Turning rotating D-branes and black holes inside out their photon-halo

Massimo Bianchi and Giorgio Di Russo

Dipartimento di Fisica, Università di Roma "Tor Vergata" and Sezione INFN Roma2, Via della ricerca scientifica 1, 00133 Rome, Italy

(Received 11 July 2022; accepted 20 September 2022; published 13 October 2022)

We extend our investigation on Couch-Torrence conformal inversions of black holes (BHs) and D-branes in various directions. We analyze asymptotically flat rotating charged BHs in $D = 4$, in particular extremal rotating BHs in STU supergravity, and find invariance for special choices of the charges. Due to the dependence of the critical radii on the impact parameter(s), the relation between the scattering angle for geodesics outside the photon-halo and the inspiraling angle for geodesics inside the photon-halo is modified by the inclusion of a boundary term. We also consider rotating BHs in $D = 5$ and rotating D3-branes and find invariance under generalized Couch-Torrence inversions for special choices of the angular momenta. Alas we don't find any similar symmetry for smooth horizonless geometries. Moreover, relying on the surprising connection between classical BH perturbation theory and quantum Seiberg-Witten curves for $\mathcal{N} = 2$ SYM theories, we study scalar wave equations in these backgrounds and identify the near superradiant modes produced in near-extremal BH mergers. Finally, we study scalar fluctuations around Kerr-Newman BHs in AdS_4 and find stringent conditions for generalized Couch-Torrence symmetry that are relaxed in the extremal case or by allowing a rescaling of the wave function.

DOI: 10.1103/PhysRevD.106.086009

I. INTRODUCTION

Many compact gravitating objects are surrounded by a ring or a halo of light formed by the quasicritical geodesics of photons surfing the potential barrier, called the photon-sphere or photon-halo, separating the horizon from the asymptotically flat region [1,2]. Perturbations of these barriers reflect into the quasinormal modes (QNMs) that dominate the gravitational wave (GW) signal during the prompt ring-down phase that follows the merger of two black holes (BHs), for instance [3,4].

In [5] we have shown that D3-branes and their bound states in lower dimensions admit a symmetry under conformal inversions that generalize the Couch-Torrence (CT) transformations known to leave the metric of extremal Reissner-Nordström BHs invariant up to a Weyl rescaling [6]. CT transformations are known to exchange the event horizon with null infinity [6–13]. Quite remarkably, we have found that the fixed loci of the (generalized) CT transformations are precisely the photon-spheres [5], thus opening new paths to explore physical implications for QNMs and other observables such as deflection angles and time delays, coded in the so-called radial action, and ultimately on a bulk-to-boundary relation proposed in [14,15].

In the present investigation we will extend our analysis to the case of rotating compact objects. In the extremal case (zero temperature), Kerr and Kerr-Newman (KN) BHs [6] as well as their cousins in STU supergravity (for special choices of the charges) [12,13] are known to admit a remnant of the CT inversion symmetry. Although the metric is not conformally invariant, the radial equation, that results from the separability of the dynamics, is invariant under transformations that depend on the angular momentum of the perturbation (or the impact parameter of probe). Even more remarkably we will show that the fixed loci of the conformal inversions are the photon-halos that form the 'asymmetric' light-ring structures [16] familiar from the images produced by the EHT Collaboration [17]. We will confirm that only special choices of the charges allow for the symmetry in its elementary form. Otherwise a Freudenthal duality transformation of the charges would be required [18,19]. We then study the implications for other physical observables and find that the radial action is formally invariant but, even in the equatorial plane $\theta = \pi/2$, the scattering angle $\Delta\phi_{\rm scatt}$ for a probe impinging from outside the photon-halo cannot be simply related to the inspiraling angle $\Delta\phi_{\rm fall}$ of a probe with the same energy E and angular momentum J falling into the horizon. We will explain this discrepancy in terms of the dependence of the extrema of integration on the impact parameter $b = J/E$ and discuss the issue for generic nonplanar motion. We also study scalar fluctuations in the (near extremal) STURBHs (rotating BH in STU supergravity) and identify the near superradiant modes that are long-lived in that their frequencies have a very small imaginary parts.

We then address the fate of the CT inversions for rotating D3-branes [20] and conclude that generalized CT transformations are symmetries of massless geodesics, at least for special choices of the rotation parameters. Extremal rotating objects do enjoy CT invariance beyond $d = 4$ in the extremal [not necessarily Bogomol'nyi-Prasad-Sommerfield (BPS)] case. We illustrate our conclusions in the case of rotating BHs in $d = 5$ of the Breckenridge-Myers-Peet-Vafa (BMPV) supersymmetric BPS kind [21] as well as their nonsupersymmetric cousins of the Cheung, Cvetic, Lu and Pope (CCLP) family [22,23]. However their smooth horizonless uplifts to $d = 6$ known as JMaRT solutions [24] or Giusto Mathur Saxena (GMS) [25,26] 'fake' fuzz balls (in the BPS case), lacking a horizon, fail to admit any symmetry under conformal inversions, even though they are endowed with a photon-halo in most cases. A similar story applies to circular fuzz balls that represent a class of microstate geometries of two charge systems like D3-D3' (or D1-D5 after T-duality), yet some embryonic form of invariance property is enjoyed by special classes of geodesics in a restricted subspace (e.g., $\theta = 0$ plane, orthogonal to the circle).

Finally we consider KN BHs in four-dimensional anti–de Sitter space (AdS_4) and determine the conditions for CT invariance exploiting the recently established connection between BH perturbation theory and quantum Seiberg-Witten (SW) curves [27–31].

The plan of the paper is as follows. We start by reviewing the prototypical rotating case of the KN BH in Sec. II. We focus on null geodesics as in [5] rather than on massless scalar wave perturbations as in [6,12,13]. Some details on the critical parameters are given in Appendix A.

In Sec. III we then pass to consider extremal rotating BHs in STU supergravity (eSTURBHs) and show that the condition on the (four electric) charges for the perturbations to be invariant under generalized Couch-Torrence transformations is to the one found in [5] for nonrotating BHs. The case $Q_1 = Q_2 = Q_3 = Q_4 = Q$ is shown to coincide with KN BHs. The details are presented in Appendix B.

Scalar wave fluctuations of near-eSTURBHs are studied in Sec. IV and the frequency of the near SR modes are determined whose expressions take a very simple form when the conditions for CT invariance are met. Rotating BHs of the CCLP kind in $D = 5$ and rotating D3-branes are discussed in Sec. V, while circular D3-D3' (or D1/D5) fuzz balls are discussed in Sec. VI together with JMaRT and GMS. The study of KN BHs in AdS and their connection with Heun equations (HE) and quantum SW curves is the subject of Sec. VII. Section VIII contains a summary of our results, the conclusions we draw from them and our outlook.

II. KERR-NEWMAN BH

Let us start with analysing CT inversions in the simplest cases of charged rotating BHs in $d = 4$, namely KN BH. Rather than studying massless neutral scalar wave perturbations of the geometry as in [6,12,13], in this section we will concentrate our attention on null geodesics as in [5]. The first important result that we find is the identification of the fixed loci of generalized CT inversions with the photon-halos, formed by the collection of photon-rings at varying impact parameters. Indeed due to the preferred axis of the geometry, represented by the angular momentum $\vec{J}_{\rm BH} = M\vec{a}$, one should introduce two independent impact parameters; $b = K/E$, where K is Carter's constant of separation [32], related to the total angular momentum of the probe, and $b_J = J/E$, where J is the projection of the angular momentum of the probe along $\vec{J}_{\rm BH}$. Due to frame-dragging one should distinguish the two cases of corotating $Ja > 0$ and counter-rotating $Ja < 0$ geodesics. After the simplified analysis of equatorial motion ($\theta = \pi/2$, $J = \pm K$), that allows to 'easily' show that the radial action as well as the deflection angle and Shapiro time delay are 'formally' invariant, we tackle the subtleties involved with the dependence of the extrema of integration on the impact parameter and eventually address the general case of nonplanar geodesics. Thanks to separability, the dynamics is encoded in an 'angular' action S_θ in addition to the radial action S_r.

The metric of KN space-time in Boyer-Lindquist coordinates and in natural units of $G_N = 1$, $c = 1$ reads [32]

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[adt - (a^2 + r^2)d\phi]^2 + \frac{\rho^2 dr^2}{\Delta_r} + \rho^2 d\theta^2, \tag{2.1}$$

where

$$\Delta_r = r^2 + a^2 - 2Mr + Q^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{2.2}$$

The horizons are the zeros of Δ_r

$$r_H^\pm = M \pm \sqrt{M^2 - Q^2 - a^2}, \tag{2.3}$$

while the singularity or rather 'ringularity', since it is a 'ring' in the equatorial plane, is located at $\rho = 0$, i.e., $r = 0$, $\theta = \pi/2$. The external horizon is surrounded by an ergoregion, delimited by the ergosphere, wherein the timelike Killing vector ∂_t becomes spacelike thus leading to interesting processes like Penrose mechanism [33] and superradiance [34], to which we will come back later on for STURBH [35].

When the extremality condition

$$M^2 = Q^2 + a^2 \tag{2.4}$$

is met, $r_+ = r_- = r_H = M = \sqrt{Q^2 + a^2}$. For later use it is convenient to perform the following change of radial coordinates,

$$\hat{r} = r - \sqrt{Q^2 + a^2}, \tag{2.5}$$

that maps the horizon into $\hat{r}_H = 0$. In this coordinate the null Hamiltonian reads

$$\mathcal{H} = \frac{1}{2\hat{\rho}^2}\left\{P_{\hat{r}}^2\hat{r}^2 - \frac{1}{\hat{r}^2}\left[E\left(a^2 + \left(\hat{r} + \sqrt{a^2+Q^2}\right)^2\right) - aJ\right]^2\right\} + \frac{1}{2\hat{\rho}^2}\left\{P_\theta^2 + \frac{(aE\sin^2\theta - J)^2}{\sin^2\theta}\right\} = 0, \tag{2.6}$$

where $E = -P_t$ and $P_\phi = J$ are conserved quantities. Following Carter [32], the null condition $\mathcal{H} = 0$ can be solved by introducing a separation constant K, representing the total angular momentum (including frame dragging) such that

$$K^2 = P_\theta^2 + \frac{(aE\sin^2\theta - J)^2}{\sin^2\theta} = -P_{\hat{r}}^2\hat{r}^2 + \frac{1}{\hat{r}^2}\left[E\left(a^2 + \left(\hat{r} + \sqrt{a^2+Q^2}\right)^2\right) - aJ\right]^2. \tag{2.7}$$

Defining the impact parameters

$$\zeta = \frac{J}{E} - a, \qquad b = \frac{K}{E}, \tag{2.8}$$

Eqs. (2.7) can be written in the form

$$P_{\hat{r}}^2 = \frac{E^2}{\hat{r}^4}\mathcal{R}(\hat{r}), \quad \mathcal{R}(\hat{r}) = \left[\left(\hat{r} + \sqrt{a^2+Q^2}\right)^2 - a\zeta\right]^2 - b^2\hat{r}^2,$$
$$P_\theta^2 = \frac{E^2}{\sin^2\theta}\Theta(\cos\theta), \quad \Theta(\cos\theta) = b^2\sin^2\theta - (\zeta + a\cos^2\theta)^2, \tag{2.9}$$

with $\mathcal{R}$ and Θ quartic polynomials of $\hat{r}$ and $\chi = \cos\theta$, respectively.

A. Geodesics in the equatorial plane

Planar motion in the equatorial plane $\theta = \pi/2$ is allowed for $P_\theta = \rho^2\dot{\theta} = 0$ i.e., for $b^2 = \zeta^2$. In these cases (2.7) reduces to[1]

$$\mathcal{R}(\hat{r}) = \frac{P_{\hat{r}}^2\hat{r}^4}{E^2} = \prod_{i=1}^4(\hat{r} - \hat{r}_i), \tag{2.10}$$

[1]For simplicity here we indicate by $b = J/E = \pm K/E$ the relevant impact parameter that has a 'sign', in that $ab > 0$ represents corotating geodesics and $ab < 0$ represents counter-rotating geodesics.

where the roots are

$$\hat{r}_1 = \hat{r}_+^{[+]}, \qquad \hat{r}_2 = \hat{r}_+^{[-]}, \qquad \hat{r}_3 = \hat{r}_-^{[+]}, \qquad \hat{r}_4 = \hat{r}_-^{[-]}, \tag{2.11}$$

with

$$\hat{r}_\pm^{[\pm]} = \frac{1}{2}\left[\pm(a-b) - 2\sqrt{a^2+Q^2}[\pm] + \sqrt{b^2 - 3a^2 + 2ab \pm 4\sqrt{a^2+Q^2}(b-a)}\right]. \tag{2.12}$$

In the extremal case under consideration, the roots satisfy

$$\hat{r}_1\hat{r}_2 = \hat{r}_3\hat{r}_4 = 2a^2 - ab + Q^2 = \hat{r}_c^2 \tag{2.13}$$

which determines the critical radius as a function of the BH parameters (Q and a) and the impact parameter $b = J/E = \pm K/E$ (in the equatorial plane). It is easy to see that

$$\frac{1}{\hat{r}^2}\mathcal{R}(\hat{r}) = \left(\frac{P_{\hat{r}}\hat{r}}{E}\right)^2 = \frac{1}{\hat{r}^2}\prod_{i=1}^4(\hat{r} - \hat{r}_i) \tag{2.14}$$

is invariant under generalized CT inversion

$$\hat{r}' = \frac{\hat{r}_c^2}{\hat{r}} \tag{2.15}$$

since $\hat{r}_c^2/\hat{r}_{1,2} = \hat{r}_{2,1}$ and $\hat{r}_c^2/\hat{r}_{3,4} = \hat{r}_{4,3}$. The fixed locus of the inversion is

$$\hat{r} = \hat{r}_c = \sqrt{2a^2 - ab + Q^2} \tag{2.16}$$

which depends on the impact parameter b and coincides with the photon-sphere after setting $b = b_c$. Indeed $\hat{r}_c$ is the radius of one of the two null critical geodesics satisfying

$$\mathcal{R}(\hat{r}_c) = 0 = \mathcal{R}'(\hat{r}_c). \tag{2.17}$$

In the equatorial plane, for corotating geodesics ($ab > 0$) one finds

$$\hat{r}_c^- = \max\{0, M - 2|a|\}, \qquad b_c^- = \max\{2M, 4(M - |a|)\},$$

while for counter-rotating geodesics ($ab < 0$)

$$\hat{r}_c^+ = M + 2|a|, \qquad b_c^+ = 4(M + 2|a|).$$

We start to see the emergence of the photon-halo, but before dwelling on that, let us consider the characteristic function S that encodes all the observables associated to (null) geodesics, such as scattering or in falling angles and Shapiro time delay.

For separable axisymmetric systems like KN BHs it reads

$$S = -Et + J\phi + S_r + S_\theta \tag{2.18}$$

with

$$S_r = \int_{r_i}^{r_f} P_r dr \tag{2.19}$$

and

$$S_\theta = \int_{\theta_i}^{\theta_f} P_\theta d\theta. \tag{2.20}$$

For geodesics in the equatorial plane $S_\theta = 0$ and the only nontrivial part is the radial action S_r. Since $P_r = P_{\hat{r}} \approx \pm E$ at large distances, S_r diverges when the initial or final points are taken to infinity. A simple regulator is to replace $r_f \to \infty$ with $r_f = \Lambda \gg a,\ M,\ Q$. A similar problem emerges when the initial or final points are taken to the horizon $\hat{r} \to 0$.

The ‘regulated’ radial action for scattering from infinity reads,

$$S_r^{\text{scatt,reg}}(E, J, \hat{r}_1, \Lambda) = \int_{\hat{r}_1}^{\Lambda} P_{\hat{r}} d\hat{r} = E \int_{\hat{r}_1}^{\Lambda} \sqrt{\prod_{i=1}^{4} (\hat{r} - \hat{r}_i)} \frac{d\hat{r}}{\hat{r}^2}. \tag{2.21}$$

Under CT inversion it transforms into the ‘regulated’ radial action for falling into the horizon with the same energy E and angular momentum J

$$S_r^{\text{fall,reg}}(E, J, \hat{r}_c^2/\Lambda, \hat{r}_2) = \int_{\hat{r}_c^2/\Lambda}^{\hat{r}_2} P_u du = E \int_{\hat{r}_c^2/\Lambda}^{\hat{r}_2} \frac{\sqrt{\prod_{i=1}^{4} (u - \hat{r}_i)}}{u^2} du \tag{2.22}$$

insofar as motion approaches the horizon within a distance $\varepsilon = \hat{r}_c^2/\Lambda$ that tends to zero when the cutoff Λ is taken to infinity.

