{10},\mathbf{3})_\mathbf{1} \cdot (\bar{\mathbf{5}},\bar{\mathbf{3}})^\mathbf{H}_\mathbf{2}$-type or $(\bar{\mathbf{5}},\bar{\mathbf{3}})_\mathbf{2} \cdot (\mathbf{10},\mathbf{3})_\mathbf{1} \cdot (\bar{\mathbf{5}},\mathbf{1})^\mathbf{H}_{-\mathbf{3}}$-type Yukawa couplings. Note that in this latter case the fields in the $(\bar{\mathbf{5}},\bar{\mathbf{3}})_\mathbf{2}$ with $Q_D \neq 0$ can also pick up an electroweak-scale mass term. Recall that, in either of these cases, the SM fermion or Higgs representation assignments will still force us to require that $Q_D = a_1\lambda_3 + a_2\lambda_8$ without there being any Q_8 contribution. For either choice of these assignments, the u-quark mass is always generated by the coupling $(\mathbf{10},\mathbf{3})_\mathbf{1} \cdot (\mathbf{10},\mathbf{3})_\mathbf{1} \cdot (\mathbf{5},\mathbf{3})^\mathbf{H}_\mathbf{2}$, where again both $Q_D = 0$ as well as the $Q_D \neq 0$ components can pick up electroweak-scale masses.

The most significant, yet as we have seen apparently common, problem one faces with case (b) is the lack of a sufficient number of mass terms for all of the PM fields which should lie in (at least) the few TeV range. The only mass terms for pairs of $SU(5)$ nonsinglet fermions that are generated by $SU(5)$-singlet, $SU(3)'$-breaking (i.e., nonsinglet) Higgs scalars with potentially large VEVs are seen to be of the general coupling structures:

$$\sim (\mathbf{10},\mathbf{3})_\mathbf{1} \cdot (\overline{\mathbf{10}},\mathbf{1})_\mathbf{9} \cdot (\mathbf{1},\bar{\mathbf{3}})^\mathbf{H}_{-\mathbf{10}} + (\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{3}} \cdot (\mathbf{5},\bar{\mathbf{3}})_{-\mathbf{7}} \cdot (\mathbf{1},\mathbf{3})^\mathbf{H}_\mathbf{10} + (\bar{\mathbf{5}},\bar{\mathbf{3}})_\mathbf{2} \cdot (\mathbf{5},\bar{\mathbf{3}})_{-\mathbf{7}} \cdot (\mathbf{1},\bar{\mathbf{3}})^\mathbf{H}_\mathbf{5} + \mathrm{H.c.} \tag{17}$$

Employing this expression, it is easily seen that it is impossible to simultaneously supply all of the desired PM mass terms while also keeping $U(1)_D$ unbroken below the few TeV scale, since all the $\mathbf{3},\bar{\mathbf{3}}$ VEVs must be aligned in a single direction, so that this case is also excluded.

From the set of analyses above, we can conclude that all of the $SU(8)$ scenarios that we have considered are excluded, thus disfavoring this unification gauge group.

D. $SU(9)$

Since in the case of $G = SU(9)$ one has $G_{\mathrm{Dark}} = SU(4)' \times U(1)_9$, here we can define the dark charge as $Q_D = a_1\lambda_3 + a_2\lambda_8 + a_3\lambda_{15}$, so that a $\mathbf{4},\bar{\mathbf{4}}$ representation can now have up to two of its members with $Q_D = 0$, thus generalizing the case of $SU(3)'$ seen above. These two potential $Q_D = 0$ elements can be easily achieved by, e.g., taking $a_2 = a_3 = 0$, although this is not a unique choice. Thus, to break the $SU(4)'$ gauge symmetry in this case, we can, in principle, employ two unaligned VEVs in different **4** or $\bar{\mathbf{4}}$'s to reduce the symmetry to $SU(2)'$ and then follow the same path as was discussed above for the $SU(8)$ scenario employing the $SU(2)'$ real triplet in the $SU(9)$ adjoint to reduce this gauge symmetry further down to $U(1)_D$. Note also that, unlike the previously considered models, the Higgs fields in the (now distinct) second-rank, antisymmetric tensor representation (or its conjugate) of $SU(4)'$, i.e., $\mathbf{6},\bar{\mathbf{6}}$, can also participate in the symmetry-breaking process, which can break $SU(4)'$ directly down to $SU(2)'$ [65]. We see that the chosen number of $Q_D = 0$ elements of the fundamental representation will then determine the corresponding number of such elements in the $\mathbf{6},\bar{\mathbf{6}}$ second-rank antisymmetric representation; i.e., for a single $Q_D = 0$ VEV to occur for such reps, here we need to have two $Q_D = 0$ elements in the fundamental. As in the case of $SU(8)$, $U(1)_9$ will break at the same scale as does $SU(4)'$, since several SM singlet dark Higgs fields will always be present with different values of Q_9.

As in the examples of both $SU(7)$ and $SU(8)$ above, there are many representative $SU(9)$ unification models that have been considered in the previous literature [44,45,48,72–75] having either $n_g = 1$ or 3, e.g.,

$$\begin{aligned}
&(\mathrm{a}) \quad 3[5(\bar{\mathbf{9}}) + \mathbf{36}],\\
&(\mathrm{b}) \quad 3[\overline{\mathbf{36}} + \mathbf{126}],\\
&(\mathrm{c}) \quad 3[\bar{\mathbf{9}} + \overline{\mathbf{84}} + \mathbf{2}(\mathbf{126})],\\
&(\mathrm{d}) \quad 4(\bar{\mathbf{9}}) + \mathbf{2}(\mathbf{36}) + \mathbf{84} + \overline{\mathbf{126}},\\
&(\mathrm{e}) \quad 9(\bar{\mathbf{9}}) + \mathbf{84},
\end{aligned} \tag{18}$$

which form a nonexhaustive but typical set of these possibilities. As before, this set is easily expandable by, e.g., adding or subtracting multiples of the anomaly free combinations [62] $6(\bar{\mathbf{9}}) + \mathbf{3}(\mathbf{36}) + \overline{\mathbf{84}}$ and/or $5(\bar{\mathbf{9}}) + \mathbf{2}(\mathbf{36}) + \overline{\mathbf{126}}$. However, when doing so, one must take care that the asymptotic freedom requirement imposed on $SU(9)$ is still satisfied, since, as noted previously, the strength of this requirement grows as N increases. Furthermore, under the $SU(9) \to SU(5) \times SU(4)' \times U(1)_9$ decomposition, one finds for the relevant representations that

$$\begin{aligned}
\bar{\mathbf{9}} &\to (\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{4}} + (\mathbf{1},\bar{\mathbf{4}})_\mathbf{5},\\
\mathbf{36} &\to (\mathbf{5},\mathbf{4})_{-\mathbf{1}} + (\mathbf{10},\mathbf{1})_\mathbf{8} + (\mathbf{1},\mathbf{6})_{-\mathbf{10}},\\
\mathbf{84} &\to (\mathbf{5},\mathbf{6})_{-\mathbf{6}} + (\mathbf{10},\mathbf{4})_\mathbf{3} + (\overline{\mathbf{10}},\mathbf{1})_\mathbf{12} + (\mathbf{1},\bar{\mathbf{4}})_{-\mathbf{15}},\\
\mathbf{126} &\to (\mathbf{5},\bar{\mathbf{4}})_{-\mathbf{11}} + (\mathbf{10},\mathbf{6})_{-\mathbf{2}} + (\overline{\mathbf{10}},\mathbf{4})_\mathbf{7} + (\bar{\mathbf{5}},\mathbf{1})_\mathbf{16} + (\mathbf{1},\mathbf{1})_{-\mathbf{20}},\\
\mathbf{80} &\to (\mathbf{24},\mathbf{1})_\mathbf{0} + (\mathbf{1},\mathbf{15})_\mathbf{0} + (\mathbf{1},\mathbf{1})_\mathbf{0} + (\mathbf{5},\bar{\mathbf{4}})_\mathbf{9} + (\bar{\mathbf{5}},\mathbf{4})_{-\mathbf{9}}.
\end{aligned} \tag{19}$$

As was also found with the $G = SU(8)$ possibility, we see that some of these models above will immediately fail the growingly powerful asymptotic freedom condition for $SU(9)$—even without any consideration of the scalar sector. Here, we observe that we no longer need to realistically consider cases (b) and (c) any further, which is rather unfortunate, as the set of three copies of the single set of representations needed to reproduce the three SM generations has just too many degrees of freedom to maintain AF above the unification scale—even before the symmetry-breaking Higgs scalar sector contributions are included in the $SU(9)$ β function. Case (d) may also be excluded by this constraint depending upon the required complexity of its scalar Higgs sector as we shall soon find. We do note in passing that in case (c), even though the SM fields are embedded in a nontrivial fashion, the Higgs field responsible for generating the u-type quark masses is an $SU(4)'$ singlet carrying $Q_9 \neq 0$, thus disallowing any contribution of Q_9 to Q_D.

