Milne spacetime with conical defect: Some holographic studies

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We initiate a holographic study of field theory in a time-dependent background with a conical defect. We focus on the Milne spacetime to which, in the absence of cosmological constant, at late time any hyperbolic Friedmann-Robertson-Walker metric flows. When the Milne vacuum is represented by the adiabatic one, we are able to compute the two point correlators of operators which are dual to the massive scalars in the bulk AdS-Milne spacetime background with a defect. We find, for both twisted and untwisted operators, the correlators can be represented as the sum over images. This sum can be carried out explicitly to write the results in compact forms.

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I. INTRODUCTION

AdS/CFT duality has been extremely useful in exploring various features of strongly coupled quantum field theories as well as in providing insights into the quantum theory of gravity, particularly in the context of black hole physics. However, progress has been somewhat limited in exploiting the duality in the cosmological context. This could be due to the fact that a cosmological spacetime typically contains a spacelike singularity and around that region, classical gravity becomes unreliable. Nonetheless, since such a singularity generally runs all the way to the boundary, one may wish to examine if the boundary gauge theory could sense this singularity. Indeed in [1–7], in the context of AdS cosmologies and in particular, for AdS-Kasner and AdS-FRW (Friedmann-Robertson-Walker) spacetime cosmologies, such questions were addressed with various degrees of success.

Among the class of hyperbolic FRW geometries, the simplest is the Milne spacetime where the scale factor has a linear dependence on time. On one hand, since by a coordinate transformation the Milne spacetime can be expressed as the future wedge of the Minkowski metric, the geometry is free of curvature. On the other hand, as the time dependence of the metric is relatively simple, one may hope to have some analytical handle [8,9]. Further, in the absence of a cosmological constant, we note that all the hyperbolic FRW geometries approach the Milne spacetime at late time as the expansions in these models drive the mass-energy density to zero. In this paper, our focus will be on the Milne spacetime. Within the holographic setup, we will represent the boundary by the Milne geometry. The corresponding bulk dual will be the AdS-Milne spacetime. The precise nature of the gauge theory on this boundary will depend on the spacetime dimensions.

A generic difficulty that arises while working with quantum field theories on a curved, especially time-dependent, background is that the choice of the vacuum becomes ambiguous. Consequently, the correlation functions constructed out of the quantum fields start to depend on the choice of the vacuum. Even though the Milne spacetime is a patch of Minkowski, there exists multiple complete sets of modes which are related by Bogoliubov transformations. Fock space built out of the corresponding creation and annihilation operators are necessarily inequivalent and corresponding vacuum states carry different physical properties. Two preferred choices for the vacuum states in the Milne spacetime are the adiabatic and the conformal vacuum. Other less commonly used vacua can also be defined, see for example [10]. Among these vacua, the adiabatic vacuum is particularly appealing because of its similarity with the familiar vacuum in the Minkowski spacetime.

In the present work, to start with, we perform several computations in the AdS-Milne spacetime background and extract the corresponding gauge theory quantities. For a massive minimally coupled scalar field we compute the

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bulk Wightman and the other Green's functions in the case when the Milne part of the vacuum is represented by the adiabatic one. Subsequently, following the holographic prescription, the corresponding boundary correlators are extracted. An important issue here is whether the subregion duality [11] holds. We find that, at least for the cases we study, the bulk and the boundary correlators seem to respect the subregion duality. Next we turn our attention to the one where the Milne part is represented by the conformal vacuum. Here, we have not been able to find a closed form expression for the Wightman function. However, we see that the retarded Green's function has the same form as the one obtained in the adiabatic vacuum. This is indeed expected as the retarded correlator is supposed to be a state-independent function. Equipped with these computations, we turn our focus on the Milne spacetime in the presence of a conical defect.

Field theory on spacetime with a conical defect has been of interest for a long time [12–18]. Lately, with the advent of the holographic correspondence, exploring the properties of a class of strongly coupled field theories on conical spacetime has become possible [19–22]. For example, this correspondence can be exploited to compute the correlation functions of operators in a strongly coupled field theory, admitting large N expansion, on a spacetime with a conical defect once the gravitational dual is known. As a simple illustration, let us consider a field theory on a three-dimensional background given by

$$ds^2 = -dT^2 + dR^2 + R^2 d\theta^2, \tag{1.1}$$

where the angular coordinate θ has a periodicity of $2\pi/q$. For any value of q other than 1, there is a singularity at $R = 0$. For $q > 1$, to which we would restrict to, the spacetime has an angle deficit. In the large N limit of the field theory, the leading contribution to the two point correlation of a scalar operator would come from examining appropriate the scalar field on the AdS spacetime with the conical singularity. Written in the Poincare coordinates, the bulk geometry in question is therefore

$$ds^2 = \frac{1}{z^2}[dz^2 - dT^2 + dR^2 + R^2 d\theta^2], \tag{1.2}$$

where $z = \infty$ represents a horizon and $z = 0$ is the boundary where the gauge theory lives. The details of this theory follow from the $M2$ brane of the M-theory compactified on S_7. Note that the singularity present on the boundary now extends for all values of the radial coordinate z. The Wightman function of a minimally coupled massive scalar Φ of mass m on this spacetime can be read out from [23]. Denoting the coordinates (z, T, R, θ) together as x and restricting to the integer values of q, it is given by

$$G_+(x, x') = \langle 0|\Phi(x)\Phi(x')|0\rangle = -\frac{1}{4\pi^2}\sum_{k=0}^{q-1}\frac{Q^1_{\nu-1/2}(w_k)}{\sqrt{w_k^2 - 1}},$$

where $Q^1_{\nu-1/2}(w_k)$ is the associated Legendre function and

$$w_k = 1 + \frac{R^2 + R'^2 - 2RR'\cos(\theta - \theta' - 2\pi k/q) + (z - z')^2 - (T - T')^2}{2zz'},$$

and $\nu = \sqrt{9/4 + m^2}$. On the boundary, this field is dual to a scalar primary operator $\mathcal{O}$ with scaling dimension $\Delta = \nu + 3/2$. The correlator can then be extracted following the Banks-Douglas-Horowitz-Martinec (BDHM) prescription [24], namely,

$$\begin{aligned}&\langle\Psi|\mathcal{O}(T,R,\theta)\mathcal{O}(T',R',\theta')|\Psi\rangle\\ &\quad= \lim_{z,z'\to 0}(zz')^{-(\nu+3/2)}\left[-\frac{1}{4\pi^2}\sum_{k=0}^{q-1}\frac{Q^1_{\nu-1/2}(w_k)}{\sqrt{u_k^2-1}}\right]\\ &\quad= \sum_{k=0}^{q-1}\frac{C_\Delta}{[-(T-T')^2+R^2+R'^2-2RR'\cos(\theta-\theta'-2\pi k/q)]^\Delta}\end{aligned} \tag{1.3}$$

where

$$C_\Delta = \frac{2^{2\Delta-4}\Gamma(\Delta)\Gamma(\Delta-1)}{\pi^2\Gamma(2\Delta-2)}.$$

Here $|\Psi\rangle$ is the appropriate boundary state of the field theory with the conformal symmetry broken by the defect. Though the sum above can be performed, written in this way, the correlator has an interpretation in terms of the sum over images.

The metric (1.2) can arise due to the presence of a cosmic string. In the weak field approximation and in the thin string limit, the parameter q gets related to the mass density of the string. If we wish to model the formation of such a defect and examine the particle creation during the formation of the defect, we need to go beyond this static spacetime and replace it by an appropriate dynamical one [25,26]. One possibility could be to consider a geometry of the form

$$ds^2 = \frac{1}{z^2}[dz^2 - dT^2 + dR^2 + f(T)R^2 d\theta^2], \quad (1.4)$$

where $f(T) = [1 - \tanh(T/T_0)]/2$, and θ is a coordinate of period 2π. It represents a quench of the angular coordinate around $T = 0$, lasting for a time T_0. One may hope to gain some insights into the behavior of the strongly coupled field theory on the boundary by examining the bulk representing this dynamic cone. Difficulty arises immediately however due to the absence of an analytical handle. For example, for a generic function $f(T)$, the Klein-Gordon equation would not admit a separation of variables of the field. In order for that to happen we need to assume that the field is independent of the coordinate θ. While, even with such a cylindrical symmetry, it might be interesting to explore the dual field theory, in this work we take a modest step. Here we intend to study aspects of strongly coupled gauge theory on the Milne geometry in the presence of a conical defect.[1] With the choice of the adiabatic vacuum, the correlators of the operators dual to the massive scalars in the bulk can be computed. The final form turns out to be similar to the one given in (1.3) written in the Milne coordinates but we work this out explicitly starting with the quantization of a minimal massive scalar in the AdS-Milne spacetime background. Because of the presence of the defect, on the boundary we can have twisted scalar operators that are dual to the bulk scalar fields with twisted boundary conditions (3.19). It follows from [28] and subsequently from [29] that the twisted scalars can be defined on a nonsimply connected spacetime. These scalars satisfy the same equations of motion as that of the untwisted scalars but differ in their boundary conditions. We end our exploration with the computation of the boundary correlators involving the twisted operators dual to these scalars.

II. ADS-MILNE SPACETIME

We start out with the computation of the bulk and the boundary correlators in the AdS-Milne spacetime. This, in turn, will set the stage for a similar, but more involved, calculation of correlators in AdS-Milne spacetime with a conical defect. This is analyzed in a subsequent section. Mode expansions of a massive scalar field in this geometry turn out to be sensitive to the spacetime dimensions. Therefore we carry out our study both in four and five dimensions. The later has been provided in the Appendix.

