Type IIB supergravity action on $M^5 \times X^5$ solutions

S. A. Kurlyand[1,*] and A. A. Tseytlin[2,†,‡]
[1]*Physics Department, Moscow State University*
[2]*Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom*

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While the ten-dimensional type IIB supergravity action evaluated on $\mathrm{AdS}_5 \times S^5$ solution vanishes, the five-dimensional effective action reconstructed from equations of motion using the $M^5 \times S^5$ compactification ansatz is proportional to the AdS_5 volume. The latter is consistent with the conformal anomaly interpretation in AdS/CFT context. We show that this paradox can be resolved if, in the case of $M^5 \times X^5$ topology, the ten-dimensional action contains an additional 5-form-dependent "topological" term $\int F_{5M} \wedge F_{5X}$. The presence of this term is suggested also by gauge-invariance considerations in the Pasti-Sorokin-Tonin formulation of type IIB supergravity action. We show that this term contributes to the ten-dimensional action evaluated on the D3-brane solution.

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I. INTRODUCTION

Many discussions of applications of the maximally supersymmetric case of AdS/CFT duality [1] start with a classical action of five-dimensional (5D) gauged supergravity or simply 5D gravity with a cosmological term

$$S_5 = -\frac{1}{2\kappa_5^2}\int d^5x\sqrt{g}(R_5 + 12L^{-2} + \cdots). \tag{1.1}$$

Evaluating this action on the AdS_5 vacuum solution with radius L gives a factor of volume of AdS_5 space. Assuming S^4 as a boundary of AdS_5, the regularized value of the volume reproduces the planar part of the UV-divergent (conformal a-anomaly) term in the free energy of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory on S^4 (see, e.g., Refs. [2–4]),[1]

$$S_5 = \frac{8L^4}{2\kappa_5^2}\mathrm{vol}(\mathrm{AdS}_5) = N^2 \log(\Lambda \mathrm{r}). \tag{1.2}$$

The action like (1.1) is also a starting point of investigations of anti-de Sitter (AdS) black hole thermodynamics [5–8].

The 5D gauged supergravity action is assumed to follow from the ten-dimensional (10D) type IIB supergravity action compactified on S^5 [9,10]. However, the actual compactification procedure involves starting with the 10D field equations [11,12], substituting there an S^5 compactification ansatz, and then reconstructing the corresponding action for the 5D fields (cf. Ref. [13]). The bosonic part of the 10D type IIB action may be written as[2]

$$\hat{S}_{10} = -\frac{1}{2\kappa_{10}^2}\left\{\int \mathrm{d}^{10}x\sqrt{G}\left(e^{-2\phi}\left[R + 4(\partial_\mu\phi)^2 - \frac{1}{2}|H_3|^2\right] - \frac{1}{2}|F_1|^2 - \frac{1}{2}|F_3|^2 - \frac{1}{4}|F_5|^2\right) - \frac{1}{2}\int B_2\wedge F_3\wedge F_5\right\} + \cdots, \tag{1.3}$$

$$F_1 = dC_0, \qquad F_3 = dC_2 - C_0H_3,$$
$$F_5 = dC_4 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3. \tag{1.4}$$

Here, as usual, the self-duality condition $F_5 = {}^*F_5$ is relaxed [14] and is imposed by hand at the level of equations of motion (alternative approaches that involve auxiliary fields where the self-duality condition follows from the equations of motion are discussed in Refs. [15–20]).

*kurliand.sa18@physics.msu.ru
†tseytlin@imperial.ac.uk
‡On leave from Institute for Theoretical and Mathematical Physics (ITMP) and Lebedev Institute.

[1]Here, we use that $\frac{1}{2\kappa_5^2} = \frac{L^4\mathrm{vol}(S^5)}{2\kappa_{10}^2}$, $2\kappa_{10}^2 = (2\pi)^7 g_s^2\alpha'^4$, and $L^4 = 4\pi g_s\alpha'^2 N$. To recall, $\mathrm{vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \to_{n=5} = \pi^3$, $\mathrm{vol}(\mathrm{AdS}_{2n+1}) = \frac{2(-1)^n\pi^n}{\Gamma(n+1)}\log(\Lambda r) \to_{2n+1=5} = \pi^2\log(\Lambda r)$, and $R_5 = -20L^{-2}$. r is the radius of boundary 4-sphere, and Λ is an IR cutoff on the AdS side (corresponding to UV cutoff on the super Yang-Mills side).

[2]Here, $|F_p|^2 = \frac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\cdots\mu_p}$. The extra $\frac{1}{2}$ in the normalization of the F_5 kinetic term has to do with the requirement that the corresponding analog of the Einstein equation should contain the contribution of the stress tensor of only the self-dual half of F_5.

Comparing (1.1) and (1.3), we arrive at the following apparent paradox: the 10D action (1.3) evaluated on the vacuum $AdS_5 \times S^5$ solution

$$ds^2_{10} = L^2(ds^2_{AdS_5} + d\Omega_5^2), \qquad F_5 = 4L^{-1}(\epsilon_5 + {}^*\epsilon_5),$$
$$L^4 = 4\pi\alpha'^2 g_s N \tag{1.5}$$

is clearly vanishing ($R = -20L^{-2} + 20L^{-2} = 0, |F_5|^2 = 0$),[3] while the value of the 5D action (1.1) on the AdS_5 solution is nonzero (1.2) and consistent with the AdS/CFT duality.

It is of course well known that substituting some special-symmetry ansatz for a subset of fields into the action is not the same as doing this in the equations of motion and then reconstructing the corresponding dimensionally reduced action for the remaining field variables. However, the values of the actions on the full solutions are expected to match. Furthermore, the problem is that the 10D action and, in particular, its on-shell value should be more fundamental; it should follow from (a properly defined) quantum string theory path integral. Thus, using the 10D approach is important if one is to go beyond the leading order in α', in particular, in the context of AdS/CFT.

One may wonder if this issue has to do with the subtlety of implementing self-duality of F_5. However, this is not the case; similar disagreement between the on-shell values of the reduced three-dimensional (3D) action and the 10D action is found in the case of $AdS_3 \times S^3 \times T^4$ background supported by a 3-form flux. Here, the 10D action is well defined off shell for a generic 3-form field, and the effective six-dimensional (6D) self-duality of the latter (implying the vanishing of the 10D action) is just a feature of a particular solution.

A natural way to resolve this problem is to assume that the 10D action (1.3) is missing some "boundary term" that restores the equivalence of its on-shell value with that of the 5D action (1.1). However, such a term cannot be one of the familiar choices like the Gibbons-Hawking-York (GHY) one [21,22][4] or boundary terms that may be added to the 5D action (1.1) to make it IR finite when evaluated on a classical solution with AdS_5 asymptotics (see, e.g., Refs. [2,7,23,24]).

An important general point is that boundary or topological terms may not be universal; they may depend on a choice of vacuum (near which one expands in order to find an effective action for fluctuations) or asymptotic boundary conditions. For example, in the type IIB string theory, there are two maximally supersymmetric vacua—the flat space $R^{1,9}$ and $AdS_5 \times S^5$ [11]—that have different asymptotic symmetries. The corresponding effective actions may, in principle, contain different boundary terms.