To go further, we note that the various mass terms that may be needed in this $SU(9)$ scenario will be generated by (some subset of) the general Yukawa terms resulting from the generic combination

$$\sim \bar{\mathbf{9}} \cdot \mathbf{36} \cdot \bar{\mathbf{9}}_{\mathbf{H}} + \bar{\mathbf{9}} \cdot \mathbf{126} \cdot \overline{\mathbf{84}}_{\mathbf{H}} + \bar{\mathbf{9}} \cdot \mathbf{84} \cdot \overline{\mathbf{36}}_{\mathbf{H}} + \mathbf{36} \cdot \mathbf{36} \cdot \overline{\mathbf{126}}_{\mathbf{H}} + \mathbf{36} \cdot \mathbf{84} \cdot \mathbf{126}_{\mathbf{H}} + \mathbf{84} \cdot \mathbf{84} \cdot \mathbf{84}_{\mathbf{H}} + \cdots + \text{perms} + \text{H.c.} \tag{20}$$

In case (d), since the fermion fields themselves lie in representations of four different dimensionalities, the Yukawa interaction expression above tells us that we will need at least one Higgs scalar in each of the **9**, **36**, **84**, and **126** (or their conjugates) representations as well as the adjoint **80** to break all the symmetries. This increases the $SU(9)$ beta function by a positive factor of $\delta\beta = (9 + n_9 + 7n_{36} + 21n_{84} + 35n_{126})/6$, so that even if only one of each type of Higgs scalar representation were introduced the additional degrees of freedom would also render this case nonasymptotically free, and so it too now becomes excluded.

For case (a), only a few of these possible mass generating terms above are relevant. From this subset we see that case (a) for $SU(9)$ is indeed quite similar to case (a) for $SU(8)$ where (for a single generation) we identify one linear combination of the five $(\bar{\mathbf{5}}, \mathbf{1})_{-\mathbf{4}}$ fields from the five $\bar{\mathbf{9}}$'s with the usual SM field yet the assignment of the $(\mathbf{10}, \mathbf{1})_{\mathbf{8}}$ from the $SU(9)$ **36** is unique. While there are then no issues with the SM fermion masses, since these fields are themselves $SU(4)'$ singlets as above, problems do arise in generating the masses for the PM fields which transform nontrivially under both $SU(5)$ and $SU(4)'$, similar to what was found in the case of $SU(8)$. One finds that the relevant mass generating couplings in the present case to be of the form

$$y_{ia}[(\bar{\mathbf{5}}, \mathbf{1})_{-\mathbf{4}}]_{\mathbf{i}}(\mathbf{5}, \mathbf{4})_{-\mathbf{1}}[(\mathbf{1}, \bar{\mathbf{4}})_{\mathbf{5}}]^{\mathbf{a}}_{\mathbf{H}} + \text{H.c.}, \tag{21}$$

where the y's are Yukawa couplings as above while the index $i = 1$–4 now labels the four remaining combinations of the $\bar{\mathbf{5}}$'s and we again allow for the possibility of more than one Higgs field, labeled by the index a. Since only the two $Q_D = 0$ components of any $SU(4)$ **4** or $\bar{\mathbf{4}}$ fields can obtain VEVs without also breaking $U(1)_D$, two of the desired fermionic PM fields will not be able to obtain TeV-scale masses this way. Thus, case (a) is found to also be excluded when $G = SU(9)$ as was expected from the same arguments made previously for case (a) of $SU(8)$.

Lastly, we must consider the interesting case (e) which benefits from having fermion fields in only two distinct types of representations but whose symmetry-breaking requirements are rather complex, since it is an $n_g = 3$ scenario with both SM and candidate PM fields lying in the same representations at the $SU(5) \times SU(4)' \times U(1)_9$, i.e., $54'1_9$ level. As can be seen from the expression above, this implies that only the $\mathbf{36}_{\mathbf{H}}, \mathbf{84}_{\mathbf{H}}$-type Higgs scalars (and/or their complex conjugates) are needed to generate all the fermion mass terms, which helps to satisfy the asymptotic freedom constraint. However, this also implies that the fermion fields obtaining their masses in the breaking of $SU(4)'$ might be "mixed" in a nontrivial manner with those receiving masses due to the usual SM electroweak symmetry breaking. Fortunately, we can remove some of this complexity by working in the approximate limit where the $SU(4)'$-breaking scale is taken to be much larger than the electroweak scale so that these two symmetry-breaking steps can be effectively decoupled. On the other hand, making matters more complicated are the two types of representations of pure dark sector sets of fields, $(\mathbf{1}, \bar{\mathbf{4}})_{\mathbf{5},-\mathbf{15}}$, which also enter into the fermion mass matrix at the same $SU(4)'$-breaking mass scale. For example, while three sets of the $(\bar{\mathbf{5}}, \mathbf{1})_{-\mathbf{4}}$'s from the nine $\bar{\mathbf{9}}$'s might be identified with usual $Q_D = 0$, $SU(5)$ SM $\bar{\mathbf{5}}$ fields, the remaining six of these must then pair up with the $(\mathbf{5}, \mathbf{6})_{-\mathbf{6}}$ in the **84** via a $[(\mathbf{1}, \mathbf{6})_{\mathbf{10}}]_{\mathbf{H}}$ in the $\mathbf{84}_{\mathbf{H}}$ to form massive PM fields. Similarly, three of the members of the $SU(4)'\mathbf{4}$ representation in the $(\mathbf{10}, \mathbf{4})_{\mathbf{3}}$'s must be identified with usual $SU(5)$ SM **10** fields while the remaining multiplet member must pair up with the corresponding $(\overline{\mathbf{10}}, \mathbf{1})_{\mathbf{12}}$ via a $[(\mathbf{1}, \bar{\mathbf{4}})_{-\mathbf{15}}]_{\mathbf{H}}$, also in the $\mathbf{84}_{\mathbf{H}}$, to form an additional PM. Once the Higgs scalar in the coupling $(\mathbf{10}, \mathbf{4})_{\mathbf{3}} \cdot (\overline{\mathbf{10}}, \mathbf{1})_{\mathbf{12}} \cdot [(\mathbf{1}, \bar{\mathbf{4}})_{-\mathbf{15}}]_{\mathbf{H}}$ obtains a (single) VEV, breaking $SU(4)'$ down to $SU(3)'$, one sees that one of these **10**'s indeed obtains a large mass above the TeV scale as is needed. While this last step was rather straightforward, the rest of the required mass generation is found to be somewhat more difficult to manage given our model-building constraints.

As we have seen above, the requirement that only the $Q_D = 0$ elements of the various Higgs representations are allowed to obtain VEVs (at the electroweak scale or above) so that $U(1)_D$ remains unbroken leads to the result that many fields remain massless at the required breaking scale. As a practical example in the present case, consider the mass term for the six sets of the $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{4}}$'s mentioned above that we want to obtain masses at the $SU(4)'$-breaking scale by pairing with up those in the single $(\mathbf{5},\mathbf{6})_{-\mathbf{6}}$ via a Yukawa coupling structure similar to what we have frequently encountered above, i.e.,

$$y_{ia}[(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{4}}]_i(\mathbf{5},\mathbf{6})_{-\mathbf{6}}[(\mathbf{1},\mathbf{6})_{\mathbf{10}}]^{\mathbf{a}}_{\mathbf{H}} + \mathrm{H.c.}, \tag{22}$$

where $i = 1$–6 now runs over these six non-SM $\bar{\mathbf{5}}$'s as before. While the matching numbers of degrees of freedom are correct for generating mass terms for all of these states, the VEV structure of the single $[(\mathbf{1},\mathbf{6})_{\mathbf{10}}]_{\mathbf{H}}$ (which has only two $Q_D = 0$ elements) is insufficient to accomplish this, allowing for at most two of these pairings to pick up masses as the multi-TeV level. Note that adding additional $[(\mathbf{1},\mathbf{6})_{\mathbf{10}}]_{\mathbf{H}}$'s will not be helpful as long as only the $Q_D = 0$ elements are allowed to obtain VEVs, so this structure is necessarily aligned. Furthermore, we note that, at the level of the multi-TeV or above $SU(4)'$-breaking scale, the additional dark sector fields in the two $(\mathbf{1},\bar{\mathbf{4}})$ do not enter into these considerations. Similarly, we face essentially the same difficulties when we want to give $SU(2)_L$-breaking masses to the three remaining 10's discussed above. Here we are faced with the Yukawa couplings

$$\sim(\mathbf{10},\mathbf{4})_{\mathbf{3}}(\mathbf{10},\mathbf{4})_{\mathbf{3}}[(\mathbf{5},\mathbf{6})_{-\mathbf{6}}]_{\mathbf{H}} + \mathrm{H.c.}, \tag{23}$$

which is again seen to be insufficient to able to perform the required role, generating only two (instead of the needed three) mass terms for the uplike quarks. Thus, apparently, case (e) is also excluded from our list of candidate scenarios due to the now common "insufficient number of VEVs" problem.

