Semileptonic B_c decays to P-wave charmonium and the nature of $\chi_{c1}(3872)$

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The hadronic form factors of B_c semileptonic decays to the P–wave charmonium 4-plet can be expressed near the zero-recoil point in terms of universal functions, performing a systematic expansion in QCD in the relative velocity of the heavy quarks and in $1/m_Q$. Such functions are independent of the member of the multiplet involved in the transitions. We present the results of a next-to-leading order calculation up to $O(1/m_Q^2)$ classifying the universal functions at this order. We work out a set of relations among the form factors of the same mode and of different modes, which should be reproduced by explicit calculations, reducing the hadronic uncertainty affecting such channels. The approach is also helpful to investigate the debated nature of $\chi_{c1}(3872)$, studying the production in B_c semileptonic decays and comparing it to the modes involving the other $2P$ charmonia.

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I. INTRODUCTION

There is a manifold interest in the study of semileptonic decays of heavy hadrons, namely those induced by the $b \to c\ell\bar{\nu}_\ell$ transition at the quark level. The first one is the possibility to precisely measure fundamental parameters of the Standard Model (SM), in this case the element $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix. The analyses of processes involving different hadrons, modes and final states (exclusive/inclusive) should provide compatible results: such a compatibility has not been achieved, yet, considering the debated $|V_{cb}|$ determinations from inclusive and exclusive B decays [1] which fuel discussions on possible explanations of the tension within the SM [2] or beyond [3]. Other processes, e.g., B_c to charmonium, give access to such a fundamental parameter which has tight correlations with flavor observables [4].

Another important interest relies on the possibility of testing fundamental features of the Standard Model, namely lepton flavor universality (LFU). Signals of LFU violation have been detected in B decays induced by this transition [5]. They hint to physics beyond SM [6], the structure of which can be constrained by studying sets of decay observables [7–14]. Related LFU violating effects should be observed in processes involving different hadrons: the first measurements are available for $B_c \to J/\psi\ell\bar{\nu}_\ell$ [15], other B_c modes can be considered for such investigations.

In addition to the above motivations concerning the structure of the theory of electroweak interaction and of its extensions, there is interest related to strong interaction effects. For decays involving hadrons comprising a single heavy quark Q, a double expansion in QCD in powers of $1/m_Q$ and of α_s provides a powerful method to classify the hadronic matrix elements, both in the exclusive and inclusive transitions. This is the basis for an efficient control of the theoretical uncertainty in the above mentioned measurements. For mesons comprising two heavy quarks such as B_c, the expansion parameter is the relative three-velocity of the heavy quarks, with counting rules given within nonrelativistic QCD (NRQCD). At the various orders in the expansion the form factors governing $B_c \to J/\psi(\eta_c)\ell\bar{\nu}_\ell$ can be given in terms of universal functions, the same for the two final states, in selected kinematical ranges [16–18]. Relations can be established among the form factors, with a reduction of the hadronic uncertainties affecting the description of the semileptonic channels. It is worth considering such an expansion also for the form factors governing the B_c transitions to the P-wave charmonia involving a spin 4-plet of final states, for which little information is available at present.[1]

There is a further reason of interest. Semileptonic decays provide us with a probe on the structure of the hadrons involved in the transitions. This is important, for example, if

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[1]The peculiar role of B_c, a weakly decaying meson comprising only heavy quarks, has been recently discussed in [19,20].

one considers the meson $\chi_{c1}(3872)$ [usually denoted as $X(3872)$]. After the observation by the Belle Collaboration [21] and the confirmation and measurements by other collaborations [22–30], such a meson, whose quantum numbers are $J^{PC} = 1^{++}$ [31,32], is under intense scrutiny due to features hinting a nonconventional charmonium structure. Puzzling properties are the closeness of the mass of $X(3872)$ to the $D^{*0}\bar{D}^0$ threshold and the large decay rate to $J/\psi\pi\pi(J/\psi\rho)$ compared to $J/\psi\pi\pi\pi(J/\psi\omega)$ [33], which can be explained invoking a multiquark structure of $X(3872)$, compact or molecular (see the discussions in Refs. [34,35] and in references therein). On the other hand, the large $\Gamma(X(3872) \to \gamma\psi(2S))/\Gamma(X(3872) \to \gamma J/\psi)$ ratio [36] and data on the production in $\gamma\gamma^*$ interactions [37] support the identification of $\chi_{c1}(3872)$ with the state $\chi_{c1}(2P)$ sitting on the $D^{*0}\bar{D}^0$ threshold [38,39]. Experimental tests have been proposed to help decipher the structure of the meson [40], in the vast literature on the subject.

Semileptonic B_c decays to $\chi_{c1}(3872)$ can provide us with information on the nature of this state. The way is inspired by our analysis in [18] and is based on the systematic comparison of the B_c transitions to the $2P$ charmonia. In a selected kinematical range the B_c and the P–wave charmonium matrix elements can be expressed as an expansion in the heavy quark relative three-velocity in the heavy hadrons, together with an expansion in the inverse heavy quark mass. In this range the B_c to the P–wave charmonia form factors (for lowest lying or radial excitations) are related. The violation or confirmation of such relations in $B_c \to \chi_{c1}(3872)\ell\bar{\nu}$ with respect to the decays to the other $2P$ charmonia would support the interpretation of $X(3872)$ as an exotic or a conventional state.[2]

In this paper we focus on the B_c to the P-wave charmonia form factors, and compute their expressions in terms of universal functions at next-to-leading order (NLO). We classify the universal functions and work out a set of relations useful both to test calculations based on nonperturbative methods and in phenomenological analyses. In Sec. II we parametrize the $B_c \to \chi_{c0,1,2}$ and $B_c \to h_c$ matrix elements of the quark currents appearing in a generalized low-energy Hamiltonian governing the $b \to c\ell\bar{\nu}_\ell$ transition, providing the definition of the set of form factors considered in our study. For the sake of completeness, in Appendix A we also report another parameterization often employed in the literature, with straightforward relations between the two. We give the decay distributions of the four $B_c \to \chi_{c0,1,2}(h_c)\ell\bar{\nu}_\ell$ modes, which can also be used for B_c decays to the $2P$ charmonium resonances. In Sec. III we briefly describe the heavy quark expansion in QCD, we introduce the (B_c, B_c^*) spin doublet and the $(\chi_{c0,1,2}, h_c)$ spin 4-plet and the trace formalism to express the relevant hadronic matrix elements. In Sec. IV we write the form factors in terms of universal functions, with the resulting formulas collected in Appendix B. In Sec. V we present relations among several form factors for single modes and for pairs of modes, other relations being collected in Appendix C. Applications to phenomenology are discussed in Sec. VI, for the lowest lying $1P$ charmonia and for the $2P$ excitations in connection with the $\chi_{c1}(3872)$ issue. Then we conclude.

[2]The comparison between B_c semileptonic and nonleptonic $B_c \to X(3872)\rho(a_1)$ decays, fixing the $X(3872)$ polarization, has been proposed as a way to investigate the structure of this meson [41]. Semileptonic B_c decays to $X(3872)$ have been considered in [42].

II. $B_c \to \chi_{c0,1,2}$ AND $B_c \to h_c$ FORM FACTORS

The semileptonic B_c decays to the charmonium states, including the positive parity $\chi_{c0,1,2}$ and h_c, are governed by a low energy Hamiltonian with general form

$$\begin{aligned}H_{\rm eff}^{b\to c\ell\bar{\nu}} = \frac{G_F}{\sqrt{2}} V_{cb}[&(1+\epsilon_V^\ell)(\bar{c}\gamma_\mu(1-\gamma_5)b)(\bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell)\\ &+\epsilon_R^\ell(\bar{c}\gamma_\mu(1+\gamma_5)b)(\bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell)\\ &+\epsilon_S^\ell(\bar{c}b)(\bar{\ell}(1-\gamma_5)\nu_\ell)+\epsilon_P^\ell(\bar{c}\gamma_5 b)(\bar{\ell}(1-\gamma_5)\nu_\ell)\\ &+\epsilon_T^\ell(\bar{c}\sigma_{\mu\nu}(1-\gamma_5)b)(\bar{\ell}\sigma^{\mu\nu}(1-\gamma_5)\nu_\ell)] \end{aligned} \tag{1}$$

considering the full set of $D=6$ semileptonic $b\to c$ operators with left-handed neutrinos. The general Hamiltonian (1) comprises the Fermi constant G_F, the element V_{cb} of the CKM matrix and the Standard Model operator $\mathcal{O}_{\rm SM} = 4(\bar{c}_L\gamma^\mu b_L)(\bar{\ell}_L\gamma_\mu\nu_{\ell L})$. It also includes the operators $\mathcal{O}_R = 4(\bar{c}_R\gamma^\mu b_R)(\bar{\ell}_L\gamma_\mu\nu_{\ell L})$, $\mathcal{O}_S = (\bar{c}b)(\bar{\ell}(1-\gamma_5)\nu_\ell)$, $\mathcal{O}_P = (\bar{c}\gamma_5 b)(\bar{\ell}(1-\gamma_5)\nu_\ell)$ and $\mathcal{O}_T = (\bar{c}\sigma_{\mu\nu}(1-\gamma_5)b)(\bar{\ell}\sigma^{\mu\nu}(1-\gamma_5)\nu_\ell)$ arising in extensions of SM, with Wilson coefficients $\epsilon^\ell_{V,R,S,P,T}$ in general complex and lepton-flavor dependent. The SM Hamiltonian corresponds to $\epsilon_i^\ell = 0$. Eq. (1) has been considered in connection with the $B\to D^{(*)}\tau\nu_\tau$, $B\to D^{(*)}\ell\nu_\ell$ LFU anomaly [3,8–13].

It is common to parametrize the $B_c\to\chi_{ci}$ ($i=0,1,2$) and $B_c\to h_c$ matrix elements of the quark currents in Eq. (1) in terms of form factors as written in Appendix A [43–46]. p and p' are the four momenta of the initial and final meson, ϵ (η) the polarization vector (tensor) of the spin 1 (spin 2) charmonium, $q = p - p'$ the lepton pair momentum. We use $\epsilon^{0123} = -1$, hence $\sigma^{\mu\nu}\gamma_5 = -\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$. In our analysis it is more convenient to use a parametrization of the matrix elements in terms of the four-velocities of B_c and of the charmonium state $C = \chi_{c0,1,2}, h_c$: $v = p/m_{B_c}$, and $v' = p'/m_C$, with $w = v\cdot v'$. They are defined as follows.

$B_c\to\chi_{c0}$:

$$\begin{aligned}\langle\chi_{c0}(v')|\bar{c}\gamma_\mu\gamma_5 b|B_c(v)\rangle &= \sqrt{m_{\chi_{c0}}m_{B_c}}[g_+(w)(v+v')_\mu\\ &\quad + g_-(w)(v-v')_\mu]\\ \langle\chi_{c0}(v')|\bar{c}\gamma_5 b|B_c(v)\rangle &= \sqrt{m_{\chi_{c0}}m_{B_c}}g_P(w)\\ \langle\chi_{c0}(v')|\bar{c}\sigma_{\mu\nu} b|B_c(v)\rangle &= \sqrt{m_{\chi_{c0}}m_{B_c}}g_T(w)\epsilon_{\mu\nu\alpha\beta}v^\alpha v'^\beta \end{aligned} \tag{2}$$

$B_c \to \chi_{c1}$:

$$\begin{aligned}
\langle \chi_{c1}(v',\epsilon)|\bar{c}\gamma_\mu b|B_c(v)\rangle &= i\sqrt{m_{\chi_{c1}} m_{B_c}}[g_{V_1}(w)\epsilon^*_\mu + (\epsilon^*\cdot v)[g_{V_2}(w)(v+v')_\mu + g_{V_3}(w)(v-v')_\mu]] \\
\langle \chi_{c1}(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B_c(v)\rangle &= \sqrt{m_{\chi_{c1}} m_{B_c}}g_A(w)\epsilon_{\mu\alpha\beta\sigma}\epsilon^{*\alpha}v^\beta v'^\sigma \\
\langle \chi_{c1}(v',\epsilon)|\bar{c}b|B_c(v)\rangle &= i\sqrt{m_{\chi_{c1}} m_{B_c}}g_S(w)(\epsilon^*\cdot v) \\
\langle \chi_{c1}(v',\epsilon)|\bar{c}\sigma_{\mu\nu} b|B_c(v)\rangle &= \sqrt{m_{\chi_{c1}} m_{B_c}}[g_{T_1}(w)(\epsilon^*_\mu(v+v')_\nu - \epsilon^*_\nu(v+v')_\mu) \\
&\quad + g_{T_2}(w)(\epsilon^*_\mu(v-v')_\nu - \epsilon^*_\nu(v-v')_\mu) + g_{T_3}(w)(\epsilon^*\cdot v)(v_\mu v'_\nu - v_\nu v'_\mu)]
\end{aligned} \tag{3}$$

$B_c \to \chi_{c2}$:

$$\begin{aligned}
\langle \chi_{c2}(v',\eta)|\bar{c}\gamma_\mu b|B_c(v)\rangle &= \sqrt{m_{\chi_{c2}} m_{B_c}}ik_V(w)\epsilon_{\mu\alpha\beta\sigma}\eta^{*\alpha\tau}v_\tau v^\beta v'^\sigma \\
\langle \chi_{c2}(v',\eta)|\bar{c}\gamma_\mu\gamma_5 b|B_c(v)\rangle &= \sqrt{m_{\chi_{c2}} m_{B_c}}[k_{A_1}(w)\eta^*_{\mu\alpha}v^\alpha + \eta^*_{\alpha\beta}v^\alpha v^\beta(k_{A_2}(w)v_\mu + k_{A_3}(w)v'_\mu)] \\
\langle \chi_{c2}(v',\eta)|\bar{c}\gamma_5 b|B_c(v)\rangle &= \sqrt{m_{\chi_{c2}} m_{B_c}}k_P(w)\eta^*_{\alpha\beta}v^\alpha v^\beta \\
\langle \chi_{c2}(v',\eta)|\bar{c}\sigma_{\mu\nu}\gamma_5 b|B_c(v)\rangle &= i\sqrt{m_{\chi_{c2}} m_{B_c}}[k_{T_1}(w)(\eta^{*\alpha}_{\ \mu}v_\alpha v_\nu - \eta^{*\alpha}_{\ \nu}v_\alpha v_\mu) \\
&\quad + k_{T_2}(w)(\eta^{*\alpha}_{\ \mu}v_\alpha v'_\nu - \eta^{*\alpha}_{\ \nu}v_\alpha v'_\mu) + k_{T_3}(w)\eta^*_{\alpha\beta}v^\alpha v^\beta(v_\mu v'_\nu - v_\nu v'_\mu)]
\end{aligned} \tag{4}$$

$B_c \to h_c$:

$$\begin{aligned}
\langle h_c(v',\epsilon)|\bar{c}\gamma_\mu b|B_c(v)\rangle &= \sqrt{m_{h_c} m_{B_c}}[f_{V_1}(w)\epsilon^*_\mu + (\epsilon^*\cdot v)(f_{V_2}(w)(v+v')_\mu + f_{V_3}(w)(v-v')_\mu)] \\
\langle h_c(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B_c(v)\rangle &= i\sqrt{m_{h_c} m_{B_c}}f_A(w)\epsilon_{\mu\alpha\beta\sigma}\epsilon^{*\alpha}v^\beta v'^\sigma \\
\langle h_c(v',\epsilon)|\bar{c}b|B_c(v)\rangle &= \sqrt{m_{h_c} m_{B_c}}(\epsilon^*\cdot v)f_S(w) \\
\langle h_c(v',\epsilon)|\bar{c}\sigma_{\mu\nu} b|B_c(v)\rangle &= i\sqrt{m_{h_c} m_{B_c}}[f_{T_1}(w)(\epsilon^*_\mu(v+v')_\nu - \epsilon^*_\nu(v+v')_\mu) \\
&\quad + f_{T_2}(w)(\epsilon^*_\mu(v-v')_\nu - \epsilon^*_\nu(v-v')_\mu) + f_{T3}(w)(\epsilon^*\cdot v)(v_\mu v'_\nu - v_\nu v'_\mu)].
\end{aligned} \tag{5}$$

The decay distributions governed by the Hamiltonian (1) can be written in the form:

$$\begin{aligned}
\frac{d\Gamma(B_c\to C\ell\bar{\nu})}{dw} = \tilde{\Gamma}\Bigg\{ & |1+\epsilon_V|^2\frac{d\Gamma^{SM}}{dw} + |\epsilon_R|^2\frac{d\Gamma^R}{dw} + |\epsilon_X|^2\frac{d\Gamma^X}{dw} + |\epsilon_T|^2\frac{d\Gamma^T}{dw} \\
& + 2\mathrm{Re}[\epsilon_R(1+\epsilon_V^*)]\frac{d\Gamma^{SMR}}{dw} + 2\mathrm{Re}[\epsilon_X(1+\epsilon_V^*)]\frac{d\Gamma^{SMX}}{dw} \\
& + 2\mathrm{Re}[\epsilon_T(1+\epsilon_V^*)]\frac{d\Gamma^{SMT}}{dw} + 2\mathrm{Re}[\epsilon_R\epsilon_T^*]\frac{d\Gamma^{RT}}{dw} \\
& + 2\mathrm{Re}[\epsilon_X\epsilon_R^*]\frac{d\Gamma^{XR}}{dw} + 2\mathrm{Re}[\epsilon_X\epsilon_T^*]\frac{d\Gamma^{XT}}{dw}\Bigg\},
\end{aligned} \tag{6}$$

with $X = P$ in case of χ_{c0}, χ_{c2}, and $X = S$ in case of χ_{c1}, h_c. In Eq. (6) we define

$$\tilde{\Gamma} = \frac{G_F^2|V_{cb}|^2 m_{B_c}^5 r^3}{48\pi^3}\sqrt{w^2-1}\left(1-\frac{\hat{m}_\ell^2}{1+r^2-2rw}\right)^2 \tag{7}$$

with $r = m_C/m_{B_c}$ ($C = \chi_{c0,c1,c2}, h_c$) and $\hat{m}_\ell = m_\ell/m_{B_c}$. For the four decay modes the expressions of the functions in (6) in terms of the form factors in (2) –(5) are as follows.