In what follows, we will be interested in the case when the topology of 10D space-time is that of a product $M^5 \times X^5$ where M^5 is noncompact and X^5 is a compact space. We will suggest a novel 5-form dependent "topological" term that should be added to the 10D action (1.3) to restore its on-shell equivalence with the reduced 5D action (1.2).[5]

Let us stress again that the reason why one would like to understand the 10D origin of the on-shell value of the reduced action like (1.2) is that it should have a string theory origin (being related to string partition function on a 2-sphere). For example, the tree-level bosonic string effective action may be written as[6]

$$S_D = S_{\rm bulk} + S_{\rm bndry}, \qquad S_{\rm bulk} = \hat{S}_D = \varkappa \int d^D x \sqrt{G}\, e^{-2\phi}\tilde{\beta}^\phi,$$
$$\varkappa = \frac{2}{\kappa_D^2 \alpha'}, \tag{1.6}$$

$$\tilde{\beta}^\phi = c_0 - \frac{1}{4}\alpha'\left(R + 4\nabla^2\phi - 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right) + \mathcal{O}(\alpha'^2),$$
$$c_0 = \frac{1}{6}(D-26), \tag{1.7}$$

$$S_{\rm bndry} = -\frac{1}{2}\varkappa\alpha' \int d^{D-1}x\sqrt{\gamma}e^{-2\phi}(K - 2\partial_n\phi)$$
$$= -\frac{1}{2}\varkappa\alpha' \int d^{D-1}x\sqrt{\gamma}\nabla_a(e^{-2\phi}n^a). \tag{1.8}$$

Here, the integrand (1.7) of the bulk part is proportional to the generalized conformal anomaly coefficient $\tilde{\beta}^\phi$ and thus must vanish on-shell[7] not only to first two leading orders [25] but also to all orders in α' [26].[8] The boundary term (1.8) which is a dilatonic generalization [28] of the standard GHY term may, in general, produce a nonzero on-shell value for the total action.

Similar remarks apply to the NS-NS part of the type IIB superstring effective action. Note that the boundary term that should be added in general to the bulk type IIB action (1.3) [with the second-derivative dilaton term in (1.7) integrated by parts and thus not automatically vanishing

[3]Note that a self-dual 5-form is real in the case of Minkowski 10D signature but is imaginary in the Euclidean signature case.

[4]This term does not contribute in the case of AdS asymptotics.

[5]In the case of solution of 6D theory (obtained by compactification on T^4) supported by self-dual 3-form flux, one will need to add a topological term built out of H_3. In the case of the $AdS_3 \times S^3 \times T^4$ solution supported by 5-form flux discussed in Sec. III, one will need the same F_5-dependent topological term.

[6]This action may be reconstructed also from scattering amplitudes near asymptotically flat vacuum, with the boundary term required for a consistent definition of the graviton/dilaton S matrix.

[7]Strictly speaking, this is true for backgrounds for which there is no source in the dilaton equation; cf. discussion of brane solutions in Appendix B.

[8]The same conclusion was reached for the on-shell value of the closed bosonic string field theory action [27].

on solutions with nonconstant dilaton] is given by (1.8) without the $\partial_n \phi$ term, i.e.,

$$S_{\rm bndry} = -\frac{1}{\kappa_{10}^2} \int d^{D-1}x\sqrt{\gamma}e^{-2\phi} K. \qquad (1.9)$$

As for the R-R terms in the second line of (1.3), they may lead to additional nontrivial boundary contributions when evaluated on a classical solution.[9] Given that the bulk $|F_5|^2$ term vanishes identically upon use of the on-shell self-duality condition, an extra F_5-dependent contribution to 10D action would be required to get a nonzero contribution for solutions with only F_5 flux being nonzero. This new term should not change the equations of motion; i.e., it should be a topological or boundary term.[10]

We shall suggest such a topological term in Sec. II. In Sec. III, we shall compute the value of the full 10Kathy action [containing the bulk term (1.3), the boundary term (1.9), and the topological term] on the extremal D3-brane solution and its nonextremal generalization. We shall also note the nonzero value of the topological term on solutions describing BPS intersections of two and four D3-branes that in the near-core limit reduce to $\mathrm{AdS}_3 \times S^3 \times T^4$ and $\mathrm{AdS}_2 \times S^2 \times T^6$ backgrounds, respectively. Section IV will contain some concluding remarks. In Appendix A, we shall argue that the presence of the same topological term is suggested also by gauge invariance requirement in the Pasti-Sorokin-Tonin (PST) formulation [16,17] of type IIB supergravity action. In Appendix B, we shall discuss the computation of the value of the 10D action on fundamental string, NS5-brane, and D5-brane solutions.

II. TOPOLOGICAL TERM

While the obvious guess for the 10D topological invariant $\int F_5 \wedge F_5$ is identically zero, a nontrivial candidate is possible if we assume that the 10D space has a particular topological structure. Namely, let us specify to the backgrounds for which the 10D space-time is a product $M^5 \times X^5$ where M^5 is noncompact (e.g., asymptotically AdS_5) while X^5 is compact and a similar factorization applies to the 5-form field strength (for simplicity, we shall ignore all other fields),

$$\mathrm{M}^{10} = M^5 \times X^5, \qquad F_5 = F_{5M} \oplus F_{5X}, \qquad (2.1)$$

and also its potential $C_4 = C_{4M} \oplus C_{4X}$. Then, consider the following topological term

$$S_{\rm top} = \gamma \int F_{5M} \wedge F_{5X}. \qquad (2.2)$$

As M^5 is noncompact and $F_{5M} = dC_{4M}$ while $dF_{5X} = 0$, this term reduces to a boundary contribution and thus does not affect the bulk equations of motion.

Integrating over the compact X^5 then gives

$$S_{\rm top} = \gamma q \int_M F_{5M}, \qquad q = \int_X F_{5X}. \qquad (2.3)$$

The integral of a 5-form F_{5M} is effectively equivalent to an extra M^5 volume term. Equivalently, using the on-shell condition of self-duality of F_5 giving $F_{5X} = {}^*F_{5M}$, we conclude that $\mathrm{S}_{\rm top} = \gamma \int F_{5M} \wedge {}^*F_{5M} \sim \mathrm{vol}(X^5) \int_M |F_{5M}|^2$, which again produces, as is well known [30], a contribution to 5D cosmological term.

More generally, the assumption of simple "$5+5$" factorization of F_5 may be relaxed: provided F_5 can be split into an "electric" part (involving time differential) and its dual magnetic part, the topological term may be written as

$$S_{\rm top} = \gamma \int F_5^{(\rm el)} \wedge F_5^{(\rm mag)}, \qquad F_5^{(\rm mag)} = {}^*F_5^{(\rm el)}. \qquad (2.4)$$

The value of the coefficient γ in (2.2), (2.4) required to match the coefficient of the cosmological term in (1.1) is[11]

$$\gamma = -\frac{1}{4(5!)^2\kappa_{10}^2}, \qquad (2.5)$$

so the topological term in (2.4) takes the form

$$S_{\rm top} = -\frac{1}{4(5!)^2\kappa_{10}^2} \int F_5^{(\rm el)} \wedge {}^*F_5^{(\rm el)} = \frac{1}{4\kappa_{10}^2} \int d^{10}x\sqrt{G}|F_5^{(\rm el)}|^2. \qquad (2.6)$$

The total 10T action is then given by the sum of the bulk term (1.3), the new topological term (2.2), and the boundary term (1.9):

$$S_{10} = \hat{S}_{10} + S_{\rm top} + S_{\rm bndry}. \qquad (2.7)$$

Note that the $|F_5|^2$ term in the bulk action (1.3) may be written (before imposing self-duality) as

[9]For example, the $|F_3|^2$ term reduces to a boundary term upon use of the field equation $\nabla_\mu F^{\mu\nu\lambda} + \cdots = 0$; cf. also Ref. [29].

[10]Note that the fact that particular topological or boundary terms may or may not be relevant depending on boundary asymptotics of the fields is not unfamiliar. For example, the GHY boundary term complementing the Einstein action is relevant in the asymptotically flat space but may not be contributing in the AdS case (e.g., it vanishes for the AdS Schwarzschild black hole because the black hole correction to the AdS metric vanishes too rapidly at infinity [5]).

[11]Note that in our notation (with Minkowski signature 10D metric) for a general 5-form one has $\int F_5 \wedge {}^*F_5 = -(5!)^2 \int d^{10}x\sqrt{G}|F_5|^2$.

