Locality of staggered overlap operators

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We give an explicit proof for the locality of staggered overlap operators. The proof covers the original two flavor construction by Adams as well as a single flavor version. As in the case of Neuberger's operator, an admissibility condition for the gauge fields is required.

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I. INTRODUCTION AND MOTIVATION

As Adams has shown [1], it is possible to construct chirally symmetric lattice fermions based on the staggered discretization. While Adams' original construction provided a two flavor operator, single flavor versions were found soon after [2,3]. These staggered chiral fermions are obtained by first adding a mass term [4] to the staggered operator, followed by an overlap construction [5], which contains an inverse square root. It is thus evident that staggered chiral fermions are not ultralocal by construction and their locality needs to be proven. Numerically, Ref. [6] found strong evidence in support of the locality of Adams' original two flavor operator. In the free case, one can furthermore show that the lifting of the doubler modes is achieved via flavor dependent mass term [1,2,7]. In addition, the index theorem has been established for the two flavor operator [8] and the correct continuum limit of the index was found in [9]. In this paper, we give an analytic proof for the locality of staggered overlap fermions, for both the single and two flavor cases. The general strategy we employ is quite similar to the one used by Hernández, Jansen and Lüscher to demonstrate the locality of the original Neuberger operator [10]. We will start in Sec. II by expanding the inverse square root as a series of Legendre polynomials, which can be shown to be local if a spectral condition of the kernel operator is fulfilled. This spectral condition involves an upper as well as a lower bound on the kernel operator. In Sec. III we will show that both bounds are fulfilled for Adams' original two flavor construction, provided an admissibility condition of the form $\|\mathbb{1} - P\| < \varepsilon$ is fulfilled by all plaquettes P of the gauge field. The exact value of ε will depend on the details of the action, specifically the negative mass parameter s and the Wilson parameter r. We then turn to a single flavor staggered operator and show that similar bounds also hold in this case.

II. LOCALITY

A. Staggered overlap Dirac operator

Let us first introduce the staggered overlap Dirac operator

$$D_{\mathrm{so}} = \frac{1}{a}(\mathbb{1} + A/\sqrt{A^\dagger A}) \tag{2.1}$$

with

$$A = aD_{\mathrm{sw}} - rs\mathbb{1} \quad D_{\mathrm{sw}} = D_{\mathrm{st}} + W_{\mathrm{st}} \tag{2.2}$$

where r is the Wilson parameter and $0 < s < 2$ is the negative mass term of the kernel operator. The staggered operator is defined as

$$D_{\mathrm{st}} = \eta_\mu \nabla_\mu \tag{2.3}$$

with

$$(\eta_\mu)_x = (-1)^{\sum_{\nu<\mu} x_\nu} \tag{2.4}$$

and the symmetric derivative operator

$$\nabla_\mu = \frac{1}{2a}(T_{\mu+} - T_{\mu-}). \tag{2.5}$$

The $T_{\mu\pm}$ are parallel transports defined as

$$(T_{\mu+})_{xy} = U_\mu(x)\delta_{x+\hat{\mu},y} \quad (T_{\mu-})_{xy} = U_\mu^\dagger(y)\delta_{x-\hat{\mu},y}. \tag{2.6}$$

The staggered Wilson term W_{st} reads

$$W_{\mathrm{st}} = \frac{r}{a}(\mathbb{1} - M^{(2)}) \tag{2.7}$$

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in the two flavor case [1,8,11] and

$$W_{\mathsf{st}} = \frac{r}{a}(2 \cdot \mathbb{1} + M^{(1)}) \tag{2.8}$$

in the one flavor case [2,3]. The operators $M^{(\mathsf{N_f})}$ are in turn given[1] by

$$M^{(2)} = \epsilon\eta_5 \quad CM^{(1)} = \mathrm{i}\eta_{12}C_{12} + \mathrm{i}\eta_{34}C_{34} \tag{2.9}$$

with the phase factors

$$\eta_5 = \eta_1\eta_2\eta_3\eta_4 \tag{2.10}$$

$$\epsilon_x = (-1)^{\sum_\nu x_\nu} \tag{2.11}$$

$$(\eta_{\mu\nu})_x = -\eta_{\nu\mu} = (-1)^{\sum_{\rho=\mu+1}^{\nu} x_\rho} \quad \text{for } \mu \le \nu \tag{2.12}$$

and the diagonal hopping terms

$$C = (C_1C_2C_3C_4)_{\mathrm{sym}} = \frac{1}{4!}P_{\alpha\beta\gamma\delta}C_\alpha C_\beta C_\gamma C_\delta \tag{2.13}$$

$$C_{\mu\nu} = \frac{1}{2}\{C_\mu, C_\nu\} \tag{2.14}$$

where

$$C_\mu = \frac{1}{2}(T_{\mu+} + T_{\mu-}) \tag{2.15}$$

and $P_{\alpha\beta\gamma\delta}$ denotes the permutation symbol

$$P_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \alpha,\beta,\gamma,\delta \text{ is a permutation of } 1,2,3,4 \\ 0 & \text{else.} \end{cases} \tag{2.16}$$

The kernel A is ultralocal, but due to the $(A^\dagger A)^{-1/2}$ term the staggered overlap Dirac operator D_{so} is not. However, if the matrix elements $(D_{\mathsf{so}})_{x,y}$ of the staggered overlap operator are decaying exponentially for large distances $\|x-y\|$ with a decay constant $\propto a^{-1}$, then we recover a local operator in the continuum limit.

