Tree-level unitarity, causality, and higher-order Lorentz and *CPT* violation

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Higher-order effects of *CPT* and Lorentz violation within the Standard-Model Extension effective framework including Myers-Pospelov dimension-five operator terms are studied. The model is canonically quantized by giving special attention to the arising of indefinite-metric states or ghosts in an indefinite Fock space. As is well-known, without a perturbative treatment that avoids the propagation of ghost modes or any other approximation, one has to face the question of whether unitarity and microcausality are preserved. In this work, we study both possible issues. We found that microcausality is preserved due to the cancellation of residues occurring in pairs or conjugate pairs when they become complex. Also, by using the Lee-Wick prescription, we prove that the *S* matrix can be defined as perturbatively unitary for tree-level $2 \to 2$ processes with an internal fermion line.

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I. INTRODUCTION

Quantum field theory (QFT) is conceptually based on locality and Lorentz invariance. Any departure from these two basic concepts will introduce serious alterations to the traditional construction of field theory and will necessarily imply new physics. Alternative theories containing Lorentz invariance violation have been widely studied to test the limits of conventional QFT. The triad of theoretical, phenomenological, and experimental work has made significant progress in the past two decades. In particular, the search for potential Lorentz violations has received special attention producing stringent limits on Lorentz violations with ultrahigh sensitive experiments [1,2].

The fundamental interplay between matter and geometry continues to be a source of conceptual issues. At the Planck mass $m_{\rm Pl} \approx 10^{19}$ GeV, various candidate theories of quantum gravity suggest the disruption of the continuum property of spacetime. If Minkowski spacetime is not the exact geometry at these energies, then it is justified to consider the standard model of particles to be an effective theory. One should expect experiments taking place at scales Λ to describe gravitational effects suppressed by $\Lambda/m_{\rm Pl}$. Nevertheless, residual gravitational effects could be detected at currently attainable energies. A possible manifestation of such disruption has been realized in the form of *CPT* and Lorentz violations [3–5]. In this way, the search for possible effects of Lorentz violation using effective field theory has been amply adopted. Effective field theory has become a natural language in high-energy phenomenology to describe possible Lorentz violations. This work focuses on the possible effects of *CPT* and Lorentz violation described within an effective framework.

The effective framework of the Standard-Model Extension (SME) describes effects of *CPT* and Lorentz violation in field theory by introducing gauge-invariant objects constructed from Standard-Model fields coupled to vectors and tensors that parametrize the Lorentz violation. It also covers the gravity sector where local Lorentz and diffeomorphism violation give rise to modified-gravity theories. The SME can be divided into a minimal sector and a nonminimal sector. The minimal sector includes renormalizable operators of mass dimensions equal to or lower than four, and it was the first sector to be proposed [6]. The natural next step was to focus on higher-order operators with mass dimensions five or higher, which has been carried out extensively in the past years, giving several bounds on the parameters that modify QFT [7,8] and linearized gravity [9]. The Myers-Pospelov model was formulated independently and focused on dimension-five operators containing Lorentz violation in the scalar, fermion, and photon sectors [10,11]. Consistency properties such as causality, stability [12–15], and unitarity in the minimal [16,17] and nonminimal sectors of the SME

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[18–21] have been studied intensively in the past years. Also, theories of fermions and photons with broken spin degeneracy have been studied in [22]. This class of theories provides the possibility to open a window to effects relying on a nonzero phase space, such as Cherenkov radiation in vacuo and decay of photons into electron-positron pairs [23,24]. Radiative corrections have also been extensively studied within the SME [25]. Recently a sector of modified gravity has been cast in canonical form [26], and Lorentz-violating cosmology has been proposed [27].

The effects introduced by higher-order operators become stronger at higher energies since they scale with higher powers of momenta. However, a notable nonperturbative effect is that they generically introduce extra degrees of freedom associated with negative-norm states in an indefinite Hilbert space. Contrary to the Gupta-Bleuler formalism in covariant QED [28] the negative-norm states associated with higher-order operators cannot be *a priori* excluded from the asymptotic state space. A treatment introduced by Lee and Wick in which a specific asymptotic space is adopted successfully proved that theories with indefinite metric can preserve unitarity, thereby respecting the probability interpretation of quantum mechanics [29,30]. Indefinite Hilbert spaces may lead to the loss of unitarity. The negative-metric part associated with ghost states can modify the amplitudes, disrupting the optical theorem, being a direct consequence of unitarity. In this work, we investigate the preservation of unitarity in a process of QED involving $2 \to 2$ particles at tree level. We have focused on the extension of the Myers and Pospelov fermion sector that is even under charge conjugation (C). In particular, the C-odd part has been studied in [21].

The organization of this work is as follows. In Sec. II we compute the dispersion relations and find the spinor solutions. In Sec. III we quantize the fermion sector, find the Hamiltonian, and compute the propagator using its definition in terms of expectation values of the fields. Furthermore, in Sec. IV we compute the Pauli-Wigner function for two separated spacetime points and verify microcausality. In Sec. V we compute unitarity at tree level in $2 \to 2$ particle processes by using the optical theorem. Section VI contains our final remarks.

II. HIGHER-ORDER LORENTZ VIOLATING MODEL

We start with the higher-order Lorentz and CPT-violating Lagrangian proposed in [10]

$$\mathcal{L}_F = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{\bar{\psi}}{m_{\mathrm{Pl}}}(\eta_1 n\!\!\!/ + \eta_2 n\!\!\!/\gamma_5)(n\cdot\partial)^2\psi, \tag{1}$$

where n^μ is a constant four-vector, η_1 and η_2 are constant couplings being charge conjugation odd and even, respectively. The Lorentz-violating term is suppressed by the Planck mass m_{Pl}.

The generalized effective Lagrangian describing fermions in the presence of Lorentz and CPT violations can be written as

$$\mathcal{L}_{\mathrm{SME}} = \bar{\psi}(i\hat{\Gamma}^\mu\partial_\mu - \hat{M})\psi. \tag{2}$$

The above SME Lagrangian contains all possible minimal and nonminimal Lorentz-violating effective terms coded within the operators $\hat{\Gamma}^\mu$ and $\hat{M}$ [8]. To make contact with the effective Lagrangian (1) we set $\hat{\Gamma}^\mu = \gamma^\mu$ and identify the CPT odd operators of mass dimension one,

$$\hat{a}^{(5)\mu} = -\frac{\eta_1}{m_{\mathrm{Pl}}}(n\cdot\partial)^2 n^\mu, \tag{3a}$$

$$\hat{b}^{(5)\mu} = \frac{\eta_2}{m_{\mathrm{Pl}}}(n\cdot\partial)^2 n^\mu, \tag{3b}$$

for the decomposition of $\hat{M}$ in terms of the basis of 16 Dirac matrices

$$\hat{M} = m + \hat{a}^{(5)\mu}\gamma_\mu + \hat{b}^{(5)\mu}\gamma_5\gamma_\mu. \tag{4}$$

The Lorentz violating terms in (3a) and (3b) have been tested with astrophysical observations and laboratory experiments imposing strong limits on their Lorentz violation [2,8,31–34].

The free equation of motion is

$$\left(i\partial\!\!\!/ - m + \frac{n^\mu n^\nu}{m_{\mathrm{Pl}}}(\eta_1 n\!\!\!/ + \eta_2 n\!\!\!/\gamma_5)(\partial_\mu\partial_\nu)\right)\psi(x) = 0. \tag{5}$$

The gauge-invariant QED Lagrangian can be obtained via minimal coupling substitution in (1), producing

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}(iD\!\!\!\!/ - m)\psi + \frac{n^\mu n^\nu}{m_{\mathrm{Pl}}}\bar{\psi}(\eta_1 n\!\!\!/ + \eta_2 n\!\!\!/\gamma_5)D_\mu D_\nu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Consider the gauge transformations on the fields

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x), \tag{7a}$$

$$\psi(x) \to e^{-ie\lambda}\psi(x), \tag{7b}$$

and one can prove they lead to

$$D_\mu\psi \to e^{-ie\lambda}D_\mu\psi, \tag{7c}$$

Thus, the gauge invariance of the Lagrangian (6) follows from the transformation

$$D_\alpha(e^{-ie\lambda}D_\mu\psi) \to \partial_\alpha(e^{-ie\lambda}D_\mu\psi) + ie(A_\alpha + \partial_\alpha\lambda) \times e^{-ie\lambda}D_\mu\psi = e^{-ie\lambda}D_\alpha D_\mu\psi. \tag{8}$$

Here we work with the Dirac matrices in the chiral representation, i.e.,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \tag{9}$$

where $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$, and $\mathbb{1}_2$ is the 2×2 identity matrix. The fields are defined in Minkowski spacetime with metric signature $(+,-,-,-)$.

A. The dispersion relation

For the rest of the work we turn off the charge conjugation odd sector setting $\eta_1 = 0$ in the Lagrangian (1).

Consider the ansatz $\psi(\vec{x}) = \int d^3\vec{p}\, u(p) e^{-ip\cdot x}$ substituted in Eq. (5). We arrive at

$$(\not{p} - m - g_2\not{n}\gamma_5(n\cdot p)^2)u(p) = 0, \tag{10}$$

with the redefined coupling $g_2 \equiv \eta_2/m_{\mathrm{Pl}}$.

Let us define the operators

$$M = \not{p} - m - g_2\not{n}\gamma_5(n\cdot p)^2, \tag{11a}$$

$$\bar{\mathcal{M}} = \not{p} + m - g_2\not{n}\gamma_5(n\cdot p)^2, \tag{11b}$$

and

$$\mathcal{N} = \not{p} + m + g_2\not{n}\gamma_5(n\cdot p)^2, \tag{11c}$$

$$\bar{\mathcal{N}} = \not{p} - m + g_2\not{n}\gamma_5(n\cdot p)^2. \tag{11d}$$

In addition we define

$$\mathcal{Q} = -\frac{[\not{p},\not{n}]\gamma_5}{2\sqrt{D}}, \tag{12}$$

where $D(n,p) \coloneqq (n\cdot p)^2 - p^2n^2$ is the Gramian of the two four-vectors n and p. The operator $\mathcal{Q}$ commutes with the equation of motion, i.e.,

$$[\mathcal{Q},\mathcal{M}] = 0, \tag{13}$$

and with any of the operators $\bar{\mathcal{M}},\mathcal{N},\bar{\mathcal{N}}$, so we expect the spinor solutions to be eigenstates of $\mathcal{Q}$.

Some useful relations follow by considering

$$\bar{\mathcal{M}}\mathcal{M} = p^2 - m^2 - g_2^2n^2(n\cdot p)^4 + 2g_2(n\cdot p)^2\sqrt{D}\mathcal{Q} \tag{14a}$$

and

$$\bar{\mathcal{N}}\mathcal{N} = p^2 - m^2 - g_2^2n^2(n\cdot p)^4 - 2g_2(n\cdot p)^2\sqrt{D}\mathcal{Q}. \tag{14b}$$

We have

$$(\bar{\mathcal{N}}\mathcal{N}\bar{\mathcal{M}}\mathcal{M})u(p) = [(p^2 - m^2 - g_2^2n^2(n\cdot p)^4)^2 - 4g_2^2(n\cdot p)^4D]u(p) = 0, \tag{15}$$

where the identities have been used,

$$[\not{p},\not{n}]\gamma_5[\not{p},\not{n}]\gamma_5 = 4D \tag{16a}$$

and

$$\mathcal{Q}^2 = 1. \tag{16b}$$

We arrive at the dispersion relation by requiring a nontrivial solution for $u(p)$, that is to say,

$$(p^2 - m^2 - g_2^2n^2(n\cdot p)^4)^2 - 4g_2^2(n\cdot p)^4D = 0. \tag{17}$$

Let us define the two quantities

$$\tilde{\Lambda}_+^2(p) = p^2 - m^2 - g_2^2n^2(n\cdot p)^4 - 2g_2(n\cdot p)^2\sqrt{D} \tag{18a}$$

and

$$\tilde{\Lambda}_-^2(p) = p^2 - m^2 - g_2^2n^2(n\cdot p)^4 + 2g_2(n\cdot p)^2\sqrt{D}. \tag{18b}$$

Their product produces the dispersion relation

$$\tilde{\Lambda}_+^2(p)\tilde{\Lambda}_-^2(p) \equiv (p^2 - m^2 - g_2^2n^2(n\cdot p)^4)^2 - 4g_2^2(n\cdot p)^4D. \tag{19}$$

B. Purely timelike model

Here we consider the background to be purely timelike with $n = (1,0,0,0)$. Hence, the Lagrangian (1) takes the form