$\frac{1}{8\kappa_{10}^2}\int d^{10}x\sqrt{G}(|F_5^{(\mathrm{el})}|^2+|F_5^{(\mathrm{mag})}|^2)$. Adding the topological term (2.6) corresponds effectively to reversing the sign of the magnetic part in $|F_5|^2$, thus doubling the contribution of the electric part once going on shell (the self-duality condition implies $|F_5^{(\mathrm{el})}|^2 = -|F_5^{(\mathrm{mag})}|^2$).

As a result, the value (1.2) of the tree-level type IIB action on the $\mathrm{AdS}_5 \times S^5$ vacuum solution comes entirely from the topological term (2.2), (2.5): using (1.5), we get

$$\begin{aligned} S_{10}|_{\mathrm{AdS}_5\times S^5} &= S_{\mathrm{top}}|_{\mathrm{AdS}_5\times S^5} \\ &= -\frac{1}{4(5!)^2\kappa_{10}^2}\int F_{\mathrm{AdS}_5}\wedge F_{S^5} \\ &= -\frac{4L^8}{\kappa_{10}^2}\mathrm{vol}(\mathrm{AdS}_5), \end{aligned} \tag{2.8}$$

which is the same result that follows from the 5D action (1.2).

This has straightforward generalization to the case of $\mathrm{AdS}_5 \times X^5$ solutions where X^5 is an Einstein manifold as in Ref. [3]: instead of (1.2), one gets $S_5 = \mathrm{k}\,N^2\log(\Lambda \mathrm{r})$, with $\mathrm{k} \equiv \frac{\mathrm{vol}(S^5)}{\mathrm{vol}(X^5)}$ and $L^4 = 4\pi\alpha'^2 g_s\,\mathrm{k}\,N$.

As we will show in Appendix A, the same term (2.2) with precisely the same coefficient (2.5) is also required for gauge invariance in the PST formulation [16,17] of the 10D supergravity action where the 5-form self-duality condition follows from the equations of motion.

To provide further evidence that adding the term (2.2) to the type IIB action (1.3) restores its on-shell equivalence with the 5D reduced action like (1.1), let us consider the following $M^5 \times S^5$ ansatz for the metric and F_5 (with its self-duality condition relaxed and all other fields set to zero):

$$\begin{aligned} ds_{10}^2 &= L^2[e^{-\frac{10}{3}\nu(x)}g_{mn}(x)dx^m dx^n + e^{2\nu(x)}d\Omega_5^2], \\ F_5 &= 4L^{-1}[a(x)w_5 + b\mathrm{w}_5]. \end{aligned} \tag{2.9}$$

Here, $x = \{x^m\}$ $(m = 0, 1, \ldots 4)$, w_5 and w_5 are the volume forms on M^5 (with metric g_{mn}) and S^5, and we extracted the factors of the overall scale L. Following Ref. [31], we introduced the warp factors depending on a "fixed scalar" $\nu(x)$.[12] The condition $dF_5 = 0$ implies that $a = a(x)$ and $b = \mathrm{const}$. Then, the $R - \frac{1}{4}|F_5|^2$ part of the 10D action (1.3) compactified on S^5 becomes

$$\hat{S}_5 = -\frac{1}{2\kappa_5^2}\int d^5x\sqrt{g}\left[R_5 - \frac{40}{3}(\partial_m\nu)^2 - V(\nu) + \cdots\right], \tag{2.10}$$

$$V(\nu) = L^{-2}(-20e^{-\frac{16}{3}\nu} - 4a^2e^{\frac{40}{3}\nu} + 4b^2e^{-\frac{40}{3}\nu}). \tag{2.11}$$

The three terms in the potential V originate from the scalar curvature of S^5 and the $|F_5|^2$ term in (1.3) (cf. Refs. [1,31]). Using the on-shell self-duality of F_5 that gives $a = e^{-\frac{40}{3}\nu}b$, we find that the last two terms in the potential (2.11) mutually cancel, and thus, as was already mentioned above, we do not reproduce the value of the cosmological constant in (1.1).

If instead one plugs the ansatz (2.9) into the 10D equations of motion for (1.3) (that imply that $b^2 = 1$, $a = e^{-\frac{40}{3}\nu}$) and then reconstructs the corresponding effective action for the remaining 5D fields $g_{mn}(x)$ and $\nu(x)$, one finds instead the action (2.10) with the following potential [31]:

$$V(\nu) = L^{-2}(-20e^{-\frac{16}{3}\nu} + 8e^{-\frac{40}{3}\nu}). \tag{2.12}$$

This potential has the minimum at $\nu = 0$ and where it reproduces the cosmological term $12L^{-2}$ in (1.1). Comparing to (2.11), the potential (2.12) has the sign of the middle a^2 term in (2.11) effectively reversed so that it doubles the coefficient of the last b^2 term upon use of the on-shell condition $a = e^{-\frac{40}{3}\nu}$.

This is precisely what happens if we add to (2.10) the contribution of the topological term (2.2), (2.5) and then use the self-duality of F_5. We conclude that adding this term to the type IIB action ensures the equivalence between the 10D and 5D actions not only for $\mathrm{AdS}_5 \times S^5$ but also for more general solutions of $M^5 \times S^5$ topology.

III. 10D ACTION ON D3-BRANE SOLUTIONS

Let us now generalize the above discussion of the on-shell value of the type IIB action (1.3) with the topological term (2.2) added to the case of the extremal and nonextremal D3-brane solutions that also have the product topology as in (2.1).

The extremal D3-brane solution is given by [33,34]

$$\begin{aligned} ds_{10}^2 &= h^{-1/2}(r)dy^\mu dy_\mu + h^{1/2}(r)(dr^2 + r^2 d\Omega_5^2), \\ h(r) &= 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi\alpha'^2 g_s N, \end{aligned} \tag{3.1}$$

$$\begin{aligned} C_4^{(\mathrm{el})} &= [h^{-1}(r) - 1]dt\wedge dy^1\wedge dy^2\wedge dy^3, \\ F_5 &= F_5^{(\mathrm{el})} + F_5^{(\mathrm{mag})}, \qquad F_5^{(\mathrm{mag})} = {}^*F_5^{(\mathrm{el})}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F_5^{(\mathrm{el})} &= \frac{4r^3L^4}{(r^4+L^4)^2}dt\wedge dy^1\wedge dy^2\wedge dy^3\wedge dr, \\ F_5^{(\mathrm{mag})} &= 4L^{-1}\mathrm{w}_5. \end{aligned} \tag{3.3}$$

Here, $y^\mu = (y^0 \equiv t, y^1, y^2, y^3)$ are coordinates along the D3-brane, and $\mathrm{w}_5 = \sqrt{g_{S5}}dz^5\wedge\ldots\wedge dz^9$ is the volume

[12]The specific dependence on ν in the metric is required to decouple ν from the 5D graviton; this generalizes the graviton mode decomposition in Refs. [10,32] where ν was identified with the zero mode of the trace of the perturbation of the metric of S^5.

form of S^5. The near-core limit $h \to \frac{L^4}{r^4}$ corresponds to the $AdS_5 \times S^5$ case.

