Pseudocomplexity of purification for free scalar field theories

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We compute the pseudocomplexity of purification corresponding to the reduced transition matrices for free scalar field theories with an arbitrary dynamical exponent. We plot the behavior of complexity with various parameters of the theory under study and compare it with the complexity of purification of the reduced density matrices of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ that constitute the transition matrix. We first find the transition matrix by reducing to a small number (1 and 2) of degrees of freedom in lattice from a lattice system with many lattice points and then purify it by doubling the degrees of freedom (2 and 4 respectively) for this reduced system. This is a primary step towards the natural extension to the idea of the complexity of purification for reduced density matrices relevant for the studies related to postselection.

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I. INTRODUCTION

Quantum entanglement continues to attract attention across multiple disciplines. Recently, a generalization of entanglement entropy called pseudo entropy was proposed in [1,2]. There, instead of starting from a usual density matrix, the authors define a matrix using two pure quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\tau^{1|2} = \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle}, \tag{1.1}$$

which is dubbed the transition matrix. The interest in this is twofold. First, in the case of postselection in quantum experiments, the transition matrix becomes important once one specifies the initial state as $|\psi_1\rangle$ and the final state as $|\psi_2\rangle$, given that they are not orthogonal to each other. In the postselection experiment, the transition matrix then plays the role of the density matrix while computing the weak expectation value $\langle\mathcal{O}\rangle = \mathrm{Tr}(\mathcal{O}\tau^{1|2})$ of an observable $\mathcal{O}$. The second motivation comes from the AdS/CFT correspondence [3,4], where pseudo entropy is proposed to be the dual CFT quantity to the area of a minimal hypersurface in Euclidean AdS spacetime. In holography, this is constructed in the path integral technique by dividing a Euclidean time slice into two regions corresponding to the two states. Dividing the entire system into A and its complement A^C, the pseudo entropy of A (relative to A^C) is defined as

$$S(\tau^{1|2}) = -\mathrm{Tr}_A[\tau_A^{1|2}\log\tau_A^{1|2}], \tag{1.2}$$

where $\tau_A^{1|2} = \mathrm{Tr}_{A^C}(\tau^{1|2})$ is called the reduced transition matrix, in analogy with existing literature on entanglement entropy derived from the reduced density matrix. When the two states are the same, the transition matrix reduces to the usual density matrix, and pseudo entropy reduces to usual entanglement entropy.

Quantum circuit complexity is another quantum information (QI) theoretic quantity that has been at the center of interest for the last few years. Recently, circuit complexity has been explored in the context of quantum field theory [5–25].[1] Complexity is usually defined as the number of elementary structural components needed to construct either an evolved state or operator from a simple initial state or operator. Here, we try to compute the circuit complexity of the reduced transition matrix. This reduced transition matrix is an analogue of a mixed state corresponding to a reduced density matrix, although it can be non-Hermitian, in general. The methods that we use for

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[1]This list is by no means exhaustive. Interested readers are referred to the reviews [26,27] and references therein for more details.

computing circuit complexity are those for the complexity of purification [28–34].

First, given a transition matrix, which is written in density matrix form, a state can be associated with this operator by writing the transition matrix in the vectorized form. Then, we compute the circuit complexity for this state using Nielsen's method [35–37]. This vectorization means changing the bra part of the transition matrix to a ket part, which amounts to a doubling of the Hilbert space [38–40]. Hence, the corresponding transition state looks like

$$\psi_{1|2} = \frac{|\psi_1\rangle|\psi_2\rangle}{\langle\psi_1|\psi_2\rangle}, \tag{1.3}$$

which, as mentioned in [2], for a transition matrix, turns the corresponding state into a tensor product of the two states. Then, the corresponding complexity, following the approach of [5], of the state dual to the transition matrix is

$$C(\tau^{1|2}) = C(|\psi_1\rangle) \oplus C(|\psi_2\rangle). \tag{1.4}$$

Note that the complexity defined using canonical purification in (1.4) is simply the direct sum of two complexities of two individual states $|\psi_{1,2}\rangle$. The direct sum is representative of the fact that the vectorized version is a tensor product. It is quite unlikely that this will have any nontrivial holographic interpretation, as typically holographic duals involve some sort of minimization. Motivated by this, we want to define some notion of pseudocomplexity of purification $[C(\tau_A^{1|2})]$ in this paper, extending the notion of complexity of purification [29–34] for this case. We first purify the reduced transition matrix and compute the minimal complexity among all possible purifications.[2] We do these computations for free scalar field theories and Lifshitz field theories in $(1+1)$ dimensions.

Physically, pseudocomplexity is related to the idea of postselection as suggested in [1,2,41] while defining pseudo entropy. The idea is that if, along some physical process, one starts with a state $|\psi_1\rangle$ and somehow the final state is postselected to be $|\psi_2\rangle$, which is *not* necessarily a result of a simple unitary evolution with a simple Hamiltonian, the process can still be approximated as a result of an operator $|\psi_2\rangle\langle\psi_1|$ operating on the initial state. In general, $|\psi_2\rangle$ might be a result of many unitary and nonunitary steps along the way.[3] Hence, this operator $|\psi_2\rangle\langle\psi_1|$ might very well be a non-Hermitian operator. This operator is considered to be a transition matrix, similar to a density matrix in this treatment. The reason is that this operator does not have any nontrivial information about the system other than the outer product of the initial and the postselected state, similar to a density matrix. Then, the reduced transition matrix, defined after a reduction in degrees of freedom, can associate a notion of entropy similar to the entanglement entropy of a reduced density matrix. Similarly, the pseudocomplexity should measure the complexity of this operator, whereas the pseudocomplexity of purification should measure the complexity of forming such a reduced transition matrix once the two states $|\psi_{1,2}\rangle$ are given. Another way of looking at this problem is to relate it to the averaged number of maximally entangled pairs of qubits to be distilled from the intermediate state once the final state is postselected [1]. While pseudo entropy measures this number, pseudocomplexity of purification measures the amount of work (of course, in terms of resources or gates) needed for this distillation starting from the reduced transition matrix.

The rest of this article is organized as follows: In Sec. II we first review a path-integral calculation of Gaussian transition matrix following [41] and propose a definition of pseudocomplexity of purification. We then use our new definition and calculate this quantity in a simple setting of coupled harmonic oscillators on a lattice. The results of our numerical analyses are collected in Sec. III. We conclude in Sec. IV with discussions and an outlook.

