# Nonlinear effective dynamics of a Brownian particle in magnetized plasma

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An effective description is presented for a Brownian particle in a magnetized plasma. In order to systematically capture various corrections to linear Langevin equation, we construct effective action for the Brownian particle, to quartic order in its position. The effective action is first derived within nonequilibrium effective field theory formalism, and then confirmed via a microscopic holographic model consisting of an open string probing magnetic fifth-dimensional anti–de Sitter ($AdS_5$) black brane. For practical usage, the non-Gaussian effective action is converted into Fokker-Planck-type equation, which is an Euclidean analog of Schrödinger equation and describes time evolution of probability distribution for particle's position and velocity.

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## I. INTRODUCTION

Brownian motion is perhaps the simplest example of nonequilibrium phenomena, which, however, has played a profound role in the development of nonequilibrium statistical mechanics [1]. In the simplest case, a Brownian particle moving in a thermal medium is effectively described by linear Langevin equation

$$M\frac{d^2}{dt^2}q(t)+\eta_0\frac{d}{dt}q(t)=\chi(t), \qquad (1.1)$$

where $q(t)$ and $M$ are the position and effective mass of the Brownian particle, $\eta_0$ is damping coefficient. A Gaussian white noise $\chi(t)$ could be characterized by one- and two-point functions,

$$\langle\chi(t)\rangle=0, \qquad \langle\chi(t)\chi(t')\rangle=2T\eta_0\delta(t-t'), \qquad (1.2)$$

where the coefficient in the second relation is due to fluctuation-dissipation theorem. Here, $T$ is the temperature of thermal medium.

For specific purpose, linear Langevin theory (1.1)–(1.2) would be recast into an alternative formalism. For instance, in order to avoid repeatedly solving stochastic equation (1.1) with (infinitely) many different samplings of noise, one could equivalently consider Fokker-Planck equation, which is a deterministic differential equation for probability distribution function $\mathcal{P}(q,\dot{q},t)$, where a dot means time derivative. Moreover, for a Gaussian distribution of noise, the Langevin equation (1.1) could be reformulated as a functional integral [2], with the weight given by the Martin-Siggia-Rose-deDominicis-Janssen (MSRDJ) action. The functional integral formalism based on MSRDJ action makes it natural to adopt modern field theoretic methods to analyze more general stochastic processes.

Indeed, linear Langevin theory (1.1)–(1.2) could be generalized in a number of ways [2–6]. First, the linear Langevin equation (1.1) could be made nonlinear by adding general polynomial terms such as $f_1(q,\dot{q},\chi)$, which contains self-interactions for the dynamical variable $q$ and nonlinear interactions between dynamical variable $q$ and noise $\xi$. One special case is multiplicative s noise, which amounts to making a replacement $\chi\to f_2(q,\dot{q})\chi$ in (1.1). Second, the noise would obey a non-Gaussian distribution, and might be colored as well, which requires us to go beyond (1.2). Third, isotropy would be broken by an external field, such as in a magnetized thermal medium [7,8]. Then, beyond the linear level, the dynamics of transverse and longitudinal modes (with respect to external field) would get mixed. These corrections may become relevant and/or important for more realistic systems. A natural question arises; what is a more systematic way of organizing these extensions? This will be pursued here through two complementary approaches.

In this work we search for an effective description for a Brownian particle in a magnetized plasma, with potential applications in heavy-ion collisions in mind. The main purpose is to reveal nonlinear corrections to

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linear Langevin theory (1.1)–(1.2) in a systematic way. This will be achieved by nonequilibrium effective field theory (EFT) formalism for a quantum many-body system at finite temperature [9–12] (see [13–15] for an alternative approach). Within such a formalism, dynamics of Brownian particle is entirely dictated by an effective action, which will be constructed on a set of symmetries. The effective action may be thought of as generalization of the MSRDJ action for a linear theory (1.1)–(1.2). Moreover, the effective action contains "free parameters" representing UV physics and information of the state as well. Generically, it is challenging to compute those free parameters from an underlying UV theory (here, it is a closed system consisting of the Brownian particle and the magnetized plasma). Given that quark-gluon plasma produced in heavy-ion collisions is strongly coupled, we turn to a microscopic holographic model and derive the effective action (including values of free parameters).

Holography [16–18] is insightful in understanding symmetry principles underlying nonequilibrium effective action. Of particular importance is the dynamical Kubo-Martin-Schwinger (KMS) symmetry [9–11] acting on dynamical variable of the effective action, which guarantees the generalized fluctuation-dissipation theorem [19,20] at full level. In [9–11], dynamical KMS symmetry is implemented in the classical statistical limit where $\hbar \to 0$, which corresponds to neglecting quantum fluctuations in the effective theory. However, for a holographic theory, the mean free path is $\sim \hbar/T$, which implies that gradient expansion would generally inevitably bear quantum fluctuations [21]. Via the example of Brownian motion, we will elaborate on this point from both nonequilibrium EFT approach and holographic calculation. Intriguingly, imposition of a constant translational invariance (i.e., $q \to q + c$ with $c$ a constant) renders the resultant effective theory to be of classical statistical nature.

While effective action formalism is more systematic in covering nonlinear corrections alluded above, it turns out to be inconvenient to convert non-Gaussian effective action into a Langevin-type equation [9]. The main obstacle stems from non-Gaussianity in the $a$-variable (to be defined below), which prohibits from carrying out Hubbard-Stratonovich transformation [2]. Interestingly, we are able to convert the non-Gaussian effective action constructed in present work into Fokker-Planck-type equation, which will be useful in a numerical study.

The rest of this paper will be structured as follows. In Sec. II we clarify the set of symmetries and construct effective action for Brownian particles, which is further put into Fokker-Planck-type equation. In Sec. III we derive effective action for a Brownian particle moving in magnetized thermal plasma from a holographic perspective. In Sec. IV we summarize and outlook future directions. Appendixes A and B provide further calculational details.

## II. EFFECTIVE DYNAMICS FROM SYMMETRY PRINCIPLE

Dynamics of a closed system consisting of a Brownian particle and a thermal medium is presumably described by an action

$$S_{\rm C} = S_{\rm p}[q] + S_{\rm th}[\Phi] + S_{\rm int}[q,\Phi], \tag{2.1}$$

where $S_{\rm p}[q]$ is action for the Brownian particle, $S_{\rm th}[\Phi]$ describes microscopic theory of the constitutes (collectively denoted as $\Phi$) for the thermal medium, and $S_{\rm int}[q,\Phi]$ is the interaction between Brownian particles and constituents of the thermal medium. In principle, the effective action for the Brownian particles would be obtained by integrating out degrees of freedom $\{\Phi\}$ for the thermal medium, as illustrated below,

$$Z = \int [Dq][D\Phi] e^{{\rm i}S_{\rm C}} = \int [Dq] e^{{\rm i}I[q]}, \tag{2.2}$$

where $I[q]$ is the desired effective action. For such a quantum many-body system, the time evolution of the state will effectively go forwards and backwards along the Schwinger-Keldysh (SK) closed-time contour (see Fig. 1). Apparently, when carrying out the "integrating out" procedure in (2.2), one shall place the closed system on the SK closed-time contour of Fig. 1. Resultantly, the degrees of freedom are doubled, $q \to (q_1, q_2)$, where the subscripts 1,2 correspond to the upper and lower branches of Fig. 1.

Except for a few simple models [3,4,12,22–24], it is very challenging to implement the "integrating out" procedure illustrated in (2.2). It is thus natural to construct the effective action based on symmetry principle, which will be pursued here.

### A. Construction of effective action

The effective action $I[q_1;q_2]$ is usually presented in $(r,a)$-basis,

$$q_r \equiv \frac{1}{2}(q_1 + q_2), \qquad q_a \equiv q_1 - q_2, \tag{2.3}$$

where $q_r$ is the physical variable and $q_a$ is an auxiliary variable [conjugate to noise $\xi(t)$]. We summarize various symmetries and constraints obeyed by the effective action

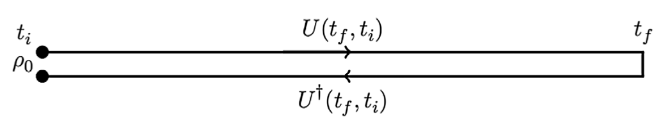


FIG. 1. The SK closed time contour: $\rho_0$ is initial density matrix, and $U(t_f, t_i)$ is the time-evolution operator from initial time $t_i$ to final time $t_f$.

$I[q_1;q_2] = I[q_r;q_a]$ for a Brownian particle moving in a magnetized plasma.

#### 1. $Z_2$-reflection symmetry

Take the complex conjugate of partition function (2.2), we find the reflection conditions,

$$I^*[q_1;q_2] = -I[q_2;q_1] \Leftrightarrow I^*[q_r;q_a] = -I[q_r;-q_a]. \quad (2.4)$$

#### 2. Normalization condition

If we set the two coordinates to be the same $q_1 = q_2 = q$, we find

$$\mathrm{Tr}(U(+\infty,-\infty;q)\rho_0 U^\dagger(+\infty,-\infty;q)) = \mathrm{Tr}\rho_0 = 1, \quad (2.5)$$

which lead to the normalization condition,

$$I[q_r;q_a=0] = 0. \quad (2.6)$$

So, it will be convenient to present the effective action as an expansion in number of $q_a$-variable.

Moreover, for the path integral based on $I[q_r;q_a]$ to be well defined, the imaginary part of $I[q_r;q_a]$ should be non-negative,

$$\mathrm{Im}(I[q_r;q_a]) \geq 0, \quad (2.7)$$

which will constrain some parameters in the effective action.

#### 3. Dynamical KMS symmetry

$$I[q_1;q_2] = I[\tilde{q}_1;\tilde{q}_2], \quad (2.8)$$

where

$$\tilde{q}_1(-t) = q_1(t), \qquad \tilde{q}_2(-t) = q_2(t-\mathrm{i}\beta). \quad (2.9)$$

Here, $\beta = 1/T$ is the inverse temperature of the plasma medium. The dynamical KMS symmetry is crucial in formulating an EFT for a quantum many-body system at finite temperature. It guarantees the generalized nonlinear fluctuation-dissipation theorem (FDT) at full quantum level [19], which originates from time-reversal invariance of underlying microscopic theory and relies on the fact that initially the system is in a thermal state.

Intriguingly, it is possible to take classical statistical limit of dynamical KMS symmetry so that only thermal fluctuations in the EFT will survive [9]. Let us properly restore Planck constant

$$\beta \to \hbar\beta, \qquad q_r \to q_r, \qquad q_a \to \hbar q_a. \quad (2.10)$$

Then, the classical statistical limit is achieved by taking $\hbar \to 0$ in the effective action. Consequently, the classical statistical limit of (2.9) becomes

$$\tilde{q}_r(-t) = q_r(t), \qquad \tilde{q}_a(-t) = q_a(t) + \mathrm{i}\beta\partial_t q_r(t). \quad (2.11)$$

On the other hand, when the mean free path is of order $\hbar\beta$ (as for a holographic theory), derivative expansion adopted in the construction of effective action would suggest considering $\hbar\beta$-expansion[1] of the dynamical KMS transformation (2.9):

$$\begin{aligned} \tilde{q}_r(-t) &= q_r(t) - \frac{\mathrm{i}}{2}\hbar\beta\partial_t q_r(t) + \frac{\mathrm{i}}{4}\hbar^2\beta\partial_t q_a(t), \\ \tilde{q}_a(-t) &= q_a(t) + \mathrm{i}\beta\partial_t q_r(t) - \frac{\mathrm{i}}{2}\hbar\beta\partial_t q_a(t). \end{aligned} \quad (2.12)$$

Interestingly, taking $\hbar \to 0$ in (2.12) will recover the classical statistical limit (2.11).

The $Z_2$-reflection symmetry and dynamical KMS symmetry are generic to nonequilibrium EFT. Specific to the problem of Brownian motion, we will impose additional symmetries.

#### 4. $Z_2$-parity

$$I[q_1;q_2] = I[-q_1;-q_2] \Leftrightarrow I[q_r;q_a] = I[-q_r;-q_a] \quad (2.13)$$

which means the action $I$ contains only even powers of $q_{r,a}$.

#### 5. Rotational symmetry

$$I[\hat{q}_1;\hat{q}_2] = I[q_1;q_2], \text{ with } \hat{q}_1 = \mathcal{R}q_1, \quad \hat{q}_2 = \mathcal{R}q_2, \quad (2.14)$$

where $\mathcal{R}$ denotes rotational transformation in space. Relevant to present work, the rotational symmetry will be reduced into $SO(2)_{xy}$ thanks to presence of a background magnetic field along $z$-direction.

#### 6. Constant translational symmetry

$$I[q_1+c;q_2+c] = I[q_1;q_2] \Leftrightarrow I[q_r+c;q_a] = I[q_r;q_a] \quad (2.15)$$

where $c$ is a constant. This symmetry is due to the homogeneous property of the plasma medium. Under this symmetry, the dependence of $I[q_r;q_a]$ on particle's position will be through the velocity $\dot{q}_r$. Interestingly, combined with the dynamical KMS symmetry, this symmetry will stringently constrain the form of $I[q_r;q_a]$, which will accidently make the dynamical KMS symmetry at quantum

[1]Here, we keep the expansion to first order in $\beta$ for later purpose.

level indistinguishable from its classical statistical limit, at least valid at first order in derivative expansion.

Now it is ready to present the effective action for Brownian particle in magnetized plasma. The effective action $I = \int dt L_{\rm SK}$ will be organized by employing amplitude expansion in $q_{r,a}$ and derivative expansion. Schematically, the effective Lagrangian is expanded as

$$L_{\rm SK} = L_{\rm SK}^{(2)} + L_{\rm SK}^{(4)} + \cdots, \tag{2.16}$$

where superscript denotes number of $q_{r,a}$ variables. Here, we have imposed the $Z_2$-parity (2.13).