The regulated deflection angle

$$\Delta\phi^{\text{reg}}(r_i, r_f) = \frac{\partial S_r^{\text{reg}}(E, J, r_i, r_j)}{\partial J} \tag{2.23}$$

can be computed by means of (2.21) and (2.22). Since the end points of the regulated integration range depend on J after CT inversion, the derivative of $S_r(E, J, r_i(J), r_f(J))$ w.r.t. J involves a boundary term[2]

$$\frac{\partial S_r(E, J, r_i(J), r_f(J))}{\partial J} = \int_{r_i(J)}^{r_f(J)} \frac{\partial P_r(r, E, J)}{\partial J} dr + \frac{\partial r_f(J)}{\partial J} P_r(r_f(J), E, J) - \frac{\partial r_i(J)}{\partial J} P_r(r_i(j), E, J).$$

[2]We thank Alfredo Grillo for discussions on this point.

As a consequence, one finds

$$\Delta\phi_{\text{scatt}}^{\text{reg}} = \int_{\hat{r}_1}^{\Lambda} \frac{\partial P_{\hat{r}}}{\partial J} d\hat{r}, \tag{2.24}$$

$$\Delta\phi_{\text{fall}}^{\text{reg}} = \int_{\frac{\hat{r}_c^2(J)}{\Lambda}}^{\hat{r}_2} \frac{\partial P_{\hat{r}}}{\partial J} d\hat{r} - \Lambda \left(\frac{\partial \hat{r}_c^2(J)}{\partial J} \right) \frac{E}{\hat{r}_c^4} \sqrt{\prod_{i=1}^{4} \left(\frac{\hat{r}_c^2(J)}{\Lambda} - \hat{r}_i \right)}. \tag{2.25}$$

We conclude that the CT inversion symmetry holds only at the level of the regulated radial actions, whereby $S_r^{\text{scatt,reg}}$ transforms into $S_r^{\text{fall,reg}}$ and vice versa, while equality between scattering and falling angles is lost to some extent, due the dependence on J of the product of the cutoffs $\varepsilon\Lambda = \hat{r}_c^2(J)$. Notice the analogies and the differences with the bulk-to-boundary formula [14,15] whereby an analytic continuation is needed.

B. Nonplanar motion, nonequatorial geodesics

When $P_\theta = \rho^2 \dot{\theta} \neq 0$ the angular dynamics becomes highly nontrivial. Yet, thanks to separability the system is integrable and one can write Hamilton principal function as in (2.18). The angular part of the action reads

$$S_\theta = \int_{\theta_0}^{\theta_f} P_\theta d\theta = E \int_{\theta_0}^{\theta_f} \sqrt{b^2 \sin^2\theta - (\zeta + a\cos^2\theta)^2} \frac{d\theta}{\sin\theta} = E \int_{\chi_0}^{\chi_f} \frac{d\chi}{1 - \chi^2} \sqrt{b^2(1 - \chi^2) - (\zeta + a\chi^2)^2}, \tag{2.26}$$

with $\chi = \cos\theta$, but will play only a marginal role in the following.

Let us focus on the radial part. In general the four roots of $\mathcal{R}$ are

$$\hat{r}_{\pm}^{[\pm]} = -\sqrt{a^2 + Q^2} + \frac{1}{2}\left[\pm b [\pm] \sqrt{b^2 + 4a\zeta \mp 4b\sqrt{a^2 + Q^2}} \right]. \tag{2.27}$$

The generalized CT inversion $\hat{r} \to \hat{r}_c^2/\hat{r}$ with

$$\hat{r}_c^2 = M^2 - a\zeta = 2a^2 + Q^2 - a\frac{J}{E} \tag{2.28}$$

formally leaves the radial action invariant, up to regularization. Due to frame dragging $\hat{r}_c$ depends on the impact parameters ζ and b. The conditions for criticality $\mathcal{R}(\hat{r}_c) = 0 = \mathcal{R}'(\hat{r}_c)$ are easier to solve for ζ_c and b_c in terms of $\hat{r}_c$ and yield

$$a\zeta_c = M^2 - \hat{r}_c^2, \qquad b_c^2 = 4(\hat{r}_c + M)^2 \tag{2.29}$$

with $M = M_e = \sqrt{a^2 + Q^2}$ for extremal KN BHs.

In Appendix A we (re)derive the allowed range for r_c for (non)extremal KN BHs that characterizes the photon-halo. In the extremal case of interest here, it turns out be

$$\mathrm{Max}\{0, M_e - 2a\} \le \hat{r}_c \le M_e + 2a.$$

Correspondingly,

$$\mathrm{Max}\{2M_e, 4(M_e - a)\} \le b_c \le 4(M_e + a),$$

with $|\zeta_c| \le b_c$, that is saturated on the equatorial plane as we have seen before. Notice that for $a > Q/\sqrt{3}$ (whereby $M_e < 2a$) the inner critical geodesics lies on the horizon. This plays a role in the phenomenon of superradiance that reflects in the presence of special (near) zero-damping modes to which we will come back later on.

In order to visualize the photon-halo, recall that for nonplanar motion $b = K/E$ and $\zeta = J/E - a$ are related by

$$\sin^2\theta_0 P^2_{\theta,0} = E^2[b^2\sin^2\theta_0 - (\zeta + a\cos^2\theta_0)^2] \tag{2.30}$$

for an observer at infinity on a plane at fixed θ, the collection of all points leading to a critical geodesics delimits the edge or rim of the BH shadow. This is encoded in the parametric curve

$$Y^2 + (X - X_0)^2 = b_c^2 \tag{2.31}$$

with $Y = P_\theta(\hat{r}_c)/E$, $X = J(\hat{r}_c)/E\sin\theta$ and $X_0 = a\sin\theta$ for $\hat{r}_c$ varying in the (sub)interval allowed by the chosen value of $\chi = \cos\theta$. See for instance [36–38] for further details and the connection with chaos and Lyapunov exponent.

III. EXTREMAL ROTATING BHS IN STU SUPERGRAVITY

In this section we extend the previous analysis of CT transformations to eSTURBHs [12,13,35,39]. For simplicity we only consider charged BHs with 4 electric charges that can be obtained adding angular momentum to bound states of four stacks of D3-branes wrapped around (intersecting) 3-cycles in a torus T^6 or a CY 3-fold.

A. Geodesic motion and CT inversion

Parametrizing the 4 charges as $Q_i = 2ms_ic_i$ with m a mass scale, $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$ and neglecting scalar and abelian vector fields, that play no role in our analysis, the metric reads [12,13]

$$ds^2 = \frac{2mr - \rho^2}{W}(dt + \mathcal{B}d\phi)^2 + W\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\sin^2\theta d\phi^2}{\rho^2 - 2mr}\right) \tag{3.1}$$

with

$$\begin{aligned}
\mathcal{B} &= 2ma\sin^2\theta\frac{r\prod_c - (r-2m)\prod_s}{\rho^2 - 2mr},\\
\rho^2 &= r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2,\\
R_i &= r + 2ms_i^2, \qquad \prod_c = \prod_i c_i, \qquad \prod_s = \prod_i s_i,\\
W^2 &= \prod_i R_i + a^4\cos^4\theta + 2a^2\cos^2\theta\Big[r^2 + mr\sum_i s_i^2\\
&\quad + 4m^2\Big(\prod_c - \prod_s\Big)\prod_s - 2m^2\sum_{i<j<k} s_i^2 s_j^2 s_k^2\Big].
\end{aligned} \tag{3.2}$$

It is straightforward but tedious to check that for $Q_1 = Q_2 = Q_3 = Q_4 = Q$ (3.1) reproduces the KN metric. The details are relegated in Appendix B. Let us stress that the mass parameter m is not precisely the BH mass. The Arnowitt-Deser-Misner mass is actually $M = \frac{1}{4}\sum_i\sqrt{m^2 + Q_i^2}$.

The condition for extremality is

$$m = a \tag{3.3}$$

so that $\rho^2 - 2mr = (r-a)^2 - a^2\sin^2\theta$ and $\Delta = (r-a)^2$ and the event horizon is then at $r = a$. Setting

$$\hat{r} = r - a \tag{3.4}$$

the horizon is located at $\hat{r} = 0$.

For neutral massless probes, the null mass-shell condition in Hamiltonian form reads

$$\hat{r}^2P_{\hat{r}}^2 + P_\theta^2 + \frac{\hat{r}^2 - a^2\sin^2\theta}{\hat{r}^2\sin^2\theta}\left\{(J + E\mathcal{B})^2 - \frac{\hat{r}^2\sin^2\theta W E^2}{(\hat{r}^2 - a^2\sin^2\theta)^2}\right\} = 0. \tag{3.5}$$

The separation is straightforward and yields

$$\frac{P_\theta^2}{E^2} + \frac{b^2}{\sin^2\theta} + a^2\sin^2\theta = \lambda^2, \qquad \frac{\hat{r}^2P_{\hat{r}}^2}{E^2} = \frac{\mathcal{R}(\hat{r})}{\hat{r}^2}, \tag{3.6}$$

with λ^2 a positive constant and $\mathcal{R}(\hat{r})$ a quartic polynomial viz.

$$\begin{aligned}
\mathcal{R}(\hat{r}) &= \prod_i(\hat{r} - \hat{r}_i) = \hat{r}^4 + 2a\Big(2 + \sum_i s_i^2\Big)\hat{r}^3\\
&\quad + \Big[2a^2\Big(4 + 3\sum_i s_i^2 + 2\sum_{i<j}s_i^2s_j^2\Big) - \lambda^2\Big]\hat{r}^2 \qquad (3.7)\\
&\quad + \Big[8a^3\Big(1 + \sum_i s_i^2 + \sum_{i<j}s_i^2s_j^2 + \sum_{i<j<k}s_i^2s_j^2s_k^2\Big)\\
&\quad - 4a^2b\Big(\prod_c - \prod_s\Big)\Big]\hat{r} + a^2b^2 - 4a^3b\Big(\prod_c + \prod_s\Big)\\
&\quad + 4a^4\Big[1 + \sum_i s_i^2 + \sum_{i<j}s_i^2s_j^2 + \sum_{i<j<k}s_i^2s_j^2s_k^2\\
&\quad + 2\Big(\prod_c + \prod_s\Big)\prod_s\Big], \qquad (3.8)
\end{aligned}$$

where $b = J/E$. Very much as for massless neutral scalar waves [12,13], symmetry of radial motion under generalized CT inversions $\hat{r} \to \hat{r}_c^2/\hat{r}$ at fixed charges requires

$$\hat{r}_1 \hat{r}_2 = \hat{r}_3 \hat{r}_4 = \hat{r}_c^2(b, \lambda), \tag{3.9}$$

where $\hat{r}_1, \hat{r}_2$ are the positive roots and $\hat{r}_3, \hat{r}_4$ are the negative ones and

$$\hat{r}_c^2(b, \lambda) = 2a^2 \left(\prod_c + \prod_s \right) - ab \tag{3.10}$$

that follows from the remarkable identity

$$1 + \sum_i s_i^2 + \sum_{i<j} s_i^2 s_j^2 + \sum_{i<j<k} s_i^2 s_j^2 s_k^2 + 2\left(\prod_c + \prod_s \right) \prod_s = \left(\prod_c + \prod_s \right)^2, \tag{3.11}$$

where the sign in (3.10) is chosen by comparing with r_c in Kerr. This is possible if and only if the charges Q_i or equivalently the "boost" parameters δ_i, satisfy the condition

$$\tilde{\delta}_i \equiv \frac{1}{2} \sum_j \delta_j - \delta_i = \delta_{\pi(i)} \tag{3.12}$$

with π a permutation of the four indices, that is allowed since the metric is permutation invariant. Taking a pair of indices i and $\pi(i)$ to be 1,2 without loss of generality, one has

$$2\delta_1 + 2\delta_2 = \sum_j \delta_j = \frac{1}{2} \sum_j \log\left(q_j + \sqrt{q_j^2 + 1} \right) \tag{3.13}$$

with $q_j = Q_j/m = Q_j/a$, that is satisfied when

$$\left(Q_1 + \sqrt{Q_1^2 + a^2} \right)\left(Q_2 + \sqrt{Q_2^2 + a^2} \right) = \left(Q_3 + \sqrt{Q_3^2 + a^2} \right)\left(Q_4 + \sqrt{Q_4^2 + a^2} \right), \tag{3.14}$$

or permutations thereof. This condition generalizes in an interesting way the condition $Q_1 Q_2 = Q_3 Q_4$ found in [5] for extremal nonrotating STU BHs. The special case $Q_1 = Q_2 = Q_3 = Q_4 = Q$, that we have shown to coincide with KN or, for $Q = 0$, with Kerr, and the special cases with pairwise equal charges, e.g., $Q_1 = Q_3$ and $Q_2 = Q_4$ are simply subcases of the general case that survive the inclusion of angular momentum a.

Although, integrality of charges and spin may not be an issue for large 'astrophysical' BHs, it is amusing to observe that a particularly simple integer solution of (3.14) is $Q_1 = 18$, $Q_2 = 32$, $Q_3 = 10$, $Q_4 = 45$, $a = 24$. It is also easy to convince oneself that (3.14) admit an infinite number of integer solutions.

For 4-charge extremal rotating BHs, satisfying the above condition and thus admitting a proper (generalized) CT symmetry, the analysis of the observables proceeds along the same lines as in the simpler KN contexts. The radial action is form invariant. Null infinity is exchanged with the horizon. Since the metric is not invariant, even if one allows Weyl rescalings, it does not make much sense to ask how the ergoregion transforms. Yet it is quite remarkable that the photon-halo is left fixed.

In the equatorial plane ($\theta = \pi/2$), the scattering angle $\Delta\phi_{\text{scatt}}(E, J)$ is mapped into the (regulated) inspiraling angle $\Delta\phi_{\text{scatt}}(E, J)$ for massless neutral probes with the same energy E and angular momentum J. While the former is finite for noncritical values, the latter diverges due to the $\hat{r}^2$ factor in the denominator. In fact the divergence may be imputed to a boundary contribution generated by the dependence on J or b of the transformed extrema $\hat{r}' = \hat{r}_c^2(J, E)/\hat{r}$.

Since geodesics are generically nonplanar as in the KN case, the relevant observable is the full action, including the nontrivial angular part S_θ, but the only part acted on by CT transformations is the radial action S_r. Even though conformal spatial inversions $\vec{x}' = -r_c^2 \vec{x}'/|\vec{x}|^2$ seem to produce an antipodal transformation $\theta \to \pi - \theta$ ($\chi = \cos\theta \to -\chi$) the metric and the geodesics are invariant under and the angular action $\int P_\theta d\theta$ is unaffected, up to exchange of the extrema of integration.

B. Critical regime

Once shown that massless geodesics in eSTURBH's metric (3.1) admit CT conformal inversion as symmetries, let us focus on the critical geodesics that form the halo, fixed under CT transformations.

Let us step back and reconsider the nonextremal case. Later on we will refocus on the extremal case.