A. (3 + 1)-dimensional AdS-Milne spacetime

In the Poincare coordinates, the AdS-Milne spacetime metric in four dimensions takes the form

$$ds^2 = \frac{1}{z^2}(-dt^2 + t^2 dr^2 + t^2 \sinh^2 r d\theta^2 + dz^2), \quad (2.1)$$

θ being a periodic coordinate with period 2π. We see that we have a Milne spacetime for every value of the bulk radial coordinate z. The fact that the AdS-Milne spacetime is a subregion or a patch in the AdS can be seen from the coordinate transformations, $T = t\cosh r, R = t\sinh r$. These transformations cover only the part $T \geq 0$, $R \geq 0$ of the AdS. Even though the Poincare AdS spacetime and AdS-Milne spacetime written in Poincare coordinates are related by coordinate transformations, we must be careful while studying field theory in these backgrounds as corresponding propagators need not be related by similar coordinate transformations. Therefore, in the holographic setup, we should independently define the Green's functions in each coordinate system.

The equation of motion of a minimally coupled scalar of mass m on AdS-Milne spacetime is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0. \quad (2.2)$$

Written explicitly, it takes the form

$$-\frac{2z^2}{t}\partial_t\phi - z^2\partial_t^2\phi + \frac{z^2}{t^2}\coth r\partial_r\phi + \frac{z^2}{t^2}\partial_r^2\phi + \frac{z^2}{t^2\sinh^2 r}\partial_\theta^2\phi - 2z\partial_z\phi + z^2\partial_z^2\phi - m^2\phi = 0. \quad (2.3)$$

For generic values of $\nu = \sqrt{m^2 + 9/4}$, the solutions are either

$$\phi_{\lambda\alpha n}(z,t,r,\theta) = C_{\lambda\alpha n}[z^{\frac{3}{2}}J_\nu(\lambda z)]\left[\frac{H^{(2)}_{i\alpha}(\lambda t)}{\sqrt{t}}\right]P^{-n}_{i\alpha-\frac{1}{2}}(\cosh r)e^{in\theta}, \quad (2.4)$$

or

$$\phi_{\lambda\alpha n}(z,t,r,\theta) = \tilde{C}_{\lambda\alpha n}[z^{\frac{3}{2}}J_\nu(\lambda z)]\left[\frac{J_{-i\alpha}(\lambda t)}{\sqrt{t}}\right]P^{-n}_{i\alpha-\frac{1}{2}}(\cosh r)e^{in\theta}. \quad (2.5)$$

In these equations, $J_\mu(x)$, $H^{(2)}_{i\alpha}(\lambda t)$. and $P^{-n}_{i\alpha-1/2}(u)$ are the Bessel function, Hankel function of the second kind, and the associated Legendre function respectively. The constants $\lambda \geq 0$, $\alpha \geq 0$, and n take integer values. In writing down these solutions, we used the boundary conditions that the solutions are regular at $z = 0$ and $r = 0$. Since these solutions are normalizable modes, they are dual to operators of conformal dimension $\Delta = \Delta_+ = 3/2 + \nu$. Looking at the large t behavior of the Bessel and the Hankel functions, it is easily seen that the choice $H^{(2)}_{i\alpha}$ is

[1] The possible occurrence of topological defects in the early universe and various cosmological consequences due to their presence have been an active area of research in the past, see for example [27].

similar to working with the Minkowskian positive energy modes. The vacuum defined with respect to these modes is called the adiabatic vacuum. On the other hand, if we take $J_{-i\alpha}(\lambda t)$, we are in the conformal vacuum [8]. We first work with the former. The normalization constants $C_{\lambda\alpha n}$ are determined by the following condition:

$$\int dzdrd\theta\sqrt{-g}g^{tt}[\phi_{\lambda\alpha n}(z,t,r,\theta)\partial_t\phi^*_{\lambda'\alpha'n'}(z,t,r,\theta) - \phi^*_{\lambda'\alpha'n'}(z,t,r,\theta)\partial_t\phi_{\lambda\alpha n}(z,t,r,\theta)] = -i\delta(\lambda-\lambda')\delta(\alpha-\alpha')\delta_{nn'}. \tag{2.6}$$

Using (2.4), we get

$$C_{\lambda\alpha n} = i\sqrt{\frac{\alpha\lambda\sinh\pi\alpha}{2\pi}}\Gamma[i\alpha+1/2+n]\frac{e^{\frac{\pi\alpha}{2}}}{2}. \tag{2.7}$$

To get to this, we have used the property (A4). The canonical quantization proceeds by defining the field operator

$$\Phi = \sum_i[\phi_i a_i + \phi_i^* a_i^\dagger],$$

where $a_l{}^\dagger$, a_l are the creation and the annihilation operators respectively and i includes the set of quantum numbers λ, α, n. The summation above represents integrations over λ, α and a sum over n.

We start by computing the Wightman function. It is defined as

$$G_+(z,t,r,\theta;z',t',r',\theta') = \langle 0|\Phi(z,t,r,\theta)\Phi(z',t',r',\theta')|0\rangle = \sum_i\phi_i(z,t,r,\theta)\phi_i^*(z',t',r',\theta'). \tag{2.8}$$

Here, $|0\rangle$ includes the Minkowski vacuum for the Milne part. The other Wightman function can be obtained from the relation $G_-(x,x') = G_+^*(x,x')$. Denoting the set of coordinates (z,t,r,θ) together as x, we get

$$\begin{aligned} G_+(x,x') &= \sum_{n=-\infty}^{\infty}\int_0^\infty d\lambda d\alpha\phi(z,r,t,\theta)\phi^*(z',r',t',\theta') \\ &= \frac{1}{4}\int d\lambda d\alpha\left(\frac{(zz')^{\frac{3}{2}}}{\sqrt{tt'}}\alpha\lambda\sinh(\pi\alpha)e^{\pi\alpha}\right)J_\nu(\lambda z)J_\nu(\lambda z')H^{(2)}_{i\alpha}(\lambda t)H^{(1)}_{-i\alpha}(\lambda t') \\ &\quad\times\sum_{n=-\infty}^{\infty}\Gamma[i\alpha+1/2+n]\Gamma[-i\alpha+1/2+n]P^{-n}_{i\alpha-1/2}(\cosh r)P^{-n}_{i\alpha-1/2}(\cosh r')e^{in(\theta-\theta')} \\ &= \frac{(zz')^{\frac{3}{2}}}{\pi\sqrt{tt'}}\int_0^\infty d\lambda d\alpha\alpha\lambda\tanh(\pi\alpha)J_\nu(\lambda z)J_\nu(\lambda z')K_{i\alpha}(i\lambda t)K_{i\alpha}(-i\lambda t')P_{i\alpha-1/2}(\cosh\chi). \end{aligned} \tag{2.9}$$

The identities used to reach here have been presented in the Appendix, see (A5). In the above, $K_\nu(z)$ is the modified Bessel function and χ is defined as

$$\cosh\chi = \cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta'). \tag{2.10}$$

Equation (2.9) can be further simplified using (A6) to arrive at

$$\begin{aligned} G_+(x,x') &= \frac{1}{\pi\sqrt{2\pi}}(zz')^{\frac{3}{2}}\int_0^\infty d\lambda\lambda^{\frac{3}{2}}J_\nu(\lambda z)J_\nu(\lambda z') \\ &\quad\times\frac{K_{\frac{1}{2}}(\lambda\sqrt{-t^2-t'^2+2tt'\cosh\chi})}{\sqrt{-t^2-t'^2+2tt'\cosh\chi}} \\ &= -\frac{1}{4\pi^2(u^2-1)^{\frac{1}{2}}}Q^1_{\nu-1/2}(u), \end{aligned} \tag{2.11}$$

where

$$u = \frac{1}{2zz'}(-t^2-t'^2+2tt'\cosh\chi+z^2+z'^2+i\epsilon\,\mathrm{sgn}(t-t')), \tag{2.12}$$

the geodesic distance between two points in AdS-Milne spacetime, and $Q^1_{\nu-1/2}(u)$ is the associated Legendre function. To arrive at the last line of (2.11), we have used a result from [30]. In the conformal limit, when the mass $m^2=-2$ (and consequently $\nu=1/2$), the correlator simplifies. Owing to the property that $Q^1_0 = -1/\sqrt{u^2-1}$, we get[2]

$$G_+(x,x') = \frac{1}{4\pi^2(u^2-1)}.$$

[2]We have calculated the Wightman function for normalizable modes that is G_{Δ_+}. There is a corresponding Wightman function G_{Δ_-} for the non-normalizable modes as shown in [31]; it is the sum $G_{\Delta_+}+G_{\Delta_-}$ which corresponds to a simple power solution in the conformal limit.