First, we consider the quadratic Lagrangian $L_{\rm SK}^{(2)}$. Recall that the constant translational symmetry (2.15) tells us that the effective Lagrangian should be a functional of $\dot{q}_r$ instead of $q_r$. After imposing the rotational symmetry (2.14), the most general quadratic Lagrangian is[2]

$$L_{\rm SK}^{(2)} = -M^{\rm T} q_a^i \ddot{q}_r^i - M^{\rm L} q_a^z \ddot{q}_r^z - q_a^i \eta^{\rm T} \dot{q}_r^i - q_a^z \eta^{\rm L} \dot{q}_r^z + \frac{\rm i}{2}[q_a^i \xi^{\rm T} q_a^i + q_a^z \xi^{\rm L} q_a^z] - QB\epsilon_{ij}\dot{q}_r^i q_a^j, \tag{2.17}$$

where constants $M^{\rm T,L}$ are identified with effective mass for transverse and longitudinal modes, respectively. $\eta^{\rm T,L}$, and $\xi^{\rm T,L}$ are functionals of the time-derivative operator $\partial_t$, and could be expanded in the hydrodynamic limit

$$\mathfrak{C}^{\rm T,L} = \mathfrak{C}_0^{\rm T,L} + \mathfrak{C}_1^{\rm T,L}\partial_t + \mathfrak{C}_2^{\rm T,L}\partial_t^2 + \cdots, \quad \text{with} \quad \mathfrak{C} = \eta, \xi. \tag{2.18}$$

The $Q$ term represents the Lorentz force (with $Q$ the charge of Brownian particle), which is not a medium effect.

The requirement (2.7) sets inequality relations for $\xi^{\rm T,L}$, e.g., in accord with derivative expansion (2.18), we have

$$\xi_0^{\rm T} \geq 0, \qquad \xi_0^{\rm L} \geq 0; \qquad \xi_{2n}^{\rm T} \leq 0, \qquad \xi_{2n}^{\rm L} \leq 0, \quad n = 1, 2, 3, \ldots. \tag{2.19}$$

Finally, we impose the dynamical KMS symmetry (2.8) at the full level (2.9). At quadratic order, this is equivalent to the familiar FDT

$$\eta^{\rm T}(\omega) = \coth\left(\frac{\beta\omega}{2}\right) {\rm Im}[{\rm i}\omega\xi^{\rm T}(\omega)], \qquad \eta^{\rm L}(\omega) = \coth\left(\frac{\beta\omega}{2}\right) {\rm Im}[{\rm i}\omega\xi^{\rm L}(\omega)], \tag{2.20}$$

where we turn to frequency domain by $\partial_t \to -{\rm i}\omega$. Since we will be interested in truncating the expansion (2.18) to leading order, the linear FDT becomes

$$\eta_0^{\rm T} = \frac{1}{2}\beta\xi_0^{\rm T}, \qquad \eta_0^{\rm L} = \frac{1}{2}\beta\xi_0^{\rm L}. \tag{2.21}$$

Through Legendre transformation [9,12], it is easy to show that the quadratic Lagrangian $L_{\rm SK}^{(2)}$ is equivalent to a linear Langevin theory.

Next we turn to quartic Lagrangian $L_{\rm SK}^{(4)}$, which contains mixing effects between transverse and longitudinal modes. With symmetries (2.6), (2.13), (2.14), and (2.15) imposed, the most general quartic Lagrangian is

$$L_{\rm SK}^{(4)} = \kappa^{\rm T}\dot{q}_r^i q_a^i (q_a^j)^2 + \kappa^{\rm L}\dot{q}_r^z (q_a^z)^3 + \kappa_2^\times \dot{q}_r^i q_a^i (q_a^z)^2 + \kappa_1^\times \dot{q}_r^z q_a^z (q_a^i)^2 + \frac{\rm i}{4!}[\zeta^{\rm T}(q_a^i)^2(q_a^j)^2 + \zeta^{\rm L}(q_a^z)^4 + \zeta^\times (q_a^i)^2 (q_a^z)^2], \tag{2.22}$$

where we have truncated at first order in the derivative expansion. Here, in contrast with (2.17), various coefficients in (2.22) are constants.

From the constraint (2.7), we have

$$\zeta^{\rm T} \geq 0, \qquad \zeta^{\rm L} \geq 0, \qquad \zeta^\times \geq 0. \tag{2.23}$$

Finally, we impose dynamical KMS symmetry (2.8). In the classical statistical limit (2.11), this implies

$$\kappa^{\rm T} = -\frac{1}{12}\beta\zeta^{\rm T}, \qquad \kappa^{\rm L} = -\frac{1}{12}\beta\zeta^{\rm L}, \qquad \kappa_1^\times = \kappa_2^\times = -\frac{1}{24}\beta\zeta^\times. \tag{2.24}$$

Intriguingly, it can be shown that if we had imposed (2.12), we would obtain the same conclusion (2.24). This seemingly implies for the Brownian motion example considered in this work, that the classical statistical limit (2.11) is indistinguishable from the high-temperature limit (2.12). This is indeed attributed to the constant translational invariance (2.15). More precisely, if this symmetry is relaxed, the following terms shall be added to (2.22) (each with an independent coefficient)

$$q_r q_a^3, \quad q_r^2 q_a^2, \quad q_r^3 q_a, \quad \dot{q}_r q_r q_a^2, \quad \dot{q}_r q_r^2 q_a, \quad \cdots. \tag{2.25}$$

Then, it is straightforward to check that under classical statistical limit (2.11), the dynamical KMS symmetry (2.8) will yield the same constraints (2.24) [plus additional constraints among added terms (2.25)]. However, imposing (2.8) under a high-temperature limit (2.12) will give different constraints. We demonstrate this claim in Appendix B.

[2] Throughout this paper, the magnetic field is along the $z$-direction, and indices $i$, $j$ denote transverse directions. Here, we have ignored terms like $B\epsilon_{ij}\dot{q}_a^i q_a^j$, $M^\times \epsilon_{ij}\ddot{q}_r^i q_a^j$ since they will not appear in our holographic model.

### B. Fokker-Planck equation from non-Gaussian effective action

As explained in [9], inclusion of non-Gaussian terms, such as $L^{(4)}_{\rm SK}$ of (2.22), would make it inconvenient to perform a Legendre transformation from variable $q_a$ to noise $\theta$, which amounts to saying that it is generically inconvenient to cast non-Gaussian effective action into a stochastic Langevin-type equation.[3] In this subsection we derive a Fokker-Planck-type equation from the non-Gaussian effective action, for the purpose of future numerical study.

With MSRDJ action for a classical stochastic system, the derivation of the Fokker-Planck equation could be found in e.g., the textbook [2]. This mainly relies on the following observation; the partition function based on MSRDJ action is proportional to the probability of finding the system at certain configurations. Here, we will take an alternative approach, by considering Hamiltonian formulation of the effective action and making an analogy with quantum mechanics. To illustrate the derivation, we truncate the expansion (2.18) at leading order so that the effective Lagrangian reads

$$L^0_{\rm SK} = -M^{\rm T} q^i_a \ddot{q}^i_r - M^{\rm L} q^z_a \ddot{q}^z_r - q^i_a \eta^{\rm T}_0 \dot{q}^i_r - q^z_a \eta^{\rm L}_0 \dot{q}^z_r + \frac{\rm i}{2}[q^i_a \xi^{\rm T}_0 q^i_a + q^z_a \xi^{\rm L}_0 q^z_a] - QB\epsilon_{ij}\dot{q}^i_r q^j_a + \kappa^{\rm T}\dot{q}^i_r q^i_a (q^j_a)^2 + \kappa^{\rm L}\dot{q}^z_r (q^z_a)^3$$
$$+ \kappa^\times_2 \dot{q}^i_r q^i_a (q^z_a)^2 + \kappa^\times_1 \dot{q}^z_r q^z_a (q^i_a)^2 + \frac{\rm i}{4!}[\zeta^{\rm T}(q^i_a)^2(q^j_a)^2 + \zeta^{\rm L}(q^z_a)^4 + \zeta^\times (q^i_a)^2 (q^z_a)^2]. \tag{2.26}$$

In the Lagrangian formulation for the effective theory, $q_r$ satisfies a second-order differential equation. Thus, to search for the Hamiltonian formulation, we need to consider particle velocity $\dot{q}_r$ as another coordinate [2], which can be achieved by introducing associated multipliers

$$\begin{aligned}\tilde{L}^0_{\rm SK} &= \tilde{L}^0_{\rm SK}[q_r, \dot{q}_r; v, \dot{v}; q_a] \equiv \lambda^i_a (v^i - \dot{q}^i_r) + \lambda^z_a (v^z - \dot{q}^z_r) + L^0_{\rm SK}|_{\dot{q}^i_r \to v^i, \dot{q}^z_r \to v^z} \\ &= \lambda^i_a (v^i - \dot{q}^i_r) + \lambda^z_a (v^z - \dot{q}^z_r) - M^{\rm T} q^i_a \dot{v}^i - M^{\rm L} q^z_a \dot{v}^z - \eta^{\rm T}_0 q^i_a v^i - \eta^{\rm L}_0 q^z_a v^z - QB\epsilon_{ij} v^i q^j_a \\ &\quad + \frac{\rm i}{2}[q^i_a \xi^{\rm T}_0 q^i_a + q^z_a \xi^{\rm L}_0 q^z_a] + \kappa^{\rm T} v^i q^i_a (q^j_a)^2 + \kappa^{\rm L} v^z (q^z_a)^3 + \kappa^\times_2 v^i q^i_a (q^z_a)^2 + \kappa^\times_1 v^z q^z_a (q^i_a)^2 \\ &\quad + \frac{\rm i}{4!}[\zeta^{\rm T}(q^i_a)^2(q^j_a)^2 + \zeta^{\rm L}(q^z_a)^4 + \zeta^\times (q^i_a)^2(q^z_a)^2].\end{aligned} \tag{2.27}$$

Then, in the path integral based on modified Lagrangian $\tilde{L}^0_{\rm SK}$, integrating over multipliers $\lambda^i_a$, $\lambda^z_a$ gives rise to delta functions $\delta(v^i - \dot{q}^i_r)$ and $\delta(v^z - \dot{q}^z_r)$ as desired. Now, the conjugate momenta for $q_r$ and $v$ are defined as

$$\begin{aligned} k^i &\equiv -{\rm i}\frac{\partial \tilde{L}^0_{\rm SK}}{\partial \dot{q}^i_r} = {\rm i}\lambda^i_a, \qquad k^z \equiv -{\rm i}\frac{\partial \tilde{L}^0_{\rm SK}}{\partial \dot{q}^z_r} = {\rm i}\lambda^z_a, \\ p^i &\equiv -{\rm i}\frac{\partial \tilde{L}^0_{\rm SK}}{\partial \dot{v}^i} = {\rm i}M^{\rm T} q^i_a, \qquad p^z \equiv -{\rm i}\frac{\partial \tilde{L}^0_{\rm SK}}{\partial \dot{v}^z} = {\rm i}M^{\rm L} q^z_a. \end{aligned} \tag{2.28}$$

Therefore, in terms of conjugate pairs $(q_r, k)$ and $(v, p)$, the effective action can be rewritten as the anticipated Hamiltonian formulation

$${\rm i}\tilde{I}^0 \equiv {\rm i}\int dt \tilde{L}^0_{\rm SK} = -\int dt[k^i \dot{q}^i_r + k^z \dot{q}^z_r + p^i \dot{v}^i + p^z \dot{v}^z - H(k, p; q_r, v)], \tag{2.29}$$

where the Fokker-Planck Hamiltonian $H$ is

$$\begin{aligned} H = {}& v^i k^i + v^z k^z - \tilde{\eta}^{\rm T}_0 p^i v^i - \tilde{\eta}^{\rm L}_0 p^z v^z - \tilde{Q}B\epsilon_{ij} p^j v^i + \frac{1}{2}[\tilde{\xi}^{\rm T}_0 (p^i)^2 + \tilde{\xi}^{\rm L}_0 (p^z)^2] \\ & - [\tilde{\kappa}^{\rm T}(p^i)^2 p^j v^j + \tilde{\kappa}^{\rm L}(p^z)^3 v^z + \tilde{\kappa}^\times_2 (p^z)^2 p^i v^i + \tilde{\kappa}^\times_1 (p^i)^2 p^z v^z] \\ & - \frac{1}{4!}[\tilde{\zeta}^{\rm T}(p^i)^2(p^j)^2 + \tilde{\zeta}^{\rm L}(p^z)^4 + \tilde{\zeta}^\times (p^i)^2(p^z)^2]. \end{aligned} \tag{2.30}$$

[3]The inverse problem, i.e., deriving the MSRDJ-type action for a nonlinear version of (1.1)–(1.2), was recently considered in [6]. However, it was found that the parameters in the MSRDJ-type action are not in simple correspondence with those in the nonlinear Langevin equation.