In general for $m \neq a$, the zero mass-shell condition in Hamiltonian form can be written as

$$\Delta P_r^2 - \frac{a^2 J^2}{\Delta} + \frac{2ma[r\prod_c - (r - 2m)\prod_s]}{\Delta} 2EJ - \frac{E^2}{2\Delta} \left\{ a^4 + 2\prod_{i=1}^4 R_i + a^2 \left[3r^2 + 2mr\left(1 + 2\sum_{i=1}^4 s_i^2 \right) - 8m^2 \left(\sum_{i<j<k} s_i^2 s_j^2 s_k^2 + 2\prod_s^2 - 2\prod_c \prod_s \right) \right] - a^2 \Delta \right\} + P_\theta^2 + \frac{J^2}{\sin^2\theta} - E^2 a^2 \cos^2\theta = 0. \tag{3.15}$$

If we introduce the separation constant λ^2, we obtain

$$
\begin{aligned}
\frac{P_\theta^2}{E^2} &= \lambda^2 - \frac{b_J^2}{\sin^2\theta} - a^2\sin^2\theta \\
\frac{\Delta P_r^2}{E^2} &= -\lambda^2 + \frac{a^2 b_J^2}{\Delta} - \frac{4mab_J[r\prod_c - (r-2m)\prod_s]}{\Delta} \\
&\quad + \frac{1}{2\Delta}\left\{a^4 + 2\prod_{i=1}^4 R_i + a^2\left[3r^2 + 2mr\left(1 + 2\sum_{i=1}^4 s_i^2\right)\right.\right. \\
&\quad \left.\left. - 8m^2\left(\sum_{i<j<k} s_i^2 s_j^2 s_k^2 + 2\prod_s{}^2 - 2\prod_c\prod_s\right)\right] + \frac{a^2\Delta}{2}\right\},
\end{aligned}
\tag{3.16}
$$

where $\lambda = K/E$ and $b_J = J/E$. For brevity we set

$$
\begin{aligned}
\mathcal{R}_4(r) &= \frac{\Delta^2 P_r^2}{E^2} = r^4 + Ar^3 + Br^2 + Cr + D - \Delta\lambda^2, \\
A &= 2m\sum_{i=4}^4 s_i^2, \qquad B = 2a^2 + 4m^2\sum_{i<j} s_i^2 s_j^2, \\
C &= C_1 b_J + C_2, \qquad C_1 = 4ma\left(\prod_s - \prod_c\right), \\
C_2 &= 2a^2 m\sum_{i=1}^4 s_i^2 + 8m^3\sum_{i<j<k} s_i^2 s_j^2 s_k^2, \\
D &= a^2 b_J^2 - 8m^2 a\prod_s b_J + D_1, \\
D_1 &= a^4 + 16m^4\prod_s{}^2 \\
&\quad - 4a^2m^2\left(\sum_{i<j<k} s_i^2 s_j^2 s_k^2 + 2\prod_s{}^2 - 2\prod_s\prod_c\right).
\end{aligned}
\tag{3.17}
$$

In the critical regime we have

$$
\begin{aligned}
&r^4 + Ar^3 + Br^2 + Cr + D = \Delta\lambda^2, \\
&4r^3 + 3Ar^2 + 2Br + C = 2(r-m)\lambda^2.
\end{aligned}
\tag{3.18}
$$

The solutions to this system are

$$
\begin{aligned}
b_J &= \frac{m}{a(r-m)}\left[(a-2m+r)(a+2m-r)\prod_s\right. \\
&\quad \left. + (r-a)(r+a)\prod_c - \frac{\sqrt{\tilde\Delta}}{4am}\right], \\
\tilde\Delta &= 16a^2\Delta^2\left[r^2 + m\sum_{i=1}^4 s_i^2 r + m^2\left(\left(\prod_s - \prod_c\right)^2\right.\right. \\
&\quad \left.\left. - 1 - \sum_{i=1}^4 s_i^2\right)\right],
\end{aligned}
\tag{3.19}
$$

and

$$
\lambda^2 = \frac{4r^3 + 3Ar^2 + 2Br + C_2}{2(r-m)} + \frac{C_1}{2(r-m)} b_J, \tag{3.20}
$$

where for b_J we choose the negative sign in order to match with KN. In order to match with the angular equation in (2.9), we have to redefine the angular momenta as follows:

$$
b^2 = \lambda^2 - 2ab_J, \qquad \zeta = b_J - a. \tag{3.21}
$$

Thanks to the condition $b \geq \zeta$ descending from the non-negativity of (2.9), we can identify the photon region.

In the extremal limit, i.e., when $m = a$, we have

$$
\begin{aligned}
\zeta &= -a + (3a-r)\prod_s + (a+r)\prod_c + \frac{a-r}{a}\sqrt{r^2 + a\sum_{i=1}^4 s_i^2 r + a^2\left(-1 - \sum_{i=1}^4 s_i^2 + \left(\prod_s - \prod_c\right)^2\right)} \\
b^2 &= 2r^2 + a\left(2 + 3\sum_{i=1}^4 s_i^2\right)r + a^2\left(2 + \sum_{i=1}^4 s_i^2 + 2\sum_{i<j} s_i^2 s_j^2 - 2\sum_{i<j<k} s_i^2 s_j^2 s_k^2 - 4\prod_s{}^2 + 4\prod_s\prod_c\right) \\
&\quad - 2a(3a-r)\prod_s - 2a(a+r)\prod_c - 2\left[a - r + a\left(\prod_s - \prod_c\right)\right] \\
&\quad \times \sqrt{r^2 + ar\sum_{i=1}^4 s_i^2 - a^2\left[1 + \sum_{i=1}^4 s_i^2 - \left(\prod_s - \prod_c\right)^2\right]}.
\end{aligned}
\tag{3.22}
$$

If we take the charges s.t. $s_1 = s_2 = \sigma$ and $s_3 = s_4 = \tau$, (3.22) drastically simplifies

$$\zeta = -\frac{1}{a}[(r-a)^2 - a^2(1+2\sigma^2)(1+2\tau^2)]$$
$$b^2 = 4[r + a(\sigma^2+\tau^2)]^2. \tag{3.23}$$

The condition for r_c^- to be outside the horizon $r_c^- \geq r_H = a$ boils down to

$$b(r_H) \leq \zeta(r_H) \Rightarrow \sigma^2\tau^2 \geq \frac{1}{4}. \tag{3.24}$$

In the case of four different charges, the condition $r_c^- \geq r_H$ becomes

$$3 + 2\sum_{i=1}^{4} s_i^2 - 4\sum_{i<j<k} s_i^2 s_j^2 s_i^2 - 8\prod_s \left(\prod_c + \prod_s\right) \leq 0. \tag{3.25}$$

If the condition for invariance under CT inversions, that can be written as

$$2 + \sum_{i=1}^{4} s_i^2 = 2\left(\prod_c - \prod_s\right) \tag{3.26}$$

is obeyed, the expression for ζ (and consequently for λ^2) can be simplified to

$$\zeta = \frac{-\Delta - a^2 + 2m^2(\prod_c + \prod_s)}{a} + \frac{2m(m^2-a^2)(\prod_c - \prod_s)}{a(r-m)}, \tag{3.27}$$

which is another bonus of CT invariance.

IV. WAVES IN 4 CHARGE STU BH BACKGROUND AND SUPERRADIANT MODES

In this section we switch gear and consider the so-called (near) superradiant modes of near-extremal STURBHs (NESTURBHs). The photon sphere, that is the fixed locus of CT inversions if allowed, plays a crucial role in the linear response of BHs and other compact objects to small perturbations. In particular the ringdown phase of BH mergers is known to be dominated by quasinormal modes (QNMs). These are fluctuations that satisfy outgoing boundary conditions at infinity and ingoing boundary conditions at the horizon. In the WKB approximation the frequency is given by

$$\omega_{\text{QNM}}^{\text{WKB}} = \omega_c - i\lambda(2n+1),$$

where ω_c is the frequency of critical circular null orbits, while λ is the Lyapunov exponent governing the chaotic behavior of nearby critical geodesics [36–38]. A detailed study of the full spectrum of QNMs of STURBHs is beyond the scope of the present investigation and we hope to report on this soon. Exact results may be obtained resorting to the surprising connection between QNMs and quantum SW curves of $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and N_f hypermultiplets in the fundamental, that we will exploit later on to some extent.

Near extremal BHs however possess a special class of QNMs; near superradiant modes, also known as zero-damping modes (ZDMs), since $\text{Im}\omega_{\text{ZDM}}$ is very small and vanishes in the extremal limit. These modes are produced by near-extremal BH mergers and thanks to their slow falloff in time provide a very peculiar feature of the ringdown phase in these cases. The superradiant threshold frequency turns out to be

$$\omega_{SR} = m_\phi \Omega_H, \tag{4.1}$$

where Ω_H is the angular velocity at the horizon. Near superradiant modes are defined by taking

$$\omega = \omega_{SR} + \nu\delta \tag{4.2}$$

with finite $\nu = \nu_1 + i\nu_2$ (to be determined) and $\delta = r_+ - r_- \ll r_\pm$, the small separation between the inner and the outer horizon.

The superradiant threshold frequency for NESTURBHs is given by the above expression with

$$\Omega_H = -\frac{g_{t\phi}(r_+)}{g_{\phi\phi}(r_+)} = -\frac{1}{\mathcal{B}_{(1)}(r_+)} = \frac{a}{2m(r_+\prod_c - (r_+ - 2m)\prod_s)}. \tag{4.3}$$

Since the horizons are located at

$$r_\pm = m \pm \sqrt{m^2 - a^2}, \tag{4.4}$$

setting

$$m^2 = a^2 + \frac{\delta^2}{4} \tag{4.5}$$

at zero order in δ, one has

$$r_+ \simeq m \simeq a,, \qquad \Omega_H \simeq \frac{1}{2a(\prod_c + \prod_s)}. \tag{4.6}$$

Starting from (3.1), it is straightforward to separate variables in the massless scalar wave equation $\Box\Phi = 0$ in this background. Setting

$$\Phi = e^{-i\omega t + im\phi}\frac{\psi(r)S(\chi)}{\sqrt{\Delta(1-\chi^2)}}, \qquad \chi = \cos\theta, \tag{4.7}$$

and introducing the separation constant λ^2 bring the two wave equations into Schödinger-like canonical form. The angular equation determines the spheroidal harmonics

$$S_{\lambda,m_\phi}''(\chi) + \frac{(1-\chi^2)(a^2\omega^2\chi^2 - a^2\omega^2 + \lambda^2) + 1 - m_\phi^2}{(1-\chi^2)^2} \times S_{\lambda,m_\phi}(\chi) = 0. \tag{4.8}$$

For small $a^2\omega^2$, $\lambda^2 = \ell(\ell+1) + \mathcal{O}(a^2\omega^2)$ and $S_{\ell,m}(\chi) = P_{\ell,m}(\chi) + \mathcal{O}(a^2\omega^2)$ are known as spheroidal harmonics. The problem can be solved using standard perturbation theory in quantum mechanics, even though the wave equation is classical, or identifying the (confluent) HE (4.8) (with two regular and one irregular singularities) with the quantum SW curves for $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and $N_f = 3$ hypermultiplets in the fundamental (six doublets) [27–31].

If we define $A = \lambda^2 - a^2\omega^2$ as in [30], the radial equation can be written as

$$\begin{aligned}\psi'' + \Bigg\{&\omega^2\Delta\Bigg[r^2 + 2m\Big(1+\sum_i s_i^2\Big)r \\ &- 4m^2\Big(\sum_{i<j<k}\sigma_i^2\sigma_j^2\sigma_k^2 - 2\prod_s\Big(\prod_c - \prod_s\Big)\Big)\Bigg] + \frac{\Delta'^2}{4} - \Delta \\ &+ 4m^2\Bigg[\omega_{SR}\Big(r_+\prod_c - (r_+ - 2m)\prod_s\Big) \\ &- \omega\Big(r\prod_c - (r-2m)\prod_s\Big)\Bigg]^2 - \Delta A\Bigg\}\frac{\psi}{\Delta^2} = 0. \end{aligned}\tag{4.9}$$

Far from the horizon $r \gg r_+ \gg \delta$ the radial equation reduces to

$$\psi''(r) + \Bigg[\omega_{SR}^2 + \frac{2r_+\omega_{SR}^2(2+\sum_i s_i^2)}{(r-r_+)} + \frac{r_+^2\omega_{SR}^2(7+6\sum_i s_i^2 + 4\sum_{i<j}s_i^2 s_j^2) - A}{(r-r_+)^2}\Bigg]\psi(r) = 0. \tag{4.10}$$

Requiring $\psi \sim e^{i\omega_{SR}\hat{r}}$ at infinity (outgoing) and as $\psi \sim \hat{r}^{1/2+\alpha}$ at the horizon (regularity) one has

$$\psi(r) = c_\infty e^{i\omega_{SR}(r-r_+)}(r-r_+)^{1/2+\alpha} U(\tilde{A},\tilde{B};z), \tag{4.11}$$

where c_∞ is a constant and U is Tricomi confluent hypergeometric function, while

$$\begin{aligned} z &= -2i\omega_{SR}(r-r_+), \\ \alpha^2 &= A + \frac{1}{4} - a^2\omega_{SR}^2\Big(7 + 6\sum_i s_i^2 + 4\sum_{i<j} s_i^2 s_k^2\Big), \\ \tilde{A} &= \alpha + \frac{1}{2} - i\omega_{SR} a\Big(2 + \sum_i s_i^2\Big), \\ \tilde{B} &= 1 + 2\alpha. \end{aligned}\tag{4.12}$$

On the other hand, the radial equation in the near-horizon limit can be approximated by defining $\tau = (r - r_+)/\delta$ with $\delta \ll r_+$. In the variable τ one finds

$$\psi''(\tau) + Q(\tau)\psi(\tau) = 0 \tag{4.13}$$

with

$$\begin{aligned} Q(\tau) = \frac{1}{\tau^2(1+\tau)^2}\Bigg\{&4a^2\Bigg[\omega_{SR}\tau\Big(\prod_c - \prod_s\Big) \\ &+ \nu a\Big(\prod_c + \prod_s\Big)\Bigg]^2 + \frac{1}{4} - \tau(1+\tau) \\ &\times\Bigg[\alpha^2 + 4\omega_{SR}^2 a^2\Big(\prod_c - \prod_s\Big)^2 - \frac{1}{4}\Bigg]\Bigg\}. \end{aligned}\tag{4.14}$$

The solution can be written in terms of hypergeometric functions. Imposing ingoing boundary conditions at the horizon, one finds

$$\psi(\tau) = c_H \tau^{\frac{1}{2} - \frac{ia\nu}{\Omega_H}}(1+\tau)^{\frac{1}{2} - \frac{ia\nu}{\Omega_H} + 2ia\omega_{SR}\left(\prod_c - \prod_s\right)} \times {}_2F_1(\bar{A},\bar{B},\bar{C};-\tau), \tag{4.15}$$

where c_H is a constant and

$$\begin{aligned} \bar{A} &= \frac{1}{2} - \alpha - \frac{2ia\nu}{\Omega_H} + 2ia\omega_{SR}\Big(\prod_c - \prod_s\Big), \\ \bar{B} &= \frac{1}{2} + \alpha - \frac{2ia\nu}{\Omega_H} + 2ia\omega_{SR}\Big(\prod_c - \prod_s\Big), \\ \bar{C} &= 1 - \frac{2ia\nu}{\Omega_H}. \end{aligned}\tag{4.16}$$

By expanding (4.11) near the horizon, one finds

$$\psi(r) \sim (r-r_+)^{\frac{1}{2}-\alpha}\Bigg[\frac{(-2i\omega)^{-2\alpha}\Gamma(2\alpha)}{\Gamma(\tilde{A})} + \frac{\Gamma(-2\alpha)(r-r_+)^{2\alpha}}{\Gamma(\tilde{A}-\tilde{B}+1)}\Bigg], \tag{4.17}$$

while far away from the horizon, (4.15) reduces to

$$\psi(r) \sim (r-r_+)^{\frac{1}{2}-\alpha}\Bigg[\frac{\Gamma(2\alpha)}{\Gamma(\bar{B})\Gamma(\bar{C}-\bar{A})}(r-r_+)^{2\alpha}\delta^{-\frac{1}{2}-\alpha} + \frac{\Gamma(-2\alpha)}{\Gamma(\bar{A})\Gamma(\bar{C}-\bar{B})}\delta^{\alpha-\frac{1}{2}}\Bigg]\Gamma(\bar{C}). \tag{4.18}$$

Matching (4.17) and (4.18) one finds

$$\frac{\Gamma^2(2\alpha)\Gamma(\tilde{A}-\tilde{B}+1)\Gamma(\bar{A})\Gamma(\bar{C}-\bar{B})}{\Gamma(\bar{B})\Gamma(\bar{C}-\bar{A})\Gamma^2(-2\alpha)\Gamma(\tilde{A})}(-2i\omega_{SR}\delta)^{-2\alpha} = 1. \tag{4.19}$$

Since $\delta \sim 0$ and $\Re\alpha > 0$, the factor $\delta^{-\alpha}$ in the left hand side diverges, so it has to be compensated by a pole of $\Gamma(\bar{B})$ in the denominator[3] i.e.,

$$\bar{B} = -n + (-2i\omega_{SR}\delta)^{2\alpha}\eta \tag{4.20}$$

with

$$\eta = \frac{(-1)^n}{n!}\frac{\Gamma^2(-2\alpha)\Gamma(\bar{C}-\bar{A})\Gamma(\tilde{A})}{\Gamma^2(2\alpha)\Gamma(\tilde{A}-\tilde{B}+1)\Gamma(\bar{A})\Gamma(\bar{C}-\bar{B})} \tag{4.21}$$

leading to

$$\omega = \omega_{SR} + \delta\left[\Omega_H\omega_{SR}\left(\prod_c - \prod_s\right) - i\frac{\Omega_H}{2a}\left(n+\frac{1}{2}+\alpha\right)\right] + \cdots. \tag{4.22}$$

The imaginary part of the quasinormal frequencies is expressed in terms of the Lyapunov exponent $\lambda_L = \frac{\Omega_H}{2a} \approx T_{\mathrm{BH}}$ [36,37,40] and the overtone number n gets shifted by $\alpha + \frac{1}{2}$.