II. PSEUDOCOMPLEXITY OF PURIFICATION

The concept of purification originates from quantum information theory (considered to be a close cousin of Schmidt decomposition as a process). The process of purification refers to constructing a pure state from a mixed reduced density (transition) matrix such that if one traces out the auxiliary degrees of freedom added while constructing the purified state, one gets back the original reduced density (transition) matrix. While both the actual and purified states generate the same reduced density matrix, the purified state only has important information about the density matrix because the extra parameters added in the process of purification are random. The only constraint on them is that the purified state should follow the standard quantum mechanical postulates and properties of a pure state. As mentioned already, we consider random purification by all possible values of parameters for which the purified state can be consistently formed. However, we choose a specific purification among them by minimizing the complexity functional [30]. This is motivated by the definition of complexity, where minimal resources play an important part. In terms of quantum gates, this means choosing the minimum number of gates so that the resources needed are minimized. Now we describe our main setup.

[2]The state in (1.3) can be obtained by using "operator-state mapping," a particular example of purification, namely, "canonical purification."

[3]By nonunitary, we mean that there might be measurements made along an evolution due to which the state might collapse to a different state, which results in an overall nonunitarity. It is worth noting that the process of purification by itself is a nonunitary one, which, however, is different from the nonunitarity caused by the postselection.

A. Setup

We consider a system of scalar fields which is governed by the Hamiltonian

$$H = \frac{1}{2}\int dx[\pi^2 + (\partial_x^z)^2 + m^{2z}\phi^2]. \tag{2.1}$$

This theory exhibits an anisotropic (Lifshitz) scaling symmetry $t \to \lambda^z t$, $x \to \lambda x$ in the $m \to 0$ limit; for $z = 1$, this is an ordinary relativistic scalar field theory. For practical purposes, it is useful to consider the discretized Hamiltonian

$$H = \sum_{i=1}^{N}\left[\frac{\pi_i^2}{2} + \frac{m^{2z}}{2}\phi_n^2 + \frac{1}{2}\left(\sum_{k=0}^{z}(-1)^{z+k}\binom{z}{k}\phi_{i-1+k}\right)^2\right], \tag{2.2}$$

with N being the total number of points in the lattice.

We consider a collection of coupled linear harmonic oscillators on a lattice of space points, labeled by capital Latin indices. Any two Gaussian states $|\psi_\alpha\rangle$ ($\alpha = 1, 2$) of this system can be generically expressed in their position representation as

$$\langle q_A|\psi_\alpha\rangle = N^{(\alpha)}\exp\left[-\frac{1}{2}q_A W_{AB}^{(\alpha)} q_B\right], \tag{2.3}$$

where q_A denotes displacement of the Ath oscillator, $W_{AB}^{(\alpha)}$ is a positive-definite matrix, and N^α is the normalization constant,

$$N^{(\alpha)} = \left(\det\left(\frac{W_{AB}^{(\alpha)}}{\pi}\right)\right)^{\frac{1}{4}}.$$

We divide the entire system into a part A and its complement A^C; the lattice points within the subsystem Ω are labeled by lowercase Latin letters, while lowercase Greek indices label those outside it. We adopt the notation

$$W_{AB}^{(\alpha)} = \begin{pmatrix} W_{ab}^{(\alpha)} & W_{a\beta}^{(\alpha)} \\ W_{\alpha b}^{(\alpha)} & W_{\alpha\beta}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} A^{(\alpha)} & B^{(\alpha)} \\ B^{(\alpha)T} & C^{(\alpha)} \end{pmatrix}, \tag{2.4}$$

to denote submatrices of $W_{AB}^{(\alpha)}$. The reduced transition matrix for the subsystem Ω in this representation is found by integrating over the rest,

$$\langle q_a|\mathrm{Tr}_{A^C}(|\psi_1\rangle\langle\psi_2|)|q_b\rangle = \int dq_\alpha \langle q_a, q_\alpha|\psi_1\rangle\langle\psi_2|q_b, q_\alpha\rangle. \tag{2.5}$$

This is a Gaussian integral. Finally, the matrix elements of the reduced transition matrix can be expressed in a simple form [2]:

$$\langle q^{(1)}|\tau_A^{1|2}|q^{(2)}\rangle = \frac{N'}{\sqrt{\det\frac{\bar{C}}{\pi}}}\times\exp\left[-\frac{1}{2}\begin{pmatrix} q^{(1)T} & q^{(2)T}\end{pmatrix}M\begin{pmatrix} q^{(1)} \\ q^{(2)}\end{pmatrix}\right], \tag{2.6}$$

where

$$M = \begin{pmatrix} X^{(1)} & 2Y \\ 2Y^T & X^{(2)}\end{pmatrix} \tag{2.7}$$

and

$$\begin{aligned} X^{(\alpha)} &= A^{(\alpha)} - \frac{1}{2}B^{(\alpha)}\bar{C}^{-1}B^{(\alpha)T}, \\ Y &= -\frac{1}{4}B^{(1)}\bar{C}^{-1}B^{(2)T}, \\ \bar{C} &= \frac{1}{2}(C^{(1)} + C^{(2)}), \\ N' &= \sqrt{\det\frac{\bar{W}}{\pi}}, \\ \bar{W} &= \frac{1}{2}(W^{(1)} + W^{(2)}). \end{aligned} \tag{2.8}$$

B. Auxiliary parameters and purification

We use Nielsen's method for the computation of circuit complexity [5,35–37] and choose the F^1 norm for the complexity functional [5,42]. For a pure (or purified) state, this is written in terms of the (unentangled) reference state frequency and the frequency of the normalized version of the coupled oscillator system in the lattice pictures. We start from various numbers of initial oscillators and reduce to transition matrices after tracing out all but one or two of the oscillators. Afterwards, we purify this reduced transition matrix $\tau_A^{1|2}$ by adding parameters corresponding to adding one or two more oscillators (similar to the doubling of degrees of freedom in qubit purifications). However, we do not choose canonical purification since it always returns the original transition state as mentioned in (1.3), the complexity of which is trivial, as explained before.[4] For the reduced transition matrix, the parameters that arise while matching the original reduced transition matrix with the reduced density matrix of the purified state take care of all possible purifications. In this optimal picture of purification, the number of unknown arbitrary parameters is much higher than the number of unknowns needed to purify ρ_1^A or ρ_2^A. This is because the information of not one but two pure states goes into the transition matrix and hence the reduced transition matrix.

[4]As mentioned in (1.4), for canonical purification the complexity of the purified state is simply the sum of the complexity of $|\psi_1\rangle$ and $|\psi_2\rangle$.