(i) $C = \chi_{c0}$:

$$\frac{d\Gamma^{SM}}{dw} = \frac{d\Gamma^{R}}{dw} = -\frac{d\Gamma^{SMR}}{dw} = [g_+(w)]^2(w+1)\left[(w-1)(1+r)^2 + \frac{\hat{m}_\ell^2}{1+r^2-2rw}((2w+1)(1+r^2)-2r(w+2))\right] + [g_-(w)]^2(w-1)\left[(w+1)(1-r)^2 + \frac{\hat{m}_\ell^2}{1+r^2-2rw}((2w-1)(1+r^2)+2r(w-2))\right] - 2g_+(w)g_-(w)(w^2-1)(1-r^2)\left(1+\frac{2\hat{m}_\ell^2}{1+r^2-2rw}\right) \tag{8}$$

$$\frac{d\Gamma^{P}}{dw} = \frac{3}{2}[g_P(w)]^2(1+r^2-2rw) \tag{9}$$

$$\frac{d\Gamma^{T}}{dw} = 8[g_T(w)]^2(1+r^2-2rw+2\hat{m}_\ell^2)(w^2-1) \tag{10}$$

$$\frac{d\Gamma^{SMP}}{dw} = -\frac{d\Gamma^{PR}}{dw} = \frac{3}{2}g_P(w)\hat{m}_\ell[(w-1)(1+r)g_-(w)-(w+1)(1-r)g_+(w)] \tag{11}$$

$$\frac{d\Gamma^{SMT}}{dw} = -\frac{d\Gamma^{RT}}{dw} = -6g_T(w)\hat{m}_\ell(w^2-1)[(1+r)g_+(w)-(1-r)g_-(w)]. \tag{12}$$

(ii) $C = \chi_{c1}$:

$$\frac{d\Gamma^{SM}}{dw} = \frac{d\Gamma^{R}}{dw} = [g_{V_1}(w)]^2\left[3(w-r)^2-2(w^2-1)+\frac{\hat{m}_\ell^2}{2(1+r^2-2rw)}(3(w-r)^2+w^2-1)\right] + [g_{V_2}(w)]^2(w^2-1)(w+1)\left[(1+r)^2(w-1)+\frac{\hat{m}_\ell^2}{1+r^2-2rw}((2w+1)(1+r^2)-2r(w+2))\right] + [g_{V_3}(w)]^2(w^2-1)(w-1)\left[(1-r)^2(w+1)+\frac{\hat{m}_\ell^2}{1+r^2-2rw}((2w-1)(1+r^2)+2r(w-2))\right] + [g_A(w)]^2(w^2-1)(2(1+r^2-2rw)+\hat{m}_\ell^2) + g_{V_1}(w)g_{V_2}(w)(w^2-1)\left[2(1+r)(w-r)-\frac{\hat{m}_\ell^2}{1+r^2-2rw}(-3+r^2-4w+2r(w+2))\right] - g_{V_1}(w)g_{V_3}(w)(w^2-1)\left[2(1-r)(w-r)+\frac{\hat{m}_\ell^2}{1+r^2-2rw}(-3+r^2+4w+2r(w-2))\right] - 2g_{V_2}(w)g_{V_3}(w)(w^2-1)^2(1-r^2)\left(1+\frac{2\hat{m}_\ell^2}{1+r^2-2rw}\right) \tag{13}$$

$$\frac{d\Gamma^{SMR}}{dw} = \frac{d\Gamma^{SM}}{dw} - 2[g_A(w)]^2(w^2-1)(2(1+r^2-2rw)+\hat{m}_\ell^2) \tag{14}$$

$$\frac{d\Gamma^{S}}{dw} = \frac{3}{2}[g_S(w)]^2(1+r^2-2rw)(w^2-1) \tag{15}$$

$$\frac{d\Gamma^{T}}{dw} = 8\left(1+\frac{2\hat{m}_\ell^2}{1+r^2-2rw}\right)\{[g_{T_1}(w)]^2(w+1)[(5w+1)(1+r^2)-2r(w^2+w+4)] + [g_{T_2}(w)]^2(w-1)[(5w-1)(1+r^2)-2r(w^2-w+4)] + [g_{T_3}(w)]^2(w^2-1)^2(1+r^2-2rw) + 2g_{T_1}(w)g_{T_2}(w)(w^2-1)(5r^2-2rw-3) - 2g_{T_1}(w)g_{T_3}(w)(w^2-1)(w+1)(1+r^2-2rw) - 2g_{T_2}(w)g_{T_3}(w)(w^2-1)(w-1)(1+r^2-2rw)\} \tag{16}$$

$$\frac{d\Gamma^{SMS}}{dw} = \frac{d\Gamma^{SR}}{dw} = \frac{3}{2} g_S(w)\hat{m}_\ell(w^2-1)[g_{V_1}(w) + (w+1)(1-r)g_{V_2}(w) - (w-1)(1+r)g_{V_3}(w)] \tag{17}$$

$$\begin{aligned}\frac{d\Gamma^{SMT}}{dw} = {} & 6\hat{m}_\ell\{-g_{T_1}(w)(w+1)[(2-3r+w)g_{V_1}(w) + (w^2-1)[(1+r)g_{V_2}(w) - (1-r)g_{V_3}(w)]] \\ & - g_{T_2}(w)(w-1)[(-2-3r+w)g_{V_1}(w) + (w^2-1)[(1+r)g_{V_2}(w) - (1-r)g_{V_3}(w)]] \\ & + g_{T_3}(w)(w^2-1)[(w-r)g_{V_1}(w) + (w^2-1)[(1+r)g_{V_2}(w) - (1-r)g_{V_3}(w)]] \\ & + 2(w^2-1)g_A(w)[(1+r)g_{T_1}(w) - (1-r)g_{T_2}(w)]\}\end{aligned} \tag{18}$$

$$\frac{d\Gamma^{RT}}{dw} = \frac{d\Gamma^{SMT}}{dw} - 24\hat{m}_\ell(w^2-1)g_A(w)[(1+r)g_{T_1}(w) - (1-r)g_{T_2}(w)]. \tag{19}$$

(iii) $C = \chi_{c2}$:

$$\begin{aligned}\frac{d\Gamma^{SM}}{dw} = \frac{d\Gamma^{R}}{dw} = \frac{w^2-1}{6}\Bigg\{ & [k_V(w)]^2 3\left(2 + \frac{\hat{m}_\ell^2}{1+r^2-2rw}\right)(w^2-1)(1+r^2-2rw) \\ & + [k_{A_1}(w)]^2\left[2(3+5r^2-10rw+2w^2) + \frac{\hat{m}_\ell^2}{1+r^2-2rw}(-3+5r^2-10rw+8w^2)\right] \\ & + [k_{A_2}(w)]^2 2(w^2-1)\left[2r^2(w^2-1) + \frac{\hat{m}_\ell^2}{1+r^2-2rw}(3-6rw+r^2(4w^2-1))\right] \\ & + [k_{A_3}(w)]^2 2(w^2-1)\left[2(w^2-1) + \frac{\hat{m}_\ell^2}{1+r^2-2rw}(-1+3r^2-6rw+4w^2))\right] \\ & + 4(w^2-1)\Bigg[k_{A_1}(w)k_{A_2}(w)\left[2r(w-r) + \frac{\hat{m}_\ell^2}{1+r^2-2rw}(3-r^2-2rw)\right] \\ & - k_{A_2}(w)k_{A_3}(w)\left[-2r(w^2-1) + \frac{\hat{m}_\ell^2}{1+r^2-2rw}[2r(w^2+2) - 3w(1+r^2)]\right] \\ & + 2k_{A_1}(w)k_{A_3}(w)\left(1 + \frac{2\hat{m}_\ell^2}{1+r^2-2rw}\right)(w-r)\Bigg]\Bigg\}\end{aligned} \tag{20}$$

$$\frac{d\Gamma^{SMR}}{dw} = -\frac{d\Gamma^{SM}}{dw} + [k_V(w)]^2(2(1+r^2-2rw) + \hat{m}_\ell^2)(w^2-1)^2 \tag{21}$$

$$\frac{d\Gamma^{P}}{dw} = [k_P(w)]^2(w^2-1)^2(1+r^2-2rw) \tag{22}$$

$$\begin{aligned}\frac{d\Gamma^{T}}{dw} = {} & \frac{8}{3}\left(1 + \frac{2\hat{m}_\ell^2}{1+r^2-2rw}\right)(w^2-1)\{[k_{T_1}(w)]^2[(3+2w^2)(1-2rw) + r^2(8w^2-3)] \\ & + [k_{T_2}(w)]^2(-1+5r^2-10rw+6w^2) + [k_{T_3}(w)]^2 2(w^2-1)^2(1+r^2-2rw) \\ & - 2k_{T_1}(w)k_{T_2}(w)[6r - 5w(1+r^2) + 4rw^2] \\ & - 4(w^2-1)(1+r^2-2rw)k_{T_3}(w)[wk_{T_1}(w) + k_{T_2}(w)]\}\end{aligned} \tag{23}$$

$$\frac{d\Gamma^{SMP}}{dw} = -\frac{d\Gamma^{PR}}{dw} = -k_P(w)\hat{m}_\ell(w^2-1)^2[k_{A_1}(w) + (1-rw)k_{A_2}(w) + (w-r)k_{A_3}(w)] \tag{24}$$

$$\frac{d\Gamma^{SMT}}{dw} = 2\hat{m}_\ell(w^2-1)\times\{k_{T_1}(w)[(3-5rw+2w^2)k_{A_1}(w)+(w^2-1)(2rwk_{A_2}(w)+2wk_{A_3}(w)+3rk_V(w))] + k_{T_2}(w)[5(w-r)k_{A_1}(w)+(w^2-1)(2rk_{A_2}(w)+2k_{A_3}(w)+3k_V(w))] - 2k_{T_3}(w)(w^2-1)[(w-r)k_{A_1}(w)+(w^2-1)(rk_{A_2}(w)+k_{A_3}(w))]\} \tag{25}$$

$$\frac{d\Gamma^{RT}}{dw} = -\frac{d\Gamma^{SMT}}{dw} + 12\hat{m}_\ell(w^2-1)^2k_V(w)[rk_{T_1}(w)+k_{T_2}(w)]. \tag{26}$$

(iv) $C = h_c$:
Looking at the matrix elements for χ_{c1} and h_c in (3) and (5), the distributions for h_c are obtained from those of χ_{c1} replacing

$$g_{V_i}g_{V_j}\to f_{V_i}f_{V_j},\qquad g_A^2\to f_A^2,\qquad g_S^2\to f_S^2,\qquad g_{T_i}g_{T_j}\to f_{T_i}f_{T_j},$$
$$g_Sg_{V_i}\to f_Sf_{V_i},\qquad g_{V_i}g_{T_j}\to -f_{V_i}f_{T_j},\qquad g_Ag_{T_i}\to f_Af_{T_i}, \tag{27}$$

with $i,\ j = 1,\ 2,\ 3$.

The above expressions also hold for B_c decays to the $2P$ (and higher) charmonium resonances. They are the subject of our analysis together with the form factors in Eqs. (2)–(5).

III. FORM FACTORS IN THE EFFECTIVE THEORY

As discussed in [18], we aim at expressing the form factors in the effective theory of QCD resulting by an expansion in the inverse heavy quark mass. The heavy quark QCD field $Q(x)$ with mass m_Q is written, with the notations in [47], as

$$Q(x) = e^{-im_Q v\cdot x}\psi(x) = e^{-im_Q v\cdot x}(\psi_+(x)+\psi_-(x)) \tag{28}$$

with $\psi_\pm = P_\pm\psi(x)$ and $P_\pm = \frac{1\pm \slashed{v}}{2}$. ψ_+ is the positive energy component of the field, v is the heavy meson (quarkonium) 4-velocity with $v^2=1$. Hence, $Q(x)$ is expressed as

$$Q(x) = e^{-im_Q v\cdot x}\left(1+\frac{i\slashed{D}_\perp}{2m_Q}+\frac{(-iv\cdot D)}{2m_Q}\frac{i\slashed{D}_\perp}{2m_Q}+\dots\right)\psi_+(x), \tag{29}$$

with $D_{\perp\mu} = D_\mu - (v\cdot D)v_\mu$. In the rest frame $v=(1,\vec{0})$ we have $v\cdot D = D_t$ and $D_{\perp\mu} = (0, D_i)$. The QCD Lagrangian is expressed in terms of ψ_+ as an expansion,

$$\mathcal{L}_{\rm QCD} = \bar\psi_+(x)\left(iv\cdot D + \frac{(iD_\perp)^2}{2m_Q} + \frac{g}{4m_Q}\sigma\cdot G_\perp + \frac{i\slashed{D}_\perp}{2m_Q}\frac{(-iv\cdot D)}{2m_Q}(i\slashed{D}_\perp)+\dots\right)\psi_+(x) = \mathcal{L}_0+\mathcal{L}_1+\dots, \tag{30}$$

with $G_{\perp\mu\nu} = (g_{\mu\alpha}-v_\mu v_\alpha)(g_{\nu\beta}-v_\nu v_\beta)G^{\alpha\beta}$.

Nonrelativistic QCD provides us with power counting of the operators in terms of the relative heavy quark 3-velocity in the hadron rest frame $\tilde{v} = |\vec{\tilde{v}}| \ll 1$, starting from $D_t\sim\tilde{v}^2$, $D_\perp\sim\tilde{v}$ and $\psi_+\sim\tilde{v}^{3/2}$ [48]. Hence, the second term in Eq. (29) is $\mathcal{O}(\tilde{v}\times\tilde{v}^{3/2})$, the third one is $\mathcal{O}(\tilde{v}^3\times\tilde{v}^{3/2})$, and so on. In the following we omit the power $\tilde{v}^{3/2}$ for each quark field in the power counting of the various operators. Then, the chromoelectric field components $E_i = G_{0i}$ and the chromomagnetic ones $B_i = \frac{1}{2}\epsilon_{ijk}G^{jk}$ are $\mathcal{O}(\tilde{v}^3)$ and $\mathcal{O}(\tilde{v}^4)$, respectively [49].