We can conclude from this discussion above that the choice $G = SU(9)$ is highly disfavored or more likely excluded.

Before continuing, we make the following observation: Interestingly, we find that as N increases one sees that the number of $Q_D = 0$ VEVs that are allowed in the (anti) fundamental and higher scalar representations also increases. However, we also see that, apparently, it does not increase fast enough (at least in the present example) to generate all the required masses for the SM fields as well as for the similarly growing number of candidate PM fermions that we encounter.

E. $SU(\mathbf{10})$

In the scenario with $G = SU(10)$ broken by the VEV of the Higgs in the adjoint representation, one has $G_{\mathrm{Dark}} = SU(5)' \times U(1)_{10}$, and thus we can define the dark charge as $Q_D = a_1\lambda_3 + a_2\lambda_8 + a_3\lambda_{15} + a_4\lambda_{24}$. This implies that a $\mathbf{5}/\bar{\mathbf{5}}$ representation can now have up to three members with $Q_D = 0$ which may (or may not) be fortuitously the same as the number of SM generations. As in previous cases, since $SU(5)'$ singlet fields carrying nonzero values of the $U(1)_{10}$ charge, Q_{10}, are involved in generating the SM fermion masses, under the assumption that $Q_D(\mathrm{SM}) = 0$, then Q_D cannot have a contribution from Q_{10}. Similar to the discussion above, $Q_D = 0$ for three members of the fundamental can be easily achieved by, e.g., taking $a_2 = a_3 = a_4 = 0$ although this is far from a unique choice. To break the $SU(5)'$ gauge symmetry in this case, we can, in principle, employ three distinct unaligned VEVs in different $\mathbf{5}$ or $\bar{\mathbf{5}}$'s reducing the symmetry to $SU(2)'$ as above and then employ the $SU(2)'$ triplet in the $SU(10)$ adjoint to reduce this even further to the desired $U(1)_D$. Here, with three $Q_D = 0$ members of the fundamental or antifundamental obtaining VEVs [and, hence, breaking the $SU(3)'$ subgroup of the $SU(5)'$], the second-rank antisymmetric $\mathbf{10}/\overline{\mathbf{10}}$ Higgs representation will then correspondingly now have *four* VEVs with $Q_D = 0$, three of which contribute to $SU(3)'$ breaking while the fourth breaks any additional remaining $U(1)'$ so that only an $SU(2)'$ remains. As was the case above, $U(1)_{10}$ will break at the same scale as does $SU(5)'$, since we always have several SM singlet dark Higgs fields present with different values of Q_{10}. The $SU(10)$ model is also unique among those previously considered, as both the visible and dark sectors are (obviously) described by the same non-Abelian gauge group. As in the models above, we note that the PM fields will be *chiral* with respect to $SU(5)'$ and, hence, with respect to the $SU(2)'$ group just as the SM fields are chiral with respect to $SU(2)_L$.

As in the cases above, many representative $SU(10)$ unification models that have been considered in the previous literature [44–50] with either $n_g = 1$ or 3, e.g.,

$$\begin{aligned}
&(\mathrm{a}) && 3[6(\overline{\mathbf{10}}) + \mathbf{45}],\\
&(\mathrm{b}) && 3[\overline{\mathbf{120}} + \mathbf{210}],\\
&(\mathrm{c}) && 2(\mathbf{10}) + \mathbf{5}(\overline{\mathbf{45}}) + \mathbf{2}(\mathbf{120}),\\
&(\mathrm{d}) && 8(\overline{\mathbf{10}}) + \overline{\mathbf{45}} + \mathbf{120},\\
&(\mathrm{e}) && 2(\overline{\mathbf{10}}) + \mathbf{2}(\overline{\mathbf{45}}) + \mathbf{210},
\end{aligned} \tag{24}$$

which form a nonexhaustive but representative set of these possibilities. As in the previous scenarios, this set is easily expandable by, e.g., adding or subtracting multiples of the combinations [62] $10(\overline{\mathbf{10}}) + \mathbf{4}(\mathbf{45}) + \overline{\mathbf{120}}$ and/or $16(\overline{\mathbf{10}}) + \mathbf{5}(\mathbf{45}) + \overline{\mathbf{210}}$. However, in using this freedom one again quickly runs into difficulties with $SU(10)$'s asymptotic freedom, as the representations are growing quite large—especially so if multiple $\mathbf{120}/\overline{\mathbf{120}}$'s are needed or if more than a single $\mathbf{210}/\overline{\mathbf{210}}$ is present as in case (b). Here, one must seriously take (in this case extra) care to ensure that the asymptotic freedom requirement imposed on the $SU(10)$

gauge coupling is still satisfied, especially now with the noted large dimensionalities of these representations.

We note that, under the $SU(10) \to SU(5) \times SU(5)' \times U(1)_{10}$ decomposition, one finds for the relevant representations that

$$\begin{aligned} \overline{\mathbf{10}} &\to (\bar{\mathbf{5}},\mathbf{1})_{-1} + (\mathbf{1},\bar{\mathbf{5}})_{1}, \\ \mathbf{45} &\to (\mathbf{5},\mathbf{5})_{0} + (\mathbf{10},\mathbf{1})_{2} + (\mathbf{1},\mathbf{10})_{-2}, \\ \mathbf{120} &\to (\mathbf{5},\mathbf{10})_{-1} + (\mathbf{10},\mathbf{5})_{1} + (\overline{\mathbf{10}},\mathbf{1})_{3} + (\mathbf{1},\overline{\mathbf{10}})_{-3}, \\ \mathbf{210} &\to (\mathbf{5},\overline{\mathbf{10}})_{-2} + (\mathbf{10},\mathbf{10})_{0} + (\overline{\mathbf{10}},\mathbf{5})_{2} + (\bar{\mathbf{5}},\mathbf{1})_{4} + (\mathbf{1},\bar{\mathbf{5}})_{-4}, \\ \mathbf{99} &\to (\mathbf{24},\mathbf{1})_{0} + (\mathbf{1},\mathbf{24})_{0} + (\mathbf{1},\mathbf{1})_{0} + (\mathbf{5},\bar{\mathbf{5}})_{2} + (\bar{\mathbf{5}},\mathbf{5})_{-2}, \end{aligned} \tag{25}$$

again showing the obvious symmetry between the visible and dark sectors. As was found with both the $G = SU(8)$ and $G = SU(9)$ possibilities above, one of these model cases immediately fails the asymptotic freedom condition for $SU(10)$ even without any contribution from the Higgs sector which would only make things worse; in this regard, we need not realistically consider case (b) any further. As above, this is the result of requiring three copies of a single set of somewhat high-dimensional representations as is needed to reproduce the three SM fermion generations. We must also be especially mindful of both cases (c) and (e) that come rather close in this regard, so some additional care is necessary when examining the content of their required Higgs scalar symmetry-breaking sectors, as this may also lead to their exclusion for similar reasons.

As before, to go any further, we must consider how the various and numerous fermion mass terms that may be needed in this $SU(10)$ scenario will be obtained from the general Yukawa coupling terms resulting from multiple (sub)combination of factors:

$$\begin{aligned} &\sim \overline{\mathbf{10}} \cdot \mathbf{45} \cdot \overline{\mathbf{10}}_{\mathbf{H}} + \overline{\mathbf{10}} \cdot \mathbf{120} \cdot \overline{\mathbf{45}}_{\mathbf{H}} + \mathbf{45} \cdot \mathbf{45} \cdot \overline{\mathbf{210}}_{\mathbf{H}} \\ &\quad + \mathbf{120} \cdot \mathbf{120} \cdot \mathbf{210}_{\mathbf{H}} + \cdots + \text{perms} + \text{H.c.} \end{aligned} \tag{26}$$

The potential contribution of the various Higgs scalar representations we find in this expression (together with the usual adjoint) to the $SU(10)$ β function is given by the sum $(10 + n_{10} + 8n_{45} + 28n_{120} + 56n_{210})/6$ from which we see that, e.g., $n_{10} = n_{45} = n_{210} \geq 1$, which is the common expectation, already yields a contribution of $\delta\beta \geq 75/6$. Given this, we can now also exclude both cases (c) and (e) immediately from further consideration, since they will also violate the bound supplied by the asymptotic freedom constraint due to even this minimal Higgs scalar breaking sector. Before we turn to the remaining cases, it is interesting to note that in cases (b) and (c), where the SM $\bar{\mathbf{5}}$'s are not embedded simply into $SU(10)$ $\overline{\mathbf{10}}$'s, one again finds that the Higgs fields responsible for generating the SM d, e-type masses are in $SU(5)'$ singlets, thus excluding any contribution of Q_{10} to Q_D in these scenarios as usual.

