Path integrals of perturbative strings on curved backgrounds from string geometry theory

Matsuo Sato,[1,*] Yuji Sugimoto,[2,†] and Kunihito Uzawa[1,‡]

[1]*Graduate School of Science and Technology, Hirosaki University, Bunkyo-cho 3, Hirosaki, Aomori 036-8561, Japan*

[2]*Department of Physics, POSTECH, Pohang 37673, Korea*

(Received 9 May 2022; accepted 21 September 2022; published 12 October 2022)

String geometry theory is one of the candidates of the nonperturbative formulation of string theory. In this paper, from the closed bosonic sector of string geometry theory, we derive path integrals of perturbative strings on all of the string backgrounds, $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$, by considering fluctuations around the string background configurations, which are parametrized by the string backgrounds.

DOI: 10.1103/PhysRevD.106.086006

I. INTRODUCTION

String geometry theory is one of the candidates of nonperturbative formulation of string theory. It is formulated by a semiclassical path integral of string manifolds, which belong to a class of infinite-dimensional manifolds, string geometry [1]. String manifolds are defined by patching open sets of the model space defined by introducing a topology to a set of strings. One of the remarkable facts concerning string geometry theory is that the path integral of perturbative superstrings on the flat background is derived including the moduli of super-Riemann surfaces, by considering fluctuations around the flat background in the theory [1–3].

Moreover, configurations of fields in string geometry theory include all configurations of fields in the ten-dimensional supergravities, namely string backgrounds [4,5]. Especially, it is shown that an infinite number of equations of motion of string geometry theory are consistently truncated to finite numbers of equations of motion of the supergravities. That is, string geometry theory includes string backgrounds not as external fields like the perturbative string theories. Dynamics of string backgrounds are a part of dynamics of the fields in the theory. It is natural to expect to derive the path integral of perturbative strings on the sting backgrounds by considering fluctuations around the corresponding configurations in string geometry theory. Furthermore, a string background that minimizes the energy of the string background configurations will be chosen spontaneously, because string geometry theory is formulated nonperturbatively [4,5].

For each background, one theory is formulated in case of a perturbative string theory, whereas perturbative string theories not only on the flat background but also on nontrivial backgrounds should be derived from a single theory in case of the nonperturbative formulation of string theory. In this paper, from the closed bosonic sector of string geometry theory, we derive the path integrals of perturbative strings on all the string backgrounds $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$.

The organization of the paper is as follows. In Sec. II, we briefly review the closed bosonic sector in string geometry theory. In Sec. III, we set string background configurations parametrized by the string backgrounds $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$, and set the classical part of fluctuations representing strings. In Sec. IV, we consider two-point correlation functions of the quantum part of the fluctuations and derive the path integrals of the perturbative strings on the string backgrounds. In Sec. V, we conclude and discuss our results. In the Appendix, we obtain a Green's function on the flat string manifold.

II. REVIEW OF CLOSED BOSONIC SECTOR IN STRING GEOMETRY THEORY

In this paper, we discuss only the closed bosonic sector of string geometry theory. One can generalize the result in this paper to the full string geometry theory in the same way as in [1]. The closed bosonic sector [4,5] is described by a partition function

$$Z = \int \mathcal{D}G\mathcal{D}\phi\mathcal{D}Be^{-S}, \tag{2.1}$$

*msato@hirosaki-u.ac.jp
†yujisugimoto@postech.ac.kr
‡kunihito.uzawa@hirosaki-u.ac.jp

where the action is given by

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\sqrt{-G}e^{-2\phi}\left[R + 4\nabla_I\phi\nabla^I\phi - \frac{1}{2}|H|^2\right], \tag{2.2}$$

where $|H|^2 = \frac{1}{3!}H_{MNP}H^{MNP}$. The path integral is defined by semiclassically integrating a metric G_{IJ}, a scalar ϕ, and a two-form B_{IJ} defined on an infinite dimensional manifold, a so-called string manifold. It will be enough to define the path integral by semiclassically integrating classical solutions and small classical and quantum fluctuations up to the second orders around them, because string manifolds themselves possess quantum corrections, and loops of the fields on them do not correspond to quantum corrections as in [1–3]. There is no UV divergence from loop integrals, by defining the path integral semiclassically. A string manifold is constructed by patching open sets in string model space E, whose definition is summarized as follows. First, a global time $\bar{\tau}$ is defined canonically and uniquely on a Riemann surface $\bar{\Sigma}$ by the real part of the integral of an Abelian differential uniquely defined on $\bar{\Sigma}$ [6,7]. We restrict $\bar{\Sigma}$ to a $\bar{\tau}$ constant line and obtain $\bar{\Sigma}|_{\bar{\tau}}$. An embedding of $\bar{\Sigma}|_{\bar{\tau}}$ to $\mathbb{R}^d$ represents a many-body state of strings in $\mathbb{R}^d$, and is parametrized by coordinates $(\bar{h}, X(\bar{\tau}), \bar{\tau})$[1] where $\bar{h}$ is a metric on $\bar{\Sigma}$ and $X(\bar{\tau})$ is a map from $\bar{\Sigma}|_{\bar{\tau}}$ to $\mathbb{R}^d$. String model space E is defined by the collection of the string states by considering all the $\bar{\Sigma}$, all the values of $\bar{\tau}$, and all the $X(\bar{\tau})$. How near the two string states is defined by how near the values of $\bar{\tau}$ and how near $X(\bar{\tau})$. $\bar{h}$ is a discrete variable in the topology of string geometry, where an ϵ-open neighborhood of $[\bar{h}, X_s(\bar{\tau}_s), \bar{\tau}_s]$ is defined by

$$\begin{aligned} &U([\bar{h},X_s(\bar{\tau}_s),\bar{\tau}_s],\epsilon) \\ &:= \left\{[\bar{h},X(\bar{\tau}),\bar{\tau}] \middle| \sqrt{|\bar{\tau}-\bar{\tau}_s|^2 + \|X(\bar{\tau})-X_s(\bar{\tau}_s)\|^2} < \epsilon\right\}. \end{aligned} \tag{2.3}$$

As a result, $d\bar{h}$ cannot be a part of the basis that spans the cotangent space in (2.4), whereas fields are functionals of $\bar{h}$ as in (2.5). The precise definition of the string topology is given in the Sec. II in [1]. By this definition, arbitrary two string states on a connected Riemann surface in E are connected continuously. Thus, there is a one-to-one correspondence between a Riemann surface in $\mathbb{R}^d$ and a curve parametrized by $\bar{\tau}$ from $-\infty$ to ∞ on E. That is, curves that represent asymptotic processes on E reproduce the right moduli space of the Riemann surfaces in $\mathbb{R}^d$. Therefore, a string geometry model possesses all-order information of the perturbative string theory. Indeed, the path integral of perturbative strings on the flat spacetime is derived from the string geometry theory as in [1,3]. We use the Einstein notation for the index I, where $I = \{d, (\mu\bar{\sigma})\}$. The cotangent space is spanned by

$$\begin{aligned} dX^d &:= d\bar{\tau}, \\ dX^{(\mu\bar{\sigma})} &:= dX^\mu(\bar{\sigma},\bar{\tau}), \end{aligned} \tag{2.4}$$

for $\mu = 0, 1, \ldots, d-1$, while $d\bar{h}_{mn}$ with $m, n = \bar{\tau}, \bar{\sigma}$ cannot be a part of the basis because $\bar{h}_{mn}$ is treated as a discrete valuable in the string topology. The summation over $\bar{\sigma}$ is defined by $\int d\bar{\sigma}\,\bar{e}(\bar{\sigma},\bar{\tau})$, where $\bar{e} := \sqrt{\bar{h}_{\bar{\sigma}\bar{\sigma}}}$. This summation is transformed as a scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X(\bar{\tau}))$, and invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$.