B. Legendre series expansion

Following the strategy employed in Ref. [10] we begin by expanding $(A^\dagger A)^{-1/2}$ in a series of Legendre polynomials. In order to make the expansion convergent we impose the following inequality, which we will show in Sec. III:

$$0 < u \le A^\dagger A \le v < \infty. \tag{2.17}$$

[1]Note that in principle more general single flavor terms are allowed [3]. These are, however, not substantially different and the generalization is straightforward.

The inequality stands for the corresponding inequality between the expectation values of the operators in arbitrary normalizable states. We also explicitly assume that $u < v$. In the following we can set $u = \lambda_{\min}$ and $v = \lambda_{\max}$ as noted in Ref. [12].

The Legendre polynomials $P_k(z)$ can be defined through the expansion of the generating function

$$(1 - 2tz + t^2)^{-1/2} = \sum_{k=0}^{\infty} t^k P_k(z). \tag{2.18}$$

We can now set

$$z = \frac{(\lambda_{\min} + \lambda_{\max})\mathbb{1} - 2A^\dagger A}{\lambda_{\max} - \lambda_{\min}} \tag{2.19}$$

and due to Eq. (2.17) find that this operator has norm $\|z\| = 1$. Here and in the following $\|\cdot\| = \|\cdot\|_2 \equiv \sigma_{\max}(\cdot)$ refers to the spectral norm and $\sigma_{\max}$ refers to the largest singular value.

Then the property $|P_k(x)| \le 1\ \forall\, x \in [-1,1]$ together with $\|z\| = 1$ translates to

$$\|P_k(z)\| \le 1. \tag{2.20}$$

It follows that Eq. (2.18) is norm convergent for our choice of z for all t satisfying $|t| < 1$. Due to Eq. (2.17), we can now introduce θ through

$$\cosh\theta = \frac{\lambda_{\max} + \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}, \quad \theta > 0, \tag{2.21}$$

and set

$$t = e^{-\theta}, \tag{2.22}$$

which implies $0 < t \le 1$, so that the series is convergent. Note that this allows us to express t as

$$t = \cosh\theta - \sqrt{\cosh^2\theta - 1} = \frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}. \tag{2.23}$$

From Eq. (2.18) we thus obtain

$$\begin{aligned}
&(1 - 2tz + t^2)^{-1/2} \\
&\quad = \left(1 - \frac{2t}{\lambda_{\max} - \lambda_{\min}}(\lambda_{\min} + \lambda_{\max} - 2A^\dagger A) + t^2\right)^{-1/2} \\
&\quad = \left(1 - 2e^{-\theta}\cosh\theta + \frac{4t}{\lambda_{\max} - \lambda_{\min}}A^\dagger A + e^{-2\theta}\right)^{-1/2} \\
&\quad = \sqrt{\frac{\lambda_{\max} - \lambda_{\min}}{4t}}(A^\dagger A)^{-1/2} \\
&\quad = \frac{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}{2}(A^\dagger A)^{-1/2}
\end{aligned} \tag{2.24}$$

and therefore

$$(A^\dagger A)^{-1/2} = \kappa \sum_{k=0}^{\infty} t^k P_k(z) \tag{2.25}$$

with $\kappa = 2/(\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}})$.

C. Locality of the inverse square root

The lack of ultralocality stems from the $(A^\dagger A)^{-1/2}$ term, hence it is sufficient to establish the locality of that term in the sense defined earlier. We start by defining the kernel $G(x, y)$ via

$$G(x, y) = ((A^\dagger A)^{-1/2})_{xy}. \tag{2.26}$$

Similarly, we define the kernels of the $P_k(z)$ via

$$G_k(x, y) = (P_k(z))_{xy} \tag{2.27}$$

and use Eq. (2.25) to obtain

$$G(x, y) = \kappa \sum_{k=0}^{\infty} t^k G_k(x, y). \tag{2.28}$$

The norm convergence of the Legendre expansion implies the absolute convergence of this series for all x and y. From Eqs. (2.20) and (2.27) we infer that

$$\|G_k(x, y)\| \le 1, \quad \forall\, x\, \forall\, y\, \forall\, k, \tag{2.29}$$

where the norm is in color space.

Because $P_k(z)$ is a polynomial in $A^\dagger A$ and A is an ultralocal operator, we find that $G_k(x, y)$ vanishes unless x and y are sufficiently close to each other. If we introduce the Manhattan distance $\|\cdot\|_1$, we have

$$G_k(x, y) = 0, \quad \forall\, k < \frac{1}{2\ell a}\|x - y\|_1, \tag{2.30}$$

where ℓ is the range of the operator A in lattice units, i.e., the maximum Manhattan distance in lattice units between points coupled by the operator. For two flavor staggered Wilson fermions we have $\ell = 4$, for one flavor staggered Wilson fermions $\ell = 2$ and for Wilson fermions $\ell = 1$. Using the shorthand notation $d = \|x - y\|_1/(2\ell a)$ we find

$$\begin{aligned}\|G(x, y)\| &= \kappa \sum_{k=d}^{\infty} t^k \|G_k(x, y)\| \\ &\le \kappa \sum_{k=d}^{\infty} t^k \\ &= \frac{\kappa}{1-t} t^d \\ &= \frac{\kappa}{1-t} \exp\left(-\frac{\theta}{2\ell a}\|x - y\|_1\right) \\ &= \frac{1}{\sqrt{\lambda_{\min}}} \exp\left(-\frac{1}{\xi}\|x - y\|_1\right)\end{aligned} \tag{2.31}$$

and thus an exponential falloff with the decay constant[2]

$$\xi^{-1} = \frac{\theta}{2\ell a} = \frac{1}{2\ell a}\log\left(\frac{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}\right) \propto \frac{1}{a}. \tag{2.32}$$

This establishes the locality of $(A^\dagger A)^{-1/2}$ providing Eq. (2.17) holds with the spectral bounds given by $u = \lambda_{\min}$ and $v = \lambda_{\max}$. The equivalent of this particular form for usual overlap fermions was derived in Ref. [12].