In order to define pseudocomplexity, we begin by introducing purification of the transition matrix $\tau_A^{1|2}$. We consider a (fictional) auxiliary system $\tilde{A}$ whose Hilbert space is of the same dimension as our original subsystem A, and we consider two Gaussian states in the enlarged Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{\tilde{A}}$,

$$\langle q_A, q_{\tilde{A}}|\Psi_\alpha\rangle = N_{A\tilde{A}} \exp\left[-\frac{1}{2}\begin{pmatrix} q_A^T & q_{\tilde{A}}^T \end{pmatrix}\begin{pmatrix} J^{(\alpha)} & K^{(\alpha)} \\ K^{(\alpha)T} & L^{(\alpha)} \end{pmatrix}\begin{pmatrix} q_A \\ q_{\tilde{A}} \end{pmatrix}\right] \quad (\alpha = 1, 2). \tag{2.9}$$

Then, proceeding exactly as above, we calculate a transition matrix in the enlarged Hilbert space. This particular choice of Gaussian purification is primarily motivated by the study done in [43] in the context of entanglement of purification. This simplifies the numerical analysis as well as the computation of circuit complexity. We consider that this is a purification of our original reduced transition matrix (2.6) for the subsystem A; in this case, the following constraints must be obeyed:

$$J^{(\alpha)} - \frac{1}{2}K^{(\alpha)}\bar{L}^{-1}K^{(\alpha)T} = A^{(\alpha)} - \frac{1}{2}B^{(\alpha)}\bar{C}^{-1}B^{(\alpha)T}, \tag{2.10}$$

$$K^{(1)}\bar{L}^{-1}K^{(2)T} = B^{(1)}\bar{C}^{-1}B^{(2)T}, \tag{2.11}$$

where $\bar{L} = \frac{1}{2}(L^{(1)} + L^{(2)})$. In our numerical analysis, we adopt the following conditions to satisfy these constraints:

$$J^{(\alpha)} = A^{(\alpha)}, \tag{2.12a}$$

$$\bar{L} = (K^{(2)-1}B^{(2)}\bar{C}^{-1}(K^{(2)}B^{(2)})^T)^{-1}, \tag{2.12b}$$

$$K^{(2)} = (\bar{L}K^{(1)-1}B^{(1)}\bar{C}^{-1}B^{(2)T})^T. \tag{2.12c}$$

This particular choice leaves only $K^{(1)}$ and $L^{(1)}$ undetermined.

For the particular model of our choice, the matrices $W_{AB}^{(\alpha)}$ are given by

$$W_{AB}^{(\alpha)} = \frac{1}{N}\sum_{C=1}^{N}\sqrt{m_\alpha^{2z_\alpha} + \left(2\left(1 - \cos\frac{2\pi n}{N}\right)\right)^{z_\alpha}} \times \exp\left(\frac{2\pi i C(A-B)}{N}\right) \quad (\alpha = 1, 2), \tag{2.13}$$

where N is the total number of lattice points. Here, m_α and z_α are the mass and dynamical exponent of the respective theory. The matching of $J, K, L^{(\alpha)}$ matrices with their corresponding counterparts introduces a few arbitrary parameters in the purified state. These parameters denote the infinitely many possible Gaussian purifications of the reduced transition matrix. Afterwards, we define the complexity of purification as the complexity of the specific purification for which the complexity functional is minimized in terms of all the parameters of the purified version of the reduced transition matrix. This involves purification parameters for both $|\psi_{1,2}\rangle$ states, constituting the transition matrix. Finally, we compare the complexity of purification of the reduced transition matrix $C(\tau_A^{(1|2)})$ with the complexity of purifications for the individual reduced density matrices (ρ_A^1 and ρ_A^2) derived from the initial and final states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively.

We propose that the pseudocomplexity of purification is given by

$$\mathcal{C}_P = \min_{\tilde{A}} \frac{1}{2}\sqrt{\sum_{j=1}^{N}\sum_{\alpha=1}^{2}\log\left(\frac{\Omega_j^{(\alpha)}}{\omega_2}\right)}, \tag{2.14}$$

where $\Omega_j^{(\alpha)}$ denotes the jth eigenvalue of the matrix $\left(\begin{smallmatrix} J^{(\alpha)} & K^{(\alpha)} \\ K^{(\alpha)T} & L^{(\alpha)} \end{smallmatrix}\right)$, and the functional is minimized over the parameters associated with the auxiliary system $\tilde{A}$. This means choosing a particular purification associated with minimizing parameter values.

Let us point out a few attributes of our method of computation. Most of our numerical analyses are performed for the simplest case of a reduced system made of only one oscillator, where the above functional is to be minimized over two unknown parameters. In general, when the reduced system consists of n number of linear harmonic oscillators, the J, K, and L matrices are of dimension $n \times n$. The choice II B always leaves two of them undetermined, and therefore we end up with a total of $2n^2$ unknown parameters. On the other hand, to calculate the more well-known complexity of purification from a reduced density matrix, the number of unknown parameter values over which one is required to perform a minimization scales as n^2 with the reduced system size. It is easy to conclude that the calculation of the new functional (2.14) is at least twice as hard since the transition matrices have information of both the initial and the final state.

However, there exists a way to purify the system mode by mode [30]. Thus, it might be possible to reduce the number of unknown parameters to $2n$. Even then, the problem is twice as hard as computing complexity of purification from density matrices, where the associated number scales as n.

It may also be of relevance to note that any purification we consider takes into account only Gaussian states, and the auxiliary system always has the same dimension as the original one, motivated by the usual doubling of Hilbert space typically done in purification using Schmidt decomposition. These are restrictions we invoke in order to keep the numerical job simple. There may exist ways to relax the conditions.

III. RESULTS AND DISCUSSION

A. Pseudocomplexity of purification

In this subsection, we report our numerical results after the minimization with different variables of the theory. In the primary Hamiltonian (2.2), we choose a general version which applies to both the ordinary free scalar QFTs ($z_1 = z_2 = 1$) and Lifshitz scalar field theories ($z_1 \neq 1 \neq z_2$). The variables considered are the dynamical exponents $z_{1,2}$, the masses $m_{1,2}$, the number of oscillators before tracing out, the number of oscillators of the reduced system, and the frequency of the unentangled reference state.

(1) *Varying z_2 and m_2:* From the plots of pseudocomplexity of purification against reference frequency in Figs. 1(a) and 1(b), we notice that as the reference frequency is increased, the pseudocomplexity of purification primarily decreases and then shows a polynomial growth and saturation. When the scaling factor z_2 is increased in the same plots, the pseudocomplexity of purification for higher values of reference frequency decreases for higher z_2. However, the behavior is opposite for small reference frequency values. Hence, there is a crossover in complexity after which the small to large behavior changes. A similar change is observed from Fig. 1(b), where the mass parameter m_2 is varied. We find that at large frequencies, the pseudocomplexity of purification decreases as the mass is increased, whereas, at very small frequencies, the behavior is again opposite.