From these definitions, we can write down the general form of the metric of the string geometry as follows:

$$\begin{aligned} &ds^2(\bar{h}, X(\bar{\tau}), \bar{\tau}) \\ &\quad = G_{dd}(\bar{h}, X(\bar{\tau}), \bar{\tau})(d\bar{\tau})^2 \\ &\qquad + 2d\bar{\tau}\int d\bar{\sigma}\,\bar{e}(\bar{\sigma},\bar{\tau})\sum_\mu G_{d(\mu\bar{\sigma})}(\bar{h}, X(\bar{\tau}), \bar{\tau})dX^\mu(\bar{\sigma},\bar{\tau}) \\ &\qquad + \int d\bar{\sigma}\,\bar{e}(\bar{\sigma},\bar{\tau})\int d\bar{\sigma}'\bar{e}(\bar{\sigma}',\bar{\tau})\sum_{\mu,\mu'} G_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}(\bar{h}, X(\bar{\tau}), \bar{\tau}) \\ &\qquad \times dX^\mu(\bar{\sigma},\bar{\tau})dX^{\mu'}(\bar{\sigma}',\bar{\tau}). \end{aligned} \tag{2.5}$$

The inverse metric $G^{IJ}(\bar{h}, X_{\hat{D}_T}(\bar{\tau}), \bar{\tau})$ is defined by $G_{IJ}G^{JK} = G^{KJ}G_{JI} = \delta_I^K$, where $\delta_d^d = 1$ and $\delta_{(\mu\bar{\sigma})}^{(\mu'\bar{\sigma}')} = \frac{1}{\bar{e}(\bar{\sigma},\bar{\tau})}\delta_\mu^{\mu'}\delta(\bar{\sigma}-\bar{\sigma}')$. In the following, we use $D := \int d\bar{\sigma}\,\bar{e}\,\delta_{(\mu\bar{\sigma})}^{(\mu\bar{\sigma})} = 2\pi d\delta(0)$, then $\delta_M^M = D + 1$. Although D is infinity, we treat D as regularization parameter and will take $D \to \infty$ later.

III. STRING BACKGROUND CONFIGURATIONS AND FLUCTUATIONS REPRESENTING STRIGS

In this paper, we consider only static configurations, including quantum fluctuations:

$$\begin{aligned} \partial_d G_{MN} &= 0, \\ \partial_d B_{MN} &= 0, \\ \partial_d \phi &= 0. \end{aligned} \tag{3.1}$$

In this section, we will set classical backgrounds including string backgrounds and consider fluctuations that represent strings around them. The Einstein equation of the action (2.2) is given by

$$\begin{aligned} &\bar{R}_{MN} - \frac{1}{4}\bar{H}_{MAB}\bar{H}_N{}^{AB} + 2\bar{\nabla}_M\bar{\nabla}_N\bar{\phi} \\ &\quad -\frac{1}{2}\bar{G}_{MN}\left(\bar{R} - 4\bar{\nabla}_I\bar{\phi}\bar{\nabla}^I\bar{\phi} + 4\bar{\nabla}_I\bar{\nabla}^I\bar{\phi} - \frac{1}{2}\bar{H}^2\right) = 0, \end{aligned} \tag{3.2}$$

[1] "–" represents a representative of the diffeomorphism and Weyl transformations on the worldsheet. Giving a Riemann surface $\bar{\Sigma}$ is equivalent to giving a metric $\bar{h}$ up to diffeomorphism and Weyl transformations.

where $\bar{R}$, $\bar{R}_{MN}$, $\bar{R}^{M}{}_{NPQ}$, and $\bar{\nabla}_M$ denote the Ricci scalar, Ricci tensor, curvature tensor, and covariant derivative constructed from the metric $\bar{G}_{MN}$, respectively. We consider a perturbation with respect to the metric $\bar{G}_{MN}$:

$$\bar{G}_{MN} = \hat{G}_{MN} + \bar{h}_{MN}, \tag{3.3}$$

where $\bar{h}_{MN}$ denotes a fluctuation around the 0th order background $\hat{G}_{MN}$.We raise and lower the indices by $\hat{G}_{MN}$ in the following. We also consider a perturbation with respect to the two-form $\bar{B}_{MN}$ and the scalar $\bar{\phi}$ around the 0th order backgrounds 0.

First, we generalize the harmonic gauge to the one when we have the dilaton. If we define $\bar{\psi}_{MN}$ as

$$\bar{\psi}_{MN} = \bar{h}_{MN} - \frac{1}{2}\hat{G}^{IJ}\bar{h}_{IJ}\hat{G}_{MN} + \Lambda\hat{G}_{MN}\bar{\phi}, \tag{3.4}$$

the Einstein equation (3.2) is expressed as

$$\begin{aligned}
&\hat{R}_{MN} - \frac{1}{2}\hat{G}_{MN}\hat{R} + \frac{1}{2}(-\hat{\nabla}_I\hat{\nabla}^I\bar{\psi}_{MN} + \hat{R}_{MA}\bar{\psi}^A_N + \hat{R}_{NA}\bar{\psi}^A_M \\
&\quad - 2\hat{R}_{MANB}\bar{\psi}^{AB} + \hat{\nabla}_M\hat{\nabla}_A\bar{\psi}^A_N + \hat{\nabla}_N\hat{\nabla}_A\bar{\psi}^A_M - \hat{\nabla}^I\hat{\nabla}^J\bar{\psi}_{IJ}\hat{G}_{MN} \\
&\quad + \hat{R}^{IJ}\bar{\psi}_{IJ}\hat{G}_{MN} - \hat{R}\bar{\psi}_{MN}) + (2-\Lambda)\hat{\nabla}_M\hat{\nabla}_N\bar{\phi} \\
&\quad - (2-\Lambda)\hat{G}_{MN}\hat{\nabla}_I\hat{\nabla}^I\bar{\phi} = 0,
\end{aligned} \tag{3.5}$$

up to the first order in the fields, $\bar{h}_{IJ}$, $\bar{B}_{IJ}$, and $\bar{\phi}$. $\hat{R}$, $\hat{R}_{MN}$, $\hat{R}^M_{NPQ}$, $\hat{\nabla}_M$ denote the Ricci scalar, Ricci tensor, curvature tensor, and covariant derivative constructed from the metric $\hat{G}_{MN}$. We set $\Lambda = 2$ so that the Einstein equation becomes only for $\bar{\psi}_{MN}$. $\bar{h}_{MN}$ is inversely expressed as

$$\bar{h}_{MN} = \bar{\psi}_{MN} + \frac{1}{D-1}(-\hat{G}^{PQ}\bar{\psi}_{PQ} + 4\bar{\phi})\hat{G}_{MN}. \tag{3.6}$$

We impose a generalization of the harmonic gauge:

$$\hat{\nabla}^M\bar{\psi}_{MN} = 0, \tag{3.7}$$

which reduces to the ordinary harmonic gauge if the dilaton is zero. Then, the Einstein equation (3.5) becomes

$$\begin{aligned}
&\hat{R}_{MN} - \frac{1}{2}\hat{G}_{MN}\hat{R} + \frac{1}{2}(-\hat{\nabla}_I\hat{\nabla}^I\bar{\psi}_{MN} + \hat{R}_{MA}\bar{\psi}^A_N + \hat{R}_{NA}\bar{\psi}^A_M \\
&\quad - 2\hat{R}_{MANB}\bar{\psi}^{AB} + \hat{R}^{IJ}\bar{\psi}_{IJ}\hat{G}_{MN} - \hat{R}\bar{\psi}_{MN}) = 0.
\end{aligned} \tag{3.8}$$