Let us finally remark that this result can be slightly generalized in the case of a single isolated zero or near zero mode $\lambda_{\min}$. As shown in Sec. 2.4 of Ref. [10], one can treat a single isolated zero or near zero mode separately and still establish locality. In that case we identify the lower spectral bound $u = \lambda_2$ with the second smallest eigenvalue of $A^\dagger A$. If $\lambda_{\min} < u/2$, locality can again be established [10].

III. BOUNDS ON $A^\dagger A$

We now need to establish the spectral bounds as defined in Eq. (2.17) for the kernel operator. We first derive some useful identities and then establish the upper bound, which is straightforward. The main task is then to establish the lower bound, which we do separately for the two and one flavor case. In both instances, the bound can be established given an admissibility condition for the gauge fields.

A. Some useful identities

We first note that the parallel transports fulfill the relations $T_{\mu-} = T_{\mu+}^\dagger = T_{\mu+}^{-1}$, which implies that the $T_{\mu\pm}$ are unitary and thus have singular values 1, i.e., $\|T_{\mu\pm}\| = 1$. The covariant second derivative operator is given by

$$\Delta_\mu = T_{\mu+} + T_{\mu-} - 2, \tag{3.1}$$

so we can recast Eq. (2.15) as

$$C_\mu = 1 + \frac{\Delta_\mu}{2}. \tag{3.2}$$

[2]Note that the log term may provide subleading corrections to this behavior only.

Using this relation we find

$$C_\mu^2 = 1 + \frac{1}{4}(T_{\mu+}^2 + T_{\mu-}^2 - 2). \tag{3.3}$$

Defining

$$V_\mu = \frac{1}{4}(T_{\mu+}^2 + T_{\mu-}^2 - 2) \tag{3.4}$$

it follows that

$$C_\mu^2 = 1 + V_\mu. \tag{3.5}$$

From Eq. (2.15) we also find

$$\|C_\mu\| \le \frac{1}{2}(\|T_{\mu+}\| + \|T_{\mu-}\|) = 1, \tag{3.6}$$

which implies

$$\|M^{(2)}\| = \|\eta_5 C\epsilon\| \le 1 \tag{3.7}$$

and, since both η_5 and ϵ commute with C and square to the identity, $M^{(2)2} = C^2$. Another useful identity is

$$a^2\nabla_\mu^2 = V_\mu \tag{3.8}$$

which, together with the anti-Hermiticity condition $\nabla_\mu^\dagger = -\nabla_\mu$, implies that

$$0 \le a^2\nabla_\mu^\dagger\nabla_\mu = -V_\mu. \tag{3.9}$$

Additionally, the Hermiticity condition $C_\mu^\dagger = C_\mu$ implies that $C_\mu^2 \ge 0$ and thus $1 + V_\mu \ge 0$.

Next, we want to find a more explicit expressions for $A^\dagger A$. Noting that

$$\nabla_\mu\eta_\nu = \begin{cases} \eta_\nu\nabla_\mu & \mu \ge \nu, \\ -\eta_\nu\nabla_\mu & \mu < \nu, \end{cases} \tag{3.10}$$

we find

$$\sum_{\mu,\nu}\eta_\mu\nabla_\mu\eta_\nu\nabla_\nu = \nabla^2 + \sum_{\mu>\nu}\eta_\mu\eta_\nu[\nabla_\mu, \nabla_\nu], \tag{3.11}$$

where we have introduced the shorthand notation

$$\nabla^2 = \sum_\mu \nabla_\mu\nabla_\mu. \tag{3.12}$$

We then find

$$A_2^\dagger A_2 = -a^2\nabla^2 - a^2\sum_{\mu>\nu}\eta_\mu\eta_\nu[\nabla_\mu, \nabla_\nu] + r^2(\mathbb{1}(1-s) - M^{(2)})^2 - ar[M^{(2)}, \eta_\mu\nabla_\mu] \tag{3.13}$$

in the two flavor case and

$$A_1^\dagger A_1 = -a^2\nabla^2 - a^2\sum_{\mu>\nu}\eta_\mu\eta_\nu[\nabla_\mu, \nabla_\nu] + r^2(\mathbb{1}(2-s) + M^{(1)})^2 + ar[M^{(1)}, \eta_\mu\nabla_\mu] \tag{3.14}$$

in the one flavor case.