One might notice that imposing the condition for CT invariance (3.26) allows to rewrite $\tilde{A}$ and $\tilde{B}$ in (4.12) in terms of $\bar{A}$, $\overline{B}$, and $\bar{C}$ in (4.16) so that

$$\tilde{A} = \bar{C} - \bar{A}, \bar{C} - \bar{B} = \tilde{A} - \tilde{B} + 1. \tag{4.23}$$

This simplification is another bonus of the CT symmetry for NESTURBHs with special charges, satisfying (3.14).

V. ROTATING BHS AND D3-BRANES IN HIGHER DIMENSIONS

Very much as for nonrotating BHs and branes, one may ask whether rotating BHs and branes in higher dimensions enjoy invariance under generalized CT transformations. As we will see the answer is positive and we manage to identify at least one class of $5d$ extremal BHs that enjoy this property. We will then address the issue in $D > 5$ and we do not find any simple solution with this property. We do not exclude that more elaborate constructions with various scalars and gauge fields can enjoy CT invariance.

A. $d = 5$ CCLP solution

Five-dimensional charged rotating BH solutions to Einstein-Maxwell theory were found by Cheung, Cvetic, Lu and Pope in [22,23]. They depend on four parameters; mass M, charge Q, and two angular momenta ℓ_1 and ℓ_2. The metric reads

$$\begin{aligned} ds_5^2 = &-dt^2 + \Delta_t(dt-\omega_1)^2 - \frac{2Q}{\rho^2}(dt-\omega_1)\omega_2 \\ &+ \rho^2\left(d\theta^2 + \frac{dr^2}{\Delta_r}\right) + (r^2+l_1^2)\sin^2\theta d\phi^2 \\ &+ (r^2+l_2^2)\cos^2\theta d\psi^2, \end{aligned} \tag{5.1}$$

while the abelian gauge field is

$$A = \frac{\sqrt{3}Q}{\rho^2}(dt-\omega_1),$$

where

$$\begin{aligned} &\rho^2 = r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta, \\ &\Delta_r = \frac{(r^2+l_1^2)(r^2+l_2^2) - 2Mr^2 + 2l_1l_2Q + Q^2}{r^2}, \\ &\Delta_t = \frac{2M}{\rho^2} - \frac{Q^2}{\rho^4}, \\ &\omega_1 = l_1\sin^2\theta d\phi + l_2\cos^2\theta d\psi, \\ &\omega_2 = l_2\sin^2\theta d\phi + l_1\cos^2\theta d\psi. \end{aligned} \tag{5.2}$$

The curvature singularity of the metric is located at $\rho = 0$, i.e., $r^2_{\sin g} = -l_1^2\cos^2\theta - l_2^2\sin^2\theta$. The horizons are located at

$$r_\pm^2 = \frac{1}{2}(2M - l_1^2 - l_2^2) \pm \frac{1}{2}\sqrt{(2M-l_1^2-l_2^2)^2 - 4(Q+l_1l_2)^2} \tag{5.3}$$

where Δ_r vanishes. There is also an ergosurface that is delimited by the larger solution of $g_{tt}(r_{\mathrm{ergo}}) = 0$ where $\Delta_t = 1$ and $\rho^2_{\mathrm{ergo}} = M \pm \sqrt{M^2 - Q^2}$:

$$r^2_{\mathrm{ergo}} = M \pm \sqrt{M^2-Q^2} - l_1^2\cos^2\theta - l_2^2\sin^2\theta. \tag{5.4}$$

Extremality ($T_{\mathrm{BH}} = 0$) requires $r_+ = r_-$, that is

$$|2M - l_1^2 - l_2^2| = 2|Q + l_1l_2| \tag{5.5}$$

while supersymmetry (BPS condition) requires $Q = M$ [21].

Geodesics of massless neutral probes can be written in Hamiltonian form $\mathcal{H} = 0$ exploiting the conservation of three momenta: $P_t = -E$, $P_\phi = J_\phi$, and $P_\psi = J_\psi$. As in the KN and STURBH cases the Hamiltonian can be separated as

$$\mathcal{H} = \frac{\mathcal{H}_r + \mathcal{H}_\theta}{2\rho^2} = 0 \tag{5.6}$$

with

[3]Since we want to determine the near superradiance parameter ν, we are forced to quantize $\bar{B}$. The arguments of the other Γ functions in the denominator are independent of ν.

$$
\begin{aligned}
\mathcal{H}_r &= \Delta_r P_r^2 + \frac{E^2}{r^2\Delta_r}[(r^2+\ell_1^2+\ell_2^2)(Q^2-r^2\Delta_r)+-2M(r^2+\ell_1^2)(r^2+\ell_2^2)] + \frac{P_\psi^2}{r^2\Delta_r}[(\ell_1^2-\ell_2^2)(r^2+\ell_1^2)\\
&\quad -2\ell_1(M\ell_1+Q\ell_2)] + \frac{P_\phi^2}{r^2\Delta_r}[(\ell_2^2-\ell_1^2)(r^2+\ell_2^2)+-2\ell_2(M\ell_2+Q\ell_1)] + \frac{2P_\psi E}{r^2\Delta_r}[(r^2+\ell_1^2)(2M\ell_2+Q\ell_1)\\
&\quad -\ell_2(Q^2+r^2\Delta_r)] + \frac{2P_\phi E}{r^2\Delta_r}[(r^2+\ell_2^2)(2M\ell_1+Q\ell_2)-\ell_1(Q^2+r^2\Delta_r)] - \frac{2P_\psi P_\phi}{r^2\Delta_r}[2M\ell_1\ell_2+Q(\ell_1^2+\ell_2^2)] = -K^2,\\
\mathcal{H}_\theta &= P_\theta^2 + \left(E\ell_1\sin\theta+\frac{P_\phi}{\sin\theta}\right)^2 + \left(E\ell_2\cos\theta+\frac{P_\psi}{\cos\theta}\right)^2 = K^2. \qquad (5.7)
\end{aligned}
$$

Once again, K^2 represents the square of the angular momentum. In a certain sense, one can then write the radial equation in the form

$$
\mathcal{R}(r) = \left(\frac{r}{2E}\frac{\partial\mathcal{H}_r}{\partial P_r}\right)^2 = \frac{r^2\Delta_r^2P_r^2}{E^2} = Ar^6+Br^4+Cr^2+D \qquad (5.8)
$$

with

$$
\begin{aligned}
A &= 1, \qquad B = 2\ell_2 b_\psi + 2\ell_1 b_\phi + 2(\ell_1^2+\ell_2^2) - b^2,\\
C &= 2Q\ell_1\ell_2 + \ell_1^4 + 3\ell_1^2\ell_2^2 + \ell_2^4 + b^2(2M-\ell_1^2-\ell_2^2) + (\ell_1^2-\ell_2^2)(b_\phi^2-b_\psi^2) + 2b_\phi(\ell_1(\ell_1^2+\ell_2^2-4M)-Q\ell_2)\\
&\quad + 2b_\psi(\ell_2(\ell_1^2+\ell_2^2-4M)-Q\ell_1),\\
D &= (Q+\ell_1\ell_2)^2(-b^2+4b_\phi\ell_1+4b_\psi\ell_2) + 2(Q+\ell_1\ell_2)(b_\phi^2\ell_1\ell_2+b_\psi^2\ell_1\ell_2+\ell_1^3\ell_2+\ell_1\ell_2^3+b_\phi b_\psi(\ell_1^2+\ell_2^2)\\
&\quad - b_\psi(\ell_1^3+2\ell_1\ell_2^2) - b_\phi(2\ell_1^2\ell_2+\ell_2^3)) + (2M-\ell_1^2-\ell_2^2)(b_\psi^2\ell_1^2+2b_\phi b_\psi\ell_1\ell_2-2b_\psi\ell_1^2\ell_2\\
&\quad + b_\phi^2\ell_2^2 - 2b_\phi\ell_1\ell_2^2 + \ell_1^2\ell_2^2), \qquad (5.9)
\end{aligned}
$$

and

$$
b = \frac{K}{E}, \qquad b_\phi = \frac{P_\phi}{E}, \qquad b_\psi = \frac{P_\psi}{E}, \qquad (5.10)
$$

are the impact parameters. Exchanging $\ell_1 \leftrightarrow \ell_2$ corresponds to exchanging $b_\psi \leftrightarrow b_\phi$ and $\theta \leftrightarrow \frac{\pi}{2}-\theta$.

Looking for extremal (not necessarily BPS) solutions with $r_+ = r_- = r_H$, that enjoy symmetry under CT inversions we further specialize to the case

$$
2M = \ell_1^2+\ell_2^2, \qquad Q = -\ell_1\ell_2, \qquad (5.11)
$$

but we allow for $Q \neq M$ and $\ell_1 \neq \ell_2$. With this choice, the horizon falls onto the origin

$$
r_H^2 = 0 \quad \text{so that} \quad \Delta_r = r^2 \qquad (5.12)
$$

and the parameters of the radial equation simplify to

$$
\begin{aligned}
A &= 1, \qquad B = 2\ell_1 b_\phi + 2\ell_2 b_\psi + 2(\ell_1^2+\ell_2^2) - b^2,\\
C &= (\ell_1^2-\ell_2^2)\left[\left(b_\phi - \frac{\ell_1^3}{\ell_1^2-\ell_2^2}\right)^2 - \left(b_\psi - \frac{\ell_2^3}{\ell_2^2-\ell_1^2}\right)^2\right],\\
D &= 0, \qquad (5.13)
\end{aligned}
$$

The radial equation can be written as

$$
\mathcal{R}(r) = \frac{r^4P_r^2}{E^2} = r^4+Br^2+C = (r^2-r_1^2)(r^2-r_2^2). \qquad (5.14)
$$

The zeros of P_r are located at

$$
\begin{aligned}
r_{1,2}^2 = \frac{1}{2}\Big\{ & b^2 - 2(\ell_1^2+\ell_2^2) - 2\ell_1 b_\phi - 2\ell_2 b_\psi\\
& \pm \Big[b^4 + -4b^2(\ell_1^2+\ell_2^2+\ell_1 b_\phi+\ell_2 b_\psi)\\
& + 4[b_\phi^2\ell_2^2 + b_\psi^2\ell_1^2 + 2b_\phi\ell_1(2\ell_1^2+\ell_2^2)\\
& + 2b_\psi\ell_2(2\ell_2^2+\ell_1^2) - 4\ell_1^2\ell_2^2\Big]^{\frac{1}{2}}\Big\} \qquad (5.15)
\end{aligned}
$$

so that

$$r_1 r_2 = \sqrt{C} = r_c^2(b_\psi^c, b_\phi^c),$$
$$b_c^2(b_\psi^c, b_\phi^c) = 2\ell_1 b_\phi^c + 2\ell_2 b_\psi^c + 2(\ell_1^2 + \ell_2^2) + 2r_c^2. \quad (5.16)$$

Indeed, the criticality conditions $\mathcal{R} = 0 = \mathcal{R}'$ allow to express r_c and b_c^2 in terms of b_ψ^c, b_ϕ^c., so much so that generalized CT inversions that exchange horizon and infinity, keep the photon-halo $r = r_c(b_\psi^c, b_\phi^c)$ fixed, as expected. For consistency one has to impose positivity of $C = r_c^4$ and that means

$$\left(b_\phi - \frac{\ell_1^3}{\ell_1^2 - \ell_2^2}\right)^2 > \left(b_\psi + \frac{\ell_2^3}{\ell_1^2 - \ell_2^2}\right)^2 \quad \text{for } \ell_1 > \ell_2 \quad (5.17)$$

or vice versa for $\ell_2 > \ell_1$. Moreover $b_c^2 \geq 0$ requires

$$\ell_1 b_\phi^c + \ell_2 b_\psi^c > -\ell_1^2 - \ell_2^2 - r_c^2.$$

For $\ell_1 = \pm\ell_2 = \ell$ only the combination $b_\pm = b_\phi \pm b_\psi$ matters. In this case, choosing the plus sign and introducing $\hat{r}_c = r_c/\ell$ and $\hat{b}_{+,c} = b_{+,c}/\ell$, one has

$$\hat{r}_c^4 = 3 - 2\hat{b}_{+,c}, \qquad \hat{b}_c^2 = 2\hat{b}_{+,c} + 4 + 2\sqrt{3 - 2\hat{b}_{+,c}}. \quad (5.18)$$

The minimal value for both $\hat{r}_c^2$ and $\hat{b}_c^2$ is zero. The former, $\hat{r}_c^2 = 0$ (horizon), is reached for $\hat{b}_{+,c} = 3/2$ whereby $\hat{b}_c^2 = 7$. The latter, $\hat{b}_c^2 = 0$, for $\hat{b}_{+,c} = -3 - \sqrt{8}$, whereby $\hat{r}_c^2 = 1 + \sqrt{8} > \sqrt{3}$.

For $\ell_1 \neq \pm\ell_2$ the situation is much more involved. Although it is easy to determine the minimal value of $\hat{r}_c^2 = 0$ (horizon) that is reached for $b_\phi = \frac{\ell_1^3}{\ell_1^2 - \ell_2^2}$ and $b_\psi = -\frac{\ell_2^3}{\ell_1^2 - \ell_2^2}$ whereby $b^2 = 4(\ell_1^2 + \ell_2^2)$, the maximum is harder to determine. In fact, for $\ell_1^2 > \ell_2^2$, taking $b_\psi = -\frac{\ell_2^3}{\ell_1^2 - \ell_2^2}$ and letting $\tilde{b}_\phi = b_\phi - \frac{\ell_1^3}{\ell_1^2 - \ell_2^2}$ grow arbitrarily, both r_c^2 and b_c^2 grow arbitrarily.

Thanks to the previous properties of the roots, the radial action is form invariant

$$S_r = \int P_r dr \quad (5.19)$$

under CT inversions $r \to r_c^2/r$ that maps the scattering regime outside the photon-sphere into the inspiraling regime inside it.

B. Rotating D3 branes

Let us now pass to consider rotating D3-branes. Thanks to the transversal $SO(6)$ symmetry, one can introduce three independent angular momentum parameters ℓ_1,ℓ_2, and ℓ_3. The metric of a rotating black D3 brane is given by [20]

$$ds^2 = f_0^{-1/2}(-h_0 dt^2 + d\mathbf{x}^2) + f_0^{1/2}\Bigg\{\frac{\Delta dr^2}{\prod_{i=1}^3 (1 + \frac{\ell_i^2}{r^2}) - \frac{2m}{r^4}} + r^2\Bigg[\Delta_1 d\theta^2 + \Delta_2 c_\theta^2 d\psi^2 - 2\frac{\ell_2^2 - \ell_3^2}{r^2} c_\theta s_\theta c_\psi s_\psi d\theta d\psi$$
$$+\left(1 + \frac{\ell_1^2}{r^2}\right) s_\theta^2 d\phi_1^2 + \left(1 + \frac{\ell_2^2}{r^2}\right) c_\theta^2 s_\psi^2 d\phi_2^2 + \left(1 + \frac{\ell_3^2}{r^2}\right) c_\theta^2 c_\psi^2 d\phi_3^2 + \frac{2m}{r^6 \Delta f_0}(\ell_1 s_\theta^2 d\phi_1 + \ell_2 c_\theta^2 s_\psi^2 d\phi_2 + \ell_3 c_\theta^2 c_\psi^2 d\phi_3)^2\Bigg] +$$
$$-\frac{4m \cosh\alpha}{r^4 \Delta f_0} dt(\ell_1 s_\theta^2 d\phi_1 + \ell_2 c_\theta^2 s_\psi^2 d\phi_2 + \ell_3 c_\theta^2 c_\psi^2 d\phi_3)\Bigg\}, \quad (5.20)$$

where $d\mathbf{x}^2$ denotes the Euclidean metric of the longitudinal $\mathbf{R}^3$ and

$$h_0 = 1 - \frac{2m}{r^4 \Delta}, \qquad f_0 = 1 + \frac{2m \sinh^2\alpha}{r^4 \Delta},$$
$$\Delta = 1 + \frac{\ell_1^2}{r^2} c_\theta^2 + \frac{\ell_2^2}{r^2}(s_\theta^2 s_\psi^2 + c_\psi^2) + \frac{\ell_3^2}{r^2}(s_\theta^2 c_\psi^2 + s_\psi^2)$$
$$+ \frac{\ell_2^2 \ell_3^2}{r^4} s_\theta^2 + \frac{\ell_1^2 \ell_3^2}{r^4} c_\theta^2 s_\psi^2 + \frac{\ell_1^2 \ell_2^2}{r^4} c_\theta^2 c_\psi^2,$$
$$\Delta_1 = 1 + \frac{\ell_1^2}{r^2} c_\theta^2 + \frac{\ell_2^2}{r^2} s_\theta^2 s_\psi^2 + \frac{\ell_3^2}{r^2} s_\theta^2 c_\psi^2,$$
$$\Delta_2 = 1 + \frac{\ell_2^2}{r^2} c_\psi^2 + \frac{\ell_3^2}{r^2} s_\psi^2. \quad (5.21)$$

The parameters α and m are related to the D3-brane charge N by

$$4\pi g_s N \alpha'^2 = 2m \cosh\alpha \sinh\alpha. \quad (5.22)$$

The horizon $r = r_H$ is given by the largest real root of

$$\prod_{i=1}^3 (r^2 + \ell_i^2) - 2mr^2 = 0. \quad (5.23)$$

The extremal configuration is reached if we take simultaneously $\alpha \to \infty$ and $m \to 0$, keeping fixed $L^4 = 2m \sinh^2\alpha$. If $\ell_i \neq 0$, the limit $m \to 0$ exposes a naked singularity [20].