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi + g_2\bar{\psi}\gamma_0\gamma_5\ddot{\psi}, \tag{20}$$

with equation of motion in momentum space

$$(\not{p} - m - g_2p_0^2\gamma_0\gamma_5)\psi(p) = 0. \tag{21}$$

The previous operators with the special choice of n turn to be

$$M = \not{p} - m - g_2p_0^2\gamma_0\gamma_5, \tag{22a}$$

$$\bar{M} = \not{p} + m - g_2p_0^2\gamma_0\gamma_5, \tag{22b}$$

$$N = \not{p} + m + g_2p_0^2\gamma_0\gamma_5, \tag{22c}$$

$$\bar{N} = \not{p} - m + g_2p_0^2\gamma_0\gamma_5. \tag{22d}$$

Furthermore, we have

$$Q = -\frac{p_i\gamma^i}{|\vec{p}|}\gamma_0\gamma_5 = -\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix} \tag{23}$$

and

$$\Lambda_+^2(p) = p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4 - 2g_2 p_0^2|\vec{p}|, \tag{24a}$$

$$\Lambda_-^2(p) = p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4 + 2g_2 p_0^2|\vec{p}|, \tag{24b}$$

which can be rewritten as

$$\Lambda_+^2 + m^2 = (p_0 + g_2 p_0^2 + |\vec{p}|)(p_0 - g_2 p_0^2 - |\vec{p}|), \tag{25a}$$

$$\Lambda_-^2 + m^2 = (p_0 + g_2 p_0^2 - |\vec{p}|)(p_0 - g_2 p_0^2 + |\vec{p}|). \tag{25b}$$

The dispersion relation for a pure timelike n is from Eq. (19),

$$(p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4)^2 - 4g_2^2 p_0^4 |\vec{p}|^2 = 0. \tag{26}$$

The eight solutions to the dispersion relations come from two sectors. We have four solutions of the dispersion relation $\Lambda_+^2 = 0$,

$$\omega_1 = \sqrt{\frac{1 - 2g_2|\vec{p}| - \sqrt{(1 - 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}, \tag{27a}$$

$$\bar{\omega}_1 = -\omega_1, \tag{27b}$$

$$W_1 = \sqrt{\frac{1 - 2g_2|\vec{p}| + \sqrt{(1 - 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}, \tag{27c}$$

$$\bar{W}_1 = -W_1, \tag{27d}$$

and four solutions of the dispersion relation $\Lambda_-^2 = 0$,

$$\omega_2 = \sqrt{\frac{1 + 2g_2|\vec{p}| - \sqrt{(1 + 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}, \tag{28a}$$

$$\bar{\omega}_2 = -\omega_2, \tag{28b}$$

$$W_2 = \sqrt{\frac{1 + 2g_2|\vec{p}| + \sqrt{(1 + 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}, \tag{28c}$$

$$\bar{W}_2 = -W_2, \tag{28d}$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2}$.

Alternatively, we can rewrite the total dispersion relation as

$$\Lambda_+^2(p)\Lambda_-^2(p) = g_2^4(p_0^2 - \omega_1^2)(p_0^2 - W_1^2)(p_0^2 - \omega_2^2) \times (p_0^2 - W_2^2) = 0. \tag{29}$$

The solutions can be analyzed individually; let us expand for small coupling, and obtain up to linear order in g_2

$$\omega_1 \approx E_p + |\vec{p}|E_p g_2, \tag{30a}$$

$$\omega_2 \approx E_p - |\vec{p}|E_p g_2, \tag{30b}$$

$$W_1 \approx \frac{1}{g_2} - |\vec{p}| - \frac{1}{2}(E_p^2 + |\vec{p}|^2)g_2, \tag{30c}$$

$$W_2 \approx \frac{1}{g_2} + |\vec{p}| - \frac{1}{2}(E_p^2. + |\vec{p}|^2)g_2. \tag{30d}$$

The low-energy modes ω_1 and ω_2 are perturbatively connected to particle propagation; however, the additional degrees of freedom corresponding to the higher-energy modes W_1 and W_2 correspond to the propagation of negative-norm states or ghosts as we will show in the next sections.

The frequencies ω_1, W_1 and $\bar{\omega}_1$, $\bar{W}_1$ can become complex for higher momenta. The condition for this to occur is

$$(1 - 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2 < 0, \tag{31}$$

from where we find a region where energies become complex $|\vec{p}| > |\vec{p}_{\max}| = \frac{1 - 4g_2^2 m^2}{g_2}$. Note that the condition for energies ω_2, W_2 and $\bar{\omega}_2$, $\bar{W}_2$,

$$(1 + 2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2 < 0, \tag{32}$$

cannot be satisfied for small values of $g_2^2 m^2$, and hence the energy remains real for any momenta. We find

$$\omega_1(|\vec{p}_{\max}|) = W_1(|\vec{p}_{\max}|) = \frac{1}{2}\sqrt{\frac{1}{g_2^2} + 4m^2}, \tag{33}$$

and $\lim_{|\vec{p}|\to\infty}\omega_2 = \lim_{|\vec{p}|\to\infty} W_2 \to \infty$. At this level, the theory establishes a maximum value for the momentum and *a priori* an energy scale for the effective region of the theory.

C. Spinor solutions

Now we focus on finding the eigenspinors of the modified Dirac equation using the energy solutions (27) and (28). Consider the field $\psi(x) = \int d^3\vec{p}\, u(p)e^{-ip\cdot x}$ in the equation of motion (21) which produces

$$Mu(p) = 0, \tag{34}$$

where M defined in Eq. (22a) has the matrix form

$$M = \begin{pmatrix} -m & p_0 - g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma}) \\ p_0 + g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}) & -m \end{pmatrix}. \tag{35}$$

We define the spinor in terms of bi-spinors

$$u(p) = \begin{pmatrix} \chi_1(p) \\ \chi_2(p) \end{pmatrix}, \tag{36}$$

and replacing the above we arrive at the equations

$$(p_0 - g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma}))\chi_2 = m\chi_1, \tag{37a}$$

$$(p_0 + g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}))\chi_1 = m\chi_2. \tag{37b}$$

The spinor solutions of the dispersion relation $\Lambda_+^2 = 0$ are

$$u^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_1}, \tag{38a}$$

$$U^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_1}, \tag{38b}$$

and the solutions of the dispersion relation $\Lambda_-^2 = 0$,

$$u^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_2}, \tag{39a}$$

$$U^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_2}. \tag{39b}$$

For the negative-energy solutions, we consider the field to be $\psi(x) = \int d^3\vec{p}\, v(p) e^{ip\cdot x}$ and the eigenvalue equation

$$Nv(p) = 0, \tag{40}$$

with

$$N = \begin{pmatrix} m & p_0 + g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma}) \\ p_0 - g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}) & m \end{pmatrix}, \tag{41}$$

given in Eq. (22c) and

$$v(p) = \begin{pmatrix} \phi_1(p) \\ \phi_2(p) \end{pmatrix}. \tag{42}$$

We have the equations

$$(p_0 + g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma}))\phi_2 = -m\phi_1, \tag{43a}$$

$$(p_0 - g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}))\phi_1 = -m\phi_2. \tag{43b}$$

We find for the negative-energy solutions associated with $\Lambda_+^2 = 0$,

$$v^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_1}, \tag{44a}$$

$$V^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_1}, \tag{44b}$$

and with $\Lambda_-^2 = 0$,

$$v^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_2}, \tag{45a}$$

$$V^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_2}. \tag{45b}$$

We can write some relations satisfied by the spinors, which do not part too much from the usual expressions. They are

$$u^{s\dagger}(p)u^r(p) = 2\omega_s\delta^{rs}, \tag{46a}$$

$$v^{s\dagger}(p)v^r(p) = 2\omega_s\delta^{rs}, \tag{46b}$$

and

$$U^{s\dagger}(p)U^r(p) = 2W_s\delta^{rs}, \tag{47a}$$

$$V^{s\dagger}(p)V^r(p) = 2W_s\delta^{rs}, \tag{47b}$$

and for the fields $\bar{u} = u^\dagger\gamma_0$ we have

$$\bar{u}^s(p)u^r(p) = 2m\delta^{rs}, \tag{48a}$$

$$\bar{v}^s(p)v^r(p) = -2m\delta^{rs}, \tag{48b}$$

and

$$\bar{U}^s(p)U^r(p) = 2m\delta^{rs}, \tag{49a}$$

$$\bar{V}^s(p)V^r(p) = -2m\delta^{rs}, \tag{49b}$$

where the indices run over $r, s = 1, 2$. The detailed derivation of the spinors, together with their complete inner and outer product relations are given in Appendix.

III. QUANTIZATION

In this section, we focus on the quantization of the Lorentz-violating fermion model. We derive the Hamiltonian and the four-dimensional representation of the Feynman propagator. In the last section, we study microcausality preservation.

A. Equal-time anticommutation relations of the fields

The Lagrangian (20) can be integrated by parts to produce

$$\mathcal{L}' = \frac{i}{2}(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) + \bar{\psi}(i\gamma^i \partial_i - m)\psi - g_2 \dot{\psi}^\dagger \gamma_5 \dot{\psi}. \tag{50}$$

The above Lagrangian is equivalent to the original one, but it is simpler in the sense of being standard-derivative order and symmetrical with respect to time derivatives. We work with this Lagrangian in the next sections.

It is convenient to decompose the field $\psi(\vec{x}, x_0)$ in terms of two fields ψ_1 and ψ_2 as

$$\psi(\vec{x}, x_0) = \psi_1(\vec{x}, x_0) + \psi_2(\vec{x}, x_0). \tag{51}$$

We take the field ψ_1 to describe standard particle states, which eventually includes perturbative corrections in the parameter g_2. On the other hand, the field ψ_2 is defined to be associated with negative-metric particles or ghosts.

We expand each field considering their plane wave and spinor solutions found earlier. The particle field is

$$\psi_1(\vec{x}, x_0) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{N_r}} \Big(a_p^r u^r(p) e^{-ip\cdot x} + b_p^{r\dagger} v^r(p) e^{ip\cdot x} \Big)_{p_0=\omega_r}, \tag{52a}$$

and the ghost field

$$\psi_2(\vec{x}, x_0) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r}} \Big(\alpha_p^r U^r(p) e^{-ip\cdot x} + \beta_p^{r\dagger} V^r(p) e^{ip\cdot x} \Big)_{p_0=W_r}. \tag{52b}$$

We have introduced the creation operators $a_p^{\dagger r}, b_p^{\dagger r}$ and the annihilation operators a_p^r, b_p^r for particle states and the set of operators $\alpha_p^{\dagger r}, \beta_p^{\dagger r}$ and α_p^r, β_p^r representing creation and annihilation operators, respectively, for ghost states.

The fields $\psi_1(\vec{x}, x_0)$ and $\psi_2(\vec{x}, x_0)$ are normalized with the constants

$$N_1 = 2\omega_1 g_2^2 (W_1^2 - \omega_1^2), \tag{53a}$$

$$N_2 = 2\omega_2 g_2^2 (W_2^2 - \omega_2^2), \tag{53b}$$

and

$$\mathcal{N}_1 = 2W_1 g_2^2 (W_1^2 - \omega_1^2), \tag{54a}$$

$$\mathcal{N}_2 = 2W_2 g_2^2 (W_2^2 - \omega_2^2). \tag{54b}$$

In Appendix, we explain how they appear associated with a modified internal product between spinor states of positive and negative energy.

From the Lagrangian (50), we compute the momenta associated with the independent fields ψ and $\psi^\dagger$,

$$\pi_\psi = \frac{\partial \mathcal{L}'}{\partial \dot{\psi}} = \frac{i}{2}\psi^\dagger - g_2 \dot{\psi}^\dagger \gamma_5, \tag{55a}$$

$$\pi_{\psi^\dagger} = \frac{\partial \mathcal{L}'}{\partial \dot{\psi}^\dagger} = -\frac{i}{2}\psi - g_2 \gamma_5 \dot{\psi}. \tag{55b}$$

We impose the equal-time anticommutation relations for the fields and their conjugate momenta fields

$$\{\psi(\vec{x}, x_0), \pi_\psi(\vec{y}, x_0)\} = i\delta^{(3)}(\vec{x} - \vec{y}), \tag{56a}$$

$$\{\psi^\dagger(\vec{x}, x_0), \pi_{\psi^\dagger}(\vec{y}, x_0)\} = i\delta^{(3)}(\vec{x} - \vec{y}), \tag{56b}$$

with the rest of commutators being zero. In order to achieve Eqs. (56a) and (56b) we take the creation and annihilation operators to obey the rules

$$\{a_p^s, a_k^{r\dagger}\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\vec{k} - \vec{p}), \tag{57a}$$

$$\{b_p^s, b_k^{r\dagger}\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\vec{k} - \vec{p}), \tag{57b}$$

and

$$\{\alpha_p^s, \alpha_k^{r\dagger}\} = -(2\pi)^3 \delta^{sr} \delta^{(3)}(\vec{k} - \vec{p}), \tag{58a}$$

$$\{\beta_p^s, \beta_k^{r\dagger}\} = -(2\pi)^3 \delta^{sr} \delta^{(3)}(\vec{k} - \vec{p}), \tag{58b}$$

with the vacuum defined by

$$a_p^s|0\rangle = b_p^s|0\rangle = \alpha_p^s|0\rangle = \beta_p^s|0\rangle = 0. \tag{59}$$

Notice that the second set of rules are defined with a nonstandard negative sign in (58) which is the first indication of having an indefinite metric in Hilbert space.