B. Upper bound

Using $\|T_{\mu\pm}\| = 1$ we find the following bounds

$$\|a\nabla_\mu\| \le \frac{1}{2}(\|T_{\mu+}\| + \|T_{\mu-}\|) \le 1, \tag{3.15}$$

$$\|a\eta_\mu\nabla_\mu\| \le 4, \tag{3.16}$$

$$\|C_\mu\| = \frac{1}{2}\|T_{\mu+} + T_{\mu-}\| \le 1, \tag{3.17}$$

$$\|C\| = \|(C_1C_2C_3C_4)_{\mathsf{sym}}\| \le \frac{1}{4!}\cdot 4!\cdot\prod_\mu\|C_\mu\| \le 1, \tag{3.18}$$

and using Eq. (3.7) we find

$$\|r(\mathbb{1}(1-s) - M^{(2)})\| \le |r|(2-s). \tag{3.19}$$

Putting all this together, we find

$$\|A_2\| = \|a\eta_\mu\nabla_\mu + r(\mathbb{1}(1-s) - M^{(2)})\| \le 4 + |r|(2-s). \tag{3.20}$$

The same bound holds for $A_2^\dagger$ and so

$$\|A_2^\dagger A_2\| \le \|A_2^\dagger\|\|A_2\| \le (4 + |r|(2-s))^2 \tag{3.21}$$

is uniformly bounded from above for all r and s and we can establish the existence of v in Eq. (2.17) in the two flavor case.

For the one flavor case we note that

$$\|C_{\mu\nu}\| = \frac{1}{2}\|\{C_\mu, C_\nu\}\| \le 1, \tag{3.22}$$

from which it follows that

$$\|M^{(1)}\| \le \|C_{12}\| + \|C_{34}\| \le 2. \tag{3.23}$$

Hence we find

$$\|\mathbb{1}(2-s) + M^{(1)}\| \le 4 - s \tag{3.24}$$

and it follows, similarly to the two flavor case, that

$$\|A_1\| \le 4 + |r|(4-s). \tag{3.25}$$

Since $A_1^\dagger$ does obey the same bound, we obtain

$$\|A_1^\dagger A_1\| \leq (4 + |r|(4-s))^2. \tag{3.26}$$

This establishes the existence of v in Eq. (2.17) in the single flavor case as well.

C. Lower bound

As $A^\dagger A$ is Hermitian and positive semidefinite we are left with showing the absence of zero-modes. However, in general this operator can have zero-modes for certain gauge configurations, therefore no uniform positive lower bound exists. Zero-modes can only be excluded if we assume the gauge field to be sufficiently smooth. In our case let us assume that

$$\|\mathbb{1} - P\| < \varepsilon \quad \text{for all plaquettes } P. \tag{3.27}$$

As a consequence of the smoothness condition, we obtain the following relations (see Appendix)

$$\|a^2[\nabla_\mu, \nabla_\nu]\| < \varepsilon, \qquad \|[C_\mu, C_\nu]\| < \varepsilon,$$
$$\|a[C_\mu, \nabla_\nu]\| < \varepsilon. \tag{3.28}$$

1. Lower bound on the two flavor operator $A_2^\dagger A_2$

There are four terms in

$$A_2^\dagger A_2 = -a^2\nabla^2 - \sum_{\mu>\nu}\eta_\mu\eta_\nu a^2[\nabla_\mu, \nabla_\nu] + r^2(\mathbb{1}(1-s) - M^{(2)})^2 - ar[M^{(2)}, \eta_\mu\nabla_\mu], \tag{3.29}$$

for which we will find bounds individually. We will consider the case $0 < r \leq 1$ first and derive a bound for $r > 1$ later.[3]

The first and third term We first look at $-a^2\nabla^2 + r^2C^2$, where $M^{(2)2} = C^2$ is used. Using inequality (3.28) we find (cf. Appendix)

$$\|C^2 - (C_1^2C_2^2C_3^2C_4^2)_{\text{sym}}\| < 9\varepsilon. \tag{3.30}$$

Using Eqs. (3.5), (3.8), and (3.9), we furthermore see that for $0 < r \leq 1$

$$\begin{aligned}
-a^2\nabla^2 + r^2C^2 &> -a^2\nabla^2 + r^2(C_1^2C_2^2C_3^2C_4^2)_{\text{sym}} - 9r^2\varepsilon \\
&= -\sum_\mu V_\mu + r^2\frac{1}{4!}P_{\alpha\beta\gamma\delta}(1+V_\alpha)(1+V_\beta)(1+V_\gamma) \\
&\quad \times (1+V_\delta) - 9r^2\varepsilon \\
&= -\sum_\mu V_\mu + r^2 + r^2\sum_\mu V_\mu + \frac{1}{2}r^2\sum_{\mu\neq\nu}V_\mu V_\nu \\
&\quad + \frac{1}{3!}r^2\sum_{\mu\neq\nu\neq\alpha\neq\mu}V_\mu V_\nu V_\alpha + r^2(V_1V_2V_3V_4)_{\text{sym}} - 9r^2\varepsilon \\
&= r^2 - (1-r^2)\sum_\mu V_\mu + \frac{1}{2}r^2\sum_{\mu\neq\nu}V_\mu V_\nu \\
&\quad + \frac{1}{3!}r^2\sum_{\mu\neq\nu\neq\alpha\neq\mu}V_\mu V_\nu V_\alpha + r^2(V_1V_2V_3V_4)_{\text{sym}} - 9r^2\varepsilon \\
&\geq r^2\Bigg(1 + \frac{1}{2}\sum_{\mu\neq\nu}V_\mu V_\nu + \frac{1}{3!}\sum_{\mu\neq\nu\neq\alpha\neq\mu}V_\mu V_\nu V_\alpha \\
&\quad + (V_1V_2V_3V_4)_{\text{sym}} - 9\varepsilon\Bigg).
\end{aligned} \tag{3.31}$$