The null Hamiltonian in the extremal case reads

$$\mathcal{H}=-f_0^{1/2}E^2+f_0^{-1/2}\left\{\frac{\prod_{i=1}^3\rho_i^2}{r^6\Delta}P_r^2+\frac{J_1^2}{\rho_1^2s^2\theta}+\frac{J_2^2}{\rho_2^2c_\theta^2s_\psi^2}+\frac{J_3^2}{\rho_3^2c_\theta^2c_\psi^2}+\frac{r^2(\Delta_2P_\theta^2+\frac{\Delta_1P_\psi^2}{c_\theta^2})+2(\ell_2^2-\ell_3^2)s_\psi c_\psi t_\theta P_\theta P_\psi}{r^4\Delta_1\Delta_2-(\ell_2^2-\ell_3^2)^2c_\psi^2s_\theta^2s_\psi^2}\right\}, \quad (5.24)$$

where $\rho_i^2 = r^2 + \ell_i^2$. In general it is not separable.

If we restrict attention on geodesics in the $\theta = \psi = 0$ plane, where $\partial\mathcal{H}/\partial\theta = \partial\mathcal{H}/\partial\psi = 0$ so that one can consistently set $P_\theta = P_\psi = 0$, the metric drastically simplifies

$$ds^2=-f_0^{-1/2}dt^2+f_0^{1/2}\Delta_3^{-1}dr^2+r^2\Delta_3f_0^{1/2}d\phi_3^2,$$
$$\Delta_3=1+\frac{\ell_3^2}{r^2},\qquad \Delta=\Delta_1\Delta_2,\qquad h_0=1,\qquad f_0=1+\frac{L^4}{\Delta r^4}. \quad (5.25)$$

In Hamiltonian form, the system is separable and the radial momentum can be expressed in terms of the radial variable r and the conserved quantities i.e., the energy E and the relevant angular momentum $J = J_3$, since $J_1 = J_2 = 0$,

$$\frac{r^4\Delta_3^2P_r^2}{E^2}=r^4\Delta_3f_0-b^2r^2=r^4\Delta+\frac{L^4}{\Delta}-b^2r^2, \quad (5.26)$$

where $b = J/E$. If we take for simplicity $\ell_1 = \ell_2 = \ell_3 = \ell \leq L$, so that $\Delta_1 = \Delta_2 = \Delta_3$, we find

$$P_r^2=\frac{E^2}{r^4(1+\frac{\ell^2}{r^2})^3}[(r^2+\ell^2)^2-b^2(r^2+\ell^2)+L^4]. \quad (5.27)$$

The radial action, in the coordinate $z = r^2 + \ell^2$ reads

$$S_r=\int P_rdr=\int\frac{E}{2}\frac{dz}{z^{3/2}}\sqrt{z^2-b^2z+L^4}. \quad (5.28)$$

This integral is invariant under the transformation $z \to L^4/z$, or in the original radial coordinate

$$r^2+\ell^2\to\frac{L^4}{r^2+\ell^2}. \quad (5.29)$$

The critical impact parameter and the critical radius (that determines the photon-sphere in this plane) are the critical points of P_r, thus looking at (5.27)

$$r_c^2+\ell^2=\frac{1}{2}[b_c^2\pm\sqrt{b_c^4-4L^4}]. \quad (5.30)$$

So the critical impact parameter is the same as in the nonrotating case

$$b_c=\sqrt{2}L,\qquad r_c^2=L^2-\ell^2. \quad (5.31)$$

The significance of the CT symmetry in the present context is not totally clear since the range of validity is limited to $\ell^2r^2 < L^4 - \ell^4$ that, reassuringly, includes the photon-sphere for $L > \ell$.

VI. NO HORIZON, NO CT SYMMETRY

As shown in [5] and above, CT transformations exchange null infinity with the horizon keeping the photon-sphere fixed. Thus we should not expect smooth horizonless geometries like JMaRT [24] (and their supersymmetric limit GMS [25,26]) nor (circular) fuzz balls in $D = 6$ to enjoy CT symmetry despite the presence of a photon-sphere. One may try to identify a generalization of CT transformations that exchange infinity with the cap but the different nature of the two regions presagite the failure of the attempt, as we will show in the following. In order to illustrate the problem, we briefly analyze massless geodesics in the smooth horizonless backgrounds of the JMaRT (and GMS) family and in the circular fuzz balls in $D = 6$.

A. JMaRT and GMS limit in $d = 6$

JMaRT solutions [24] are smooth horizonless geometries of 3-charge systems. They are 'over-rotating' to be considered *bona fide* microstates of BHs. They are known to undergo an ergoregion instability [41,42] and a Penrose process [43] that should produce a BPS supersymmetric GMS solution [25,26] as stable remnant [44]. The six-dimensional metric of JMaRT can be written as

$$ds_6^2=ds_5^2+\left(dy-\frac{A}{\sqrt{3}}\right)^2, \quad (6.1)$$

where in the extremal limit and in the equatorial plane ($\theta = \pi/2$)

$$ds_5^2=-Z^{-2}(dt+\alpha)^2+Zds_4^2, \quad (6.2)$$

$$A=\frac{Q}{Zf}\left[-dt+\frac{\sqrt{Q}}{2}\left(\frac{a_\psi}{a_\phi}+\frac{a_\phi}{a_\psi}+\frac{1}{Q}a_\phi a_\psi\right)d\phi\right],$$
$$Z=1+\frac{Q}{f},\qquad f=r^2-a_\psi^2,$$
$$\alpha=\frac{\sqrt{Q}}{2f}\left[Q\left(\frac{a_\psi}{a_\phi}+\frac{a_\phi}{a_\psi}\right)+(1+2Z)a_\psi a_\phi\right]d\phi, \quad (6.3)$$

$$ds_{4,\mathrm{extr}}^2=\frac{fdr^2}{r^2+a_\phi^2-a_\psi^2}+\left(\frac{a_\phi^2a_\psi^2}{f}+r^2+a_\phi^2\right)d\phi^2. \quad (6.4)$$

The components of the metric are easily identified

$$
\begin{aligned}
g_{tt} &= -Z^{-2} + \frac{Q^2}{3Z^2 f^2}, \qquad g_{yy} = 1, \qquad g_{rr} = \frac{Zf}{r^2 + a_\phi^2 - a_\psi^2},\\
g_{\phi\phi} &= \frac{Q^3}{12Z^2 f^2}\left(\frac{a_\psi}{\alpha_\phi} + \frac{a_\phi}{\alpha_\psi} + \frac{a_\phi a_\psi}{Q}\right)^2 + Z\left(\frac{a_\phi^2 a_\psi^2}{f} + r^2 + a_\phi^2\right) - \frac{Z^{-2}Q}{4f^2}\left[Q\left(\frac{a_\psi}{\alpha_\phi} + \frac{a_\phi}{\alpha_\psi}\right) + (1+2Z)a_\psi a_\phi\right]^2,\\
g_{t\phi} &= g_{\phi t} = -\frac{Q^{5/2}}{6Z^2 f^2}\left(\frac{a_\psi}{\alpha_\phi} + \frac{a_\phi}{\alpha_\psi} + \frac{a_\phi a_\psi}{Q}\right) - \frac{Z^{-2}\sqrt{Q}}{2f}\left[Q\left(\frac{a_\psi}{\alpha_\phi} + \frac{a_\phi}{\alpha_\psi}\right) + (1+2Z)a_\psi a_\phi\right],\\
g_{ty} &= g_{yt} = \frac{Q}{\sqrt{3}Zf},\\
g_{\phi y} &= g_{y\phi} = -\frac{Q^{3/2}}{2\sqrt{3}Zf}\left(\frac{a_\psi}{\alpha_\phi} + \frac{a_\phi}{\alpha_\psi} + \frac{a_\phi a_\psi}{Q}\right). \qquad (6.5)
\end{aligned}
$$

Setting for simplicity $P_y = 0$, the relevant element of the inverse metric are also easy to determine and the Hamiltonian for null geodesics reads

$$
\begin{aligned}
P_r^2 &= \frac{E^2\left(\sum_{k=0}^3 A_{2k} r^{2k}\right)}{4a_\phi^2 a_\psi^2 r^2 (r^2 - a_\psi^2 + a_\phi^2)^2},\\
A_6 &= 4a_\phi^2 a_\psi^2,\\
A_4 &= 4a_\phi^2 a_\psi^2 (a_\phi^2 - 2a_\psi^2 - b^2 + 3Q),\\
A_2 &= 4a_\phi a_\psi [-a_\psi^2 b Q^{3/2} - a_\phi^2 b\sqrt{Q}(3a_\psi^2 + Q)\\
&\quad - a_\phi^3 (a_\psi^3 - 3a_\psi Q)\\
&\quad + a_\phi (a_\psi^5 + a_\psi^3 (2b^2 - 3Q) + 3a_\psi Q^2)],\\
A_0 &= -[2a_\phi a_\psi^3 b - a_\psi^2 Q^{3/2} + a_\phi^2 \sqrt{Q}(-3a_\psi^2 + Q)]^2. \qquad (6.6)
\end{aligned}
$$

So the asymptotics of the radial action at infinity and near the cap $r_0^2 = a_\psi^2 - a_\phi^2$ are very different

$$
\begin{aligned}
&S_r \sim E\int^\infty dr \quad vs \quad S_r \sim E\beta^2 \int_{r_0} d\log(r - r_0)\\
&\beta^2 = \frac{2ba_\phi^3 a_\psi + Q^{3/2} a_\psi^2 - \sqrt{Q} a_\phi^2 (Q + 3a_\psi^2)}{4a_\phi a_\psi r_0^2}, \qquad (6.7)
\end{aligned}
$$

which seems to exclude any conceivable CT symmetry acting by conformal inversions. The only chance would be taking the BH limit $a_\phi = a_\psi = 0$ keeping their ratio fixed.

B. D1/D5 fuzz ball

With little hope, we now explore case of circular fuzz balls of 2-charge BHs and identify a CT-like transformation that gets close to being a symmetry, but we should recognize our inability that can be ascribed to the absence of the horizon that may reflect into new features in the GW ringdown signal such as echoes [1,45–49] and modified memories [50–55] or else in their multipoles [56–59].

The 6-d metric of a circular D1/D5 or, equivalently, D3-D3' fuzz ball reads [60–63]

$$
\begin{aligned}
ds^2 &= H^{-1}[-(dt + \omega_\phi d\phi)^2 + (dz + \omega_\psi d\psi)^2]\\
&\quad + H\left[(\rho^2 + a^2\cos^2\theta)\left(\frac{d\rho^2}{\rho^2 + a^2} + d\theta^2\right) + \rho^2\cos^2\theta d\psi^2\right.\\
&\quad \left. + (\rho^2 + a^2)\sin^2\theta d\phi^2\right], \qquad (6.8)
\end{aligned}
$$

where $H = \sqrt{H_1 H_5}$, with

$$
H_i = 1 + \frac{L_i^2}{\rho^2 + a^2 c_\theta^2}, \qquad \omega_\phi = \frac{aL_1L_5 s_\theta^2}{\rho^2 + a^2 c_\theta^2}, \qquad \omega_\psi = \frac{aL_1L_5 c_\theta^2}{\rho^2 + a^2 c_\theta^2}. \qquad (6.9)
$$

The metric (6.9) has no explicit dependence on t, z, ϕ, and ψ. As a result, the conjugate momenta $P_t = -E$, $P_z = P$, $P_\phi = J_\phi$ and $P_\psi = J_\psi$ are conserved.

Quite remarkably, very much as for KN or STURBHs, the system is separable [64–66]. In order to expose this property, one introduces the conjugate momenta

$$
P_\rho = \frac{H(\rho^2 + a^2 c_\theta^2)\dot{\rho}}{\rho^2 + a^2}, \qquad P_\theta = H(\rho^2 + a^2 c_\theta^2)\dot{\theta}, \qquad (6.10)
$$

and writes the (zero) mass-shell condition in Hamiltonian form $\mathcal{H} = g^{\mu\nu} P_\mu P_\nu = 0$ that can be separated into two equations

$$
\begin{aligned}
K^2 &= P_\theta^2 + (E^2 - P^2)a^2\sin^2\theta + \frac{J_\phi^2}{\sin^2\theta} + \frac{J_\psi^2}{\cos^2\theta},\\
-K^2 &= (\rho^2 + a^2)P_\rho^2 + \frac{(PL_1L_5 - aJ_\psi)^2}{\rho^2} +\\
&\quad - \frac{(EL_1L_5 - aJ_\phi)^2}{\rho^2 + a^2} - (\rho^2 + a^2 + L_1^2 + L_5^2)(E^2 - P^2), \qquad (6.11)
\end{aligned}
$$

where K^2 is the separation constant.

For brevity we will set

$$B_\psi^2 = \frac{(PL_1L_5 - aJ_\psi)^2}{\mathcal{E}^2 a^2}, \qquad B_\phi^2 = \frac{(EL_1L_5 - aJ_\phi)^2}{\mathcal{E}^2 a^2},$$
$$\mathcal{E}^2 = E^2 - P^2, \qquad b^2 = \frac{K^2}{\mathcal{E}^2}. \tag{6.12}$$

By the second (radial) equation in (6.11) the radial action is:

$$S_\rho = \int d\rho \mathcal{E} \sqrt{1 + \frac{L_1^2 + L_5^2 - b^2}{\rho^2 + a^2} - \frac{a^2 B_\psi^2}{\rho^2(\rho^2 + a^2)} + \frac{a^2 B_\phi^2}{(\rho^2 + a^2)^2}} \tag{6.13}$$

In terms of the variable $\rho^2 = x$ and if $B_\psi^2 = 0$ it becomes

$$S_\rho = \int \frac{\mathcal{E} dx}{2\sqrt{x}(x + a^2)} \Big[x^2 + x(L_1^2 + L_5^2 - b^2 + 2a^2) + a^2(a^2 + L_1^2 + L_5^2 + B_\phi^2 - b^2)\Big]^{\frac{1}{2}}. \tag{6.14}$$

One can envisage a generalized CT inversion of the form

$$x + a^2 \to \frac{x_0 + a^2}{x + a^2} \tag{6.15}$$

with some x_0; alas, this does not leave the radial action invariant for any choice of x_0, except for the case $a = 0$ that produces a 2-charge small BH.

VII. (A)DS KN VS HEUN EQUATION AND CT INVARIANCE

Let us pass on to consider AdS_4 KN BHs and study scalar wave propagation in this backgrounds. As shown in [30], in order to have the minimal number (i.e., four) of regular singular points in the radial equation, it is convenient to choose a tachyonic scalar with mass $\mu^2 L^2 = -2$. The mass is above the Breitenlohner-Freedman (BF) bound $\mu_{BF}^2 L^2 = -9/4$. Scalar fields of this kind are dual to boundary conformal operators of dimension $\Delta = 1, 2$ such as ϕ^2 or ψ^2 [67,68], and correspond to internal fluctuations of the metric in the case of maximal supergravity in S^7. We spell out the conditions under which the relevant Heun equation (HE) with four regular singular points, corresponding to real and complex horizons, admit CT symmetry, exchanging horizons in pairs. This leads to a SHE (special Heun equation) with restricted parameters (but no confluence of singularities). However, quite disappointingly, when we translate the parameters of the SHE into BH parameters we find a trivial configuration with zero mass. We then switch to the extremal case as well as to generalized CT transformation that rescale the wave function.