(2) *Pseudocomplexity of purification vs z_2 plots:* From Fig. 2(a) we observe that as z_2 is increased, the pseudocomplexity of purification primarily decreases and then grows linearly. By looking at multiple plots for various values of z_1, we find that with increasing z_1 as z_2 is varied, the pseudocomplexity of purification decreases. In this case, there is no change of behavior or crossover between different plots, which we saw earlier.

(3) *Pseudocomplexity of purification vs m_2 plots:* As shown in Fig. 2(b), we find that one of the mass parameters is varied from zero to one, and the pseudocomplexity of purification does not change much. Although the plots are not completely parallel to the x axis, their variance is relatively small compared to what we have observed for other parameters. On the other hand, if we vary one of the scaling parameters z_2, we find that with increasing values of z_2, pseudocomplexity of purification increases.

(4) *Varying number of oscillators before tracing out:* All the above plots are given for the case where the total number of oscillators associated with the states $\psi_{1,2}$ before tracing out degrees of freedom is 2. So we trace out one of the two oscillators in each of the states. This begs the question of how the results might possibly change if the number of oscillators is increased. This is more relevant when thinking about quantum field theories, which can be written as the lattice of infinitely many coupled harmonic oscillators. However, we find that the pseudocomplexity of purification does not vary much at all when the number of oscillators of the actual system is varied. This can be seen from Fig. 3.

(5) *Varying number of oscillators in the reduced system:* All of our results so far are for the case when there is one oscillator in the reduced transition matrix. The reason is twofold: First and foremost, as we have mentioned before, the number of unknown parameters for pseudocomplexity of purification in terms of which the complexity functional is minimized is much more than the case of usual complexity of purification. This makes the problem much more challenging numerically as the number of

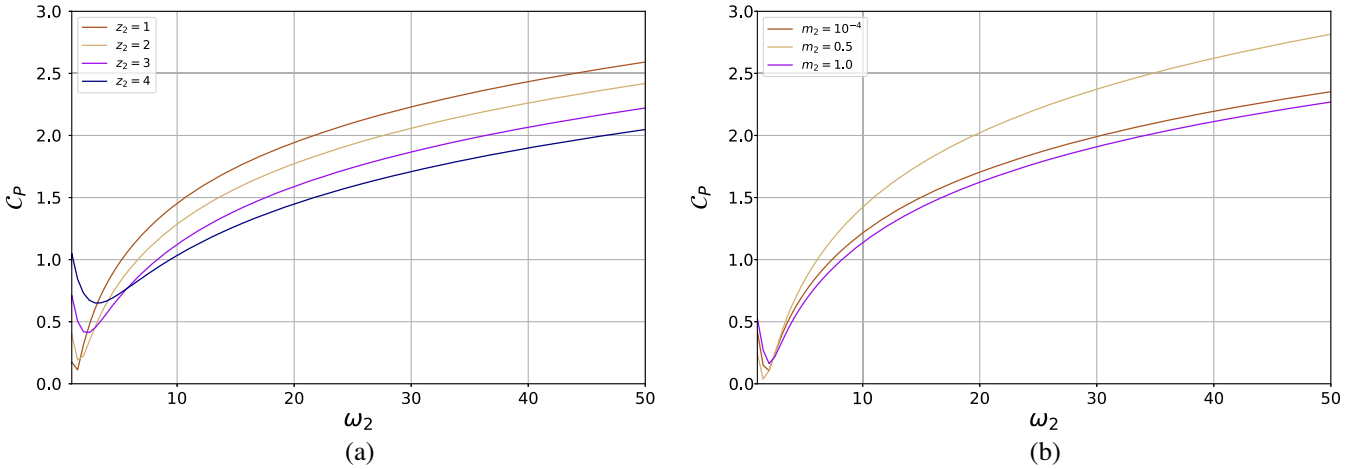

FIG. 1. Behavior of $\mathcal{C}_P$ with increasing ω_2 for (a) $m_1 = m_2 = 0.5$, $z_1 = 1$ and different z_2 and (b) $m_1 = 1.0$, $z_1 = 1$, $z_2 = 2$ and different m_2.

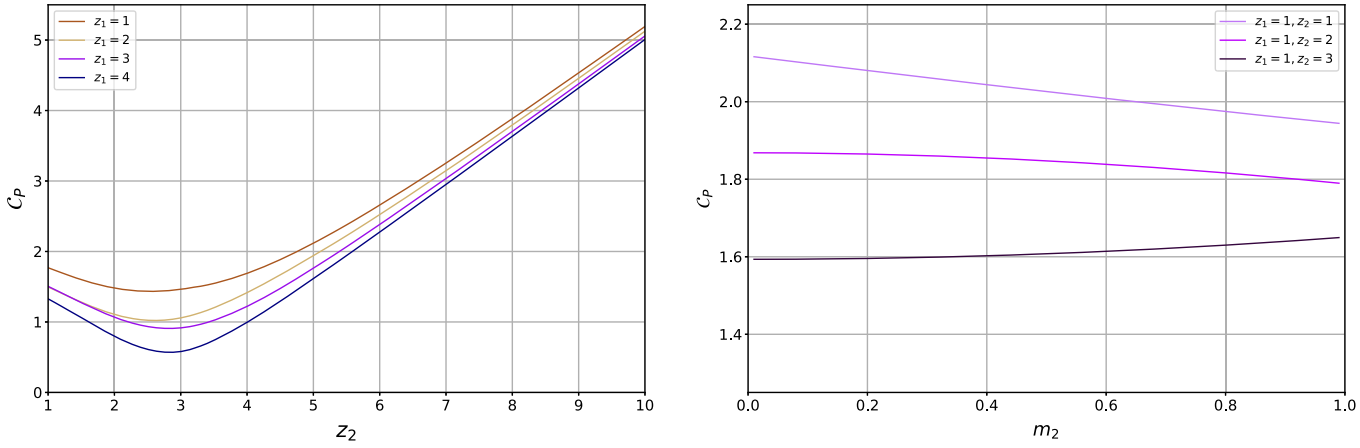

FIG. 2. Behavior of $\mathcal{C}_P$ with increasing (a) z_2 and (b) m_2 for $\omega_2 = 25$.

parameters grows very quickly as one increases the number of oscillators in the reduced system. Second, from a physical point of view, with the motivation of relating the entanglement entropy to some thermal notion of entropy, one typically chooses the number of oscillators in the reduced system to be small. Although there is no clear motivation for complexity, we stick to this particular choice.

The qualitative behavior of the pseudocomplexity of purification with the varying reference frequency remains similar when the subsystem size is increased to include two oscillators as shown in Fig. 4. Hence, we expect all of our results to be universal and hold for larger subsystem sizes.