The first two terms in Eq. (30) are $\mathcal{O}(\tilde{v}^2)$. They give the leading order effective Lagrangian

$$\mathcal{L}_0 = \bar\psi_+(x)\left(iv\cdot D+\frac{(iD_\perp)^2}{2m_Q}\right)\psi_+(x). \tag{31}$$

The third and fourth term in Eq. (30) are $\mathcal{O}(\tilde{v}^4)$. They correspond to the NLO term

$$\mathcal{L}_1 = \mathcal{L}_{1,1}+\mathcal{L}_{1,2}, \tag{32}$$

where

$$\mathcal{L}_{1,1} = \bar\psi_+(x)\frac{g\sigma\cdot G_\perp}{4m_Q}\psi_+(x)$$
$$\mathcal{L}_{1,2} = \bar\psi_+(x)\frac{i\slashed{D}_\perp}{2m_Q}\frac{(-iv\cdot D)}{2m_Q}(i\slashed{D}_\perp)\psi_+(x). \tag{33}$$

$\mathcal{L}_{1,2}$ can be written as

$$\begin{aligned}\mathcal{L}_{1,2} = &-\frac{1}{4m_Q^2}\left(\bar{\psi}_+(x)\left(-\frac{(iD_\perp)^4}{2m_Q}\right)\psi_+(x)\right.\\&+\bar{\psi}_+(x)\frac{g}{2}\sigma\cdot G_\perp\left(-\frac{(iD_\perp)^2}{2m_Q}\right)\psi_+(x)\\&+igv^\alpha\bar{\psi}_+(x)iD_\perp^\sigma G_{\alpha\sigma}\psi_+(x)\\&\left.+gv^\alpha\bar{\psi}_+(x)iD_{\perp\tau}\sigma^{\tau\sigma}G_{\alpha\sigma}\psi_+(x)\right)\\=&\,\mathcal{L}_{1,2}^{(1)}+\mathcal{L}_{1,2}^{(2)}+\mathcal{L}_{1,2}^{(3)}+\mathcal{L}_{1,2}^{(4)},\end{aligned} \tag{34}$$

with $\mathcal{L}_{1,2}^{(2)}$ of higher order in $\tilde{v}$. $\mathcal{L}_1$ can be arranged as

$$\mathcal{L}_1 = \mathcal{L}_1^A + \mathcal{L}_1^B \tag{35}$$

with

$$\mathcal{L}_1^A = \mathcal{L}_{1,1} + \mathcal{L}_{1,2}^{(4)} = \frac{1}{4m_Q}\bar{\psi}_+(x)A_{\tau\sigma}\sigma^{\tau\sigma}\psi_+(x) \tag{36}$$

$$\mathcal{L}_1^B = \mathcal{L}_{1,2}^{(1)} + \mathcal{L}_{1,2}^{(3)} = \frac{1}{4m_Q^2}\bar{\psi}_+(x)B\psi_+(x), \tag{37}$$

factorizing in (36) the leading $1/m_Q$ power.

An analogous expansion describes the antiquark: $Q(x)$ is written as

$$Q(x) = e^{im_Q v\cdot x}X(x) = e^{im_Q v\cdot x}(X_+(x)+X_-(x)) \tag{38}$$

giving

$$\mathcal{L}_{\rm QCD} = \bar{X}_-(x)\left(-iv\cdot D+\frac{(iD_\perp)^2}{2m_Q}+\dots\right)X_-(x). \tag{39}$$

To express the heavy meson form factors in the effective QCD theory, we expand the weak current involving two heavy quarks $\bar{Q}'\Gamma Q$, with Γ a generic Dirac matrix:

$$\begin{aligned}&\bar{Q}'(x)\Gamma Q(x)\\&=\bar{\psi}'_+(x)\left(1-\frac{i\overleftarrow{\not{D}}'_\perp}{2m_{Q'}}-\frac{1}{4m_{Q'}^2}(i\overleftarrow{\not{D}}'_\perp)(iv'\cdot\overleftarrow{D})+\dots\right)\\&\quad\Gamma\left(1+\frac{i\vec{\not{D}}_\perp}{2m_Q}+\frac{1}{4m_Q^2}(-iv\cdot\vec{D})i\vec{\not{D}}_\perp+\dots\right)\psi_+(x),\end{aligned} \tag{40}$$

where $D'_{\perp\mu} = D_\mu-(v'\cdot D)v'_\mu$. Up to $\mathcal{O}(1/m_Q^2)$ the expansion reads:

$$\begin{aligned}\bar{Q}'(x)\Gamma Q(x) = &\,J_0+\left(\frac{J_{1,0}}{2m_Q}+\frac{J_{0,1}}{2m_{Q'}}\right)\\&+\left(-\frac{J_{2,0}}{4m_Q^2}-\frac{J_{0,2}}{4m_{Q'}^2}+\frac{J_{1,1}}{4m_Qm_{Q'}}\right),\end{aligned} \tag{41}$$

with

$$\begin{aligned}J_0 &= \bar{\psi}'_+\Gamma\psi_+\\J_{1,0} &= \bar{\psi}'_+\Gamma i\vec{\not{D}}_\perp\psi_+\\J_{0,1} &= \bar{\psi}'_+(-i\overleftarrow{\not{D}}'_\perp)\Gamma\psi_+\\J_{2,0} &= \bar{\psi}'_+\Gamma(iv\cdot\vec{D})i\vec{\not{D}}_\perp\psi_+\\J_{0,2} &= \bar{\psi}'_+i\overleftarrow{\not{D}}'_\perp(iv'\cdot\overleftarrow{D})\Gamma\psi_+\\J_{1,1} &= \bar{\psi}'_+(-i\overleftarrow{\not{D}}'_\perp)\Gamma(i\vec{\not{D}}_\perp)\psi_+.\end{aligned} \tag{42}$$

Equation (41) comprises terms up to $\mathcal{O}(\tilde{v}^3)$. $\mathcal{O}(1/m_Q^3)$ terms with three derivatives have not been included, even though they can be of the same order in $\tilde{v}$ of some terms in the corrections discussed in the following: we assume that their contribution is small.

The hadronic matrix elements of the operators in Eq. (41) can be written using the trace formalism [50]. For states comprising a heavy quark and a heavy antiquark, spin symmetry is expected to hold in the preasymptotic mass range which includes beauty and charm, outside the QCD Coulombic regime [16,51]. Hence, the two states (B_c, B_c^*) and the four states $(\chi_{c0,1,2}, h_c)$ can be organized in a negative parity spin doublet and in a positive parity spin 4-plet. They are described by 4×4 matrices, for the spin doublet [16]

$$\mathcal{M}(v) = P_+(v)[B_c^{*\mu}\gamma_\mu - B_c\gamma_5]P_-(v), \tag{43}$$

for the spin 4-plet [52]

$$\begin{aligned}\mathcal{M}'^\mu(v') = P_+(v')\Big[&\chi_{c2}^{\mu\nu}\gamma_\nu+\frac{1}{\sqrt{2}}\chi_{c1,\gamma}\epsilon^{\mu\alpha\beta\gamma}v'_\alpha\gamma_\beta\\&+\frac{1}{\sqrt{3}}\chi_{c0}(\gamma^\mu-v'^\mu)+h_c^\mu\gamma_5\Big]P_-(v')\end{aligned} \tag{44}$$

with the condition

$$v'_\mu\mathcal{M}'^\mu = 0. \tag{45}$$

The normalization is $\sqrt{m_M}$, with M a meson in the spin multiplet. Radial excitations belong to multiplets analogous to (43) and (44). The trace formalism has been used to obtain the effective Lagrangians governing strong and radiative heavy quarkonium transitions in the soft-exchange approximation [51–53]. We apply it to express the various form factors in terms of (universal) functions independent of the particular decay mode.

IV. FORM FACTORS IN TERMS OF UNIVERSAL FUNCTIONS

Applying the trace formalism, the matrix elements of the leading order term of the expansion of the quark current (41) are written as

$$\langle M'(v')|J_0|M(v)\rangle = -\Xi(w) v_\mu \mathrm{Tr}[\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] \quad (46)$$

involving the single universal function $\Xi(w)$.

At $1/m_Q$ the relevant matrix elements are parametrized as

$$\langle M'(v')|\bar{\psi}'_+ \Gamma i\vec{D}_\alpha \psi_+|M(v)\rangle = -\mathrm{Tr}[\Sigma^{(b)}_{\mu\alpha}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}], \quad (47a)$$

$$\langle M'(v')|\bar{\psi}'_+(-i\overleftarrow{D}_\alpha)\Gamma \psi_+|M(v)\rangle = -\mathrm{Tr}[\Sigma^{(c)}_{\mu\alpha}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}]. \quad (47b)$$

The general expression of the functions $\Sigma^{(Q)}_{\mu\alpha}$ is

$$\Sigma^{(Q)}_{\mu\alpha} = \Sigma^{(Q)}_1 g_{\mu\alpha} + \Sigma^{(Q)}_2 v_\mu v_\alpha + \Sigma^{(Q)}_3 v_\mu v'_\alpha + \Sigma^{(Q)}_4 v_\mu \gamma_\alpha + \Sigma^{(Q)}_5 \gamma_\mu v_\alpha + \Sigma^{(Q)}_6 \gamma_\mu v'_\alpha + \Sigma^{(Q)}_7 i\sigma_{\mu\alpha}. \quad (48)$$

The terms proportional to v'^μ vanish due to (45). $\Sigma^{(Q)}_i(w)$ ($i = 1,\ldots 7$) are the universal functions at this order. Upon integration by parts, relations can be worked out. Using

$$i\partial_\alpha(\bar{\psi}'_+\Gamma\psi_+) = \bar{\psi}'_+ i\overleftarrow{D}_\alpha \Gamma\psi_+ + \bar{\psi}'_+\Gamma i\vec{D}_\alpha \psi_+, \quad (49)$$

we obtain:

$$-(\tilde{\Lambda} v_\alpha - \tilde{\Lambda}' v'_\alpha)\Xi v_\mu \mathrm{Tr}[\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] = \mathrm{Tr}[\Sigma^{(c)}_{\mu\alpha}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] - \mathrm{Tr}[\Sigma^{(b)}_{\mu\alpha}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}]. \quad (50)$$

For the $Q\bar{Q}'$ mesons in the initial and final state the parameters $\tilde{\Lambda}$, $\tilde{\Lambda}'$ are given by

$$\tilde{\Lambda} = m_H - m_Q - m_{\bar{Q}'} \quad (51)$$

and analogously for $\tilde{\Lambda}'$. The relations follow:

$$\Sigma^{(b)}_i(w) - \Sigma^{(c)}_i(w) = 0 \qquad i = 1,4,5,6,7, \quad (52a)$$

$$\Sigma^{(b)}_2(w) - \Sigma^{(c)}_2(w) = \tilde{\Lambda}\Xi, \quad (52b)$$

$$\Sigma^{(b)}_3(w) - \Sigma^{(c)}_3(w) = -\tilde{\Lambda}'\Xi(w). \quad (52c)$$

The same procedure allows us to parametrize the matrix elements relevant for the $1/m_Q^2$ terms in (41):

$$\langle M'(v')|\bar{\psi}'_+\Gamma i\vec{D}_\alpha i\vec{D}_\beta \psi_+|M(v)\rangle = -\mathrm{Tr}[\Omega^{(b)}_{\mu\alpha\beta}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] \quad (53a)$$

$$\langle M'(v')|\bar{\psi}'_+ i\overleftarrow{D}_\alpha i\overleftarrow{D}_\beta \Gamma \psi_+|M(v)\rangle = -\mathrm{Tr}[\Omega^{(c)}_{\mu\alpha\beta}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}]. \quad (53b)$$

$\Omega^{(Q)}_{\mu\alpha\beta}$ have the general expression

$$\begin{aligned}\Omega^{(Q)}_{\mu\alpha\beta} = {} & \Omega^{(Q)}_1 g_{\mu\alpha} v_\beta + \Omega^{(Q)}_2 g_{\alpha\beta} v_\mu + \Omega^{(Q)}_3 g_{\beta\mu} v_\alpha + \Omega^{(Q)}_4 g_{\mu\alpha} v'_\beta + \Omega^{(Q)}_5 g_{\beta\mu} v'_\alpha + \Omega^{(Q)}_6 g_{\mu\alpha}\gamma_\beta \\ & + \Omega^{(Q)}_7 g_{\alpha\beta}\gamma_\mu + \Omega^{(Q)}_8 g_{\beta\mu}\gamma_\alpha + \Omega^{(Q)}_9 v_\mu v_\alpha v_\beta + \Omega^{(Q)}_{10} v_\mu v_\alpha v'_\beta + \Omega^{(Q)}_{11} v_\mu v'_\alpha v_\beta \\ & + \Omega^{(Q)}_{12} v_\mu v'_\alpha v'_\beta + \Omega^{(Q)}_{13} v_\mu v_\alpha \gamma_\beta + \Omega^{(Q)}_{14} v_\mu \gamma_\alpha v_\beta + \Omega^{(Q)}_{15} \gamma_\mu v_\alpha v_\beta + \Omega^{(Q)}_{16} v_\mu v'_\alpha \gamma_\beta \\ & + \Omega^{(Q)}_{17} \gamma_\mu v_\alpha v'_\beta + \Omega^{(Q)}_{18} v_\mu \gamma_\alpha v'_\beta + \Omega^{(Q)}_{19} \gamma_\mu v'_\alpha v_\beta + \Omega^{(Q)}_{20} \gamma_\mu v'_\alpha v'_\beta + \Omega^{(Q)}_{21} i\sigma_{\mu\alpha} v_\beta \\ & + \Omega^{(Q)}_{22} i\sigma_{\alpha\beta} v_\mu + \Omega^{(Q)}_{23} i\sigma_{\beta\mu} v_\alpha + \Omega^{(Q)}_{24} i\sigma_{\mu\alpha} v'_\beta + \Omega^{(Q)}_{25} i\sigma_{\beta\mu} v'_\alpha. \end{aligned} \quad (54)$$

The terms proportional to v'^μ vanish. The matrix element with two derivatives, one acting on ψ_+, the other one on $\bar{\psi}'_+$, can be parametrized integrating by parts either Eq. (53a),

$$\langle M'(v')|\bar{\psi}'_+(-i\overleftarrow{D}_\alpha)\Gamma i\vec{D}_\beta \psi_+|M(v)\rangle = (\tilde{\Lambda} v_\alpha - \tilde{\Lambda}' v'_\alpha)\mathrm{Tr}[\Sigma^{(b)}_{\mu\beta}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] - \mathrm{Tr}[\Omega^{(b)}_{\mu\alpha\beta}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}], \quad (55)$$

or Eq. (53b),

$$\langle M'(v')|\bar{\psi}'_+(-i\overleftarrow{D}_\alpha)\Gamma i\vec{D}_\beta \psi_+|M(v)\rangle = -(\tilde{\Lambda} v_\beta - \tilde{\Lambda}' v'_\beta)\mathrm{Tr}[\Sigma^{(c)}_{\mu\alpha}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}] - \mathrm{Tr}[\Omega^{(c)}_{\mu\alpha\beta}\overline{\mathcal{M}}'^\mu \Gamma \mathcal{M}]. \quad (56)$$

This gives

$$\Omega^{(b)}_{\mu\alpha\beta} - \Omega^{(c)}_{\mu\alpha\beta} = (\tilde{\Lambda} v_\alpha - \tilde{\Lambda}' v'_\alpha)\Sigma^{(b)}_{\mu\beta} + (\tilde{\Lambda} v_\beta - \tilde{\Lambda}' v'_\beta)\Sigma^{(c)}_{\mu\alpha}. \tag{57}$$

To express the form factors in the effective theory we have also to take into account the corrections from the expansion of the states. They can be parametrized as

$$\begin{aligned}\langle M'(v')|i\int d^4x \mathrm{T}[J_0(0),\mathcal{L}_1(x)]|M(v)\rangle \\ = -\frac{1}{4m_b}\underbrace{\left(-\frac{i}{2}\right)\mathrm{Tr}[\Upsilon^{(b)}_{2\mu\alpha\beta}\overline{\mathcal{M}}'^{\mu}\Gamma P_+\sigma^{\alpha\beta}\mathcal{M}]}_{G^{(b)}} \\ -\frac{1}{2m_b^2}\underbrace{\mathrm{Tr}[\Upsilon^{(b)}_{1\mu}\overline{\mathcal{M}}'^{\mu}\Gamma\mathcal{M}]}_{K^{(b)}},\end{aligned} \tag{58}$$

$$\begin{aligned}\langle M'(v')|i\int d^4x \mathrm{T}[J_0(0),\mathcal{L}'_1(x)]|M(v)\rangle \\ = -\frac{1}{4m_c}\underbrace{\left(-\frac{i}{2}\right)\mathrm{Tr}[\Upsilon^{(c)}_{2\mu\alpha\beta}\overline{\mathcal{M}}'^{\mu}\sigma^{\alpha\beta}P'_+\Gamma\mathcal{M}]}_{G^{(c)}} \\ -\frac{1}{2m_c^2}\underbrace{\mathrm{Tr}[\Upsilon^{(c)}_{1\mu}\overline{\mathcal{M}}'^{\mu}\Gamma\mathcal{M}]}_{K^{(c)}},\end{aligned} \tag{59}$$

where $\mathcal{L}_1$, $\mathcal{L}'_1$ are given in (32) with $m_Q \to m_b, m_c$, respectively. The functions $\Upsilon^{(Q)}$ have the general expression

$$\Upsilon^{(Q)}_{1\mu} = \Upsilon^{(Q)}_{1A} v_\mu + \Upsilon^{(Q)}_{1B}\gamma_\mu, \tag{60}$$

$$\begin{aligned}\Upsilon^{(Q)}_{2\mu\alpha\beta} = {} & \Upsilon^{(Q)}_{2A}(g_{\mu\alpha}v_\beta - g_{\mu\beta}v_\alpha) + \Upsilon^{(Q)}_{2B}(g_{\mu\alpha}v'_\beta - g_{\mu\beta}v'_\alpha) + \Upsilon^{(Q)}_{2C}(g_{\mu\alpha}\gamma_\beta - g_{\mu\beta}\gamma_\alpha) \\ & + \Upsilon^{(Q)}_{2D}v_\mu(v_\alpha v'_\beta - v'_\alpha v_\beta) + \Upsilon^{(Q)}_{2E}v_\mu(v_\alpha\gamma_\beta - \gamma_\alpha v_\beta) + \Upsilon^{(Q)}_{2F}v_\mu(v'_\alpha\gamma_\beta - \gamma_\alpha v'_\beta) \\ & + \Upsilon^{(Q)}_{2G}\gamma_\mu(v_\alpha v'_\beta - v'_\alpha v_\beta) + \Upsilon^{(Q)}_{2H}i(\sigma_{\mu\alpha}v_\beta - \sigma_{\mu\beta}v_\alpha) + \Upsilon^{(Q)}_{2I}i(\sigma_{\mu\alpha}v'_\beta - \sigma_{\mu\beta}v'_\alpha) \\ & + \Upsilon^{(Q)}_{2J}iv_\mu\sigma_{\alpha\beta}.\end{aligned} \tag{61}$$

Since the terms proportional to v_α or v_β do not contribute to $G^{(b)}$, and those proportional to v'_α or v'_β do not contribute to $G^{(c)}$, we have: $\Upsilon^{(Q)}_{2D} = \Upsilon^{(Q)}_{2G} = 0$ (for $Q = c$, b), $\Upsilon^{(c)}_{2B} = \Upsilon^{(c)}_{2F} = \Upsilon^{(c)}_{2I} = 0$ and $\Upsilon^{(b)}_{2A} = \Upsilon^{(b)}_{2E} = \Upsilon^{(b)}_{2H} = 0$.