In fact, we can write down the metric for each sector in the indefinite Hilbert space. We define the n-particle states of polarization s to appear by applying repeatedly creation operators on the vacuum state. For particle states

$$|n_{1,s}\rangle = \frac{1}{\sqrt{(n_{1,s})!}} (a_p^{s\dagger})^{n_{1,s}} |0\rangle, \tag{60a}$$

and for ghost states

$$|n_{2,s}\rangle = \frac{1}{\sqrt{(n_{2,s})!}}(\alpha_p^{s\dagger})^{n_{2,s}}|0\rangle, \tag{60b}$$

where $n_{1,s}$ and $n_{2,s}$ are the eigenvalues of the number operators $\hat{N}_{1,s} = a_p^{s\dagger}a_p^s$ and $\hat{N}_{2,s} = \alpha_p^{s\dagger}\alpha_p^s$, respectively. Hence, for particles we have the positive metric

$$\eta_{1,s} = \langle n_{1,s}|n_{1,s}\rangle = 1, \tag{61a}$$

and for ghost states the indefinite metric

$$\eta_{2,s} = \langle n_{2,s}|n_{2,s}\rangle = (-1)^{n_{2,s}}. \tag{61b}$$

From (52a) and (52b) we have

$$\psi^\dagger(\vec{x},x_0) = \psi_1^\dagger(\vec{x},x_0) + \psi_2^\dagger(\vec{x},x_0), \tag{62}$$

where

$$\psi_1^\dagger(\vec{x},x_0) = \sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r}}\Big(a_p^{r\dagger}u^{r\dagger}(p)e^{ip\cdot x} + b_p^r v^{r\dagger}(p)e^{-ip\cdot x}\Big)_{p_0=\omega_r}, \tag{63a}$$

$$\psi_2^\dagger(\vec{x},x_0) = \sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{\mathcal{N}_r}}\Big(\alpha_p^{r\dagger}U^{r\dagger}(p)e^{ip\cdot x} + \beta_p^r V^{r\dagger}(p)e^{-ip\cdot x}\Big)_{p_0=W_r}. \tag{63b}$$

We introduce momenta with respect to the decomposed fields in the form

$$\pi_1 = \frac{\partial\mathcal{L}'}{\partial\dot{\psi}_1} = \frac{i}{2}\psi_1^\dagger - g_2\dot{\psi}_1^\dagger\gamma_5, \tag{64a}$$

$$\pi_2 = \frac{\partial\mathcal{L}'}{\partial\dot{\psi}_2} = \frac{i}{2}\psi_2^\dagger - g_2\dot{\psi}_2^\dagger\gamma_5, \tag{64b}$$

and

$$\pi_1^\dagger = \frac{\partial\mathcal{L}'}{\partial\dot{\psi}_1{}^\dagger} = -\frac{i}{2}\psi_1 - g_2\gamma_5\dot{\psi}_1, \tag{65a}$$

$$\pi_2^\dagger = \frac{\partial\mathcal{L}'}{\partial\dot{\psi}_2{}^\dagger} = -\frac{i}{2}\psi_2 - g_2\gamma_5\dot{\psi}_2. \tag{65b}$$

Therefore, we can write

$$\pi_\psi = \pi_1 + \pi_2, \tag{66a}$$

$$\pi_{\psi^\dagger} = \pi_1^\dagger + \pi_2^\dagger. \tag{66b}$$

With these simplifications, we start computing the commutator (56a). We can write the first commutator as the sum

$$\{\psi(\vec{x},x_0),\pi_\psi(\vec{y},x_0)\} = \{\psi_1(\vec{x},x_0),\pi_1(\vec{y},x_0)\} + \{\psi_2(\vec{x},x_0),\pi_2(\vec{y},x_0)\}, \tag{67}$$

and momenta (64a) and (64b) as

$$\pi_1(\vec{x},x_0) = i\sum_s\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_s}}\left[a_p^{s\dagger}u^{s\dagger}(p)\left(\frac{1}{2}-g_2\omega_s\gamma_5\right)\times e^{ip\cdot x} + b_p^s v^{s\dagger}(p)\left(\frac{1}{2}+g_2\omega_s\gamma_5\right)e^{-ip\cdot x}\right]_{p_0=\omega_s} \tag{68a}$$

and

$$\pi_2(\vec{x},x_0) = i\sum_s\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{\mathcal{N}_s}}\left[\alpha_p^{s\dagger}U^{s\dagger}(p)\left(\frac{1}{2}-g_2W_s\gamma_5\right)\times e^{ip\cdot x}+\beta_p^s V^{s\dagger}(p)\left(\frac{1}{2}+g_2W_s\gamma_5\right)e^{-ip\cdot x}\right]_{p_0=W_s}. \tag{68b}$$

The first commutator in (67) can be shown to be

$$\{\psi_1(\vec{x},x_0),\pi_1(\vec{y},x_0)\} = \sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{i}{N_r}\left[u^r(p)u^{r\dagger}(p)\left(\frac{1}{2}-g_2\omega_r\gamma_5\right) + v^r(-p)v^{r\dagger}(-p)\left(\frac{1}{2}+g_2\omega_r\gamma_5\right)\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{69}$$

We can proceed analogously, and by considering the minus sign due to the minus in the anticommutation relations (58) we obtain

$$\{\psi_2(\vec{x},x_0),\pi_2(\vec{y},x_0)\} = -\sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{i}{N_r}\left[U^r(p)U^{r\dagger}(p)\left(\frac{1}{2}-g_2\omega_r\gamma_5\right) + V^r(-p)V^{r\dagger}(-p)\left(\frac{1}{2}+g_2\omega_r\gamma_5\right)\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{70}$$

For the first commutator involving particles we use Eqs. (A58)–(A61) and for the second ghost commutator Eqs. (A62)–(A65) given in the third subsection of Appendix, and we arrive at

$$\{\psi_1(\vec{x},x_0),\pi_1(\vec{y},x_0)\} = i\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{\omega_1}{N_1}\left[\frac{1}{2}(\mathbb{1}_4-Q)-g_2(\gamma^i p_i+m-g_2\omega_1^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4-Q)\gamma_5\right]\right.$$
$$\left.+\frac{\omega_2}{N_2}\left[\frac{1}{2}(\mathbb{1}_4+Q)-g_2(\gamma^i p_i+m-g_2\omega_2^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4+Q)\gamma_5\right]\right)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}, \tag{71}$$

and to

$$\{\psi_2(\vec{x},x_0),\pi_2(\vec{y},x_0)\} = -i\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{W_1}{\mathcal{N}_1}\left[\frac{1}{2}(\mathbb{1}_4-Q)-g_2(\gamma^i p_i+m-g_2W_1^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4-Q)\gamma_5\right]\right.$$
$$\left.+\frac{W_2}{\mathcal{N}_2}\left[\frac{1}{2}(\mathbb{1}_4+Q)-g_2(\gamma^i p_i+m-g_2W_2^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4+Q)\gamma_5\right]\right)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{72}$$

We use the relations

$$\frac{\omega_1}{N_1}=\frac{W_1}{\mathcal{N}_1}=\frac{1}{2g_2^2(W_1^2-\omega_1^2)}, \tag{73}$$

and by adding (71) and (72) produce

$$\begin{aligned}&\{\psi(\vec{x},x_0),\pi(\vec{y},x_0)\}\\&=i\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{\gamma_0\gamma_5\gamma_0}{2g_2^2(W_1^2-\omega_1^2)}(g_2^2(\omega_1^2-W_1^2)(\mathbb{1}_4-Q)\gamma_5)\right.\\&\left.+\frac{\gamma_0\gamma_5\gamma_0}{2g_2^2(W_2^2-\omega_2^2)}(g_2^2(\omega_2^2-W_2^2)(\mathbb{1}_4+Q)\gamma_5)\right]e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\end{aligned} \tag{74}$$

and

$$\begin{aligned}&\{\psi(\vec{x},x_0),\pi(\vec{y},x_0)\}\\&=-i\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{1}{2}\gamma_0\gamma_5\gamma_0(\mathbb{1}_4-Q)\gamma_5+\frac{1}{2}\gamma_0\gamma_5\gamma_0\right.\\&\left.\times(\mathbb{1}_4+Q)\gamma_5\right)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}.\end{aligned} \tag{75}$$

Finally,

$$\begin{aligned}\{\psi(\vec{x},x_0),\pi(\vec{y},x_0)\}&=-i\int\frac{d^3\vec{p}}{(2\pi)^3}(\gamma_0\gamma_5\gamma_0\gamma_5)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\\&=i\delta^{(3)}(\vec{x}-\vec{y}).\end{aligned} \tag{76}$$

In a similar way the commutator (56b) is also satisfied.

B. The Hamiltonian

The Legendre transformation of the Lagrangian (50) produces the Hamiltonian

$$H=\int d^3\vec{x}\left(\pi_\psi\dot{\psi}+\dot{\psi}^\dagger\pi_{\psi^\dagger}-\mathcal{L}'\right). \tag{77}$$

Considering momenta in Eqs. (55a) and (55b) the Hamiltonian can be cast into the form

$$H=\int d^3\vec{x}\left(-g_2\dot{\psi}^\dagger\gamma_5\dot{\psi}+\overline{\psi}(-i\gamma^i\partial_i+m)\psi\right). \tag{78}$$

With the decomposition of fields (51) let us write

$$H\equiv\sum_{a,b=1,2}H_{ab}=\sum_{a,b=1,2}\int d^3\vec{x}\mathcal{H}_{ab}(x), \tag{79}$$

where

$$\begin{aligned}\mathcal{H}_{ab}(x)=&-g_2\dot{\psi}_a^\dagger(x)\gamma_5\dot{\psi}_b(x)\\&+\bar{\psi}_a(x)(-i\gamma^k\partial_k+m)\psi_b(x).\end{aligned} \tag{80}$$

We write the contributions coming from both fields separately.

The contributions coming from ψ_1 are

$$\begin{aligned}-g_2\gamma_5\dot{\psi}_1=-g_2\gamma_5\sum_s\int\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{N_s'}}&\left(-i\omega_s'u^s(p')a_{p'}^s e^{-ip'\cdot x}\right.\\&\left.+i\omega_s'v^s(p')b_{p'}^{s\dagger}e^{ip'\cdot x}\right)_{p_0'=\omega_s'}\end{aligned} \tag{81}$$

and

$$\begin{aligned}&(-i\gamma^i\partial_i+m)\psi_1(x)\\&=\sum_s\int\frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{N_s'}}\left((-\gamma^i p_i'+m)u^s(p')a_{p'}^s e^{-ip'\cdot x}\right.\\&\left.+(\gamma^i p_i'+m)v^s(p')b_{p'}^{s\dagger}e^{ip'\cdot x}\right)_{p_0'=\omega_s'}.\end{aligned} \tag{82}$$

And the ones coming from ψ_2 are

$$\begin{aligned}
&- g_2\gamma_5\dot{\psi}_2 \\
&\quad = -g_2\gamma_5\sum_s \int \frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{\mathcal{N}'_s}}\Big(-iW'_s U^s(p')\alpha^s_{p'}e^{-ip'\cdot x} \\
&\qquad + (iW'_s)V^s(p')\beta^{s\dagger}_{p'}e^{ip'\cdot x}\Big)_{p'_0=W'_s}
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
&(-i\gamma^i\partial_i + m)\psi_2(x) \\
&\quad = \sum_s \int \frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{\mathcal{N}'_s}}\Big((-\gamma^i p'_i + m)U^s(p')\alpha^s_{p'}e^{-ip'\cdot x} \\
&\qquad + (\gamma^i p'_i + m)V^s(p')\beta^{s\dagger}_{p'}e^{ip'\cdot x}\Big)_{p'_0=W'_s}.
\end{aligned} \tag{84}$$

We can rewrite the terms involving space derivatives (82) and (84) using the equations of motion (34) and (40), i.e.,

$$(-\gamma^i p'_i + m)u^s(p') = \gamma_0(\omega'_s - g_2\gamma_5\omega'^2_s)u^s(p'), \tag{85a}$$

$$(\gamma^i p'_i + m)v^s(p') = -\gamma_0(\omega'_s + g_2\gamma_5\omega'^2_s)v^s(p'), \tag{85b}$$

and

$$(-\gamma^i p'_i + m)U^s(p') = \gamma_0(W'_s - g_2\gamma_5 W'^2_s)U^s(p'), \tag{86a}$$

$$(\gamma^i p'_i + m)V^s(p') = -\gamma_0(W'_s + g_2\gamma_5 W'^2_s)V^s(p'). \tag{86b}$$

This yields

$$\begin{aligned}
(-i\gamma^i\partial_i + m)\psi_1(x) = \sum_s \int \frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{N'_s}}\Big[&(\gamma_0(\omega'_s - g_2\omega'^2_s\gamma_5)u^s(p')a^s_{p'}e^{-i\omega'_s x_0})e^{i\vec{p}'\cdot\vec{x}} \\
&- (\gamma_0(\omega'_s + g_2\omega'^2_s\gamma_5)v^s(p')b^{s\dagger}_{p'}e^{i\omega'_s x_0})e^{-i\vec{p}'\cdot\vec{x}}\Big]
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
(-i\gamma^i\partial_i + m)\psi_2(x) = \sum_s \int \frac{d^3\vec{p}'}{(2\pi)^3}\frac{1}{\sqrt{\mathcal{N}'_s}}\Big[&(\gamma_0(W'_s - g_2 W'^2_s\gamma_5)U^s(p')\alpha^s_{p'}e^{-iW'_s x_0})e^{i\vec{p}'\cdot\vec{x}} \\
&- (\gamma_0(W'_s + g_2 W'^2_s\gamma_5)V^s(p')\beta^{s\dagger}_{p'}e^{iW'_s x_0})e^{-i\vec{p}'\cdot\vec{x}}\Big].
\end{aligned} \tag{88}$$