B. Difference ΔC: Mutual pseudocomplexity of purification

The reduced transition matrix carries complicated mixed information of not one but two pure states ψ_1 and ψ_2. Hence, while studying the complexity of purification for the reduced transition matrix, it is natural to ask how different it is from the complexities of individual reduced density matrices of those two states. Inspired by the definition of mutual complexity ($\triangle C(\rho_{AB})$) [30,44,45], which is defined as the difference of complexity between the full state ρ_{AB} and the sum of complexities of the two reduced density matrices ρ_A and ρ_B, we define the difference as the mutual pseudocomplexity of purification. In the following, we report some properties of the mutual pseudocomplexity of purification based on our numerical analysis.

(1) *Subadditivity:* From Fig. 5, it is easy to observe that

$$\triangle C(\tau_A^{1|2}) = C(\tau_A^{1|2}) - \frac{C(\rho_A^1) + C(\rho_A^2)}{2} \leq 0. \quad (3.1)$$

Therefore, we find that the pseudocomplexity of purification is always subadditive to the sum of the individual complexities of purification of the two relevant states. Our numerical results suggest that the term $\triangle C(\tau_A^{1|2})$ is always negative or zero. The exact

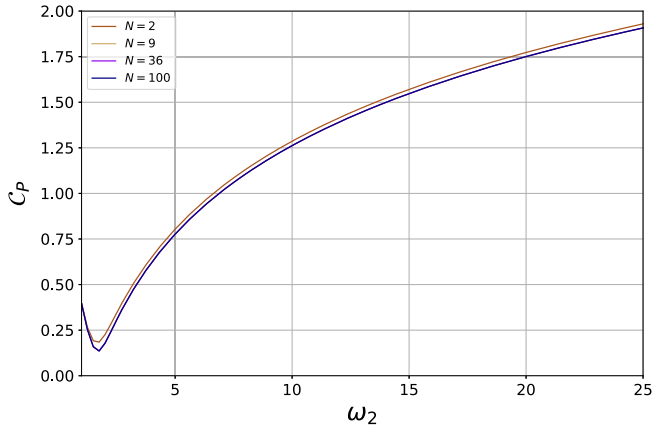

FIG. 3. Comparison of $\mathcal{C}_P$ for different total numbers of lattice sites (N). In all graphs $m_1 = m_2 = 0.5$ and $z_1 = 1$, $z_2 = 2$.

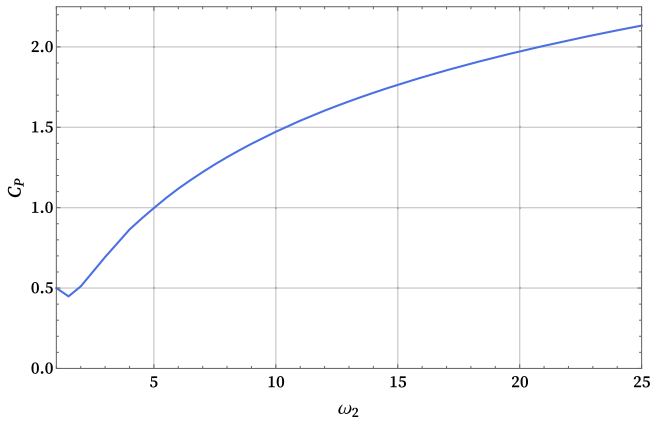

FIG. 4. Pseudocomplexity $\mathcal{C}_P$ when the reduced system contains two LHOs.

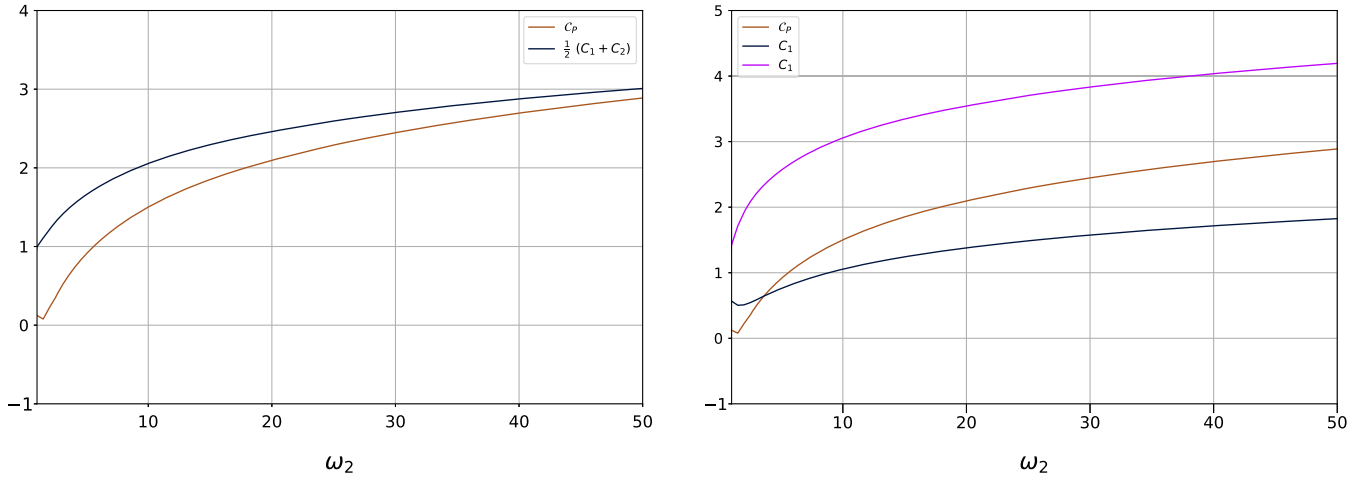

FIG. 5. Comparison of $\mathcal{C}_P$ with individual complexities of purification of the two relevant states ψ_1 and ψ_2. We let $z_1 = 1$, $z_2 = 2$ and $m_1 = m_2 = 0.5$.

equality is expected for the case when the two states become the same and hence $\rho_A^1 = \rho_A^2$.

(2) *Near saturation behavior:* It is observed that both the complexity of purification and the pseudocomplexity of purification tend to saturate at a very large reference frequency. By looking at the saturation values of the corresponding plots from Fig. 5, we make the following comment on the saturation of pseudocomplexity of purification in terms of complexity of purification of the individual reduced density matrices,

$$C(\tau_A^{1|2}) \sim \frac{C(\rho_A^1) + C(\rho_A^2)}{2}. \tag{3.2}$$

This means that the quantity $\Delta C(\tau_A^{1|2})$ approaches zero at large reference frequencies. This indicates that the dependence of $C(\tau_A^{1|2})$ on the reference frequency ω is such that for large enough values of ω, the complexity does not distinguish between a transition matrix and the density matrix.