We can easily make short work of case (a), as it falls into the same familiar pattern that we observed earlier for the (a) cases of both $SU(8)$ and $SU(9)$ above; as before, it is sufficient for our purposes to consider a single SM generation. Also as before, the realization of the SM fermions masses goes through without any issues [since the SM fermions lie in $SU(5)'$ singlet representations] via the now familiar couplings

$$\begin{aligned} &\sim (\bar{\mathbf{5}},\mathbf{1})_{-1}(\mathbf{10},\mathbf{1})_{2}[(\bar{\mathbf{5}},\mathbf{1})_{-1}]_{\mathbf{H}} \\ &\quad + (\mathbf{10},\mathbf{1})_{2}(\mathbf{10},\mathbf{1})_{2}[(\mathbf{5},\mathbf{1})_{-4}]_{\mathbf{H}} + \text{H.c.}, \end{aligned} \tag{27}$$

after selecting out one linear combination of the six $(\bar{\mathbf{5}},\mathbf{1})_{-1}$'s to play the role of the SM $\bar{\mathbf{5}}$. As expected, however, the Yukawa term generating masses for the PM in this scenario is found to be inadequate to the task, i.e., with the familiar structure

$$y_{ia}[(\bar{\mathbf{5}},\mathbf{1})_{-1}]_{\mathbf{i}}(\mathbf{5},\mathbf{5})_{\mathbf{0}}[(\mathbf{1},\bar{\mathbf{5}})_{\mathbf{1}}]^{\mathbf{a}}_{\mathbf{H}} + \text{H.c.}, \tag{28}$$

where the y's are Yukawa couplings as before, the index i runs over the five remaining $(\bar{\mathbf{5}},\mathbf{1})_{-1}$'s, and where, in principle, we can also allow for multiple Higgs scalars each with three $Q_D = 0$ VEVs that can all be different. We correspondingly recall, however, that there are only three distinct linearly independent sets of elements or directions in the group or field space for these multiple sets of scalar VEVs [thus breaking $SU(5)' \to SU(2)'$] to maintain an unbroken $U(1)_D$. This leaves us with two of the five potential PM $SU(5)$ $\mathbf{5}$ fermion fields remaining massless at the desired stage of symmetry breaking. As before, adding further additional Higgs scalars, i.e., beyond three, is redundant and of no help due to the required limitation on the number of independent sets of VEVs, since there are only $Q_D = 0$ directions allowed in field space. *Without* this restriction, of course, it would be straightforward to generate the masses of all five of these vectorlike fermion PM representations. This eliminates the set of $n_g = 1$ models from any further consideration, and so we must turn our attention to the one remaining case (d).

In model (d), which is an $n_g = 3$ scenario, both SM as well as PM fermions are found to lie in some common representations of the $55'1_{10}$ subgroup, and many of the aspects of its symmetry breaking and mass generation aspects will be familiar from the previously examined cases above, albeit now with a bit more complexity. We begin by reconsidering the set of Yukawa couplings that can produce the masses for the various fermions which is quite rich in this particular scenario, since all of the possible Higgs scalars are involved and some of the SM fermion fields now can lie in a $\overline{\mathbf{45}}$ instead of a $\mathbf{45}$ as they did in case (a):

$$\sim \overline{\mathbf{10}}\cdot(\overline{\mathbf{10}}\cdot\mathbf{45_H}+\overline{\mathbf{45}}\cdot\mathbf{120_H}+\mathbf{120}\cdot\overline{\mathbf{45}}_\mathbf{H})+\overline{\mathbf{45}}\cdot(\overline{\mathbf{45}}\cdot\mathbf{210_H}+\mathbf{120}\cdot\overline{\mathbf{10}}_\mathbf{H})+\mathbf{120}\cdot\mathbf{120}\cdot\mathbf{210_H}+\mathrm{H.c.} \qquad (29)$$

This being the case, we are reminded that the full $SU(10)$ one-loop β function for this model is given by $\beta_d = -62/3 + (10 + n_{10} + 8n_{45} + 28n_{120} + 56n_{210})/6$, where the first term is the sum of the fixed gauge and fermion contributions. Note that the number of the larger Higgs scalar representations, **120**, **210**, is therefore quite restricted here. We further observe that if we analyze the 245, two-component fermion degrees of freedom in case (d), we see that, in addition to the usual 45 SM fields, 60 of them are purely dark sector fields while the remaining 140 are (hopefully) to be identified with 70, four-component vectorlike PM fermions, all of which must obtain masses significantly in excess of the electroweak scale via suitable Higgs choices. The mass requirements for the purely dark sector fields are not as strict except that we would like to identify the lightest such field having $Q_D \neq 0$ with DM, and so it should have a mass roughly of the order of the $U(1)_D$-breaking scale. Of course, a purely dark scalar is also just as likely, and may even be favored, to be the light DM when all of this model's details and constraints are fully examined.

Because of the complexities of the relevant couplings in this scenario, we will consider the generation of the fermion masses in the different sectors in some further detail here. In this model, we can consider two quite distinct ways of embedding the SM and PM fermions into the various $SU(10)$ representations. The first, more obvious and more familiar approach [here called subcase (α)], is where three linear combinations of the eight $\bar{\mathbf{5}}$'s among the eight $\overline{\mathbf{10}}$'s of $SU(10)$ are identified as the familiar SM fields. In a somewhat more complex embedding, the SM fields can instead [termed subcase (β)] correspond to the three $Q_D = 0$ members of the $(\bar{\mathbf{5}},\bar{\mathbf{5}})_0$ in the $\overline{\mathbf{45}}$. In subcase (α), since at least some of the SM fields are to be identified with $SU(5)'$ singlets, it is imperative that Q_D cannot receive any contribution from Q_{10}. In either of these subcases, the three familiar SM $SU(5)$ **10**'s can be identified only with the three $Q_D = 0$ members of the $(\mathbf{10},\mathbf{5})_\mathbf{1}$ that lie in the **120**, while the other two members of this representation must pair up with the $(\mathbf{10},\mathbf{1})_\mathbf{2}$ in the $\overline{\mathbf{45}}$ and the $(\overline{\mathbf{10}},\mathbf{1})_\mathbf{3}$ in the **120** to obtain a total of 20 vectorlike masses for the resulting four-component fermions and then be identified with some of the PM. Failure at this step would immediately exclude this model independently of the choice of subcase.

Both of these possible scenarios have some rather nontrivial mass generation requirements, but, as is immediately apparent, subcase (α) is somewhat more straightforward. For either of these possibilities, at the $55'1_{10}$ level, one finds that there are 22 allowed Yukawa couplings: four for the purely PM sector involving fields which transform nontrivially under both $SU(5)$'s, five which solely concern dark sector fields with only nontrivial $SU(5)'$ transformation properties (both of which we can imagine obtaining masses at multi-TeV scales), and 13 which result in, e.g., the breaking of the SM gauge symmetry and the generation of SM fermion masses. Interestingly, also among the latter are many "extra" terms which are responsible for admixtures between the fields in the various sectors in addition to those producing the SM fermion masses themselves. This makes the overall fermion mass generation problem difficult to analyze in this particular setup partially due to, e.g., the previously discussed "leftover" $\bar{\mathbf{5}}$'s after the SM fields have obtain their masses. However, since the $SU(5)'$ and electroweak breaking scales are very different, we may, at least approximately, treat them independently and roughly decoupled. Note that, as before, since either fields that will be identified as SM fermions or as $SU(2)_L$-breaking Higgs are to be found in $SU(5)'$ singlets in either of these subcases, then Q_D cannot have a contribution from Q_{10} as is the norm.