As discussed in the Introduction, the bulk part of the type IIB action (1.3) has zero on-shell value (once again, the self-duality of F_5 implies $|F_5|^2 = 0$ and thus also $R = 0$). A nontrivial contribution may come from the topological term (2.2) and also from the GHY boundary term (1.9) that may be nonvanishing in this asymptotically flat case. From (2.2) and (2.3), we find [cf. (2.8)][13]

$$\begin{aligned} S_{\rm top}|_{\rm D3} &= -\frac{1}{4(5!)^2\kappa_{10}^2}\int F_{5M}\wedge F_{5X} \\ &= -\frac{1}{4(5!)^2\kappa_{10}^2}\int F_5^{(\rm el)}\wedge F_5^{(\rm mag)} \\ &= -\frac{{\rm vol}(S^5)}{2\kappa_{10}^2}\int_0^\infty \frac{dr 8L^8 r^3}{(r^4+L^4)^2}\int d^4y \\ &= -\frac{{\rm vol}(S^5)}{\kappa_{10}^2}L^4\int d^4y = -\frac{1}{2}N\mu_3\int d^4y. \end{aligned} \tag{3.4}$$

Here,

$$\mu_3 = \frac{2{\rm vol}(S^5)L^4}{N\kappa_{10}^2} = \frac{1}{(2\pi)^3 g_s \alpha'^2} \tag{3.5}$$

is tension of a unit-charge D3-brane (cf. Footnote 1 and $\int d^4y$ is the integral over the D3 world volume directions. Compactifying (y^1, y^2, y^3) on a torus with volume V_3, we get

$$S_{\rm top}|_{\rm D3} = -\frac{1}{2}NM_3\int dt, \quad M_3 = \mu_3 V_3, \quad V_3 = \int d^3y, \tag{3.6}$$

where M_3 is the mass of a single D3-brane.

The GHY boundary term (1.9) (that did not contribute in the $AdS_5 \times S^5$ case) happens to give the same result as in (3.4) (here, the asymptotic boundary is at $r = \infty$)[14]:

$$\begin{aligned} S_{\rm bndry}|_{\rm D3} &= -\frac{{\rm vol}(S^5)}{\kappa_{10}^2}\frac{L^4}{1+\frac{L^4}{r^4}}\bigg|_{r\to\infty}\int d^4y \\ &= -\frac{1}{2}N\mu_3\int d^4y. \end{aligned} \tag{3.7}$$

Then, the on-shell value of the 10D action (2.7) on the D3-brane solution is given by

$$S_{10}|_{\rm D3} = (S_{\rm top} + S_{\rm bndry})|_{\rm D3} = -N\mu_3\int d^4y. \tag{3.8}$$

In addition, one may consider the value of the D3-brane source action that provides the delta function in the equation for the harmonic function $h(r)$,

$$S_{\rm source} = -N\mu_3\int d^4y\sqrt{G_4} + N\mu_3\int C_4. \tag{3.9}$$

More generally, considering this as an action of a static probe D3-branes placed at distance r parallel to the source branes at $r = 0$, one finds from (3.1) and (3.2) that the h^{-1} factors from the two terms in (3.9) cancel each other,[15] leaving simply

$$S_{\rm source}|_{\rm D3} = -N\mu_3\int d^4y \tag{3.10}$$

coming from the -1 in C_4 in (3.2). This is equal to the free brane action at $r = \infty$, and the same expression is thus also at $r \to 0$.

As a result, the total action on D3-brane solution is given by

$$\begin{aligned} &S_{\rm tot} \equiv S_{\rm bulk} + S_{\rm top} + S_{\rm bndry} + S_{\rm source}, \\ &S_{\rm tot}|_{\rm D3} = -2N\mu_3\int d^4y. \end{aligned} \tag{3.11}$$

Similar computations of the value of 10The action on some other p-brane solutions are presented in Appendix B.

Next, let us consider the nonextremal (black) D3-brane solution [33] generalizing (3.1)–(3.3),[16]

$$\begin{aligned} ds_{10}^2 = h^{-1/2}(r)[-f(r)dt^2 + dy^i dy^i] \\ + h^{1/2}(r)[f^{-1}(r)dr^2 + r^2 d\Omega_5^2], \end{aligned} \tag{3.12}$$

$$\begin{aligned} &h(r) = 1 + \frac{\tilde{L}^4}{r^4}, \qquad f(r) = 1 - \frac{r_0^4}{r^4}, \\ &\tilde{L}^4 = \sqrt{L^8 + \frac{1}{4}r_0^8} - \frac{1}{2}r_0^4, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &C_4^{(\rm el)} = \sigma[h^{-1}(r) - 1]dy^0\wedge\ldots\wedge dy^3, \quad \sigma \equiv \frac{L^4}{\tilde{L}^4} = \sqrt{1 + \frac{r_0^4}{\tilde{L}^4}}, \\ &F_5^{(\rm el)} = \frac{4\sigma\tilde{L}^4 r^3}{(r^4+\tilde{L}^4)^2}dy^0\wedge\ldots\wedge dy^3\wedge dr, \quad F_5^{(\rm mag)} = 4\sigma\tilde{L}^{-1}{\rm w}_5, \\ &F_5 = F_5^{(\rm el)} + F_5^{(\rm mag)}, \end{aligned} \tag{3.14}$$

[13] If we focus on the near-core limit ($r \ll L$) of (3.1) and (3.2), we get the same expression as in (2.8) with the volume of AdS_5 written in Poincaré coordinates.

[14] Here and below, when evaluating the boundary term (1.9), we neglect contributions that are independent of the parameters of the solution.

[15] This is, of course, a manifestation of the BPS condition of the vanishing force; see, e.g., Ref. [35].

[16] We use the same parametrization as in Ref. [36].

where L is the same as in (3.1). We shall consider this solution for $r_0 \leq r < \infty$ and should not introduce an explicit brane source.

The value of the topological term (2.3) is found as in (3.4):

$$\begin{aligned} S_{\rm top}|_{\rm black\,D3} &= -\frac{1}{4(5!)^2\kappa_{10}^2}\int F_5^{(\rm el)}\wedge F_5^{(\rm mag)} \\ &= -\frac{{\rm vol}(S^5)}{2\kappa_{10}^2}\sigma^2\int_{r_0}^{\infty}\frac{dr 8\tilde{L}^8 r^3}{(r^4+\tilde{L}^4)^2}\int d^4y \\ &= -\frac{{\rm vol}(S^5)}{\kappa_{10}^2}\tilde{L}^4\int d^4y. \end{aligned} \tag{3.15}$$

Once again, we see that the topological term gives a nontrivial contribution to the action.

The expression (3.15) may be written also as

$$S_{\rm top}|_{\rm black\,D3} = \frac{1}{2}N\mu_3 C_4^{(el)}(r_0)\int d^4y, \tag{3.16}$$

$$\begin{aligned} C_4^{(\rm el)}(r_0) &= -\frac{\sigma\tilde{L}^4}{r_0^4+\tilde{L}^4}, \\ N\mu_3 &= \frac{2{\rm vol}(S^5)L^4}{\kappa_{10}^2} = \frac{2{\rm vol}(S^5)\sigma\tilde{L}^4}{\kappa_{10}^2}, \end{aligned} \tag{3.17}$$

i.e., proportional to a product of the electric potential $C_4^{(\rm el)}$ at the horizon and the black D3-brane charge. This is analogous to what one finds in the case of the Reissner-Nordstrom black hole [22].[17]

The calculation of the asymptotic $r\to\infty$ boundary GHY term (1.9) here gives [cf. Eq. (3.7)]

$$\begin{aligned} S_{\rm bndry}|_{\rm black\,D3} &= -\frac{{\rm vol}(S^5)}{\kappa_{10}^2}\left[\tilde{L}^4\frac{1-\frac{r_0^4}{r^4}}{1+\frac{\tilde{L}^4}{r^4}}+3r_0^4\right]_{r\to\infty}\int d^4y \\ &= -\frac{{\rm vol}(S^5)}{\kappa_{10}^2}(\tilde{L}^4+3r_0^4)\int d^4y. \end{aligned} \tag{3.18}$$

As the bulk 10D action (1.3) is again vanishing, the total action (2.7) computed on the nonextremal D3-brane solution then follows by combining (3.15) and (3.18):

$$\begin{aligned} S_{10}|_{\rm black D3} &= (S_{\rm top}+S_{\rm bndry})|_{\rm black D3} \\ &= -\frac{2{\rm vol}(S^5)}{\kappa_{10}^2}\left(\tilde{L}^4+\frac{3}{2}r_0^4\right)\int d^4y \\ &= -\frac{2{\rm vol}(S^5)}{\kappa_{10}^2}\left(\sqrt{L^8+\frac{1}{4}r_0^8}+r_0^4\right)\int d^4y. \end{aligned} \tag{3.19}$$

The same result should be found by first compactifying on S^5, finding the reduced 5D action generalizing (1.1), and then evaluating it on the corresponding 5D black brane solution.[18]

A similar discussion can be repeated for the type IIB solutions describing BPS intersections of D3-branes—D3⊥D3 [37] and D3⊥D3⊥D3⊥D3 [38]. In the near-core limit, they reduce (in the extremal case) to $\mathrm{AdS}_3\times S^3\times T^4$ and $\mathrm{AdS}_2\times S^2\times T^6$ backgrounds, respectively. Here, the bulk part of type IIB action is again vanishing, with possible nonzero contribution coming from the topological term (2.4) defined in terms of

$$F_5^{(\rm el)} = dC_4^{(\rm el)}, \quad F_5^{(\rm mag)} = {}^*F_5^{(\rm el)}, \quad F_5 = F_5^{(\rm el)}+F_5^{(\rm mag)} \tag{3.20}$$

and also the GHY term (in the case of the full asymptotically flat solution).