Next, we set the 0th order background $\hat{G}_{MN}$ as a flat background:

$$\hat{G}_{MN} = a_M\eta_{MN}, \tag{3.9}$$

where $a_d = 1$ and $a_{(\mu\bar{\sigma})} = \frac{\bar{e}^3(\bar{\sigma})}{\sqrt{\bar{h}(\bar{\sigma})}}$. Then, the gauge fixing condition (3.7) becomes

$$\int d\bar{\sigma}\,\bar{e}\,\partial^{(\mu\bar{\sigma})}\bar{\psi}_{(\mu\bar{\sigma})M} = 0, \tag{3.10}$$

the Einstein equation (3.8) becomes a Laplace equation,

$$\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{MN} = 0, \tag{3.11}$$

and the components of (3.6) read

$$\begin{aligned}
\bar{h}_{dd} &= \frac{D-2}{D-1}\bar{\psi}_{dd} + \frac{1}{D-1}\int d\bar{\sigma}''\bar{e}''\bar{\psi}^{(\mu''\bar{\sigma}'')}_{(\mu''\bar{\sigma}'')} - \frac{4}{D-1}\bar{\phi}, \\
\bar{h}_{d(\mu\bar{\sigma})} &= \bar{\psi}_{d(\mu\bar{\sigma})}, \\
\bar{h}_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')} &= \bar{\psi}_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')} + \frac{\bar{e}^3}{\sqrt{\bar{h}}}\delta_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}\Bigg(\frac{1}{D-1}\bar{\psi}_{dd} \\
&\quad - \frac{1}{D-1}\int d\bar{\sigma}''\bar{e}''\bar{\psi}^{(\mu''\bar{\sigma}'')}_{(\mu''\bar{\sigma}'')} + \frac{4}{D-1}\bar{\phi}\Bigg).
\end{aligned} \tag{3.12}$$

Next, the equation of motion of the scalar of the action (2.2)

$$\bar{R} - 4\bar{\nabla}_M\bar{\phi}\bar{\nabla}^M\bar{\phi} + 4\bar{\nabla}_M\bar{\nabla}^M\bar{\phi} - \frac{1}{2}|\bar{H}|^2 = 0 \tag{3.13}$$

is written as

$$\hat{R} + \hat{\nabla}^M\hat{\nabla}^N\bar{h}_{MN} - \hat{\nabla}^M\hat{\nabla}_M\bar{h}^N{}_N + 4\hat{G}^{MN}\hat{\nabla}_M\hat{\nabla}_N\bar{\phi} = 0, \tag{3.14}$$

up to the first order in the fields, $\bar{h}_{IJ}$, $\bar{B}_{IJ}$, and $\bar{\phi}$. Furthermore, this can be written as

$$\begin{aligned}
&\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\phi} + \frac{1}{4}\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd} \\
&\quad - \frac{1}{4}\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\int d\bar{\sigma}'\bar{e}'\bar{\psi}^{(\mu'\bar{\sigma}')}_{(\mu'\bar{\sigma}')} = 0,
\end{aligned} \tag{3.15}$$

around the flat 0th order background (3.9) under the static condition (3.1) in the generalized harmonic gauge (3.7). This becomes a Laplace equation,

$$\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\phi} = 0, \tag{3.16}$$

if the metric satisfies the Einstein equation (3.11).

On the other hand, the equation of motion of the two-form field

$$\bar{\nabla}_M(e^{-2\bar{\phi}}\bar{H}^{MNP}) = 0, \tag{3.17}$$

is written as

$$\hat{\nabla}_M\bar{H}^{MNP} = 0, \tag{3.18}$$

up to the first order in the fields, $\bar{h}_{IJ}$, $\bar{B}_{IJ}$, and $\bar{\phi}$. Furthermore, this becomes a Laplace equation

$$\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{B}_{MN}=0, \tag{3.19}$$

around the flat 0th order background (3.9) under the static condition (3.1) in Lorentz gauge,

$$\hat{\nabla}_M\bar{B}^{MN}=0, \tag{3.20}$$

which is written as

$$\partial_{(\mu\bar{\sigma})}\bar{B}^{(\mu\bar{\sigma})N}=0. \tag{3.21}$$

We consider classical backgrounds corresponding to the string background configurations:

$$\bar{\psi}_{dd}=0, \tag{3.22}$$

$$\bar{\psi}_{d(\mu\bar{\sigma})}=0, \tag{3.23}$$

$$\bar{h}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')}=\frac{\bar{e}^3}{\sqrt{\bar{h}}}g_{\mu\nu}(X(\bar{\sigma}))\delta_{\bar{\sigma}\bar{\sigma}'}, \tag{3.24}$$

$$\bar{B}_{d(\mu\bar{\sigma})}=0, \tag{3.25}$$

$$\bar{B}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')}=\frac{\bar{e}^3}{\sqrt{\bar{h}}}B_{\mu\nu}(X(\bar{\sigma}))\delta_{\bar{\sigma}\bar{\sigma}'}, \tag{3.26}$$

$$\bar{\phi}=\int d\bar{\sigma}\,\bar{e}\,\Phi(X(\bar{\sigma})), \tag{3.27}$$

where $g_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$ satisfy gauge fixing conditions,

$$\begin{aligned}\partial^\mu\psi_{\mu\nu}(x)&=0,\\ \partial^\mu B_{\mu\nu}(x)&=0,\end{aligned} \tag{3.28}$$

where

$$\psi_{\mu\nu}=g_{\mu\nu}-\frac{1}{2}\delta^{\alpha\beta}g_{\alpha\beta}\delta_{\mu\nu}+2\delta_{\mu\nu}\Phi, \tag{3.29}$$

which imply (3.10) and (3.21). Indeed, these are equivalent to

$$\bar{G}_{dd}=-1, \tag{3.30}$$

$$\bar{G}_{d(\mu\bar{\sigma})}=0, \tag{3.31}$$

$$\bar{G}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')}=\frac{\bar{e}^3}{\sqrt{\bar{h}}}G_{\mu\nu}(X(\bar{\sigma}))\delta_{\bar{\sigma}\bar{\sigma}'}, \tag{3.32}$$

$$\bar{B}_{d(\mu\bar{\sigma})}=0, \tag{3.33}$$

$$\bar{B}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')}=\frac{\bar{e}^3}{\sqrt{\bar{h}}}B_{\mu\nu}(X(\bar{\sigma}))\delta_{\bar{\sigma}\bar{\sigma}'}, \tag{3.34}$$

$$\bar{\phi}=\int d\bar{\sigma}\,\bar{e}\,\Phi(X(\bar{\sigma})), \tag{3.35}$$

where

$$G_{\mu\nu}(x)=\delta_{\mu\nu}+g_{\mu\nu}(x). \tag{3.36}$$

These are the string background configurations themselves [4,5]. If we impose that $g_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$ satisfy the Laplace equations,

$$\begin{aligned}\partial_\rho\partial^\rho g_{\mu\nu}(x)&=0,\\ \partial_\rho\partial^\rho B_{\mu\nu}(x)&=0,\\ \partial_\rho\partial^\rho\Phi(x)&=0,\end{aligned} \tag{3.37}$$

$\bar{G}_{MN}$, $\bar{B}_{MN}$, and $\bar{\phi}$ satisfy their equations of motion in string geometry theory,[2] (3.11), (3.16), and (3.19), and $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ also satisfy their equations of motion of the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector in the supergravity. Therefore, these string background configurations in string geometry theory represent perturbative string vacua parametrized by the on-shell fields in the supergravity as string backgrounds.