For either of these subcases, however, in the PM sector we must generate the vectorlike fermion masses for the remaining two non-SM elements of the **10** representation (which are not given any electroweak masses and to which we will return below) by coupling them to the two different $\overline{\mathbf{10}}$'s as noted by employing the $55'1_{10}$-level mass terms, i.e.,

$$\mathrm{y}_{-2}(\overline{\mathbf{10}},\mathbf{1})_{\mathbf{-2}}(\mathbf{10},\mathbf{5})_\mathbf{1}[(\mathbf{1},\bar{\mathbf{5}})_\mathbf{1}]_\mathbf{H} + \mathrm{y}_3(\overline{\mathbf{10}},\mathbf{1})_\mathbf{3}(\mathbf{10},\mathbf{5})_\mathbf{1}[(\mathbf{1},\bar{\mathbf{5}})_{\mathbf{-4}}]_\mathbf{H} + \mathrm{H.c.}, \qquad (30)$$

with the coefficients $\mathrm{y}_{-2,3}$ now being the two relevant Yukawa couplings corresponding to these distinct $\overline{\mathbf{10}}$'s. The VEVs of the two scalar fields, $[(\mathbf{1},\bar{\mathbf{5}})_{\mathbf{1},\mathbf{-4}}]_\mathbf{H}$, which are responsible for these mass terms, essentially consist of two distinct sets of [the $SU(5)'$ subgroup] $SU(3)'$, $Q_D = 0$, antitriplets, each with the three nonzero VEVs, i.e., $\sim v_i$, v'_i, $i = 1$–3, with i being an $SU(3)'$ index. Thus, working in the $SU(5)'$ subspace and denoting the fields there as *uppercase and primed* versions of their familiar SM $SU(5)$ analogs and with the two corresponding $SU(5)'$ singlets denoted by $S'^c_{-2,3}$, respectively, only one fermion bilinear arises from each of these two coupling terms, i.e., $\sim S'^c_{-2,(3)} D'_i v_i (v'_i)$.[9] So, now going back to the full $55'1_{10}$ space, we observe that each term in the equation above leads to a different set of ten degenerate PM masses, "eating up" two of the five **10** fields in the $(\mathbf{10},\mathbf{5})_\mathbf{1}$ and leaving us the three sets of fields that are necessary to form the corresponding masses for the three generations of the u-type SM fermions. This Yukawa structure thus accounts for $(2 \times 10 =)20$, out of the total 70 needed PM mass terms; so far, so good.

To go further, we need to choose between the two subcases, (α, β), and this can be most easily done by

[9]Repeated indices are summed over as usual.

examining the $55'1_{10}$-level mass term directly coupling five sets of intended SM 5's to the corresponding $\bar{\mathbf{5}}$'s, i.e.,

$$\tau(\bar{\mathbf{5}},\bar{\mathbf{5}})_{\mathbf{0}}(\mathbf{10},\mathbf{5})_{\mathbf{1}}[(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{1}}]_{\mathbf{H}} + \mathrm{H.c.}, \tag{31}$$

where τ is just another Yukawa coupling parameter. Analyzing this in the $SU(3)'$ subspace as we did above and employing the same suggestive notation, we see that five field bilinears can be formed:

$$\sim\epsilon_{ijk}(D_i^c)'(U_j^c)'v_k, \qquad \sim N'U_i'v_i, \qquad \sim E'D_i'v_i, \tag{32}$$

where i, j, $k = 1$–3 are $SU(3)'$ indices, so that going back to the full $55'1_{10}$ space we see that $(5\times 5 =)25$ of the additional PM mass terms (out of the total of 70 required) can be generated in this way. Note that since, apart from the VEVs, the only freedom here is the overall Yukawa τ, this construction directly leads to 25 degenerate PM mass terms. Now if we were in subcase (β), we would require *only* ten mass terms out of this set, as the other fields in the $\bar{5}$ would need to be identified with the SM ones and this does not naturally happen here unless we reduce the number of nonzero VEVs in the Higgs field—but this is something we cannot do, as they are already needed to generate masses elsewhere. Given this, it appears that, in this setup, the subcase (β) is excluded and this leaves us only with subcase (α), which we will now assume is realized in what follows.

At this point, one might be somewhat concerned about the SM particle masses themselves; we note that, at the SM $SU(2)_L$-breaking level, this scenario will not only (hopefully) generate all the required SM fermion mass terms but will *also* lead to a rather complex mixing between the SM, PM, and dark sector fields as well. If we were to consider the three generations of the SM fermion masses in *isolation* from these considerations (which is sufficient for the present discussion), in subcase (α), the relevant Yukawa coupling terms are given by

$$y_{\sigma a}[(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{1}}]^{\sigma}(\mathbf{10},\mathbf{5})_{\mathbf{1}}[(\bar{\mathbf{5}},\bar{\mathbf{5}})_{\mathbf{0}}]^{\mathbf{a}}_{\mathbf{H}} + \rho(\mathbf{10},\mathbf{5})_{\mathbf{1}}(\mathbf{10},\mathbf{5})_{\mathbf{1}}(\bar{\mathbf{5}},\overline{\mathbf{10}})_{-\mathbf{2}} + \mathrm{H.c.}, \tag{33}$$

with $y_{\sigma a}, \rho$ being Yukawa parameters and, as above, we have considered the possibility of additional scalar fields, labeled by the index a as above, for the different SM $\bar{\mathbf{5}}$ fermions as labeled by σ. For convenience, we can label the corresponding Higgs VEVs of the $(\bar{\mathbf{5}},\bar{\mathbf{5}})_{\mathbf{0}}]^{\mathbf{a}}_{\mathbf{H}}$'s as w_{ai}, $i = 1$–3, similar to the above [although remember that here these are now $SU(2)_L$-violating VEVs]. Let us consider the second term first which links two identical SM **10**'s to the corresponding $\bar{\mathbf{5}}_{\mathbf{H}}$. In the $SU(5)'$ subspace, we can denote the four allowed $SU(2)_L$-breaking, $Q_D = 0$ VEVs of the $(\bar{\mathbf{5}},\overline{\mathbf{10}})_{-\mathbf{2}}$ as $u_{i=1-3}, u_4$. Because of the identical nature of the fermions, this leads to only three nontrivial bilinears: $\sim\epsilon_{ijk}(D_i^c)'(D_j^c)'u_k$ with three different VEVs. Projecting this back into the full $55'1_{10}$ space and scaling by ρ, we see that this will correspond to the three nondegenerate SM up-type quark masses that are required —another success for this model. Turning to the first term in the above expression, we recall that we wish to identify three linear combinations of the $(\bar{5},1)_{-1}$'s with those of the SM, and, further recalling that the $[(\bar{\mathbf{5}},\bar{\mathbf{5}})_{\mathbf{0}}]_{\mathbf{H}}$ Higgs scalar arises from the $\overline{\mathbf{45}}_{\mathbf{H}}$, we again find it useful to consider the coupling structure solely in the $SU(5)'$ subspace, here denoting the set of $SU(5)'$ singlets in the $[(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{1}}]^{\mathbf{a}}$'s simply as S_a^c. We then find that the only bilinears we can construct are of the form $\sim S_a^c(D_i)'w_{ai}$, i.e., *one* for each value of a; allowing for $a = 1$–3 then produces the distinct masses for the three generations of d, e-type SM fermions. Now this implementation requires that $n_{45} = 3$; assuming that $n_{10} = n_{120} = n_{210} = 1$ still holds, then this yields $\beta_d = -5/6$ so our asymptotic freedom constraint still remains satisfied. The generation of three distinct masses for the three generations of SM fermions is another success of this model.

Outside of the purely dark sector fields, there now remains only the issue of the five unpaired SM $SU(5)$ $\bar{\mathbf{5}}$ fermions that we have already encountered for which we must generate vectorlike masses so that they can also be identified as PM; these are the last of the $(70-20-25=)25$ mass terms that we noted above. In general, these couplings will take the form

$$z_{\mathrm{ab}}(\bar{\mathbf{5}},\mathbf{1})^{\mathbf{a}}_{-\mathbf{1}}(\mathbf{5},\mathbf{10})_{-\mathbf{1}}[(\mathbf{1},\overline{\mathbf{10}})_{\mathbf{2}}]_{\mathbf{H}}{}^{\mathbf{b}} + \mathrm{H.c.}, \tag{34}$$

with z_{ab} being Yukawa couplings and in subcase (α), which we are considering, the index $a = 1$–5, just labels these remaining unaccounted for $\bar{\mathbf{5}}$'s while b runs over the set of possible Higgs scalars. Recall that the $(\mathbf{1},\overline{\mathbf{10}})_{\mathbf{2}}]_{\mathbf{H}}$ originates from a $\overline{\mathbf{45}}_{\mathbf{H}}$ representation that we have already introduced three of in the previous discussion to generate the three generations of d, e-like SM fermion masses. As above, we turn to the $SU(5)'$ subspace to do the accounting of the number of independent mass terms generated in the present case; we will denote for simplicity the four, $Q_D = 0$ VEVs of the $[(\mathbf{1},\overline{\mathbf{10}})_{\mathbf{2}}]_{\mathbf{H}}$ by $u_{i=1-3,4}$, but these are not to be confused with those employed above arising from a different a $SU(5)'$ **10** representation. For a *single* $[(\mathbf{1},\overline{\mathbf{10}})_{\mathbf{2}}]_{\mathbf{H}}$ and a single $(\bar{\mathbf{5}},\mathbf{1})_{-\mathbf{1}}$ in this subspace, one correspondingly finds only a single bilinear, i.e., the combination $S^c[(U_i^c)'u_i + (E^c)'u_4]$, which then would generalize to $z_{ab}S_a^c[(U_i^c)'u_{bi} + (E^c)'u_{b4}]$. Momentarily forgetting the AF and β-function constraint (which tells us $b\le 3$), this leads to at most four linearly independent combinations of VEVs and, thus, only $4\times 5 = 20$ additional vectorlike fermion PM mass terms beyond those constructed above. Once the β-function constraint is imposed, this is reduced to only $3\times 5 = 15$ mass terms based on the three allowed and previously introduced $\overline{\mathbf{45}}_{\mathbf{H}}$'s, implying that, in SM $SU(5)$ language, two pairs of $\mathbf{5}+\bar{\mathbf{5}}$ PM fields will remain massless at the scale of $SU(5)'$ breaking. It is important to recall that the absence of the necessary mass terms can be partly traced to the requirement that we cannot give VEVs to the $SU(2)'$ doublets within the various $SU(5)'$ Higgs representations, as they necessarily must carry $Q_D \neq 0$ and would result in the breaking of $U(1)_D$. Such Higgs fields are mandatory as, although the PM fermion fields are vectorlike with respect to the SM, they are chiral under $SU(2)'$ gauge group as was discussed earlier. Thus, at least at tree level, these fields will remain massless even after $SU(2)' = SU(2)_D$ breaks down to $U(1)_D$ so, no matter what happens in the purely dark sector, this model is now also excluded, thus highly disfavoring $SU(10)$ scenarios. It is important to note that this problem would still persist, although with a somewhat reduced severity, even if the AF requirement were to be relaxed, although we were brought rather quite close to success in this scenario.