Now, it is convenient to decompose further by considering

$$H_{11} = H^{uu} + H^{uv} + H^{vv} + H^{vu}, \tag{89a}$$

$$H_{12} = H^{uU} + H^{uV} + H^{vU} + H^{vV}, \tag{89b}$$

$$H_{21} = H^{Uu} + H^{Uv} + H^{Vu} + H^{Vv}, \tag{89c}$$

$$H_{22} = H^{UU} + H^{UV} + H^{VU} + H^{VV}. \tag{89d}$$

After some algebra we find the particle contributions

$$\begin{aligned}
H^{uu} = \sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r N_s}}a^{r\dagger}_p a^s_p e^{i(\omega_r-\omega_s)x_0} \\
\times \omega_s u^{r\dagger}(p)(1 - g_2\gamma_5(\omega_s + \omega_r))u^s(p),
\end{aligned} \tag{90}$$

$$\begin{aligned}
H^{uv} = -\sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r N_s}}a^{r\dagger}_p b^{s\dagger}_{-p} e^{i(\omega_r+\omega_s)x_0} \\
\times \omega_s u^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s - \omega_r))v^s(-p),
\end{aligned} \tag{91}$$

$$\begin{aligned}
H^{vu} = \sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r N_s}}b^r_p a^s_{-p} e^{-i(\omega_r+\omega_s)x_0} \\
\times \omega_s v^{r\dagger}(p)(1 - g_2\gamma_5(\omega_s - \omega_r))u^s(-p),
\end{aligned} \tag{92}$$

$$\begin{aligned}
H^{vv} = -\sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r N_s}}b^r_p b^{s\dagger}_p e^{-i(\omega_r-\omega_s)x_0} \\
\times \omega_s v^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s + \omega_r))v^s(p),
\end{aligned} \tag{93}$$

the mixed ones

$$\begin{aligned}
H^{uU} = \sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r\mathcal{N}_s}}a^{r\dagger}_p \alpha^s_p e^{i(\omega_r-W_s)x_0} \\
\times W_s u^{r\dagger}(p)(1 - g_2\gamma_5(W_s + \omega_r))U^s(p),
\end{aligned} \tag{94}$$

$$\begin{aligned}
H^{uV} = -\sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r\mathcal{N}_s}}a^{r\dagger}_p \beta^{s\dagger}_{-p} e^{i(\omega_r+W_s)x_0} \\
\times W_s u^{r\dagger}(p)(1 + g_2\gamma_5(W_s - \omega_r))V^s(-p),
\end{aligned} \tag{95}$$

$$\begin{aligned}
H^{vU} = \sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r\mathcal{N}_s}}b^r_p \alpha^s_{-p} e^{-i(\omega_r+W_s)x_0} \\
\times W_s v^{r\dagger}(p)(1 - g_2\gamma_5(W_s - \omega_r))U^s(-p),
\end{aligned} \tag{96}$$

$$\begin{aligned}
H^{vV} = -\sum_{r,s}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{N_r\mathcal{N}_s}}b^r_p \beta^{s\dagger}_p e^{-i(\omega_r-W_s)x_0} \\
\times W_s v^{r\dagger}(p)(1 + g_2\gamma_5(W_s + \omega_r))V^s(p),
\end{aligned} \tag{97}$$

$$H^{Uu} = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r N_s}} \alpha_p^{r\dagger} a_p^s e^{i(W_r-\omega_s)x_0} \times \omega_s U^{r\dagger}(p)(1 - g_2\gamma_5(\omega_s + W_r))u^s(p), \tag{98}$$

$$H^{Uv} = -\sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r N_s}} \alpha_p^{r\dagger} b_{-p}^{s\dagger} e^{i(W_r+\omega_s)x_0} \times \omega_s U^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s - W_r))v^s(-p), \tag{99}$$

$$H^{Vu} = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r N_s}} \beta_p^r a_{-p}^s e^{-i(W_r+\omega_s)x_0} \times \omega_s V^{r\dagger}(p)(1 - g_2\gamma_5(\omega_s - W_r))u^s(-p), \tag{100}$$

$$H^{Vv} = -\sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r N_s}} \beta_p^r b_p^{s\dagger} e^{-i(W_r-\omega_s)x_0} \times \omega_s V^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s + W_r))v^s(p), \tag{101}$$

and the ghost contributions

$$H^{UU} = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r \mathcal{N}_s}} \alpha_p^{r\dagger} \alpha_p^s e^{i(W_r-W_s)x_0} \times W_s U^{r\dagger}(p)(1 - g_2\gamma_5(W_s + W_r))U^s(p), \tag{102}$$

$$H^{UV} = -\sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r \mathcal{N}_s}} \alpha_p^{r\dagger} \beta_{-p}^{s\dagger} e^{i(W_r+W_s)x_0} \times W_s U^{r\dagger}(p)(1 + g_2\gamma_5(W_s - W_r))V^s(-p), \tag{103}$$

$$H^{VU} = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r \mathcal{N}_s}} \beta_p^r \alpha_{-p}^s e^{-i(W_r+W_s)x_0} \times W_s V^{r\dagger}(p)(1 - g_2\gamma_5(W_s - W_r))U^s(-p), \tag{104}$$

$$H^{VV} = -\sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{\mathcal{N}_r \mathcal{N}_s}} \beta_p^r \beta_p^{s\dagger} e^{-i(W_r-W_s)x_0} \times W_s V^{r\dagger}(p)(1 + g_2\gamma_5(W_s + W_r))V^s(p). \tag{105}$$

After considering the 16 terms and using Eqs. (A45)–(A47) of Appendix the only nonzero contributions are

$$H^{uu} = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_s a_p^{s\dagger} a_p^s, \tag{106a}$$

$$H^{vv} = -\sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_s b_p^s b_p^{s\dagger}, \tag{106b}$$

and

$$H^{UU} = -\sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} W_s \alpha_p^{s\dagger} \alpha_p^s, \tag{106c}$$

$$H^{VV} = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} W_s \beta_p^s \beta_p^{s\dagger}. \tag{106d}$$

Finally, adding all the parts we arrive at

$$H = \sum_{s=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \Big(\omega_s a_p^{s\dagger} a_p^s - \omega_s b_p^s b_p^{s\dagger} - W_s \alpha_p^{s\dagger} \alpha_p^s + W_s \beta_p^s \hat{\beta}_p^{s\dagger} \Big), \tag{107}$$

and the normal ordering gives

$$:H: = \sum_{s=1,2} \int \frac{d^3 p}{(2\pi)^3} \Big(\omega_s (a_p^{s\dagger} \hat{a}_p^s + b_p^{s\dagger} b_p^s) - W_s (\alpha_p^{s\dagger} \alpha_p^s + \beta_p^{s\dagger} \beta_p^s) \Big). \tag{108}$$

The Hamiltonian is stable, and in the presence of interaction we can always redefine the vacuum in order to produce a well bounded Hamiltonian. For fermions this is always possible due to the invariance of the algebra (58) under a vacuum redefinition [29]. However, it is noted that for energies higher than $\frac{1}{2g_2}\sqrt{1 + 4m^2 g_2^2}$ at which the solutions $\pm\omega_1$ and $\pm W_1$ become complex, the Hamiltonian is no longer Hermitian.

C. The Feynman propagator

We compute the modified propagator starting from its definition

$$S_F(x-y) = \langle 0|T\{\psi(x), \bar{\psi}(y)\}|0\rangle, \tag{109}$$

and in terms of theta functions and vacuum expectation values of fields we have

$$S_F(x-y) = \theta(x_0 - y_0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle - \theta(y_0 - x_0)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle. \tag{110}$$

To simplify the calculation and without loss of generality we set $y = 0$.

We start with the case $x_0 > 0$ and define

$$S_F(x) = S_F^{(>)}(x) \equiv \langle 0|\psi(x)\bar{\psi}(0)|0\rangle. \tag{111}$$

Using the decomposition of fields in Eq. (51) we can write

$$S_F^{(>)}(x) = \langle 0|\psi_1(x)\bar{\psi}_1(0)|0\rangle + \langle 0|\psi_2(x)\bar{\psi}_2(0)|0\rangle. \tag{112}$$

Consider

$$\begin{aligned}\langle 0|\psi_1(x)\bar{\psi}_1(0)|0\rangle &= \sum_{r,s=1,2}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{k}}{(2\pi)^3} \\ &\times \langle 0|\frac{1}{\sqrt{N_r}}\left(a_p^r u^r(p)e^{-ipx} + b_p^{r\dagger}v^r(p)e^{ip\cdot x}\right)_{p_0=\omega_r} \\ &\times \frac{1}{\sqrt{N_s}}\left(a_k^{s\dagger}\bar{u}^s(k) + b_k^s\bar{v}^s(k)\right)_{k_0=\omega_s}|0\rangle. \end{aligned} \tag{113}$$

The action of the annihilation operators on the vacuum produces

$$\begin{aligned}\langle 0|\psi_1(x)\bar{\psi}_1(0)|\rangle 0 &= \sum_{r,s=1,2}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{N_r}\sqrt{N_s}} \\ &\times u^r(p)\bar{u}^s(k)\langle 0|a_p^r a_k^{s\dagger}|0\rangle e^{-ip_r x},\end{aligned} \tag{114}$$

where $p_r = (\omega_r, \vec{p})$ and from the anticommutation relations (57) one has

$$\langle 0|\psi_1(x)\bar{\psi}_1(0)|0\rangle = \sum_{r=1,2}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{N_r}u^r(p)\bar{u}^r(p)e^{-ip_r x}. \tag{115}$$

Now we use the expression (A50) and (A51) to arrive at

$$\langle 0|\psi_1(x)\bar{\psi}_1(0)|0\rangle \tag{116}$$

$$\begin{aligned}&= \int \frac{d^3\vec{p}}{(2\pi)^3}\Big((\gamma_0\omega_1 + \gamma^i p_i + m - g_2\omega_1^2\gamma_0\gamma_5) \\ &\times \frac{1}{2}(\mathbb{1}_4 - Q)\frac{e^{-i\omega_1 x_0}}{N_1} + (\gamma_0\omega_2 + \gamma^i p_i + m - g_2\omega_2^2\gamma_0\gamma_5) \\ &\times \frac{1}{2}(\mathbb{1}_4 + Q)\frac{e^{-i\omega_2 x_0}}{N_2}\Big)e^{i\vec{p}\cdot\vec{x}},\end{aligned} \tag{117}$$

and we factorize the global operator

$$\begin{aligned}&\langle 0|\psi_1(x)\bar{\psi}_1(0)|0\rangle \\ &= (i\partial\!\!\!/ + m + g_2\gamma_0\gamma_5\partial_0^2)\int \frac{d^3\vec{p}}{(2\pi)^3}\Big[\frac{1}{2}(\mathbb{1}_4 - Q) \\ &\times \frac{e^{-i\omega_1 x_0}}{N_1} + \frac{1}{2}(\mathbb{1}_4 + Q)\frac{e^{-i\omega_2 x_0}}{N_2}\Big]e^{i\vec{p}\cdot\vec{x}}.\end{aligned} \tag{118}$$

Analogously, for the ghost field we find

$$\begin{aligned}\langle 0|\psi_2(x)\bar{\psi}_2(0)|0\rangle &= -(i\partial\!\!\!/ + m + g_2\gamma_0\gamma_5\partial_0^2) \\ &\times \int \frac{d^3\vec{p}}{(2\pi)^3}\Big[\frac{1}{2}(\mathbb{1}_4 - Q)\frac{e^{-iW_1 x_0}}{\mathcal{N}_1} \\ &+ \frac{1}{2}(\mathbb{1}_4 + Q)\frac{e^{-iW_2 x_0}}{\mathcal{N}_2}\Big]e^{i\vec{p}\cdot\vec{x}},\end{aligned} \tag{119}$$

where a minus sign has appeared due to the ghost oscillators anticommutation relations.