1. Boundary correlator

Having obtained the bulk Wightman function, we can use the BDHM prescription to construct the boundary correlator,

$$\lim_{z,z'\to 0}(zz')^{-(\nu+3/2)}\langle\Phi(z,r,t,\theta)\Phi(z',r',t',\theta')\rangle_{\mathrm{AdS-Milne}} = \langle\Psi|\mathcal{O}(r,t,\theta)\mathcal{O}(r',t',\theta')|\Psi\rangle.$$

Here $\mathcal{O}$ is the scalar primary operator of dimension $\Delta = 3/2+\nu$, dual to the bulk scalar Φ. On the left the subscript indicates that the bulk computation is done on AdS-Milne spacetime, and on the right, $|\Psi\rangle$ is the corresponding state of the boundary conformal theory. Implementing the above, we get

$$\begin{aligned}\langle\Psi|\mathcal{O}(t,r,\theta)\mathcal{O}(t',r',\theta')|\Psi\rangle &= \lim_{z,z'\to 0}(zz')^{-(\nu+3/2)}G_+(x,x') \\ &= \frac{(\Delta-2)2^\Delta}{4\pi^2(-t^2-t'^2+2tt'\cosh\chi)^\Delta}\int_0^\infty \frac{\cosh\beta}{(1+\cosh\beta)^{\Delta-1}}d\beta \\ &= \frac{C_\Delta}{(-t^2-t'^2+2tt'\cosh\chi)^\Delta},\end{aligned} \tag{2.13}$$

where

$$C_\Delta = \frac{2^{2\Delta-4}\Gamma(\Delta)\Gamma(\Delta-1)}{\pi^2\Gamma(2\Delta-2)}. \tag{2.14}$$

To arrive at the final line, we have used (A7).

A similar computation can be carried out for the AdS-Milne spacetime in five dimensions where the metric is

$$\begin{aligned}ds^2 = \frac{1}{z^2}[&-dt^2+t^2dr^2+t^2\sinh^2 r d\theta^2 \\ &+ t^2\sinh^2 r\sin^2\theta d\phi^2+dz^2].\end{aligned} \tag{2.15}$$

The details of the computation is provided in the Appendix. The Wightman function turns out to be

$$G_+(x,x') = \frac{i}{(2\pi)^{5/2}(u^2-1)^{3/4}}Q^{3/2}_{\nu-1/2}(u), \tag{2.16}$$

where

$$u = \frac{1}{2zz'}(-t^2-t'^2+2tt'\cosh\gamma+z^2+z'^2),$$

leading to the boundary correlator

$$\begin{aligned}\langle\Psi|\mathcal{O}(t,r,\theta,\phi),\mathcal{O}(t',r',\theta',\phi')|\Psi\rangle &= \frac{2^{2\nu-1}\Gamma(\frac{1}{2}+\nu)\Gamma(\nu+2)}{(\pi)^{\frac{5}{2}}\Gamma(2\nu+1)}\frac{1}{a^{\nu+2}} \\ &= \frac{2^{2\nu-1}\Gamma(\frac{1}{2}+\nu)\Gamma(\nu+2)}{(\pi)^{\frac{5}{2}}\Gamma(2\nu+1)}\frac{1}{(-t^2-t'^2+2tt'\cosh\gamma)^{\nu+2}}.\end{aligned} \tag{2.17}$$

Before proceeding to the next subsection, we end with the following note. In a new coordinate system τ, ρ, defined as $t = be^{\tau/b}$ and $r = \rho/b$ with b being constant, the metric in (2.1) becomes

$$ds^2 = \frac{1}{z^2}[dz^2+e^{2\tau/b}(-d\tau^2+d\rho^2+b^2\sinh^2(\rho/b)d\theta^2)].$$

Now, in the limit $b\to\infty$, the boundary becomes flat. The quantity u defined in (2.12) reduces to $u = \frac{1}{2zz'}[-(\tau-\tau')^2+\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')]$ and the boundary correlator becomes the one in the Minkowski spacetime.

B. Comparison with Poincare-AdS

Since the metric as well as the vacuum state are invariant under full Poincare-AdS symmetries, the final expression can be written in terms of the geodesic distance, which facilitates a comparison with the Green's functions in Poincare-AdS.

For AdS-Milne spacetime in $3+1$ dimensions we get the geodesic distance $d(z,w) = \int_z^w ds$ as

$$d(z_1,t_1,r_1,\theta_1;z_2,t_2,r_2,\theta_2) = \ln\frac{1+\sqrt{1-\xi^2}}{\xi},$$

where ξ is given by

$$\xi = \frac{2z_1z_2}{z_1^2+z_2^2-t_1^2-t_2^2+2t_1t_2\cosh\chi} = \frac{1}{u}.$$

Using the relation between the hypergeometric function and the associated Legendre function of the second kind, namely

$$\begin{aligned}Q^\mu_\nu(z) = &\frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)\Gamma(1/2)}{2^{\nu+1}\Gamma(\nu+3/2)}(z^2-1)^{\mu/2}z^{-\nu-\mu-1} \\ &\times F\left(\frac{\nu+\mu+1}{2},\frac{\nu+\mu+2}{2};\nu+3/2;\frac{1}{z^2}\right),\end{aligned}$$

we find

$$Q^1_{\nu-1/2}\left(\frac{1}{\xi}\right) = -4\pi^2\left(\frac{1}{\xi^2}-1\right)^{1/2}G_{\nu+\frac{3}{2}}(\xi),$$

where

$$G_{\nu+\frac{3}{2}}(\xi) = \frac{2^{-\nu-5/2}\Gamma(\nu+3/2)}{\pi^{3/2}\Gamma(\nu+1)}\xi^{\nu+3/2} \times F\left(\frac{3}{4}+\frac{\nu}{2}, \frac{5}{4}+\frac{\nu}{2}; \nu+1; \xi^2\right)$$

is the scalar propagator for the normalizable modes in Poincare-AdS as given in [32].

For AdS-Milne spacetime in $4+1$ dimensions we can write the geodesic distance in a similar way,

$$d(z_1, t_1, r_1, \theta_1, \phi_1; z_2, t_2, r_2, \theta_2, \phi_2) = \ln\frac{1+\sqrt{1-\xi^2}}{\xi},$$

where ξ is now given by

$$\xi = \frac{2z_1z_2}{z_1^2+z_2^2-t_1^2-t_2^2+2t_1t_2\cosh\chi} = \frac{1}{u}.$$

Again, we can convert from associated Legendre functions to hypergeometric function

$$Q^{3/2}_{\nu-1/2}\left(\frac{1}{\xi}\right) = -i(2\pi)^{5/2}\left(\frac{1}{\xi^2}-1\right)^{3/4}G_{\nu+2}(\xi),$$

where

$$G_{\nu+2}(\xi) = \frac{2^{-\nu-3}\Gamma(\nu+2)}{\pi^2\Gamma(\nu+1)}\xi^{\nu+2}F\left(1+\frac{\nu}{2}, \frac{3}{2}+\frac{\nu}{2}; \nu+1; \xi^2\right)$$

is the scalar Green's function given in [32].

C. AdS-Milne spacetime in conformal vacuum

As we discussed previously, working in conformal vacuum is equivalent to choosing $J_{-i\alpha}(\lambda t)$ in the t part instead of the Hankel function. The basis of the mode expansion is therefore given by (2.5) rather than (2.4)

$$\phi(z,t,r,\theta) = \tilde{C}_{\lambda\alpha n}[z^{3/2}J_\nu(\lambda z)]\left[\frac{J_{-i\alpha}(\lambda t)}{\sqrt{t}}\right]P^{-n}_{i\alpha-1/2}(\cosh r)e^{in\theta}, \tag{2.18}$$

where the normalization constant, up to a constant phase factor, is given by

$$\tilde{C}_{\lambda\alpha n} = i\sqrt{\frac{\alpha\lambda}{4\pi}}\Gamma[i\alpha+1/2+n].$$

Now doing manipulations similar to what was done in the Minkowski vacuum, we arrive at

$$\begin{aligned} G_+(x,x') &= \frac{1}{4\pi}\int d\lambda d\alpha\left[\frac{(zz')^{3/2}}{\sqrt{tt'}}\alpha\lambda\right]J_\nu(\lambda z)J_\nu(\lambda z')J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t') \\ &\quad\times\sum_{n=-\infty}^{+\infty}\Gamma(i\alpha+1/2+n)\Gamma(-i\alpha+1/2+n)P^{-n}_{i\alpha-1/2}(\cosh r)P^{-n}_{i\alpha-1/2}(\cosh r')e^{in(\theta-\theta')} \\ &= \frac{(zz')^{3/2}}{4\sqrt{tt'}}\int d\lambda d\alpha\, \alpha\lambda J_\nu(\lambda z)J_\nu(\lambda z')J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t')\frac{1}{\cosh\pi\alpha}P_{i\alpha-1/2}(\cosh\chi) \end{aligned} \tag{2.19}$$

where χ is defined by (2.10). The other Wightman function is given by $G_-(x,x') = G_+^*(x,x')$.

In $4+1$ dimensions, the right solution of the scalar equation of motion turns out to be

$$\phi(z,t,r,\theta,\phi) = \tilde{C}_{\lambda\alpha}[z^2J_\nu(\lambda z)]\left[\frac{J_{-i\alpha}(\lambda t)}{t}\right]Y_{\alpha lm}(r,\theta,\phi)$$

where the normalization constant is given by

$$\tilde{C}_{\lambda\alpha} = \sqrt{\frac{\pi\lambda}{2\sinh\pi\alpha}}.$$

The Wightman function is then

$$G_+(x,x') = \sum_{l,m}\int_0^\infty \lambda.d\lambda d\alpha\frac{\pi}{2}\frac{(zz')^2}{(tt')^2\sinh\pi\alpha}J_\nu(\lambda z)J_\nu(\lambda z') \times J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t')Y_{\alpha lm}(r,\theta,\phi)Y_{\alpha lm}(r',\theta',\phi').$$

As was done for the Minkowski vacuum, we can use the completeness relation of the spherical harmonics (A2) to cast the above function as

$$G_+(x,x') = \int_0^\infty \lambda d\lambda d\alpha\frac{1}{4\pi}\frac{(zz')^2}{(tt')^2\sinh\pi\alpha}J_\nu(\lambda z)J_\nu(\lambda z') \times J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t')\frac{\alpha\sin\alpha\gamma}{2\pi^2\sinh\gamma},$$

where γ is given, as before, by (A3). Unfortunately, we have not succeeded in carrying out all the integrals in (2.19) and (2.20). However, we could make some progress while considering the retarded propagator. This is given by $G_R(x,x') = \theta(t-t')G(x,x')$ where $G(x,x') = G_+(x,x') - G_-(x,x')$ is the difference between the positive and the negative frequency Wightman functions.