For notational simplicity, we introduced tilde coefficients

$$\tilde{\eta}_0^{\rm T}=\frac{\eta_0^{\rm T}}{M^{\rm T}},\ \tilde{\eta}_0^{\rm L}=\frac{\eta_0^{\rm L}}{M^{\rm L}},\ \tilde{Q}=\frac{Q}{M^{\rm T}},\ \tilde{\xi}_0^{\rm T}=\frac{\xi_0^{\rm T}}{(M^{\rm T})^2},\ \tilde{\xi}_0^{\rm L}=\frac{\xi_0^{\rm L}}{(M^{\rm L})^2},$$
$$\tilde{\kappa}^{\rm T}=\frac{\kappa^{\rm T}}{(M^{\rm T})^3},\ \tilde{\kappa}^{\rm T}=\frac{\kappa^{\rm L}}{(M^{\rm L})^3},\ \tilde{\kappa}_2^{\times}=\frac{\kappa_2^{\times}}{M^{\rm T}(M^{\rm L})^2},\ \tilde{\kappa}_1^{\times}=\frac{\kappa_1^{\times}}{M^{\rm L}(M^{\rm T})^2},$$
$$\tilde{\zeta}^{\rm T}=\frac{\zeta^{\rm T}}{(M^{\rm T})^4},\ \tilde{\zeta}^{\rm L}=\frac{\zeta^{\rm L}}{(M^{\rm L})^4},\ \tilde{\zeta}^{\times}=\frac{\zeta^{\times}}{(M^{\rm T}M^{\rm L})^2}. \tag{2.31}$$

In analogy with quantum mechanics, we will quantize the action (2.29) by promoting conjugate pairs to operators and impose canonical commutation relations

$$[\hat{q}_r^i,\hat{k}^j]=\delta^{ij},\quad [\hat{q}_r^z,\hat{k}^z]=1,\quad [\hat{v}^i,\hat{p}^j]=\delta^{ij},\quad [\hat{v}^z,\hat{p}^z]=1, \tag{2.32}$$

which, in coordinate representation $(\hat{q}_r\to q_r,\hat{v}\to v)$, could be realized by the replacement rule

$$\hat{k}\to-\vec{\nabla}\equiv\left(-\frac{\partial}{\partial q_r^i},-\frac{\partial}{\partial q_r^z}\right),$$
$$\hat{p}\to-\vec{\nabla}_v\equiv\left(-\frac{\partial}{\partial v^i},-\frac{\partial}{\partial v^z}\right). \tag{2.33}$$

Meanwhile, we obtain the Fokker-Planck type equation

$$\partial_t\mathcal{P}(q_r,v,t)=\hat{H}(\vec{\nabla},\vec{\nabla}_v;q_r,v)\mathcal{P}(q_r,v,t), \tag{2.34}$$

where $\mathcal{P}(q_r,v,t)$, analogous to the wave function of quantum mechanics, is the probability of finding the system at configuration $(q_r(t),v(t))$ at time $t$. The Fokker-Planck Hamiltonian operator $\hat{H}$ is obtained from (2.30) by making the replacement rule (2.33). We split $\hat{H}$ into three parts

$$\hat{H}=\hat{H}_0+\hat{H}_1+\hat{H}_2. \tag{2.35}$$

Here, the familiar part $\hat{H}_0$ is

$$\hat{H}_0\bullet=-v^i\frac{\partial}{\partial q_r^i}\bullet-v^z\frac{\partial}{\partial q_r^z}\bullet+\tilde{\eta}_0^{\rm T}\frac{\partial}{\partial v^i}(v^i\bullet)+\tilde{\eta}_0^{\rm L}\frac{\partial}{\partial v^z}(v^z\bullet)$$
$$-\tilde{Q}B\epsilon_{ij}\frac{\partial}{\partial v^j}(v^i\bullet)+\frac{1}{2}\left(\tilde{\xi}_0^{\rm T}\frac{\partial^2}{\partial v^i\partial v^i}+\tilde{\xi}_0^{\rm L}\frac{\partial^2}{\partial v^z\partial v^z}\right)\bullet, \tag{2.36}$$

which corresponds to a classical stochastic model, such as anisotropic version of (1.1)–(1.2). The rest of the pieces $\hat{H}_1,\hat{H}_2$ contain higher-order velocity derivatives, and could be viewed as non-Gaussian corrections to a classical stochastic model. Specifically, $\hat{H}_1$ represents nonlinear interactions between noise and dynamical variables

$$\hat{H}_1\bullet=\tilde{\kappa}^{\rm T}\left(\frac{\partial}{\partial v^i}\right)^2\frac{\partial}{\partial v^j}(v^j\bullet)+\tilde{\kappa}^{\rm L}\left(\frac{\partial}{\partial v^z}\right)^3(v^z\bullet)$$
$$+\tilde{\kappa}_2^{\times}\left(\frac{\partial}{\partial v^z}\right)^2\frac{\partial}{\partial v^i}(v^i\bullet)+\tilde{\kappa}_1^{\times}\left(\frac{\partial}{\partial v^i}\right)^2\frac{\partial}{\partial v^z}(v^z\bullet). \tag{2.37}$$

The last piece $\hat{H}_2$ corresponds to non-Gaussianity for noise

$$\hat{H}_2\bullet=-\frac{1}{4!}\left[\tilde{\zeta}^{\rm T}\left(\frac{\partial}{\partial v^i}\right)^2\left(\frac{\partial}{\partial v^j}\right)^2+\tilde{\zeta}^{\rm L}\left(\frac{\partial}{\partial v^z}\right)^4\right.$$
$$\left.+\tilde{\zeta}^{\times}\left(\frac{\partial}{\partial v^i}\right)^2\left(\frac{\partial}{\partial v^z}\right)^2\right]\bullet. \tag{2.38}$$

Finally, we briefly discuss the strategy of solving the generalized Fokker-Planck equation (2.34). When the non-Gaussian parts $\hat{H}_1$, $\hat{H}_2$ are neglected (2.34) becomes standard Fokker-Planck equation and has been widely studied in the literature, see e.g., [25]. Basically, the idea is to first search for the stationary solution and then correct it by introducing time dependence perturbatively. On top of this, it is possible to explore consequences of the non-Gaussian corrections $\hat{H}_1$, $\hat{H}_2$ perturbatively. A detailed study along this direction is beyond the scope of present work and is left as a future project.

## III. STUDY IN A MICROSCOPIC MODEL

In this section we reveal systematic corrections to linear Langevin theory (1.1)–(1.2) from a holographic perspective, hopefully shedding light on understanding properties of strongly coupled quark-gluon plasma. On the one hand, this study will confirm results presented in Sec. II A. On the other hand, we compute various coefficients in (2.17) and (2.22) as functions of the magnetic field, which might be useful for heavy-ion collisions.

Generally, the expectation value of the Wilson-loop operator is identified with the partition function of the dual string world sheet [26]. In the large-$N_c$ and large-$\lambda$ limit the duality is greatly simplified to

$$\langle W(\mathcal{C})\rangle=e^{iS(\mathcal{C})}, \tag{3.1}$$

where $S(\mathcal{C})$ is the Nambu-Goto action, whose boundary condition is that the world sheet ends on curve $\mathcal{C}$ of the Wilson loop. In our work, we are interested in an unconfined heavy quark, which means we should set the probe D7-brane at the boundary of fifth-dimensional anti–de Sitter space ($AdS_5$). The heavy quark is dual to an open string stretching from the horizon up to the probe D7-brane. Then the Wilson loop turns to the Wilson line, which represents the world line of the heavy quark. As in [27–32], we will consider the open string moving in a target space of magnetic $AdS_5$ black brane, which is holographic dual of a heavy quark in strongly coupled magnetized

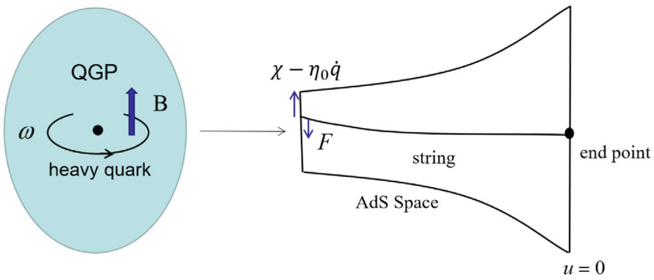


FIG. 2. Heavy quark in strongly coupled magnetized plasma (left) and its gravity dual (right).

plasma,[4] see Fig. 2 for illustration. The friction and noise forces felt by the boundary quark correspond to the ingoing mode and the outgoing one (Hawking mode) in open string's profile [29,44–48], respectively. Here, adopting the holographic prescription for SK closed time contour [49],[5] we will extend this picture to nonlinear level (see [5] for the situation without magnetic field) by analyzing dynamics of a Nambu-Goto string in magnetic $AdS_5$ black brane.

The partition function for the bulk theory is

$$Z_{\rm bulk} = \int [DX][Dg_{MN}] e^{iS_{\rm bulk}[X, g_{MN}]}, \tag{3.2}$$

where $S_{\rm bulk}$ is the total action for the bulk theory, $g_{MN}$ is the metric of target space (magnetic brane in $AdS_5$), and $X$ describes embedding profile of open string in the target space. In probe limit, the target space does not fluctuate. Then, the bulk partition function $Z_{\rm bulk}$ gets reduced into that of an open string in the magnetic $AdS_5$ brane,

$$Z_{\rm bulk} \simeq Z_{\rm string} = \int [DX] e^{iS[X]}, \tag{3.3}$$

where $S$ is the total string action. It will be clear that the string embedding profile $X$ is a functional of quark's position $q$, i.e., $X = X[q]$. Thus, the bulk path integral (3.3) will be eventually cast into a path integral over the position $q$. We will work in the saddle point approximation:

$$Z_{\rm string} = \int [Dq] e^{iS[X[q]]}, \tag{3.4}$$

where $S[X[q]]$ is the on-shell classical string action. The AdS/CFT conjectures that $Z$ of (2.2) is equivalent to $Z_{\rm bulk}$. Thus, in the probe limit, the on shell string action $S[X[q]]$ will be identified with the effective action $I[q]$ for Brownian particle in plasma medium. Therefore, holographic derivation of $I[q]$ boils down to solving the classical equation of motion (EOM) for an open string in magnetic $AdS_5$ brane.

### A. Magnetic $AdS_5$ black brane and its field theory dual

Consider a five dimensional Einstein-Maxwell theory with a negative cosmological constant (the AdS radius is set to unity)

$$S_{\rm EM} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R - \frac{1}{4} F^2 + 12 \right). \tag{3.5}$$

The equations of motion (EOMs) for bulk theory (3.5) read

$$R_{MN} + 4g_{MN} - \frac{1}{2} F_{PM} F^P_N + \frac{1}{12} g_{MN} F^2 = 0, \qquad \nabla_M F^{MN} = 0. \tag{3.6}$$

The theory (3.5) admits a magnetic brane solution [61]. To utilize the prescription to integrate out the radius coordinate, we work in the ingoing Eddington-Finkelstein coordinates [49]. Thus the magnetic brane solution ansatz is

$$ds^2 = r_h^2 \left[ -\frac{2}{r_h} du dt - \frac{U(u)}{u^2} dt^2 + \frac{V(u)}{u^2} \delta_{ij} dx^i dx^j + \frac{W(u)}{u^2} dz^2 \right], \qquad i, j = 1, 2,$$
$$A = B x^1 dx^2 \Rightarrow F = B dx^1 \wedge dx^2, \tag{3.7}$$

where the AdS boundary is located at $u = 0$ and the event horizon is at $u = 1$. For simplicity, $r_h$ will be set to unity, and could be restored by dimensional analysis. The Hawking temperature of the magnetic brane (3.7) is

$$T = -\frac{U'(u)}{4\pi} \bigg|_{u=1}, \tag{3.8}$$

where $T$ should be understood as in unit of $r_h$. With the ansatz (3.7), bulk EOMs (3.6) consist of three dynamical components (3.9) and one constraint (3.10),

[4]In holographic context, dynamics of heavy quark in magnetized plasma was considered in [33–40]. Anisotropic effects on heavy quark dynamics were considered, for e.g., in [41–43].

[5]In recent years, the holographic SK contour [49] attracted a lot of attention in various holographic settings [5,50–60].

$$
\begin{aligned}
0 &= U''(u) + U'(u)\left(\frac{V'(u)}{V(u)} + \frac{W'(u)}{2W(u)} - \frac{5}{u}\right) + U(u)\left(\frac{8}{u^2} - \frac{2V'(u)}{uV(u)} - \frac{W'(u)}{uW(u)}\right) - \frac{B^2u^2}{3V(u)^2} - \frac{8}{u^2},\\
0 &= V''(u) + V'(u)\left(\frac{U'(u)}{U(u)} + \frac{W'(u)}{2W(u)} - \frac{5}{u}\right) + V(u)\left(-\frac{8}{u^2U(u)} + \frac{8}{u^2} - \frac{2U'(u)}{uU(u)} - \frac{W'(u)}{uW(u)}\right) + \frac{2B^2u^2}{3U(u)V(u)},\\
0 &= W''(u) + W'(u)\left(\frac{U'(u)}{U(u)} + \frac{V'(u)}{V(u)} - \frac{4}{u}\right) - \frac{W'(u)^2}{2W(u)} + W(u)\left(\frac{-\frac{B^2u^2}{3V(u)^2} - \frac{8}{u^2}}{U(u)} + \frac{8}{u^2} - \frac{2U'(u)}{uU(u)} - \frac{2V'(u)}{uV(u)}\right),
\end{aligned} \tag{3.9}
$$

$$
0 = W''(u) + W(u)\left(\frac{2V''(u)}{V(u)} - \frac{V'(u)^2}{V(u)^2}\right) - \frac{W'(u)^2}{2W(u)}. \tag{3.10}
$$

where, since we have set $r_h = 1$ above, $B$ should be understood as $B/r_h^2$.

The bulk metric shall demonstrate asymptotic AdS behavior near $u = 0$, which requires

$$
U(u) \to 1, \qquad V(u) \to 1, \qquad W(u) \to 1, \quad \text{as } u \to 0. \tag{3.11}
$$

Indeed, near AdS boundary $u = 0$, the metric functions $U$, $V$, $W$ are expanded as

$$
\begin{aligned}
U(u\to 0) &= 1 + U_b^1 u + \frac{1}{4}(U_b^1)^2u^2 + \frac{B^2}{6(V_b^0)^2}u^4\log u + U_b^4u^4 + \cdots,\\
V(u\to 0) &= V_b^0 + V_b^0U_b^1u + \frac{1}{4}V_b^0(U_b^1)^2u^2 - \frac{B^2}{12V_b^0}u^4\log u + V_b^4u^4 + \cdots,\\
W(u\to 0) &= W_b^0 + W_b^0U_b^1u + \frac{1}{4}W_b^0(U_b^1)^2u^2 + \frac{W_b^0B^2}{6(V_b^0)^2}u^4\log u + W_b^4u^4\cdots,
\end{aligned} \tag{3.12}
$$

where we have made use of bulk EOMs (3.9)–(3.10). Obviously, the asymptotic boundary conditions (3.11) only give rise to two effective requirements. The regularity requirements will yield another three conditions. Here, as in [62] we can utilise the freedom of redefining the radial coordinate $u$ and set $U_b^1 = 0$. Therefore, the boundary conditions at $u = 0$ are

$$
U'(u=0) = 0, \qquad V(u=0) = W(u=0) = 1. \tag{3.13}
$$

At the horizon $u = 1$, we impose regularity condition

$$
\begin{aligned}
U(u=1) &= 0,\\
U'(1)V'(1) - 8V(1) - 2U'(1)V(1) + \frac{2B^2}{3V(1)} &= 0,\\
U'(1)W'(1) - 2W(1)U'(1) - 8W(1) - \frac{B^2W(1)}{3V(1)^2} &= 0.
\end{aligned} \tag{3.14}
$$

Then, near the horizon $u = 1$ the metric functions are expanded as

$$
\begin{aligned}
U(u\to 1) &= 0 + U_h^1(u-1) + U_h^2(u-1)^2 + \cdots,\\
V(u\to 1) &= V_h^0 + V_h^1(u-1) + V_h^2(u-1)^2 + \cdots,\\
W(u\to 1) &= W_h^0 + W_h^1(u-1) + W_h^2(u-1)^2 + \cdots,
\end{aligned} \tag{3.15}
$$

where $U_h^1$, $V_h^0$, $W_h^0$ are the horizon data and all the rest of the coefficients are fully fixed in terms of the horizon data. In terms of horizon data, the black hole temperature (3.8) is

$$
T = -\frac{U_h^1}{4\pi}. \tag{3.16}
$$

In order to determine the metric functions $U$, $V$, $W$, we shall solve bulk EOMs (3.9)–(3.10) under boundary conditions (3.13) and (3.14).