Adding both contributions produces

$$\begin{aligned}S_F^{(>)}(x) &= (i\partial\!\!\!/ + m + g_2\gamma_0\gamma_5\partial_0^2)\int \frac{d^3\vec{p}}{(2\pi)^3}\Big[\frac{1}{2}(\mathbb{1}_4 - Q) \\ &\times \Big[\frac{e^{-i\omega_1 x_0}}{N_1} - \frac{e^{-iW_1 x_0}}{\mathcal{N}_1}\Big] + \frac{1}{2}(\mathbb{1}_4 + Q) \\ &\times \Big[\frac{e^{-i\omega_2 x_0}}{N_2} - \frac{e^{-iW_2 x_0}}{\mathcal{N}_2}\Big]\Big]e^{i\vec{p}\cdot\vec{x}}.\end{aligned} \tag{120}$$

Now we proceed with $x_0 < 0$ and compute

$$S_F(x) = S_F^{(<)}(x) \equiv -\langle 0|\bar{\psi}(0)\psi(x)|0\rangle. \tag{121}$$

After some work similar to the one above, we find

$$\begin{aligned}S_F^{(<)}(x) &= (i\partial\!\!\!/ + m + g_2\gamma_0\gamma_5\partial_0^2)\int \frac{d^3\vec{p}}{(2\pi)^3}\Big[\frac{1}{2}(\mathbb{1}_4 - Q) \\ &\times \Big[\frac{e^{i\omega_1 x_0}}{N_1} - \frac{e^{iW_1 x_0}}{\mathcal{N}_1}\Big] + \frac{1}{2}(\mathbb{1}_4 + Q) \\ &\times \Big[\frac{e^{i\omega_2 x_0}}{N_2} - \frac{e^{iW_2 x_0}}{\mathcal{N}_2}\Big]\Big]e^{i\vec{p}\cdot\vec{x}}.\end{aligned} \tag{122}$$

We are interested in making contact with the four-dimensional representation of the propagator with the pole prescription. Recall the inverse of the operator in the equation of motion (21)

$$M^{-1} = \frac{i\bar{M}N\bar{N}}{g_2^4(p_0^2 - \omega_1^2)(p_0^2 - W_1^2)(p_0^2 - \omega_2^2)(p_0^2 - W_2^2)}, \tag{123}$$

with

$$\begin{aligned}\bar{M}N\bar{N} &= (p\!\!\!/ + m - g_2 p_0^2\gamma_0\gamma_5) \\ &\times (p^2 - m^2 - g_2^2 p_0^4 + 2g_2 p_0^2 p_i\gamma^i\gamma_0\gamma_5).\end{aligned} \tag{124}$$

In order to find the four-dimensional representation of the propagator we need the $i\epsilon$ prescription in the denominator of (123) or the definition of the Feynman contour C_F. We select a prescription for the propagator based on the contour C_F; see Fig. 1.

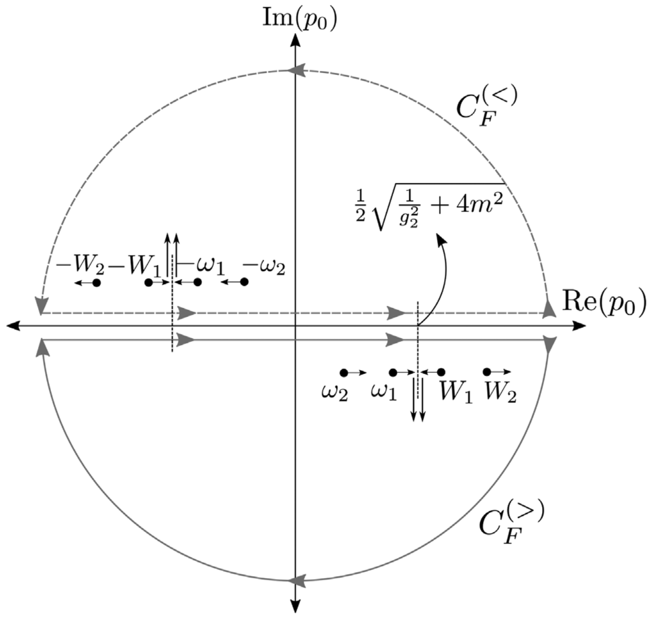

FIG. 1. The contour C_F encloses the poles $\omega_1, \omega_2, W_1, W_2$ in the lower half plane while it encloses the poles $-\omega_1, -\omega_2, -W_1, -W_2$ in the upper half plane. At momentum $|\vec{p}|_{\max} = \frac{1-4g_2^2m^2}{4g_2}$, the two poles ω_1 and W_1 have the same value and from then both move downwards parallel to the imaginary axis as momentum increases. The two poles ω_2 and W_2 go to infinity as the momentum increases, and all the opposite sign poles have a similar behavior.

Hence, let us write the Feynman propagator as

$$S_F(x) = \int_{C_F} \frac{d^4p}{(2\pi)^4} S_F(p) e^{-ip\cdot x}, \tag{125}$$

with

$$S_F(p) = \frac{i\bar{M}N\bar{N}}{\Lambda_+^2(p+i\epsilon)\Lambda_-^2(p+i\epsilon)}, \tag{126}$$

where from the expressions (24), we are defining

$$\begin{aligned}\Lambda_+^2(p+i\epsilon) &= -g_2^2(p_0+\omega_1-i\varepsilon)(p_0-\omega_1+i\varepsilon)\\ &\quad\times(p_0+W_1-i\varepsilon)(p_0-W_1+i\varepsilon),\\ \Lambda_-^2(p+i\epsilon) &= -g_2^2(p_0+\omega_2-i\varepsilon)(p_0-\omega_2+i\varepsilon)\\ &\quad\times(p_0+W_2-i\varepsilon)(p_0-W_2+i\varepsilon).\end{aligned} \tag{127}$$

To compare with the previous calculation, let us consider $x_0 > 0$ and close the contour from below with the curve $C_F^>$; see Fig. 1 to obtain

$$S_F(x) = \int_{C_F^>} \frac{dp_0}{(2\pi)} \int \frac{d^3\vec{p}}{(2\pi)^3} S_F(p) e^{-ip_0x_0+i\vec{p}\cdot\vec{x}}. \tag{128}$$

Integrating in p_0 produces

$$S_F(x) = -\frac{(2\pi i)}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\sum_{i=1}^{4}(\mathrm{Res}(S_F(p)e^{-ip_0x_0},q_i))\times e^{i\vec{p}\cdot\vec{x}}, \tag{129}$$

where the sum runs over the residues at the poles $q_1 = \omega_1, q_2 = \omega_2, q_3 = W_1, q_4 = W_2$, and $i = 1, \ldots, 4$.

The evaluation of the residues are

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},\omega_1) = \frac{i(\bar{M}N\bar{N})_{p_0=\omega_1}}{g_2^2(\omega_1^2-\omega_2^2)(W_2^2-\omega_1^2)} \times\frac{e^{-i\omega_1x_0}}{N_1}, \tag{130}$$

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},\omega_2) = -\frac{i(\bar{M}N\bar{N})_{p_0=\omega_2}}{g_2^2(\omega_1^2-\omega_2^2)(W_1^2-\omega_2^2)} \times\frac{e^{-i\omega_2x_0}}{N_2}, \tag{131}$$

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},W_1) = -\frac{i(\bar{M}N\bar{N})_{p_0=W_1}}{g_2^2(W_1^2-\omega_2^2)(W_2^2-W_1^2)} \times\frac{e^{-iW_1x_0}}{\mathcal{N}_1}, \tag{132}$$

and

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},W_2) = \frac{i(\bar{M}N\bar{N})_{p_0=W_2}}{g_2^2(W_2^2-\omega_1^2)(W_2^2-W_1^2)} \times\frac{e^{-iW_2x_0}}{\mathcal{N}_2}. \tag{133}$$

Considering the identities

$$(\bar{M}N\bar{N})_{p_0=\omega_1} = (4g_2\omega_1^2|\vec{p}|)(\omega_1\gamma_0+p_i\gamma^i+m-g_2\omega_1^2\gamma_0\gamma_5) \times\frac{1}{2}(\mathbb{1}_4-Q), \tag{134}$$

$$(\bar{M}N\bar{N})_{p_0=\omega_2} = (-4g_2\omega_2^2|\vec{p}|)(\omega_2\gamma_0+p_i\gamma^i+m-g_2\omega_2^2\gamma_0\gamma_5) \times\frac{1}{2}(\mathbb{1}_4+Q), \tag{135}$$

$$(\bar{M}N\bar{N})_{p_0=W_1} = (4g_2W_1^2|\vec{p}|)(W_1\gamma_0+p_i\gamma^i+m-g_2W_1^2\gamma_0\gamma_5) \times\frac{1}{2}(\mathbb{1}_4-Q), \tag{136}$$

$$(\bar{M}N\bar{N})_{p_0=W_2} = (-4g_2W_2^2|\vec{p}|)(W_2\gamma_0+p_i\gamma^i+m-g_2W_2^2\gamma_0\gamma_5) \times\frac{1}{2}(\mathbb{1}_4-Q), \tag{137}$$

and using the identities

$$g_2^2(\omega_1^2-\omega_2^2)(W_2^2-\omega_1^2)=4g_2\omega_1^2|\vec{p}|,$$
$$g_2^2(\omega_1^2-\omega_2^2)(W_1^2-\omega_2^2)=4g_2\omega_2^2|\vec{p}|,$$
$$g_2^2(W_1^2-\omega_2^2)(W_2^2-W_1^2)=4g_2W_1^2|\vec{p}|,$$
$$g_2^2(W_2^2-\omega_1^2)(W_2^2-W_1^2)=4g_2W_2^2|\vec{p}|, \tag{138}$$

we can verify

$$\begin{aligned}S_F(x)=\int\frac{d^3\vec{p}}{(2\pi)^3}\bigg[&(\omega_1\gamma_0+p_i\gamma^i+m-g_2\omega_1^2\gamma_0\gamma_5)\\ &\times\frac{1}{2}(\mathbb{1}_4-Q)\frac{e^{-i\omega_1x_0}}{N_1}+(\omega_2\gamma_0+p_i\gamma^i+m-g_2\omega_2^2\gamma_0\gamma_5)\\ &\times\frac{1}{2}(\mathbb{1}_4+Q)\frac{e^{-i\omega_2x_0}}{N_2}-(W_1\gamma_0+p_i\gamma^i+m-g_2W_1^2\gamma_0\gamma_5)\\ &\times\frac{1}{2}(\mathbb{1}_4-Q)\frac{e^{-iW_1x_0}}{\mathcal{N}_1}-(W_2\gamma_0+p_i\gamma^i+m-g_2W_2^2\gamma_0\gamma_5)\\ &\times\frac{1}{2}(\mathbb{1}_4+Q)\frac{e^{-iW_2x_0}}{\mathcal{N}_2}\bigg].\end{aligned}\tag{139}$$

Factorizing a global operator we arrive at

$$\begin{aligned}S_F(x)=&\,(i\partial\!\!\!/+m+g_2\gamma_0\gamma_5\partial_0^2)\\ &\times\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{1}{2}(\mathbb{1}_4-Q)\left[\frac{e^{-i\omega_1x_0}}{N_1}-\frac{e^{-iW_1x_0}}{\mathcal{N}_1}\right]\right.\\ &\left.+\frac{1}{2}(\mathbb{1}_4+Q)\left[\frac{e^{-i\omega_2x_0}}{N_2}-\frac{e^{-iW_2x_0}}{\mathcal{N}_2}\right]\right]e^{i\vec{p}\cdot\vec{x}}.\end{aligned}\tag{140}$$

By comparing we arrive at the same result as the one obtained from the definition Eq. (120).

Now we consider $x_0<0$, and we close the contour in the upper half plane

$$\begin{aligned}S_F(x)&=\int_{C_F^<}\frac{dp_0}{(2\pi)}\int\frac{d^3\vec{p}}{(2\pi)^3}S_F(p)e^{-ip_0x_0+i\vec{p}\cdot\vec{x}}\\ &=\frac{(2\pi i)}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\sum_{i=5}^{8}(\mathrm{Res}(S_F(p)e^{-ip_0x_0},q_i))e^{i\vec{p}\cdot\vec{x}},\end{aligned}\tag{141}$$

where now $q_5=-\omega_1, q_6=-\omega_2, q_7=-W_1, q_8=-W_2$, and $i=5,\ldots,8$.