To explicitly calculate the retarded propagator, we first consider the $(3+1)$-dimensional AdS-Milne spacetime in conformal vacuum. In this case, we are able to write

$$G(x,x') = \int d\lambda d\alpha \frac{(zz')^{3/2}}{4\sqrt{tt'}} J_\nu(\lambda z)J_\nu(\lambda z')(J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t') - J_{-i\alpha}(\lambda t')J_{i\alpha}(\lambda t))\frac{\alpha\lambda P_{i\alpha-1/2}(\cosh\chi)}{\cosh\pi\alpha}.$$

We can straightforwardly show that the $G(x,x')$ in this vacuum is the same as that in the Minkowski vacuum. To this end, we first write the Bessel function in terms of the Hankel function using $J_\alpha(x) = \frac{1}{2}(H_\alpha^{(1)}(x) + H_\alpha^{(2)}(x))$. Further, since $H_{-\alpha}^{(1,2)} = e^{\pm i\pi\alpha}H_\alpha^{(1,2)}$, we can then write

$$4[J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t') - J_{-i\alpha}(\lambda t')J_{i\alpha}(\lambda t)] = \sinh\pi\alpha(H_{i\alpha}^{(2)}(\lambda t)H_{i\alpha}^{(1)}(\lambda t') - H_{i\alpha}^{(2)}(\lambda t')H_{i\alpha}^{(1)}(\lambda t)). \quad (2.20)$$

Therefore,

$$G(x,x') = \int d\lambda d\alpha \frac{(zz')^{3/2}}{\sqrt{tt'}}\alpha\lambda J_\nu(\lambda z)J_\nu(\lambda z') \times \tanh\pi\alpha P_{i\alpha-1/2}(\cosh\chi)(H_{i\alpha}^{(2)}(\lambda t)H_{i\alpha}^{(1)}(\lambda t') - H_{i\alpha}^{(2)}(\lambda t')H_{i\alpha}^{(1)}(\lambda t)).$$

This is same as the one we get for the Minkowski vacuum. Likewise, in $4+1$ dimensions, in the Minkowski vacuum $G(x,x')$ is

$$G(x,x') = \frac{\pi(zz')^2}{4tt'}\int \lambda d\lambda d\alpha J_\nu(\lambda z)J_\nu(\lambda z')(H_{i\alpha}^{(2)}(\lambda t)H_{i\alpha}^{(1)}(\lambda t') - H_{i\alpha}^{(2)}(\lambda t')H_{i\alpha}^{(1)}(\lambda t))\frac{\alpha\sin\alpha\xi}{2\pi^2\sinh\xi},$$

and for conformal vacuum, the corresponding expression is

$$G(x,x') = \frac{\pi(zz')^2}{4tt'}\int \lambda d\lambda d\alpha J_\nu(\lambda z)J_\nu(\lambda z')(J_{-i\alpha}(\lambda t)J_{i\alpha}(\lambda t') - J_{i\alpha}(\lambda t)J_{-i\alpha}(\lambda t'))\frac{\alpha\sin\alpha\gamma}{2\pi^2\sinh\pi\alpha\sinh\gamma}.$$

Using (2.20), we immediately see that the last two expressions coincide. This is not surprising. $G(x,x')$, also known as the Pauli-Jordan function, arises from a commutator of the fields and is supposed to be a state-independent function.

We can also give a general check that the retarded propagators or the Pauli-Jordan functions are the same in both vacua. We start with

$$G^+(x,y) = \sum_i \phi_i^*(x)\phi_i(y),$$
$$G^-(x,y) = \sum_i \phi_i^*(y)\phi_i(x).$$

Here ϕ_i are modes calculated in one vacuum (e.g., Minkowski vacuum) and summation represents the integration over all continuous indices and the sum over the discrete indices. Using the Bogoliubov coefficients, we can express one set of modes ϕ_i in terms of another set $\tilde{\phi}$ as

$$\phi_i(y) = \sum_j (A_{ji}^*\tilde{\phi}_j(y) - B_{ji}\tilde{\phi}_j^*(y)),$$
$$\phi_i^*(x) = \sum_k (A_{ki}\tilde{\phi}_k^*(x) - B_{ki}^*\tilde{\phi}_k(x)), \quad (2.21)$$

where Bogoliubov coefficients satisfy [8]

$$\sum_k (A_{ik}A_{jk}^* - B_{ik}B_{jk}^*) = \delta_{ij},$$
$$\sum_k (A_{ik}B_{jk} - B_{ik}A_{jk}) = 0. \quad (2.22)$$

It is easy to check that Bogoliubov coefficients in our case indeed satisfy these identities.[3] Further, putting the expansion (2.21) in the Wightman functions, we get

$$G_+(x,y) = \sum_{i,j,k}(A_{ki}A_{ji}^*\tilde{\phi}_k^*(x)\tilde{\phi}_j(y) + B_{ji}B_{ki}^*\tilde{\phi}_j^*(y)\tilde{\phi}_k(x) - A_{ki}B_{ji}\tilde{\phi}_k^*(x)\tilde{\phi}_j^*(y) - B_{ki}^*A_{ji}^*\tilde{\phi}_k(x)\tilde{\phi}_j(y)),$$

and

$$G_-(x,y) = \sum_{i,j,k}(A_{ki}A_{ji}^*\tilde{\phi}_k^*(y)\tilde{\phi}_j(x) + B_{ji}B_{ki}^*\tilde{\phi}_j^*(x)\tilde{\phi}_k(y) - A_{ki}B_{ji}\tilde{\phi}_k^*(y)\tilde{\phi}_j^*(x) - B_{ki}^*A_{ji}^*\tilde{\phi}_k(y)\tilde{\phi}_j(x)).$$

Subtracting the above two expressions and using (2.22), we get the Pauli-Jordan function $G(x,y)$,

$$G(x,y) = \sum_i(\phi_i^*(x)\phi_i(y) - \phi_i^*(y)\phi_i(x)) = \sum_i(\tilde{\phi}_i^*(x)\tilde{\phi}_i(y) - \tilde{\phi}_i^*(y)\tilde{\phi}_i(x)).$$

[3] In $3+1$ dimensions, we see that the transformation equation is

$$\tilde{\phi}_{\lambda\alpha n} = \sum_{\lambda'\alpha' n'}(A_{\lambda\alpha n\lambda'\alpha' n'}\phi_{\lambda'\alpha' n'} + B_{\lambda\alpha n\lambda'\alpha' n'}\phi_{\lambda'\alpha' n'}^*).$$

Using (2.4) and (2.5) we see that the Bogoliubov coefficients are

$$A_{\lambda\alpha n\lambda'\alpha' n'} = \delta_{\lambda\lambda'}\delta_{nn'}\delta_{\alpha\alpha'}c(\alpha), \qquad B_{\lambda\alpha n\lambda'\alpha' n'} = \delta_{\lambda\lambda'}\delta_{n,-n'}\delta_{\alpha\alpha'}d(\alpha).$$

Using the relationship between Hankel and Bessel functions, along with $H_{i\alpha'}^{(2)} = e^{-\pi\alpha'}H_{-i\alpha'}^{(2)}$, we get

$$c(\alpha) = \frac{e^{\frac{\pi\alpha}{2}}}{\sqrt{2\sinh\pi\alpha}}, \qquad d(\alpha) = -\frac{e^{-\frac{\pi\alpha}{2}}}{\sqrt{2\sinh\pi\alpha}}.$$

Equipped now with these results, in the next section, we analyze scalar field theory on the AdS-Milne spacetime containing a conical defect. Here this defect runs all the way to the boundary. Our primary aim would be to find a boundary two point correlator of operators dual to the bulk field in a closed form.

D. AdS-Milne spacetime in other vacua

So far we have focused on the adiabatic and the conformal vacua. Recently, in [10], a more general class of vacua, similar to the alpha-vacua of the de Sitter spacetime, was considered. We end this section with a computation of the Wightman function in these vacua. Our result is similar to the one of [10], namely, the Wightman function picks up a dependence on the coordinates in a non-Poincare invariant manner.