When magnetic field is weak, metric functions $U$, $V$, $W$ can be solved analytically [62,63]

$$
\begin{aligned}
U(u) &= 1 - u^4 + \frac{1}{6}B^2u^4\log(u) + \cdots,\\
V(u) &= 1 + \frac{B^2}{48}\mathrm{Li}_2(u^4) - \frac{B^2(1-u^4)\log(1-u^4)}{12(u^4+3)} + \cdots,\\
W(u) &= 1 - \frac{B^2}{24}\mathrm{Li}_2(u^4) + \frac{B^2(1-u^4)\log(1-u^4)}{6(u^4+3)} + \cdots.
\end{aligned} \tag{3.17}
$$

Meanwhile, the black hole temperature (3.16) is expanded as

$$T = \frac{1}{\pi}\left(1 - \frac{B^2}{24}\right) + \mathcal{O}(B^4). \qquad (3.18)$$

For a generic value of $B$, the metric functions are known numerically only [61,62,64–67]. Practically, instead of solving bulk EOMs (3.9)–(3.10) under the boundary conditions (3.13) and (3.14), one could take a set of convenient horizon data and evolve bulk EOMs (3.9)–(3.10). More precisely, we will solve bulk EOMs (3.9)–(3.10) as an initial value problem (3.15) with horizon data taken as (in unit of $r_h = 1$)

$$U_h^1 = -4, \qquad V_h^0 = 1, \qquad W_h^0 = 1. \qquad (3.19)$$

Notice that the choice of $U_h^1 = -4$ will set $\pi T = r_h(B)$. Consequently, near the AdS boundary $u = 0$ the bulk metric behaves as

$$ds^2|_{u\to 0} = \frac{1}{u^2}[-dt^2 + v(b)\delta_{ij}d\hat{x}^i d\hat{x}^j + w(b)d\hat{z}^2], \qquad (3.20)$$

which is not the required one (3.11). Finally, one obtains correct solution by rescaling of boundary coordinate

$$\hat{x}^i \to x^i/\sqrt{v(b)}, \qquad \hat{z} \to z/\sqrt{w(b)}. \qquad (3.21)$$

Here, we use $b$ to denote the magnetic field in the incorrect boundary metric (3.20). Then the physical magnetic field $B$ (in unit of $r_h^2$) should be

$$B = \frac{b}{v(b)}. \qquad (3.22)$$

Finally, we would like to point out that the background solution obtained with initial conditions (3.15) and (3.19) does not necessarily satisfy $U_b^1 = 0$ [cf. (3.12)].

The magnetic brane solution (3.7) is dual to strongly coupled $\mathcal{N} = 4$ SYM plasma exposed to an external magnetic field. In order to add an external magnetic field for boundary theory, we could think of gauging a $U(1)$ subgroup of $R$-symmetry of $\mathcal{N} = 4$ SYM theory [68]. Schematically, the microscopic Lagrangian for the magnetized $\mathcal{N} = 4$ SYM plasma is [69]

$$S_{\rm th}[\Phi] = S_{\rm SYM,min.coupled} + S_{\rm e.m.}, \qquad (3.23)$$

where $S_{\rm SYM,min\,.coupled}$ represents action for $\mathcal{N} = 4$ SYM theory minimally coupled to a U(1) gauge field, and $S_{\rm e.m.}$ is Maxwell action for U(1) gauge field. Apparently, the thermal bath described by (3.23) preserves time-reversal symmetry, which plays a crucial role in formulating EFT for a quantum many-body system [9]. From the bulk perspective, the Einstein-Maxwell theory (3.5) transparently preserves time-reversal invariance. However, thanks to usage of ingoing Eddington–Finkelstein coordinate system in (3.7), the time-reversal symmetry is not simply realized as $t \to -t$, which will become clear in the linearized string solution. The microscopic time-reversal symmetry will be translated into dynamical KMS symmetry (2.8) for effective theory for Brownian particles.

### B. Dynamics of open string in magnetic brane

Classical dynamics of open string is described by Nambu-Goto action

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-h(X)}, \qquad (3.24)$$

where $h$ is determinant of the induced metric $h_{ab}$ on string world sheet,

$$ds^2_{\rm WS} = h_{ab}d\sigma^a\sigma^b = g_{MN}\frac{\partial X^M}{\partial\sigma^a}\frac{\partial X^N}{\partial\sigma^b}. \qquad (3.25)$$

Here, we use $X^M$ to denote embedding of string in the target space (3.7). We will take a static gauge so that string world sheet coordinate is $\sigma^a \equiv (\sigma, \tau) = (u, t)$. Then, embedding of the open string is specified by spatial coordinates $X^i(\sigma^a)$, $X^z(\sigma^a)$. In presence of an external Maxwell field, we shall supplement the Nambu-Goto action (3.24) by a boundary term

$$S_{\rm bdy} = Q\int d\tau A_M(X)\frac{dX^M}{d\tau}\bigg|_{\rm bdy}. \qquad (3.26)$$

Imagine a static string with $X^i = X^z = 0$, for which the world sheet spacetime is

$$d\bar{s}^2_{\rm WS} = -\frac{1}{u^2}[2dudt + U(u)dt^2], \qquad (3.27)$$

which has an event horizon identical to that of the target space (3.7). While such a static string will lose energy into the horizon of the target space, it will also receive Hawking radiation emitted from the horizon. Resultantly, we will have a fluctuating string around a static configuration. These two modes and their interactions will be translated into effective dynamics of the quark in the boundary plasma medium. Following [49], we double the background world sheet spacetime (3.27), and analytically continue it around the event horizon $u = 1$, so that the radial coordinate $u$ now varies along the contour of Fig. 3.

The specific symmetries postulated for effective action of Brownian particles, say (2.13)–(2.15), can be understood from action of open string. Without a nontrivial background, the Nambu-Goto action for open string's fluctuation (still denoted as $X^i$, $X^z$) preserves the following symmetries independently,

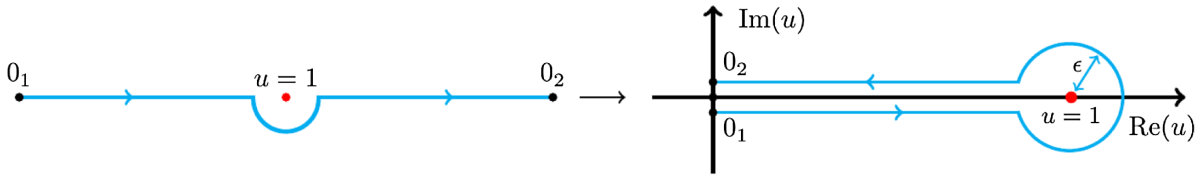


FIG. 3. From complexified (analytically continued near horizon) double AdS (left) [70] to the holographic SK contour (right) [49]. The holographic contour infinitely close to the horizon, $\epsilon \to 0$. Indeed, the two horizontal legs overlap with the real axis.

$$(1)\ X^i \to -X^i, \qquad X^z \to -X^z;$$
$$(2)\ X^i \to R^{ij} X^j, \qquad X^z \to X^z;$$
$$(3)\ X^i \to X^i + c^i, \qquad X^z \to X^z + c^z, \tag{3.28}$$

which simply translate into (2.13)–(2.15) at the boundary.

The classical EOM for open string, obtained from (3.24), is highly nonlinear. Thus, we expand (3.24) to quartic order in open string's fluctuations $X^i$, $X^z$,

$$S_{\rm NG} = S_{\rm NG}^{(2)} + S_{\rm NG}^{(4)} + \cdots. \tag{3.29}$$

The quadratic part of Nambu-Goto action is

$$S_{\rm NG}^{(2)} = -\frac{1}{2\pi\alpha'}\int dt \int_{0_1}^{0_2}\frac{du}{u^2}\left\{V(u)\left[-\partial_u X^i \partial_t X^i + \frac{1}{2}U(u)(\partial_u X^i)^2\right] + W(u)\left[-\partial_u X^z \partial_t X^z + \frac{1}{2}U(u)(\partial_r X^z)^2\right]\right\}. \tag{3.30}$$

The quartic part of Nambu-Goto action is

$$S_{\rm NG}^{(4)} = -\frac{1}{2\pi\alpha'}\int dt \int_{0_1}^{0_2}\frac{du}{u^2}\left\{-\frac{1}{8}V(u)^2[2\partial_t X^i - U(u)\partial_u X^i]^2(\partial_u X^j)^2 - \frac{1}{8}W(u)^2[2\partial_t X^z - U(u)\partial_u X^z]^2(\partial_u X^z)^2 \right.$$
$$\left. -\frac{1}{8}V(u)W(u)[(2\partial_t X^i - U(u)\partial_u X^i)^2(\partial_u X^z)^2 + (2\partial_t X^z - U(u)\partial_u X^z)^2(\partial_u X^i)^2]\right\}. \tag{3.31}$$

Based on truncated action $S_{\rm NG}^{(2)} + S_{\rm NG}^{(4)}$, the EOMs for string fluctuations are

$$0 = \mathrm{EOMT}_i = \partial_u\left(\frac{\mathrm{UV}}{u^2}\partial_u X^i\right) - \frac{2V}{u^2}\partial_u\partial_t X^i - \left(\frac{\partial_u V}{u^2} - \frac{2V}{u^3}\right)\partial_t X^i - f_i[X^i, X^z],$$
$$0 = \mathrm{EOML}_z = \partial_u\left(\frac{UW}{u^2}\partial_u X^z\right) - \frac{2W}{u^2}\partial_u\partial_t X^z - \left(\frac{\partial_u W}{u^2} - \frac{2W}{u^3}\right)\partial_t X^z - f_z[X^i, X^z], \tag{3.32}$$

Here, $f_i$, $f_z$ are cubic terms in $X^i$, $X^z$, whose exact forms will not be relevant in subsequent calculations. The string's EOMs (3.32) will be solved under doubled AdS boundary conditions

$$X^i(t, u = 0_s) = q_s^i, \qquad X^z(t, u = 0_s) = q_s^z, \quad \text{with} \quad s = 1 \quad \text{or} \quad 2. \tag{3.33}$$

The coupled nonlinear system (3.32) will be further linearized as

$$X^i = \alpha X_{(1)}^i + \alpha^3 X_{(3)}^i + \cdots, \qquad X^z = \alpha X_{(1)}^z + \alpha^3 X_{(3)}^z + \cdots, \tag{3.34}$$

where $\alpha$ is a formal book-keeping parameter. Accordingly, the boundary conditions (3.33) are implemented as

$$X_{(1)}^i(t, u = 0_s) = q_s^i, \qquad X_{(1)}^z(t, u = 0_s) = q_s^z, \quad \text{with} \quad s = 1 \quad \text{or} \quad 2,$$
$$X_{(n)}^i(t, u = 0_s) = 0, \qquad X_{(n)}^z(t, u = 0_s) = 0, \quad \text{with} \quad s = 1 \quad \text{or} \quad 2, \quad \text{when } n \geq 3. \tag{3.35}$$

Then, $X_{(1)}^{i,z}$ satisfy linearized EOMs

$$0 = \partial_u\left(\frac{UV}{u^2}\partial_u X^i_{(1)}\right) + \mathrm{i}\omega\frac{2V}{u^2}\partial_u X^i_{(1)} + \mathrm{i}\omega\left(\frac{\partial_u V}{u^2} - \frac{2V}{u^3}\right)X^i_{(1)},$$
$$0 = \partial_u\left(\frac{UW}{u^2}\partial_u X^z_{(1)}\right) + \mathrm{i}\omega\frac{2W}{u^2}\partial_u X^z_{(1)} + \mathrm{i}\omega\left(\frac{\partial_u W}{u^2} - \frac{2W}{u^3}\right)X^z_{(1)}, \tag{3.36}$$

where we have turned to frequency domain via Fourier transformation,

$$X^{i,z}_{(1)}(u,t) = \int\frac{d\omega}{2\pi}X^{i,z}_{(1)}(u,\omega)e^{-\mathrm{i}\omega t}. \tag{3.37}$$

It turns out that, under AdS boundary conditions (3.35), both $S^{(2)}_{\mathrm{NG}}$ and $S^{(4)}_{\mathrm{NG}}$ are fully determined by linearized fluctuations $X^{i,z}_{(1)}$ [5,58]. Indeed, via integration by part, $S^{(2)}_{\mathrm{NG}}$ of (3.30) is reduced into a surface term,

$$S^{(2)}_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int dt\left[\frac{U(u)V(u)}{2u^2}X^i_{(1)}\partial_u X^i_{(1)} + \frac{U(u)W(u)}{2u^2}X^z_{(1)}\partial_u X^z_{(1)}\right]^{0_2}_{0_1}. \tag{3.38}$$

The quadratic order action is simply obtained from (3.31) by replacement rule $X^{i,z} \to X^{i,z}_{(1)}$,

$$S^{(4)}_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int dt\int_{0_1}^{0_2}\frac{du}{u^2}\left\{-\frac{1}{8}V(u)^2[2\partial_t X^i_{(1)} - U(u)\partial_u X^i_{(1)}]^2(\partial_u X^j)^2 - \frac{1}{8}W(u)^2[2\partial_t X^z_{(1)} - U(u)\partial_u X^z_{(1)}]^2(\partial_u X^z_{(1)})^2\right.$$
$$\left.-\frac{1}{8}V(u)W(u)[(2\partial_t X^i_{(1)} - U(u)\partial_u X^i_{(1)})^2(\partial_u X^z_{(1)})^2 + (2\partial_t X^z_{(1)} - U(u)\partial_u X^z_{(1)})^2(\partial_u X^i_{(1)})^2]\right\}. \tag{3.39}$$

Therefore, once linearized profiles $X^{i,z}_{(1)}$ are obtained, evaluating (3.38) and (3.39) will give effective action for boundary quark.