We have

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},-\omega_1)=-\frac{i(\bar{M}N\bar{N})_{p_0=\omega_1}}{g_2^2(\omega_1^2-\omega_2^2)(W_2^2-\omega_1^2)}\times\frac{e^{i\omega_1x_0}}{N_1},\tag{142}$$

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},-\omega_2)=\frac{i(\bar{M}N\bar{N})_{p_0=\omega_2}}{g_2^2(\omega_1^2-\omega_2^2)(W_1^2-\omega_2^2)}\times\frac{e^{i\omega_2x_0}}{N_2},\tag{143}$$

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},-W_1)=\frac{i(\bar{M}N\bar{N})_{p_0=W_1}}{g_2^2(W_1^2-\omega_2^2)(W_2^2-W_1^2)}\times\frac{e^{iW_1x_0}}{\mathcal{N}_1},\tag{144}$$

$$\mathrm{Res}(S_F(p)e^{-ip_0x_0},-W_2)=-\frac{i(\bar{M}N\bar{N})_{p_0=W_2}}{g_2^2(W_2^2-\omega_1^2)(W_2^2-W_1^2)}\times\frac{e^{iW_2x_0}}{\mathcal{N}_2}.\tag{145}$$

Consider

$$\begin{aligned}&(\bar{M}N\bar{N})_{p_0=-\omega_1}\\ &=(4g_2\omega_1^2|\vec{p}|)(-\omega_1\gamma_0+p_i\gamma^i+m-g_2\omega_1^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4-Q),\end{aligned}\tag{146}$$

$$\begin{aligned}&(\bar{M}N\bar{N})_{p_0=-\omega_2}\\ &=(-4g_2\omega_2^2|\vec{p}|)(-\omega_2\gamma_0+p_i\gamma^i+m-g_2\omega_2^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4+Q),\end{aligned}\tag{147}$$

$$\begin{aligned}&(\bar{M}N\bar{N})_{p_0=-W_1}\\ &=(4g_2W_1^2|\vec{p}|)(-W_1\gamma_0+p_i\gamma^i+m-g_2W_1^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4-Q),\end{aligned}\tag{148}$$

$$\begin{aligned}&(\bar{M}N\bar{N})_{p_0=-W_2}\\ &=(-4g_2W_2^2|\vec{p}|)(-W_2\gamma_0+p_i\gamma^i+m-g_2W_2^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4+Q).\end{aligned}\tag{149}$$

We finally verify that

$$S_F(x)=\int\frac{d^3\vec{p}}{(2\pi)^3}\bigg[(-\omega_1\gamma_0+p_i\gamma^i+m-g_2\omega_1^2\gamma_0\gamma_5)\times\frac{1}{2}(\mathbb{1}_4-Q)\frac{e^{i\omega_1x_0}}{N_1}+(-\omega_2\gamma_0+p_i\gamma^i+m-g_2\omega_2^2\gamma_0\gamma_5)\times\frac{1}{2}(\mathbb{1}_4+Q)\frac{e^{i\omega_2x_0}}{N_2}-(-W_1\gamma_0+p_i\gamma^i+m-g_2W_1^2\gamma_0\gamma_5)\times\frac{1}{2}(\mathbb{1}_4-Q)\frac{e^{iW_1x_0}}{\mathcal{N}_1}-(-W_2\gamma_0+p_i\gamma^i+m-g_2W_2^2\gamma_0\gamma_5)\times\frac{1}{2}(\mathbb{1}_4+Q)\frac{e^{iW_2x_0}}{\mathcal{N}_2}\bigg]. \tag{150}$$

Again factorizing global operators, we arrive at

$$S_F(x)=(i\partial\!\!\!/+m+g_2\gamma_0\gamma_5\partial_0^2)\int\frac{d^3\vec{p}}{(2\pi)^3}\bigg[\frac{1}{2}(\mathbb{1}_4-Q)\times\bigg[\frac{e^{i\omega_1x_0}}{N_1}-\frac{e^{iW_1x_0}}{\mathcal{N}_1}\bigg]+\frac{1}{2}(\mathbb{1}_4+Q)\times\bigg[\frac{e^{i\omega_2x_0}}{N_2}-\frac{e^{iW_2x_0}}{\mathcal{N}_2}\bigg]\bigg]e^{i\vec{p}\cdot\vec{x}}, \tag{151}$$

which is the same as obtained in (122) with the definition.

IV. MICROCAUSALITY

In quantum mechanics the property of causality means that local observables commute at causally disconnected regions. In relativistic field theory this assumption called microcausality is translated into the condition

$$[O(x),O(x')]=0,\quad \text{for } (x-x')^2<0. \tag{152}$$

For a fermion theory, since observables are constructed from bilinear forms, it is enough to impose

$$iS(x-x')=\{\psi(x),\bar{\psi}(x')\},\quad \text{for } (x-x')^2<0. \tag{153}$$

In the model we are studying we can identify two sources of possible microcausality violations. The first one is related to the breaking of Lorentz symmetry where the notion of light cone loses some of its properties due to superluminal propagation. The second one involves an indefinite metric leading to acausal propagation that has been extensively discussed in the literature by Lee and Wick and also in posterior works.

We begin the study of microcausality by considering the decomposition (51), and we obtain

$$\{\psi(x),\bar{\psi}(x')\}=\{\psi_1(x),\bar{\psi}_1(x')\}+\{\psi_2(x),\bar{\psi}_2(x')\}. \tag{154}$$

We compute first

$$\{\psi_1(x),\bar{\psi}_1(x')\}=\sum_{r,s=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{N_r\bar{N}_s}}\{a_p^r u^r(p)e^{-i\omega_r x_0+i\vec{p}\cdot\vec{x}}+b_p^{r\dagger}v^r(p)e^{i\omega_r x_0-i\vec{p}\cdot\vec{x}},a_k^{s\dagger}u^{s\dagger}(k)\times\gamma_0 e^{i\bar{\omega}_s x_0'-i\vec{k}\cdot\vec{x}'}+b_k^s v^{s\dagger}(k)\gamma_0 e^{-i\bar{\omega}_s x_0'+i\vec{k}\cdot\vec{x}'}\}. \tag{155}$$

We use the algebra (57) and the outer relations in (A50) and (A51) to arrive at

$$\{\psi_1(x),\bar{\psi}_1(x')\}=\int\frac{d^3\vec{p}}{(2\pi)^3}\bigg[\frac{1}{N_1}\bigg((\gamma_0\omega_1+\gamma^ip_i+m-g_2\omega_1^2\gamma_0\gamma_5)\gamma_0\frac{1}{2}(\mathbb{1}_4-Q)\gamma_0e^{-i\omega_1(x_0-x_0')}+(\gamma_0\omega_1-\gamma^ip_i-m+g_2\omega_1^2\gamma_0\gamma_5)\gamma_0\frac{1}{2}(\mathbb{1}_4-Q)\gamma_0e^{i\omega_1(x_0-x_0')}\bigg)+\frac{1}{N_2}\bigg((\gamma_0\omega_2+\gamma^ip_i+m-g_2\omega_2^2\gamma_0\gamma_5)\gamma_0\frac{1}{2}(\mathbb{1}_4+Q)\gamma_0e^{-i\omega_2(x_0-x_0')}+(\gamma_0\omega_2-\gamma^ip_i-m+g_2\omega_2^2\gamma_0\gamma_5)\gamma_0\frac{1}{2}(\mathbb{1}_4+Q)\gamma_0e^{i\omega_2(x_0-x_0')}\bigg)\bigg]e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}. \tag{156}$$

Taking $x'=0$ we get

$$\begin{aligned}\{\psi_1(x),\bar{\psi}_1(0)\} = \int \frac{d^3\vec{p}}{(2\pi)^3}\bigg[\frac{1}{N_1}\bigg(&(\gamma_0\omega_1+\gamma^i p_i+m-g_2\omega_1^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4-Q)e^{-i\omega_1 x_0}\\ &+(\gamma_0\omega_1-\gamma^i p_i-m+g_2\omega_1^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4-Q)e^{i\omega_1 x_0}\bigg)\\ &+\frac{1}{N_2}\bigg((\gamma_0\omega_2+\gamma^i p_i+m-g_2\omega_2^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4+Q)e^{-i\omega_2 x_0}\\ &+(\gamma_0\omega_2-\gamma^i p_i-m+g_2\omega_2^2\gamma_0\gamma_5)\frac{1}{2}(\mathbb{1}_4+Q)e^{i\omega_2 x_0}\bigg)\bigg]e^{i\vec{p}\cdot\vec{x}},\end{aligned} \tag{157}$$

and hence

$$\{\psi_1(x),\bar{\psi}_1(0)\} = (i\partial\!\!\!/+m+g_2\partial_0^2\gamma_0\gamma_5)\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{1}{N_1}(e^{-i\omega_1x_0}-e^{i\omega_1x_0})\frac{1}{2}(\mathbb{1}_4-Q)+\frac{1}{N_2}(e^{-i\omega_2x_0}-e^{i\omega_2x_0})\frac{1}{2}(\mathbb{1}_4+Q)\right]e^{i\vec{p}\cdot\vec{x}}. \tag{158}$$

Similar calculations lead to

$$\begin{aligned}\{\psi_2(x),\bar{\psi}_2(0)\} = (-1)(i\partial\!\!\!/+m+g_2\partial_0^2\gamma_0\gamma_5)\int\frac{d^3\vec{p}}{(2\pi)^3}\bigg[&\frac{1}{\mathcal{N}_1}(e^{-iW_1x_0}-e^{iW_1x_0})\frac{1}{2}(\mathbb{1}_4-Q)\\ &+\frac{1}{\mathcal{N}_2}(e^{-iW_2x_0}-e^{iW_2x_0})\frac{1}{2}(\mathbb{1}_4+Q)\bigg]e^{i\vec{p}\cdot\vec{x}}.\end{aligned} \tag{159}$$

We have the four-dimensional representation of the anticommutator $\{\psi(x),\bar{\psi}(x')\}$ by using the curve C which encloses the eight poles. From 1, where $C=C_F^{<}-C_F^{>}$, we can write

$$S(x)=\hat{\hat{M}}\,\hat{N}\,\hat{\hat{N}}\int_C\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot x}}{\Lambda_+^2(p+i\epsilon)\Lambda_-^2(p+i\epsilon)}, \tag{160}$$

where

$$\begin{aligned}\hat{\hat{M}} &= i\partial\!\!\!/+m+g_2\partial_0^2\gamma_0\gamma_5,\\ \hat{N} &= i\partial\!\!\!/+m-g_2\partial_0^2\gamma_0\gamma_5,\\ \hat{\hat{N}} &= i\partial\!\!\!/-m-g_2\partial_0^2\gamma_0\gamma_5.\end{aligned} \tag{161}$$

We can always perform an observer transformation when both points are spacelike separated, leaving us with $x=(0,\vec{x})$. In this way we can integrate and obtain an integral proportional to

$$\begin{aligned}&\int_C\frac{dp_0}{(p_0^2-\omega_1^2)(p_0^2-\omega_2^2)(p_0^2-W_1^2)(p_0^2-W_2^2)}\\ &=2\pi i\bigg[\frac{1}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1^2-W_1^2)(\omega_1^2-W_2^2)}-\frac{1}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1^2-W_1^2)(\omega_1^2-W_2^2)}+\frac{1}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)}\\ &\quad-\frac{1}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)}+\frac{1}{2W_1(W_1^2-\omega_1^2)(W_1^2-\omega_2^2)(W_1^2-W_2^2)}-\frac{1}{2W_1(W_1^2-\omega_1^2)(W_1^2-\omega_2^2)(W_1^2-W_2^2)}\\ &\quad+\frac{1}{2W_2(W_2^2-\omega_1^2)(W_2^2-\omega_2^2)(W_2^2-W_1^2)}-\frac{1}{2W_2(W_2^2-\omega_1^2)(W_2^2-\omega_2^2)(W_2^2-W_1^2)}\bigg]\\ &=0.\end{aligned} \tag{162}$$

The combination is always zero even when the poles ω_1 and W_1 become complex as can be seen in Fig. 1. and therefore microcausality is preserved.

V. TREE-LEVEL UNITARITY

Recapitulating, we have found $\eta_{2,s}$ the metric associated with the indefinite Fock space which is not positive defined and will produce negative-norm states for an odd occupation number of particles. Generally, an indefinite metric η can lead to a pseudo-unitary relation for the S-matrix

$$S^{\dagger}\eta S = \eta, \tag{163}$$

which is not satisfactory to describe probability amplitudes. However, as was shown by Lee and Wick an indefinite-metric theory can have a chance to develop a fully unitary S-matrix. In particular, they showed that by restricting the asymptotic space to contain only particles with positive metric, it is possible to have a unitary condition for the S-matrix [29,30].

To study unitarity at tree level we will use the tool of the optical theorem and adopt the Lee-Wick prescription. The optical theorem provides an important constraint equation to test perturbative unitarity based on individual diagrams, which is well suited for our analysis. Moreover, adopting the Lee-Wick prescription in our model means that ghost states are unstable, and so they will not appear in external legs in any Feynman diagram. However, internal fermion lines propagating ghost modes are perfectly acceptable, leading to possible violations of unitarity. Therefore to test these possible sources of unitarity violation, we focus our analysis on the class of diagrams describing $2 \to 2$ processes at tree level with an internal fermion line.

Recall, the optical theorem has a simple expression

$$2\mathrm{Im}(M_{ii}) = \sum_m \int d\Pi_m |M_{im}|^2, \tag{164}$$

where M_{ii} is the amplitude for a forward scattering process. The sum runs over all possible intermediate states, and the integral over the phase space $d\Pi_m$ is restricted by momentum conservation.

We study the process of Compton scattering of electrons and positrons. We consider the incoming fermion or antifermion of spin r to have momentum p and the photon to have momentum k. The final states are other photon-electron or positron-electron pairs, as shown in Fig. 2.