F. Possible future directions: Going beyond $SU(10)$ and other options

When N is small, e.g., $N \leq 7$, G_{Dark} is clearly not large enough to embed a $U(1)_D$ in a successful manner. However, the previously studied models have shown us that, as $8 \leq N \leq 10$ increases, our other constraints can quickly become more difficult to satisfy due to the tensions between the ever rapidly growing dimensionality of the nonfundamental representations for both the fermion and Higgs scalar fields, the requirement of asymptotic freedom above the $SU(N)$ unification scale, and the need to generate all of the SM and non-SM fermion masses with a restricted number of VEVs so that $U(1)_D$ remains unbroken. This latter problem, however, was seen to be somewhat alleviated by the presence of new types of Higgs fields in the second-rank antisymmetric representation of G_{Dark} once $N \geq 9$, although not sufficiently so as to provide for a successful result in the examined cases. A significant obstacle in the construction of a viable model was observed to be that a $SU(2)'$ survives after the initial dark gauge group breaking and that $U(1)_D$ must lie (at least in the examples discussed above) wholly within this $SU(2)'$. When $N > 10$, these conflicting requirements will make finding a potentially successful model even more difficult. For example, consider two scenarios in $SU(11)$: first, an $n_g = 3$ model [59] with a relatively low number of fermion degrees of freedom (i.e., having "only" 341) given by $\mathbf{6}(\overline{\mathbf{11}}) + \mathbf{2}(\overline{\mathbf{55}}) + \mathbf{165}$. This setup requires at least one Higgs scalar in each of the $\overline{\mathbf{11}}$, $\overline{\mathbf{55}}$, $\mathbf{165}$, $\mathbf{330}$, and $\mathbf{462}$ representations to at least make at attempt at generating the necessary masses and so easily fails the test of asymptotic freedom. Similarly, a second, simpler $n_g = 1$ scenario [48] in $SU(11)$, with only 351 degrees of freedom, $\mathbf{3}[\mathbf{7}(\overline{\mathbf{11}}) + \mathbf{55}]$ with a correspondingly simpler Higgs scalar sector, while possibly asymptotically free, now ends up following the well-known pattern of the case (a)-type models for $N = 8$, 9, 10 discussed above and can be easily shown to not allow for all the required mass terms due to the remaining $SU(2)'$. Even this class of (a)-type scenarios, assuming the simplest of Higgs sectors, will also fail the asymptotic freedom constraint once $N > 12$. Apparently, these multiple requirements, taken together, do not allow us to achieve our desired goal if the other model-building assumptions above are not altered, and apparently going to even larger values of N will fail to buy much that is new.

While the asymptotic freedom requirement is clearly of some significant impact as we have seen, it is the fact that the $U(1)_D$ must lie totally within $SU(2)'$ that inhibits a larger number of scalar VEVs from appearing, preventing a successful outcome.[10] This requirement, fundamentally, arose from the assumed gauge symmetry-breaking structure of $SU(N) \to SU(5) \times SU(N-5)' \times U(1)_N$ and that Q_D is found to be independent of Q_N, i.e., without any contributions arising from an additional $U(1)$ factor and, in particular, that we can always choose a basis where $Q_D \sim T_3'$. In models where the relevant Higgs scalars are only in the (anti)fundamental representation of G_{Dark}, this obstacle would require circumventing, or at least substantial softening, if we are to find an amiable solution. One possibility that we might imagine is to slightly modify the dark gauge group breaking pattern to allow for an extra $U(1)$ factor, thus providing additional freedom to embed Q_D. For example, one might consider beginning with the group $SU(N+1)$, selecting a model with a fixed representation content which is both anomaly and asymptotically free. Then we consider the breaking path $SU(N+1) \to SU(N) \times U(1)_{N+1} \to SU(5) \times SU(N-5)' \times U(1)_N \times U(1)_{N+1}$ which apparently seems to provide us with an additional $U(1)$ factor. To see if such an approach can be remotely viable, let us consider a pair of examples, both based on $SU(9)$ models previously examined above: (i) case (a) and (ii) case (e).

In example (i), we consider a single generation consisting of the anomaly free set of $5(\bar{\mathbf{9}}) + \mathbf{36}$ representations which now under the breaking $SU(8) \times U(1)_9$ is just $[4(\bar{\mathbf{8}}_{-1}) + \mathbf{28}_2] + [\bar{\mathbf{8}}_{-1} + \mathbf{8}_{-7}] + \mathbf{1}_8$ where the set of representations in the first bracket we recognize as a single generation of the $SU(8)$ case (a) [albeit now with a set of additional $U(1)$ quantum number attached], while the second bracket, with respect to $SU(8)$, is a further pair of PM fields that are *not* present in the version of this now $SU(8)$ model previously considered above. Completing the picture, an additional $SU(8)$ singlet field is now also seen to be present. In comparison to the $SU(8)$ model analysis above, we have gained the additional wanted freedom associated with the new $U(1)$ factor but, simultaneously,

[10] As noted, at large values of N this is alleviated to some extent by there also being distinct Higgs fields in the second-rank, antisymmetric representation of $SU(N-5)'$ thus increasing the number of allowed $Q_D = 0$ VEVs.

we have needed to add some new fermion fields whose masses we must also unfortunately now generate. This supplies new constraints that need to be satisfied in addition to those already encountered for the SM and PM fields examined previously. Recall that, in general, we now find that $Q_D = a_1\lambda_3 + a_2\lambda_8 + bQ_8 + cQ_9$, where $Q_{8,9}$ are just the $U(1)_{8,9}$ quantum number assignments of the various representations. Some algebra tells us that while the condition $c = -3b$ will allow for the generation of both the $\bar{\mathbf{5}} \cdot \mathbf{10}$ and $\mathbf{10} \cdot \mathbf{10}$ SM particle masses, the corresponding requirements for all of the PM fields (including these newly introduced fields) cannot be met simultaneously even though the mass terms for the SM $\bar{\mathbf{5}}$ and $\mathbf{10}$ fields yield identical constraints on both of the $U(1)$'s individual contributions to $U(1)_D$. In this example, we have gained no ground by having this additional $U(1)$ factor due to the presence of the additional PM.

Perhaps, if we could remove the additional constraints arising from the new PM states, then greater success would become possible. To that end, we next, in example (ii), revisit $SU(9)$ case (e) wherein the three generations of SM fermions are assigned as $9(\bar{\mathbf{9}}) + \mathbf{84}$, i.e., an example of an $n_g = 3$ scenario. Again going to $SU(8) \times U(1)_9$ level, this corresponds to the representations $9(\bar{\mathbf{8}}_{-\mathbf{1}}) + \mathbf{28}_{-\mathbf{6}} + \mathbf{56}_{\mathbf{3}} + 9(\mathbf{1}_{\mathbf{8}})$, which appears quite similar to the previously examined $SU(8)$ case (e) apart from the extra singlets and the overall additional $U(1)$ factor as we wanted. In this case, we see that there are *no* additional PM-like states resulting from this procedure which is just what we desired to achieve, apparently alleviating the additional constraint found in the previous example. However, in this scenario, unlike in (i), the generation of both the $\bar{\mathbf{5}} \cdot \mathbf{10}$ and $\mathbf{10} \cdot \mathbf{10}$ SM particle masses leads to *different* requirements on the values of the b and c parameters so that the *total* number of constraints in this setup is found to be the same as in (i). In fact, one finds that the generation of the SM masses requires $b = c = 0$ so that we are led back to the (failing) analysis in the previous subsection. Again, we find that we have gained nothing by adding this additional new $U(1)$ factor. Indeed, one finds by further analysis that there is nothing special about these two examples and that these results are, unfortunately, rather general. Although several possible scenarios seem to be opened by this initial idea, the additional constraints are found to more than outweigh any gains associated with the additional $U(1)$ degrees of freedom so that this approach is seen to fail.