Next, we consider fluctuations around these vacua. The scalar fluctuation ψ_{dd} represents the degrees of freedom of perturbative strings in the case of the flat background as in [1–3]. Thus, we also consider the scalar fluctuation ψ_{dd} around the general perturbative vacua. We set the classical part of ψ_{dd} as

$$\begin{aligned}\bar{\psi}_{dd}=\int\mathcal{D}X'(\bar{\tau})G(X;X')\int d\bar{\sigma}\sqrt{\bar{h}}\Big[&\alpha' R_{\bar{h}}\Phi(X'(\bar{\sigma}))\\ &+\frac{1}{\bar{e}^2}G_{\mu\nu}(X'(\bar{\sigma}))\partial_{\bar{\sigma}}X'^\mu\partial_{\bar{\sigma}}X'^\nu\Big],\end{aligned} \tag{3.38}$$

where $R_{\bar{h}}$ is the scalar curvature of the two-dimensional metric $\bar{h}_{mn}$ and $G(X;X')$ is a Green's function on the flat string manifold given by

$$G(X;X')=\mathcal{N}\left[\int d\bar{\sigma}'\frac{\bar{e}'^2}{\sqrt{\bar{h}'}}(X^\mu(\bar{\sigma}')-X'^\mu(\bar{\sigma}'))^2\right]^{\frac{2-D}{2}}, \tag{3.39}$$

which satisfies

$$\int d\bar{\sigma}\sqrt{\bar{h}}\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu(\bar{\sigma})}\frac{1}{\bar{e}}\frac{\partial}{\partial X_\mu(\bar{\sigma})}G(X;X')=\delta(X-X'), \tag{3.40}$$

where $\mathcal{N}$ is a normalizing constant. A derivation is given in the Appendix. As a result, $\bar{\psi}_{dd}$ is not on-shell but satisfies

[2]Under (3.16), (3.11) is equivalent to $\int d\bar{\sigma}\,\bar{e}\,\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{h}_{MN}=0$, because (3.6).

$$\int d\bar{\sigma}\sqrt{\bar{h}}\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\mu}(\bar{\sigma})}\frac{1}{\bar{e}}\frac{\partial}{\partial X_{\mu}(\bar{\sigma})}\bar{\psi}_{dd}$$
$$= \int d\bar{\sigma}\sqrt{\bar{h}}\left[\alpha' R_{\bar{h}}\Phi(X(\bar{\sigma})) + \frac{1}{\bar{e}^2}G_{\mu\nu}(X(\bar{\sigma}))\partial_{\bar{\sigma}}X^{\mu}\partial_{\bar{\sigma}}X^{\nu}\right]. \tag{3.41}$$

Furthermore, we consider the quantum part of ψ_{dd},

$$\tilde{\psi}_{dd} = \frac{D-1}{D-2}\tilde{\phi}, \tag{3.42}$$

where $\frac{D-1}{D-2}$ is introduced for later convenience. Totally,

$$G_{MN} = \hat{G}_{MN} + \bar{h}_{MN} + \tilde{G}_{MN}, \tag{3.43}$$

where $\hat{G}_{MN}$ is given by (3.9), $\bar{h}_{MN}$ is given by (3.12) with (3.38), (3.23), (3.24), and (3.27), and $\tilde{G}_{MN}$ is given by

$$\tilde{G}_{dd} = \tilde{\phi}, \quad \tilde{G}_{d(\mu\bar{\sigma})} = 0, \quad \tilde{G}_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')} = \frac{1}{D-2}\frac{\bar{e}^3}{\sqrt{\bar{h}}}\tilde{\phi}\delta_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}. \tag{3.44}$$

IV. DERIVING THE PATH INTEGRALS OF THE PERTURBATIVE STRINGS ON CURVED BACKGROUNDS

In this section, we will derive the path integrals of the perturbative strings up to any order from the two-point correlation functions of the quantum scalar fluctuations of the metric. In order to obtain a propagator, we add a gauge fixing term corresponding to (3.7) into the action (2.2) and obtain

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\sqrt{-G}e^{-2\phi}\left[R + 4\nabla_I\phi\nabla^I\phi - \frac{1}{2}|H|^2 - \frac{1}{2}\left\{\bar{\nabla}^N\left(\tilde{G}_{MN} - \frac{1}{2}\bar{G}^{IJ}\tilde{G}_{IJ}\bar{G}_{MN} + 2\bar{G}_{MN}\tilde{\phi}\right)\right\}^2\right]. \tag{4.1}$$

As explained in Sec. II, the path integral of string geometry theory is defined semiclassically. That is, the theory is a free theory because quantum fluctuations are defined up to only the second order. The Faddeev-Popov ghost term does not contribute to the two-point correlation functions of the metrics because the theory is free. Thus, we abbreviate the Faddeev-Popov ghost term in the action. By substituting Eqs. (3.43), (3.25), (3.26), and (3.27) into (4.1), this is expressed as

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\Big(c_0 + c_1\tilde{\phi} + \tilde{\phi}c_2\tilde{\phi} + \tilde{\phi}\int d\bar{\sigma}\,\bar{e}\int d\bar{\sigma}'\,\bar{e}' c^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}\partial_{(\mu\bar{\sigma})}\partial_{(\mu'\bar{\sigma}')}\tilde{\phi}\Big), \tag{4.2}$$

where

$$c_0 = -\frac{1}{D-1}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd} - \frac{4D}{D-1}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\phi} + \frac{1}{D-1}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\int d\bar{\sigma}'\,\bar{e}'\bar{\psi}^{(\mu'\bar{\sigma}')}_{(\mu'\bar{\sigma}')}, \tag{4.3a}$$

$$c_1 = \frac{1}{2}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd} + \frac{1}{2(D-2)}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\int d\bar{\sigma}'\,\bar{e}'\bar{\psi}^{(\mu'\bar{\sigma}')}_{(\mu'\bar{\sigma}')}, \tag{4.3b}$$

$$c_2 = \frac{1}{4}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd} - \frac{1}{4(D-2)^2}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\int d\bar{\sigma}'\,\bar{e}'\bar{\psi}^{(\mu'\bar{\sigma}')}_{(\mu'\bar{\sigma}')}, \tag{4.3c}$$

$$c^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')} = \left[\frac{D-1}{4(D-2)} + \frac{1}{2}\bar{\psi}_{dd} + \frac{1}{2(D-2)}\int d\bar{\sigma}''\bar{e}''\bar{\psi}^{(\mu''\bar{\sigma}'')}_{(\mu''\bar{\sigma}'')} - \frac{2}{D-2}\bar{\phi}\right]\delta^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')} - \frac{D-1}{4(D-2)}\bar{\psi}^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}, \tag{4.3d}$$

up to the first order in the classical fields and the second order in $\tilde{\phi}$. Here, we take the regularization parameter $D \to \infty$. Then, (4.2) becomes

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\Big[-4\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\phi} + \frac{1}{2}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd}\tilde{\phi} + \tilde{\phi}\frac{1}{4}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd}\tilde{\phi} + \tilde{\phi}\left(\frac{1}{4} + \frac{1}{2}\bar{\psi}_{dd}\right)\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\tilde{\phi} - \frac{1}{4}\tilde{\phi}\int d\bar{\sigma}\,\bar{e}\int d\bar{\sigma}'\,\bar{e}'\bar{\psi}^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}\partial_{(\mu\bar{\sigma})}\partial_{(\mu'\bar{\sigma}')}\tilde{\phi}\Big]. \tag{4.4}$$

By shifting the field $\tilde{\phi}$ as $\tilde{\phi} = \tilde{\phi}' - \frac{2}{3}$, the first order term in $\tilde{\phi}'$ vanishes as

$$S = \int \mathcal{D}\bar{\tau}\mathcal{D}\,\bar{h}\mathcal{D}X(\bar{\tau})\Big[\tilde{\phi}'\frac{1}{4}\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd}\tilde{\phi}' + \tilde{\phi}'\left(\frac{1}{4} + \frac{1}{2}\bar{\psi}_{dd} + \frac{1}{8}\hat{G}^{IJ}\bar{h}_{IJ}\right)\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\tilde{\phi}' - \frac{1}{4}\tilde{\phi}'\int d\bar{\sigma}\,\bar{e}\int d\bar{\sigma}'\,\bar{e}'\bar{h}^{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}\partial_{(\mu\bar{\sigma})}\partial_{(\mu'\bar{\sigma}')}\tilde{\phi}'\Big], \tag{4.5}$$

where surface terms are dropped and the gauge fixing condition in (3.28) and a relation (3.4) are applied.