Using the relation (3.9), we conclude that

$$V_\mu V_\nu = (-V_\mu)(-V_\nu) \geq 0, \tag{3.32}$$

so that each contribution to the two-product term as well as the four-product term is positive semidefinite. We use these properties and $1 + V_\mu \geq 0$ to obtain

$$\begin{aligned}
&-a^2\nabla^2 + r^2C^2 \\
&\quad > r^2\left(1 + \frac{1}{2}\sum_{\mu\neq\nu}V_\mu V_\nu + \frac{1}{3!}\sum_{\mu\neq\nu\neq\alpha\neq\mu}V_\mu V_\nu V_\alpha - 9\varepsilon\right) \\
&\quad > r^2\left(1 + \frac{1}{3!}\sum_{\mu\neq\nu\neq\alpha\neq\mu}V_\mu V_\nu(V_\alpha + 1) - 9\varepsilon\right) \\
&\quad \geq r^2(1 - 9\varepsilon).
\end{aligned} \tag{3.33}$$

Using Eq. (3.7), we finally obtain

$$\begin{aligned}
&-a^2\nabla^2 + r^2(\mathbb{1}(1-s) - M^{(2)})^2 \\
&\quad = -a^2\nabla^2 + r^2C^2 - 2r^2(1-s)M^{(2)} + r^2(1-s)^2\mathbb{1} \\
&\quad \geq r^2(1 - 9\varepsilon - 2|1-s| + |1-s|^2) \\
&\quad = r^2(1 - |1-s|)^2 - 9r^2\varepsilon
\end{aligned} \tag{3.34}$$

for $0 < r \leq 1$. For the case $r > 1$ we can decompose

$$\begin{aligned}
&-a^2\nabla^2 + r^2(\mathbb{1}(1-s) - M^{(2)})^2 \\
&\quad = -a^2\nabla^2 + (\mathbb{1}(1-s) - M^{(2)})^2 \\
&\qquad + (r^2 - 1)(\mathbb{1}(1-s) - M^{(2)})^2
\end{aligned} \tag{3.35}$$

[3]The $r < 0$ case can be covered by the simple replacement of $r \to |r|$ in the bounds. However, negative r do not represent a physically different system compared to positive r and will therefore not be considered further.

and, since $r^2 - 1 > 0$, observe that the last term is positive semidefinite. The first two terms, however, just correspond to the $r = 1$ case, so the $r = 1$ lower bound also applies for the $r > 1$ case. All together we thus have

$$-a^2\nabla^2 + r^2(\mathbb{1}(1-s) - M^{(2)})^2 > \begin{cases} r^2(1-|1-s|)^2 - 9r^2\varepsilon & 0 < r \leq 1, \\ (1-|1-s|)^2 - 9\varepsilon & r > 1. \end{cases} \tag{3.36}$$

The second term As a result of Eq. (3.28) we find

$$\left\| \sum_{\mu>\nu} \eta_\mu \eta_\nu a^2 [\nabla_\mu, \nabla_\nu] \right\| \leq \sum_{\mu>\nu} \| a^2 [\nabla_\mu, \nabla_\nu] \| < 6\varepsilon, \tag{3.37}$$

so that we obtain the lower bound

$$-\sum_{\mu>\nu} \eta_\mu \eta_\nu a^2 [\nabla_\mu, \nabla_\nu] > -6\varepsilon \tag{3.38}$$

for the second term.

The fourth term From the commutation properties

$$C_\mu \eta_\nu = \begin{cases} \eta_\nu C_\mu & \mu \geq \nu, \\ -\eta_\nu C_\mu & \mu < \nu, \end{cases} \tag{3.39}$$

it follows that $C\eta_\mu = (-1)^{\mu+1}\eta_\mu C$. Similarly one can show that $\nabla_\mu \eta_5 = (-1)^\mu \eta_5 \nabla_\mu$. Using these relations we find

$$\begin{aligned} [M^{(2)}, \eta_\mu \nabla_\mu] &= (\epsilon\eta_5 C\eta_\mu \nabla_\mu - \eta_\mu \nabla_\mu \epsilon \eta_5 C) \\ &= \epsilon(\eta_5 C\eta_\mu \nabla_\mu + \eta_\mu \nabla_\mu \eta_5 C) \\ &= \epsilon(\eta_5 \eta_\mu (-1)^{\mu+1} C\nabla_\mu + \eta_5 \eta_\mu (-1)^\mu \nabla_\mu C) \\ &= \epsilon\eta_5 \eta_\mu (-1)^{\mu+1} [C, \nabla_\mu]. \end{aligned} \tag{3.40}$$

From Eqs. (3.6) and (3.28) we can then conclude that

$$\begin{aligned} \| a[M^{(2)}, \eta_\mu \nabla_\mu] \| &\leq a \sum_\mu \| [C, \nabla_\mu] \| \\ &\leq a \sum_{\mu\nu} \| [C_\nu, \nabla_\mu] \| \\ &< \sum_{\mu\neq\nu} \varepsilon \\ &= 12\varepsilon \end{aligned} \tag{3.41}$$

and thus we obtain the lower bound

$$ar[M^{(2)}, \eta_\mu \nabla_\mu] > -12r\varepsilon \tag{3.42}$$

for all $r > 0$.