Another possible path that one can imagine is to give up on the identification $G_{\mathrm{SM}} = SU(5)$ and assume a somewhat larger SM gauge group such as, e.g., $SU(6)$. In doing so, still beginning with $G = SU(N)$, one trivially finds that the rank of G_{Dark} is now reduced to $G_{\mathrm{Dark}} = SU(N-6)' \times U(1)'_N$, which one might think could be advantageous. However, in doing so for fixed G this reduces the number of dark sector diagonal generators on which Q_D can depend (i.e., the rank of G_{Dark} is now smaller by unity); hence, the number of diagonal generators out of which Q_D can be constructed is also reduced. This then implies that the number of possible VEVs of the various Higgs fields having $Q_D = 0$ to maintain an unbroken $U(1)_D$ is also reduced, making it more difficult to simultaneously generate masses for the set of SM $Q_D = 0$ fermions as well as those for the remaining additional PM fermion fields. Thus, if anything, for fixed G, one would like to *increase* the rank of G_{Dark} or at least increase the number of its diagonal generators contributing to Q_D to allow for more possible Higgs VEVs to generate the needed fermion mass terms.

In the current study, we have considered models which are in either of the $n_g = 1$ or $n_g = 3$ classes; there is also the possibility, not entertained above, of $n_g = 2 + 1$, wherein, by construction, one of the generations is embedded into the $SU(N)$ group fermion representation structure asymmetrically from the other two. Such an occurrence, at least partially, already happens in some of the $n_g = 3$ models, e.g., in $SU(7)$ model (c), where two of the SM $\mathbf{10}$'s lie in a $\mathbf{21}$ and the other in a $\mathbf{35}$ or vice versa. In principle, such an approach could lead to some additional model-building flexibility, but, based upon what we have seen above, this would most likely occur in setups wherein none of the SM fermions of the two "common" generations lie in the (anti)fundamental representation of $SU(N)$. Clearly, we would be most interested in scenarios which are not just simple deconstructions of previously examined $n_g = 3$ models as we gain nothing by doing this; e.g., the $SU(7) n_g = 3$ model (c) is seen to be composed of two sets of representations as are occurring in model (a) plus a single set from model (b). Similarly, we see that the corresponding $n_g = 3$ model (d) is observed to combine one set of representations from model (a) and two from model (b) [76]. The fact that both of these two models failed to satisfy all of our constraints gives some indication that the general $n_g = 21$ set of models will also not meet with much success. However, a detailed study of this possibility in more realistic scenarios lies beyond the scope of the present work.

In the analysis above, we have concentrated on the generation of the Dirac masses for the charged SM and PM fields; of course, if we were to find a successful scenario, we would want to explore how the light neutrino masses might be generated and the nature of the associated phenomenology. Clearly, once we go beyond the standard $SU(5)$, the opportunities are many to generate interesting neutrino mass and mixing structures due to the existence of new neutral heavy vectorlike leptons (some of which also may carry dark charges and can be identified as PM) which transform as SM singlets or as parts of one or more sets of $\mathbf{5} + \bar{\mathbf{5}}$'s under $SU(5)$ which are lie in SM isodoublets. For example, even in the simplest case of $SU(6)$, one sees that a single family will contain an additional such set of $\mathbf{5} + \bar{\mathbf{5}}$'s as well as two additional SM singlet fermions which open many possibilities for interesting scenarios [77]. Of course, the present model-building structure will add extra complexities to the usual approaches, as the $U(1)_D$ subgroup needs to remain unbroken down to $\lesssim 1$ GeV, and this may have a significant influence on the resulting neutrino mass spectra. However, to pursue this matter further we would first need a model that successfully generated all of the SM and PM masses, and it might be more useful to first explore some of these possibilities from a bottom-up perspective.

There are, of course, many other potential directions and modifications to the approach followed above that one might try to explore to alleviate the problems we encountered some of which we will reserve for future work.

IV. DISCUSSION AND CONCLUSIONS

The Abelian KM portal model wherein a new gauge boson, the dark photon, associated with the gauge group $U(1)_D$, kinetically mixes with the hypercharge gauge boson of the SM offers a very attractive and well-studied scenario for thermal DM at the $\lesssim 1$ GeV mass scale. This KM is generated by loops of a set of portal matter fields which carry both dark and SM quantum numbers and should have masses above the electroweak scale. Specifically, fermionic PM must be vectorlike with respect to the SM if it is to satisfy constraints arising from electroweak precision measurements, unitarity, and direct collider searches as well as the values of the Higgs boson loop-induced $gg, \gamma\gamma$ partial widths. It is natural to ask how the physics of the SM, dark, and PM sectors might be combined into a single unified framework. Bottom-up approaches [22,25] in our class of models seem to indicate that an important first step in this direction is to embed the $U(1)_D$ group into a non-Abelian structure at a mass scale similar to or perhaps not too far above that associated with the PM fields themselves. An obvious issue is that we unfortunately lack a sufficiently large enough set of examples of this idea that would help guide us in a top-down approach. With this limitation in mind, in this paper, we have attempted to construct, employing such a top-down approach, a unified scenario that combines the SM, dark, and PM sectors into an overarching framework based on the assumption that all of the relevant gauge forces can be unified within a single $G = SU(N)$ gauge group. It was further assumed that G then decomposes into the product $G_{\rm SM} \times G_{\rm Dark}$ with, for simplicity, the familiar $SU(5)$ playing the role of a stand-in for $G_{\rm SM}$ with $U(1)_D$ assumed to be a diagonal subgroup of $G_{\rm Dark}$.

Given this overall structural assumption, our model building was subsequently subjected to a rather large set of requirements, each of which—on its own—seems to be quite reasonable but which used collectively led to interesting and powerful constraints on the overall nature of the resulting model. For example, when these model-building constraints were combined, it was found to be impossible to generate all of the necessary electroweak scale or above mass terms for both the fermionic SM and PM fields while also simultaneously allowing a $U(1)_D$, under which the SM fields were singlets, to remain unbroken far below the electroweak scale as required. However, we were able to get reasonably close to a satisfactory solution in the case of one of the $G = SU(10)$-based setups, especially so if the rather powerful $SU(N)$ asymptotic freedom requirement were to be dropped. The cases with larger values of N, though generally more constrained by the AF requirements, allowed for $G_{\rm Dark}$ breaking and PM mass generation not only via the Higgs fields in the (anti)fundamental representations of this group but also by second-rank antisymmetric tensor representations as well which provided an additional source of the needed scalar VEVs. Quite generally, this difficulty mostly resulted from the constraint, which followed directly from the model-building requirements, that the $U(1)_D$ must solely originate from the unbroken diagonal subgroup of an $SU(2)'$ at the next to final stage of the symmetry breaking for $G_{\rm Dark}$ and the assumption that $G_{\rm Dark}$ itself had the largest rank possible given that $G_{\rm SM} = SU(5)$ was assumed. This problem is then further accentuated by the inability of Q_D to obtain a dependence of the $U(1)_N$ charge, Q_N. It would appear that more work on the bottom-up approach would also be useful to help clarify the physics at the intermediate breaking scale as well as the varieties of possible group representations of G that can be shared by both SM and PM fermion fields.

Several possible hopeful directions to overcome these model-building difficulties were discussed and identified and will become the subject of future work.