With this set of relations, the form factors in Eqs. (2)–(5) are expressed in terms of universal functions up to $\mathcal{O}(1/m_Q^2)$. The formulas are collected in Appendix B.

V. RELATIONS AMONG FORM FACTORS

At the leading order all form factors in $B_c \to \chi_{c0,1,2}, h_c$ matrix elements are expressed near zero recoil in terms of the single function $\Xi(w)$, see Appendix B. Increasing the order of the expansion, relations among the form factors can be worked out exploiting the results in Appendix C.

The expansion in the relative quark velocity $\tilde{v}$ involves a large number of universal functions, which renders difficult the derivations of the relations among form factors. It is interesting to consider the various orders in $1/m_Q$. In particular, at $\mathcal{O}(1/m_Q)$ there are relations among form factors in the same decay mode, in pairs of decay modes and in more than two modes, near the zero-recoil point.

(1) Relations among the form factors in the same channel

For the $B_c \to \chi_{ci}$ (i = 0, 1, 2) and $B_c \to h_c$ form factors we have:

(a) $B_c \to \chi_{c0}$:

$$g_T(w) = -\frac{1}{w+1}[2g_-(w) + g_P(w)]. \tag{62}$$

(b) $B_c \to \chi_{c1}$:

$$g_{T_2}(w) = -\frac{1}{2}[g_{V_1}(w) - (1+w)g_A(w)] \tag{63}$$

$$\begin{aligned}g_{T_3}(w) = {} & -\frac{1}{2(w-1)}[g_{V_1}(w) + 4g_{V_2}(w)] \\ & +\frac{1}{2}g_A(w) + \frac{1}{w-1}[g_S(w) + g_{T_1}(w)]\end{aligned} \tag{64}$$

with the condition

$$-\frac{1}{2}[g_{V_1}(1) + 4g_{V_2}(1)] + g_S(1) + g_{T_1}(1) = 0. \tag{65}$$

(c) $B_c \to \chi_{c2}$:

$$k_{T_1}(w) = -wk_V(w) + k_{A_2}(w) + wk_{A_3}(w) + k_P(w) \tag{66}$$

$$k_{T_2}(w) = k_V(w) - k_{A_1}(w) - k_{A_2}(w) - wk_{A_3}(w) - k_P(w) \quad (67)$$

$$k_{T_3}(w) = -k_V(w) + k_{A_3}(w). \quad (68)$$

(d) $B_c \to h_c$:

$$f_{T_2}(w) = \frac{1}{2}[f_{V_1}(w) + (1+w)f_A(w)] \quad (69)$$

$$f_{T_3}(w) = \frac{1}{2(w-1)}[f_{V_1}(w) + 4f_{V_2}(w)] + \frac{1}{2}f_A(w) - \frac{1}{w-1}[f_S(w) - f_{T_1}(w)] \quad (70)$$

with the condition

$$\frac{1}{2}[f_{V_1}(1) + 4f_{V_2}(1)] - [f_S(1) - f_{T_1}(1)] = 0. \quad (71)$$

(2) Relations among form factors of pairs of decay channels

We have:

(a) $B_c \to \chi_{c0}$ and $B_c \to \chi_{c1}$:

$$(w+1)g_+(w) - (w-1)g_-(w) + g_P(w) = \frac{w+1}{\sqrt{6}}\{2g_{V_1}(w) + (w+1)g_{V_2}(w) - (w-1)[g_{V_3}(w) + g_A(w)] - g_S(w) + 2g_{T_1}(w)\}. \quad (72)$$

(b) $B_c \to h_c$ and $B_c \to \chi_{c1}$:

$$f_{V_1}(w) + (w-1)f_A(w) - 2f_{T_1}(w) = \sqrt{2}\{g_{V_1}(w) + (w+1)g_{V_2}(w) - (w-1)g_{V_3}(w) - g_S(w)\} \quad (73)$$

$$3f_{V_1}(w) + 2(w+1)f_{V_2}(w) - (w-1)[2f_{V_3}(w) - f_A(w)] - 2[f_S(w) + f_{T_1}(w)] = \sqrt{2}\{g_{V_1}(w) - (w-1)g_A(w) + 2g_{T_1}(w)\}. \quad (74)$$

VI. PHENOMENOLOGY

We now discuss some consequences of the relations found in the previous sections. We use $m_{B_c} = 6274.47 \pm 0.28 \pm 0.17$ MeV, $\tau_{B_c} = 0.510 \pm 0.009$ ps, $m_{\chi_{c0}(1P)} = 3414.71 \pm 0.30$ MeV, $m_{\chi_{c1}(1P)} = 3510.67 \pm 0.05$ MeV, $m_{\chi_{c2}(1P)} = 3556.17 \pm 0.07$ MeV, and $m_{h_c(1P)} = 3525.38 \pm 0.11$ MeV [33]. We first focus on the LO relations, then on NLO, mainly considering the Standard Model. The form factors are expressed in terms of universal functions in a selected kinematic range, close to the zero recoil point $w = 1$. In the numerical analyses we extrapolate the relations to the full kinematic range in the various channels. The range is not wide ($w_{\rm max} \sim 1.16$–1.09 for $B_c \to \chi_{c0}$, $B_c \to \chi_{c2}$ and light leptons or τ, $w_{\rm max} \sim 1.11$–1.05 for $2P$ charmonia). This allows to get information on the results in the full kinematical range. In general, the extrapolation can be constrained by making use of dispersive matrices, which allow to reconstruct the form factors knowing their values in few kinematical points [54]: the application of such methods is beyond the purposes of the present study.

The LO relations obtained from the formulas in appendix B connect all form factors of B_c transitions to the P–wave charmonium 4-plet to the single function $\Xi(w)$, which is different for the $B_c \to 1P$ and the $B_c \to 2P$ modes. In ratios of decay distributions, for the same value of w the form factor dependence cancels out. The ratios are depicted in Fig. 1 both for $1P$ and the $2P$ channels, in the range of w common to all modes. For the $2P$ mesons we use the masses $m_{\chi_{c0}}(2P) = 3860$ MeV and $m_{\chi_{c2}}(2P) = 3930$ MeV [33], even though another $J^{PC} = 0^{++}$ state is also reported by the Particle Data Group, $\chi_{c0}(3915)$ [33]. The results in the figure can be understood considering the connection among the decay distributions, holding at this order if the χ_{ci} mass differences are neglected:

$$2\frac{d\Gamma}{dw}(B_c \to \chi_{c0}\ell\bar{\nu}_\ell) + \frac{d\Gamma}{dw}(B_c \to \chi_{c1}\ell\bar{\nu}_\ell) - \frac{d\Gamma}{dw}(B_c \to \chi_{c2}\ell\bar{\nu}_\ell) = 0. \quad (75)$$

Notice that this relation holds in the SM and also including the full set of NP operators in the generalized Hamiltonian (1) regardless of the Wilson coefficients ϵ_i.

A simple parametrization of the function $\Xi(w)$ involves the intercept, slope, and curvature at the zero recoil point,

$$\Xi(w) = \Xi_0 + \Xi_1(w-1) + \Xi_2(w-1)^2. \quad (76)$$

The three parameters Ξ_0, Ξ_1, Ξ_2 can be constrained, namely $\Xi_0 \in [0.05, 1]$, $\Xi_1 \in [-1, 1]$ and $\Xi_2 \in [-1, 1]$, requiring that the measurement $\mathcal{B}(B_c^- \to \chi_{c0}\pi^-) = (2.4\pm^{0.9}_{0.8}) \times 10^{-5}$ [33] is reproduced at 1σ by naive factorization. Correlations between ratios of branching fractions are found varying Ξ_1/Ξ_0 and Ξ_2/Ξ_0 in the selected regions. We choose the same ranges also for $2P$ excitations in the

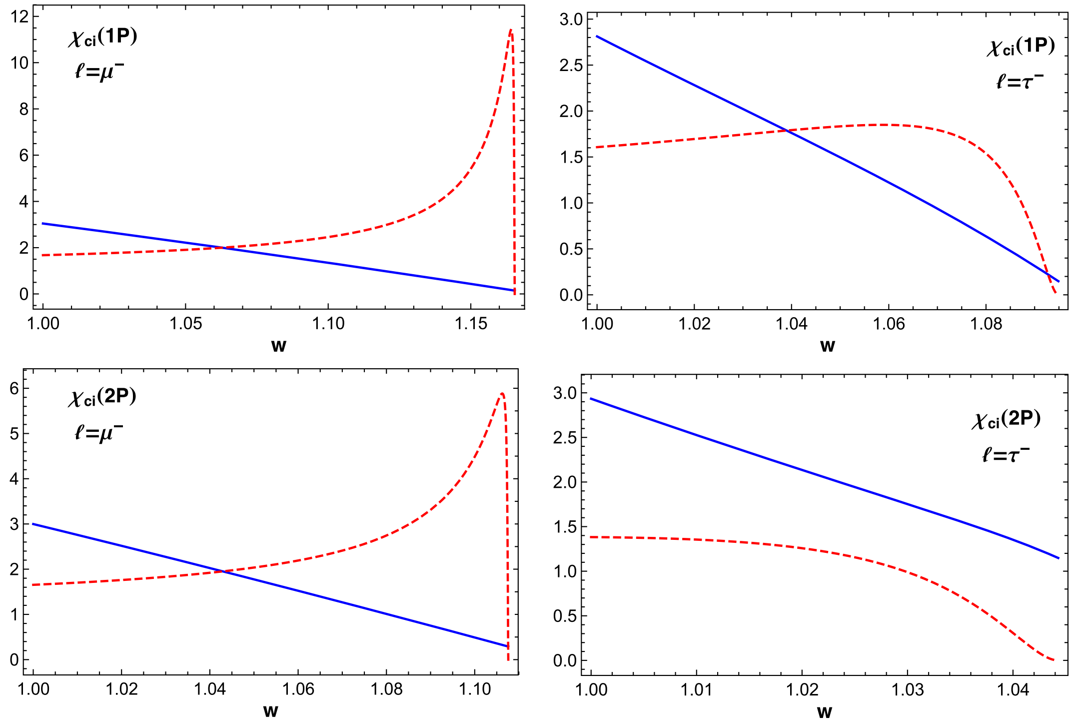

FIG. 1. Ratios of decay distributions $\frac{d\Gamma(B_c\to\chi_{c1}\ell\bar{\nu})/dw}{d\Gamma(B_c\to\chi_{c0}\ell\bar{\nu})/dw}$ (continuous blue line) and $\frac{d\Gamma(B_c\to\chi_{c2}\ell\bar{\nu})/dw}{d\Gamma(B_c\to\chi_{c1}\ell\bar{\nu})/dw}$ (dashed red line) in the Standard Model, in the case $\ell=\mu$ (left) and $\ell=\tau$ (right) for the $1P$ (top row) and $2P$ final charmonia (bottom row), with the meson masses quoted in the text. The LO relations among form factors are extrapolated to the full kinematical range.

final state. For the B_c transitions to the $1P$ charmonium a negative (positive) correlation is found between the ratios $\frac{\Gamma(B_c\to\chi_{c2}\ell\bar{\nu}_\ell)}{\Gamma(B_c\to\chi_{c1}\ell\bar{\nu}_\ell)}$ and $\frac{\Gamma(B_c\to\chi_{c1}\ell\bar{\nu}_\ell)}{\Gamma(B_c\to\chi_{c0}\ell\bar{\nu}_\ell)}$ for $\ell=\mu(\tau)$, as in fig. 2. There are positive correlations also between the lepton flavor universality ratios $R(C)=\Gamma(B_c\to C\tau\bar{\nu}_\tau)/\Gamma(B_c\to C\ell\bar{\nu}_\ell)$ which compare the decay rates to τ and to light leptons, see Fig. 3. At the same LO, the results for B_c decays to the $2P$ charmonium are shown in Figs. 1, 4, and 5, the behavior expected for $\chi_{c1}(3872)$ if it corresponds to $\chi_{c1}(2P)$.

The above results are independent of the function Ξ, a benefit of the expansion of the form factors dominated by the leading order. Including the NLO terms the phenomenology is more involved: nevertheless, the expansion allows us to perform systematic analyses and improvements. An important feature is that the obtained relations connect the form factors to the matrix elements of operators in the effective theory: they should be verified in explicit calculations, and represent a testing ground for the various

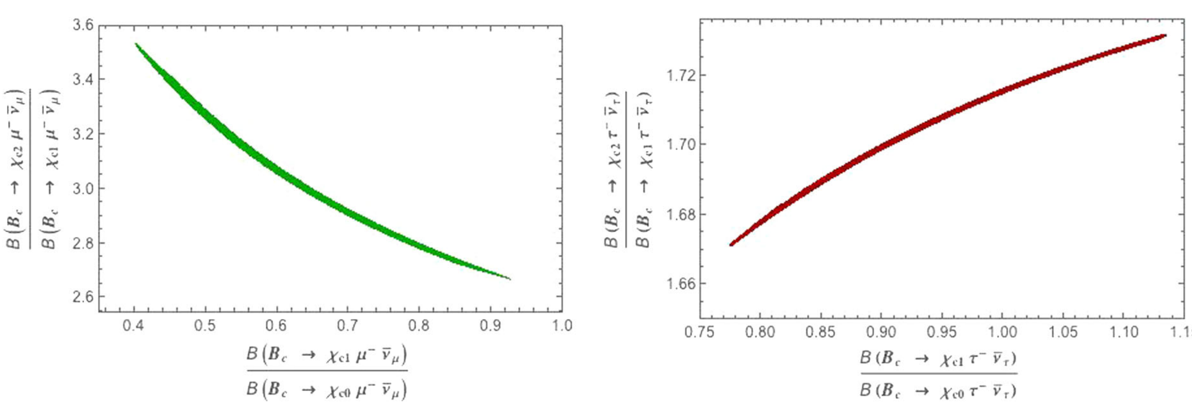

FIG. 2. Correlations between the ratios of branching fractions $\frac{\mathcal{B}(B_c\to\chi_{c2}\ell\bar{\nu})}{\mathcal{B}(B_c\to\chi_{c1}\ell\bar{\nu})}$ and $\frac{\mathcal{B}(B_c\to\chi_{c1}\ell\bar{\nu})}{\mathcal{B}(B_c\to\chi_{c0}\ell\bar{\nu})}$ for $\ell=\mu$ (left) and $\ell=\tau$ (right). The LO expression of the form factors is considered, with $\Xi(w)$ parametrized as in Eq. (76).