To proceed, we start with the general solution of (2.3)

$$\phi_{\lambda\alpha n}(z,t,r,\theta) = [z^{\frac{3}{2}}J_\nu(\lambda z)]\left[\frac{C_{\lambda\alpha n}H^{(2)}_{i\alpha}(\lambda t) + \tilde{C}_{\lambda\alpha n}H^{(1)}_{i\alpha}(\lambda t')}{\sqrt{t}}\right] \times P^{-n}_{i\alpha-\frac{1}{2}}(\cosh r)e^{in\theta}. \tag{2.23}$$

The normalization condition (II A) for the above solution gives

$$|C_{\lambda\alpha n}|^2 e^{-\pi\alpha} - |\tilde{C}_{\lambda\alpha n}|^2 e^{\pi\alpha} = -\frac{\alpha\lambda}{8\pi}\left|\Gamma\left(n+\frac{1}{2}+i\alpha\right)\right|^2 \sinh\pi\alpha. \tag{2.24}$$

It can be checked that the choice

$$C_{\lambda\alpha n} = \frac{i}{2}\sqrt{\frac{\alpha\lambda}{4\pi}}e^{\pi\alpha}\Gamma\left(i\alpha+\frac{1}{2}+n\right),$$
$$\tilde{C}_{\lambda\alpha n} = \frac{i}{2}\sqrt{\frac{\alpha\lambda}{4\pi}}e^{-\pi\alpha}\Gamma\left(i\alpha+\frac{1}{2}+n\right) \tag{2.25}$$

gives the solution (2.18) in the conformal vacuum with the correct normalization factor. Related to the adiabatic modes is a two-parameter family of modes, called alpha modes. We can write the normalization constants of these in terms of a two-parameter family of constants labeled by ρ, σ as

$$C_{\lambda\alpha n} = i\cosh\rho\sqrt{\frac{\alpha\lambda\sinh\pi\alpha}{8\pi}}e^{\frac{\pi\alpha}{2}}\Gamma\left(i\alpha+\frac{1}{2}+n\right),$$
$$\tilde{C}_{\lambda\alpha n} = i\sinh\rho e^{i\sigma}\sqrt{\frac{\alpha\lambda\sinh\pi\alpha}{8\pi}}e^{-\frac{\pi\alpha}{2}}\Gamma\left(i\alpha+\frac{1}{2}+n\right)\sqrt{\sinh\pi\alpha}. \tag{2.26}$$

For ρ, σ set to zero, we get back the adiabatic modes (2.4) with the correct normalization constant. The above normalization constants satisfy the relation (2.24). Since ρ, σ are constants, independent of α, we can see that this family does not contain the conformal modes given by (2.25). We can calculate the Wightman function for these alpha modes by putting (2.23) in the relation (2.8) along with the normalization constants in (2.26) and using the appropriate alpha vacua states. The Wightman function for the field in the alpha vacua then comes out as

$$G_+(x,x') = -\frac{\cosh^2\rho Q^1_{\nu-1/2}(u_1)}{4\pi^2(u_1^2-1)^{\frac{1}{2}}} - \frac{\sinh^2\rho Q^1_{\nu-1/2}(u_2)}{4\pi^2(u_2^2-1)^{\frac{1}{2}}} + i\sinh\rho\cosh\rho\frac{e^{-i\sigma}Q^1_{\nu-1/2}(u_3)}{4\pi^2(u_3^2-1)^{\frac{1}{2}}} + \frac{i\sinh\rho\cosh\rho e^{i\sigma}Q^1_{\nu-1/2}(u_4)}{4\pi^2(u_4^2-1)^{\frac{1}{2}}},$$

where

$$u_1 = \frac{1}{2zz'}(-t^2-t'^2+2tt'\cosh\chi+z^2+z'^2+i\epsilon\,\mathrm{sgn}(t-t')),$$
$$u_2 = \frac{1}{2zz'}(-t^2-t'^2+2tt'\cosh\chi+z^2+z'^2-i\epsilon\,\mathrm{sgn}(t-t')),$$
$$u_3 = \frac{1}{2zz'}(-t^2-t'^2-2tt'\cosh\chi+z^2+z'^2+i\epsilon),$$
$$u_4 = \frac{1}{2zz'}(-t^2-t'^2-2tt'\cosh\chi+z^2+z'^2-i\epsilon)).$$

The first term is just $\cosh^2\rho$ times the Wightman function in the adiabatic vacuum (2.11). The second term is $\sinh^2\rho$ times the Wightman function in the adiabatic vacuum, but with $t\leftrightarrow t'$. The other two terms are not Poincare invariant and are analogous to the dependence on antipodal distance as discussed in [10].

III. ADS-MILNE SPACETIME WITH A CONICAL DEFECT

The AdS-Milne spacetime background in the presence of a conical defect has the same form as before except that the angular coordinate has a different periodicity. In $3+1$ dimensions, the metric is given by

$$ds^2 = \frac{1}{z^2}[dz^2 - dt^2 + t^2dr^2 + t^2\sin\mathrm{h}^2\mathrm{r}d\theta^2], \tag{3.1}$$

where $0\leq\theta<2\pi/q$. We will take $q>1$.

The above geometry may arise in the presence of an infinite string in a four-dimensional AdS-Milne spacetime. This can be seen by closely following the treatment of [22]. Consider the Nambu-Goto action

$$S_{NG} = -\mu\int d\sigma_0 d\sigma_1\sqrt{-detP[g_{ab}]},$$
$$P[g_{ab}] = g_{\mu\nu}(X)\frac{\partial X^\mu(\sigma)}{\partial\sigma^a}\frac{\partial X^\nu(\sigma)}{\partial\sigma^b}. \tag{3.2}$$

Here μ is the tension associated with the string. We choose a gauge such that $\sigma^0,\sigma^1 = t,\ z$ and consider the string extended along the z direction. Embedding coordinates will then be $X^\mu(t,z) = (t, z, r(t,z), \theta(t,z))$. In general, we can vary the Nambu-Goto action with respect to embedding coordinates to get the equations of motion. But we are interested in a particular solution corresponding to a string at the origin of the (r,θ) plane, represented by a delta-function source $\delta(r)$.

The corresponding Nambu-Goto action and its variation with respect to the metric is given by

$$S = -\frac{\mu}{\pi}\int dtdzdrd\theta\sqrt{-g_{tt}g_{zz}}\delta(r),$$
$$\delta S = \frac{\mu}{2\pi}\int dtdzdrd\theta\left(\frac{g_{zz}\delta g_{tt} + g_{tt}\delta g_{zz}}{\sqrt{-g_{tt}g_{zz}}}\right)\delta(r). \quad (3.3)$$

Comparing this with the general relation between metric variation and stress tensor

$$\delta S = \frac{1}{2}\int dtdzdrd\theta\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}, \quad (3.4)$$

we can get a read off the stress tensor corresponding to the string configuration. Equivalently, we can do a coordinate transformation of the result given in [22]. We get

$$T^{\mu\nu} = -\frac{\mu\delta(r)t}{\pi\sqrt{g_{rr}g_{\theta\theta}}}\begin{pmatrix} g_{tt}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{zz}^{-1}\end{pmatrix}. \quad (3.5)$$

Taking this as the stress tensor for the string, we solve the coupled Einstein-Hilbert and the Nambu-Goto action

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2}\right)$$
$$- \mu\int d\sigma_0 d\sigma_1\sqrt{-detP[g_{ab}]}. \quad (3.6)$$

Variation of the above leads to

$$R^{\mu\nu} - \frac{R}{2}g^{\mu\nu} - \frac{3}{L^2}g^{\mu\nu} = 8\pi G_4 T^{\mu\nu}. \quad (3.7)$$

Solving this system of equations we get the metric

$$ds^2 = \frac{1}{z^2}[dz^2 - dt^2 + t^2dr^2 + t^2\sinh^2 r\, d\theta^2], \quad (3.8)$$

with $0 \le t, r \le \infty$ and $0 \le \theta \le 2\pi(1 - 4\mu G_4)$. This is the same as (3.1) once we identify $q = (1 - 4\mu G_4)$.

We now turn to the scalar propagator on this geometry in the Minkowski vacuum. In the following we will primarily restrict q to be an integer. The scalar field now needs to satisfy $\phi(z,t,r,\theta) = \phi(z,t,r,\theta + 2\pi/q)$. Solving the equation of motion, we find

$$\phi(z,t,r,\theta) = \frac{i}{2}\sqrt{\frac{q\alpha\lambda\sinh\pi\alpha}{2\pi}}\Gamma\left(i\alpha + \frac{1}{2} + qn\right)e^{\pi\alpha/2}$$
$$\times[z^{\frac{3}{2}}J_\nu(\lambda z)]\left[\frac{H_{i\alpha}^{(2)}(\lambda t)}{\sqrt{t}}\right]P_{i\alpha-\frac{1}{2}}^{-qn}(\cosh r)e^{iqn\theta}.$$

Therefore, now the Wightman function takes the form

$$G_+^q(x,x') = \frac{q(zz')^{\frac{3}{2}}}{8\pi\sqrt{tt'}}\int\sum_{n=-\infty}^{\infty}\lambda\,d\lambda\,\alpha\,d\alpha\,e^{\pi\alpha}\sinh(\pi\alpha)J_\nu(\lambda z)$$
$$\times J_\nu(\lambda z')H_{i\alpha}^{(2)}(\lambda t)H_{-i\alpha}^{(1)}(\lambda t')$$
$$\times|\Gamma(i\alpha + 1/2 + qn)|^2 P_{i\alpha-1/2}^{-qn}(\cosh r)$$
$$\times P_{i\alpha-1/2}^{-qn}(\cosh r')e^{iqn(\theta-\theta')}.$$

Further, using

$$H_{i\alpha}^{(2)}(\lambda t)H_{-i\alpha}^{(1)}(\lambda t') = \frac{4}{\pi^2}e^{-\pi\alpha}K_{i\alpha}(i\lambda t)K_{i\alpha}(-i\lambda t'),$$

we can rewrite

$$G_+^q(x,x') = \frac{q(zz')^{\frac{3}{2}}}{2\pi^3\sqrt{tt'}}\int\sum_{n=-\infty}^{\infty}\lambda\,d\lambda\,\alpha\,d\alpha\,e^{\pi\alpha}\sinh(\pi\alpha)J_\nu(\lambda z)$$
$$\times J_\nu(\lambda z')K_{i\alpha}(i\lambda t)K_{i\alpha}(-i\lambda t')$$
$$\times|\Gamma(i\alpha + 1/2 + qn)|^2 P_{i\alpha-1/2}^{-qn}(\cosh r)$$
$$\times P_{i\alpha-1/2}^{-qn}(\cosh r')e^{iqn(\theta-\theta')}. \quad (3.9)$$