(3) *Varying masses:* We also study the behavior of $\frac{C(\rho_A^1)+C(\rho_A^2)}{2} - C(\tau_A^{1|2})$ $(=\triangle C(\tau_A^{1|2}))$ with a change in one of the masses m_2 corresponding to the state ψ_2 while keeping the other parameters $(m_1,\ z_1,\ z_2)$ fixed. We look at the behavior in both a small mass difference and a large mass difference regime. In both cases, we find that the difference decreases as one increases the difference between mass parameters; see Fig. 6. In the small mass difference regime, the slope of the plot is linear. This means $|\triangle C(\tau_A^{1|2})| \propto -a(m_2 - m_1)$ in this range. On the

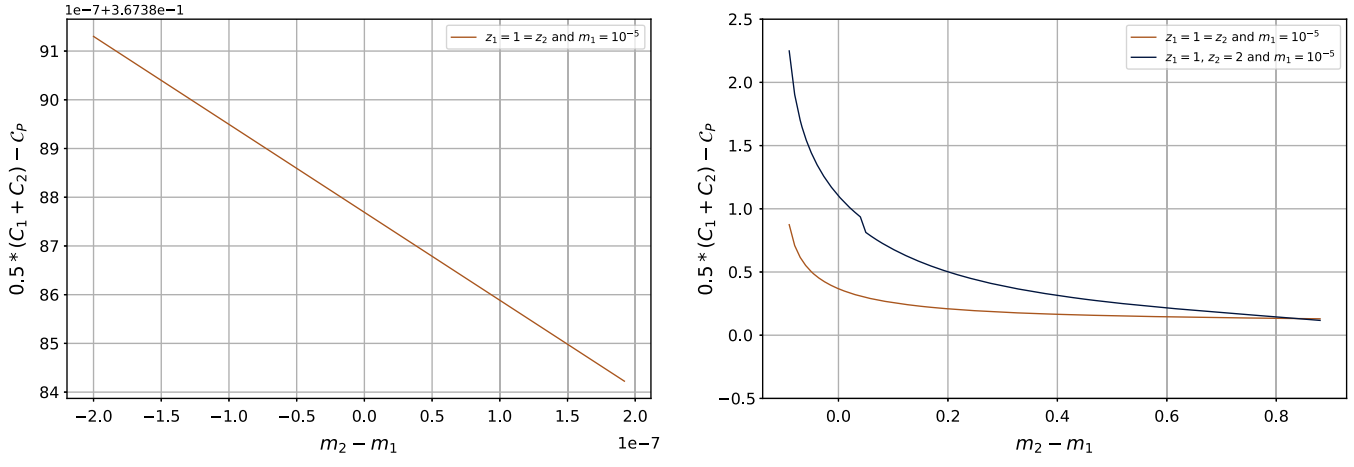

FIG. 6. Behavior of $\frac{1}{2}(C_1 + C_2) - \mathcal{C}_P$. (a) Small mass difference and (b) large mass difference. In all results, $\omega_2 = 50$.

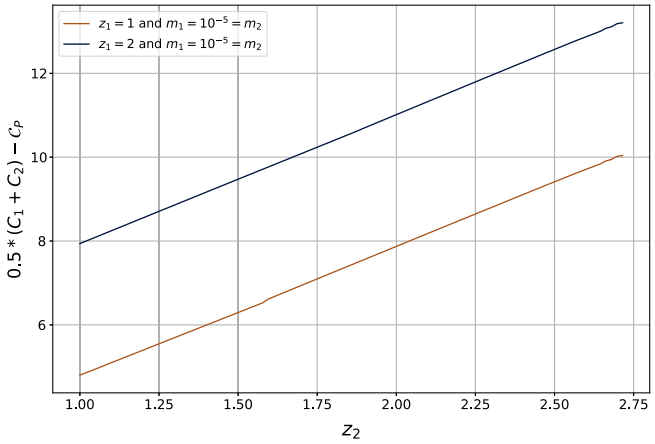

FIG. 7. Behavior of $\frac{1}{2}(C_1 + C_2) - C_P$ with z_2 in the small mass regime.

other hand, in the large mass difference regime, the decay of the slope is close to exponential inspiring us to write $|\triangle C(\tau_A^{1|2})| \propto e^{-a(m_2-m_1)}$, with lower saturation close to zero as $(m_2 - m_1) \sim 1$.

(4) *Varying z_2:* In the same figure, we see that as the difference between $z_2 - z_1$ is increased, the absolute value of $\triangle C(\tau_A^{1|2})$ increases. However, the lower saturation $|\triangle C(\tau_A^{1|2})| \sim 0$, near $(m_2 - m_1) \sim 1$, remains unchanged. Hence, the numerics definitely suggests that the saturation is universal and independent of $z_2 - z_1$.

(5) $|\triangle C|$ vs z_2 *plot:* As shown in Fig. 7, if we keep z_1, m_1, and m_2 fixed while varying z_2, $|\triangle C|$ increases linearly:

$$|\triangle C| \propto \alpha z_1 + \beta z_2. \tag{3.3}$$

The above expression is written in this way because our treatment is symmetric with respect to $z_{1,2}$. In the same plot, we also notice that once z_1 is chosen to be 2 instead of 1, $|\triangle C|$ increases, but the linear behavior remains unchanged.

IV. CONCLUSION

We have studied the pseudocomplexity of purification of the reduced transition matrix (2.14) numerically for free scalar and Lifshitz scalar field theories. We have also defined the mutual pseudocomplexity of purification by comparing the pseudocomplexity of purification with the complexity of purification of the individual reduced density matrices of the states $|\psi_1\rangle$ and $|\psi_2\rangle$. The main findings of this work are listed below.

(i) Although the pseudocomplexity is supposed to be a simple sum of the two states involved in the construction of the transition matrix, the pseudocomplexity of purification can show nontrivial behavior compared to with individual pseudocomplexities of purification associated with the reduced density matrices of the two states involved.

(2) We also find that the mutual pseudocomplexity of purification satisfies the generally expected inequality $\Delta C(\tau_A^{1|2}) \leq 0$ [30].[5] Also, for all scaling exponents and masses corresponding to the two states $|\psi_1\rangle$ and $|\psi_2\rangle$, the saturation of the pseudocomplexity of purification is similar to the saturation of half the sum of individual complexities of purification for the reduced density matrices ρ_A^1 and ρ_A^2 associated with the initial and final states. More details on the behavior of mutual pseudocomplexity of purification can be found in Sec. III B.