When $u$ varies along the radial contour of Fig. 3, linearized EOMs (3.36) have been studied in [5,58] when $B = 0$. The basic idea [58] is as follows: First, one cuts the radial contour of Fig. 3 at the rightmost point $u = 1 + \epsilon$. It is direct to find out generic solutions when $u$ varies either on upper branch or lower branch of Fig. 3. Then, the generic solution on the upper branch and the generic solution on the lower branch will be properly glued at $u = 1 + \epsilon$. The gluing conditions can be derived from the requirement that variational problem of (3.30) is well defined at $u = 1 + \epsilon$. Finally, one imposes the AdS boundary conditions (3.35). This strategy can be directly applied to solve (3.36) when $B \neq 0$. Below we skip details and present the final solutions.

Under boundary conditions (3.35), the linearized EOMs (3.36) are solved as

$$X^i_{(1)}(u,\omega) = \mathfrak{A}_{\mathrm{T}}(u,\omega)q^i_r(\omega) + \mathfrak{B}_{\mathrm{T}}(u,\omega)q^i_a(\omega),\quad u\in[0_2,0_1],$$
$$X^z_{(1)}(u,\omega) = \mathfrak{A}_{\mathrm{L}}(u,\omega)q^z_r(\omega) + \mathfrak{B}_{\mathrm{L}}(u,\omega)q^z_a(\omega),\quad u\in[0_2,0_1], \tag{3.40}$$

where (S = T, L)

$$\mathfrak{A}_{\mathrm{S}}(u,\omega) = \frac{\Phi^{\mathrm{ig}}_{\mathrm{S}}(u,\omega)}{\Phi^{\mathrm{ig}(0)}_{\mathrm{S}}(\omega)},$$
$$\mathfrak{B}_{\mathrm{S}}(u,\omega) = \frac{1}{2}\coth\frac{\beta\omega}{2}\frac{\Phi^{\mathrm{ig}}_{\mathrm{S}}(u,\omega)}{\Phi^{\mathrm{ig}(0)}_{\mathrm{S}}(\omega)} - \frac{e^{2\mathrm{i}\omega\chi(u)}}{1-e^{-\beta\omega}}\frac{\Phi^{\mathrm{ig}}_{\mathrm{S}}(u,-\omega)}{\Phi^{\mathrm{ig}(0)}_{\mathrm{S}}(-\omega)}, \tag{3.41}$$

where the function $\chi$ is

$$\chi(u) \equiv -\int_{0_2}^{u}\frac{1}{U(u)}\qquad u\in[0_2,0_1]. \tag{3.42}$$

Obviously, the task of solving linearized EOMs (3.36) reduces to searching for ingoing solutions $\Phi^{\mathrm{ig}}_{\mathrm{T,L}}(u,\omega)$. Expressed in the form (3.40), the linearized solutions demonstrate explicit time-reversal symmetry [56,58]. Importantly, ingoing modes $\Phi^{\mathrm{ig}}_{\mathrm{T,L}}(u,\omega)$ are regular over the entire radial contour and can be constructed for $u$ varying on either upper branch or lower branch. In the low frequency limit, we have formally constructed $\Phi^{\mathrm{ig}}_{\mathrm{T,L}}(u,\omega)$,

$$\Phi^{\mathrm{ig}}_{\mathrm{S}}(u,\omega) = \Phi^{\mathrm{ig}}_{\mathrm{S},0}(u) + \omega\Phi^{\mathrm{ig}}_{\mathrm{S},1}(u) + \omega^2\Phi^{\mathrm{ig}}_{\mathrm{S},2}(u) + \cdots,\quad \mathrm{S} = \mathrm{T},\mathrm{L}, \tag{3.43}$$

where

$$\Phi^{\rm ig}_{\rm S,0}(u)=1,$$
$$\Phi^{\rm ig}_{\rm L,n}(u)=-\int_0^u d\tilde u\frac{\tilde u^2}{UW}\int_1^{\tilde u}d\hat u\frac{2{\rm i}W}{\hat u^2}\partial_{\hat u}\Phi^{\rm ig}_{{\rm L},n-1}+\left(\frac{{\rm i}\partial_{\hat u}W}{\hat u^2}-\frac{2{\rm i}W}{\hat u^3}\right)\Phi^{\rm ig}_{{\rm L},n-1},\quad n\geq1,\tag{3.44}$$

$$\Phi^{\rm ig}_{\rm T,n}(u)=-\int_0^u d\tilde u\frac{\tilde u^2}{\rm UV}\int_1^{\tilde u}d\hat u\frac{2{\rm i}V}{\hat u^2}\partial_{\hat u}\Phi^{\rm ig}_{{\rm T},n-1}+\left(\frac{{\rm i}\partial_{\hat u}V}{\hat u^2}-\frac{2{\rm i}V}{\hat u^3}\right)\Phi^{\rm ig}_{{\rm T},n-1},\qquad n\geq1.\tag{3.45}$$

### C. Effective action for a Brownian quark: Quadratic order

Quadratic effective action $I^{(2)}$ for boundary quark is related to string action via

$$I^{(2)}=S^{(2)}_{\rm NG}+S_{\rm bdy},\tag{3.46}$$

where $S^{(2)}_{\rm NG}$ is presented in (3.38). Near two AdS boundaries, linearized string fluctuations behave as

$$X^{i,z}_{(1)}(u\to0_s,\omega)=q^{i,z}_s(\omega)-{\rm i}\omega q^{i,z}_s(\omega)u+\mathbb{O}^{i,z}_s(\omega)u^3+\cdots,\qquad s=1\quad\text{or}\quad2,\tag{3.47}$$

where the normalizable modes $\mathbb{O}^{i,z}_s(\omega)$ are

$$\mathbb{O}^{i(z)}_2(\omega)=\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(\omega)}q^{i(z)}_r(\omega)+\frac{1}{2}\coth\frac{\beta\omega}{2}\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(\omega)}q^{i(z)}_a(\omega)-\frac{1}{1-e^{-\beta\omega}}\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(-\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(-\omega)}q^{i(z)}_a(\omega)+\frac{2{\rm i}\omega^3}{3(1-e^{-\beta\omega})}q^{i(z)}_a(\omega),$$
$$\mathbb{O}^{i(z)}_1(\omega)=\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(\omega)}q^{i(z)}_r(\omega)+\frac{1}{2}\coth\frac{\beta\omega}{2}\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(\omega)}q^{i(z)}_a(\omega)-\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\frac{\Phi^{\rm ig(3)}_{\rm T(L)}(-\omega)}{\Phi^{\rm ig(0)}_{\rm T(L)}(-\omega)}q^{i(z)}_a(\omega)+\frac{2{\rm i}\omega^3e^{-\beta\omega}}{3(1-e^{-\beta\omega})}q^{i(z)}_a(\omega).\tag{3.48}$$

where pairing $(i,{\rm T})$ or $({\rm z},{\rm L})$ is assumed over indices. Here, $\Phi^{\rm ig(0)}$ and $\Phi^{\rm ig(3)}$ are read off from near boundary expansion of ingoing solution

$$\Phi^{\rm ig}(u\to0,\omega)=\Phi^{\rm ig(0)}(\omega)+\cdots+\Phi^{\rm ig(3)}(\omega)u^3+\cdots.\tag{3.49}$$

Immediately, (3.46) is computed as

$$I^{(2)}=\frac{1}{2\pi\alpha'}\int\frac{d\omega}{2\pi}\left\{\frac{\rm i}{2}q^i_a(-\omega)G^{\rm T}_{rr}(\omega)q^i_a(\omega)+q^i_a(-\omega)[M_0\omega^2+G^{\rm T}_{ra}(\omega)]q^i_r(\omega)\right\}$$
$$+\frac{1}{2\pi\alpha'}\int\frac{d\omega}{2\pi}\left\{\frac{\rm i}{2}q^3_a(-\omega)G^{\rm L}_{rr}(\omega)q^3_a(\omega)+q^3_a(-\omega)[M_0\omega^2+G^{\rm L}_{ra}(\omega)]q^3_r(\omega)\right\}$$
$$-\int\frac{d\omega}{2\pi}QBi\omega[q^1_r(-\omega)q^2_a(\omega)+q^1_a(-\omega)q^2_r(\omega)],\tag{3.50}$$

where $({\rm S}={\rm T},{\rm L})$

$$G^{\rm S}_{rr}(\omega)=-{\rm i}\coth\frac{\beta\omega}{2}\left[\frac{3}{2}\frac{\Phi^{\rm ig(3)}_{\rm S}(\omega)}{\Phi^{\rm ig(0)}_{\rm S}(\omega)}-\frac{3}{2}\frac{\Phi^{\rm ig(3)}_{\rm S}(-\omega)}{\Phi^{\rm ig(0)}_{\rm S}(-\omega)}+{\rm i}\omega^3\right],$$
$$G^{\rm S}_{ra}(\omega)=3\frac{\Phi^{\rm ig(3)}_{\rm S}(\omega)}{\Phi^{\rm ig(0)}_{\rm S}(\omega)}+{\rm i}\omega^3.\tag{3.51}$$

In (3.50) the bare quark mass $M_0$ is related to location of probe D7-brane by $M_0=\lim_{u\to\Lambda}1/u$. The holographic result (3.50) is identical to (2.17) via the following identification (with $2\pi\alpha'=1$),

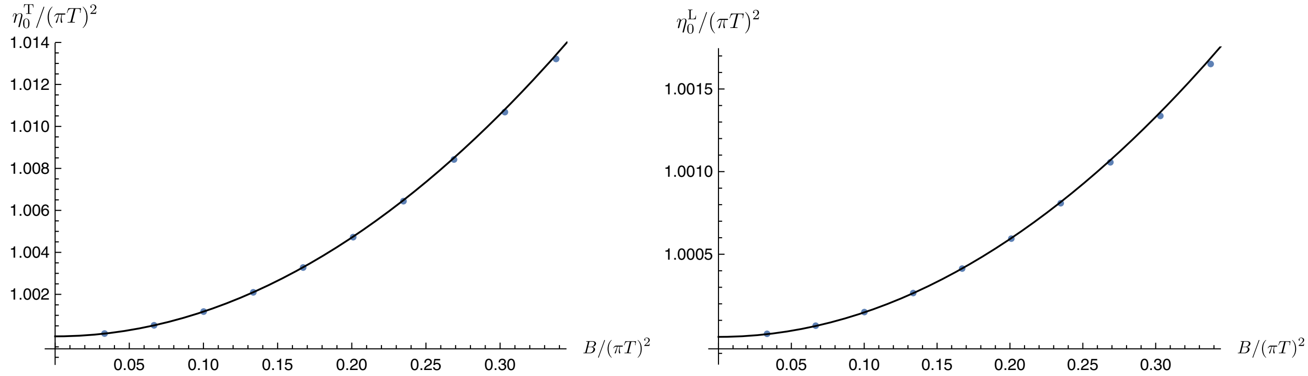


FIG. 4. Demonstration of perfect agreement between analytical (solid lines) and numerical (dots) results for $\eta_0^{\rm T,L}$ when $B/T^2 \ll 1$.

$$M^{\rm S}\omega^2 + {\rm i}\omega\eta^{\rm S}(\omega) = M_0\omega^2 + G_{ra}^{\rm S},$$
$$\xi^{\rm S} = -2{\rm i}G_{rr}^{\rm S}, \quad \text{with} \quad {\rm S} = {\rm T} \quad \text{or} \quad {\rm L}. \tag{3.52}$$

The familiar FDT (2.20) is equivalent to

$$G_{rr}^{\rm T,L} = \coth\frac{\beta\omega}{2}{\rm Im}[G_{ra}^{\rm T,L}(\omega)], \tag{3.53}$$

which is automatically satisfied.

While it is straightforward to numerically compute $G_{rr}^{\rm T,L}$ and $G_{ra}^{\rm T,L}$ by scanning over $(\omega, B)$, we will be limited to the leading order results $\eta_0^{\rm T,L}$ (equivalently $\xi_0^{\rm T,L}$), which are related to metric functions by

$$\eta_0^{\rm T} = 1 + \int_1^0 du\left[\frac{2}{u^3}(W(u)-1) - \frac{W'(u)}{u^2}\right],$$
$$\eta_0^{\rm L} = 1 + \int_1^0 du\left[\frac{2}{u^3}(V(u)-1) - \frac{V'(u)}{u^2}\right]. \tag{3.54}$$

When $B/T^2 \ll 1$, metric functions $U$, $V$, $W$ are known analytically, see (3.17). Thus, we have perturbative expansions for $\eta_0^{\rm T,L}$,

$$\frac{\eta_0^{\rm T}}{(\pi T)^2} = 1 + \frac{B^2}{12(\pi T)^4} + \frac{\pi^2 B^2}{288(\pi T)^4} + \cdots,$$
$$\frac{\eta_0^{\rm L}}{(\pi T)^2} = 1 + \frac{B^2}{12(\pi T)^4} - \frac{\pi^2 B^2}{144(\pi T)^4} + \cdots, \tag{3.55}$$

which are in perfect agreement with numerical results, as demonstrated in Fig. 4. Here we restore the $r_h$ in $\eta_0^{\rm T,L}$, $B$ and transform the $r_h$ to the temperature of plasma.

For generic value of $B/T^2$, we show numerical results for $\eta_0^{\rm T,L}$ in Figs. 5 and 6. Given that a strong magnetic field is produced in off-center heavy-ion collisions, we examine $\eta_0^{\rm T,L}$ when $B/T^2 \gg 1$, which are well fitted as

$$\frac{\eta_0^{\rm T}}{(\pi T)^2} \to 0.707 + 0.0303\frac{B}{T^2},$$
$$\frac{\eta_0^{\rm L}}{(\pi T)^2} \to 1.386 - 0.524\frac{\log(B/T^2)}{\sqrt{B/T^2}}, \quad \text{as } B/T^2 \gg 1. \tag{3.56}$$

Apparently, the magnetic field strengths damping effects for both transverse and longitudinal sectors. While the enhancement in transverse sector is linear in

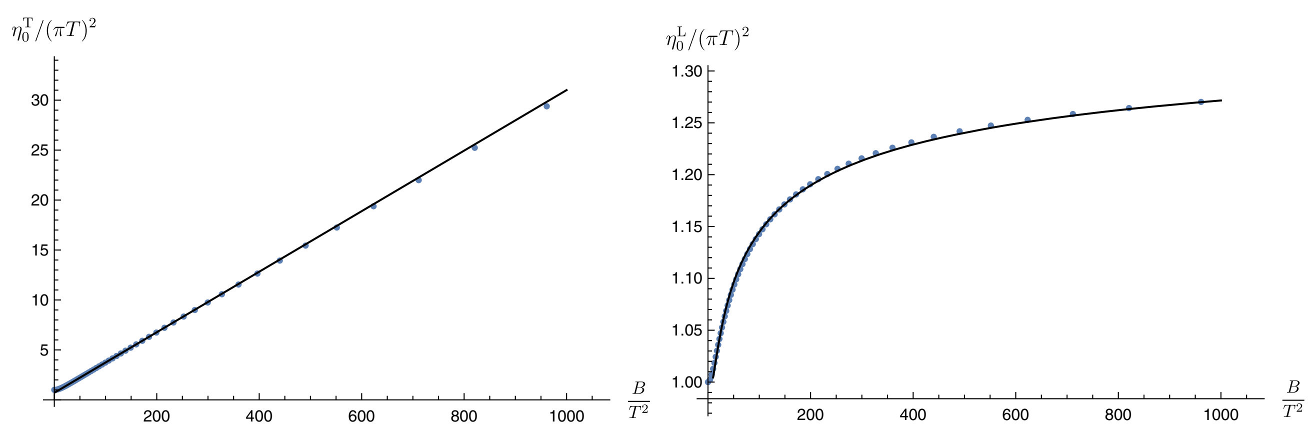


FIG. 5. Numerical (dots) results for $\eta_0^{\rm T,L}$, which are fitted to simple functions (3.56) (solid lines) when $B/T^2 \gg 1$.