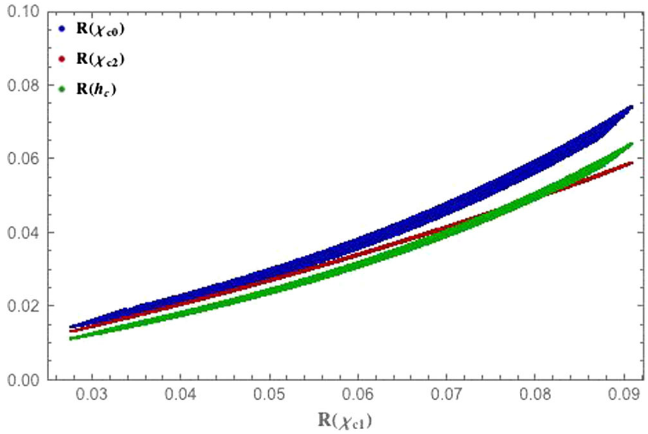

FIG. 3. Correlations between lepton flavor universality ratios $R(\chi_{c0,2})$ and $R(h_c)$ with $R(\chi_{c1})$ in the SM. The LO expression of the form factors is considered, with $\Xi(w)$ parametrized in Eq. (76).

computations based on QCD methods. This is the case of the relations among the form factors in the same channel, such as (62) for $J^{PC} = 0^{++}$, and (64), (67) for the $J^{PC} = 1^{++}, 2^{++}$ modes.

At $\mathcal{O}(1/m_Q)$, once the relations in Appendix C are taken into account, the number of independent structures is 13 in terms of universal functions. For χ_{c0}, χ_{c1}, h_c the decay distributions in SM are related for $\frac{1}{\tilde{\Gamma}}\frac{d\Gamma}{dw}|_{w\to 1}$, with $\tilde{\Gamma}$ in (7). The expressions only involve $\Sigma^{(b)}_{\chi_{c1},1}$ and $\Sigma^{(c)}_{\chi_{c1},1}$ defined in Appendix B:

$$\lim_{w\to 1}\frac{1}{\tilde{\Gamma}}\frac{d\Gamma}{dw}(B_c \to \chi_{c0}\ell\bar{\nu}_\ell) = 18\hat{m}_\ell^2(\epsilon_b+\epsilon_c)^2[\Sigma^{(b)}_{\chi_{c1},1}(1)]^2 \tag{77}$$

$$\lim_{w\to 1}\frac{1}{\tilde{\Gamma}}\frac{d\Gamma}{dw}(B_c \to \chi_{c1}\ell\bar{\nu}_\ell) = 12[2(1-r_1)^2+\hat{m}_\ell^2][\epsilon_b\Sigma^{(b)}_{\chi_{c1},1}(1) - \epsilon_c\Sigma^{(c)}_{\chi_{c1},1}(1)]^2 \tag{78}$$

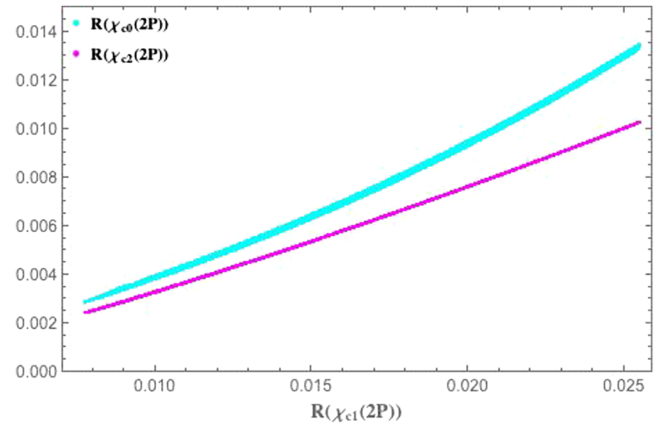

FIG. 5. Correlations between lepton flavor universality ratios $R(\chi_{c0,2}(2P))$ with $R(\chi_{c1}(2P))$ in the SM. The LO expression of the form factors is considered, with $\Xi(w)$ parametrized in Eq. (76).

$$\lim_{w\to 1}\frac{1}{\tilde{\Gamma}}\frac{d\Gamma}{dw}(B_c \to h_c\ell\bar{\nu}_\ell) = 6[2(1-r_h)^2+\hat{m}_\ell^2][(\epsilon_b-\epsilon_c)\Sigma^{(b)}_{\chi_{c1},1}(1) + 2\epsilon_c\Sigma^{(c)}_{\chi_{c1},1}(1)]^2. \tag{79}$$

Such relations also hold for the decays into the $2P$ resonances, and can be used to compare the mode involving $\chi_{c1}(3872)$ to other modes in the $2P$ 4-plet. Admittedly, this is a difficult measurement.

As for the NP extension in Eq. (1) with the pseudoscalar and tensor operators included, the form factors parametrizing the scalar and tensor matrix elements give contributions related to the SM ones as in Eq. (62), a useful connection for phenomenological analyses.

Other improvements are possible if reliable results are available even for a single form factor. As shown in [18] using input from lattice NRQCD, details on a form factor can be employed to get information on universal functions, predicting other form factors and establishing connections among observables.

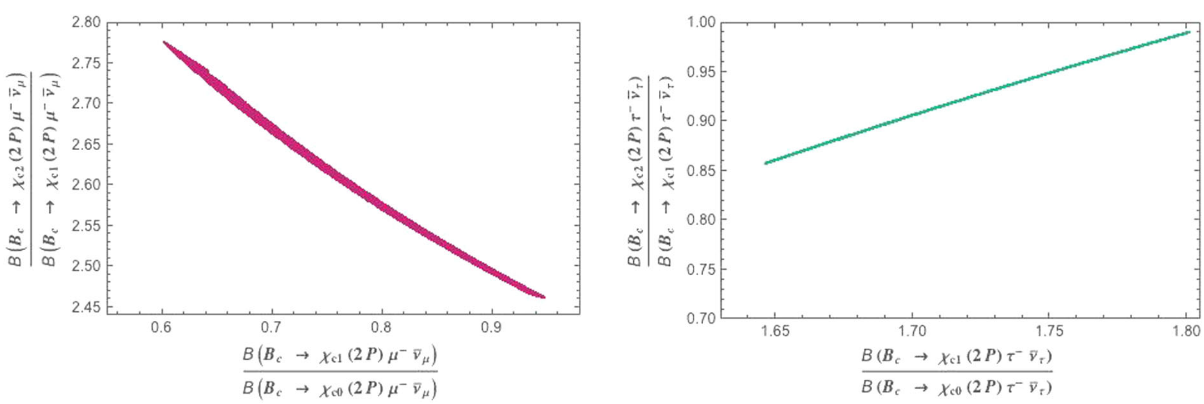

FIG. 4. Correlation between ratios of branching fractions of B_c decays to $2P$ charmonia and $\ell = \mu$ (left), $\ell = \tau$ (right), with the LO expression of the form factors and $\Xi(w)$ parametrized in Eq. (76).

VII. CONCLUSIONS

We have derived the expressions of the form factors of the semileptonic B_c decays to the P-wave charmonium 4-plet as an expansion in the relative velocity of the heavy quarks in the charmonium and of $1/m_Q$. The expressions involve universal functions, independent of the specific channel, and allow to connect different modes. They are a testing ground for explicit calculations of the form factors and can be used in studying LFU ratios and the effects of SM extensions. They are useful in analyzing the semileptonic B_c decays to the $2P$ charmonia: the comparison of measurements of B_c transitions to $\chi_{c1}(3872)$ with other states in the $2P$ charmonium 4-plet is a tool to gain new information on the nature of $X(3872)$.

ACKNOWLEDGMENTS

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APPENDIX A: ANOTHER PARAMETRIZATION OF THE $B_c \to \chi_{c0,1,2}$ AND $B_c \to h_c$ MATRIX ELEMENTS

For the sake of completeness we report another parametrization of the $B_c \to \chi_{c0,1,2}$ and $B_c \to h_c$ matrix elements in terms of form factors, often used in the literature:

$$\begin{aligned}
\langle S(p')|\bar c\gamma^\mu\gamma_5 b|B_c(p)\rangle &= f_0^S(q^2)\frac{m_{B_c}^2-m_S^2}{q^2}q^\mu + f_+^S(q^2)\left(p^\mu+p'^\mu-\frac{m_{B_c}^2-m_S^2}{q^2}q^\mu\right),\\
\langle S(p')|\bar c\sigma^{\mu\nu} b|B_c(p)\rangle &= -\frac{2f_T^S(q^2)}{m_{B_c}+m_S}\varepsilon^{\mu\nu\rho\sigma}p_\rho p'_\sigma,\\
\langle S(p')|\bar c\sigma^{\mu\nu}\gamma_5 b|B_c(p)\rangle &= -i\frac{2f_T^S(q^2)}{m_{B_c}+m_S}(p^\mu p'^\nu-p^\nu p'^\mu),
\end{aligned}\tag{A1}$$

$$\begin{aligned}
\langle A(p',\epsilon)|\bar c\gamma^\mu b|B_c(p)\rangle &= -i\Bigg[2m_A A_0^A(q^2)\frac{\epsilon^*\cdot q}{q^2}q^\mu + (m_{B_c}+m_A)A_1^A(q^2)\left(\epsilon^{*\mu}-\frac{\epsilon^*\cdot q}{q^2}q^\mu\right)\\
&\quad - A_2^A(q^2)\frac{\epsilon^*\cdot q}{m_{B_c}+m_A}\left(p^\mu+p'^\mu-\frac{m_{B_c}^2-m_A^2}{q^2}q^\mu\right)\Bigg],\\
\langle A(p',\epsilon)|\bar c\gamma^\mu\gamma_5 b|B_c(p)\rangle &= \frac{2V^A(q^2)}{m_{B_c}+m_A}\varepsilon^{\mu\nu\rho\sigma}\epsilon^*_\nu p'_\rho p_\sigma,\\
\langle A(p',\epsilon)|\bar c\sigma^{\mu\nu} b|B_c(p)\rangle &= iT_0^A(q^2)\frac{\epsilon^*\cdot q}{(m_{B_c}+m_A)^2}(p^\mu p'^\nu-p^\nu p'^\mu) + iT_1^A(q^2)(p^\mu\epsilon^{*\nu}-\epsilon^{*\mu}p^\nu) + iT_2^A(q^2)(p'^\mu\epsilon^{*\nu}-\epsilon^{*\mu}p'^\nu),\\
\langle A(p',\epsilon)|\bar c\sigma^{\mu\nu}\gamma_5 b|B_c(p)\rangle &= T_0^A(q^2)\frac{\epsilon^*\cdot q}{(m_{B_c}+m_A)^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta + T_1^A(q^2)\varepsilon^{\mu\nu\alpha\beta}p_\alpha\epsilon^*_\beta + T_2^A(q^2)\varepsilon^{\mu\nu\alpha\beta}p'_\alpha\epsilon^*_\beta,
\end{aligned}\tag{A2}$$

$$\begin{aligned}
\langle T(p',\eta)|\bar c\gamma^\mu b|B_c(p)\rangle &= \frac{2V^T(q^2)}{m_{B_c}(m_{B_c}+m_T)}\varepsilon^{\mu\nu\rho\sigma}\eta^*_{\nu\alpha}p'_\rho p_\sigma q^\alpha,\\
\langle T(p',\eta)|\bar c\gamma^\mu\gamma_5 b|B_c(p)\rangle &= \Bigg[2m_T A_0^T(q^2)\frac{\eta^{*\alpha\beta}q_\beta}{q^2}q^\mu + (m_{B_c}+m_T)A_1^T(q^2)\left(\eta^{*\mu\alpha}-\frac{\eta^{*\alpha\beta}q_\beta}{q^2}q^\mu\right)\\
&\quad - A_2^T(q^2)\frac{\eta^{*\alpha\beta}q_\beta}{m_{B_c}+m_T}\left(p^\mu+p'^\mu-\frac{m_{B_c}^2-m_T^2}{q^2}q^\mu\right)\Bigg]\frac{-iq_\alpha}{m_{B_c}},\\
\langle T(p',\eta)|\bar c\sigma^{\mu\nu} b|B(p)\rangle &= \Bigg[T_0^T(q^2)\frac{\eta^*_{\rho\tau}q^\rho}{(m_{B_c}+m_T)^2}\varepsilon^{\mu\nu\alpha\beta}p'_\alpha p_\beta + T_1^T(q^2)\varepsilon^{\mu\nu\alpha\beta}\eta^*_{\alpha\tau}p_\beta + T_2^T(q^2)\varepsilon^{\mu\nu\alpha\beta}\eta^*_{\alpha\tau}p'_\beta\Bigg]\frac{q^\tau}{m_{B_c}},\\
\langle T(p',\eta)|\bar c\sigma^{\mu\nu}\gamma_5 b|B(p)\rangle &= \Bigg[T_0^T(q^2)\frac{\eta^{*\alpha\beta}q_\alpha}{(m_{B_c}+m_T)^2}(p^\mu p'^\nu-p^\nu p'^\mu)\\
&\quad + T_1^T(q^2)(p^\mu\eta^{*\nu\beta}-\eta^{*\mu\beta}p^\nu) + T_2^T(q^2)(p'^\mu\eta^{*\nu\beta}-\eta^{*\mu\beta}p'^\nu)\Bigg]\frac{-iq_\beta}{m_{B_c}}.
\end{aligned}\tag{A3}$$

In Eqs. (A1)–(A3) we denote $S=\chi_{c_0}$, $A=\chi_{c_1},h_c$, and $T=\chi_{c_2}$. The conditions

$$f_0^S(0) = f_+^S(0)$$
$$A_0^{A(T)}(0) = \frac{m_B + m_{A(T)}}{2m_{A(T)}} A_1^{A(T)}(0) - \frac{m_B - m_{A(T)}}{2m_{A(T)}} A_2^{A(T)}(0) \tag{A4}$$

remove the singularity at $q^2 = 0$.

APPENDIX B: FORM FACTORS IN TERMS OF UNIVERSAL FUNCTIONS AT $\mathcal{O}(1/m_Q^2)$

In this Appendix we collect the relations between the form factors in Eqs. (2)–(5) and those expressed in the effective theory at $\mathcal{O}(1/m_Q^2)$. We define $\epsilon_{b(c)} = \frac{1}{2m_{b(c)}}$ and $\Omega_i^{(Q),\mathrm{mix}} = \Omega_i^{(b),\mathrm{mix}} + \Omega_i^{(c),\mathrm{mix}}$ (for $i = 0, 1, 2$), with $\Omega_i^{(b,c),\mathrm{mix}}$ given below.

1. $B_c \to \chi_{c0}$

We define:

$$\Sigma_{\chi_{c0}}^{(b)} = (2+w)\Sigma_1^{(b)} + (w^2-1)\Sigma_3^{(b)} - 3(w-1)[\Sigma_4^{(b)} + \Sigma_6^{(b)}] - (w-7)\Sigma_7^{(b)}, \tag{B1}$$

$$\Sigma_{\chi_{c0}}^{(c)} = 3\Sigma_1^{(c)} - (w^2-1)\Sigma_2^{(c)} - (w-1)[\Sigma_4^{(c)} - 3\Sigma_5^{(c)}] + 6\Sigma_7^{(c)}, \tag{B2}$$

$$\begin{aligned}\Omega_{\chi_{c0}}^{(b)} = {}& (w+2)[\Omega_3^{(b)} + w\Omega_5^{(b)} - \Omega_8^{(b)}] + (w^2-1)[\Omega_4^{(b)} + \Omega_{10}^{(b)} + w\Omega_{12}^{(b)} - \Omega_{18}^{(b)}] \\ & - 3(w-1)[\Omega_6^{(b)} + \Omega_{13}^{(b)} + w\Omega_{16}^{(b)} + \Omega_{17}^{(b)} + w\Omega_{20}^{(b)} + \Omega_{22}^{(b)}] + (w-7)[\Omega_{23}^{(b)} + w\Omega_{25}^{(b)}] - (w-1)(w-2)\Omega_{24}^{(b)},\end{aligned} \tag{B3}$$

$$\begin{aligned}\Omega_{\chi_{c0}}^{(c)} = {}& -3[w\Omega_1^{(c)} + \Omega_4^{(c)} - \Omega_6^{(c)} + 2w\Omega_{21}^{(c)} + 2\Omega_{24}^{(c)}] + (w^2-1)[w\Omega_9^{(c)} + \Omega_{10}^{c} - \Omega_{13}^{(c)}] \\ & + (w-1)[w\Omega_{14}^{(c)} - 3w\Omega_{15}^{(c)} - 3\Omega_{17}^{(c)} + \Omega_{18}^{(c)} + \Omega_{22}^{(c)} + 3\Omega_{23}^{(c)}],\end{aligned} \tag{B4}$$

$$\begin{aligned}\Omega_{\chi_{c0}}^{(Q),\mathrm{mix}} = {}& (w+1)(w-2)\Omega_2^{(Q)} + (w+1)(w+2)\Omega_3^{(Q)} - 3(w+1)[\Omega_4^{(Q)} + 2\Omega_{24}^{Q}] + 9\Omega_6^{(Q)} \\ & - (w-2)[3\Omega_7^{(Q)} - \Omega_8^{(Q)}] + (w-1)(w+1)^2\Omega_{10}^{(Q)} - (w^2-1)[3\Omega_{13}^{(Q)} + 3\Omega_{17}^{(Q)} - \Omega_{18}^{(Q)}] \\ & - (w+1)^2\Omega_{22}^{(Q)} + (w+1)(w-7)\Omega_{23}^{(Q)},\end{aligned} \tag{B5}$$

$$\Upsilon_{2,\chi_{c0}}^{(b)} = (w+1)[2\Upsilon_{2B}^{(b)} + 2(w-1)\Upsilon_{2F}^{(b)} + 4\Upsilon_{2I}^{(b)} + 3\Upsilon_{2J}^{(b)}] + 2(w-2)\Upsilon_{2C}^{(b)}, \tag{B6}$$

$$\Upsilon_{2,\chi_{c0}}^{(c)} = (w+1)[2\Upsilon_{2A}^{(c)} + 2(w-1)\Upsilon_{2E}^{(c)} + 4\Upsilon_{2H}^{(c)} - \Upsilon_{2J}^{(c)}] - 6\Upsilon_{2C}^{(c)}. \tag{B7}$$