(3) From the behavior of pseudocomplexity of purification, it appears that the qualitative behavior remains similar to that of the usual complexity of purification of a reduced density matrix. Hence, we expect the general dependence of the quantity in terms of the system parameters to be similar to that of the complexity of purification of the reduced density matrices. However, the pseudocomplexity of purification, since it carries information of both states $|\psi_1\rangle$ and $|\psi_2\rangle$, is expected to depend on parameters of two states instead of one. This also results in an increment in the number of auxiliary parameters to be added and minimized for finding the most optimal purification. We find that the dependence on the two states should be symmetric. It is not surprising since the transition matrix, or more specifically, the reduced transition matrix by structure, does not differentiate in any way between the two states used to define it.

(4) For the most general purification of the reduced transition matrix with n oscillators, the number of unknown parameters scales as $2n^2$. This may be understood from Eq. II B: For a reduced system made of n linear harmonic oscillators, the J, K, and L matrices would be square matrices of rank n. The choice II B always leaves any two of the six matrices undetermined, and thus the number of unknown parameters in the final optimization problem is $2n^2$.

Our analysis is mostly done for a reduced transition matrix of one oscillator after integrating the rest. This analysis can be extended to more reduced oscillators in many ways. However, due to the increase of unknown parameters, the complexity optimization problem becomes numerically more challenging, in general. However, one can perform a mode-by-mode purification as done in [30] on a physical basis. There, instead of optimizing all the modes together, one purifies each of the modes individually and then optimizes the sum of the complexities for all the modes. This decreases the number of unknowns

[5]Note that the equality sign is different from [30] due to our choice of definition.

significantly. For mode-by-mode purifications, the number of unknowns will scale as $2n$ as each mode involves 2 auxiliary parameters. However, since we get similar quantitative behavior for pseudocomplexity of purification as for usual complexity of purification for the reduced system of one oscillator, we expect this to hold for the reduced system with more modes. With this assumption, we can therefore use a similar form for the pseudocomplexity of purification, in terms of variables of two states instead of one,

$$\mathcal{C}_P(\tau_A^{1|2}) = \frac{N}{2}\log(\omega_2\delta) + \frac{1}{2}\log\left(\frac{1}{f_1(m_1,z_1,m_2,z_2)N\delta}\right) - \frac{f_2(m_1,z_1,m_2,z_2)N^2\delta^2}{48} + \cdots, \tag{4.1}$$

where δ is the gap between two points on the lattice.[6] This expression should work for very small masses, and the ellipses denote higher order terms. Usually, $f_1 \sim \mathcal{O}(m_{1,2})$ and $f_2 \sim \mathcal{O}(m_{1,2}^2)$. The functions f_1 and f_2 are symmetric under the exchange $1 \leftrightarrow 2$ but arbitrary otherwise.[7] Apart from the symmetric dependence, the otherwise full form of the complexity has to follow all the necessary properties, as mentioned in Sec. III for both small and large mass regimes. Also, the nontrivial dependence on the scaling exponents has to abide by the plots we found and reported in the previous section. A more detailed study for subsystems considering larger size (by taking mode-by-mode purification), disjoint nature, etc., can yield more exact behavior, which is a very natural extension of this work.

Many interesting questions arise in this direction. Since the postselected states might be, in general, the result of a nonunitary evolution, it might be related to studies of complexity for open systems where the Lindbladian evolution is nonunitary. One recent study in this direction is [46], where initial steps towards studying Krylov complexity have been taken. It would be interesting if one could relate the notion of Nielsen complexity with the Krylov complexity along these lines. Another natural direction is to study the pseudocomplexity of purification for spin systems, e.g., in the transverse field Ising model, and check if the complexity can probe phase transitions like pseudo entropy.[8] It would also be interesting to study conformal systems as done for usual complexity of purification in [32]. Finally, some holographic notions of pseudo entropy by computing minimal surfaces in Euclidean setups were proposed in [1]. Therefore, it would be natural to compute the volumes below those minimal surfaces and compute a holographic pseudosubregion complexity (similar to holographic subregion complexity [49–53]) in those geometries. We hope to address some of these problems later.

[6]This variable is not used in our study since we did not consider any disjoint systems.

[7]One possibility is $f_1 = \frac{1}{2}\sqrt{m_1^{2z_1} + m_2^{2z_2}}$ and $f_2 = f_1^2$.

[8]This can be further compared with the study of entanglement of purification for the transverse field Ising model as done in [47], as well as with the corresponding study of phase transitions using circuit complexity. Interested readers are referred to [48] and the citations therein for the application of circuit complexity to detect phase transitions in various quantum systems.

[1] Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka, and Z. Wei, New holographic generalization of entanglement entropy, Phys. Rev. D **103**, 026005 (2021).

[2] A. Mollabashi, N. Shiba, T. Takayanagi, K. Tamaoka, and Z. Wei, Pseudo Entropy in Free Quantum Field Theories, Phys. Rev. Lett. **126**, 081601 (2021).

[3] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).

[4] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[5] R. Jefferson and R. C. Myers, Circuit complexity in quantum field theory, J. High Energy Phys. 10 (2017) 107.

[6] S. Chapman, M. P. Heller, H. Marrochio, and F. Pastawski, Toward a Definition of Complexity for Quantum Field Theory States, Phys. Rev. Lett. **120**, 121602 (2018).

[7] L. Hackl and R. C. Myers, Circuit complexity for free fermions, J. High Energy Phys. 07 (2018) 139.

[8] R. Khan, C. Krishnan, and S. Sharma, Circuit complexity in fermionic field theory, Phys. Rev. D **98**, 126001 (2018).

[9] A. Bhattacharyya, A. Shekar, and A. Sinha, Circuit complexity in interacting QFTs and RG flows, J. High Energy Phys. 10 (2018) 140.

[10] A. Bhattacharyya, P. Nandy, and A. Sinha, Renormalized Circuit Complexity, Phys. Rev. Lett. **124**, 101602 (2020).

[11] T. Ali, A. Bhattacharyya, S. Shajidul Haque, E. H. Kim, and N. Moynihan, Time evolution of complexity: A critique of three methods, J. High Energy Phys. 04 (2019) 087.

[12] J. M. Magán, Black holes, complexity and quantum chaos, J. High Energy Phys. 09 (2018) 043.

[13] P. Caputa and J. M. Magan, Quantum Computation as Gravity, Phys. Rev. Lett. **122**, 231302 (2019).

[14] A. Bhattacharyya, W. Chemissany, S. Shajidul Haque, and B. Yan, Towards the web of quantum chaos diagnostics, Eur. Phys. J. C **82**, 87 (2022).

[15] T. Ali, A. Bhattacharyya, S. S. Haque, E. H. Kim, N. Moynihan, and J. Murugan, Chaos and complexity in quantum mechanics, Phys. Rev. D **101**, 026021 (2020).