Final lower bound Combining Eqs. (3.36), (3.38), and (3.42), we get a lower bound for the two flavor operator

$$A_2^\dagger A_2 > \begin{cases} r^2(1-|1-s|)^2 - (6+12r+9r^2)\varepsilon & 0 < r \leq 1, \\ (1-|1-s|)^2 - (15+12r)\varepsilon & r > 1. \end{cases} \tag{3.43}$$

2. Lower bound on the one flavor operator $A_1^\dagger A_1$

We will now try to find a lower bound on the operator

$$A_1^\dagger A_1 = -a^2\nabla^2 - \sum_{\mu>\nu} \eta_\mu \eta_\nu a^2 [\nabla_\mu, \nabla_\nu] + r^2(2\cdot\mathbb{1} + M^{(1)} - s\mathbb{1})^2 + ar[M^{(1)}, \eta_\mu \nabla_\mu], \tag{3.44}$$

by finding a bound of each term separately. Since the second term is the same as in the two flavor case, we can take the previous result Eq. (3.38). Once again, we consider the case $0 < r \leq 1$ first.

The first and third terms We start by observing that

$$\begin{aligned} C_{\mu\nu}^2 &= \frac{1}{4}(C_\mu C_\nu + C_\nu C_\mu)^2 \\ &= \frac{1}{4}(C_\mu C_\nu C_\mu C_\nu + C_\mu C_\nu C_\nu C_\mu + C_\nu C_\mu C_\mu C_\nu + C_\nu C_\mu C_\nu C_\mu) \\ &> \frac{1}{4}(C_\mu^2 C_\nu^2 - \varepsilon + C_\mu^2 C_\nu^2 - 2\varepsilon + C_\nu^2 C_\mu^2 - 2\varepsilon + C_\nu^2 C_\mu^2 - \varepsilon) \\ &= \frac{C_\mu^2 C_\nu^2 + C_\nu^2 C_\mu^2 - 3\varepsilon}{2}, \end{aligned} \tag{3.45}$$

where we have used Eq. (3.28). For $0 < r \leq 1$ we thus obtain the bound

$$
\begin{aligned}
-a^2\nabla^2 + r^2(\mathbb{1} + M^{(1)})^2 &= -a^2\nabla^2 + r^2(1 + i\eta_{12}C_{12} + i\eta_{34}C_{34})^2 \\
&= -\sum_\mu V_\mu + r^2(C_{12}^2 + C_{34}^2 + \{(1 + i\eta_{12}C_{12}), (1 + i\eta_{34}C_{34})\} - 1) \\
&> -\sum_\mu V_\mu + r^2\left(\frac{C_1^2C_2^2 + C_2^2C_1^2 - 3\varepsilon}{2} + \frac{C_3^2C_4^2 + C_4^2C_3^2 - 3\varepsilon}{2} - 1\right) \\
&= -\sum_\mu V_\mu + \frac{r^2}{2}(\{(1 + V_1), (1 + V_2)\} + \{(1 + V_3), (1 + V_4)\} - 2 - 6\varepsilon) \\
&= -\sum_\mu V_\mu + \frac{r^2}{2}\left(2 + 2\sum_\mu V_\mu + \{V_1, V_2\} + \{V_3, V_4\} - 6\varepsilon\right) \\
&\geq -(1 - r^2)\sum_\mu V_\mu + r^2(1 - 6\varepsilon) \\
&\geq r^2 - 6r^2\varepsilon.
\end{aligned} \tag{3.46}
$$

For the general case of $0 < s < 2$ we use

$$
\mathbb{1} + M^{(1)} \geq -1, \tag{3.47}
$$

which follows from $\|M^{(1)}\| \leq 2$, to find

$$
\begin{aligned}
&- a^2\nabla^2 + r^2((2 - s)\mathbb{1} + M^{(1)})^2 \\
&\quad = -a^2\nabla^2 + r^2((1 - s)\mathbb{1} + (\mathbb{1} + M^{(1)}))^2 \\
&\quad = -a^2\nabla^2 + r^2(\mathbb{1} + M^{(1)})^2 + r^2(1 - s)^2\mathbb{1} \\
&\qquad + 2r^2(1 - s)(\mathbb{1} + M^{(1)}) \\
&\quad > r^2 - 6r^2\varepsilon + r^2|1 - s|^2 - 2r^2|1 - s| \\
&\quad = r^2(1 - |1 - s|)^2 - 6r^2\varepsilon.
\end{aligned} \tag{3.48}
$$

The lower bound of the first and third term for $0 < r \leq 1$ is thus given by

$$
-a^2\nabla^2 + r^2((2 - s)\mathbb{1} + M^{(1)})^2 > r^2(1 - |1 - s|)^2 - 6r^2\varepsilon. \tag{3.49}
$$

For the $r > 1$ case we can again show that the $r = 1$ bound holds with the same argument used in Eq. (3.35). We thus obtain the general lower bound

$$
\begin{aligned}
&- a^2\nabla^2 + r^2((2 - s)\mathbb{1} + M^{(1)})^2 \\
&\quad > \begin{cases} r^2(1 - |1 - s|)^2 - 6r^2\varepsilon & 0 < r \leq 1, \\ (1 - |1 - s|)^2 - 6\varepsilon & r > 1. \end{cases}
\end{aligned} \tag{3.50}
$$