By normalizing the leading part of the kinetic term as $\tilde{\phi}' = 2(1 - \bar{\psi}_{dd} - \frac{1}{4}\hat{G}^{IJ}\bar{h}_{IJ})\tilde{\phi}''$, we have

$$S = \int \mathcal{D}\bar{\tau}\,\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\bigg[\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\bar{\psi}_{dd}(\tilde{\phi}'')^2 + \tilde{\phi}''\int d\bar{\sigma}\,\bar{e}\partial_{(\mu\bar{\sigma})}\partial^{(\mu\bar{\sigma})}\tilde{\phi}'' - \tilde{\phi}''\int d\bar{\sigma}\,\bar{e}\int d\bar{\sigma}'\,\bar{e}'\bar{h}_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}\partial^{(\mu\bar{\sigma})}\partial^{(\mu'\bar{\sigma}')}\tilde{\phi}''\bigg]. \tag{4.6}$$

This can be written as

$$S = -2\int \mathcal{D}\bar{\tau}\,\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\tilde{\phi}''H\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h}\right)\tilde{\phi}'', \tag{4.7}$$

where

$$H(p_X, X, \bar{h}) = \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,(p_X^\mu)^2 - \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,g_{\mu\nu}(X(\bar{\sigma}))p_X^\mu p_X^\nu - \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\frac{1}{\bar{e}}\frac{\partial}{\partial X^\mu}\frac{1}{\bar{e}}\frac{\partial}{\partial X_\mu}\bar{\psi}_{dd} + \int d\bar{\sigma}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu\bar{e}p_{\mu X} + \int d\bar{\sigma}\,i\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\nu B_\nu{}^\mu\bar{e}p_{\mu X}. \tag{4.8}$$

Here we have added terms

$$0 = -2\int \mathcal{D}\bar{\tau}\,\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\tilde{\phi}''\left(-i\int d\bar{\sigma}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu\frac{\partial}{\partial X^\mu} + \int d\bar{\sigma}\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\nu B_\nu{}^\mu\frac{\partial}{\partial X^\mu},\right)\tilde{\phi}'', \tag{4.9}$$

which is true because of the gauge fixing condition (3.28).

The propagator for $\tilde{\phi}$ defined by

$$\Delta_F(\bar{h}, X(\bar{\tau}); \bar{h}{,}'X'(\bar{\tau}')) = \langle\tilde{\phi}(\bar{h}, X(\bar{\tau}))\tilde{\phi}(\bar{h}{,}'X'(\bar{\tau}'))\rangle \tag{4.10}$$

satisfies

$$H\left(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X(\bar{\tau})}, X(\bar{\tau}), \bar{h}\right)\Delta_F(\bar{h}, X(\bar{\tau}); \bar{h}', X'(\bar{\tau}')) = \delta(\bar{h} - \bar{h}')\delta(X(\bar{\tau}) - X'(\bar{\tau}')). \tag{4.11}$$

In order to obtain a Schwinger representation of the propagator, we use the operator formalism $(\hat{\bar{h}}, \hat{X}(\bar{\tau}))$ of the first quantization, whereas the conjugate momentum is written as $(\hat{p}_{\bar{h}}, \hat{p}_X(\bar{\tau}))$. The eigenstate is given by $|\bar{h}, X(\bar{\tau})\rangle$.

Since (4.11) means that Δ_F is an inverse of H, Δ_F can be expressed by a matrix element of the operator $\hat{H}^{-1}$ as

$$\Delta_F(\bar{h}, X(\bar{\tau}); \bar{h}', X'(\bar{\tau}')) = \langle\bar{h}, X(\bar{\tau})|\hat{H}^{-1}(\hat{p}_X(\bar{\tau}), \hat{X}(\bar{\tau}), \hat{\bar{h}})|\bar{h}', X'(\bar{\tau}')\rangle. \tag{4.12}$$

On the other hand,

$$\hat{H}^{-1} = i\int_0^\infty dT e^{-iT\hat{H}}, \tag{4.13}$$

because

$$\lim_{\epsilon\to 0+}\int_0^\infty dT e^{-T(i\hat{H}+\epsilon)} = \lim_{\epsilon\to 0+}\left[\frac{1}{-(i\hat{H}+\epsilon)}e^{-T(i\hat{H}+\epsilon)}\right]_0^\infty = -i\hat{H}^{-1}. \tag{4.14}$$

This fact and (4.12) imply

$$\Delta_F(\bar{h}, X(\bar{\tau}); \bar{h}', X'(\bar{\tau}')) = i\int_0^\infty dT\langle\bar{h}, X(\bar{\tau})|e^{-iT\hat{H}}|\bar{h}', X'(\bar{\tau}')\rangle. \tag{4.15}$$

In order to define two-point correlation functions that are invariant under the general coordinate transformations in the string geometry, we define in and out states as

$$\|X_i|h_f, ; h_i\rangle_{\text{in}} \coloneqq \int_{h_i}^{h_f}\mathcal{D}h'|\bar{h}', X_i \coloneqq X'(\bar{\tau}' = -\infty)\rangle,$$

$$\langle X_f|h_f, ; h_i\|_{\text{out}} \coloneqq \int_{h_i}^{h_f}\mathcal{D}h\langle\bar{h}, X_f \coloneqq X(\bar{\tau} = \infty)|, \tag{4.16}$$

where h_i and h_f represent the metrics of the cylinders at $\bar{\tau} = \pm\infty$, respectively. $\int$ in $\int\mathcal{D}h$ includes $\sum_{\text{compact topologies}}$, where $\mathcal{D}h$ is the invariant measure[3] of the metrics h_{mn} on the two-dimensional Riemannian manifolds Σ. h_{mn} and $\bar{h}_{mn}$ are related to each other by the diffeomorphism and the Weyl transformations. When we insert asymptotic states, we integrate out X_f, X_i, h_f, and h_i in the two-point correlation function for these states;

$$\Delta_F(X_f; X_i|h_f, ; h_i) \coloneqq i\int_0^\infty dT\langle X_f|h_f, ; h_i\|_{\text{out}}e^{-iT\hat{H}}\|X_i|h_f, ; h_i\rangle_{\text{in}}. \tag{4.17}$$

[3]The invariant measure is defined implicitly by the most general invariant norm without derivatives for elements δh_{mn} of the tangent space of the metric, $\|\delta h\|^2 = \int d^2\sigma\sqrt{h}(h^{mp}h^{nq} + Ch^{mn}h^{pq})\delta h_{mn}\delta h_{pq}$ with C an arbitrary constant, and a normalization $\int\mathcal{D}\delta h\exp^{-\frac{1}{2}\|\delta h\|^2} = 1$.

This can be written as in [1],[4]

$$\begin{aligned}
&\Delta_F(X_f;X_i|h_f,;h_i)\\
&\quad := i\int_0^\infty dT\langle X_f|h_f,;h_i\|_{\mathrm{out}}e^{-iT\hat{H}}\|X_i|h_f,;h_i\rangle_{\mathrm{in}}\\
&\quad = i\int_0^\infty dT\lim_{N\to\infty}\int_{h_i}^{h_f}\mathcal{D}h\int_{h_i}^{h_f}\mathcal{D}h'\prod_{n=1}^{N}\int d\bar{h}_n dX_n(\bar{\tau}_n)\\
&\prod_{m=0}^{N}\langle\bar{h}_{m+1},X_{m+1}(\bar{\tau}_{m+1})|e^{-i\frac{1}{N}T\hat{H}}|\bar{h}_m,X_m(\bar{\tau}_m)\rangle\\
&\quad = i\int_0^\infty dT_0\lim_{N\to\infty}\int dT_{N+1}\int_{h_i}^{h_f}\mathcal{D}h\int_{h_i}^{h_f}\mathcal{D}h'\prod_{n=1}^{N}\int dT_n d\bar{h}_n dX_n(\bar{\tau}_n)\\
&\prod_{m=0}^{N}\langle X_{m+1}(\bar{\tau}_{m+1})|e^{-i\frac{1}{N}T_m\hat{H}}|X_m(\bar{\tau}_m)\rangle\delta(\bar{h}_m-\bar{h}_{m+1})\delta(T_m-T_{m+1})\\
&\quad = i\int_0^\infty dT_0\lim_{N\to\infty}dT_{N+1}\int_{h_i}^{h_f}\mathcal{D}h\prod_{n=1}^{N}\int dT_n dX_n(\bar{\tau}_n)\prod_{m=0}^{N}\int dp_{T_m}dp_{X_m}(\bar{\tau}_m)\\
&\qquad\times\exp\left(i\sum_{m=0}^{N}\Delta t\left(p_{T_m}\frac{T_m-T_{m+1}}{\Delta t}+p_{X_m}(\bar{\tau}_m)\cdot\frac{X_m(\bar{\tau}_m)-X_{m+1}(\bar{\tau}_{m+1})}{\Delta t}-T_m H(p_{X_m}(\bar{\tau}_m),X_m(\bar{\tau}_m),\bar{h})\right)\right)\\
&\quad = i\int_{h_iX_i}^{h_f,X_f}\mathcal{D}h\mathcal{D}X(\bar{\tau})\int\mathcal{D}T\int\mathcal{D}p_T\mathcal{D}p_X(\bar{\tau})\\
&\qquad\times\exp\left(i\int_{-\infty}^{\infty}dt\left(p_T(t)\frac{d}{dt}T(t)+p_X(\bar{\tau}(t),t)\cdot\frac{d}{dt}X(\bar{\tau}(t),t)-T(t)H(p_X(\bar{\tau}(t),t),X(\bar{\tau}(t),t),\bar{h})\right)\right),
\end{aligned} \tag{4.18}$$