$B$ (qualitatively similar to weakly-coupled QGP [7]), it seems to saturate for longitudinal mode with an upper bound $\eta_0^{\rm L} \le 1.386(\pi T)^2$.

Finally, we compare damping coefficients for transverse and longitudinal modes by plotting the ratio $\eta_0^{\rm L}/\eta_0^{\rm T}$ in Fig. 6, which is in perfect agreement with that of [34]. Interestingly, the ratio $\eta_0^{\rm L}/\eta_0^{\rm T}$ shows a reasonably slow decrease as $B$ becomes large.

### D. Effective action for Brownian quark: Quartic order

The quartic effective action $I^{(4)}$ for Brownian quark is related to string action via

$$I^{(4)} = S_{\rm NG}^{(4)}|_{X^{i,z}\to X^{i,z}_{(1)}}, \tag{3.57}$$

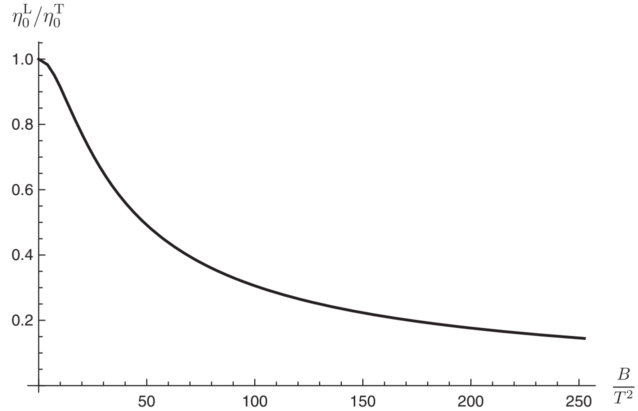


FIG. 6. Ratio $\eta_0^{\rm L}/\eta_0^{\rm T}$ as a function of $B$, in perfect agreement with that of [34].

with $S_{\rm NG}^{(4)}$ presented in (3.30). In frequency domain, quartic action becomes convolution

$$I^{(4)} = \frac{1}{2\pi\alpha'}\int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3}\delta(\omega_1+\omega_2+\omega_3+\omega_4)\sum_{m=1}^{4}\mathcal{I}_m(\omega_1,\omega_2,\omega_3,\omega_4), \tag{3.58}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_{0_1}^{0_2}\frac{du}{u^2}\frac{V(u)^2}{8}[2i\omega_1 X^i_{(1)}(u,\omega_1) + U(u)\partial_u X^i_{(1)}(u,\omega_1)] \\
&\quad\times[2i\omega_2 X^i_{(1)}(u,\omega_2) + U(u)\partial_u X^i_{(1)}(u,\omega_2)]\partial_u X^j_{(1)}(u,\omega_3)\partial_u X^j_{(1)}(u,\omega_4), \\
\mathcal{I}_2 &= \int_{0_1}^{0_2}\frac{du}{u^2}\frac{W(u)^2}{8}[2i\omega_1 X^z_{(1)}(u,\omega_1) + U(u)\partial_u X^z_{(1)}(u,\omega_1)] \\
&\quad\times[2i\omega_2 X^z_{(1)}(u,\omega_2) + U(u)\partial_u X^z_{(1)}(u,\omega_2)]\partial_u X^z_{(1)}(u,\omega_3)\partial_u X^z_{(1)}(u,\omega_4), \\
\mathcal{I}_3 &= \int_{0_1}^{0_2}\frac{du}{u^2}\frac{W(u)V(u)}{8}[2i\omega_1 X^i_{(1)}(u,\omega_1) + U(u)\partial_u X^i_{(1)}(u,\omega_1)] \\
&\quad\times[2i\omega_2 X^i_{(1)}(u,\omega_2) + U(u)\partial_u X^i_{(1)}(u,\omega_2)]\partial_u X^z_{(1)}(u,\omega_3)\partial_u X^z_{(1)}(u,\omega_4), \\
\mathcal{I}_4 &= \int_{0_1}^{0_2}\frac{du}{u^2}\frac{W(u)V(u)}{8}[2i\omega_1 X^z_{(1)}(u,\omega_1) + U(u)\partial_u X^z_{(1)}(u,\omega_1)] \\
&\quad\times[2i\omega_2 X^z_{(1)}(u,\omega_2) + U(u)\partial_u X^z_{(1)}(u,\omega_2)]\partial_u X^i_{(1)}(u,\omega_3)\partial_u X^i_{(1)}(u,\omega_4).
\end{aligned} \tag{3.59}$$

In contrast to computation of (3.46), quartic action (3.58) inevitably involves contour integrals (3.59), which are generically hard to compute. We proceed by examining the singular behavior for the integrands of (3.59) when $u$ is in the region enclosed by the radial contour. Recall that $X^{i,z}_{(1)}$ is the linear superposition of the ingoing mode and the Hawking mode (3.40). While the former is regular when $u$ is inside the contour, the latter shows logarithmic singularity near horizon due to the oscillating factor $e^{2i\omega\chi(u)}$ [see (3.41)]

$$e^{2i\omega\chi(u)} = (u-1)^{i\beta\omega/(2\pi)}g(u,\omega), \tag{3.60}$$

where $g(u,\omega)$ is a regular function of $u$. This type of singularity raises potential subtlety [49] regarding the order of taking $\omega\to 0$ limit and taking the limit $\epsilon\to 0$ (cf. Fig. 3 for $\epsilon$)

$$\lim_{\omega\to 0}\lim_{\epsilon\to 0} e^{2i\omega\chi(u)} \ne \lim_{\epsilon\to 0}\lim_{\omega\to 0} e^{2i\omega\chi(u)}, \qquad \text{as } u\to 1+\epsilon. \tag{3.61}$$

However, for the purpose of calculating (3.58) up to $\mathcal{O}(\omega^1)$, it turns out that noncommutativity issue of (3.61) is accidentally washed away, as demonstrated in Appendix A. Thus, it becomes valid to proceed as follows: Expand integrands of (3.59) in low frequency limit, and then evaluate radial integrals at each order in $\omega$, and finally take the limit $\epsilon \to 0$.

To facilitate discussion of derivative expansion for (3.58), we introduce compact notations for each piece in (3.59)

$$\begin{aligned} &2\mathrm{i}\omega X^i_{(1)} + U(u)\partial_u X^i_{(1)} \equiv \hat{\mathcal{A}}_\mathrm{T} q^i_r + \hat{\mathcal{B}}_\mathrm{T} q^i_a, \quad \partial_u X^i_{(1)} \equiv \tilde{\mathcal{A}}_\mathrm{T} q^i_r + \tilde{\mathcal{B}}_\mathrm{T} q^i_a, \\ &2\mathrm{i}\omega X^z_{(1)} + U(u)\partial_u X^z_{(1)} \equiv \hat{\mathcal{A}}_\mathrm{L} q^z_r + \hat{\mathcal{B}}_\mathrm{L} q^z_a, \qquad \partial_u X^z_{(1)} \equiv \tilde{\mathcal{A}}_\mathrm{L} q^z_r + \tilde{\mathcal{B}}_\mathrm{L} q^z_a, \end{aligned} \tag{3.62}$$

where $(\mathrm{S} = \mathrm{T}, \mathrm{L})$,

$$\hat{\mathcal{A}}_\mathrm{S} = 2\mathrm{i}\omega\mathfrak{A}_\mathrm{S} + U(u)\partial_u\mathfrak{A}_\mathrm{S}, \qquad \hat{\mathcal{B}}_\mathrm{S} = 2\mathrm{i}\omega\mathfrak{B}_\mathrm{S} + U(u)\partial_u\mathfrak{B}_\mathrm{S}, \qquad \tilde{\mathcal{A}}_\mathrm{S} = \partial_u\mathfrak{A}_\mathrm{S}, \qquad \tilde{\mathcal{B}}_\mathrm{S} = \partial_u\mathfrak{B}_\mathrm{S}. \tag{3.63}$$

In terms of $\hat{\mathcal{A}}_\mathrm{T,L}$, $\hat{\mathcal{B}}_\mathrm{T,L}$, $\tilde{\mathcal{A}}_\mathrm{T,L}$, $\tilde{\mathcal{B}}_\mathrm{T,L}$, the contour integrals in (3.59) become

$$\begin{aligned} \mathcal{I}_1 = \int_{0_1}^{0_2} du \frac{V(u)^2}{8u^2} \{ &\hat{\mathcal{B}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{B}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^i_a(\omega_1) q^i_a(\omega_2) q^j_a(\omega_3) q^j_a(\omega_4) \\ &+ 2\hat{\mathcal{A}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{B}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^i_r(\omega_1) q^i_a(\omega_2) q^j_a(\omega_3) q^j_a(\omega_4) \\ &+ 2\hat{\mathcal{B}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{A}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^j_a(\omega_1) q^j_a(\omega_2) q^i_r(\omega_3) q^i_a(\omega_4) \}, \\ \mathcal{I}_2 = \int_{0_1}^{0_2} du \frac{W(u)^2}{8u^2} \{ &\hat{\mathcal{B}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{B}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^3_a(\omega_1) q^3_a(\omega_2) q^3_a(\omega_3) q^3_a(\omega_4) \\ &+ 2\hat{\mathcal{A}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{B}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^3_r(\omega_1) q^3_a(\omega_2) q^3_a(\omega_3) q^3_a(\omega_4) \\ &+ 2\hat{\mathcal{B}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{A}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^3_a(\omega_1) q^3_a(\omega_2) q^3_r(\omega_3) q^3_a(\omega_4) \}, \\ \mathcal{I}_3 = \int_{0_1}^{0_2} du \frac{W(u)V(u)}{8u^2} \{ &\hat{\mathcal{B}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{B}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^i_a(\omega_1) q^i_a(\omega_2) q^3_a(\omega_3) q^3_a(\omega_4) \\ &+ 2\hat{\mathcal{A}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{B}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^i_r(\omega_1) q^i_a(\omega_2) q^3_a(\omega_3) q^3_a(\omega_4) \\ &+ 2\hat{\mathcal{B}}_\mathrm{T}(\omega_1)\hat{\mathcal{B}}_\mathrm{T}(\omega_2)\tilde{\mathcal{A}}_\mathrm{L}(\omega_3)\tilde{\mathcal{B}}_\mathrm{L}(\omega_4) q^i_a(\omega_1) q^i_a(\omega_2) q^3_r(\omega_3) q^3_a(\omega_4) \}, \\ \mathcal{I}_4 = \int_{0_1}^{0_2} du \frac{W(u)V(u)}{8u^2} \{ &\hat{\mathcal{B}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{B}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^3_a(\omega_1) q^3_a(\omega_2) q^i_a(\omega_3) q^i_a(\omega_4) \\ &+ 2\hat{\mathcal{A}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{B}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^3_r(\omega_1) q^3_a(\omega_2) q^i_a(\omega_3) q^i_a(\omega_4) \\ &+ 2\hat{\mathcal{B}}_\mathrm{L}(\omega_1)\hat{\mathcal{B}}_\mathrm{L}(\omega_2)\tilde{\mathcal{A}}_\mathrm{T}(\omega_3)\tilde{\mathcal{B}}_\mathrm{T}(\omega_4) q^3_a(\omega_1) q^3_a(\omega_2) q^i_r(\omega_3) q^i_a(\omega_4) \}, \end{aligned} \tag{3.64}$$

where we omitted terms that are explicitly beyond $\mathcal{O}(\omega^1)$.

To leading order in $\omega$, the coefficients in (3.40) are expanded as $(\mathrm{S} = \mathrm{T}, \mathrm{L})$

$$\begin{aligned} &\mathfrak{A}_S = 1 + \mathcal{O}(\omega), \\ &\mathfrak{B}_S = -\frac{1}{2} + \frac{2}{\beta}\frac{\Phi^\mathrm{ig}_{\mathrm{S},1}(u)}{\Phi^\mathrm{ig}_{\mathrm{S},0}(u)} - \frac{2\mathrm{i}}{\beta}\chi(u) + \mathcal{O}(\omega). \end{aligned} \tag{3.65}$$

where $\Phi^\mathrm{ig}_{\mathrm{S},0}(u)$ and $\Phi^\mathrm{ig}_{\mathrm{S},1}(u)$ are introduced in the low-frequency expansion of ingoing solution $\Phi^\mathrm{ig}(u, \omega)$, see (3.43). So, in the low-frequency limit, (3.63) scale as

$$\tilde{\mathcal{A}}_\mathrm{T,L}, \quad \hat{\mathcal{A}}_\mathrm{T,L} \sim \mathcal{O}(\omega^1), \quad \tilde{\mathcal{B}}_\mathrm{T,L}, \quad \hat{\mathcal{B}}_\mathrm{T,L} \sim \mathcal{O}(\omega^0). \tag{3.66}$$

To extract $\mathcal{O}(\omega^1)$ part of $I^{(4)}$, in (3.64) it is sufficient to retain the $\hat{\mathcal{B}}\hat{\mathcal{B}}\tilde{\mathcal{B}}\tilde{\mathcal{B}}$ type terms to $\mathcal{O}(\omega^0)$ while retaining the $\hat{\mathcal{A}}\hat{\mathcal{B}}\tilde{\mathcal{B}}\tilde{\mathcal{B}}$ type terms to $\mathcal{O}(\omega^1)$. This means that we only need lowest-order terms of $\tilde{\mathcal{A}}_{T,L}$, $\hat{\mathcal{A}}_{T,L}$, $\tilde{\mathcal{B}}_{T,L}$, $\hat{\mathcal{B}}_{T,L}$, all of which are regular. Therefore, expressed in the form (3.64), it is transparent that the contour integrals can be computed by residue theorem. Eventually, the result (2.22) is recovered with various coefficients given as

$$\kappa^{\rm T} = \frac{B^2 - 24(V_h^0)^2}{32\pi^3\alpha'},$$
$$\zeta^{\rm T} = \frac{-12\beta[B^2 - 24(V_h^0)^2]}{32\pi^3\alpha'},$$
$$\kappa^{\rm L} = -\frac{3(W_h^0)^2[B^2 + 8(V_h^0)^2]}{32\pi^3\alpha'(V_h^0)^2},$$
$$\zeta^{\rm L} = \frac{36\beta(W_h^0)^2[B^2 + 8(V_h^0)^2]}{32\pi^3\alpha'(V_h^0)^2},$$
$$\kappa_{1,2}^{\times} = -\frac{W_h^0[B^2 + 24(V_h^0)^2]}{32\pi^3\alpha' V_h^0},$$
$$\zeta^{\times} = \frac{24\beta W_h^0[B^2 + 24(V_h^0)^2]}{32\pi^3\alpha' V_h^0}, \tag{3.67}$$

where $V_h^0$, $W_h^0$ are horizon data, cf. (3.15), and we have transformed $U_h^1$ to the inverse temperature $\beta$ by (3.16). Interestingly, the KMS conditions (2.24) are perfectly satisfied even without knowledge of exact solution for metric functions.