The fourth term Let us first decompose the mass term

$$
a[M^{(1)}, \eta_\mu\nabla_\mu] = ai([\eta_{12}C_{12}, \eta_\mu\nabla_\mu] + [\eta_{34}C_{34}, \eta_\mu\nabla_\mu]) \tag{3.51}
$$

and look at the first of the two commutators. We have

$$
\begin{aligned}
&ai[\eta_{12}C_{12}, \eta_\mu\nabla_\mu] \\
&\quad = ai(\eta_{12}[C_{12}, \eta_\mu]\nabla_2 + \eta_\mu\eta_{12}[C_{12}, \nabla_\mu] + \eta_\mu[\eta_{12}, \nabla_\mu]C_{12}) \\
&\quad = ai(-2\eta_{12}\eta_2C_{12}\nabla_2 + \eta_\mu\eta_{12}[C_{12}, \nabla_\mu] + 2\eta_2\eta_{12}\nabla_2C_{12}) \\
&\quad = ai(-1)^{\delta_{\mu,2}}\eta_\mu\eta_{12}[C_{12}, \nabla_\mu],
\end{aligned} \tag{3.52}
$$

which results in

$$
\begin{aligned}
&\|ai[\eta_{12}C_{12}, \eta_\mu\nabla_\mu]\| \\
&\quad = \|ai(-1)^{\delta_{\mu,2}}\eta_\mu\eta_{12}[C_{12}, \nabla_\mu]\| \\
&\quad \leq \frac{a}{2}(\|[C_1C_2, \nabla_\mu]\| + \|[C_2C_1, \nabla_\mu]\|) \\
&\quad \leq \frac{a}{2}(\|C_1[C_2, \nabla_\mu]\| + \|[C_1, \nabla_\mu]C_2\| \\
&\qquad + \|C_2[C_1, \nabla_\mu]\| + \|[C_2, \nabla_\mu]C_1\|).
\end{aligned} \tag{3.53}
$$

With Eqs. (3.28) and (3.17) we thus obtain the upper bound

$$
\|ai[\eta_{12}C_{12}, \eta_\mu\nabla_\mu]\| < 2\varepsilon \tag{3.54}
$$

for the first term. Similarly, we obtain for the second term

$$
\|ai[\eta_{34}C_{34}, \eta_\mu\nabla_\mu]\| = \|ai(-1)^{\delta_{\mu,4}}\eta_\mu\eta_{34}[C_{34}, \nabla_\mu]\| < 2\varepsilon \tag{3.55}
$$

and thus conclude

$$
ar[M^{(1)}, \eta_\mu\nabla_\mu] > -4r\varepsilon. \tag{3.56}
$$

Final lower bound Combining Eqs. (3.50), (3.38), and (3.56), we get a lower bound for the single flavor operator

$$
A_1^\dagger A_1 > \begin{cases} r^2(1 - |1 - s|)^2 - (6 + 4r + 6r^2)\varepsilon & 0 < r \leq 1 \\ (1 - |1 - s|)^2 - (12 + 4r)\varepsilon & r > 1 \end{cases} \tag{3.57}
$$

IV. CONCLUSION

In this note we have proven that, when the admissibility condition $\|\mathbb{1} - P\| < \varepsilon$ is imposed on every plaquette P, both one and two flavor staggered overlap operators are local. In particular, we can perform a Legendre expansion of the inverse square root of $A^\dagger A$, which is convergent if the spectral condition of Eq. (2.17) is fulfilled. From Eqs. (3.43) and (3.57), we find that this is the case when

$$\varepsilon < \frac{r^2(1-|1-s|)^2}{6+12r+9r^2} \qquad \text{two flavor,} \quad 0 < r \leq 1, \tag{4.1}$$

$$\varepsilon < \frac{(1-|1-s|)^2}{15+12r} \qquad \text{two flavor,} \quad r > 1, \tag{4.2}$$

$$\varepsilon < \frac{r^2(1-|1-s|)^2}{6+4r+6r^2} \qquad \text{single flavor,} \quad 0 < r \leq 1, \tag{4.3}$$

$$\varepsilon < \frac{(1-|1-s|)^2}{12+4r} \qquad \text{single flavor,} \quad r > 1, \tag{4.4}$$

which is dependent on the projection point s and the Wilson parameter r. The staggered overlap operator is thus conceptually on the same footing as the standard overlap operator with a Wilson kernel.

APPENDIX: PLAQUETTE DEPENDENT COMMUTATORS

1. Representations of the plaquette

Since it is essential for the proof to have a bound on the plaquette, we first want to show how the plaquette can be represented. Let us define the plaquette as the operator

$$(P_{\mu\nu})_{xy} = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)\delta_{x,y}. \tag{A1}$$

We find that

$$\begin{aligned}(T_{\mu+}T_{\nu+}T_{\mu-}T_{\nu-})_{xy} &= U_\mu(x)\delta_{x+\hat{\mu},z}U_\nu(z)\delta_{z+\hat{\nu},t}U_\mu^\dagger(u)\delta_{t-\hat{\mu},u}U_\nu^\dagger(y)\delta_{u-\hat{\nu},y} \\ &= U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(y)\delta_{x,y} \\ &= (P_{\mu\nu})_{xy}\end{aligned} \tag{A2}$$

or equivalently

$$P_{\mu\nu} = T_{\mu+}T_{\nu+}T_{\mu-}T_{\nu-}. \tag{A3}$$

Similarly, we can define plaquettes into negative coordinate directions as

$$P_{(-\mu)\nu} = T_{\mu-}T_{\nu+}T_{\mu+}T_{\nu-}, \tag{A4}$$

$$P_{\mu(-\nu)} = T_{\mu+}T_{\nu-}T_{\mu-}T_{\nu+}, \tag{A5}$$

$$P_{(-\mu)(-\nu)} = T_{\mu-}T_{\nu-}T_{\mu+}T_{\nu+}. \tag{A6}$$

With these, we can find commutation relations among the $T_{\mu\pm}$ $(\mu \neq \nu)$ as

$$\begin{aligned}[T_{\mu+}, T_{\nu+}] &= T_{\mu+}T_{\nu+} - T_{\nu+}T_{\mu+} \\ &= T_{\mu+}T_{\nu+}(1 - T_{\nu-}T_{\mu-}T_{\nu+}T_{\mu+}) \\ &= T_{\mu+}T_{\nu+}(1 - P_{(-\nu)(-\mu)})\end{aligned} \tag{A7}$$

and similarly for other combinations.