A. Heun equation for CT invariant AdS KN BHs

The AdS-KN metric in $4d$ is given by

$$ds^2 = -\frac{\Delta_r[dt - ad\phi(1-\chi^2)]^2}{\alpha^2\rho^2} + \frac{\Delta_\chi[adt - a\phi(a^2 + r^2)]^2}{\alpha^2\rho^2} + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\chi^2}{\Delta_\chi}\right), \tag{7.1}$$

where $\chi = \cos\theta$ and

$$\Delta_r = (r^2 + a^2)\left(1 + \frac{r^2}{L^2}\right) - 2Mr + Q^2$$
$$= \frac{1}{L^2}(r - r_+)(r - r_-)(r - \rho)(r - \bar\rho),$$
$$\Delta_\chi = (1 - \chi^2)\left(1 - \frac{a^2\chi^2}{L^2}\right), \qquad \rho^2 = r^2 + a^2\chi^2,$$
$$\alpha = 1 - \frac{a^2}{L^2}. \tag{7.2}$$

The four roots of Δ_r satisfy

$$r_+ + r_- + \rho + \bar\rho = 0 \tag{7.3}$$

and the relations with the BH parameters a, Q, and M,

$$r_+r_- + |\rho|^2 - (r_+ + r_-)^2 = a^2 + L^2,$$
$$(|\rho|^2 - r_+r_-)(r_+ + r_-) = 2ML^2,$$
$$r_+r_-|\rho|^2 = L^2(a^2 + Q^2), \tag{7.4}$$

that easily allow us to express M, Q and a in terms of $r_\pm$, ρ, and $\bar\rho$ at fixed L, while the inverse relations require solving a cubic equation and are more involved. For large $L \gg M$, a, Q two roots are real

$$r_\pm \approx M \pm \sqrt{M^2 - a^2 - Q^2} + \mathcal{O}(L^{-2}) \tag{7.5}$$

and two are complex conjugates

$$\rho, \bar\rho = \pm iL - M \pm i\mathcal{O}(L^{-1}) \tag{7.6}$$

The wave equation for a scalar can be separated. Setting

$$\Phi(t, r, \chi, \phi) = e^{-i\omega t + im_\phi \phi} \frac{\psi(r)S(\chi)}{\sqrt{\Delta_r \Delta_\chi}} \tag{7.7}$$

brings both radial and angular equations in canonical form. The eigenvalue $K^2 = \ell(\ell + 1) + \mathcal{O}(a^2\omega^2)$ of the spheroidal-harmonics equation for the angular part plays the role of a generalized angular momentum and enters the radial equation as a separation constant

$$\psi''(r)+Q_r(r)\psi(r)=0,$$
$$Q_r(r)=\frac{(\hat\omega(a^2+r^2)-a\hat m)^2}{\Delta_r^2}-\frac{K^2+\mu^2r^2}{\Delta_r}-\frac{1}{2}\frac{\Delta_r''}{\Delta_r}+\frac{1}{4}\left(\frac{\Delta_r'}{\Delta_r}\right)^2, \tag{7.8}$$

with $\hat\omega=\alpha\omega$, $\hat m=\alpha m_\phi$. For $\mu^2L^2=-2$ (7.8) has four regular singularities where $\Delta_r=0$. For different choices of μ^2—including 0—the resulting wave equation has more than four regular singular points. Performing the coordinate transformation

$$z=\frac{(r-r_+)(\rho-r_-)}{(r-r_-)(\rho-r_+)} \tag{7.9}$$

brings (7.8) in the standard form

$$\left[\partial_z^2+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-t}\right)\partial_z+\frac{\alpha\beta z-p}{z(z-1)(z-t)}\right]w(z)=0 \tag{7.10}$$

which has four regular singular points at 0, 1, t, and ∞ that correspond to r_+, ρ, $\bar\rho$, and r_- respectively.

Since the curvature singularity at $r=0$ and the boundary $r=\infty$ are regular points of the radial equation, we should look for generalized CT transformations exchanging $r_+\leftrightarrow r_-$ and $\rho\leftrightarrow\bar\rho$ for instance. In the standard form we should impose invariance under $z\to t/z$ which requires

$$\gamma=1-\delta,\qquad \epsilon=\delta,\qquad \alpha\beta=0, \tag{7.11}$$

and arbitrary q and

$$t=\frac{(\bar\rho-r_+)(\rho-r_-)}{(\bar\rho-r_-)(\rho-r_+)} \tag{7.12}$$

(7.10) can be put in canonical form if we introduce

$$w(z)=P(z)\psi(z),\qquad P(z)=z^{-\gamma/2}(z-1)^{-\delta/2}(z-t)^{-\varepsilon/2}, \tag{7.13}$$

and taking into account the restrictions on the parameters, it becomes

$$\psi''(z)+\left[\frac{1-\delta^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{2\delta-\delta^2}{4(z-t)^2}+\frac{\delta^2-\delta+2p+\delta^2t-\delta t+(2\delta-\delta^2)z}{2(z-1)z(t-z)}\right]\psi(z)=0. \tag{7.14}$$

Inverting (7.9) and comparing (7.14) with the radial equation for a scalar with mass $\mu^2L^2=-2$ in the AdS-KN background put severe constraints on the BH parameters as well as on ω and m_ϕ. We derive the dictionary in Appendix C, by exploiting the recently established correspondence between BH perturbation theory and quantum Seiberg-Witten curves [27–31]. In the present case of AdS_4-KN BHs, the relevant $\mathcal{N}=2$ SYM theory has gauge group $SU(2)$ and $N_f=4$ flavors of fundamental hypermultiplets with masses m_f. Moreover, imposing $z\to t/z$ invariance (à la CT) requires $m_0=m_\infty$ and $m_1=m_t$.

Although beyond the scope of the present investigation, let us stress here that the analysis of QNMs and other observables associated to fluctuations around a variety of BHs, D-branes, and fuzz balls with flat or AdS asymptotics governed by the standard Heun equation of various kinds, can be successfully tackled within this approach. Without dwelling on the details, let us pause and simply list, for the curious reader, the various interesting cases for the radial equation[4] following[5] [27–31]

(i) Heun equation (HE) with four regular singular points $\sim N_f=4=(2_L,2_R)$ SW $\sim$ AdS_4 KN with $\mu^2L^2=-2$ (no CT symmetry, in general).
(ii) Confluent Heun equation (CHE) with 2 regular and 1 irregular singular points $\sim N_f=3=(2_L,1_R)$ SW $\sim$ flat KN or STURBH (no CT symmetry, in general).
(iii) Reduced confluent Heun equation (RCHE) with two regular and one irregular singular points $\sim N_f=2=(2_L,0)$ SW $\sim$ CCLP BHs and D1/D5 or D3/D3' circular fuzz balls (no CT symmetry, in general).
(iv) Doubly confluent Heun equation (DCHE) with two irregular singular points $\sim N_f=2=(1_L,1_R)$ SW $\sim$ Extremal KN BHs, ESTURBHs and D3-D3-D3-D3 (CT symmetry).
(v) Reduced doubly confluent Heun equation (RDCHE) with two irregular singular points $\sim N_f=1=(1_L,0)$ SW $\sim$ extremal CCLP or BMPV BHs (CT symmetry).
(vi) Doubly reduced doubly confluent Heun equation (DRDCHE) with two irregular singular points $\sim N_f=0=(0_L,0_R)$ SW $\sim$ extremal D3-branes, D1/D5 or D3/D3' BHs (CT symmetry).

Going back to our problem, in the end it turns out that only trivial values for the BH parameters (as for example $M=0$) are compatible with CT symmetry.

We then have two options; to consider extremal KN BH (in Sec. VII B) or gain some flexibility by performing a Weyl transformation of the HE so as to introduce new parameters in the equation (in Sec. VII C).

[4]Spheroidal harmonics in $D=4$ (S^2 sphere) and $D=4$ (S^3 sphere) are related to CHE and RCHE respectively. Nothing special happens to these angular equations in the extremal limit.

[5]For convenience we split the flavors into $N_f=(N_L,N_R)$ as they appear in the Hanany-Witten setup of the $\mathcal{N}=2$ SYM theory [69].

B. Extremal AdS KN vs restricted confluent Heun equation

Some simplifications occur for extremal BHs with $r_+ = r_- = r_H$ still with $\rho \neq \bar{\rho}$ since we consider $L \gg M$. For $\rho = \bar{\rho}$ one would get a doubly confluent HE (DCHE) that can be easily shown to admit CT symmetry in general. The extremal mass is [70]

$$M_{\rm extr} = \frac{L}{3\sqrt{6}}\left(\sqrt{\left(1+\frac{a^2}{L^2}\right)^2 + \frac{12}{L^2}(a^2+Q^2)} + \frac{2a^2}{L^2} + 2\right) \times \left(\sqrt{\left(1+\frac{a^2}{L^2}\right)^2 + \frac{12}{L^2}(a^2+Q^2)} - \frac{a^2}{L^2} - 1\right)^{\frac{1}{2}}. \quad (7.15)$$

The relevant equation turns out to be the CHE (confluent Heun equation) that reads

$$\left[\partial_z^2 + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)\partial_z + \frac{\alpha z - q}{z(z-1)}\right] W(z) = 0. \quad (7.16)$$

If we define $W(z) = e^{-\frac{\varepsilon z}{2}} F(z)$ and impose symmetry under $z \to 1-z$, the parameters of the equation must obey to $\gamma = \delta$ and $\alpha = \gamma\varepsilon$. So the equation becomes

$$\left[\partial_z^2 + \gamma\left(\frac{1}{z} + \frac{1}{z-1}\right)\partial_z + \left(-\frac{\varepsilon^2}{4} + \frac{\tilde{q}}{z(z-1)}\right)\right] F(z) = 0,$$
$$\tilde{q} = \frac{\gamma\varepsilon}{2} - q. \quad (7.17)$$

In order to establish a dictionary with extremal AdS KN BH, we perform the mapping

$$z = \frac{(r-\rho)(\bar{\rho}-r_H)}{(r-r_H)(\bar{\rho}-\rho)}. \quad (7.18)$$

In canonical form the radial equation reads

$$\psi''(r) + Q_r(r)\psi(r) = 0,$$
$$Q_r(r) = \frac{\tilde{q}+\gamma-\gamma^2}{(r-r_H)^2} + \frac{2\gamma-\gamma^2}{4(r-\rho)^2} + \frac{2\gamma-\gamma^2}{4(r-\bar{\rho})^2} - \frac{\varepsilon^2(3\rho+\bar{\rho})^2(\rho+3\bar{\rho})^2}{64(\rho-\bar{\rho})^2(r-r_H)^4} + \frac{8(\tilde{q}+\gamma-\gamma^2)(\rho+\bar{\rho})}{(3\rho+\bar{\rho})(\rho+3\bar{\rho})(r-r_H)}$$
$$+ \frac{(\gamma-4)\gamma\rho + (4-5\gamma)\gamma\bar{\rho} + 2\tilde{q}(\rho+3\bar{\rho})}{2(\rho-\bar{\rho})(3\rho+\bar{\rho})(r-\rho)} + \frac{(\gamma-4)\gamma\bar{\rho} + (4-5\gamma)\gamma\rho + 2\tilde{q}(3\rho+\bar{\rho})}{2(\bar{\rho}-\rho)(\rho+3\bar{\rho})(r-\bar{\rho})}. \quad (7.19)$$

In terms of radial coordinates, the exchange symmetry $z \to 1-z$ translates into

$$r - r_H \to \frac{(r-r_H)(r_H-\rho)(r_H-\bar{\rho})}{2(\rho+\bar{\rho})r + r_H^2 - \rho\bar{\rho}}. \quad (7.20)$$

We start the comparison with (7.8) by studying the residues of the double poles in ρ and $\bar{\rho}$. Imposing equality of the two residues we obtain

$$a^2 - ab_\phi = \frac{1}{2}\left[\rho\bar{\rho} + \frac{(\rho+\bar{\rho})^2}{4}\right] = \frac{1}{2}(a^2 + L^2 + 4r_H^2),$$
$$(\gamma-1)^2 = -\frac{L^4\hat{\omega}^2}{(\rho-\bar{\rho})^2}. \quad (7.21)$$

By comparing the residues of the order four poles in r_H, we obtain an expression for ε

$$\varepsilon = \frac{\pm 4i\hat{\omega}L^2(\rho-\bar{\rho})}{(3\rho+\bar{\rho})(3\bar{\rho}+\rho)}. \quad (7.22)$$

If we compare the residues of the double and the single poles in r_H, we obtain

$$2\frac{L^4\hat{\omega}^2 - 2K^2L^2 + (\rho+\bar{\rho})^2}{(3\rho+\bar{\rho})(3\bar{\rho}+\rho)} = \tilde{q} + \gamma - \gamma^2. \quad (7.23)$$

The simple pole in ρ,

$$-\frac{8K^2L^2(\rho-\bar{\rho})^2 + (\rho-\bar{\rho})^3(5\rho+3\bar{\rho}) + L^4(5\rho-\bar{\rho})(\rho+3\bar{\rho})\omega^2}{2(\rho-\bar{\rho})^3(3\rho+\bar{\rho})^2} = \frac{(\gamma-4)\gamma\rho + (4-5\gamma)\gamma\bar{\rho} + 2\tilde{q}(\rho+3\bar{\rho})}{2(\rho-\bar{\rho})(3\rho+\bar{\rho})}. \quad (7.24)$$

The relation that holds for $\bar{\rho}$ is the same as the previous one after exchanging ρ and $\bar{\rho}$. As a result, $\tilde{q}$ can be written (up to $\rho \leftrightarrow \bar{\rho}$ exchange) as

$$\tilde{q} = \pm\frac{iL^2\hat{\omega}}{\rho-\bar{\rho}} - \frac{L^4\hat{\omega}^2}{(\rho-\bar{\rho})^2} + \frac{2[-2K^2L^2 + (\rho+\bar{\rho})^2 + L^4\hat{\omega}^2]}{(3\rho+\bar{\rho})(3\bar{\rho}+\rho)}. \quad (7.25)$$

Thus ω and K remain free parameters, while the mass is dictated by extremality (7.15).

Notice that the critical radii in the equatorial plane are the extremal points of

$$P_4(r) = [r^2 - a(\alpha_L b - a)]^2 - (\alpha_L b - a)^2\Delta_r(r) \quad (7.26)$$

so that

$$r_c = \pm r_H \quad \text{with} \quad \alpha_L b_c = a + \frac{r_H^2}{a}. \tag{7.27}$$

In the end, CT invariance works as much as possible as in the asymptotically flat case in that it keeps the photon-sphere fixed ($r_c = r_H$ in the equatorial plane) and exchanges the two complex horizons. Recall that both the AdS boundary ($r \to \infty$) and the curvature singularity ($r = 0$) are regular points of the wave equation instead.

C. Transformed Heun cs AdS KN with rescalings

So far we have implicitly assumed that CT transformations ($z \to t/z$ or $z \to 1 - z$) preserved the wave function in the standard HE or CHE form. In principle this is not necessary and the function may undergo some (Weyl) rescaling. In order to account for this flexibility let us go back to HE in (7.10) and perform the transformation[6] $W(z) = z^a(z-1)^b(z-t)^b f(z)$. One gets a new HE with parameters

$$\gamma \to 2a + \gamma, \qquad \delta \to 2b + \delta, \qquad \varepsilon \to 2b + \varepsilon,$$
$$p \to (2ab + b\gamma)(1+t) + a\delta t + a\varepsilon + p,$$
$$\alpha\beta \to \alpha\beta + 2b(2a + b + \gamma) + (\delta + \varepsilon)(a + b). \tag{7.28}$$

Imposing symmetry under $z \to t/z$ requires

$$\varepsilon = \delta, \quad \gamma = 1 - 2(a+b) - \delta, \quad \alpha\beta = -2(a+b)\delta. \tag{7.29}$$

In the coordinates (7.9) SHE can be put in canonical form by introducing

$$f(r) = [(r-r_+)(r-r_-)]^{-\frac{1}{2}(1-2b-\delta)}[(r-\rho)(r-\bar{\rho})]^{-\frac{1}{2}(2b+\delta)}\psi(r) \tag{7.30}$$

so that

$$\psi''(r) + \sum_{i=1}^{4}\left[\frac{A_{2,i}}{(r-r_i)^2} + \frac{A_{1,i}}{(r-r_i)}\right]\psi(r) = 0$$

with

$$A_{2,+} = A_{2,-} = -\frac{1}{4}(-1+\sigma+\delta)(1+\sigma+\delta),$$
$$A_{2,\rho} = A_{2,\bar{\rho}} = \frac{1}{4}(2-\delta)\delta,$$
$$A_{1,+} = \frac{1}{2(r_- - r_+)(r_+ - \rho)(r_+ - \bar{\rho})}\Big[2p(r_+ - \rho)(r_- - \bar{\rho}) - (\delta+\sigma-1)[-r_-^2\delta + r_- r_+(1-3\delta+\sigma) + \rho\bar{\rho}(1-\delta+\sigma) + r_+^2(2+\delta+2\sigma)]\Big],$$
$$A_{1,\rho} = \frac{1}{2}\left[\frac{2p+\delta(\delta+2\sigma)}{\bar{\rho}-\rho} + \frac{\delta(\delta+\sigma-1)}{\rho - r_+} + \frac{2p+\delta(\delta+\sigma-1)}{\rho - r_-}\right],$$
$$A_{1,-} = A_{1,+}(r_+ \leftrightarrow r_-, \rho \leftrightarrow \bar{\rho}),$$
$$A_{1,\rho} = A_{1,\bar{\rho}}(r_+ \leftrightarrow r_-, \rho \leftrightarrow \bar{\rho}), \tag{7.31}$$

where $\sigma = 2(a+b)$. From the perspective of the AdS KN wave equation (7.8), the residues of the double poles are

$$B_{2,i} = \lim_{r\to r_i}[(r-r_i)^2 Q_r(r)] = \frac{L^4[\hat{\omega}(r_i^2 + a_J^2) - a_J\hat{m}]^2}{\prod_{j\neq i}(r_j - r_i)^2} + \frac{1}{4}. \tag{7.32}$$