where $p_X(\bar{\tau}(t),t)\cdot\frac{d}{dt}X(\bar{\tau}(t),t) := \int d\bar{\sigma}\bar{e}\,p_X^\mu(\bar{\tau}(t),t)\frac{d}{dt}X_\mu(\bar{\tau}(t),t)$. $\bar{h}_0=\bar{h}'$, $X_0(\bar{\tau}_0)=X_i$, $\bar{\tau}_0=-\infty$, $\bar{h}_{N+1}=\bar{h}$, $X_{N+1}(\bar{\tau}_{N+1})=X_f$, $\bar{\tau}_{N+1}=\infty$, and $\Delta t := \frac{1}{\sqrt{N}}$. A trajectory of points $[\bar{\Sigma},X(\bar{\tau})]$ is necessarily continuous in $\mathcal{M}_D$ so that the kernel $\langle\bar{h}_{m+1},X_{m+1}(\bar{\tau}_{m+1})|e^{-i\frac{1}{N}T_m\hat{H}}|\bar{h}_m,X_m(\bar{\tau}_m)\rangle$ in the fourth line is nonzero when $N\to\infty$.

By integrating out $p_X(\bar{\tau}(t),t)$, we move from the canonical formalism to the Lagrange formalism. Because the exponent of (4.18) is at most the second order in $p_X(\bar{\tau}(t),t)$, integrating out $p_X(\bar{\tau}(t),t)$ is equivalent to substituting into (4.18) the solution $p_X(\bar{\tau}(t),t)$ of

$$\begin{aligned}
&-i\bar{e}\frac{d}{dt}X^\mu+iT\bar{e}\left(\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\mu+i\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\nu B_\nu{}^\mu\right)+iT\sqrt{\bar{h}}p_X^\mu\\
&\quad-iT\sqrt{\bar{h}}g^{\mu\nu}(X)p_{\nu X}=0,
\end{aligned} \tag{4.19}$$

which is obtained by differentiating the exponent of (4.18) with respect to $p_X(\bar{\tau}(t),t)$. The solution is given by

$$p_{\mu X}=[\delta_{\mu\nu}+g_{\mu\nu}(X)]\frac{1}{T}\frac{\bar{e}}{\sqrt{\bar{h}}}\left[\frac{d}{dt}X^\nu-T\left(\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^\nu+i\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^\gamma B_\gamma{}^\nu(X)\right)\right], \tag{4.20}$$

up to the first order in the classical backgrounds $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$. By substituting this, we obtain

$$\begin{aligned}
\Delta_F(X_f;X_i|h_f;h_i) = i\int_{h_iX_i}^{h_f,X_f}&\mathcal{D}T\mathcal{D}h\mathcal{D}X(\bar{\tau})\mathcal{D}p_T\\
&\times\exp\left(i\int_{-\infty}^{\infty}dt\left(p_T(t)\frac{d}{dt}T(t)+\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\left(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\tau}(t),t)\partial_t X^\nu(\bar{\tau}(t),t)\right.\right.\right.
\end{aligned}$$

[4]The correlation function is zero if h_i and h_f of the in state do not coincide with those of the out states, because of the delta functions in the sixth line.

$$+\bar{h}^{01}\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)\Big)$$
$$+\int d\bar{\sigma} iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,T(t)\alpha' R_{\bar{h}}\Phi(X(\bar{\tau}(t),t))\Big)\Big), \quad (4.21)$$

where we use (3.41) and the ADM decomposition of the two-dimensional metric,

$$\bar{h}_{mn}=\begin{pmatrix}\bar{n}^2+\bar{n}_{\bar{\sigma}}\bar{n}^{\bar{\sigma}} & \bar{n}_{\bar{\sigma}}\\ \bar{n}_{\bar{\sigma}} & \bar{e}^2\end{pmatrix}, \qquad \sqrt{\bar{h}}=\bar{n}\,\bar{e}, \qquad \bar{h}^{mn}=\begin{pmatrix}\frac{1}{\bar{n}^2} & -\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}^2}\\ -\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}^2} & \bar{e}^{-2}+\left(\frac{\bar{n}^{\bar{\sigma}}}{\bar{n}}\right)^2\end{pmatrix}. \quad (4.22)$$

In this way, the Green's function can generate all the terms without $\bar{\tau}$ derivatives in the string action as in (3.41), but cannot do those with $\bar{\tau}$ derivatives, which need to be derived nontrivially, because the coordinates $X^\mu(\bar{\tau})$ in string geometry theory are defined on the $\bar{\tau}$ constant lines. We should note that the time derivative in (4.21) is in terms of t, not $\bar{\tau}$ at this moment. In the following, we will see that t can be fixed to $\bar{\tau}$ by using a reparametrization of t that parametrizes a trajectory.

By inserting $\int \mathcal{D}c\mathcal{D}b e^{\int_0^1 dt(\frac{db(t)}{dt}\frac{dc(t)}{dt})}$, where $b(t)$ and $c(t)$ are a bc ghost, we obtain

$$\Delta_F(X_f;X_i|h_f;h_i)=Z_0\int_{h_iX_i}^{h_f,X_f}\mathcal{D}T\mathcal{D}h\mathcal{D}X(\bar{\tau})\mathcal{D}p_T\mathcal{D}c\mathcal{D}b$$
$$\times\exp\Big(-\int_{-\infty}^{\infty}dt\Big(-ip_T(t)\frac{d}{dt}T(t)+\frac{db(t)}{dt}\frac{d(T(t)c(t))}{dt}$$
$$+\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\Big(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\tau}(t),t)\partial_t X^\nu(\bar{\tau}(t),t)$$
$$+\bar{h}^{01}\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)\Big)$$
$$+\int d\bar{\sigma} iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,T(t)\alpha' R_{\bar{h}}\Phi(X(\bar{\tau}(t),t))\Big)\Big), \quad (4.23)$$

where we redefine as $c(t)\to T(t)c(t)$, and Z_0 represents an overall constant factor. In the following, we will rename it $Z_1, Z_2, \cdots$ when the factor changes. The integrand variable $p_T(t)$ plays the role of the Lagrange multiplier providing the following condition:

$$F_1(t)\coloneqq\frac{d}{dt}T(t)=0, \quad (4.24)$$

which can be understood as a gauge fixing condition. Indeed, by choosing this gauge in

$$\Delta_F(X_f;X_i|h_f;h_i)=Z_1\int_{h_iX_i}^{h_f,X_f}\mathcal{D}T\mathcal{D}h\mathcal{D}X(\bar{\tau})$$
$$\times\exp\Big(-\int_{-\infty}^{\infty}dt\Big(\int d\bar{\sigma}\sqrt{\bar{h}}\,G_{\mu\nu}(X(\bar{\tau}(t),t))\Big(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\tau}(t),t)\partial_t X^\nu(\bar{\tau}(t),t)$$
$$+\bar{h}^{01}\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}}X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)\Big)$$
$$+\int d\bar{\sigma} iB_{\mu\nu}(X(\bar{\tau}(t),t))\partial_t X^\mu(\bar{\tau}(t),t)\partial_{\bar{\sigma}}X^\nu(\bar{\tau}(t),t)+\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,T(t)\alpha' R_{\bar{h}}\Phi(X(\bar{\tau}(t),t))\Big)\Big), \quad (4.25)$$

we obtain (4.23). The expression (4.25) has a manifest one-dimensional diffeomorphism symmetry with respect to t, where $T(t)$ is transformed as an einbein [8].