Using (A8), we can simplify (3.9) as

$$G_+^q(x,x') = \frac{q(zz')^{\frac{3}{2}}}{4\pi\sqrt{2\pi tt'}}\int_0^\infty\lambda d\lambda\sum_{n=-\infty}^{\infty}\int_0^\infty dx x^{-1/2}J_\nu(\lambda z)$$
$$\times J_\nu(\lambda z')e^{(\frac{t^2+t'^2}{2tt'}x - \frac{\lambda^2 tt'}{2x})}I_{|n|q}(x\sinh r\sinh r')$$
$$\times e^{-x\cosh r\cosh r'}e^{iqn(\theta-\theta')}. \quad (3.10)$$

Now the λ integral can be performed using (A9) to get, after a little algebra,[4]

[4]Some of our manipulations here are similar to that of [23].

$$G_+^q(x,x') = \frac{q(zz'/tt')^{\frac{3}{2}}}{4\pi\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{iqn(\theta-\theta')} \int_0^\infty dx\sqrt{x} I_{|n|q}(x\sinh r\sinh r') I_\nu\left(\frac{zz'x}{tt'}\right)$$
$$\times \exp\left[-\frac{x}{2tt'}(z^2+z'^2-t^2-t'^2+2tt'\cosh r\cosh r')\right]$$
$$= q\left(\frac{zz'}{4\pi}\right)^{\frac{3}{2}} \int_0^\infty \frac{ds}{s^4} e^{-\frac{\mathcal{V}^2}{4s^2}} S_q\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right) I_\nu\left(\frac{zz'}{2s^2}\right). \quad (3.11)$$

In arriving at the last equation, we have used $s^2 = \frac{tt'}{2x}$ and

$$S_q\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right) = \sum_{n=-\infty}^{\infty} e^{inq(\theta-\theta')} I_{|n|q}\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right)$$
$$= 2\sum_{n=0}^{\infty\,\prime} \cos[nq(\theta-\theta')] I_{nq}\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right) \quad (3.12)$$

$$= \frac{1}{q}\sum_{k=0}^{\infty} \exp\left[\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right)\cos\left(\theta-\theta'-\frac{2k\pi}{q}\right)\right]. \quad (3.13)$$

In (3.11),

$$\mathcal{V}^2 = z^2+z'^2-t^2-t'^2+2tt'\cosh r\cosh r'.$$

The prime on the sum in Eq. (3.12) means that the contribution from the $n=0$ term should be halved. Since we are working with integer values of q, the k sum will take integer values from 0 to $q-1$. We can now carry out the integration over s using (A10) to get the final expression of the Wightman function

$$G_+^q(x,x') = -\frac{1}{4\pi^2}\sum_{k=0}^{q-1} \frac{Q^1_{\nu-1/2}(u_k)}{\sqrt{u_k^2-1}}. \quad (3.14)$$

Here u_k is defined as

$$u_k = \frac{1}{2zz'}(z^2+z'^2-t^2-t'^2+2tt'\cosh\chi_{kq}), \quad (3.15)$$

and

$$\cosh\chi_{k,q} = \cosh r\cosh r' - \sinh r\sinh r'\cos(\theta-\theta'-2\pi k/q).$$

The expression for the Wightman function is reminiscent of the method of images. In the coincident limit, $G_+^q(x,x')$ diverges. This divergence comes from the $k=0$ term of the sum. We can define a renormalized function subtracting this contribution and write

$$G_{+R}^q(x,x') = -\frac{1}{4\pi^2}\sum_{k=1}^{q-1} \frac{Q^1_{\nu-1/2}(u_k)}{\sqrt{u_k^2-1}}. \quad (3.16)$$

A. Boundary correlator

The boundary correlator is constructed as before

$$\langle\Psi_q|\mathcal{O}(t,r,\theta)\mathcal{O}(t',r',\theta')|\Psi_q\rangle$$
$$= \lim_{z,z'\to 0}(zz')^{-(\nu+3/2)} G_+^q(x,x')$$
$$= \sum_{k=0}^{q-1} \frac{C_\Delta}{(-t^2-t'^2+2tt'\cosh\chi_{k,q})^\Delta}, \quad (3.17)$$

where C_Δ is a constant defined in (2.14).

The summation over k can be performed and the final result comes out as

$$\langle\Psi_q|\mathcal{O}(t,r,\theta)\mathcal{O}(t',r',\theta')|\Psi_q\rangle$$
$$= \frac{(-1)^{\Delta-1}C_\Delta}{\Gamma(\Delta)(2tt'\sinh r\sinh r')^\Delta}\left(\frac{\partial^{\Delta-1}}{\partial\gamma^{\Delta-1}}\right) I_1(q,\gamma,\theta-\theta'), \quad (3.18)$$

where

$$I_1(q,\gamma,\theta-\theta') = \left[\frac{q[(\gamma+\sqrt{\gamma^2-1})^{2q}-1]}{\sqrt{\gamma^2-1}[1+(\gamma+\sqrt{\gamma^2-1})^{2q}-2\cos q(\theta-\theta')(\gamma+\sqrt{\gamma^2-1})^q]}\right],$$

and

$$\gamma = \frac{-t^2 - t'^2 + 2tt' \cosh r \cosh r'}{2tt' \sinh r \sinh r'}.$$

The details are provided in the Appendix.

B. Twisted scalar

Having come thus far, we end this section with a study of the correlators involving twisted fields. These fields satisfy the same equations as the untwisted scalars but differ in their boundary conditions. The quasiperiodic boundary condition that the twisted scalars obey is given by

$$\phi\left(z, t, r, \theta + \frac{2\pi}{q}\right) = e^{-2\pi i\beta}\phi(z, t, r, \theta), \qquad (3.19)$$

where $0 \leq \beta \leq 1$. Such twisted scalar fields arise, for example, when we consider a charged scalar field in AdS in the presence of a cosmic string carrying internal magnetic flux. As is well known (see, for example [33] and references therein), the corresponding gauge field component can be eliminated by a gauge transformation and then one is left with a scalar with twisted boundary conditions. For earlier studies along this direction, see for example [17,34].

The normalized solution of the equation of motion is now

$$\phi(z,t,r,\theta) = C_{\lambda\alpha n} z^{\frac{3}{2}} J_\nu(\lambda z) \frac{H^2_{i\alpha}(\lambda t)}{\sqrt{t}} P^{q(n-\beta)}_{i\alpha-1/2}(\cosh r) e^{iq(n-\beta)\theta},$$

where

$$C_{\lambda\alpha n} = \frac{ie^{\pi\alpha/2}}{2}\sqrt{\frac{q\alpha\sinh(\pi\alpha)}{2\pi}}\Gamma[q(n-\beta) + i\alpha + 1/2].$$

The calculation is quite similar to the previous case of $\beta = 0$ and as before the Wightman function takes the form

$$W^q_\beta(x,x') = q\left(\frac{zz'}{4\pi}\right)^{\frac{3}{2}} \int_0^\infty \frac{ds}{s^4} e^{-\frac{\mathcal{V}^2}{4s^2}} S_{q\beta}\left(\frac{tt'\sinh r \sinh r'}{2s^2}\right) \times I_\nu\left(\frac{zz'}{2s^2}\right), \qquad (3.20)$$

where now

$$S_{q\beta}\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right) = \sum_{n=-\infty}^{\infty} e^{i(n-\beta)q(\theta-\theta')} I_{|n-\beta|q}\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right),$$

and

$$\mathcal{V}^2 = z^2 + z'^2 - t^2 - t'^2 + 2tt'\cosh r\cosh r'.$$

We can now use the following relation to carry out the s integration [35]:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} & e^{iq(n-\beta)(\theta-\theta')} I_{|n-\beta|q}\left(\frac{tt'\sinh r\sinh r'}{2s^2}\right) \\ &= \sum_n \left[\frac{1}{q} e^{[tt'\sinh r\sinh r'\cos(2\pi n/q - \theta + \theta')]/2s^2} e^{i\beta(2\pi n - q\theta + q\theta')} \right. \\ &\quad \left. - \frac{1}{2\pi i}\sum_{j=+,-} je^{ji\pi q\beta}\int_0^\infty dy e^{-tt'\sinh r\sinh r'\cosh y/2s^2} f(y)\right] \\ &\quad \times e^{-iq\beta(\theta-\theta')}, \end{aligned} \qquad (3.21)$$

where

$$f(y) = \frac{\cosh[qy(1-\beta)] - \cosh(q\beta y)e^{-iq(\theta-\theta'+j\pi)}}{\cosh(qy) - \cos[q(\theta - \theta' + j\pi)]}.$$

Here the sum over n runs as

$$-\frac{q}{2} + \frac{(\theta-\theta')q}{2\pi} \leq n \leq +\frac{q}{2} + \frac{(\theta-\theta')q}{2\pi}. \qquad (3.22)$$

Once we have substituted (3.21) into (3.20), we can integrate over s using

$$\int_0^\infty \frac{ds}{s^4} e^{-[\frac{\mathcal{V}^2}{2} - tt'\sinh r\sinh r'\cos(2\pi n/q - \theta + \theta')]/(2s^2)} I_\nu\left(\frac{zz'}{2s^2}\right) = -\frac{2}{(zz')^{3/2}\sqrt{\pi}\sqrt{u_n^2 - 1}} Q^1_{\nu-1/2}(u_n),$$

where u_n is defined as in (3.15), to finally arrive at

$$\begin{aligned} W^q_\beta(x,x') = &-\frac{1}{(2\pi)^2}\sum_n e^{-2i\pi\beta n}\frac{Q^1_{\nu-1/2}(u_n)}{\sqrt{u_n^2-1}} \\ &- \frac{iq}{(2\pi)^3} e^{-iq\beta(\theta-\theta')}\sum_{j=+,-} je^{ji\pi q\beta} \\ &\times \int_0^\infty dy f(y) \frac{Q^1_{\nu-1/2}(u_y)}{\sqrt{u_y^2-1}}, \end{aligned} \qquad (3.23)$$

where u_y has the same structure of (3.15) with $\cosh\chi_{kq}$ replaced by $\cosh y$. This is the general form of the Wightman function.