Using these definitions we obtain the expressions of the form factors in terms of universal functions:

$$g_+ = -\frac{1}{\sqrt{3}}[\epsilon_b\Sigma_{\chi_{c0}}^{(b)} + \epsilon_c\Sigma_{\chi_{c0}}^{(c)}] + \frac{1}{\sqrt{3}}[\epsilon_b^2\Omega_{\chi_{c0}}^{(b)} - \epsilon_c^2\Omega_{\chi_{c0}}^{(c)}], \tag{B8}$$

$$\begin{aligned}g_- = {}& \frac{w+1}{\sqrt{3}}\Xi + \frac{1}{2\sqrt{3}}[\epsilon_b\Upsilon_{2,\chi_{c0}}^{(b)} + \epsilon_c\Upsilon_{2,\chi_{c0}}^{(c)}] + \frac{2}{\sqrt{3}}[\epsilon_b^2[(w+1)\Upsilon_{1A}^{(b)} - 3\Upsilon_{1B}^{(b)}] + \epsilon_c^2[(w+1)\Upsilon_{1A}^{(c)} - 3\Upsilon_{1B}^{(c)}]] \\ & + \frac{1}{2\sqrt{3}}\epsilon_b\epsilon_c[(w+1)(\tilde\Lambda\Sigma_{\chi_{c0}}^{(b)} - \tilde\Lambda'\Sigma_{\chi_{c0}}^{(c)}) - \Omega_{\chi_{c0}}^{(Q),\mathrm{mix}}],\end{aligned} \tag{B9}$$

$$\begin{aligned}g_P = {}& \frac{w^2-1}{\sqrt{3}}\Xi - \frac{w+1}{\sqrt{3}}[\epsilon_b\Sigma_{\chi_{c0}}^{(b)} - \epsilon_c\Sigma_{\chi_{c0}}^{(c)}] + \frac{w-1}{2\sqrt{3}}[\epsilon_b\Upsilon_{2,\chi_{c0}}^{(b)} + \epsilon_c\Upsilon_{2,\chi_{c0}}^{(c)}] \\ & + \frac{2(w-1)}{\sqrt{3}}[\epsilon_b^2[(w+1)\Upsilon_{1A}^{(b)} - 3\Upsilon_{1B}^{(b)}] + \epsilon_c^2[(w+1)\Upsilon_{1A}^{(c)} - 3\Upsilon_{1B}^{(c)}]] \\ & + \frac{w+1}{\sqrt{3}}[\epsilon_b^2\Omega_{\chi_{c0}}^{(b)} + \epsilon_c^2\Omega_{\chi_{c0}}^{(c)}] - \frac{w-1}{2\sqrt{3}}\epsilon_b\epsilon_c[(w+1)(\tilde\Lambda\Sigma_{\chi_{c0}}^{(b)} - \tilde\Lambda'\Sigma_{\chi_{c0}}^{(c)}) - \Omega_{\chi_{c0}}^{(Q),\mathrm{mix}}],\end{aligned} \tag{B10}$$

$$g_T = -\frac{w+1}{\sqrt{3}}\Xi + \frac{1}{\sqrt{3}}[\epsilon_b \Sigma^{(b)}_{\chi_{c0}} - \epsilon_c \Sigma^{(c)}_{\chi_{c0}}] - \frac{1}{2\sqrt{3}}[\epsilon_b \Upsilon^{(b)}_{2,\chi_{c0}} + \epsilon_c \Upsilon^{(c)}_{2,\chi_{c0}}]$$
$$- \frac{2}{\sqrt{3}}[\epsilon_b^2[(w+1)\Upsilon^{(b)}_{1A} - 3\Upsilon^{(b)}_{1B}] + \epsilon_c^2[(w+1)\Upsilon^{(c)}_{1A} - 3\Upsilon^{(c)}_{1B}]]$$
$$- \frac{1}{\sqrt{3}}[\epsilon_b^2 \Omega^{(b)}_{\chi_{c0}} + \epsilon_c^2 \Omega^{(c)}_{\chi_{c0}}] + \frac{1}{2\sqrt{3}}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}\Sigma^{(b)}_{\chi_{c0}} - \tilde{\Lambda}'\Sigma^{(c)}_{\chi_{c0}}) - \Omega^{(Q),\mathrm{mix}}_{\chi_{c0}}]. \tag{B11}$$

2. $B_c \to \chi_{c1}$

We define:

$$\Sigma^{(b)}_{\chi_{c1},1} = \Sigma^{(b)}_1 - (w-1)\Sigma^{(b)}_6 + 2\Sigma^{(b)}_7, \tag{B12}$$

$$\Sigma^{(b)}_{\chi_{c1},2} = \Sigma^{(b)}_1 + (w+1)\Sigma^{(b)}_3 - 3\Sigma^{(b)}_4 - \Sigma^{(b)}_7, \tag{B13}$$

$$\Sigma^{(c)}_{\chi_{c1},1} = \Sigma^{(c)}_1 - (w-1)\Sigma^{(c)}_5, \tag{B14}$$

$$\Sigma^{(c)}_{\chi_{c1},2} = (w+1)\Sigma^{(c)}_2 - \Sigma^{(c)}_4 - 4\Sigma^{(c)}_5, \tag{B15}$$

$$\Omega^{(b)}_{\chi_{c1},1} = -\Omega^{(b)}_3 - w\Omega^{(b)}_5 + \Omega^{(b)}_8 + (w-1)[\Omega^{(b)}_{17} + w\Omega^{(b)}_{20} - \Omega^{(b)}_{24}] + 2\Omega^{(b)}_{23} + 2w\Omega^{(b)}_{25}, \tag{B16}$$

$$\Omega^{(b)}_{\chi_{c1},2} = \Omega^{(b)}_3 + (w+1)[\Omega^{(b)}_4 + \Omega^{(b)}_{10} + w\Omega^{(b)}_{12} - \Omega^{(b)}_{18} - \Omega^{(b)}_{24}]$$
$$+ w\Omega^{(b)}_5 - 3\Omega^{(b)}_6 - \Omega^{(b)}_8 - 3\Omega^{(b)}_{13} - 3w\Omega^{(b)}_{16} - 3\Omega^{(b)}_{22} + \Omega^{(b)}_{23} + w\Omega^{(b)}_{25}, \tag{B17}$$

$$\Omega^{(c)}_{\chi_{c1},1} = w\Omega^{(c)}_1 + \Omega^{(c)}_4 - \Omega^{(c)}_6 - (w-1)[w\Omega^{(c)}_{15} + \Omega^{(c)}_{17} - \Omega^{(c)}_{23}], \tag{B18}$$

$$\Omega^{(c)}_{\chi_{c1},2} = -(w+1)[w\Omega^{(c)}_9 + \Omega^{(c)}_{10} - \Omega^{(c)}_{13}] + w[\Omega^{(c)}_{14} + 4\Omega^{(c)}_{15}]$$
$$+ 4\Omega^{(c)}_{17} + \Omega^{(c)}_{18} + \Omega^{(c)}_{22} - 4\Omega^{(c)}_{23}, \tag{B19}$$

$$\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} = (w+1)[\Omega^{(Q)}_3 + \Omega^{(Q)}_4 - 2\Omega^{(Q)}_{23}] - 3\Omega^{(Q)}_6 - (w+2)\Omega^{(Q)}_7 - \Omega^{(Q)}_8 - (w^2-1)\Omega^{Q}_{17}, \tag{B20}$$

$$\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2} = w\Omega^{(Q)}_2 + (w+3)\Omega^{(Q)}_3 - 4\Omega^{(Q)}_7 - \Omega^{(Q)}_8 + (w^2-1)\Omega^{(Q)}_{10}$$
$$- (w-1)[3\Omega^{(Q)}_{13} + 4\Omega^{(Q)}_{17} + \Omega^{(Q)}_{18}] - (w-3)\Omega^{(Q)}_{22} + (w-9)\Omega^{(Q)}_{23}, \tag{B21}$$

$$\Upsilon^{(b)}_{2,\chi_{c1},1} = (1+w)[\Upsilon^{(b)}_{2B} + \Upsilon^{(b)}_{2I}] - 2\Upsilon^{(b)}_{2C}, \tag{B22}$$

$$\Upsilon^{(b)}_{2,\chi_{c1},2} = \Upsilon^{(b)}_{2B} - 2\Upsilon^{(b)}_{2C} - 2(w-1)\Upsilon^{(b)}_{2F} - \Upsilon^{(b)}_{2I} - 3\Upsilon^{(b)}_{2J}, \tag{B23}$$

$$\Upsilon^{(c)}_{2,\chi_{c1},1} = \Upsilon^{(c)}_{2A} + \Upsilon^{(c)}_{2H}, \tag{B24}$$

$$\Upsilon^{(c)}_{2,\chi_{c1},2} = 3\Upsilon^{(c)}_{2A} + \Upsilon^{(c)}_{2H} - \Upsilon^{(c)}_{2J}. \tag{B25}$$

With these definitions, we express the form factors in terms of universal functions:

$$
\begin{aligned}
g_{V_1} = {} & \frac{w^2-1}{\sqrt{2}}\Xi - \frac{w+1}{\sqrt{2}}[\epsilon_b[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \epsilon_c[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]] \\
& + \frac{w-1}{2\sqrt{2}}[\epsilon_b[2\Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2}] - \epsilon_c(w+1)[2\Upsilon^{(c)}_{2,\chi_{c1},1} - \Upsilon^{(c)}_{2,\chi_{c1},2}]] \\
& + \sqrt{2}(w-1)[\epsilon_b^2[(w+1)\Upsilon^{(b)}_{1A} - 2\Upsilon^{(b)}_{1B})] + \epsilon_c^2[(w+1)\Upsilon^{(c)}_{1A} - 2\Upsilon^{(c)}_{1B}]] \\
& - \frac{w+1}{\sqrt{2}}[\epsilon_b^2[2\Omega^{(b)}_{\chi_{c1},1} - (w-1)\Omega^{(b)}_{\chi_{c1},2}] + \epsilon_c^2[2\Omega^{(c)}_{\chi_{c1},1} + (w-1)\Omega^{(c)}_{\chi_{c1},2}]] \\
& - \frac{w-1}{2\sqrt{2}}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \tilde{\Lambda}'[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]) \\
& + [2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} - (w+1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2}]],
\end{aligned}
\tag{B26}
$$

$$
\begin{aligned}
g_{V_2} = {} & -\frac{w-1}{2\sqrt{2}}\Xi + \frac{1}{2\sqrt{2}}[\epsilon_b[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \epsilon_c[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]] \\
& - \frac{1}{4\sqrt{2}}[\epsilon_b[2\Upsilon^{(b)}_{2,\chi_{c1},1} - (w-1)\Upsilon^{(b)}_{2,\chi_{c1},2}] - \epsilon_c[2(w+1)\Upsilon^{(c)}_{2,\chi_{c1},1} - (w-1)\Upsilon^{(c)}_{2,\chi_{c1},2}]] \\
& - \frac{1}{\sqrt{2}}[\epsilon_b^2[(w-1)\Upsilon^{(b)}_{1A} - 2\Upsilon^{(b)}_{1B}] + \epsilon_c^2[(w-1)\Upsilon^{(c)}_{1A} - 2\Upsilon^{(c)}_{1B}]] \\
& + \frac{1}{2\sqrt{2}}[\epsilon_b^2[2\Omega^{(b)}_{\chi_{c1},1} - (w-1)\Omega^{(b)}_{\chi_{c1},2}] + \epsilon_c^2[2\Omega^{(c)}_{\chi_{c1},1} + (w-1)\Omega^{(c)}_{\chi_{c1},2}]] \\
& + \frac{1}{4\sqrt{2}}\epsilon_b\epsilon_c[\tilde{\Lambda}[2(w-3)\Sigma^{(b)}_{\chi_{c1},1} + (w-1)^2\Sigma^{(b)}_{\chi_{c1},2}] - \tilde{\Lambda}'[2(w+1)\Sigma^{(c)}_{\chi_{c1},1} - (w-1)^2\Sigma^{(c)}_{\chi_{c1},2}] \\
& + [2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} - (w-1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2}]],
\end{aligned}
\tag{B27}
$$

$$
\begin{aligned}
g_{V_3} = {} & \frac{w+1}{2\sqrt{2}}\Xi - \frac{1}{2\sqrt{2}}[\epsilon_b[2\Sigma^{(b)}_{\chi_{c1},1} + (w+1)\Sigma^{(b)}_{\chi_{c1},2}] - \epsilon_c[2\Sigma^{(c)}_{\chi_{c1},1} - (w+1)\Sigma^{(c)}_{\chi_{c1},2}]] \\
& + \frac{1}{4\sqrt{2}}[\epsilon_b[2\Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2}] - \epsilon_c(w+1)[2\Upsilon^{(c)}_{2,\chi_{c1},1} - \Upsilon^{(c)}_{2,\chi_{c1},2}]] \\
& + \frac{1}{\sqrt{2}}[\epsilon_b^2[(w+1)\Upsilon^{(b)}_{1A} - 2\Upsilon^{(b)}_{1B}] + \epsilon_c^2[(w+1)\Upsilon^{(c)}_{1A} - 2\Upsilon^{(c)}_{1B}]] \\
& - \frac{1}{2\sqrt{2}}[\epsilon_b^2[2\Omega^{(b)}_{\chi_{c1},1} - (w+1)\Omega^{(b)}_{\chi_{c1},2}] + \epsilon_c^2[2\Omega^{(c)}_{\chi_{c1},1} + (w+1)\Omega^{(c)}_{\chi_{c1},2}]] \\
& - \frac{1}{4\sqrt{2}}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \tilde{\Lambda}'[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]) \\
& + [2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} - (w+1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2}]],
\end{aligned}
\tag{B28}
$$

$$
\begin{aligned}
g_A = {} & \frac{w+1}{\sqrt{2}}\Xi - \frac{1}{\sqrt{2}}[\epsilon_b[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \epsilon_c[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]] \\
& + \frac{1}{2\sqrt{2}}[\epsilon_b[2\Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2}] - \epsilon_c(w+1)[2\Upsilon^{(c)}_{2,\chi_{c1},1} - \Upsilon^{(c)}_{2,\chi_{c1},2}]] \\
& + \sqrt{2}[\epsilon_b^2[(w+1)\Upsilon^{(b)}_{1A} - 2\Upsilon^{(b)}_{1B}] + \epsilon_c^2[(w+1)\Upsilon^{(c)}_{1A} - 2\Upsilon^{(c)}_{1B}]] \\
& - \frac{1}{\sqrt{2}}[\epsilon_b^2[2\Omega^{(b)}_{\chi_{c1},1} - (w-1)\Omega^{(b)}_{\chi_{c1},2}] + \epsilon_c^2[2\Omega^{(c)}_{\chi_{c1},1} + (w-1)\Omega^{(c)}_{\chi_{c1},2}]] \\
& - \frac{1}{2\sqrt{2}}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \tilde{\Lambda}'[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}])] \\
& + (2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} - (w+1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2})],
\end{aligned}
\tag{B29}
$$

$$
\begin{aligned}
g_S = {} & \sqrt{2}[\epsilon_b \Sigma^{(b)}_{\chi_{c1},1} + \epsilon_c \Sigma^{(c)}_{\chi_{c1},1}] - \frac{1}{\sqrt{2}}[\epsilon_b \Upsilon^{(b)}_{2,\chi_{c1},1} - \epsilon_c (w+1) \Upsilon^{(c)}_{2,\chi_{c1},1}] \\
& + 2\sqrt{2}[\epsilon_b^2 \Upsilon^{(b)}_{1B} + \epsilon_c^2 \Upsilon^{(c)}_{1B}] + \sqrt{2}[\epsilon_b^2 \Omega^{(b)}_{\chi_{c1},1} - \epsilon_c^2 \Omega^{(c)}_{\chi_{c1},1}] \\
& + \frac{1}{\sqrt{2}} \epsilon_b \epsilon_c [(w+1)(\tilde{\Lambda} \Sigma^{(b)}_{\chi_{c1},1} + \tilde{\Lambda}' \Sigma^{(c)}_{\chi_{c1},1}) - \Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1}],
\end{aligned} \tag{B30}
$$