The D3⊥D3 solution is the following generalization of the D3 background (3.1)–(3.3):

$$\begin{aligned} ds_{10}^2 &= (h_1h_2)^{1/2}[(h_1h_2)^{-1}(-dt^2+dy_1^2)+h_1^{-1}(dy_2^2+dy_3^2) \\ &\quad + h_2^{-1}(dy_4^2+dy_5^2)+dr^2+r^2d\Omega_3^2], \qquad h_i = 1+\frac{L_i^2}{r^2}, \\ C_4^{(\rm el)} &= [h_1^{-1}-1]dt\wedge dy^1\wedge dy^2\wedge dy^3 \\ &\quad + [h_2^{-1}-1]dt\wedge dy^1\wedge dy^4\wedge dy^5. \end{aligned} \tag{3.21}$$

Here, (y^1,y^2,y^3) and (y^1,y^3,y^4) are spatial coordinates along the two D3-branes intersecting over the y^1 direction. In the near-core limit $h_i\to\frac{L_i^2}{r^2}$, this background reduces to $\mathrm{AdS}_3\times S^3\times T^4$ with $ds^2_{\mathrm{AdS}_3} = \frac{r^2}{L^2}(-dt^2+dy_1^2)+\frac{L^2}{r^2}dr^2$, $ds^2_{S^3} = L^2 d\Omega_3^2$ (where $L^2 = L_1L_2$), and $ds^2_{T^4} = \frac{L_2}{L_1}(dy_2^2+dy_3^2)+\frac{L_1}{L_2}(dy_4^2+dy_5^2)$.

Note that here F_5 does not have a simple $5+5$ decomposition so the topological term is defined by Eq. (2.4) or, equivalently, Eq. (2.6). Computing it gives a nonzero value consistent with the one of the dimensionally reduced 3D analog of the action (1.1) that admits AdS_3 as its solution. Explicitly, in the $\mathrm{AdS}_3\times S^3\times T^4$ limit, we find that $S_{\rm top} = -\frac{2}{\kappa_{10}^2}{\rm vol}(\mathrm{AdS}_3){\rm vol}(S^3){\rm vol}(T^4)$ (where we did not extract the dependence on the scale $L=\sqrt{L_1L_2}$).

[17]In the context of black brane thermodynamics, the topological term will thus contribute to the part of the "thermodynamic potential" related to the product of the chemical potential and the corresponding conserved charge.

[18]For comparison with the extremal case (3.6), let us note that the ADM mass of black D3-brane is given by $\tilde{M}_3 = \mu_3 V_3\frac{\tilde{L}^4+\frac{5}{4}r_0^4}{L^4} = M_3[\sqrt{1+\frac{r_0^8}{4L^8}}+\frac{3r_0^4}{4L^4}]$.

Similarly, the four D3-brane solution is given by

$$\begin{aligned} ds_{10}^2 &= (h_1h_2h_3h_4)^{1/2}[-(h_1h_2h_3h_4)^{-1}dt^2 + (h_1h_2)^{-1}dy_1^2 \\ &\quad + (h_1h_3)^{-1}dy_2^2 + (h_1h_4)^{-1}dy_3^2 + (h_2h_3)^{-1}dy_4^2 \\ &\quad + (h_2h_4)^{-1}dy_5^2 + (h_3h_4)^{-1}dy_6^2 + dr^2 + r^2 d\Omega_2^2], \\ h_i &= 1 + \frac{L_i}{r}, \\ C_4^{(\mathrm{el})} &= [h_1^{-1} - 1]dt \wedge dy^1 \wedge dy^2 \wedge dy^3 \\ &\quad + [h_2^{-1} - 1]dt \wedge dy^1 \wedge dy^4 \wedge dy^5 \\ &\quad + [h_3^{-1} - 1]dt \wedge dy^2 \wedge dy^4 \wedge dy^6 \\ &\quad + [h_4^{-1} - 1]dt \wedge dy^3 \wedge dy^5 \wedge dy^6. \end{aligned} \tag{3.22}$$

This reduces to $\mathrm{AdS}_2 \times S^2 \times T^6$ with the 6-torus formed by $(y_1, \ldots, y_6)$. Here, the topological term (2.4), (2.6) produces again a nonzero contribution to 10D action.

IV. CONCLUDING REMARKS

Depending on topology of space-time or asymptotic boundary conditions, the 10D supergravity action (or, more generally, string effective action) may need to be supplemented by particular boundary or topological terms specific to a type of backgrounds considered.

Here, we considered the case of $\mathrm{M}^{10} = M^5 \times X^5$ with 5-form flux and showed that adding the topological term (2.2) or (2.4) to the bulk type IIB action (1.3) restores its equivalence with the 5D reduced action (obtained via equations of motion by compactifying on X^5). This leads to consistent on-shell values of the full 10D action (e.g., for $\mathrm{AdS}_5 \times X^5$ or D3-brane solution). Similar terms are to be added in cases of other topologies, e.g., $\int_6 F_{3M} \wedge F_{3X}$ for $\mathrm{M}^{10} = M^3 \times X^3 \times T^4$.[19]

The string theory origin of the term (2.2) and whether it may receive α' corrections remains to be understood. One particular case when the contribution of this term may be important is the computation of α' corrections to near-extremal D3-brane entropy as in Ref. [39].

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APPENDIX A: TOPOLOGICAL TERM IN F_5 ACTION ON $M^5 \times X^5$ FROM PST FORMULATION

In the PST formulation [16,17] of the 5-form action, the condition of self-duality is derived from an action. This is achieved by introducing an extra scalar field $a(x)$ along with extra gauge invariance so that the number of dynamical degrees of freedom is unchanged. For a closed 5-form F_5, let us consider the following action[20]:

$$\begin{aligned} S_{\mathrm{PST}} &= \int (F_5 \wedge {}^*F_5 + \mathfrak{i}_v \mathcal{F} \wedge {}^* \mathfrak{i}_v \mathcal{F}) \\ &= -\int 2v \wedge F_5 \wedge \mathfrak{i}_v (F_5 - {}^*F_5), \quad \mathcal{F} \equiv F_5 - {}^*F_5. \end{aligned} \tag{A1}$$

We assume that F_5 can be expressed locally as $F_5 = dC_4$ [we ignore all the other fields that may contribute to F_5 in (1.4)]. $v = v_\mu dx^\mu$ is defined in terms of a scalar $a(x)$ as

$$v_\mu = \frac{1}{\sqrt{-|\partial a|^2}} \partial_\mu a, \qquad v^\mu v_\mu = -1. \tag{A2}$$

The variation of (A1) over C_4 and a then leads to equations that imply the self-duality condition $F_5 = {}^*F_5$. The dependence on the scalar a drops out of the equations of motion.