When q and $q\beta$ are both integers, the second term vanishes and we get

$$W^q_\beta(x,x') = -\frac{1}{(2\pi)^2}\sum_n e^{-2i\beta\pi n}\frac{Q^1_{\nu-1/2}(u_n)}{\sqrt{u_n^2-1}}. \qquad (3.24)$$

The twisted bulk scalar is dual to a twisted scalar primary $\mathcal{O}_\beta$ of dimension Δ with the same periodicity (3.19) along θ inherited from ϕ. The boundary correlator is therefore,

$$\langle\Psi_{\beta q}|\mathcal{O}_\beta(t,r,\theta)\mathcal{O}_\beta(t',r',\theta')|\Psi_{\beta q}\rangle = \sum_n \frac{C_\Delta e^{-2i\beta\pi n}}{(-t^2-t'^2+2tt'\cosh\chi_{n,q})^\Delta}, \qquad (3.25)$$

where the constant C_Δ has been defined earlier. This series can be summed to get

$$\langle\Psi_{\beta q}|\mathcal{O}_\beta(t,r,\theta)\mathcal{O}_\beta(t',r',\theta')|\Psi_{\beta q}\rangle = \frac{(-1)^{\Delta-1}C_\Delta}{\Gamma(\Delta)(2tt'\sinh r\sinh r')^\Delta}\left(\frac{\partial^{\Delta-1}}{\partial\gamma^{\Delta-1}}\right)J_\beta^q(\gamma,\theta-\theta'), \qquad (3.26)$$

where $J_\beta^q(\gamma,\theta-\theta') = J$ is given by

$$J = \frac{qe^{-iq\beta(\theta-\theta')}[(\gamma+\sqrt{\gamma^2-1})^{2q}-(\gamma+\sqrt{\gamma^2-1})^{q\beta}+2e^{iq(\theta-\theta')}(\gamma+\sqrt{\gamma^2-1})^q\sinh(q\beta\ln(\gamma+\sqrt{\gamma^2-1}))]}{\sqrt{\gamma^2-1}[1+(\gamma+\sqrt{\gamma^2-1})^{2q}-2\cos q(\theta-\theta')(\gamma+\sqrt{\gamma^2-1})^q]},$$

and

$$\gamma = \frac{-t^2-t'^2+2tt'\cosh r\cosh r'}{2tt'\sinh r\sinh r'}.$$

The details are provided in the Appendix.

IV. CONCLUSIONS

We have initiated a holographic study of field theory on the time-dependent background with a conical defect. In this work, our focus has been on the Milne spacetime to which, in the absence of a cosmological constant, any hyperbolic FRW metric flows to at late times. When the Milne vacuum is chosen to be the adiabatic one, we are able to compute the two point correlators of operators which are dual to the massive scalars in the bulk AdS-Milne spacetime background with defect. We find, for both twisted and untwisted operators, the correlators can be represented as the sum over images in the covering space. This sum can be carried out explicitly to write the results in compact forms. If we restrict ourselves to the adiabatic vacuum, our computations suggest that the field theory defined on a part of the boundary of AdS is dual to a subregion in the AdS bulk. Though it may not be entirely obvious, there are indications that the subregion duality may hold in general [36–38].

Evidently, our exploration is incomplete. First of all, Milne spacetime offers another natural vacuum known as the conformal vacuum. We have not been able get a closed form expression of correlators in this vacuum. Further, the renormalized stress tensors, which one calculates from the two point correlators, for the conformal vacuum are known to be nontrivial in Milne spacetime. This remains to be computed in the presence of the defect in this holographic framework.

In case of the AdS-Rindler foliation of AdS, interesting progress has been made in understanding the relationship between entanglement and spacetime [39]. Since AdS-Milne spacetime is another foliation of AdS, albeit, time dependent, it will be instructive to study the relationship between spacetime and entanglement structure for the time-dependent situation. Apart from these considerations, our results also connect with recent studies of conical defects in AdS_3, for example, [40]. We believe our results will also be useful in the study of entanglement entropy in time-dependent situations in a holographic context.

We are currently exploring some of these issues and hope to report on them in the near future.

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APPENDIX 5-DIMENSIONAL ADS-MILNE SPACETIME AND ADDITIONAL DETAILS

1. (4 + 1)-dimensional AdS-Milne spacetime

Since mode expansions of a massive scalar are sensitive to the spacetime dimensions, it is instructive to carry out a computation in $4+1$ dimensions. The metric is given in (2.15). The relevant solution of the equation of motion for the Minkowski vacuum turns out to be

$$\phi(z,t,r,\theta,\phi) = C_{\alpha lm}z^2 J_\nu(\lambda z)\frac{H^{(2)}_{i\alpha}(\lambda t)}{t}Y_{\alpha lm}(r,\theta,\phi),$$

where $\nu = \sqrt{4+m^2}$ and we have defined [41,42]

$$Y_{\alpha lm}(r,\theta,\phi) = \frac{\Gamma(i\alpha+l+1)}{\Gamma(i\alpha+1)}\frac{\alpha}{\sqrt{\sinh r}} \times P^{-l-1/2}_{i\alpha-1/2}(\cosh r)Y_{lm}(\theta,\phi).$$

The spherical harmonics $Y_{lmp}(\chi,\theta,\phi)$, for $0<p<\infty$, form a complete orthonormal set of square integrable functions on the unit hyperboloid [41,42]

$$\int_0^\infty d\chi\sinh^2\chi\int d\Omega Y_{lmp}(\chi,\Omega)Y^*_{l'm'p'}(\chi,\Omega) = \delta(p-p')\delta_{ll'}\delta_{mm'},$$

where the harmonic functions satisfy

$$\int_0^\infty d\chi \sinh\chi P^{-l-1/2}_{ip-1/2}(\cosh\chi)P^{-l'-1/2}_{ip'-1/2}(\cosh\chi) = \frac{|\Gamma(ip)|^2}{|\Gamma(ip+l+1)|^2}\delta(p-p').$$

The normalization constant $C_{\alpha lm}$ turns out to be

$$C_{\alpha lm} = i\frac{\sqrt{\pi\lambda}}{2}e^{\pi\alpha/2}.$$

Putting everything together, we therefore have

$$\phi(z,t,r,\theta,\phi) = i\int_0^\infty d\lambda d\alpha \sum_{l,m}\frac{\sqrt{\pi\lambda}}{2}e^{\pi\alpha/2}z^2 J_\nu(\lambda z)\frac{H^{(2)}_{i\alpha}(\lambda t)}{t} \times Y_{\alpha lm}(r,\theta,\phi). \tag{A1}$$

Consequently the Wightman function is

$$G_+(x,x') = -\sum_{l,m}\int_0^\infty d\lambda d\alpha \frac{\pi\lambda}{4}\frac{(zz')^2}{tt'}J_\nu(\lambda z)J_\nu(\lambda z')H^{(2)}_{i\alpha}(\lambda t) \times H^{(1)}_{i\alpha}(\lambda t')Y_{\alpha lm}(r,\theta,\phi)Y_{\alpha lm}(r',\theta',\phi').$$

Using the completeness of the spherical harmonics

$$\sum_{lm} Y_{\alpha lm}(\chi,\Omega)Y_{\alpha lm}(\chi',\Omega') = \frac{\alpha\sin\alpha\gamma}{2\pi^2\sinh\gamma}, \tag{A2}$$

we can write the Wightman function as

$$G_+(x,x') = \frac{\pi(zz')^2}{4tt'}\int \lambda d\lambda d\alpha J_\nu(\lambda z)J_\nu(\lambda z')H^{(2)}_{i\alpha}(\lambda t) \times H^{(1)}_{i\alpha}(\lambda t')\frac{\alpha\sin\alpha\gamma}{2\pi^2\sinh\gamma},$$

where

$$\cosh\gamma = \cosh\chi\cosh\chi' - \sinh\chi\sinh\chi'\cos\omega, \\ \cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{A3}$$

Further, performing the α integral, we get

$$G_+(x,x') = -\frac{i(zz')^2}{4\pi^2}\int_0^\infty \lambda^2 d\lambda J_\nu(\lambda z)J_\nu(\lambda z') \times \frac{K_1(\lambda\sqrt{-t^2-t'^2+2tt'\cosh\gamma})}{\sqrt{t^2+t'^2-2tt'\cosh\gamma}}.$$

Finally, after completing the λ integral we reach (2.16).