$$
\begin{aligned}
g_{T_1} = {} & -\frac{1}{\sqrt{2}}[\epsilon_b [2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] + \epsilon_c [2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]] \\
& - \frac{1}{\sqrt{2}}[\epsilon_b^2 [2\Omega^{(b)}_{\chi_{c1},1} - (w-1)\Omega^{(b)}_{\chi_{c1},2}] - \epsilon_c^2 [2\Omega^{(c)}_{\chi_{c1},1} + (w-1)\Omega^{(c)}_{\chi_{c1},2}]],
\end{aligned} \tag{B31}
$$

$$
\begin{aligned}
g_{T_2} = {} & \frac{w+1}{\sqrt{2}} \Xi + \frac{1}{2\sqrt{2}}[\epsilon_b [2\Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2}] - \epsilon_c (w+1)[2\Upsilon^{(c)}_{2,\chi_{c1},1} - \Upsilon^{(c)}_{2,\chi_{c1},2}]] \\
& + \sqrt{2}[\epsilon_b^2 [(w+1)\Upsilon^{(b)}_{1A} - 2\Upsilon^{(b)}_{1B}] + \epsilon_c^2 [(w+1)\Upsilon^{(c)}_{1A} - 2\Upsilon^{(c)}_{1B}]] \\
& + \frac{1}{2\sqrt{2}} \epsilon_b \epsilon_c [(w+1)(\tilde{\Lambda}[2\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] - \tilde{\Lambda}'[2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2}]) \\
& + [2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} - (w+1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2}]],
\end{aligned} \tag{B32}
$$

$$
\begin{aligned}
g_{T_3} = {} & \frac{1}{\sqrt{2}} \Xi - \frac{1}{\sqrt{2}}[\epsilon_b \Sigma^{(b)}_{\chi_{c1},2} - \epsilon_c \Sigma^{(c)}_{\chi_{c1},2}] - \frac{1}{2\sqrt{2}}[\epsilon_b \Upsilon^{(b)}_{2,\chi_{c1},2} - \epsilon_c \Upsilon^{(c)}_{2,\chi_{c1},2}] \\
& + \sqrt{2}[\epsilon_b^2 \Upsilon^{(b)}_{1A} + \epsilon_c^2 \Upsilon^{(c)}_{1A}] + \frac{1}{\sqrt{2}}[\epsilon_b^2 \Omega^{(b)}_{\chi_{c1},2} + \epsilon_c^2 \Omega^{(c)}_{\chi_{c1},2}] \\
& + \frac{1}{2\sqrt{2}} \epsilon_b \epsilon_c [\tilde{\Lambda}[4\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2}] + \tilde{\Lambda}'(w-1)\Sigma^{(c)}_{\chi_{c1},2} - \Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2}].
\end{aligned} \tag{B33}
$$

3. $B_c \to \chi_{c2}$

We define:

$$\Sigma^{(b)}_{\chi_{c2}} = \Sigma^{(b)}_1 + (w+1)\Sigma^{(b)}_3 - 3\Sigma^{(b)}_4 - \Sigma^{(b)}_7, \tag{B34}$$

$$\Sigma^{(c)}_{\chi_{c2},1} = (w+1)\Sigma^{(c)}_2 - \Sigma^{(c)}_4, \tag{B35}$$

$$\Sigma^{(c)}_{\chi_{c2},2} = (w+1)\Sigma^{(c)}_2 + \Sigma^{(c)}_4, \tag{B36}$$

$$
\begin{aligned}
\Omega^{(b)}_{\chi_{c2}} = {} & \Omega^{(b)}_3 + (w+1)[\Omega^{(b)}_4 + \Omega^{(b)}_{10} + w\Omega^{(b)}_{12} - \Omega^{(b)}_{18} - \Omega^{(b)}_{24}] + w\Omega^{(b)}_5 \\
& - 3[\Omega^{(b)}_6 + \Omega^{(b)}_{13} + w\Omega^{(b)}_{16} + \Omega^{(b)}_{22}] - \Omega^{(b)}_8 + \Omega^{(b)}_{23} + w\Omega^{(b)}_{25},
\end{aligned} \tag{B37}
$$

$$\Omega^{(c)}_{\chi_{c2},1} = -(w+1)[w\Omega^{(c)}_9 + \Omega^{(c)}_{10} - \Omega^{(c)}_{13}] + w\Omega^{(c)}_{14} + \Omega^{(c)}_{18} + \Omega^{(c)}_{22}, \tag{B38}$$

$$\Omega^{(c)}_{\chi_{c2},2} = (w+1)[w\Omega^{(c)}_9 + \Omega^{(c)}_{10} - \Omega^{(c)}_{13}] + w\Omega^{(c)}_{14} + \Omega^{(c)}_{18} + \Omega^{(c)}_{22}, \tag{B39}$$

$$\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},1} = \Omega^{(Q)}_8 - w\Omega^{(Q)}_2 + (1-w)[\Omega^{(Q)}_3 + (1+w)\Omega^{(Q)}_{10} - 3\Omega^{(Q)}_{13} - \Omega^{(Q)}_{18} + \Omega^{(Q)}_{23}] + (w-3)\Omega^{(Q)}_{22}, \tag{B40}$$

$$\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},2} = -\Omega^{(Q)}_8 + (2-w)\Omega^{(Q)}_2 + (1-w)[\Omega^{(Q)}_3 + (1+w)\Omega^{(Q)}_{10} - 3\Omega^{(Q)}_{13} + \Omega^{(Q)}_{18} + \Omega^{(Q)}_{23}] + (1+w)\Omega^{(Q)}_{22}, \tag{B41}$$

$$\Upsilon^{(b)}_{2,\chi_{c2}} = \Upsilon^{(b)}_{2B} - 2\Upsilon^{(b)}_{2C} - 2(w-1)\Upsilon^{(b)}_{2F} - \Upsilon^{(b)}_{2I} - 3\Upsilon^{(b)}_{2J}, \tag{B42}$$

$$\Upsilon^{(c)}_{2,\chi_{c2},1} = \Upsilon^{(c)}_{2A} - \Upsilon^{(c)}_{2H} + \Upsilon^{(c)}_{2J}, \tag{B43}$$

$$\Upsilon^{(c)}_{2,\chi_{c2},2} = \Upsilon^{(c)}_{2A} - 2(w-1)\Upsilon^{(c)}_{2E} - \Upsilon^{(c)}_{2H} + \Upsilon^{(c)}_{2J}. \tag{B44}$$

With these definitions the form factors are expressed in terms of universal functions:

$$\begin{aligned} k_V = {}& -\Xi + [\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} + \epsilon_c\Sigma^{(c)}_{\chi_{c_2},1}] + \frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \epsilon_c\Upsilon^{(c)}_{2,\chi_{c_2},1}] - 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] \\ & - [\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} - \epsilon_c^2\Omega^{(c)}_{\chi_{c_2},1}] + \frac{1}{2}\epsilon_b\epsilon_c[(w-1)(\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'\Sigma^{(c)}_{\chi_{c_2},1}) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1}], \end{aligned} \tag{B45}$$

$$\begin{aligned} k_{A_1} = {}& (w+1)\Xi - (w-1)[\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} + \epsilon_c\Sigma^{(c)}_{\chi_{c_2},1}] - \frac{w+1}{2}[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \epsilon_c\Upsilon^{(c)}_{2,\chi_{c_2},1}] + 2(w+1)[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] \\ & + (w-1)[\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} - \epsilon_c^2\Omega^{(c)}_{\chi_{c_2},1}] - \frac{w+1}{2}\epsilon_b\epsilon_c[(w-1)(\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'\Sigma^{(c)}_{\chi_{c_2},1}) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1}], \end{aligned} \tag{B46}$$

$$\begin{aligned} k_{A_2} = {}& -\frac{\epsilon_c}{w+1}[\Sigma^{(c)}_{\chi_{c_2},1} + \Sigma^{(c)}_{\chi_{c_2},2}] - \frac{\epsilon_c}{2(w-1)}[\Upsilon^{(c)}_{2,\chi_{c_2},1} - \Upsilon^{(c)}_{2,\chi_{c_2},2}] - \frac{\epsilon_c^2}{w+1}[\Omega^{(c)}_{\chi_{c_2},1} - \Omega^{(c)}_{\chi_{c_2},2}] \\ & - \frac{\epsilon_b\epsilon_c}{2(w-1)}[(w-1)(2\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'(\Sigma^{(c)}_{\chi_{c_2},1} + \Sigma^{(c)}_{\chi_{c_2},2})) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1} + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},2}], \end{aligned} \tag{B47}$$

$$\begin{aligned} k_{A_3} = {}& -\Xi + \left[\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} + \frac{\epsilon_c}{w+1}(w\Sigma^{(c)}_{\chi_{c_2},1} - \Sigma^{(c)}_{\chi_{c_2},2})\right] + \frac{1}{2}\left[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \frac{\epsilon_c}{w-1}(w\Upsilon^{(c)}_{2,\chi_{c_2},1} - \Upsilon^{(c)}_{2,\chi_{c_2},2})\right] \\ & - 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] - \left[\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} - \frac{\epsilon_c^2}{w+1}(w\Omega^{(c)}_{\chi_{c_2},1} + \Omega^{(c)}_{\chi_{c_2},2})\right] \\ & + \frac{\epsilon_b\epsilon_c}{2(w-1)}[(w-1)((w+1)\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'(w\Sigma^{(c)}_{\chi_{c_2},1} + \Sigma^{(c)}_{\chi_{c_2},2})) + w\Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1} + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},2}], \end{aligned} \tag{B48}$$

$$\begin{aligned} k_P = {}& -\Xi + [\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} + \epsilon_c\Sigma^{(c)}_{\chi_{c_2},2}] + \frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \epsilon_c\Upsilon^{(c)}_{2,\chi_{c_2},2}] - 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] \\ & - [\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} + \epsilon_c^2\Omega^{(c)}_{\chi_{c_2},2}] + \frac{1}{2}\epsilon_b\epsilon_c[(w-1)(\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'\Sigma^{(c)}_{\chi_{c_2},2}) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},2}], \end{aligned} \tag{B49}$$

$$\begin{aligned} k_{T_1} = {}& -\Xi + [\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} - \epsilon_c\Sigma^{(c)}_{\chi_{c_2},1}] + \frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \epsilon_c\Upsilon^{(c)}_{2,\chi_{c_2},1}] - 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] \\ & - [\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} + \epsilon_c^2\Omega^{(c)}_{\chi_{c_2},1}] - \frac{1}{2}\epsilon_b\epsilon_c[(w-1)(\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'\Sigma^{(c)}_{\chi_{c_2},1}) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1}], \end{aligned} \tag{B50}$$

$$\begin{aligned} k_{T_2} = {}& -\Xi - [\epsilon_b\Sigma^{(b)}_{\chi_{c_2}} - \epsilon_c\Sigma^{(c)}_{\chi_{c_2},1}] + \frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,\chi_{c_2}} + \epsilon_c\Upsilon^{(c)}_{2,\chi_{c_2},1}] - 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] \\ & + [\epsilon_b^2\Omega^{(b)}_{\chi_{c_2}} + \epsilon_c^2\Omega^{(c)}_{\chi_{c_2},1}] - \frac{1}{2}\epsilon_b\epsilon_c[(w-1)(\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'\Sigma^{(c)}_{\chi_{c_2},1}) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1}], \end{aligned} \tag{B51}$$

$$\begin{aligned} k_{T_3} = {}& -\frac{\epsilon_c}{w+1}[\Sigma^{(c)}_{\chi_{c_2},1} + \Sigma^{(c)}_{\chi_{c_2},2}] + \frac{\epsilon_c}{2(w-1)}[\Upsilon^{(c)}_{2,\chi_{c_2},1} - \Upsilon^{(c)}_{2,\chi_{c_2},2}] - \frac{\epsilon_c^2}{w+1}[\Omega^{(c)}_{\chi_{c_2},1} - \Omega^{(c)}_{\chi_{c_2},2}] \\ & - \frac{\epsilon_b\epsilon_c}{2(w-1)}[(w-1)(2\tilde\Lambda\Sigma^{(b)}_{\chi_{c_2}} + \tilde\Lambda'(\Sigma^{(c)}_{\chi_{c_2},1} + \Sigma^{(c)}_{\chi_{c_2},2})) + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},1} + \Omega^{(Q),\mathrm{mix}}_{\chi_{c_2},2}]. \end{aligned} \tag{B52}$$

4. $B_c \to h_c$

We define:

$$\Sigma^{(b)}_{h_c,1} = w\Sigma^{(b)}_1 + (w^2-1)\Sigma^{(b)}_3 - (w-1)[3\Sigma^{(b)}_4 + \Sigma^{(b)}_6] - (w-3)\Sigma^{(b)}_7, \tag{B53}$$

$$\Sigma^{(b)}_{h_c,2} = \Sigma^{(b)}_1 + (w+1)\Sigma^{(b)}_3 - 3\Sigma^{(b)}_4 - \Sigma^{(b)}_7, \tag{B54}$$

$$\Sigma^{(c)}_{h_c,1} = \Sigma^{(c)}_1 - (w^2-1)\Sigma^{(c)}_2 + (w-1)[3\Sigma^{(c)}_4 + \Sigma^{(c)}_5] - 2\Sigma^{(c)}_7, \tag{B55}$$

$$\Sigma^{(c)}_{h_c,2} = (w+1)\Sigma^{(c)}_2 - 3\Sigma^{(c)}_4 - 2\Sigma^{(c)}_5, \tag{B56}$$

$$\begin{aligned}\Omega^{(b)}_{h_c,1} = {} & w[\Omega^{(b)}_3 + w\Omega^{(b)}_5 - \Omega^{(b)}_8] + (w^2-1)[\Omega^{(b)}_4 + \Omega^{(b)}_{10} + w\Omega^{(b)}_{12} - \Omega^{(b)}_{18}] \\ & - (w-1)[3\Omega^{(b)}_6 + 3\Omega^{(b)}_{13} + 3w\Omega^{(b)}_{16} + \Omega^{(b)}_{17} + w\Omega^{(b)}_{20} + 3\Omega^{(b)}_{22} + w\Omega^{(b)}_{24}] \\ & + (w-3)[\Omega^{(b)}_{23} + w\Omega^{(b)}_{25}],\end{aligned} \tag{B57}$$

$$\begin{aligned}\Omega^{(b)}_{h_c,2} = {} & \Omega^{(b)}_3 + (w+1)[\Omega^{(b)}_4 + \Omega^{(b)}_{10} + w\Omega^{(b)}_{12} - \Omega^{(b)}_{18} - \Omega^{(b)}_{24}] + w\Omega^{(b)}_5 - 3\Omega^{(b)}_6 - \Omega^{(b)}_8 - 3\Omega^{(b)}_{13} \\ & - 3w\Omega^{(b)}_{16} - 3\Omega^{(b)}_{22} + \Omega^{(b)}_{23} + w\Omega^{(b)}_{25},\end{aligned} \tag{B58}$$

$$\begin{aligned}\Omega^{(c)}_{h_c,1} = {} & w\Omega^{(c)}_1 + \Omega^{(c)}_4 - \Omega^{(c)}_6 - (w^2-1)[w\Omega^{(c)}_9 + \Omega^{(c)}_{10} - \Omega^{(c)}_{13}] \\ & + (w-1)[3w\Omega^{(c)}_{14} + w\Omega^{(c)}_{15} + \Omega^{(c)}_{17} + 3\Omega^{(c)}_{18} + 3\Omega^{(c)}_{22} - \Omega^{(c)}_{23}] - 2w\Omega^{(c)}_{21} - 2\Omega^{(c)}_{24},\end{aligned} \tag{B59}$$

$$\Omega^{(c)}_{h_c,2} = (w+1)[w\Omega^{(c)}_9 + \Omega^{(c)}_{10} - \Omega^{(c)}_{13}] - 3w\Omega^{(c)}_{14} - 2w\Omega^{(c)}_{15} - 2\Omega^{(c)}_{17} - 3\Omega^{(c)}_{18} - 3\Omega^{(c)}_{22} + 2\Omega^{(c)}_{23}, \tag{B60}$$

$$\begin{aligned}\Omega^{(Q),\text{mix}}_{h_c,1} = {} & (w+1)(w+2)\Omega^{(Q)}_2 + (w+1)[w\Omega^{(Q)}_3 - \Omega^{(Q)}_4 + 2\Omega^{(Q)}_{24}] + 3\Omega^{(Q)}_6 - (w+2)\Omega^{(Q)}_7 \\ & - 3w\Omega^{(Q)}_8 + (w-1)(w+1)^2\Omega^{(Q)}_{10} - (w^2-1)[3\Omega^{(Q)}_{13} + \Omega^{Q}_{17} + 3\Omega^{(Q)}_{18}] \\ & - (w-7)(w+1)\Omega^{(Q)}_{22} + (w-3)(w+1)\Omega^{(Q)}_{23},\end{aligned} \tag{B61}$$

$$\begin{aligned}\Omega^{(Q),\text{mix}}_{h_c,2} = {} & (w+2)\Omega^{(Q)}_2 + (w+1)\Omega^{(Q)}_3 - 2\Omega^{(Q)}_7 - 3\Omega^{(Q)}_8 + (w^2-1)\Omega^{(Q)}_{10} \\ & - (w-1)[3\Omega^{(Q)}_{13} + 2\Omega^{(Q)}_{17} + 3\Omega^{(Q)}_{18}] - (w-7)\Omega^{(Q)}_{22} + (w-5)\Omega^{(Q)}_{23},\end{aligned} \tag{B62}$$

$$\Upsilon^{(b)}_{2,h_c,1} = 2w\Upsilon^{(b)}_{2C} + (w+1)[2(w-1)\Upsilon^{(b)}_{2F} + 2\Upsilon^{(b)}_{2I} + 3\Upsilon^{(b)}_{2J}], \tag{B63}$$

$$\Upsilon^{(b)}_{2,h_c,2} = \Upsilon^{(b)}_{2B} - 2\Upsilon^{(b)}_{2C} - 2(w-1)\Upsilon^{(b)}_{2F} - \Upsilon^{(b)}_{2I} - 3\Upsilon^{(b)}_{2J}, \tag{B64}$$

$$\Upsilon^{(c)}_{2,h_c,1} = 2\Upsilon^{(c)}_{2C} - (w+1)[2(w-1)\Upsilon^{(c)}_{2E} + 2\Upsilon^{(c)}_{2H} - 3\Upsilon^{(c)}_{2J}], \tag{B65}$$

$$\Upsilon^{(c)}_{2,h_c,2} = \Upsilon^{(c)}_{2A} - 2(w-1)\Upsilon^{(c)}_{2E} - 3\Upsilon^{(c)}_{2H} + 3\Upsilon^{(c)}_{2J}. \tag{B66}$$