We begin with the process involving the electron and denote the process by $e^-(p)\gamma(k) \to e^-(p)\gamma(k)$. According to the standard Feynman rules the matrix element $\mathcal{M} \equiv \mathcal{M}(e^-\gamma \to e^-\gamma)$ can be written as

$$\mathcal{M} = (-ie)^2 \int \frac{d^4p'}{(2\pi)^4} \times (2\pi)^4 \delta^{(4)}(p+k-p') \times \bar{U}^{r,\lambda}(p,k) S_{\mathrm{F}}(p') U^{r,\lambda}(p,k), \tag{165}$$

where

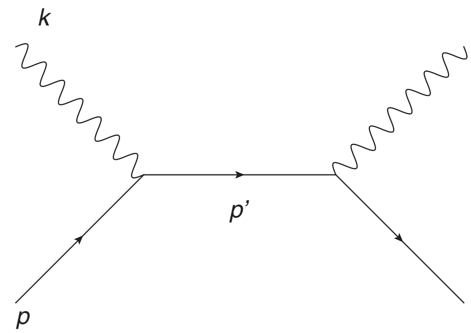

FIG. 2. The Compton scattering diagram in the analysis of tree-level order unitarity.

$$\bar{U}^{r,\lambda}(p,k) = N_p^r N_k \bar{u}^r(p) \varepsilon_\mu^{*(\lambda)}(k) \gamma^\mu, \tag{166a}$$

$$U^{r,\lambda}(p,k) = N_p^r N_k \gamma^\mu u^r(p) \varepsilon_\mu^{(\lambda)}(k), \tag{166b}$$

and $N_k = \sqrt{\frac{1}{2\omega_k}}$, with $\omega_k = |\vec{k}|$ as the usual photon normalization, $N_p^r = \sqrt{\frac{1}{N_r}}$ are the normalization constants of Eqs. (53), and the modified fermion propagator S_F is given in Eq. (126).

To compute the imaginary part we consider the decomposition in the propagator

$$\frac{1}{(p_0' - \Omega + i\epsilon)(p_0' + \Omega - i\epsilon)} = \frac{1}{2\Omega}\left[\frac{1}{(p_0' - \Omega + i\epsilon)} + \frac{1}{(p_0' + \Omega - i\epsilon)}\right], \tag{167}$$

and use the identity

$$\frac{1}{p_0' - \Omega + i\epsilon} = P\frac{1}{p_0' - \Omega} - i\pi\delta(p_0' - \Omega), \tag{168}$$

where $\mathcal{P}$ is the principal value.

Now, focusing on (165), we obtain

$$2\mathrm{Im}(M) = (2\pi)e^2 \int \frac{d^3\vec{p}'}{2N_r\omega_k} \delta^{(4)}(p+k-p') \times \bar{u}^r(p)\varepsilon_\mu^{*(\lambda)}(k)\gamma^\mu \times \sum_{s=1,2}\left(\frac{\bar{M}'N'\bar{N}'}{2\omega_s' g_2^4(\omega_s'^2 - \omega_2'^2)(\omega_s'^2 - W_1'^2)(\omega_s'^2 - W_2'^2)}\right)_{p_0'=\omega_s'} \times \gamma^\mu u^r(p)\varepsilon_\mu^{(\lambda)}(k), \tag{169}$$

where the prime reminds us that it is evaluated in $p_s' = (\omega_s(\vec{p}'), \vec{p}')$. Note that the ghost states do not appear in the sum since by momentum conservation their contribution vanishes when going on-shell.

Now, we will relate the amplitude with the total cross section σ of the process $e^-\gamma \to e^-$. We denote the total cross section by $\hat{\mathcal{M}} \equiv \mathcal{M}(e^-\gamma \to e^-)$ and write

$$\sigma = \sum_{s=1,2} \int \frac{d^3\vec{p}'}{(2\pi)^3} \times (2\pi)^4 \delta^{(4)}(p+k-p')|\hat{\mathcal{M}}_s|^2, \tag{170}$$

with

$$\hat{\mathcal{M}}_s = ie\frac{1}{\sqrt{N'_s}}\frac{1}{\sqrt{N_r}}\frac{1}{\sqrt{2\omega_k}}\bar{u}^s(p')\gamma^\nu u^r(p)\varepsilon_\nu^{(\lambda)}(k). \tag{171}$$

The integral in phase space selects only particles which have the chance to satisfy momentum conservation. We arrive at

$$\begin{aligned}\sigma = (2\pi)\sum_{s=1,2}\int \frac{d^3\vec{p}'}{2N_r\omega_k}\delta^{(4)}(p+k-p') \\ \times \left(ie\frac{1}{\sqrt{N'_s}}\bar{u}^s(p')\gamma^\nu u^r(p)\varepsilon_\nu^{(\lambda)}(k)\right)^\dagger \\ \times \left(ie\frac{1}{\sqrt{N'_s}}\bar{u}^s(p')\gamma^\nu u^r(p)\varepsilon_\nu^{(\lambda)}(k)\right),\end{aligned} \tag{172}$$

and then

$$\begin{aligned}\sigma = (2\pi)e^2\int \frac{d^3\vec{p}'}{2N_r\omega_k}\delta^{(4)}(p+k-p')\bar{u}^r(p)\gamma^\nu\varepsilon_\nu^{*(\lambda)}(k) \\ \times \left[\sum_{s=1,2}\frac{u^s(p')\bar{u}^s(p')}{N'_s}\right]\gamma^\mu u^r(p)\varepsilon_\mu^{(\lambda)}(k).\end{aligned} \tag{173}$$

To connect with the left-hand side, consider the relations

$$u^{(1)}(p)\bar{u}^{(1)}(p) = \left(\frac{\bar{M}N\bar{N}}{2(p^2-m^2-g_2^2p_0^4)}\right)_{p_0=\omega_1}, \tag{174a}$$

$$u^{(2)}(p)\bar{u}^{(2)}(p) = \left(\frac{\bar{M}N\bar{N}}{2(p^2-m^2-g_2^2p_0^4)}\right)_{p_0=\omega_2}, \tag{174b}$$

and the identities

$$2(p^2-m^2-g_2^2p_0^4)_{p_0=\omega_1} = -g_2^2(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2), \tag{175a}$$

$$2(p^2-m^2-g_2^2p_0^4)_{p_0=\omega_2} = -g_2^2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2). \tag{175b}$$

Hence we can write

$$\begin{aligned}\frac{u^{(1)}(p')\bar{u}^{(1)}(p')}{N'_1} \\ = \left(\frac{\bar{M}'N'\bar{N}'}{2\omega'_1g_2^4(\omega_1'^2-\omega_2'^2)(\omega_1'^2-W_1'^2)(\omega_1'^2-W_2'^2)}\right)_{p'_0=\omega'_1}\end{aligned} \tag{176a}$$

and

$$\begin{aligned}\frac{u^{(2)}(p')\bar{u}^{(2)}(p')}{N'_2} \\ = \left(\frac{\bar{M}'N'\bar{N}'}{2\omega'_2g_2^4(\omega_2'^2-\omega_1'^2)(\omega_2'^2-W_1'^2)(\omega_2'^2-W_2'^2)}\right)_{p'_0=\omega'_2},\end{aligned} \tag{176b}$$

Finally, we have

$$\begin{aligned}\sum_{s=1,2}\frac{u^s(p')\bar{u}^s(p')}{N_s} \\ = \sum_{s=1,2}\left(\frac{\bar{M}'N'\bar{N}'}{2\omega'_sg_2^4(\omega_s'^2-\omega_2'^2)(\omega_s'^2-W_1'^2)(\omega_s'^2-W_2'^2)}\right)_{p'_0=\omega'_s}.\end{aligned} \tag{177}$$

In this way we have proven the identity and thereby the validity of the optical theorem showing that unitarity is preserved for these processes at tree level. The Compton scattering of a positron follows by similar arguments.

VI. FINAL REMARKS

We have studied a modified QED model containing Lorentz-violating dimension-five operators of Myers-Pospelov type in the fermion sector. The effective model, also a subset of the nonminimal SME framework, introduces Lorentz violation through a four-vector n. We have set n to be purely timelike with a resulting Lagrangian coupling the effective terms to higher-order time derivatives. We have quantized the nonminimal Lorentz-violating model and distinguished at each step in the calculations between the corrected particle fields versus the new degrees of freedom that enter through the higher-order operators. We have identified the positive and negative metrics that characterize the indefinite Fock space and found that ghost states with odd occupation numbers have a negative norm.

The charge conjugation even sector of higher-order modified fermions has been less explored than the charge conjugation odd sector, making it an excellent arena to explore kinematic modifications. In particular, we have found that the theory doubles the usual number of spinors and energy solutions of the dispersion relation concerning the standard theory. We have found that the Hamiltonian is stable and Hermitian in the effective region, although it can develop complex eigenvalues for higher energies and lose its Hermitian property.

The new pole structure is essential to construct the propagator and fix the prescription for the curve C_F in the p_0-complex plane. We have seen that the poles related to negative energies ω_2, W_2 remain in the real axis while the poles ω_1, W_1 can move vertically in the imaginary axis for energies above $|p_{\max}| = \frac{1-4g_2^2m^2}{g_2}$. We have studied microcausality by focusing on an anticommutator between fields. We have found that microcausality can be preserved by considering the pole structure and its evolution properties in the complex p_0-plane. We have considered the forward scattering process involving fermion (antifermion) and photon pairs with an internal fermion line to study unitarity. We have found that unitarity is preserved at tree level by applying the Lee-Wick prescription and using the optical theorem to test perturbative unitarity.

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APPENDIX: MODIFIED KINEMATICS

Here we derive the spinor solutions of the equations of motion (34) and (40). We give various types of orthogonality and outer product relations satisfied by the spinors.

1. Spinor solutions

We start with the set of equations (37b) and multiply the second equation by $p_0 - g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma})$ to obtain

$$m^2\chi_1 = (p_0 - g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma})) \times (p_0 + g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}))\chi_1. \tag{A1}$$

To solve this equation we introduce the two bi-spinors $\xi^{(\pm)}(\vec{p})$, given by

$$\xi^{(+)}(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p^3)}}\begin{pmatrix} |\vec{p}|+p^3 \\ p^1+ip^2 \end{pmatrix}, \tag{A2}$$

$$\xi^{(-)}(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|-p^3)}}\begin{pmatrix} p^1-ip^2 \\ |\vec{p}|-p^3 \end{pmatrix}, \tag{A3}$$

which satisfy the properties

$$(\vec{p}\cdot\vec{\sigma})\xi^{(\pm)}(\vec{p}) = |\vec{p}|\xi^{(\pm)}(\vec{p}), \tag{A4}$$

$$(\vec{p}\cdot\vec{\sigma})\xi^{(\pm)}(-\vec{p}) = -|\vec{p}|\xi^{(\pm)}(-\vec{p}), \tag{A5}$$

and the orthogonality relations

$$\xi^{(+)\dagger}(\vec{p})\xi^{(+)}(\vec{p}) = \xi^{(-)\dagger}(\vec{p})\xi^{(-)}(\vec{p}) = 1, \tag{A6}$$

$$\xi^{(+)\dagger}(\vec{p})\xi^{(-)}(-\vec{p}) = \xi^{(-)\dagger}(-\vec{p})\xi^{(+)}(\vec{p}) = 0. \tag{A7}$$

In addition, we list the relations

$$\xi^{(+)}(\vec{p})\xi^{(+)\dagger}(\vec{p}) = \xi^{(-)}(\vec{p})\xi^{(-)\dagger}(\vec{p}) = \frac{1}{2}\left(1+\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right), \tag{A8}$$

$$\xi^{(+)}(-\vec{p})\xi^{(+)\dagger}(-\vec{p}) = \xi^{(-)}(-\vec{p})\xi^{(-)\dagger}(-\vec{p}) = \frac{1}{2}\left(1-\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right). \tag{A9}$$

Returning to our derivation, we select $\chi_1^{(+)}(\vec{p}) = A_1\xi^{(+)}(\vec{p})$ in Eq. (A1), and using the property (A4), it can be shown that the bi-spinor solves the equation of motion given that its momentum satisfies the dispersion relation $\Lambda_+^2(p) = 0$.

According to (37b), we have $\chi_2^{(+)}(\vec{p}) = \frac{A_1}{m}(p_0 + g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma}))\xi^{(+)}(\vec{p})$ which produces the two energy-dependent solutions

$$u^{(1)}(p) = A_1\begin{pmatrix} \xi^{(+)}(\vec{p}) \\ (\frac{p_0+g_2p_0^2+\vec{p}\cdot\vec{\sigma}}{m})\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_1} \tag{A10}$$

and

$$U^{(1)}(p) = \mathcal{A}_1\begin{pmatrix} \xi^{(+)}(\vec{p}) \\ (\frac{p_0+g_2p_0^2+\vec{p}\cdot\vec{\sigma}}{m})\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_1}. \tag{A11}$$

In a similar fashion, let us choose a different bi-spinor $\chi_1^{(-)}(\vec{p}) = A_2\xi^{(-)}(-\vec{p})$ with its momentum satisfying the dispersion relation $\Lambda_-^2(p) = 0$. The bi-spinor produces the two solutions

$$u^{(2)}(p) = A_2\begin{pmatrix} \xi^{(-)}(-\vec{p}) \\ (\frac{p_0+g_2p_0^2+\vec{p}\cdot\vec{\sigma}}{m})\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_2} \tag{A12}$$

and

$$U^{(2)}(p) = \mathcal{A}_2\begin{pmatrix} \xi^{(-)}(-\vec{p}) \\ (\frac{p_0+g_2p_0^2+\vec{p}\cdot\vec{\sigma}}{m})\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_2}. \tag{A13}$$

For positive-energy spinors associated with particle and ghost modes we choose the normalization constants as

$$A_1 = \mathcal{A}_1 = \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}, \tag{A14}$$

$$A_2 = \mathcal{A}_2 = \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}. \tag{A15}$$

In this way we obtain the spinors given in (38) and (39).