2. Implications for some commutators

We will need the commutator

$$\begin{aligned}a^2[\nabla_\mu, \nabla_\nu] &= \frac{1}{4}([T_{\mu+}, T_{\nu+}] + [T_{\mu-}, T_{\nu-}] - [T_{\mu+}, T_{\nu-}] \\ &\quad - [T_{\mu-}, T_{\nu+}]) \\ &= \frac{1}{4}(T_{\mu+}T_{\nu+}(1 - P_{(-\nu)(-\mu)}) + T_{\mu-}T_{\nu-}(1 - P_{\nu\mu}) \\ &\quad - T_{\mu+}T_{\nu-}(1 - P_{\nu(-\mu)}) - T_{\mu-}T_{\nu+}(1 - P_{(-\nu)\mu})),\end{aligned} \tag{A8}$$

where we used Eq. (2.5). Imposing a smoothness condition

$$\|\mathbb{1} - (P_{\mu\nu})_{xx}\| < \varepsilon \tag{A9}$$

on every plaquette and remembering that all $\|T_{\mu\pm}\| = 1$, we find that

$$\begin{aligned}a^2\|[\nabla_\mu, \nabla_\nu]\| &< \frac{\varepsilon}{4}(\|T_{\mu+}T_{\nu+}\| + \|T_{\mu-}T_{\nu-}\| \\ &\quad + \|T_{\mu+}T_{\nu-}\| + \|T_{\mu-}T_{\nu+}\|) \\ &= \varepsilon.\end{aligned} \tag{A10}$$

Similarly we find

$$\begin{aligned}[C_\mu, C_\nu] &= \frac{1}{4}([T_{\mu+}, T_{\nu+}] + [T_{\mu-}, T_{\nu+}] \\ &\quad + [T_{\mu+}, T_{\nu-}] + [T_{\mu-}, T_{\nu-}])\end{aligned} \tag{A11}$$

$$\begin{aligned}&= \frac{1}{4}(T_{\mu+}T_{\nu+}(1 - P_{(-\nu)(-\mu)}) + T_{\mu-}T_{\nu+}(1 - P_{(-\nu)\mu}) \\ &\quad + T_{\mu+}T_{\nu-}(1 - P_{\nu(-\mu)}) + T_{\mu-}T_{\nu-}(1 - P_{\nu\mu}))\end{aligned} \tag{A12}$$

and thus

$$\|[C_\mu, C_\nu]\| < \varepsilon. \tag{A13}$$

Using the fact that $\|C_\mu\| \leq 1$, we can also infer that

$$\left\| [C_\mu, C_\nu] \prod_{i=1}^{n} C_{\alpha_i} \right\| < \varepsilon \tag{A14}$$

for any number n of additional C_α terms. We thus see that

$$\|C^2 - (C_1^2 C_2^2 C_3^2 C_4^2)_{\mathrm{sym}}\| < N\varepsilon, \tag{A15}$$

where N is determined by the number of commutations we have to perform to bring the terms in C^2 into the correct order. Let us first rewrite

$$\begin{aligned} &C^2 - (C_1^2 C_2^2 C_3^2 C_4^2)_{\mathrm{sym}} \\ &= \frac{1}{4!} P_{\alpha\beta\gamma\delta} (C_\alpha C_\beta C_\gamma C_\delta C - C_\alpha^2 C_\beta^2 C_\gamma^2 C_\delta^2). \end{aligned} \tag{A16}$$

For each term in the symmetrization bracket we now perform the commutations in two steps. First we bring the terms in C into order, so we are left with $(C_\alpha C_\beta C_\gamma C_\delta)^2$. For each of the 4! products in C this requires a different number of commutations, namely

Number of commutations:	0	1	2	3	4	5	6
Number of products:	1	3	5	6	5	3	1

On average we thus have 3 commutations in this first step. From there on it takes 6 more commutations to obtain $C_\alpha^2 C_\beta^2 C_\gamma^2 C_\delta^2$, so we have performed 9 commutations on average. Since we average over all permutations, we have

$$\|C^2 - (C_1^2 C_2^2 C_3^2 C_4^2)_{\mathrm{sym}}\| < 9\varepsilon. \tag{A17}$$

In order to find $a\|[C_\mu, \nabla_\nu]\|$ we use Eqs. (2.15) and (2.5) to determine

$$\begin{aligned} a[C_\mu, \nabla_\nu] &= \frac{1}{4}([T_{\mu+}, T_{\nu+}] + [T_{\mu-}, T_{\nu+}] - [T_{\mu+}, T_{\nu-}] \\ &\quad - [T_{\mu-}, T_{\nu-}]) \\ &= \frac{1}{4}(T_{\mu+} T_{\nu+} (1 - P_{(-\nu)(-\mu)}) + T_{\mu-} T_{\nu+} (1 - P_{(-\nu)\mu}) \\ &\quad - T_{\mu+} T_{\nu-} (1 - P_{\nu(-\mu)}) - T_{\mu-} T_{\nu-} (1 - P_{\nu\mu})), \end{aligned} \tag{A18}$$

from which it follows that

$$\|a[C_\mu, \nabla_\nu]\| < \varepsilon. \tag{A19}$$

Also, for $\mu = \nu$ the commutator trivially vanishes.

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