In weak field limit $B/T^2 \ll 1$, we analytically compute all coefficients in (3.67)

$$\frac{\zeta^{\rm T}}{\zeta_0^{\rm T}} = 1 + \frac{18+\pi^2}{144}\frac{B^2}{\pi^4 T^4} + \cdots,$$
$$\frac{\zeta^{\rm L}}{\zeta_0^{\rm L}} = 1 + \frac{21-\pi^2}{72}\frac{B^2}{\pi^4 T^4} + \cdots,$$
$$\frac{\zeta^{\times}}{\zeta_0^{\times}} = 1 + \frac{60-\pi^2}{288}\frac{B^2}{\pi^4 T^4} + \cdots, \tag{3.68}$$

where $\zeta_0^{\rm T}$, $\zeta_0^{\rm L}$, $\zeta_0^{\times}$ are values of $\zeta^{\rm T}$, $\zeta^{\rm L}$, $\zeta^{\times}$ when $B = 0$,

$$\zeta_0^{\rm T} = \frac{9\pi}{\alpha'\beta^5}, \qquad \zeta_0^{\rm L} = \frac{9\pi}{\alpha'\beta^5}, \qquad \zeta_0^{\times} = \frac{18\pi}{\alpha'\beta^5}. \tag{3.69}$$

Here, we have restored the $r_h$ in $B$, $\zeta^{\rm T,L\times}$, $\zeta_0^{\rm T,L\times}$ and transformed it to temperature in (3.68) and (3.69). When $B = 0$, our result (3.69) is in agreement with [5], up to an overall sign. However, we are confident that our result (3.69) is more reasonable once the condition (2.23) is concerned. As shown in Fig. 7, our analytical result (3.68) is perfectly consistent with the numerical study.

For generic value of magnetic field, we show numerical results for $\zeta^{\rm T,L}, \zeta^{\times}$ in Fig. 8. Obviously, all the coefficients

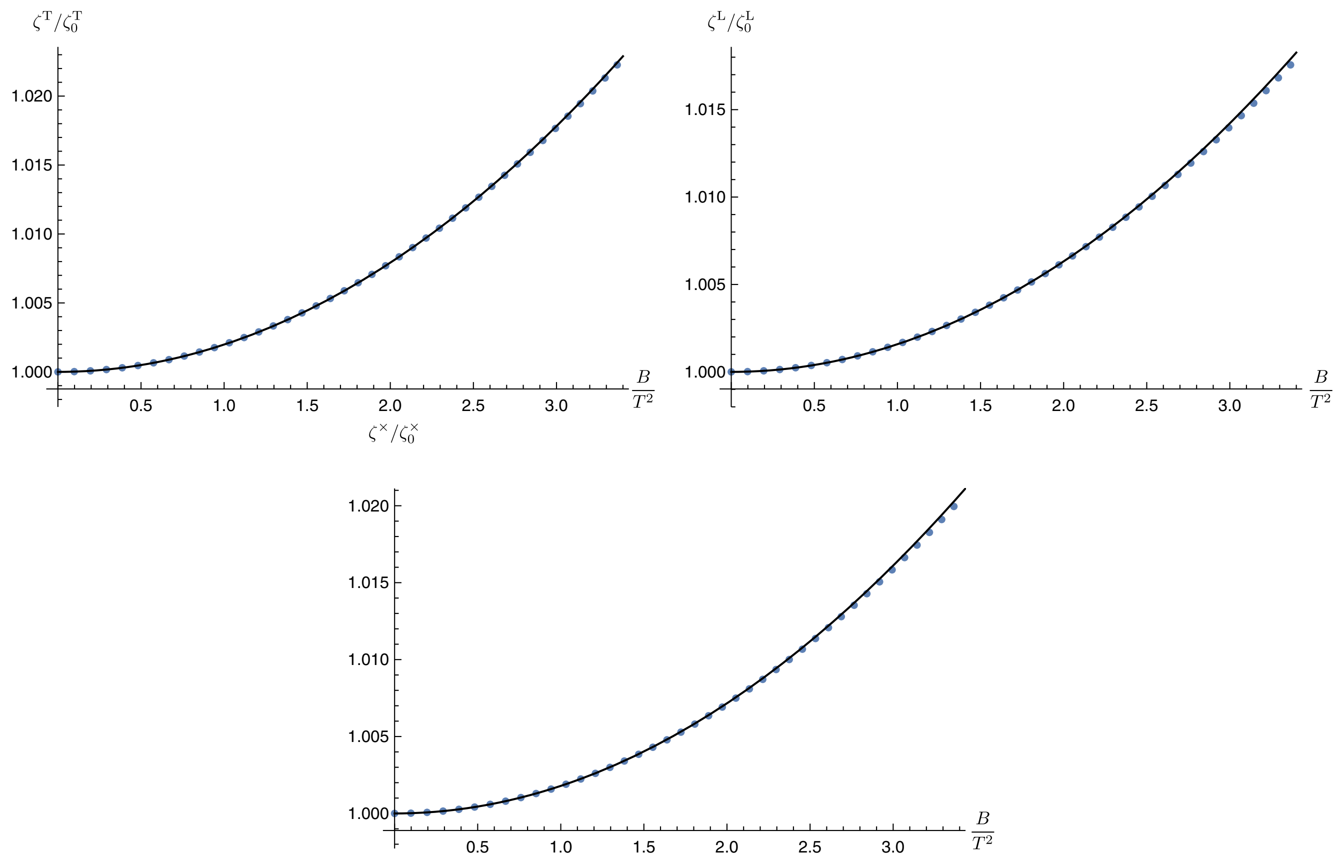


FIG. 7. Demonstration of perfect agreement between analytical (solid lines) and numerical (dots) results for $\zeta^{\rm T,L,\times}/\zeta_0^{\rm T,L,\times}$, when $B/T^2 \ll 1$.

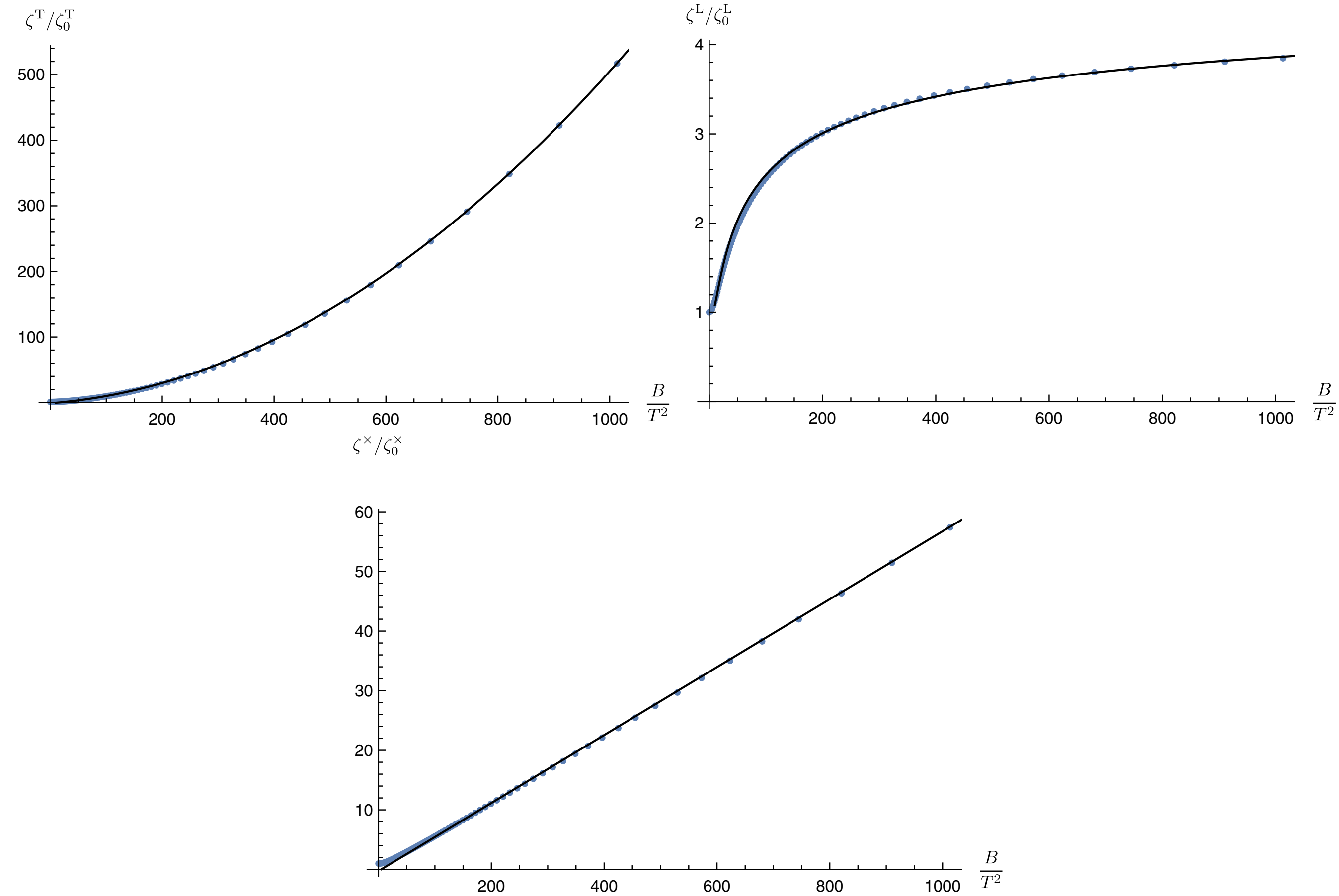


FIG. 8. The numerical values (dots) of $\zeta^{\mathrm{T,L},\times}/\zeta_0^{\mathrm{T,L},\times}$ when $B/T^2$ is generic. The solid lines are fitting functions of (3.70).

grow as magnetic field is increased. In the strong magnetic field limit $B/T^2 \gg 1$, $\zeta^{\mathrm{T,L},\times}/\zeta_0^{\mathrm{T,L},\times}$ are well fitted as

$$\frac{\zeta^{\mathrm{T}}}{\zeta_0^{\mathrm{T}}} \to -0.95 + 0.066\frac{B}{T^2} + 0.00044\left(\frac{B}{T^2}\right)^2,$$
$$\frac{\zeta^{\mathrm{L}}}{\zeta_0^{\mathrm{L}}} \to 5.05 - 5.45\frac{\log(B/T^2)}{\sqrt{B/T^2}},$$
$$\frac{\zeta^{\times}}{\zeta_0^{\times}} \to -0.25 + 0.057\frac{B}{T^2}, \tag{3.70}$$

which hold for a reasonably wide range of $B/T^2$. The coefficient $\zeta^{\mathrm{L}}$ behaves similar as its quadratic counterpart $\eta_0^{\mathrm{L}}$, and shows a mild growth as $B/T^2$ is increased, and eventually saturates as $\zeta^{\mathrm{L}}/\zeta_0^{\mathrm{L}} \lesssim 5.05$. However, the coefficients $\zeta^{\mathrm{T}}$, $\zeta^{\times}$ increase more dramatically for a strong magnetic field.

## IV. SUMMARY AND DISCUSSION

From the perspective of action principle, we presented a comprehensive study on effective description of a Brownian particle moving in a magnetized plasma. First, within the framework of the nonequilibrium EFT [9,11,12], we identify all the symmetries and construct effective action for Brownian particle, up to quartic order in particle's position. Then, we confirm the result through a model study based on holographic prescription for the SK contour. Moreover, in the holographic model, we compute various coefficients in the effective action as functions of magnetic field and temperature, focusing on the strong magnetic field limit.

Due to presence of non-Gaussian terms, it becomes inconvenient to cast the effective action into stochastic Langevin-type equation [9]. Nevertheless, we successfully convert the non-Gaussian effective action into a deterministic Fokker-Planck-type equation, which corresponds to the truncated Kramers-Moyal master equation at quartic order in derivatives. The Fokker-Planck-type equation is more efficient for computing observables, such as moments of position/velocity of Brownian particle. It will be interesting to carry out a numerical study based on our non-Gaussian theory and clarify phenomenological consequences of non-Gaussian interactions [6].

While dynamical KMS symmetry (2.8)–(2.9) is on the quantum level, the constant translational symmetry (2.15) renders the effective thoery to be entirely classical, in which quantum fluctuation is switched off. Relaxing the symmetry (2.15), we have realized that, from nonequilibrium EFT perspective, the classical statistical limit (2.11) and quantum level (2.9) of dynamical KMS symmetry will give rise to different KMS relations among coefficients in the effective action. It will be interesting to investigate on this point via a direct holographic calculation, by considering an open string moving in a slowly-varying AdS black hole [71–74] of fluid-gravity correspondence [75]. Moreover, this new setup is supposed to yield more realistic [76] effective description for Brownian motion.