2. Deriving (3.18) and (3.26)

We start with the formula [43]

$$\frac{1-p^2}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} = 1+2\sum_{m=1}^\infty p^m\cos\left[m\left(\Delta\theta-\frac{2\pi k}{q}\right)\right].$$

From the above, it follows that

$$\sum_{k=0}^{q-1}\frac{1-p^2}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} = q+2\sum_{k=0}^{q-1}\sum_{m=1}^\infty p^m\cos\left[m\left(\Delta\theta-\frac{2\pi k}{q}\right)\right].$$

Now since

$$\sum_{m=-\infty,\neq 0}^\infty p^{|m|}e^{im(\Delta\theta-2\pi k/q)} = 2\sum_{m=1}^\infty p^m\cos\left[m\left(\Delta\theta-\frac{2\pi k}{q}\right)\right],$$

we get

$$\sum_{k=0}^{q-1}\frac{1-p^2}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} = q+\sum_{k=0}^{q-1}\sum_{m=-\infty,\neq 0}^\infty p^{|m|}e^{im\Delta\theta}e^{-\frac{2i\pi mk}{q}}.$$

Further, using

$$\sum_{k=0}^{q-1}e^{-\frac{2i\pi mk}{q}} = q\sum_{n=-\infty}^\infty \delta_{-m,nq},$$

we arrive at

$$\begin{aligned}\sum_{k=0}^{q-1}\frac{1-p^2}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} &= q+q\sum_{n=-\infty}^\infty\sum_{m=-\infty,\neq 0}^\infty p^{|m|}e^{im\Delta\theta}\delta_{-m,nq} \\ &= q+q\sum_{n=-\infty,\neq 0}^\infty p^{q|n|}e^{-inq\Delta\theta} \\ &= q+2q\sum_{n=1}^\infty p^{qn}\cos(qn\Delta\theta) \\ &= q+q\left[\frac{1-p^{2q}}{1-2p^q\cos(q\Delta\theta)+p^{2q}}-1\right] \\ &= \frac{q(1-p^{2q})}{1-2p^q\cos(q\Delta\theta)+p^{2q}}.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{q-1}\frac{1}{\frac{1+p^2}{2p}-\cos(\Delta\theta-\frac{2\pi k}{q})} =\frac{2qp(1-p^{2q})}{(1-p^2)(1-2p^q\cos(q\Delta\theta)+p^{2q})}.$$

Taking $\frac{1+p^2}{2p}=\gamma$, we get (3.18).

Similarly, we can sum the series for the twisted scalar to get (3.26). Since $\beta=0$, 1 cases reduce to an untwisted scalar case, we will restrict to $0<\beta<1$. Again, we start with

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}(1-p^2)}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} =\sum_{k=0}^{q-1}e^{-2\pi i\beta k}+2\sum_{k=0}^{q-1}\sum_{m=1}^{\infty}e^{-2\pi i\beta k}p^m\cos\left[m\left(\Delta\theta-\frac{2\pi k}{q}\right)\right].$$

The first sum on the right-hand side vanishes for the range of β that we are considering while the second term can be evaluated using the same methods as before to get

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}(1-p^2)}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} =q\sum_{n=-\infty,\neq-\beta}^{\infty}p^{q|-n-\beta|}e^{-i(nq+q\beta)\Delta\theta}.$$

Since n takes integer values while β is not an integer, there is no restriction on the sum. Expanding the sum, we get

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}(1-p^2)}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} =qp^{q\beta}e^{-i(q\beta)\Delta\theta}+q\sum_{n=-\infty}^{1}p^{q|-n-\beta|}e^{-i(nq+q\beta)\Delta\theta} +q\sum_{n=1}^{\infty}p^{q|-n-\beta|}e^{-i(nq+q\beta)\Delta\theta}$$

Relabeling $n\to-n$ in the second term, we get

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}(1-p^2)}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} =qp^{q\beta}e^{-iq\beta\Delta\theta}+qe^{-iq\beta\Delta\theta}\left(p^{-q\beta}\sum_{n=1}^{\infty}p^{qn}e^{inq\Delta\theta} +p^{q\beta}\sum_{n=1}^{\infty}p^{qn}e^{-inq\Delta\theta}\right).$$

Now we can do the final sums using the formula

$$\sum_{n=0}^{\infty}s^ne^{inx}=\frac{1-se^{-ix}}{1-2s\cos x+s^2},$$

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}(1-p^2)}{1-2p\cos(\Delta\theta-\frac{2\pi k}{q})+p^2} =qp^{q\beta}e^{-iq\beta\Delta\theta} +qp^{-q\beta}e^{-iq\beta\Delta\theta}\left(\frac{1-p^qe^{-iq\Delta\theta}}{1-2p^q\cos(q\Delta\theta)+p^{2q}}-1\right) +qp^{q\beta}e^{-iq\beta\Delta\theta}\left(\frac{1-p^qe^{iq\Delta\theta}}{1-2p^q\cos(q\Delta\theta)+p^{2q}}-1\right).$$

Simplifying we finally get

$$\sum_{k=0}^{q-1}\frac{e^{-2\pi i\beta k}}{\frac{1+p^2}{2p}-\cos(\Delta\theta-\frac{2\pi k}{q})} =\frac{2qpe^{-iq\beta\Delta\theta}(p^{q\beta}-p^{2q}-2p^qe^{iq\Delta\theta}\sinh(q\beta\ln p))}{(1-p^2)(1-2p^q\cos(q\Delta\theta)+p^{2q})}.$$

Taking $\frac{1+p^2}{2p}=\gamma$, we get (3.26).

3. Identities used in the main text

Here we collect a number of identities used in the main text.

First, to get the normailzation of $C_{\lambda\alpha n}$, we have used the following properties of $P^{-n}_{i\alpha-1/2}(x)$ [44]:

$$P^m_{iz-1/2}(u)=\frac{\Gamma[-iz+m+1/2]}{\Gamma[-iz-m+1/2]}P^{-m}_{iz-1/2}(u) =\frac{\Gamma[iz+m+1/2]}{\Gamma[iz-m+1/2]}P^{-m}_{iz-1/2}(u)$$

and

$$\int_1^{\infty}duP^{-m}_{iz'-1/2}(u)P^{-m}_{iz-1/2}(u) =\frac{\pi}{z\sinh(\pi z)}\frac{\delta(z-z')}{|\Gamma(m+1/2+iz)|^2}. \tag{A4}$$

To get to (2.9), we have used

$$H^{(2)*}_{i\alpha}(x)=H^{(1)}_{-i\alpha},H^{(2)}_{i\alpha}(\lambda t)H^{(1)}_{-i\alpha}(\lambda t') =\frac{4}{\pi^2}e^{-\pi\alpha}K_{i\alpha}(i\lambda t)K_{-i\alpha}(-i\lambda t'),$$

and

$$2\sum_{n=1}^{\infty}(-)^n\cos n(\theta-\theta')P^{-n}_{i\alpha-1/2}(\cosh r)P^{n}_{i\alpha-1/2}(\cosh r') +P_{i\alpha-1/2}(\cosh r)P_{i\alpha-1/2}(\cosh r')=P_{i\alpha-1/2}(\cosh\chi). \tag{A5}$$

To reach to Eq. (2.11), we used [43]

$$\int_0^\infty d\alpha\alpha\sinh(\pi\alpha)\frac{\pi}{\cosh\pi\alpha}K_{i\alpha}(a)K_{i\alpha}(b)P_{i\alpha-\frac{1}{2}}(\cosh\chi) = \sqrt{\frac{\pi}{2}}\left(\frac{ab}{\sqrt{a^2+b^2+2ab\cosh\chi}}\right)^{\frac{1}{2}} \times K_{\frac{1}{2}}\left(\sqrt{a^2+b^2+2ab\cosh\chi}\right). \quad \text{(A6)}$$

This identity is valid when $|\arg(a)|<\pi/2$ and so we replace t with $te^{-i\epsilon}$, with limit $\epsilon\to 0$.

Equation (2.14) uses the identity

$$\int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} bt}dt = \frac{4^{x-1}}{b}B(x+y/b, x-y/b). \quad \text{(A7)}$$

Here $B(x,y)$ represents the beta function.

We used the relation [45]

$$|\Gamma(i\alpha+nq+1/2)|^2 P^{-qn}_{i\alpha-1/2}(\cosh r)P^{-qn}_{i\alpha-1/2}(\cosh r') = \sqrt{\frac{2}{\pi}}\int_0^\infty dx x^{-1/2}I_{|n|q}(x\sinh r\sinh r')K_{i\alpha}(x)e^{-x\cosh r\cosh r'},$$

where $I_\nu(x)$ is the modified Bessel function and [46]

$$\int_0^\infty x\sinh(\pi x)K_{ix}(a)K_{ix}(b)K_{ix}(y)dx = \frac{\pi^2}{4}e^{-\frac{y}{2}\left(\frac{a}{b}+\frac{b}{a}+\frac{ab}{y^2}\right)} \quad \text{(A8)}$$

to get (3.10).

Using [43]

$$\int_0^\infty e^{-\alpha x}J_\nu(2\beta\sqrt{x})J_\nu(2\gamma\sqrt{x})dx = \frac{1}{\alpha}I_\nu\left(\frac{2\beta\gamma}{\alpha}\right)e^{-\left(\frac{\beta^2+\gamma^2}{\alpha}\right)}, \quad \text{(A9)}$$

we can perform the λ integral in (3.10) to get (3.11).

Using the known result [43]

$$\int_0^\infty e^{-xu}I_\nu(x)x^{\mu-1}dx = \sqrt{\frac{2}{\pi}}\frac{e^{-i(\mu-1/2)\pi}}{\sqrt{u^2-1}}Q^{\mu-1/2}_{\nu-1/2}(u), \quad \text{(A10)}$$

we perform the integral over s to go from (3.11))–(3.14).

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