Now we search for negative-energy solutions which satisfy the equation of motion (40). We multiply the first equation in (43a) by $p_0 - g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma})$ and obtain

$$m^2\phi_2 = (p_0 - g_2 p_0^2 + (\vec{p}\cdot\vec{\sigma})) \times (p_0 + g_2 p_0^2 - (\vec{p}\cdot\vec{\sigma}))\phi_2. \tag{A16}$$

The equation can be satisfied by choosing $\phi_2(\vec{p}) = B_1\xi^{(-)}(-\vec{p})$ with on-shell momentum satisfying $\Lambda_+^2 = 0$. In an analogous form we have

$$v^{(1)}(p) = B_1\begin{pmatrix} -(\frac{p_0+g_2p_0^2-\vec{p}\cdot\vec{\sigma}}{m})\xi^{(-)}(-\vec{p}) \\ \xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_1} \tag{A17}$$

and

$$V^{(1)}(p) = \mathcal{B}_1\begin{pmatrix} -(\frac{p_0+g_2p_0^2-\vec{p}\cdot\vec{\sigma}}{m})\xi^{(-)}(-\vec{p}) \\ \xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_1}. \tag{A18}$$

Now, we choose $\phi_2(\vec{p}) = B_2\xi^{(+)}(\vec{p})$ in (A16), with momentum solving $\Lambda_-^2 = 0$, which produces the two spinor solutions

$$v^{(2)}(p) = B_2\begin{pmatrix} -(\frac{p_0+g_2p_0^2-\vec{p}\cdot\vec{\sigma}}{m})\xi^{(+)}(\vec{p}) \\ \xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_2} \tag{A19}$$

and

$$V^{(2)}(p) = \mathcal{B}_2\begin{pmatrix} -(\frac{p_0+g_2p_0^2-\vec{p}\cdot\vec{\sigma}}{m})\xi^{(+)}(\vec{p}) \\ \xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_2}. \tag{A20}$$

For this set of negative-energy spinors, we choose the normalization constants to be

$$B_1 = \mathcal{B}_1 = -\sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}, \tag{A21}$$

$$B_2 = \mathcal{B}_2 = -\sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}, \tag{A22}$$

and we obtain the solutions (A25) and (A26).

2. Inner product relations

For the many expressions it is convenient to introduce the notation for the positive-energy spinors as

$$u^{(1)}(p) = \begin{pmatrix} A\xi^{(+)}(\vec{p}) \\ B\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_1}, \quad U^{(1)}(p) = \begin{pmatrix} A\xi^{(+)}(\vec{p}) \\ B\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_1}; \tag{A23}$$

$$u^{(2)}(p) = \begin{pmatrix} C\xi^{(-)}(-\vec{p}) \\ D\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_2}, \quad U^{(2)}(p) = \begin{pmatrix} C\xi^{(-)}(-\vec{p}) \\ D\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_2}; \tag{A24}$$

and also the negative-energy spinors

$$v^{(1)}(p) = \begin{pmatrix} B\xi^{(-)}(-\vec{p}) \\ -A\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_1}, \quad V^{(1)}(p) = \begin{pmatrix} B\xi^{(-)}(-\vec{p}) \\ -A\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_1}; \tag{A25}$$

$$v^{(2)}(p) = \begin{pmatrix} D\xi^{(+)}(\vec{p}) \\ -C\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_2}, \quad V^{(2)}(p) = \begin{pmatrix} D\xi^{(+)}(\vec{p}) \\ -C\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_2}; \tag{A26}$$

with

$$A = \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}, \tag{A27}$$

$$B = \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}, \tag{A28}$$

$$C = \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}, \tag{A29}$$

$$D = \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}. \tag{A30}$$

In particular, with the property (A6) we find

$$u^{(1)}(p)u^{(1)\dagger}(p) = (A^2 + B^2)_{p_0=\omega_1}, \tag{A31}$$

resulting in

$$u^{(1)\dagger}(p)u^{(1)}(p) = 2\omega_1. \tag{A32}$$

The same occurs for $U^{(1)}(p)$ leading to the expressions in (46a) and (47a).

Now consider

$$\bar{u}^{(1)}(p)u^{(1)}(p) = 2(AB)_{p_0=\omega_1} = 2m, \quad \text{(A33)}$$

$$\bar{v}^{(1)}(p)v^{(1)}(p) = -2(AB)_{p_0=\omega_1} = -2m, \quad \text{(A34)}$$

and again we get the relations listed in (48a) and (49a).

Let us define the operators

$$q_{rs}^{(+)}(p) = \mathbb{1}_4 - g_2(\omega_r + \omega_s)\gamma_5, \quad \text{(A35)}$$

$$q_{rs}^{(-)}(p) = \mathbb{1}_4 + g_2(\omega_r + \omega_s)\gamma_5, \quad \text{(A36)}$$

and

$$Q_{rs}^{(+)}(p) = \mathbb{1}_4 - g_2(W_r + W_s)\gamma_5, \quad \text{(A37)}$$

$$Q_{rs}^{(-)}(p) = \mathbb{1}_4 + g_2(W_r + W_s)\gamma_5, \quad \text{(A38)}$$

where $\mathbb{1}_4$ is the unit 4×4 matrix and $r, s = 1, 2$.

To prove the next relations we follow a trick. Consider the element

$$u^{r\dagger}(p)\gamma_0(\gamma^i p_i - m)u^s(p), \quad \text{(A39)}$$

which can be written using the equations of motion as

$$u^{r\dagger}(p)(-\omega_s + g_2\gamma_5(\omega_s)^2)u^s(p) \quad \text{(A40)}$$

or

$$u^{r\dagger}(p)(-\omega_r + g_2\gamma_5(\omega_r)^2)u^s(p), \quad \text{(A41)}$$

we arrive at

$$u^{r\dagger}(p)((\omega_s - \omega_r) - g_2\gamma_5((\omega_s)^2 - (\omega_r)^2))u^s(p) = 0, \quad \text{(A42)}$$

and in the case $\omega_r \neq \omega_s$, we have

$$u^{r\dagger}(p)q_{rs}^{(+)}u^s(p) = 0. \quad \text{(A43)}$$

We can write

$$u^{r\dagger}(p)q_{rs}^{(+)}u^s(p) = C_r\delta^{rs}, \quad \text{(A44)}$$

where C_r is a constant that has to be determined. Doing the same with all other contributions, and computing directly for the same energies, i.e., $\omega_r = \omega_s$, we find for particle spinors

$$\begin{aligned} u^{(1)\dagger}(p)q_{11}^{(+)}u^{(1)}(p) &= N_1, \\ u^{(2)\dagger}(p)q_{22}^{(+)}u^{(2)}(p) &= N_2, \\ v^{(1)\dagger}(p)q_{11}^{(-)}v^{(1)}(p) &= N_1, \\ v^{(2)\dagger}(p)q_{22}^{(-)}v^{(2)}(p) &= N_2, \end{aligned} \quad \text{(A45)}$$

and for ghost spinors

$$\begin{aligned} U^{(1)\dagger}(p)Q_{11}^{(+)}U^{(1)}(p) &= -\mathcal{N}_1, \\ U^{(2)\dagger}(p)Q_{22}^{(+)}U^{(2)}(p) &= -\mathcal{N}_2, \\ V^{(1)\dagger}(p)Q_{11}^{(-)}V^{(1)}(p) &= -\mathcal{N}_1, \\ V^{(2)\dagger}(p)Q_{22}^{(-)}V^{(2)}(p) &= -\mathcal{N}_2. \end{aligned} \quad \text{(A46)}$$

We define positive normalization constants (53a) and (54a) with respect to those inner products, where for negative-metric states we have taken the absolute value.

In the same way one can prove that for any r, s one has the expressions

$$\begin{aligned} u^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s - \omega_r))v^s(-p) &= 0, \\ u^{r\dagger}(p)(1 - g_2\gamma_5(W_s + \omega_r))U^s(p) &= 0, \\ u^{r\dagger}(p)(1 + g_2\gamma_5(W_s - \omega_r))V^s(-p) &= 0, \\ U^{r\dagger}(p)(1 + g_2\gamma_5(\omega_s - W_r))v^s(-p) &= 0, \\ U^{r\dagger}(p)(1 + g_2\gamma_5(W_s - W_r))V^s(-p) &= 0, \\ v^{r\dagger}(-p)(1 + g_2\gamma_5(W_s + \omega_r))V^s(-p) &= 0. \end{aligned} \quad \text{(A47)}$$

3. Outer product relations

Here we prove outer product relations that are used for the quantization. We start to consider

$$u^{(1)}\bar{u}^{(1)} = \begin{pmatrix} m & (\omega_1 - g_2\omega_1^2 - (\vec{p}\cdot\vec{\sigma})) \\ (\omega_1 + g_2\omega_1^2 + (\vec{p}\cdot\vec{\sigma})) & m \end{pmatrix} \otimes \frac{1}{2}\left(1 + \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right), \quad \text{(A48)}$$

where we have used the property of the bi-spinors (A8).

Noting that

$$\bar{M}(\omega_1, \vec{p}) = \begin{pmatrix} m & \omega_1 - g_2\omega_1^2 - (\vec{p}\cdot\vec{\sigma}) \\ \omega_1 + g_2\omega_1^2 + (\vec{p}\cdot\vec{\sigma}) & m \end{pmatrix}, \quad \text{(A49)}$$

and using (22b), we can write

$$u^{(1)}(p)\bar{u}^{(1)}(p) = (\gamma_0\omega_1 + \gamma^i p_i + m - g_2\omega_1^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 - Q), \quad \text{(A50)}$$

$$u^{(2)}(p)\bar{u}^{(2)}(p) = (\gamma_0\omega_2 + \gamma^i p_i + m - g_2\omega_2^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 + Q), \tag{A51}$$

$$U^{(1)}(p)\bar{U}^{(1)}(p) = (\gamma_0 W_1 + \gamma^i p_i + m - g_2 W_1^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 - Q), \tag{A52}$$

$$U^{(2)}(p)\bar{U}^{(2)}(p) = (\gamma_0 W_2 + \gamma^i p_i + m - g_2 W_2^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 + Q), \tag{A53}$$

$$v^{(1)}(-p)\bar{v}^{(1)}(-p) = (\gamma_0\omega_1 - \gamma^i p_i - m + g_2\omega_1^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 - Q), \tag{A54}$$

$$v^{(2)}(-p)\bar{v}^{(2)}(-p) = (\gamma_0\omega_2 - \gamma^i p_i - m + g_2\omega_2^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 + Q), \tag{A55}$$

$$V^{(1)}(-p)\bar{V}^{(1)}(-p) = (\gamma_0 W_1 - \gamma^i p_i - m + g_2 W_1^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 - Q), \tag{A56}$$

$$V^{(2)}(-p)\bar{V}^{(2)}(-p) = (\gamma_0 W_2 - \gamma^i p_i - m + g_2 W_2^2\gamma_0\gamma_5) \times \frac{1}{2}(\mathbb{1}_4 + Q), \tag{A57}$$

where the operator Q is defined in (23).

Let us multiply the above identities by the left with γ_0, and adding conveniently, we obtain

$$u^{(1)}(p)u^{(1)\dagger}(p) + v^{(1)}(-p)v^{(1)\dagger}(-p) = \omega_1(\mathbb{1}_4 - Q), \tag{A58}$$

$$u^{(1)}(p)u^{(1)\dagger}(p) - v^{(1)}(-p)v^{(1)\dagger}(-p) = (\gamma^i p_i + m - g_2\omega_1^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4 - Q), \tag{A59}$$

$$u^{(2)}(p)u^{(2)\dagger}(p) + v^{(2)}(-p)v^{(2)\dagger}(-p) = \omega_2(\mathbb{1}_4 + Q), \tag{A60}$$

$$u^{(2)}(p)u^{(2)\dagger}(p) - v^{(2)}(-p)v^{(2)\dagger}(-p) = (\gamma^i p_i + m - g_2\omega_2^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4 + Q), \tag{A61}$$

$$U^{(1)}(p)U^{(1)\dagger}(p) + V^{(1)}(-p)V^{(1)\dagger}(-p) = W_1(\mathbb{1}_4 - Q), \tag{A62}$$

$$U^{(1)}(p)U^{(1)\dagger}(p) - V^{(1)}(-p)V^{(1)\dagger}(-p) = (\gamma^i p_i + m - g_2 W_1^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4 - Q), \tag{A63}$$

$$U^{(2)}(p)U^{(2)\dagger}(p) + V^{(2)}(-p)V^{(2)\dagger}(-p) = W_2(\mathbb{1}_4 + Q), \tag{A64}$$

$$U^{(2)}(p)U^{(2)\dagger}(p) - V^{(2)}(-p)V^{(2)\dagger}(-p) = (\gamma^i p_i + m - g_2 W_2^2\gamma_0\gamma_5)\gamma_0(\mathbb{1}_4 + Q). \tag{A65}$$

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