Under $\frac{d\bar{\tau}}{d\bar{\tau}'} = T(t)$, which implies

$$\bar{h}^{00} = T^2 \bar{h}'^{00}, \qquad \bar{h}^{01} = T\bar{h}'^{01}, \qquad \bar{h}^{11} = \bar{h}'^{11}, \qquad \sqrt{\bar{h}} = \frac{1}{T}\sqrt{\bar{h}'}, \qquad X^\mu(\bar{\tau}(t), t) = X'^\mu(\bar{\tau}'(t), t). \tag{4.26}$$

$T(t)$ disappears in (4.25) and we obtain

$$\begin{aligned}
\Delta_F(X_f; X_i | h_f; h_i) = Z_2 \int_{h_i X_i}^{h_f, X_f} \mathcal{D}h\mathcal{D}X(\bar{\tau}) \\
&\times \exp\bigg(-\int_{-\infty}^{\infty} dt \bigg(\int d\bar{\sigma} \sqrt{\bar{h}}\, G_{\mu\nu}(X(\bar{\tau}(t), t)) \bigg(\frac{1}{2} \bar{h}^{00} \partial_t X^\mu(\bar{\tau}(t), t) \partial_t X^\nu(\bar{\tau}(t), t) \\
&+ \bar{h}^{01} \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) + \frac{1}{2} \bar{h}^{11} \partial_{\bar{\sigma}} X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) \bigg) \\
&+ \int d\bar{\sigma}\, i B_{\mu\nu}(X(\bar{\tau}(t), t)) \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) + \frac{1}{2} \int d\bar{\sigma} \sqrt{\bar{h}}\, \alpha' R_{\bar{h}} \Phi(X(\bar{\tau}(t), t)) \bigg) \bigg).
\end{aligned} \tag{4.27}$$

This action is still invariant under the diffeomorphism with respect to t if $\bar{\tau}$ transforms in the same way as t.

If we choose a different gauge

$$F_2(t) \coloneqq \bar{\tau}(t) - t = 0, \tag{4.28}$$

in (4.27), we obtain

$$\begin{aligned}
\Delta_F(X_f; X_i | h_f; h_i) &= Z_3 \int_{h_i X_i}^{h_f, X_f} \mathcal{D}h\mathcal{D}X(\bar{\tau})\mathcal{D}\alpha\mathcal{D}c\mathcal{D}b \\
&\times \exp\bigg(-\int_{-\infty}^{\infty} dt \bigg(+\alpha(t)(\bar{\tau} - t) + b(t)c(t)\bigg(1 - \frac{d\bar{\tau}(t)}{dt}\bigg) \\
&+ \int d\bar{\sigma} \sqrt{\bar{h}}\, G_{\mu\nu}(X(\bar{\tau}(t), t)) \bigg(\frac{1}{2} \bar{h}^{00} \partial_t X^\mu(\bar{\tau}(t), t) \partial_t X^\nu(\bar{\tau}(t), t) + \bar{h}^{01} \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) \\
&+ \frac{1}{2} \bar{h}^{11} \partial_{\bar{\sigma}} X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) \bigg) \\
&+ \int d\bar{\sigma}\, i B_{\mu\nu}(X(\bar{\tau}(t), t)) \partial_t X^\mu(\bar{\tau}(t), t) \partial_{\bar{\sigma}} X^\nu(\bar{\tau}(t), t) + \frac{1}{2} \int d\bar{\sigma} \sqrt{\bar{h}}\, \alpha' R_{\bar{h}} \Phi(X(\bar{\tau}(t), t)) \bigg) \bigg) \\
&= Z \int_{h_i, X_i}^{h_f, X_f} \mathcal{D}h\mathcal{D}X(\bar{\tau}) \\
&\times \exp\bigg(-\int_{-\infty}^{\infty} d\bar{\tau} \int d\bar{\sigma} \sqrt{\bar{h}}\, G_{\mu\nu}(X(\bar{\sigma}, \bar{\tau})) \bigg(\frac{1}{2} \bar{h}^{00} \partial_{\bar{\tau}} X^\mu(\bar{\sigma}, \bar{\tau}) \partial_{\bar{\tau}} X^\nu(\bar{\sigma}, \bar{\tau}) + \bar{h}^{01} \partial_{\bar{\tau}} X^\mu(\bar{\sigma}, \bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma}, \bar{\tau}) \\
&+ \frac{1}{2} \bar{h}^{11} \partial_{\bar{\sigma}} X^\mu(\bar{\sigma}, \bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma}, \bar{\tau}) \bigg) \\
&+ \int d\bar{\sigma}\, i B_{\mu\nu}(X(\bar{\sigma}, \bar{\tau})) \partial_{\bar{\tau}} X^\mu(\bar{\sigma}, \bar{\tau}) \partial_{\bar{\sigma}} X^\nu(\bar{\sigma}, \bar{\tau}) + \frac{1}{2} \int d\bar{\sigma} \sqrt{\bar{h}}\, \alpha' R_{\bar{h}} \Phi(X(\bar{\sigma}, \bar{\tau})) \bigg).
\end{aligned} \tag{4.29}$$

The path integral is defined over all possible two-dimensional Riemannian manifolds with fixed punctures in the manifold $\mathcal{M}$ defined by the metric $G_{\mu\nu}$, as in Fig. 1. The diffeomorphism times Weyl invariance of the action in (4.29) implies that the correlation function is given by

$$\Delta_F(X_f; X_i | h_f; h_i) = Z \int_{h_i, X_i}^{h_f, X_f} \mathcal{D}h\mathcal{D}X e^{-S_s}, \tag{4.30}$$

where

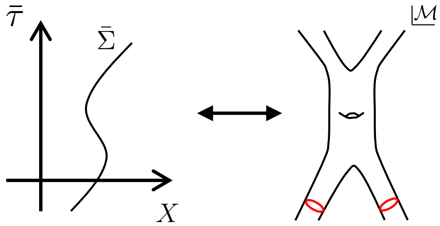

FIG. 1. A path and a Riemann surface. The line on the left is a trajectory in the path integral. The trajectory parametrized by $\bar{\tau}$ from $-\infty$ to ∞, represents a Riemann surface with fixed punctures in $\mathcal{M}$ on the right.

$$S_s = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \int d\sigma \sqrt{h(\sigma,\tau)} \times ((h^{mn}(\sigma,\tau)G_{\mu\nu}(X(\sigma,\tau)) + i\varepsilon^{mn}(\sigma,\tau)B_{\mu\nu}(X(\sigma,\tau))) \times \partial_m X^\mu(\sigma,\tau)\partial_n X^\nu(\sigma,\tau) + \alpha' R_{\bar{h}}\Phi(X(\sigma,\tau))). \quad (4.31)$$

For regularization, we divide the correlation function by Z and the volume of the diffeomorphism and the Weyl transformation $V_{\mathrm{diff}\times\mathrm{Weyl}}$, by renormalizing $\tilde{\phi}$. Equation (4.30) is the path integrals of perturbative strings on an arbitrary background that possess the moduli in the string theory themselves [9]. Especially, in string geometry, the consistency of the perturbation theory around the background (3.3), (3.25), (3.26), and (3.27) determines $d = 26$ (the critical dimension).