Since from (7.31) $A_{2,+} = A_{2,-}$, we must require that $B_{2,+} = B_{2,-}$ and the same for the residues of the double poles at ρ and $\bar{\rho}$,

$$\frac{r_-^2 + a_J^2 - b_\phi a_J}{r_+^2 + a_J^2 - b_\phi a_J} = \pm\frac{(r_- - \rho)(r_- - \bar{\rho})}{(r_+ - \rho)(r_+ - \bar{\rho})},$$
$$\frac{\rho^2 + a_J^2 - b_\phi a_J}{\bar{\rho}^2 + a_J^2 - b_\phi a_J} = \pm\frac{(\rho - r_+)(\rho - r_-)}{(\bar{\rho} - r_+)(\bar{\rho} - r_-)}, \tag{7.33}$$

where $b_\phi = \hat{m}/\hat{\omega} = m/\omega$. By comparison, from the previous relations we can write a constraint on b_ϕ (or equivalently ω),

$$a_J^2 - b_\phi a_J = -(\rho+\bar{\rho})r_+ + |\rho|^2$$
$$= -(r_+ + r_-)\rho + r_+ r_- = \frac{1}{2}(r_+ r_+ + |\rho|^2). \tag{7.34}$$

On the other hand, exploiting the first two relations in (7.31) and (7.32) and plugging (7.34), we have

$$\delta + \sigma = \pm\frac{2iL^2\hat{\omega}}{r_+ - r_-}, \qquad 1 - \delta = \pm\frac{2iL^2\hat{\omega}}{\rho - \bar{\rho}}. \tag{7.35}$$

Equations (7.35) can be used to reexpress δ,

$$\delta = \frac{\rho - \bar{\rho} - \zeta\sigma(r_+ - r_-)}{\rho - \bar{\rho} + \zeta(r_+ - r_-)} \quad \text{with} \quad \zeta = \pm 1. \tag{7.36}$$

The residues of the simple poles are

[6] We allow only rescalings that keep the symmetry between ρ and $\bar{\rho}$, i.e., $z = 1$ and $z = t$.

$$B_{1,+} = \frac{1}{2(r_+ - r_-)^3(r_+ - \rho)(2r_+ + r_- + \rho)} \Big\{ (r_+ - r_-)^2[-2K^2L^2 + r_-^2 - r_+^2 + (r_- + \rho)(r_+ + \rho)] + L^4[r_-^2 - 3r_-r_+ - r_+^2 + (r_- + r_+)\rho + \rho^2]\hat{\omega}^2 \Big\},$$
$$B_{1,\rho} = \frac{1}{2(r_- - \rho)(r_+ - \rho)(r_- + r_+ + 2\rho)^3} \Big\{ (r_- + r_+ + 2\rho)^2[r_-^2 - 2K^2L^2 - \rho^2 + (r_+ + \rho)(r_+ + r_-)] + L^4[r_-^2 + r_-r_+ + r_+^2 + 5(r_- + r_+)\rho + 3\rho^2]\hat{\omega}^2 \Big\},$$
$$B_{1,-} = B_{1,+}(r_+ \leftrightarrow r_-), \qquad B_{1,\bar{\rho}} = B_{1,\rho}(\rho \leftrightarrow \bar{\rho}). \tag{7.37}$$

Now it is possible to compute the parameter p in two different ways, i.e., $A_{1,+}(p) = B_{1,+}, A_{1,-}(p) = B_{1,-}$. The two expressions for p must be equal so, if we use (7.36), one finds

$$(r_- + r_+)[(r_+ - r_-)^2(r_- + r_+ + 2\rho)^2(1 + \sigma)^2 + L^4(r_- + r_+ + (r_+ - r_-)\zeta + 2\rho)^2\hat{\omega}^2] = 0. \tag{7.38}$$

Excluding $r_- = -r_+$ that would imply zero mass BH and using (7.35), (7.36), as well as (7.3) the previous expression becomes

$$(r_- - r_+)(\rho - \bar{\rho})^2(1 + \sigma)^2 = 0. \tag{7.39}$$

Excluding extremality $r_- = r_+ = r_H$, that would require $M = M_{\rm extr}$ already considered before, and $\rho = \bar{\rho}$, that would require $L < M$, eventually requires $\sigma = -1$. As a result of (7.35) $\hat{\omega} = 0$ and $\delta = 1$. So the last parameter to determine is

$$p = \frac{K^2L^2 + r_+r_- + \rho\bar{\rho}}{(r_+ - \rho)(r_- - \bar{\rho})}. \tag{7.40}$$

K remains a free parameter, while there are no conditions on the roots or, in other words, there are not constraints on M, a, and Q.

Once again we get a reduced form of CT invariance, valid only for static waves with $\omega = 0$ and $m_\phi = 0$.

VIII. CONCLUSIONS

We have extended our investigation on CT conformal inversions [6,12,13] of BHs and D-branes [5] in various directions.

First, we have analyzed asymptotically flat rotating charged BHs in $D = 4$, in particular (near) extremal rotating BHs in STU supergavity (n-eSTURBHs) [35], and found invariance for special choices of the charges.

Second, we have studied scalar wave equations in these backgrounds and identified the near superradiant modes a.k.a. zero damping modes in that display a very slow falloff at late times and thus represent a rather distinctive feature of the GW signal emitted in the mergers of near-extremal BHs.

We have then considered rotating BHs in $D = 5$ [22,23] and rotating D3-branes [20] and found invariance under generalized CT inversions for special choices of the angular momenta. Not surprisingly we have not found any similar symmetry for smooth horizonless geometries such as JMaRT [24], their supersymmetric GMS version [25,26] or circular fuzz balls [60,61]. Notwithstanding the presence of photon-rings, the lack of a horizon and the very different behaviors at infinity and at the cap are the reasons behind the failure.

Finally we have considered scalar perturbations of KN BHs in AdS_4 and found that the conditions for CT invariance are too stringent for generic AdS KN BHs unless one allows for a rescaling of the wave function and focuses on static waves. In the extremal case (7.15), m_ϕ is related to ω and K, very much as the 'critical' impact parameters of a massless probe are related to the critical radii of the photon-rings.

In all cases CT invariance keeps the photon-halo fixed and exchanges either the horizon and null infinity (for flat asymptotics) or two complex horizons (for KN BHs in AdS).

The relevant scalar wave equations can be separated and both the radial and angular parts, that generalize the celebrated Regge-Wheeler-Zerilli and Teukolsky equations, can be brought in the form of Heun equation or confluent versions thereof. At various points, we have also exploited the surprising connection with quantum SW curves for $\mathcal{N} = 2$ SYM with $SU(2)$ gauge group and N_f fundamental hypermultiplets [27–31].

Since for rotating objects generalized CT transformations depend on the impact parameter(s), the relation between scattering angle for geodesics outside the photon-sphere and inspiraling angle for geodesics inside the photon-sphere includes a boundary term $\Delta\phi_{bdry}(E,J) = \Delta\phi_{\rm fall}(E,J) - \Delta\phi_{\rm scatt}(E,J)$, determined by $\partial r_c/\partial J \neq 0$. In general, the observables are encoded in the full action that involves both a radial and a nontrivial angular part, although the latter plays only a marginal role in CT inversions.

The dynamics of rotating BHs at higher orders in G_N (post-Minkowskian) or in v/c (post-Newtonian)—the two being related by virial theorem—is under very active investigation [71–77]. Classical gravity may be extracted from quantum amplitudes [78,79]. New soft theorems [52,80,81], valid even in the string context [82,83], can help understanding radiation reaction, GW production [84,85] and memory effects [50–55].

Identifying (discrete) symmetries even in very special cases, such as eSTURBHs, can prove very useful in order to make further progress in this endeavor as well as in the scattering off D-branes, their bound states [86–88] or highly excited (coherent) string states [89–91] that can expose chaotic behavior. Moreover, the surprising connection with $\mathcal{N}=2$ SYM and, through the AGT correspondence, with $2d$ Liouville CFTs may shed further light on the holographic correspondence in this contexts [92–94].

ACKNOWLEDGMENTS

We acknowledge fruitful scientific exchange with G. Bonelli, G. Bossard, D. Consoli, M. Cvetic, M. Firrotta, F. Fucito, A. Grillo, F. Morales, J. Russo, R. Savelli and A. Tanzini. We would like to thank for the kind hospitality. We thank the MIUR PRIN Contract No. 2020KR4KN2 “String Theory as a bridge between Gauge Theories and Quantum Gravity” and the INFN project ST&FI “String Theory and Fundamental Interactions” for partial support.

APPENDIX A: CRITICAL PARAMETERS FOR ASYMPTOTICALLY FLAT KN BHs

The critical conditions read

$$R(r_c,\omega_c)=(r_c^2-a\zeta_c)^2-b_c^2\Delta_r(r_c)=0,$$
$$R'(r_c,\omega_c)=4r_c(r_c^2-a\zeta_c)-b_c^2\Delta_r'(r_c)=0,\tag{A1}$$

and must be supplemented with the condition

$$b^2\sin^2\theta-(\zeta+a\cos^2\theta)^2\geq 0.\tag{A2}$$

The solutions of (A1) are

$$\zeta_c=\frac{r_c^2}{a}-\frac{4r_c\Delta_r(r_c)}{a\Delta_r'(r_c)},\qquad b_c^2=\frac{16r_c^2\Delta_r(r_c)}{[\Delta_r'(r_c)]^2}.\tag{A3}$$

Setting $X=\cos 2\theta$, $b_c/a=\beta$, $\zeta_c/a=\gamma$ and using

$$\sin^2\theta=\frac{1}{2}(1-X),\qquad \cos^2\theta=\frac{1}{2}(1+X),\tag{A4}$$

the condition (A2) becomes

$$2\beta^2(1-X)\geq(2\gamma+1+X)^2\tag{A5}$$

or

$$X^2+2X(\beta^2+2\gamma+1)+(2\gamma+1)^2-2\beta^2\leq 0.\tag{A6}$$

Since

$$\Delta=\beta^2(\beta^2+4\gamma+4)\geq 0,\tag{A7}$$

then, setting $x=r/M$, $q=Q/M$ and $\alpha=a/M$, we obtain the following constraints,

$$\beta^2\geq 0\Rightarrow\begin{cases}x\geq x_H^+=1+\sqrt{1-\alpha^2-q^2}\\ x\leq x_H^-=1-\sqrt{1-\alpha^2-q^2},\end{cases}$$
$$\beta^2+4\gamma+4\geq 0\Rightarrow 2x^3-(q^2+3)x^2+2q^2x+\alpha^2\equiv f(x)\geq 0.$$

For consistency we must require (Schwarzschild) $0<\alpha^2+q^2<1$ (extremal KN).

1. Nonextremal case

The extremal points of $f(x)$ are

$$f'(x_e)=6x_e^2-2(q^2+3)x_e+2q^2=0\Rightarrow x_e^+=1,$$
$$x_e^-=\frac{q^2}{3}.\tag{A8}$$

It is very easy to see that x_e^+ is a minimum and x_e^- is a maximum. It is crucial to note that in the allowed range of the parameters, $f(x)$ has always one negative and two positive real roots. Furthermore, if we denote the roots with x_1, x_2 and x_3 with $x_1<0<x_2<x_3$, the outer horizon x_H^+ is always larger the biggest root x_3 and the inner horizon x_H^- is always smaller than x_2. So the positivity range of the discriminant (A7) is

$$0\leq x\leq x_H^-,\qquad x_2\leq x\leq x_3,\qquad x\geq x_H^+.\tag{A9}$$

The solutions of (A6) are

$$X_\pm=-(\beta^2+2\gamma+1)\pm\sqrt{\beta^2(\beta^2+4\gamma+4)},\tag{A10}$$

are real with $X_-<X_+$ so that the allowed range is $[X_-,X_+]$ but one has to make sure that it intersects with $[-1,+1]$ since $X=\cos 2\theta$, after all. The condition $X_+\leq 1$ is equivalent to

$$\beta^2\geq\gamma^2.\tag{A11}$$

Setting $m^2=\alpha^2+q^2$, β is defined for $x>x_H^+$ and for $x<x_H^-$. The derivative of β is

$$\beta'=\frac{2(x^3-3x^2+3x-m^2)}{|\alpha|(x-1)^2\sqrt{\Delta_r}}.\tag{A12}$$

In the range $0 < m^2 < 1$, β has only one extremal point in

$$x_e = 1 - \sqrt[3]{1 - m^2}, \tag{A13}$$

which is a minimum and is always located between zero and x_H^-. Now we focus on

$$|\gamma| = \left| -\frac{x(x^2 - 3x + 2m^2)}{\alpha^2(x-1)} \right|. \tag{A14}$$

The zeros of this function are

$$x_0^\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2m^2}. \tag{A15}$$

It is very easy to show that

$$x_0^+ > x_H^+, \qquad x_0^- < x_H^-, \quad \text{for } 0 < m^2 < 1. \tag{A16}$$

In the nonextremal case is always possible to find a photon region outside the outer horizon.

2. Extremal case

The situation changes drastically in the extremal case in which $m^2 = 1$. The functions β and γ reduce to

$$\beta = \frac{2x}{|\alpha|}, \qquad |\gamma| = \frac{|x(2-x)|}{\alpha^2}, \tag{A17}$$

$$\begin{cases} 2x|\alpha| = x(2-x) & 0 \le x \le 2 \to x = 0, \quad x = x_c^- = 2 - 2|\alpha| \\ 2x|\alpha| = x(x-2) & x \ge 2, x \le 0 \to x = 0, \quad x = x_c^+ = 2 + 2|\alpha|, \end{cases} \tag{A18}$$

where obviously $x_c^- < x_c^+$. However, for consistency, we must require that the lower bound of the photon region is greater than the horizon

$$2 - 2|\alpha| > 1 \Rightarrow |\alpha| = \frac{|a|}{\sqrt{a^2 + Q^2}} < \frac{1}{2} \Rightarrow |a| \le \frac{Q}{\sqrt{3}}. \tag{A19}$$

APPENDIX B: DICTIONARY eSTURBH TO eKN

In this appendix we discuss the map between the parameters in the eKN metric and the ones in the eSTURBH. Denoting by M, Q, a, r, ρ the parameters and variables in KN description and by m, q, α, u, ζ the corresponding parameters and variables in CPS-STU description and setting $s = \sinh\delta$ one finds (for $Q_1 = Q_2 = Q_3 = Q_4 = Q$)

$$M = m(1 + 2s^2), \qquad Q = q = 2ms\sqrt{1 + s^2},$$
$$a = \alpha, \qquad m^2 = M^2 - Q^2, \tag{B1}$$

and

$$r = u + 2ms^2,$$
$$\Delta = r^2 - 2Mr + a^2 + Q^2 = u^2 - 2mu + a^2 = \Delta_{CPS}, \tag{B2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta = W = (u + 2ms^2)^2 + a^2\cos^2\theta, \tag{B3}$$

$$\begin{aligned} \zeta^2 - 2mu &= u^2 + a^2\cos^2\theta - 2mu \\ &= \Delta - a^2 + a^2\cos^2\theta = \Delta - a^2\sin^2\theta. \end{aligned} \tag{B4}$$

Then

$$\begin{aligned} u - u_H &= u - \left[m + \sqrt{m^2 - a^2}\right] \\ &= r - m(1 + 2s^2) - \sqrt{M^2 - Q^2 - a^2} \\ &= r - M - \sqrt{M^2 - Q^2 - a^2} = r - r_H \end{aligned} \tag{B5}$$

that is,

$$r = u - u_H + r_H. \tag{B6}$$

In the extremal limit $M^2 = a^2 + Q^2$ one has $m^2 = a^2$ with $u_H = m = a$ and $r_H = M = \sqrt{a^2 + Q^2}$.