The reason for this is that apart from the standard gauge symmetry of a 4-form potential $C_4 \to C_4 + d\varepsilon_3$ the action (A1) is invariant (up to boundary terms; see below) under the following gauge transformations:

$$\delta_\eta a = \eta, \qquad \delta_\eta C_4 = -\frac{1}{\sqrt{-|\partial a|^2}} \mathfrak{i}_v (F_5 - {}^*F_5)\eta, \tag{A3}$$

$$\delta_\xi a = 0, \quad \delta_\xi C_4 = \xi_3 \wedge da, \quad \delta_\xi F_5 = d\xi_3 \wedge da. \tag{A4}$$

Here, the scalar $\eta(x)$ and the 3-form $\xi_3(x)$ are the gauge parameters. The first symmetry (A3) implies that a is a pure gauge field. The second is effectively reflecting the fact that the number of degrees of freedom of C_4 is halved on shell (where F_5 becomes self-dual).

Let us consider the variation of the action under arbitrary δC_4 and δa:

[19] Note also that an analogous example is found in the case of the extremal dyonic black hole in four-dimensional Einstein-Maxwell theory. Here, the $\mathrm{AdS}_2 \times S^2$ vacuum is supported by $F_2 = F_2^{(\mathrm{el})} + F_2^{(\mathrm{mag})}$, and the on-shell value of the action is zero $(R_4 = 0, |F_2|^2 = 0)$. Adding the standard topological term $\int F_2 \wedge F_2$ then produces a cosmological term in the effective two-dimensional action. This example is related to the near-core limit of the four D3-brane background discussed in the previous section.

[20] As usual, $\mathfrak{i}_v$ denotes the contraction of a differential form with a vector field, obtained from the coefficient of 1-form v by raising the index with the help of the metric. In (A1), we ignore an overall normalization factor.

$$\delta S_{\rm PST} = -\int \frac{2v}{\sqrt{-|\partial a|^2}} \wedge d\delta a \wedge \mathbf{i}_v\mathcal{F} \wedge \mathbf{i}_v\mathcal{F} - \int 4v \wedge \delta F_5 \wedge \mathbf{i}_v\mathcal{F} - \int 2F_5 \wedge \delta F_5$$
$$= -\int 2\delta a d\left[\frac{v}{\sqrt{-|\partial a|^2}} \wedge \mathbf{i}_v\mathcal{F} \wedge \mathbf{i}_v\mathcal{F}\right] - \int 4\delta C_4 \wedge d[v \wedge \mathbf{i}_v\mathcal{F}] + \int_\partial \left[\frac{2v}{\sqrt{-|\partial a|^2}}\delta a \wedge \mathbf{i}_v\mathcal{F} \wedge \mathbf{i}_v\mathcal{F} + 4\delta C_4 \wedge v \wedge \mathbf{i}_v\mathcal{F}\right] - \int 2F_5 \wedge \delta F_5. \tag{A5}$$

Assuming that $\delta C_4 = 0$ at the boundary, the resulting equations of motion may be written as

$$\delta a\colon\ d\left[\frac{v}{\sqrt{-|\partial a|^2}} \wedge \mathbf{i}_v(F_5 - {}^*F_5) \wedge \mathbf{i}_v(F_5 - {}^*F_5)\right] = 0, \tag{A6}$$

$$\delta C_4\colon\ d[v \wedge \mathbf{i}_v(F_5 - {}^*F_5)] = 0. \tag{A7}$$

Under the transformation (A4), the expression in brackets in (A7) changes as

$$\delta_\xi[v \wedge \mathbf{i}_v(F_5 - {}^*F_5)] = -\delta_\xi F_5 = -d\xi_3 \wedge da, \tag{A8}$$

so (A4) is a symmetry of (A7). Furthermore, using (A2), we may choose such ξ_3 that $\mathbf{i}_v(F_5 - {}^*F_5) = 0$. Then,

$$F_5 - {}^*F_5 = -v \wedge \mathbf{i}_v(F_5 - {}^*F_5) +{}^* (v \wedge \mathbf{i}_v(F_5 - {}^*F_5)) = 0. \tag{A9}$$

Therefore, the symmetry (A4) makes all solutions of (A7) equivalent to the self-dual solution $F_5 = {}^*F_5$ (and all of them lead to the vanishing on-shell value of $S_{\rm PST}$).

Under (A4), the integrand of (A1) changes as[21]

$$\delta_\xi\mathcal{L} = -2[v \wedge \mathbf{i}_v(\delta_\xi F_5 - {}^*\delta_\xi F_5) \wedge F_5 + v \wedge \mathbf{i}_v(F_5 - {}^*F_5) \wedge \delta_\xi F_5 + v \wedge \mathbf{i}_v(\delta_\xi F_5 - {}^*\delta_\xi F_5) \wedge \delta_\xi F_5]. \tag{A10}$$

Using that $\delta_\xi F_5 \wedge da = 0$, the variation of the action (A1) may be written as

$$\delta_\xi S_{\rm PST} = -2\int F_5 \wedge \delta_\xi F_5. \tag{A11}$$

This vanishes if 10D space has no boundary [as $dF_5 = 0$ we have $F_5 \wedge d\xi_3 \wedge da = -d(F_5 \wedge \xi_3 \wedge da)$] but otherwise produces a boundary term.

Let us now assume as in (2.1) that the 10D space has a product structure, i.e., $\mathrm{M}^{10} = M^5 \times X^5$, where X^5 is a compact Euclidean space with no boundary while M^5 (with Minkowski signature metric) may be noncompact, and also that a similar factorization applies to the 4-form potential and the parameters of the transformations in (A4), i.e.,

$$C_4 = C_{4M} \oplus C_{4X}, \qquad F_5 = F_{5M} \oplus F_{5X},$$
$$\delta_\xi C_4 = \delta_\xi C_{4M} \oplus \delta_\xi C_{4X}. \tag{A12}$$

In this case, Eq. (A11) takes the form

$$\delta_\xi S_{\rm PST} = -2\int F_{5X} \wedge \delta_\xi F_{5M} = 2\int \delta_\xi F_M \wedge F_{5X} = 2\int_X F_{5X}\int_M \delta_\xi F_{5M}, \tag{A13}$$

where we used that $\delta_\xi F_{5X}$ is exact so its integral over X^5 vanishes. The integral $\int_M \delta_\xi F_{5M} = \int_{\partial M} \xi_3 \wedge da$ depends on the boundary values of the gauge parameter ξ_3 and the scalar field a. If these are nontrivial and if F_5 has a nontrivial value of the "magnetic" charge $\int_X F_{5X} \neq 0$, then the variation (A13) may be nonzero.

A way to maintain the invariance of the action (A1) under (A4) is to add to (A1) the topological term defined in (2.2),

$$S_{\rm top} = -2\int_M F_{5M} \wedge F_{5X} = -2\int_X F_{5X}\int_M F_{5M}. \tag{A14}$$

The variation of this term under the gauge transformation (A4) will then cancel the change (A13) of the PST action. Assuming $F_{5M} = dC_{5M}$ is valid globally on M^5, the term (A14) may be expressed as an integral over the boundary $\partial M^5 \times X^5$ and thus does not affect the equations of motion for F_5. Let us note that a similar argument suggesting to add the term (A14) to maintain gauge invariance can be given [20] also in the formulation of self-dual F_5 field suggested in Ref. [19].