[16] A. Bhattacharyya, S. Das, S. Shajidul Haque, and B. Underwood, Cosmological complexity, Phys. Rev. D **101**, 106020 (2020).
[17] A. Bhattacharyya, S. Das, S. S. Haque, and B. Underwood, Rise of cosmological complexity: Saturation of growth and chaos, Phys. Rev. Res. **2**, 033273 (2020).
[18] A. Bhattacharyya, W. Chemissany, S. S. Haque, J. Murugan, and B. Yan, The multi-faceted inverted harmonic oscillator: Chaos and complexity, SciPost Phys. Core **4**, 002 (2021).
[19] J. Erdmenger, M. Gerbershagen, and A.-L. Weigel, Complexity measures from geometric actions on Virasoro and Kac-Moody orbits, J. High Energy Phys. 11 (2020) 003.
[20] M. Flory and M. P. Heller, Geometry of complexity in conformal field theory, Phys. Rev. Res. **2**, 043438 (2020).
[21] M. Flory and M. P. Heller, Conformal field theory complexity from Euler-Arnold equations, J. High Energy Phys. 12 (2020) 091.
[22] J. Erdmenger, M. Flory, M. Gerbershagen, M. P. Heller, and A.-L. Weigel, Exact gravity duals for simple quantum circuits, arXiv:2112.12158.
[23] N. Chagnet, S. Chapman, J. de Boer, and C. Zukowski, Complexity for Conformal Field Theories in General Dimensions, Phys. Rev. Lett. **128**, 051601 (2022).
[24] R. d. M. Koch, M. Kim, and H. J. R. Van Zyl, Complexity from spinning primaries, J. High Energy Phys. 12 (2021) 030.
[25] A. Bhattacharyya, G. Katoch, and S. R. Roy, Complexity of warped conformal field theory, arXiv:2202.09350.
[26] S. Chapman and G. Policastro, Quantum computational complexity—from quantum information to black holes and back, Eur. Phys. J. C **82**, 128 (2022).
[27] A. Bhattacharyya, Circuit complexity and (some of) its applications, Int. J. Mod. Phys. E **30**, 2130005 (2021).
[28] M. Ghodrati, X.-M. Kuang, B. Wang, C.-Y. Zhang, and Y.-T. Zhou, The connection between holographic entanglement and complexity of purification, J. High Energy Phys. 09 (2019) 009.
[29] C. A. Agon, M. Headrick, and B. Swingle, Subsystem complexity and holography, J. High Energy Phys. 02 (2019) 145.
[30] E. Caceres, S. Chapman, J. D. Couch, J. P. Hernandez, R. C. Myers, and S.-M. Ruan, Complexity of mixed states in QFT and holography, J. High Energy Phys. 03 (2020) 012.
[31] H. A. Camargo, P. Caputa, D. Das, M. P. Heller, and R. Jefferson, Complexity as a Novel Probe of Quantum Quenches: Universal Scalings and Purifications, Phys. Rev. Lett. **122**, 081601 (2019).
[32] H. A. Camargo, L. Hackl, M. P. Heller, A. Jahn, T. Takayanagi, and B. Windt, Entanglement and complexity of purification in $(1+1)$-dimensional free conformal field theories, Phys. Rev. Res. **3**, 013248 (2021).
[33] A. Bhattacharyya, S. S. Haque, and E. H. Kim, Complexity from the reduced density matrix: A new diagnostic for chaos, J. High Energy Phys. 10 (2021) 028.
[34] A. Bhattacharyya, T. Hanif, S. S. Haque, and M. K. Rahman, Complexity for an open quantum system, Phys. Rev. D **105**, 046011 (2022).
[35] M. A. Nielsen, A geometric approach to quantum circuit lower bounds, arXiv:quant-ph/0502070.
[36] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, Quantum computation as geometry, Science **311**, 1133 (2006).
[37] M. R. Dowling and M. A. Nielsen, The geometry of quantum computation, arXiv:quant-ph/0701004.
[38] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rep. Math. Phys. **3**, 275 (1972).
[39] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. **10**, 285 (1975).
[40] M. Jiang, S. Luo, and S. Fu, Channel-state duality, Phys. Rev. A **87**, 022310 (2013).
[41] A. Mollabashi, N. Shiba, T. Takayanagi, K. Tamaoka, and Z. Wei, Aspects of pseudo entropy in field theories, Phys. Rev. Res. **3**, 033254 (2021).
[42] M. Guo, J. Hernandez, R. C. Myers, and S.-M. Ruan, Circuit complexity for coherent states, J. High Energy Phys. 10 (2018) 011.
[43] A. Bhattacharyya, T. Takayanagi, and K. Umemoto, Entanglement of purification in free scalar field theories, J. High Energy Phys. 04 (2018) 132.
[44] M. Alishahiha, K. Babaei Velni, and M. R. Mohammadi Mozaffar, Black hole subregion action and complexity, Phys. Rev. D **99**, 126016 (2019).
[45] E. Cáceres, J. Couch, S. Eccles, and W. Fischler, Holographic purification complexity, Phys. Rev. D **99**, 086016 (2019).
[46] A. Bhattacharya, P. Nandy, P. P. Nath, and H. Sahu, Operator growth and Krylov construction in dissipative open quantum systems, arXiv:2207.05347.
[47] A. Bhattacharyya, A. Jahn, T. Takayanagi, and K. Umemoto, Entanglement of Purification in Many Body Systems and Symmetry Breaking, Phys. Rev. Lett. **122**, 201601 (2019).
[48] T. Ali, A. Bhattacharyya, S. Shajidul Haque, E. H. Kim, and N. Moynihan, Post-quench evolution of complexity and entanglement in a topological system, Phys. Lett. B **811**, 135919 (2020).
[49] A. Bhattacharya, K. T. Grosvenor, and S. Roy, Entanglement entropy and subregion complexity in thermal perturbations around pure-AdS spacetime, Phys. Rev. D **100**, 126004 (2019).
[50] A. Bhattacharya, A. Chanda, S. Maulik, C. Northe, and S. Roy, Topological shadows and complexity of islands in multiboundary wormholes, J. High Energy Phys. 02 (2021) 152.
[51] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Islands and complexity of eternal black hole and radiation subsystems for a doubly holographic model, J. High Energy Phys. 05 (**2021**) 135.
[52] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Partial islands and subregion complexity in geometric secret-sharing model, J. High Energy Phys. 12 (2021) 091.
[53] A. Bhattacharya, A. Bhattacharyya, P. Nandy, and A. K. Patra, Bath deformations, islands, and holographic complexity, Phys. Rev. D **105**, 066019 (2022).