V. CONCLUSION AND DISCUSSION

In this paper, in the closed bosonic sector of string geometry theory, we fix the classical part of the scalar fluctuation of the metric around the string background configurations, which are parametrized by the string backgrounds, $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$, and $\Phi(x)$. We showed that the two-point correlation functions of the quantum parts of the scalar fluctuation are path integrals of the perturbative strings on the string backgrounds. In this derivation, we move from the second quantization formalism to the first one, where the coordinates of the two fields in the correlation functions become the asymptotic fields that represent the initial state $X^\mu(\tau=-\infty,\sigma)$ and the final state $X^\mu(\tau=\infty,\sigma)$, respectively. All the paths on the string manifolds from $X^\mu(\tau=-\infty,\sigma)$ to $X^\mu(\tau=\infty,\sigma)$ are summed up in the first quantization representation of the two-point correlation functions. Because the paths on the string manifolds are worldsheets with genera as shown in Sec. II in [1], they reproduce the path integrals of the perturbative strings up to any order, although the correlation functions are at tree level.

The next task is a supersymmetric generalization of our result. It is known to be too difficult to describe the action of the perturbative strings on the Ramond-Ramond (R-R) backgrounds in the Neveu–Schwarz-Ramond (NS-R) formalism. Because string geometry theory is formulated in the NS-R formalism, we should derive the path integrals of the perturbative strings on the NS-NS backgrounds.

ACKNOWLEDGMENTS

We would like to thank H. Kawai, K. Kikuchi, T. Yoneya, and especially A. Tsuchiya for discussions. The research of Y. S. is supported by Samsung Science and Technology Foundation under Project No. SSTF-BA2002-05 and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2018R1D1A1B07042934).

APPENDIX: GREEN'S FUNCTION ON STRING GEOMETRY

In this appendix, we will show that (3.39) is indeed a Green's function on the flat string manifold. If $X^\mu(\bar{\sigma}) \not\equiv X'^\mu(\bar{\sigma})$, we have

$$\frac{1}{\bar{e}'}\frac{\partial}{\partial X_\nu(\bar{\sigma}')}\mathcal{N}\left[\int d\bar{\sigma}\frac{\bar{e}^2}{\sqrt{\bar{h}}}(X^\mu(\bar{\sigma})-X'^\mu(\bar{\sigma}))^2\right]^{\frac{2-D}{2}} = (2-D)\mathcal{N}\left[\int d\bar{\sigma}\frac{\bar{e}^2}{\sqrt{\bar{h}}}(X^\mu(\bar{\sigma})-X'^\mu(\bar{\sigma}))^2\right]^{-\frac{D}{2}}\frac{\bar{e}'}{\sqrt{\bar{h}'}}(X^\nu(\bar{\sigma}')-X'^\nu(\bar{\sigma}')), \quad (A1)$$

and then,

$$\begin{aligned}\frac{1}{\bar{e}''}\frac{\partial}{\partial X^\nu(\bar{\sigma}'')}\frac{1}{\bar{e}'}\frac{\partial}{\partial X_\nu(\bar{\sigma}')}\mathcal{N}\left[\int d\bar{\sigma}\frac{\bar{e}^2}{\sqrt{\bar{h}}}(X^\mu(\bar{\sigma})-X'^\mu(\bar{\sigma}))^2\right]^{\frac{2-D}{2}} &= d(2-D)\frac{1}{\sqrt{\bar{h}'}}\frac{\bar{e}'}{\bar{e}''}\mathcal{N}\left[\int d\bar{\sigma}\frac{\bar{e}^2}{\sqrt{\bar{h}}}(X^\mu(\bar{\sigma})-X'^\mu(\bar{\sigma}))^2\right]^{-\frac{D}{2}}\delta(\bar{\sigma}'-\bar{\sigma}'') \\ &\quad - D(2-D)\frac{\bar{e}'}{\sqrt{\bar{h}'}}\frac{\bar{e}''}{\sqrt{\bar{h}''}}\mathcal{N}\left[\int d\bar{\sigma}\frac{\bar{e}^2}{\sqrt{\bar{h}}}(X^\mu(\bar{\sigma})-X'^\mu(\bar{\sigma}))^2\right]^{-\frac{D+2}{2}} \\ &\quad \times (X^\nu(\bar{\sigma}')-X'^\nu(\bar{\sigma}'))(X_\nu(\bar{\sigma}'')-X'_\nu(\bar{\sigma}'')). \quad (A2)\end{aligned}$$

Thus,

$$\begin{aligned}
&\int d\bar{\sigma}' \sqrt{\bar{h}'} \frac{1}{\bar{e}'} \frac{\partial}{\partial X^\nu(\bar{\sigma}')} \frac{1}{\bar{e}'} \frac{\partial}{\partial X_\nu(\bar{\sigma}')} \mathcal{N}\left[\int d\bar{\sigma} \frac{\bar{e}^2}{\sqrt{\bar{h}}} (X^\mu(\bar{\sigma}) - X'^\mu(\bar{\sigma}))^2\right]^{\frac{2-D}{2}} \\
&\quad = d \int d\bar{\sigma}' \delta(0)(2-D)\mathcal{N}\left[\int d\bar{\sigma} \frac{\bar{e}^2}{\sqrt{\bar{h}}} (X^\mu(\bar{\sigma}) - X'^\mu(\bar{\sigma}))^2\right]^{-\frac{D}{2}} \\
&\qquad - D(2-D)\mathcal{N}\left[\int d\bar{\sigma} \frac{\bar{e}^2}{\sqrt{\bar{h}}} (X^\mu(\bar{\sigma}) - X'^\mu(\bar{\sigma}))^2\right]^{-\frac{D+2}{2}} \int d\bar{\sigma}' \frac{\bar{e}'^2}{\sqrt{\bar{h}'}} (X^\nu(\bar{\sigma}') - X'^\nu(\bar{\sigma}'))^2 \\
&\quad = 0,
\end{aligned} \tag{A3}$$

where we use $D = d \int d\bar{\sigma}' \delta(0)$. Hence, we find

$$\int d\bar{\sigma}' \sqrt{\bar{h}'} \frac{1}{\bar{e}'} \frac{\partial}{\partial X^\nu(\bar{\sigma}')} \frac{1}{\bar{e}'} \frac{\partial}{\partial X_\nu(\bar{\sigma}')} \mathcal{N}\left[\int d\bar{\sigma} \frac{\bar{e}^2}{\sqrt{\bar{h}}} (X^\mu(\bar{\sigma}) - X'^\mu(\bar{\sigma}))^2\right]^{\frac{2-D}{2}} = \delta(X - X'), \tag{A4}$$

where $\mathcal{N}$ is a normalizing constant.

[1] M. Sato, String geometry and nonperturbative formulation of string theory, Int. J. Mod. Phys. A **34**, 1950126 (2019).
[2] M. Sato and Y. Sugimoto, Topological string geometry, Nucl. Phys. **B956**, 115019 (2020).
[3] M. Sato and Y. Sugimoto, Perturbative string theory from Newtonian limit of string geometry theory, Eur. Phys. J. C **80**, 789 (2020).
[4] M. Honda and M. Sato, String backgrounds in string geometry, Int. J. Mod. Phys. A **35**, 2050176 (2020).
[5] M. Honda, M. Sato, and T. Tohshima, Superstring backgrounds in string geometry, Adv. High Energy Phys. **2021**, 9993903 (2021).
[6] I. M. Krichever and S. P. Novikov, Algebras of Virasoro type, Riemann surfaces and the structure of soliton theory, Funct. Anal. Appl. **21**, 126 (1987).
[7] I. M. Krichever and S. P. Novikov, Virasoro-type algebras, Riemann surfaces and strings in Minkowsky space, Funct. Anal. Appl. **21**, 294 (1987).
[8] E. S. Fradkin and D. M. Gitman, Path integral representation for the relativistic particle propagators and BFV quantization, Phys. Rev. D **44**, 3230 (1991).
[9] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998), Vols. 1 and 2.