With these definitions we obtain:

$$
\begin{aligned}
f_{V_1} = &-(w+1)[\epsilon_b(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\epsilon_c(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})] \\
&+\frac{w-1}{2}[\epsilon_b(\Upsilon^{(b)}_{2,h_c,1}+(w+1)\Upsilon^{(b)}_{2,h_c,2})+\epsilon_c(\Upsilon^{(c)}_{2,h_c,1}-(w+1)\Upsilon^{(c)}_{2,h_c,2})] \\
&-2(w-1)[\epsilon_b^2\Upsilon^{(b)}_{1B}+\epsilon_c^2\Upsilon^{(c)}_{1B}] \\
&+(w+1)[\epsilon_b^2(\Omega^{(b)}_{h_c,1}-(w-1)\Omega^{(b)}_{h_c,2})+\epsilon_c^2(\Omega^{(c)}_{h_c,1}+(w-1)\Omega^{(c)}_{h_c,2})] \\
&-\frac{w-1}{2}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\tilde{\Lambda}'(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})) \\
&+\Omega^{(Q),\mathrm{mix}}_{h_c,1}-(w+1)\Omega^{(Q),\mathrm{mix}}_{h_c,2}], \qquad \text{(B67)}
\end{aligned}
$$

$$
\begin{aligned}
f_{V_2} = &-\Xi+\frac{1}{2}[\epsilon_b(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\epsilon_c(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})] \\
&-\frac{1}{4}[\epsilon_b(\Upsilon^{(b)}_{2,h_c,1}+(w-1)\Upsilon^{(b)}_{2,h_c,2})+\epsilon_c(\Upsilon^{(c)}_{2,h_c,1}-(w-1)\Upsilon^{(c)}_{2,h_c,2})] \\
&-[\epsilon_b^2(2\Upsilon^{(b)}_{1A}-\Upsilon^{(b)}_{B})+\epsilon_c^2(2\Upsilon^{(c)}_{1A}-\Upsilon^{(c)}_{1B})] \\
&-\frac{1}{2}[\epsilon_b^2(\Omega^{(b)}_{h_c,1}-(w-1)\Omega^{(b)}_{h_c,2})+\epsilon_c^2(\Omega^{(c)}_{h_c,1}+(w-1)\Omega^{(c)}_{h_c,2})] \\
&+\frac{1}{4}\epsilon_b\epsilon_c[\tilde{\Lambda}((w-3)\Sigma^{(b)}_{h_c,1}-(w-1)^2\Sigma^{(b)}_{h_c,2})+\tilde{\Lambda}'((w+1)\Sigma^{(c)}_{h_c,1}+(w-1)^2\Sigma^{(c)}_{h_c,2}) \\
&+\Omega^{(Q),\mathrm{mix}}_{h_c,1}-(w-1)\Omega^{(Q),\mathrm{mix}}_{h_c,2}], \qquad \text{(B68)}
\end{aligned}
$$

$$
\begin{aligned}
f_{V_3} = &-\frac{1}{2}[\epsilon_b(\Sigma^{(b)}_{h_c,1}-(w+1)\Sigma^{(b)}_{h_c,2})+\epsilon_c(\Sigma^{(c)}_{h_c,1}+(w+1)\Sigma^{(c)}_{h_c,2})] \\
&+\frac{1}{4}[\epsilon_b(\Upsilon^{(b)}_{2,h_c,1}+(w+1)\Upsilon^{(b)}_{2,h_c,2})+\epsilon_c(\Upsilon^{(c)}_{2,h_c,1}-(w+1)\Upsilon^{(c)}_{2,h_c,2})] \\
&-[\epsilon_b^2\Upsilon^{(b)}_{1B}+\epsilon_c^2\Upsilon^{(c)}_{1B}] \\
&+\frac{1}{2}[\epsilon_b^2(\Omega^{(b)}_{h_c,1}-(w+1)\Omega^{(b)}_{h_c,2})+\epsilon_c^2(\Omega^{(c)}_{h_c,1}+(w+1)\Omega^{(c)}_{h_c,2})] \\
&-\frac{1}{4}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\tilde{\Lambda}'(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})) \\
&+\Omega^{(Q),\mathrm{mix}}_{h_c,1}-(w+1)\Omega^{(Q),\mathrm{mix}}_{h_c,2}], \qquad \text{(B69)}
\end{aligned}
$$

$$
\begin{aligned}
f_{A} = &[\epsilon_b(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\epsilon_c(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})] \\
&-\frac{1}{2}[\epsilon_b(\Upsilon^{(b)}_{2,h_c,1}+(w+1)\Upsilon^{(b)}_{2,h_c,2})+\epsilon_c(\Upsilon^{(c)}_{2,h_c,1}-(w+1)\Upsilon^{(c)}_{2,h_c,2})]+2[\epsilon_b^2\Upsilon^{(b)}_{1B}+\epsilon_c^2\Upsilon^{(c)}_{1B}] \\
&-[\epsilon_b^2(\Omega^{(b)}_{h_c,1}-(w-1)\Omega^{(b)}_{h_c,2})+\epsilon_c^2(\Omega^{(c)}_{h_c,1}+(w-1)\Omega^{(c)}_{h_c,2})] \\
&+\frac{1}{2}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}(\Sigma^{(b)}_{h_c,1}-(w-1)\Sigma^{(b)}_{h_c,2})+\tilde{\Lambda}'(\Sigma^{(c)}_{h_c,1}+(w-1)\Sigma^{(c)}_{h_c,2})) \\
&+\Omega^{(Q),\mathrm{mix}}_{h_c,1}-(w+1)\Omega^{(Q),\mathrm{mix}}_{h_c,2}], \qquad \text{(B70)}
\end{aligned}
$$

$$
\begin{aligned}
f_{S} = &-(w+1)\Xi+[\epsilon_b\Sigma^{(b)}_{h_c,1}-\epsilon_c\Sigma^{(c)}_{h_c,1}]-\frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,h_c,1}+\epsilon_c\Upsilon^{(c)}_{2,h_c,1}] \\
&-2[\epsilon_b^2(w+1)\Upsilon^{(b)}_{1A}-\Upsilon^{(b)}_{1B})+\epsilon_c^2((w+1)\Upsilon^{(c)}_{1A}-\Upsilon^{(c)}_{1B})]-[\epsilon_b^2\Omega^{(b)}_{h_c,1}-\epsilon_c^2\Omega^{(c)}_{h_c,1}] \\
&+\frac{1}{2}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}\Sigma^{(b)}_{h_c,1}-\tilde{\Lambda}'\Sigma^{(c)}_{h_c,1})-\Omega^{(Q),\mathrm{mix}}_{h_c,1}], \qquad \text{(B71)}
\end{aligned}
$$

$$f_{T_1} = [\epsilon_b(\Sigma^{(b)}_{h_c,1} - (w-1)\Sigma^{(b)}_{h_c,2}) - \epsilon_c(\Sigma^{(c)}_{h_c,1} + (w-1)\Sigma^{(c)}_{h_c,2})] - [\epsilon_b^2(\Omega^{(b)}_{h_c,1} - (w-1)\Omega^{(b)}_{h_c,2}) - \epsilon_c^2(\Omega^{(c)}_{h_c,1} + (w-1)\Omega^{(c)}_{h_c,2})], \tag{B72}$$

$$f_{T_2} = -\frac{1}{2}[\epsilon_b(\Upsilon^{(b)}_{2,h_c,1} + (w+1)\Upsilon^{(b)}_{2,h_c,2}) + \epsilon_c(\Upsilon^{(c)}_{2,h_c,1} - (w+1)\Upsilon^{(c)}_{2,h_c,2})] + 2[\epsilon_b^2\Upsilon^{(b)}_{1B} + \epsilon_c^2\Upsilon^{(c)}_{1B}] - \frac{1}{2}\epsilon_b\epsilon_c[(w+1)(\tilde{\Lambda}(\Sigma^{(b)}_{h_c,1} - (w-1)\Sigma^{(b)}_{h_c,2}) + \tilde{\Lambda}'(\Sigma^{(c)}_{h_c,1} + (w-1)\Sigma^{(c)}_{h_c,2})) + \Omega^{(Q),\mathrm{mix}}_{h_c,1} - (w+1)\Omega^{(Q),\mathrm{mix}}_{h_c,2}], \tag{B73}$$

$$f_{T_3} = \Xi - [\epsilon_b\Sigma^{(b)}_{h_c,2} + \epsilon_c\Sigma^{(c)}_{h_c,2}] - \frac{1}{2}[\epsilon_b\Upsilon^{(b)}_{2,h_c,2} - \epsilon_c\Upsilon^{(c)}_{2,h_c,2}] + 2[\epsilon_b^2\Upsilon^{(b)}_{1A} + \epsilon_c^2\Upsilon^{(c)}_{1A}] + [\epsilon_b^2\Omega^{(b)}_{h_c,2} + \epsilon_c^2\Omega^{(c)}_{h_c,2}] - \frac{1}{2}\epsilon_b\epsilon_c[\tilde{\Lambda}(2\Sigma^{(b)}_{h_c,1} - (w-1)\Sigma^{(b)}_{h_c,2}) + (w-1)\tilde{\Lambda}'\Sigma^{(c)}_{h_c,2} - \Omega^{(Q),\mathrm{mix}}_{h_c,2}]. \tag{B74}$$

APPENDIX C: A SET OF RELATIONS AMONG UNIVERSAL FUNCTIONS

In this Appendix we use the definitions in Eqs. (48), (54), and (61), together with the constraints (52), (57), to relate the various structures entering in the expressions (B1)–(B7), (B12)–(B25), (B34)–(B44), and (B53)–(B66). The results allow us to obtain the relations among form factors in Sec. V.

$$\begin{aligned}
\Sigma^{(b)}_{\chi_{c0}} &= 3\Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2} \\
\Sigma^{(b)}_{\chi_{c2}} &= \Sigma^{(b)}_{\chi_{c1},2} \\
\Sigma^{(b)}_{h_c,1} &= \Sigma^{(b)}_{\chi_{c1},1} + (w-1)\Sigma^{(b)}_{\chi_{c1},2} \\
\Sigma^{(b)}_{h_c,2} &= \Sigma^{(b)}_{\chi_{c1},2}
\end{aligned} \tag{C1}$$

$$\begin{aligned}
\Sigma^{(c)}_{h_c,1} &= -\frac{1}{3}\Sigma^{(c)}_{\chi_{c0}} + 2\Sigma^{(c)}_{\chi_{c1},1} - (w-1)\Sigma^{(c)}_{\chi_{c1},2} - \frac{w-1}{3}[3\Sigma^{(c)}_{\chi_{c2},1} - 2\Sigma^{(c)}_{\chi_{c2},2}] \\
\Sigma^{(c)}_{h_c,2} &= \frac{1}{2}\Sigma^{(c)}_{\chi_{c1},2} + \frac{1}{2}[3\Sigma^{(c)}_{\chi_{c2},1} - 2\Sigma^{(c)}_{\chi_{c2},2}]
\end{aligned} \tag{C2}$$

$$\begin{aligned}
\Upsilon^{(b)}_{2,\chi_{c0}} &= 3\Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2} \\
\Upsilon^{(b)}_{2,\chi_{c2}} &= \Upsilon^{(b)}_{2,\chi_{c1},2} \\
\Upsilon^{(b)}_{2,h_c,1} &= \Upsilon^{(b)}_{2,\chi_{c1},1} - (w+1)\Upsilon^{(b)}_{2,\chi_{c1},2} \\
\Upsilon^{(b)}_{2,h_c,2} &= \Upsilon^{(b)}_{2,\chi_{c1},2}
\end{aligned} \tag{C3}$$

$$\begin{aligned}
\Upsilon^{(c)}_{2,h_c,1} &= -\frac{1}{3}\Upsilon^{(c)}_{2,\chi_{c0}} + (w+1)[2\Upsilon^{(c)}_{2,\chi_{c1},1} - \Upsilon^{(c)}_{2,\chi_{c1},2}] + \frac{w+1}{3}[3\Upsilon^{(c)}_{2,\chi_{c2},1} + 2\Upsilon^{(c)}_{2,\chi_{c2},2}] \\
\Upsilon^{(c)}_{2,h_c,2} &= -\frac{1}{2}\Upsilon^{(c)}_{2,\chi_{c1},2} + \frac{1}{2}[3\Upsilon^{(c)}_{2,\chi_{c2},1} + 2\Upsilon^{(c)}_{2,\chi_{c2},2}]
\end{aligned} \tag{C4}$$

$$\begin{aligned}
\Omega^{(b)}_{\chi_{c0}} &= -3\Omega^{(b)}_{\chi_{c1},1} + (w-1)\Omega^{(b)}_{\chi_{c1},2} \\
\Omega^{(b)}_{\chi_{c2}} &= \Omega^{(b)}_{\chi_{c1},2} \\
\Omega^{(b)}_{h_c,1} &= -\Omega^{(b)}_{\chi_{c1},1} + (w-1)\Omega^{(b)}_{\chi_{c1},2} \\
\Omega^{(b)}_{h_c,2} &= \Omega^{(b)}_{\chi_{c1},2}
\end{aligned} \tag{C5}$$

$$\Omega^{(c)}_{h_c,1} = \frac{1}{3}\Omega^{(c)}_{\chi_{c0}} + 2\Omega^{(c)}_{\chi_{c1},1} + (w-1)\Omega^{(c)}_{\chi_{c1},2} + \frac{w-1}{3}[3\Omega^{(c)}_{\chi_{c2},1} + 2\Omega^{(c)}_{\chi_{c2},2}]$$
$$\Omega^{(c)}_{h_c,2} = -\frac{1}{2}\Omega^{(c)}_{\chi_{c1},2} - \frac{1}{2}[3\Omega^{(c)}_{\chi_{c2},1} + 2\Omega^{(c)}_{\chi_{c2},2}] \tag{C6}$$

$$\Omega^{(Q),\mathrm{mix}}_{h_c,1} = -\frac{1}{3}\Omega^{(Q),\mathrm{mix}}_{\chi_{c0}} - 2\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},1} + (w+1)\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2} - \frac{w+1}{3}[3\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},1} - 2\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},2}]$$
$$\Omega^{(Q),\mathrm{mix}}_{h_c,2} = \frac{1}{2}\Omega^{(Q),\mathrm{mix}}_{\chi_{c1},2} - \frac{1}{2}[3\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},1} - 2\Omega^{(Q),\mathrm{mix}}_{\chi_{c2},2}]. \tag{C7}$$

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