Using that the equations of motion for (A1) imply the self-duality of F_5, i.e., $F_{5M} = {}^*F_{5X}$, the on-shell value of (A1) plus (A14) may be written also as

$$(S_{\rm PST} + S_{\rm top})|_{F_5={}^*F_5} = S_{\rm top}|_{F_5={}^*F_5} = 2\int F_{5X} \wedge {}^*F_{5X} = -2\int F_{5M} \wedge {}^*F_{5M}. \tag{A15}$$

Replacing the $|F_5|^2$ term in the 10D action (1.3) by (A1), one gets the corresponding PST analog of the type IIB action to which now we should add also (A14) with the corresponding coefficient being as in (2.5). It is interesting

[21] Here, $v \wedge \mathbf{i}_v\delta_\xi F_5 \wedge F_5 = -\delta_\xi F_5 \wedge F_5 + v \wedge \delta_\xi F_5 \wedge \mathbf{i}_v F_5$ and $v \wedge \mathbf{i}_v{}^*\delta_\xi F_5 \wedge F_5 = v \wedge^* (\delta_\xi F_5 \wedge v) \wedge F_5 = -\delta_\xi F_5 \wedge v \wedge \mathbf{i}_v{}^*F_5$.

to note that the condition of the symmetry under (A4) fixes also the relative coefficient between the kinetic 5-form term (A1) and the Chern-Simons type term [$\int B_2 \wedge F_3 \wedge F_5$ in (1.3)] in the resulting version of type IIB action [17].

APPENDIX B: 10D ACTION ON F1, NS5, AND D5 BRANE SOLUTIONS

For comparison with the case of the D3-brane solution discussed in Sec. III, here we will discuss the values of the 10D action (3.11) on the fundamental string, NS5-brane, and D5-brane extremal solutions. In these cases, $F_5 = 0$, so the topological term (2.2) will not play a role. These p-brane solutions are supported by sources given by the corresponding brane actions that have the structure (see, e.g., Ref. [40])

$$S_{\rm source} = -NT_p \int d^{p+1} y e^{-q\phi} \sqrt{G_{p+1}} + NT_p \int A_{p+1}, \quad (B1)$$

where T_p is a tension of a single brane and A_{p+1} is the corresponding NS-NS or R-R potential. The dilaton coupling constant is $q = 0, 2$, and 1 for F1, NS5, and D5, cases respectively. The total action will be

$$S_{\rm tot} = S_{\rm bulk} + S_{\rm bndry} + S_{\rm source}, \qquad S_{\rm bulk} = \hat{S}_{10}, \quad (B2)$$

where the bulk part is given by (1.3) and the boundary one by (1.9).

The F1 solution [41] is electrically charged with the respect to the B_2 field ($T_1 = \frac{1}{2\pi\alpha'}$),

$$ds^2 = H^{-1}(r)(-dy_0^2 + dy_1^2) + dx^a dx^a, \qquad H(r) = 1 + \frac{Q}{r^6},$$
$$Q = \frac{NT_1\kappa_{10}^2}{3\mathrm{vol}(S^7)} = 32N\pi^2\alpha'^3 g_s^2,$$
$$B_2 = [H^{-1}(r) - 1] dy^0 \wedge dy^1, \qquad e^{2\phi} = H^{-1}(r). \quad (B3)$$

Substituting this solution into (B2), we find

$$S_{\rm bulk}|_{\rm F1} = S_{\rm bndry}|_{\rm F1} = 0, \qquad S_{\rm source}|_{\rm F1} = -NT_1 \int d^2 y,$$
$$S_{\rm tot}|_{\rm F1} = -NT_1 \int d^2 y. \quad (B4)$$

The magnetic dual of F1-brane—the NS5-brane solution [42]—may be considered as electrically charged with respect to the dual field $\tilde{B}_6$, $d\tilde{B}_6 = e^{-2\phi} {}^*H_3$, i.e.,

$$S_{\rm source} = -NT_5 \int d^6 y e^{-2\phi} \sqrt{G_6} + NT_5 \int \tilde{B}_6,$$
$$T_5 = \frac{1}{(2\pi)^5 \alpha'^3 g_s^2}. \quad (B5)$$

The corresponding background is ($\mu = 0, \ldots, 5; a = 6, 7, 8, 9$; $r^2 = x^a x^a$)

$$ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu + H(r) dx^a dx^a, \qquad H(r) = 1 + \frac{Q}{r^2},$$
$$Q = \frac{NT_5\kappa_{10}^2}{\mathrm{vol}(S^3)} = \alpha' N,$$
$$\tilde{B}_6 = [H^{-1}(r) - 1] dy^0 \wedge \ldots \wedge dy^6, \qquad e^{2\phi} = H(r). \quad (B6)$$

Here, we find

$$S_{\rm bulk}|_{\rm NS5} = 0, \qquad S_{\rm bndry}|_{\rm NS5} = -NT_5 \int d^6 y,$$
$$S_{\rm source}|_{\rm NS5} = -NT_5 \int d^6 y, \quad (B7)$$

$$S_{\rm tot}|_{\rm NS5} = -2NT_5 \int d^6 y. \quad (B8)$$

Evaluating the bulk term here and in (B4), we used the explicit form of the solution; note that the NS-NS part of the bulk action (1.3) or (1.6) automatically vanishes only for solutions without a source term in the dilaton equation.

In the case of D5-brane solution that has magnetic charge with respect to the RR 3-form F_3, we may again introduce the dual electric potential $\tilde{C}_6$ ($d\tilde{C}_6 = {}^*F_3$) and consider

$$S_{\rm source} = -N\mu_5 \int d^6 y e^{-\phi} \sqrt{G_6} + N\mu_5 \int \tilde{C}_6,$$
$$\mu_5 = \frac{1}{(2\pi)^5 \alpha'^3 g_s}. \quad (B9)$$

The D5-solution supported by the corresponding source at $x^a = 0$ is [33]

$$ds^2 = H^{-\frac{1}{2}}(r)\eta_{\mu\nu} dy^\mu dy^\nu + H^{\frac{1}{2}}(r) dx^a dx^a,$$
$$H(r) = 1 + \frac{Q}{r^2}, \quad Q = \frac{N\mu_5\kappa_{10}^2}{\mathrm{vol}(S^3)} = \alpha' N g_s,$$
$$\tilde{C}_6 = [H^{-1}(r) - 1] dy^0 \wedge \ldots \wedge dy^6, \quad e^{-2\phi} = H(r). \quad (B10)$$

The resulting contributions to the total action (B2) here are

$$S_{\rm bulk}|_{\rm D5} = -\frac{1}{2} N\mu_5 \int d^6 y, \quad S_{\rm bndry}|_{\rm D5} = -\frac{1}{2} N\mu_5 \int d^6 y,$$
$$S_{\rm source}|_{\rm D5} = -N\mu_5 \int d^6 y, \quad S_{\rm tot}|_{\rm D5} = -2N\mu_5 \int d^6 y. \quad (B11)$$

The values of the total actions for NS5 (B8) and D5 (B11) cases have the same structure as for the D3-brane solution in (3.8) and also are consistent with the S-duality relation between the two 5-branes.

Note that the bulk and boundary contributions match only in sum: one can show that the S-duality transformation in the formulation using the string frame metric leaves invariant only the sum of the bulk (1.3) and boundary (1.9) terms in the type IIB action.

[1] J. M. Maldacena, TASI 2003 lectures on AdS/CFT, arXiv:hep-th/0309246.

[2] H. Liu and A. A. Tseytlin, $D=4$ super Yang-Mills, $D=5$ gauged supergravity, and $D=4$ conformal supergravity, Nucl. Phys. **B533**, 88 (1998); M. Henningson and K. Skenderis, The holographic Weyl anomaly, J. High Energy Phys. 07 (1998) 023.

[3] S. S. Gubser, Einstein manifolds and conformal field theories, Phys. Rev. D **59**, 025006 (1999).

[4] M. Blau, K. S. Narain, and E. Gava, On subleading contributions to the AdS/CFT trace anomaly, J. High Energy Phys. 09 (1999) 018; C. P. Burgess, N. R. Constable, and R. C. Myers, The free energy of $N=4$ superYang-Mills and the AdS/CFT correspondence, J. High Energy Phys. 08 (1999) 017; J. G. Russo and K. Zarembo, Large N limit of $N=2$ SU(N) gauge theories from localization, J. High Energy Phys. 10 (2012) 082; S. Giombi and B. Offertaler, Wilson loops in $\mathcal{N}=4$ SO(N) SYM and D-branes in AdS5 × RP5, J. High Energy Phys. 10 (2021) 016.