APPENDIX C: HEUN EQUATION VS QUANTUM SW CURVE

In order to find the dictionary between HE for scalar fluctuations with $\mu^2 L^2 = -2$ around AdS KN in $D = 4$ and quantum SW curve for $\mathcal{N} = 2$ SYM theory with $G = SU(2)$ and $N_f = 4 = (2_L, 2_R)$ we have to introduce the coordinate $y = -z$ and rewrite the second term in (7.14) as follows:

$$\begin{aligned} Q_H(y) = \frac{1 - \delta^2}{4y^2} + \frac{2\delta - \delta^2}{4(1+y)^2} + \frac{2\delta - \delta^2}{4(t+y)^2} \\ + \frac{\delta^2 - \delta + 2p + \delta^2 t - \delta t + y(\delta^2 - 2\delta)}{2ty(y+1)(1 + \frac{1}{t}y)}. \end{aligned} \tag{C1}$$

From quantum SW curve, we have

$$Q_{SW}(y) = \sum_{i=1}^{3} \frac{\sigma_i}{(y - y_i)^2} + \frac{\nu_1 + qy(\sigma_4 - \sigma_1 - \sigma_2 - \sigma_3)}{y(1+y)(1+qy)}, \tag{C2}$$

where $\{y_i = 0, -1, -1/q\}$ and

$$\sigma_1 = \frac{1}{4} - \frac{(m_1-m_2)^2}{4\hbar^2}, \qquad \sigma_2 = \frac{1}{4} - \frac{(m_1+m_2)^2}{4\hbar^2},$$
$$\sigma_3 = \frac{1}{4} - \frac{(m_3+m_4)^2}{4\hbar^2}, \qquad \sigma_4 = \frac{1}{4} - \frac{(m_3-m_4)^2}{4\hbar^2},$$
$$4\hbar^2\nu_1 = (q-1)(\hbar^2+4u) + 2(m_1^2+m_2^2) + 2q\left[2m_3m_4 + (m_1+m_2)(m_3+m_4) - \hbar\sum_i m_i\right]. \tag{C3}$$

By comparing the various terms of (C1) and (C2), we find the following dictionary

$$\delta^2 = \frac{(m_1-m_2)^2}{\hbar^2} = \frac{(m_3-m_4)^2}{\hbar^2},$$
$$(\delta-1)^2 = \frac{(m_1+m_2)^2}{\hbar^2} = \frac{(m_3+m_4)^2}{\hbar^2},$$
$$t = \frac{1}{q}. \tag{C4}$$

If we take the square root of the first two relations in (C4), we have to introduce the signs $\varepsilon_i = \pm 1$ for $i = 1, \ldots, 4$. Finally the last element of the dictionary is

$$u = \frac{\hbar^2}{2(t-1)}\{-2p + (1-\delta)[1+\delta+\varepsilon_3\varepsilon_4(1-\delta)+\varepsilon_3+\varepsilon_4)]\}. \tag{C5}$$

[1] V. Cardoso and P. Pani, Tests for the existence of black holes through gravitational wave echoes, Nat. Astron. **1**, 586 (2017).

[2] V. Cardoso and P. Pani, Testing the nature of dark compact objects: a status report, Living Rev. Relativity **22**, 4 (2019).

[3] L. Barack *et al.*, Black holes, gravitational waves and fundamental physics: A roadmap, Classical Quantum Gravity **36**, 143001 (2019).

[4] E. Barausse *et al.*, Prospects for fundamental physics with LISA, Gen. Relativ. Gravit. **52**, 81 (2020).

[5] M. Bianchi and G. Di Russo, Turning black-holes and D-branes inside out their photon-spheres, Phys. Rev. D **105**, 126007 (2022).

[6] W. E. Couch and R. J. Torrence, Conformal invariance under spatial inversion of extreme reissner-nordström black holes, Gen. Relativ. Gravit. **16**, 789 (1984).

[7] S. Aretakis, Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations I, Commun. Math. Phys. **307**, 17 (2011).

[8] S. Aretakis, Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations II, Ann. Henri Poincare **12**, 1491 (2011).

[9] S. Aretakis, Horizon instability of extremal black holes, Adv. Theor. Math. Phys. **19**, 507 (2015).

[10] H. Godazgar, M. Godazgar, and C. N. Pope, Aretakis charges and asymptotic null infinity, Phys. Rev. D **96**, 084055 (2017).

[11] H. Godazgar, M. Godazgar, and C. N. Pope, Dual gravitational charges and soft theorems, J. High Energy Phys. 10 (2019) 123.

[12] M. Cvetic, C. N. Pope, and A. Saha, Generalized Couch-Torrence symmetry for rotating extremal black holes in maximal supergravity, Phys. Rev. D **102**, 086007 (2020).

[13] M. Cvetic, C. N. Pope, and A. Saha, Conformal symmetries for extremal black holes with general asymptotic scalars in STU supergravity, J. High Energy Phys. 09 (2021) 188.

[14] G. Kälin and R. A. Porto, From boundary data to bound states, J. High Energy Phys. 01 (2020) 072.

[15] G. Kälin and R. A. Porto, From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist), J. High Energy Phys. 02 (2020) 120.

[16] E. Himwich, M. D. Johnson, A. Lupsasca, and A. Strominger, Universal polarimetric signatures of the black hole photon ring, Phys. Rev. D **101**, 084020 (2020).

[17] K. Akiyama *et al.*, First M87 Event Horizon Telescope results. I. The shadow of the supermassive black hole, Astrophys. J. Lett. **875**, L1 (2019).

[18] L. Borsten, M. J. Duff, and A. Marrani, Freudenthal duality and conformal isometries of extremal black holes, Report No. DFPD/2018/TH/04, DIAS-STP-18-18, Imperial-TP-2018-MJD-02, 2018.

[19] L. Borsten, M. J. Duff, J. J. Fernández-Melgarejo, A. Marrani, and E. Torrente-Lujan, Black holes and general Freudenthal transformations, J. High Energy Phys. 07 (2019) 070.

[20] J. G. Russo and K. Sfetsos, Rotating D3-branes and QCD in three-dimensions, Adv. Theor. Math. Phys. **3**, 131 (1999).

[21] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, D-branes and spinning black holes, Phys. Lett. B **391**, 93 (1997).

[22] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, Charged rotating black holes in four-dimensional gauged and ungauged supergravities, Nucl. Phys. **B717**, 246 (2005).

[23] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity, Phys. Rev. Lett. **95**, 161301 (2005).

[24] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener, Non-supersymmetric smooth geometries and D1-D5-P bound states, Phys. Rev. D **71**, 124030 (2005).

[25] S. Giusto, S. D. Mathur, and A. Saxena, Dual geometries for a set of 3-charge microstates, Nucl. Phys. **B701**, 357 (2004).

[26] S. Giusto, S. D. Mathur, and A. Saxena, 3-charge geometries and their CFT duals, Nucl. Phys. **B710**, 425 (2005).

[27] G. Aminov, A. Grassi, and Y. Hatsuda, Black hole quasinormal modes and Seiberg–Witten theory, Ann. Henri Poincare **23**, 1951 (2022).

[28] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, QNMs of branes, BHs and fuzzballs from quantum SW geometries, Phys. Lett. B **824**, 136837 (2022).

[29] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, Exact solution of Kerr black hole perturbations via CFT2 and instanton counting: Greybody factor, quasinormal modes, and Love numbers, Phys. Rev. D **105**, 044047 (2022).

[30] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, More on the SW-QNM correspondence, J. High Energy Phys. 01 (2022) 024.

[31] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, Irregular Liouville correlators and connection formulae for Heun functions, 2022, arXiv:2201.04491.

[32] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, International Series of Monographs on Physics Vol. 69 (Oxford University Press, 1992), ISBN: 0198520506 and 9780198520504.

[33] R. Penrose and R. M. Floyd, Extraction of rotational energy from a black hole, Nature (London) Phys. Sci. **229**, 177 (1971).

[34] R. Brito, V. Cardoso, and P. Pani, Superradiance: New frontiers in black hole physics, Lect. Notes Phys. **906**, 1 (2015).

[35] M. Cvetic and D. Youm, General rotating five-dimensional black holes of toroidally compactified heterotic string, Nucl. Phys. **B476**, 118 (1996).

[36] M. Bianchi, A. Grillo, and J. F. Morales, Chaos at the rim of black hole and fuzzball shadows, J. High Energy Phys. 05 (2020) 078.

[37] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, Light rings of five-dimensional geometries, J. High Energy Phys. 03 (2021) 210.

[38] F. Bacchini, D. R. Mayerson, B. Ripperda, J. Davelaar, H. Olivares, T. Hertog, and B. Vercnocke, Fuzzball Shadows: Emergent Horizons from Microstructure, Phys. Rev. Lett. **127**, 171601 (2021).

[39] M. Cvetic, G. W. Gibbons, C. N. Pope, and B. F. Whiting, Supergravity black holes, Love numbers, and harmonic coordinates, Phys. Rev. D **105**, 084035 (2022).

[40] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.

[41] V. Cardoso, O. J. C. Dias, J. L. Hovdebo, and R. C. Myers, Instability of non-supersymmetric smooth geometries, Phys. Rev. D **73**, 064031 (2006).

[42] E. Maggio, V. Cardoso, S. R. Dolan, and P. Pani, Ergoregion instability of exotic compact objects: electromagnetic and gravitational perturbations and the role of absorption, Phys. Rev. D **99**, 064007 (2019).

[43] M. Bianchi, M. Casolino, and G. Rizzo, Accelerating strangelets via Penrose process in non-BPS fuzzballs, Nucl. Phys. **B954**, 115010 (2020).

[44] B. Chakrabarty, D. Ghosh, and A. Virmani, Quasinormal modes of supersymmetric microstate geometries from the D1-D5 CFT, J. High Energy Phys. 10 (2019) 072.

[45] B. Guo, S. Hampton, and S. D. Mathur, Can we observe fuzzballs or firewalls?, J. High Energy Phys. 07 (2018) 162.

[46] I. Bena, P. Heidmann, R. Monten, and N. P. Warner, Thermal decay without information loss in horizonless microstate geometries, SciPost Phys. **7**, 063 (2019).

[47] D. R. Mayerson, Fuzzballs and observations, Gen. Relativ. Gravit. **52**, 115 (2020).

[48] T. Ikeda, M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, Black-hole microstate spectroscopy: Ringdown, quasinormal modes, and echoes, Phys. Rev. D **104**, 066021 (2021).

[49] I. Bah, I. Bena, P. Heidmann, Y. Li, and D. R. Mayerson, Gravitational footprints of black holes and their microstate geometries, J. High Energy Phys. 10 (2021) 138.

[50] A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, J. High Energy Phys. 01 (2016) 086.

[51] S. Pasterski, A. Strominger, and A. Zhiboedov, New gravitational memories, J. High Energy Phys. 12 (2016) 053.

[52] A. Strominger, Lectures on the infrared structure of gravity and gauge theory, 2017, arXiv:1703.05448.

[53] A. Addazi, M. Bianchi, M. Firrotta, and A. Marcianò, String memories … lost and regained, Nucl. Phys. **B965**, 115356 (2021).

[54] A. Aldi, M. Bianchi, and M. Firrotta, String memories... openly retold, Phys. Lett. B **813**, 136037 (2021).

[55] A. Aldi, M. Bianchi, and M. Firrotta, Spinning-off stringy electro-magnetic memories, Nucl. Phys. **B974**, 115625 (2022).

[56] I. Bena and D. R. Mayerson, Multipole Ratios: A New Window into Black Holes, Phys. Rev. Lett. **125**, 221602 (2020).

[57] M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, Distinguishing Fuzzballs from Black Holes through Their Multipolar Structure, Phys. Rev. Lett. **125**, 221601 (2020).

[58] I. Bena and D. R. Mayerson, Black holes lessons from multipole ratios, J. High Energy Phys. 03 (2021) 114.

[59] M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, The multipolar structure of fuzzballs, J. High Energy Phys. 01 (2021) 003.

[60] O. Lunin and S. D. Mathur, AdS/CFT duality and the black hole information paradox, Nucl. Phys. **B623**, 342 (2002).

[61] S. D. Mathur, The information paradox: A pedagogical introduction, Classical Quantum Gravity **26**, 224001 (2009).

[62] M. Bianchi, J. F. Morales, and L. Pieri, Stringy origin of 4d black hole microstates, J. High Energy Phys. 06 (2016) 003.

[63] M. Bianchi, J. F. Morales, L. Pieri, and N. Zinnato, More on microstate geometries of 4d black holes, J. High Energy Phys. 05 (2017) 147.

[64] I. Bena, S. Giusto, E. J. Martinec, R. Russo, M. Shigemori, D. Turton, and N. P. Warner, Smooth Horizonless Geometries Deep Inside the Black-Hole Regime, Phys. Rev. Lett. **117**, 201601 (2016).

[65] M. Bianchi, D. Consoli, and J. F. Morales, Probing fuzzballs with particles, waves and strings, J. High Energy Phys. 06 (2018) 157.

[66] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, The dark side of fuzzball geometries, J. High Energy Phys. 05 (2019) 126.

[67] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, $N = 6$ superconformal Chern-Simons-matter theories,

M2-branes and their gravity duals, J. High Energy Phys. 10 (2008) 091.
[68] M. Bianchi, R. Poghossian, and M. Samsonyan, Precision spectroscopy and higher spin symmetry in the ABJM model, J. High Energy Phys. 10 (2010) 021.
[69] A. Hanany and E. Witten, Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics, Nucl. Phys. **B492**, 152 (1997).
[70] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity **17**, 399 (2000).
[71] D. Bini, T. Damour, and A. Geralico, Radiative contributions to gravitational scattering, Phys. Rev. D **104**, 084031 (2021).
[72] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at O(G4), Phys. Rev. Lett. **128**, 161103 (2022).
[73] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, SUSY in the sky with gravitons, J. High Energy Phys. 01 (2022) 027.
[74] U. Kol, D. O'connell, and O. Telem, The radial action from probe amplitudes to all orders, J. High Energy Phys. 03 (2022) 141.
[75] G. Cho, G. Kälin, and R. A. Porto, From boundary data to bound states. Part III. Radiative effects, J. High Energy Phys. 04 (2022) 154.
[76] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.
[77] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vines, From scattering in black hole backgrounds to higher-spin amplitudes: Part I, 2021, arXiv:2107.10179.
[78] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. **121**, 171601 (2018).
[79] A. Koemans Collado, P. Di Vecchia, R. Russo, and S. Thomas, The subleading eikonal in supergravity theories, J. High Energy Phys. 10 (**2018**) 038.
[80] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg's soft graviton theorem, J. High Energy Phys. 05 (2015) 151.
[81] F. Cachazo and A. Strominger, Evidence for a new soft graviton theorem, 2014, arXiv:1404.4091.
[82] M. Bianchi and A. L. Guerrieri, On the soft limit of open string disk amplitudes with massive states, J. High Energy Phys. 09 (2015) 164.
[83] M. Bianchi and A. L. Guerrieri, On the soft limit of closed string amplitudes with massive states, Nucl. Phys. **B905**, 188 (2016).
[84] A. Addazi, M. Bianchi, and G. Veneziano, Soft gravitational radiation from ultra-relativistic collisions at sub- and sub-sub-leading order, J. High Energy Phys. 05 (2019) 050.
[85] A. Addazi, M. Bianchi, and G. Veneziano, Glimpses of black hole formation/evaporation in highly inelastic, ultra-planckian string collisions, J. High Energy Phys. 02 (2017) 111.
[86] G. D'Appollonio, P. Di Vecchia, R. Russo, and G. Veneziano, High-energy string-brane scattering: Leading eikonal and beyond, J. High Energy Phys. 11 (2010) 100.
[87] M. Bianchi and P. Teresi, Scattering higher spins off D-branes, J. High Energy Phys. 01 (2012) 161.
[88] G. D'Appollonio, P. Di Vecchia, R. Russo, and G. Veneziano, Regge behavior saves String Theory from causality violations, J. High Energy Phys. 05 (2015) 144.
[89] M. Bianchi and M. Firrotta, DDF operators, open string coherent states and their scattering amplitudes, Nucl. Phys. **B952**, 114943 (2020).
[90] M. Bianchi and L. Lopez, Pair production of small black holes in heterotic string theories, J. High Energy Phys. 07 (2010) 065.
[91] D. J. Gross and V. Rosenhaus, Chaotic scattering of highly excited strings, J. High Energy Phys. 05 (2021) 048.
[92] M. Kulaxizi, G. S. Ng, and A. Parnachev, Black holes, heavy states, phase shift and anomalous dimensions, SciPost Phys. **6**, 065 (2019).
[93] M. Kulaxizi, G. S. Ng, and A. Parnachev, Subleading eikonal, AdS/CFT and double stress tensors, J. High Energy Phys. 10 (2019) 107.
[94] A. Parnachev and K. Sen, Notes on AdS-Schwarzschild eikonal phase, J. High Energy Phys. 03 (2021) 289.