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[1] N. Aghanim *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A6 (2020).
[2] M. Kawasaki and K. Nakayama, Annu. Rev. Nucl. Part. Sci. **63**, 69 (2013).
[3] P. W. Graham, I. G. Irastorza, S. K. Lamoreaux, A. Lindner, and K. A. van Bibber, Annu. Rev. Nucl. Part. Sci. **65**, 485 (2015).
[4] I. G. Irastorza and J. Redondo, Prog. Part. Nucl. Phys. **102**, 89–159 (2018).
[5] G. Arcadi, M. Dutra, P. Ghosh, M. Lindner, Y. Mambrini, M. Pierre, S. Profumo, and F. S. Queiroz, Eur. Phys. J. C **78**, 203 (2018).
[6] L. Roszkowski, E. M. Sessolo, and S. Trojanowski, Rep. Prog. Phys. **81**, 066201 (2018).
[7] K. Pachal, Dark Matter Searches at ATLAS and CMS, in *Proceedings of the 8th Edition of the Large Hadron Collider Physics Conference* (2020).
[8] E. Aprile *et al.* (XENON Collaboration), Phys. Rev. Lett. **121**, 111302 (2018).
[9] A. Albert *et al.* (Fermi-LAT and DES Collaborations), Astrophys. J. **834**, 110 (2017).
[10] C. Amole *et al.* (PICO Collaboration), Phys. Rev. D **100**, 022001 (2019).
[11] J. Alexander *et al.*, arXiv:1608.08632.
[12] M. Battaglieri *et al.*, arXiv:1707.04591.
[13] G. Bertone and Tim M. P. Tait, Nature (London) **562**, 51 (2018).
[14] P. Schuster, N. Toro, and K. Zhou, Phys. Rev. D **105**, 035036 (2022).
[15] B. Holdom, Phys. Lett. **166B**, 196 (1986); **178**, 65 (1986); K. R. Dienes, C. F. Kolda, and J. March-Russell, Nucl. Phys. **B492**, 104 (1997); F. Del Aguila, Acta Phys. Pol. B **25**, 1317 (1994); K. S. Babu, C. F. Kolda, and J. March-Russell, Phys. Rev. D **54**, 4635 (1996); T. G. Rizzo, Phys. Rev. D **59**, 015020 (1998).
[16] There has been a huge amount of work on this subject; see, for example, D. Feldman, B. Kors, and P. Nath, Phys. Rev. D **75**, 023503 (2007); D. Feldman, Z. Liu, and P. Nath, Phys. Rev. D **75**, 115001 (2007); M. Pospelov, A. Ritz, and M. B. Voloshin, Phys. Lett. B **662**, 53 (2008); M. Pospelov, Phys. Rev. D **80**, 095002 (2009); H. Davoudiasl, H. S. Lee, and W. J. Marciano, Phys. Rev. Lett. **109**, 031802 (2012); Phys. Rev. D **85**, 115019 (2012); R. Essig *et al.*, arXiv:1311.0029; E. Izaguirre, G. Krnjaic, P. Schuster, and N. Toro, Phys. Rev. Lett. **115**, 251301 (2015); M. Khlopov, Int. J. Mod. Phys. A **28**, 1330042 (2013); For a general overview and introduction to this framework, see D. Curtin, R. Essig, S. Gori, and J. Shelton, J. High Energy Phys. 02 (2015) 157.
[17] G. Steigman, Phys. Rev. D **91**, 083538 (2015).
[18] K. Saikawa and S. Shirai, J. Cosmol. Astropart. Phys. 08 (2020) 011.
[19] M. Fabbrichesi, E. Gabrielli, and G. Lanfranchi, *The Physics of the Dark Photon* (Springer Nature, Switzerland, 2021).
[20] M. Graham, C. Hearty, and M. Williams, Annu. Rev. Nucl. Part. Sci. **71**, 37 (2021).
[21] T. G. Rizzo, Phys. Rev. D **99**, 115024 (2019).
[22] T. D. Rueter and T. G. Rizzo, Phys. Rev. D **101**, 015014 (2020).
[23] J. H. Kim, S. D. Lane, H. S. Lee, I. M. Lewis, and M. Sullivan, Phys. Rev. D **101**, 035041 (2020).
[24] T. D. Rueter and T. G. Rizzo, arXiv:2011.03529.
[25] G. N. Wojcik and T. G. Rizzo, Phys. Rev. D **105**, 015032 (2022).
[26] T. G. Rizzo, J. High Energy Phys. 11 (2021) 035.
[27] T. G. Rizzo, arXiv:2202.02222.
[28] G. N. Wojcik, arXiv:2205.11545.
[29] T. G. Rizzo, Phys. Rev. D **106**, 035024 (2022).
[30] T. Gherghetta, J. Kersten, K. Olive, and M. Pospelov, Phys. Rev. D **100**, 095001 (2019).
[31] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438 (1974).
[32] H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. **33**, 451 (1974).
[33] T. R. Slatyer, Phys. Rev. D **93**, 023527 (2016).
[34] H. Liu, T. R. Slatyer, and J. Zavala, Phys. Rev. D **94**, 063507 (2016).
[35] R. K. Leane, T. R. Slatyer, J. F. Beacom, and K. C. Ng, Phys. Rev. D **98**, 023016 (2018).
[36] See, for example, M. Gockeler, R. Horsley, V. Linke, P. E. L. Rakow, G. Schierholz, and H. Stuben, Phys. Rev. Lett. **80**, 4119 (1998).
[37] C. Murgui and K. M. Zurek, Phys. Rev. D **105**, 095002 (2022).
[38] See, for example, F. Gursey, P. Ramond, and P. Sikivie, Phys. Lett. **60B**, 177 (1976); Y. Achiman and B. Stech, Phys. Lett. **77B**, 389 (1978); Q. Shafi, Phys. Lett. **79B**, 301 (1978).
[39] J. C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974); **11**, 703(E) (1975).
[40] S. L. Glashow, *Proceedings of the 5th Workshop on Grand Unification, Providence RI, USA* (1984) [Report No. Print-84-0577 (BOSTON)].
[41] S. Dey and M. K. Parida, Phys. Rev. D **52**, 518 (1995).
[42] S. J. Lonsdale and R. R. Volkas, Phys. Rev. D **90**, 083501 (2014); **91**, 129906(E) (2015).
[43] See, for example, A. Hebecker and M. Ratz, Nucl. Phys. **B670**, 3 (2003); C. Csaki, C. Grojean, H. Murayama, L. Pilo, and J. Terning, Phys. Rev. D **69**, 055006 (2004); C. Csaki, C. Grojean, L. Pilo, and J. Terning, Phys. Rev. Lett. **92**, 101802 (2004).
[44] H. Georgi, Nucl. Phys. **B156**, 126 (1979).
[45] P. H. Frampton, Phys. Lett. **88B**, 299 (1979).
[46] P. Frampton and S. Nandi, Phys. Rev. Lett. **43**, 1460 (1979).
[47] P. H. Frampton, Phys. Lett. **89B**, 352 (1980).
[48] M. T. Vaughn, J. Phys. G **5**, 1317 (1979).
[49] S. Kashibayashi, K. Fukuma, and T. Ueda, Phys. Rev. D **26**, 2529 (1982).
[50] P. Langacker, Phys. Rep. **72**, 185 (1981).
[51] R. P. Feger and T. W. Kephart, Phys. Rev. D **92**, 035005 (2015).
[52] See, for example, the case made instead for employing asymptotic safety: D. F. Litim and F. Sannino, J. High Energy Phys. 12 (2014) 178; D. F. Litim, M. Mojaza, and F. Sannino, J. High Energy Phys. 01 (2016) 081.
[53] N. D. Christensen and R. Shrock, Phys. Rev. D **72**, 035013 (2005).
[54] C. T. Hill, Phys. Lett. **135B**, 47 (1984).

[55] See, for example, D. Feldman, Z. Liu, and P. Nath, Phys. Rev. D **75**, 115001 (2007).
[56] J. L. Hewett and T. G. Rizzo, Phys. Rep. **183**, 193 (1989).
[57] K. Kang and I. G. Koh, Phys. Rev. D **25**, 1700 (1982).
[58] J. L. Chkareuli, C. D. Froggatt, I. G. Gogoladze, and A. B. Kobakhidze, Nucl. Phys. **B594**, 23 (2001).
[59] N. Chen, Chin. Phys. C **46**, 053107 (2022).
[60] L. L. Everett and O. Popov, arXiv:2108.09316.
[61] E. Eichten, K. Kang, and I. G. Koh, J. Math. Phys. (N.Y.) **23**, 2529 (1982).
[62] R. M. Fonseca, Nucl. Phys. **B897**, 757 (2015).
[63] R. Slansky, Phys. Rep. **79**, 1 (1981).
[64] N. Yamatsu, arXiv:1511.08771.
[65] L. F. Li, Phys. Rev. D **9**, 1723 (1974).
[66] J. L. Chkareuli, Phys. Lett. B **300**, 361 (1993).
[67] S. M. Barr, Phys. Rev. D **78**, 075001 (2008).
[68] C. W. Kim and C. Roiesnel, Phys. Lett. **93B**, 343 (1980).
[69] J. E. Kim and H. S. Song, Phys. Rev. D **23**, 2102 (1981).
[70] N. G. Deshpande and P. D. Mannheim, Phys. Rev. D **24**, 2923 (1981).
[71] See, for example, J. F. Gunion, H. E. Haber, G. L. Kane, and S. Dawson, Front. Phys. **80**, 1 (2000).
[72] Y. Fujimoto and P. Sodano, Phys. Rev. D **23**, 1657 (1981).
[73] K. Kang, C. K. Kim, and J. K. Kim, Phys. Rev. D **33**, 260 (1986).
[74] P. H. Frampton and T. W. Kephart, Phys. Lett. B **681**, 343 (2009).
[75] J. B. Dent, R. Feger, T. W. Kephart, and S. Nandi, Phys. Lett. B **697**, 367 (2011).
[76] N. Chen, Y. n. Mao, Z. Teng, B. Wang, and X. Zhao, arXiv:2209.11446.
[77] As an example, see the analysis as presented in Z. Chacko, P. S. B. Dev, R. N. Mohapatra, and A. Thapa, Phys. Rev. D **102**, 035020 (2020).