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## APPENDIX A: SUBTLETY DUE TO NONCOMMUTATIVITY OF $\epsilon \to 0$ VERSUS $\omega \to 0$

By adopting the method of [58], we now show that subtlety arising from noncommutativity (3.61) becomes accidentally irrelevant for the purpose of evaluating (3.59) up to $\mathcal{O}(\omega^1)$. With the linearized string profile $X^{i,z}_{(1)}$ presented in (3.40), it is straightforward to show that the contour integrals in (3.59) could be classified into three distinguished pieces

$$\mathcal{I}_m = \mathcal{I}_{\rm ana} + \mathcal{I}_{\rm poles} + \mathcal{I}_{\rm non-ana}. \tag{A1}$$

Here, the first piece $\mathcal{I}_{\rm ana}$ vanishes since its integrand does not contain any singularity near the horizon. The second piece $\mathcal{I}_{\rm poles}$ could be simply computed by residue theorem as its integrand contains simple poles (no branch cuts) at the horizon. The last piece $\mathcal{I}_{\rm non-ana}$ involves logarithmic branch cut (maybe poles as well) at the horizon, which has a schematic form

$$\mathcal{I}_{\rm non-ana} = \int_{0_1}^{0_2} \frac{du}{u^2}(u-1)^{\pm i\hat{\omega}} \mathcal{H}(u,\hat{\omega}), \tag{A2}$$

where a potential factor $1/u^2$ is factorized, which would bring in UV divergence. Here, we use $\hat{\omega}$ to denote certain linear combination of $\omega_{1,2,3,4}$. For generic value of $\hat{\omega}$, we do not have analytical expression for $\mathcal{H}(u,\hat{\omega})$ for generic $\hat{\omega}$. However, we do know that $(u-1)^2\mathcal{H}(u,\hat{\omega})$ is finite, nonsingular and continuous inside the radial contour of Fig. 3. Thanks to the Weierstrass approximation theorem, it is legal to represent $(u-1)^2\mathcal{H}$ by Taylor series $\sum_{l=-2}^{\infty}\mathcal{H}_l(\omega)(u-1)^{l+2}$ when $u$ is inside the radial contour. Thus,

$$\mathcal{I}_{\rm non-ana} = \sum_{l=-2}^{\infty}\mathcal{H}_l(\hat{\omega})\mathfrak{I}_l, \quad \text{with}$$
$$\mathfrak{I}_l \equiv \int_{0_1}^{0_2}\frac{du}{u^2}(u-1)^{\pm i\hat{\omega}}(u-1)^l. \tag{A3}$$

Therefore, the original task of computing (3.59) boils down to calculating simpler contour integrals $\mathfrak{I}_l$ of (A3), which could be worked out analytically for generic value of $\hat{\omega}$. Afterwards, we extract low-frequency limit of $\mathfrak{I}_l$ (see Appendix B of [58]),

$$\begin{aligned}
&\mathfrak{I}_l = \mp\,\hat{\omega}\frac{{}_2F_1(2,n+1;n+2;1-\Lambda)}{(n+1)T} + \mathcal{O}(\hat{\omega}^2), \quad l\geq 1,\\
&\mathfrak{I}_l = \mp\frac{\hat{\omega}}{\Lambda T} + \mathcal{O}(\hat{\omega}^2), \qquad l=0,\\
&\mathfrak{I}_n = 2i\pi \mp \frac{\hat{\omega}}{\Lambda T} + \mathcal{O}(\hat{\omega}^2), \qquad l=-1,\\
&\mathfrak{I}_l = 4i\pi + \mathcal{O}(\hat{\omega}), \qquad l=-2,
\end{aligned} \tag{A4}$$

where $\Lambda$ represents a UV cutoff near the AdS boundary $u=0$, and ${}_2F_1$ is a hypergeometric function. It is direct to check that the results (A4) could be correctly recovered by first expanding the integrand of (A3) in small $\hat{\omega}$ and then computing the radial integral. However, this latter treatment cannot correctly cover higher-order terms omitted in (A4), which correspond to higher derivative terms in the effective action. Therefore, in order to extracting order $\mathcal{O}(\omega^1)$ part of the quartic effective action (3.58), it is valid to first expand the integrands [including the oscillating factor like $(u-1)^{\pm i\hat{\omega}}$] in (3.59) in small $\hat{\omega}$, and then implement the radial integral.

## APPENDIX B: KMS RELATIONS WHEN (2.15) IS RELAXED

In this appendix we show that once the constant translational invariance (2.15) is relaxed, the classical statistical limit (2.11) and the high-temperature limit (2.12) will give different KMS relations among coefficients in the effective action.

First, the quadratic Lagrangian (2.17) receives corrections

$$\begin{aligned}
L_{\rm SK}^{(2),\rm new} = L_{\rm SK}^{(2)} + \theta_1 q_a^z q_r^z + \theta_2 q_a^i q_r^i\\
+\theta_3 B\epsilon_{ij}q_r^i q_a^j + \theta_4 B\epsilon_{ij}\dot{q}_a^i q_a^j.
\end{aligned} \tag{B1}$$

Imposing dynamical KMS symmetry (2.8) under the classical statistical limit (2.11) and high-temperature limit (2.12), we find the same KMS relations

$$\eta_0^{\rm T} = \frac{1}{2}\beta\xi_0^{\rm T}, \qquad \eta_0^{\rm L} = \frac{1}{2}\beta\xi_0^{\rm L}, \qquad \theta_3 = 0. \tag{B2}$$

Next, we turn to corrections of the quartic Lagrangian (2.22)

$$L_{\rm SK}^{(4),\rm new} = L_{\rm SK}^{(4)} + \delta L_{\rm SK}^{(4)}, \tag{B3}$$

where,[6] due to breaking of isotropy invariance, $\delta L^{(4)}_{\rm SK}$ looks lengthy,

$$\begin{aligned}\delta L^{(4)}_{\rm SK} =& \lambda_{0,1}\dot q^i_a q^i_a (q^z_a)^2 + \kappa_{1,1} q^i_r q^i_a (q^j_a)^2 + \kappa_{1,2} q^i_r q^i_a (q^z_a)^2 + \kappa_{1,3} q^z_r q^z_a (q^i_a)^2 + \kappa_{1,4} q^z_r (q^z_a)^3\\ &+ \lambda_{1,1} q^i_r \dot q^i_a (q^j_a)^2 + \lambda_{1,2} q^i_r \dot q^i_a (q^z_a)^2 + \lambda_{1,3} q^z_r \dot q^z_a (q^i_a)^2 + \frac{\rm i}{2!}[\kappa_{2,1}(q^i_r)^2 (q^j_a)^2 + \kappa_{2,2} q^i_r q^i_a q^j_r q^j_a\\ &+ \kappa_{2,3}(q^i_r)^2 (q^z_a)^2 + \kappa_{2,4}(q^z_r)^2 (q^i_a)^2 + \kappa_{2,5} q^i_r q^i_a q^a_r q^z_a + \kappa_{2,6}(q^z_r)^2 (q^z_a)^2] + \lambda_{2,1}\dot q^i_r q^i_r (q^j_a)^2\\ &+ \lambda_{2,2}\dot q^i_r q^i_a q^j_r q^j_a + \lambda_{2,3}\dot q^i_r q^i_r (q^z_a)^2 + \lambda_{2,4}(q^i_a)^2 \dot q^z_r q^z_r + \lambda_{2,5}\dot q^i_r q^i_a q^z_r q^z_a + \lambda_{2,6} q^i_r \dot q^i_a q^z_r q^z_a\\ &+ \lambda_{2,7} q^i_r q^i_a \dot q^z_r q^z_a + \lambda_{2,8} q^z_r \dot q^z_r (q^z_a)^2 + \kappa_{3,1}(q^i_r)^2 q^j_r q^j_a + \kappa_{3,2} q^i_a q^i_r (q^z_r)^2 + \kappa_{3,3} q^z_r q^z_a (q^i_r)^2\\ &+ \kappa_{3,4} q^z_r q^z_a (q^z_r)^2 + \lambda_{3,1}(q^i_r)^2 \dot q^j_r q^j_a + \lambda_{3,2}\dot q^i_r q^i_r q^j_r q^j_a + \lambda_{3,3} q^i_a q^i_r (q^z_r)^2 + \lambda_{3,4} q^i_a q^i_r \dot q^z_r q^z_r\\ &+ \lambda_{3,5}\dot q^z_r q^z_a (q^i_r)^2 + \lambda_{3,6} q^z_r \dot q^z_a (q^i_r)^2 + \lambda_{3,7}\dot q^z_r q^z_a (q^z_r)^2. \end{aligned} \tag{B4}$$

Imposing dynamical KMS symmetry (2.8) in the high-temperature limit (2.12), we find

$$\begin{aligned} &\lambda_{0,1} = \frac{1}{8} i(\beta\hbar^2\kappa_{1,2} - \beta\hbar^2\kappa_{1,3}), \qquad \kappa^{\rm T} = \frac{1}{48}(3\beta\hbar^2\kappa_{2,2} - 4\beta\zeta^T),\\ &\lambda_{1,1} = \frac{1}{16}(\beta\hbar^2\kappa_{2,2} - 2\beta\hbar^2\kappa_{2,1}), \qquad \kappa^\times_2 = \frac{1}{96}(3\beta\hbar^2\kappa_{2,5} - 4\beta\zeta^\times),\\ &\lambda_{1,2} = \frac{1}{32}(\beta\hbar^2\kappa_{2,5} - 4\beta\hbar^2\kappa_{2,3}), \qquad \kappa^\times_1 = \frac{1}{96}(3\beta\hbar^2\kappa_{2,5} - 4\beta\zeta^0),\\ &\lambda_{1,3} = \frac{1}{32}(\beta\hbar^2\kappa_{25} - 4\beta\hbar^2\kappa_{2,4}), \qquad \kappa^{\rm L} = \frac{1}{24}(\beta\hbar^2\kappa_{2,6} - 2\beta\zeta^L),\\ &\lambda_{2,1} = \frac{\rm i}{8}(4\beta\kappa_{1,1} - \beta\hbar^2\kappa_{3,1}), \qquad \lambda_{2,2} = \frac{\rm i}{4}(4\beta\kappa_{1,1} - \beta\hbar^2\kappa_{3,1}),\\ &\lambda_{2,3} = \frac{\rm i}{8}(4\beta\kappa_{1,2} - \beta\hbar^2\kappa_{3,3}), \qquad \lambda_{2,4} = \frac{\rm i}{8}(4\beta\kappa_{1,3} - \beta\hbar^2\kappa_{3,3}),\\ &\lambda_{2,5} = \frac{\rm i}{4}(4\beta\kappa_{1,3} - \beta\hbar^2\kappa_{3,3}), \qquad \lambda_{2,7} = \frac{\rm i}{4}(4\beta\kappa_{1,2} - \beta\hbar^2\kappa_{3,3}),\\ &\lambda_{2,8} = \frac{3\rm i}{8}(4\beta\kappa_{1,4} - \beta\hbar^2\kappa_{3,4}), \qquad \lambda_{2,6} = 0,\\ &\lambda_{3,1} = -\frac{1}{2}\beta\kappa_{2,1}, \qquad \lambda_{3,2} = -\frac{1}{2}\beta\kappa_{2,2}, \qquad \lambda_{3,3} = -\frac{1}{2}\beta\kappa_{2,4},\\ &\lambda_{3,4} = -\frac{1}{4}\beta\kappa_{2,5}, \qquad \lambda_{3,5} = \frac{1}{8}(\beta\kappa_{2,5} - 4\beta\kappa_{2,3}), \qquad \lambda_{3,6} = \frac{1}{8}\beta\kappa_{2,5},\\ &\lambda_{3,7} = -\frac{1}{2}\beta\kappa_{2,6}, \qquad \kappa_{3,2} = \kappa_{3,3}. \end{aligned} \tag{B5}$$

On the other hand, if we impose dynamical KMS symmetry (2.8) in the classical statistical limit (2.11), we would get

$$\begin{aligned} &\lambda_{0,1} = \lambda_{1,1} = \lambda_{1,2} = \lambda_{1,3} = \lambda_{2,6} = 0,\\ &\kappa^{\rm T} = -\frac{1}{12}\beta\zeta^{\rm T}, \qquad \kappa^\times_1 = \kappa^\times_2 = -\frac{1}{24}\beta\zeta^\times, \qquad \kappa^{\rm L} = -\frac{1}{12}\beta\zeta^{\rm L}\\ &\lambda_{2,1} = \frac{\rm i}{2}\beta\kappa_{1,1}, \qquad \lambda_{2,2} = {\rm i}\beta\kappa_{1,1}, \qquad \lambda_{2,3} = \frac{\rm i}{2}\beta\kappa_{1,2}, \qquad \lambda_{2,4} = \frac{\rm i}{2}\beta\kappa_{1,3},\\ &\lambda_{2,5} = {\rm i}\beta\kappa_{1,3}, \qquad \lambda_{2,7} = {\rm i}\beta\kappa_{1,2}, \qquad \lambda_{2,8} = \frac{3\rm i}{2}\beta\kappa_{1,4}, \qquad \lambda_{3,1} = -\frac{1}{2}\beta\kappa_{2,1}, \end{aligned}$$

[6]For simplicity we have ignored terms containing an antisymmetric tensor $\epsilon_{ij}$, which, under KMS transformation (2.9), will not get interference with $\delta L^{(4)}_{\rm SK}$. Thus, inclusion of them will not modify the main conclusion.

$$\lambda_{3,2} = -\frac{1}{2}\beta\kappa_{2,2}, \qquad \lambda_{3,3} = -\frac{1}{2}\beta\kappa_{2,4}, \qquad \lambda_{3,4} = -\frac{1}{4}\beta\kappa_{2,5}, \qquad \kappa_{3,2} = \kappa_{3,3},$$
$$\lambda_{3,5} = \frac{1}{8}(\beta\kappa_{2,5} - 4\beta\kappa_{2,3}), \qquad \lambda_{3,6} = \frac{1}{8}\beta\kappa_{2,5}, \qquad \lambda_{3,7} = -\frac{1}{2}\beta\kappa_{2,6}, \tag{B6}$$

which is actually $\hbar \to 0$ limit of (B5).

Obviously, if we require the constant translational invariance (2.15), by setting all the coefficients in (B4) to be zero, we will immediately see that the KMS relations (B5) and (B6) will collapse to (2.24).

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