[5] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-se Sitter space, Commun. Math. Phys. **87**, 577 (1983); E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[6] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D **60**, 064018 (1999).

[7] R. Emparan, C. V. Johnson, and R. C. Myers, Surface terms as counterterms in the AdS/CFT correspondence, Phys. Rev. D **60**, 104001 (1999).

[8] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, Phys. Rev. D **60**, 104026 (1999).

[9] M. Gunaydin, L. J. Romans, and N. P. Warner, Gauged $N=8$ supergravity in five-dimensions, Phys. Lett. **154B**, 268 (1985); M. Pernici, K. Pilch, and P. van Nieuwenhuizen, Gauged $N=8$ $D=5$ supergravity, Nucl. Phys. **B259**, 460 (1985).

[10] H. Kim, L. Romans, and P. van Nieuwenhuizen, The mass spectrum of chiral $\mathcal{N}=2$ $D=10$ supergravity on S^5, Phys. Rev. D **32**, 389 (1985).

[11] J. H. Schwarz, Covariant field equations of chiral $N=2$ $D=10$ supergravity, Nucl. Phys. **B226**, 269 (1983).

[12] P. S. Howe and P. C. West, The complete $N=2$, $D=10$ supergravity, Nucl. Phys. **B238**, 181 (1984).

[13] M. Cvetic, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati, and T. A. Tran, Embedding AdS black holes in ten-dimensions and eleven-dimensions, Nucl. Phys. **B558**, 96 (1999); H. Lu, C. N. Pope, and T. A. Tran, Five-dimensional $N=4$, $SU(2)\times U(1)$ gauged supergravity from type IIB, Phys. Lett. B **475**, 261 (2000); M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh, and T. A. Tran, Consistent SO(6) reduction of type IIB supergravity on S5, Nucl. Phys. **B586**, 275 (2000); A. Baguet, O. Hohm, and H. Samtleben, Consistent Type IIB reductions to maximal 5D supergravity, Phys. Rev. D **92**, 065004 (2015).

[14] E. Bergshoeff, H. J. Boonstra, and T. Ortin, S duality and dyonic p-brane solutions in type II string theory, Phys. Rev. D **53**, 7206 (1996).

[15] M. Henneaux and C. Teitelboim, Dynamics of chiral (Selfdual) P forms, Phys. Lett. B **206**, 650 (1988); J. H. Schwarz and A. Sen, Duality symmetric actions, Nucl. Phys. **B411**, 35 (1994).

[16] P. Pasti, D. P. Sorokin, and M. Tonin, On Lorentz invariant actions for chiral p forms, Phys. Rev. D **55**, 6292 (1997).

[17] G. Dall'Agata, K. Lechner, and D. P. Sorokin, Covariant actions for the bosonic sector of $d=10$ IIB supergravity, Classical Quantum Gravity **14**, L195 (1997); G. Dall'Agata, K. Lechner, and M. Tonin, $D=10$, N = IIB supergravity: Lorentz invariant actions and duality, J. High Energy Phys. 07 (1998) 017.

[18] D. Belov and G. W. Moore, Holographic action for the self-dual field, arXiv:hep-th/0605038; A. Sen, Covariant action for type IIB supergravity, J. High Energy Phys. 07 (2016) 017.

[19] K. Mkrtchyan, On covariant actions for chiral p—forms, J. High Energy Phys. 12 (**2019**) 076.

[20] K. Mkrtchyan and F. Valach, Actions for type II supergravities and democracy, arXiv:2207.00626.

[21] J. W. York, Jr., Role of Conformal Three Geometry in the Dynamics of Gravitation, Phys. Rev. Lett. **28**, 1082 (1972).

[22] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).

[23] G. E. Arutyunov and S. A. Frolov, On the origin of supergravity boundary terms in the AdS/CFT correspondence, Nucl. Phys. **B544**, 576 (1999).

[24] S. de Haro, S. N. Solodukhin, and K. Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. **217**, 595 (2001).

[25] C. G. Callan, Jr., I. R. Klebanov, and M. J. Perry, String theory effective actions, Nucl. Phys. **B278**, 78 (1986); A. A. Tseytlin, Vector field effective action in the open superstring theory, Nucl. Phys. **B276**, 391 (1986).

[26] A. A. Tseytlin, Conformal anomaly in two-dimensional sigma model on curved background and strings, Phys. Lett. B **178**, 34 (1986); A. A. Tseytlin, Conditions of Weyl invariance of two-dimensional σ model from equations of stationarity of 'central charge' action, Phys. Lett. B **194**, 63 (1987); A. A. Tseytlin, Mobius infinity subtraction and effective action in σ model approach to closed string theory, Phys. Lett. B **208**, 221 (1988); H. Osborn, General bosonic σ models and string effective actions, Ann. Phys. (N.Y.) **200**, 1 (1990).

[27] T. Erler, The closed string field theory action vanishes, arXiv:2204.12863.

[28] V. A. Kazakov and A. A. Tseytlin, On free energy of 2-D black hole in bosonic string theory, J. High Energy Phys. 06 (2001) 021.

[29] Y. Chen, J. Maldacena, and E. Witten, On the black hole/string transition, arXiv:2109.08563.

[30] A. Aurilia, H. Nicolai, and P. K. Townsend, Hidden constants: The theta parameter of QCD and the cosmological constant of $N=8$ supergravity, Nucl. Phys. **B176**, 509 (1980); M. J. Duff and P. van Nieuwenhuizen, Quantum inequivalence of different field representations, Phys. Lett. **94B**, 179 (1980); M. J. Duncan and L. G. Jensen, Four

forms and the vanishing of the cosmological constant, Nucl. Phys. **B336**, 100 (1990).

[31] H. Liu and A. A. Tseytlin, Dilaton—fixed scalar correlators and AdS(5) x S5—SYM correspondence, J. High Energy Phys. 10 (1999) 003.

[32] M. Gunaydin and N. Marcus, The spectrum of the S^5 compactification of the chiral $\mathcal{N} = 2$, $D = 10$ supergravity and the unitary supermultiplets of $U(2, 2|4)$, Classical Quantum Gravity **2**, L11 (1985).

[33] G. T. Horowitz and A. Strominger, Black strings and P-branes, Nucl. Phys. **B360**, 197 (1991).

[34] M. J. Duff and J. X. Lu, The selfdual type IIB superthree-brane, Phys. Lett. B **273**, 409 (1991).

[35] A. A. Tseytlin, No force condition and BPS combinations of p-branes in eleven-dimensions and ten-dimensions, Nucl. Phys. **B487**, 141 (1997).

[36] A. A. Tseytlin and S. Yankielowicz, Free energy of $N = 4$ superYang-Mills in Higgs phase and nonextremal D3-brane interactions, Nucl. Phys. **B541**, 145 (1999).

[37] A. A. Tseytlin, Harmonic superpositions of M-branes, Nucl. Phys. **B475**, 149 (1996).

[38] I. R. Klebanov and A. A. Tseytlin, Intersecting M-branes as four-dimensional black holes, Nucl. Phys. **B475**, 179 (1996).

[39] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory, Nucl. Phys. **B534**, 202 (1998); J. Pawelczyk and S. Theisen, $\mathrm{AdS}(5) \times$ S5 black hole metric at $\mathrm{O}(\alpha'^3)$, J. High Energy Phys. 09 (1998) 010.

[40] J. Simon, Brane effective actions, kappa-symmetry and applications, Living Rev. Relativity **15**, 3 (2012); T. Ortin, *Gravity and Strings*, 2nd ed. (Cambridge University Press, Cambridge, England, 2015).

[41] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, Superstrings and Solitons, Nucl. Phys. **B340**, 33 (1990).

[42] C. G. Callan, Jr., J. A. Harvey, and A. Strominger, World sheet approach to heterotic instantons and solitons, Nucl. Phys. **B359**, 611 (1991).