Holographic Meissner effect

Makoto Natsuume[*,‡]

KEK Theory Center, Institute of Particle and Nuclear Studies,
High Energy Accelerator Research Organization, Tsukuba, Ibaraki 305-0801, Japan

Takashi Okamura[†]

Department of Physics and Astronomy, Kwansei Gakuin University, Sanda, Hyogo 669-1330, Japan

The holographic superconductor is the holographic dual of superconductivity, but there is no Meissner effect in the standard holographic superconductor. This is because the boundary Maxwell field is added as an external source and is not dynamical. We show the Meissner effect analytically by imposing the semiclassical Maxwell equation on the AdS boundary. Unlike in the Ginzburg-Landau (GL) theory, the extreme type I limit cannot be reached even in the $e \to \infty$ limit where e is the $U(1)$ coupling of the boundary Maxwell field. This is due to the bound current which is present even in the pure bulk Maxwell theory. In the bulk 5-dimensional case, the GL parameter and the dual GL theory are obtained analytically for the order parameter of scaling dimension 2.

I. INTRODUCTION

The AdS/CFT duality or holography [1–4] is a useful tool to study strongly coupled systems (see, e.g., Refs. [5–10]). Let us consider the $(p + 2)$-dimensional AdS_{p+2} spacetime and the $(p + 1)$-dimensional boundary theory. In the boundary theory, one can add a curved metric and a $U(1)$ Maxwell field, but in most applications, they are not dynamical: one adds them as external sources to the boundary theory. The procedure to promote them to classical dynamical fields was discussed in [11]. Consider the $(p + 1)$-dimensional Einstein equation and the Maxwell equation[1]:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{R} = 8\pi G \langle \mathcal{T}_{\mu\nu} \rangle, \qquad (1.1a)$$

[*]makoto.natsuume@kek.jp
[†]tokamura@kwansei.ac.jp
[‡]Also at Department of Particle and Nuclear Physics, SOKENDAI (The Graduate University for Advanced Studies), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan and Department of Physics Engineering, Mie University, Tsu 514-8507, Japan.
[1]We use uppercase Latin indices $M, N, \ldots$ for the $(p + 2)$-dimensional bulk spacetime coordinates and use Greek indices $\mu, \nu, \ldots$ for the $(p + 1)$-dimensional boundary coordinates. The boundary coordinates are written as $x^\mu = (t, x^i) = (t, \vec{x}) = (t, x, y, \cdots)$.

$$\nabla_\nu \mathcal{F}^{\mu\nu} = e^2 \langle \mathcal{J}^\mu \rangle. \qquad (1.1b)$$

All quantities are the $(p + 1)$-dimensional ones. The Newton's constant G and the coupling e are the ones for the boundary theory. Here, $\langle \mathcal{T}_{\mu\nu} \rangle$ and $\langle \mathcal{J}^\mu \rangle$ are expectation values of the boundary energy-momentum tensor and the boundary $U(1)$ current computed by a standard AdS/CFT procedure [Eq. (2.7)]. In other words, one adds the following action to the boundary CFT:

$$S_{\text{bdy}} = \int d^{p+1}x \sqrt{-\mathcal{G}} \left(\frac{1}{16\pi G} \mathcal{R} - \frac{1}{4e^2} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right). \qquad (1.2)$$

In standard applications, one imposes the Dirichlet boundary condition on the AdS boundary. For example, for the bulk Maxwell field A_M, one imposes

$$\mathcal{A}_\mu = A_\mu|_{u=0} \qquad (1.3)$$

on the AdS boundary $u \to 0$. Instead, we impose the holographic semiclassical equation (1.1) as the boundary condition: we impose the "mixed" boundary condition.

While the procedure has been known, it has not been studied extensively. One has to consider the bulk equations of motion and the mixed boundary condition simultaneously, and the latter is now a differential equation. In general, it is a difficult task (see, e.g., Ref. [12] for a recent application). In this paper, we impose the holographic semiclassical equation on the holographic superconductors and show the Meissner effect analytically.

A holographic superconductor is typically an Einstein-Maxwell-scalar system [13–15]. For $T > T_c$, the solution is a standard black hole with no scalar, but for $T < T_c$, the solution becomes unstable and is replaced by a solution with scalar hair. Thus, the scalar corresponds to the order parameter of the phase transition. This is a superconducting transition. For example, the dc conductivity diverges and the London equation holds.

However, in the standard discussion, the boundary Maxwell field is added as a source so is not dynamical. As a result, there is no Meissner effect. The Meissner effect arises from the London equation and the Maxwell equation:

$$e^2 \mathcal{J}_i = -\frac{1}{\lambda^2} \mathcal{A}_i, \tag{1.4a}$$

$$\partial_j \mathcal{F}^{ij} = e^2 \mathcal{J}^i. \tag{1.4b}$$

Because the latter is absent in the standard holographic superconductor, the Meissner effect does not arise, and a magnetic field can penetrate the holographic superconductor. In a sense, the standard holographic superconductor is the "extreme" type II superconductors. Or one would regard the system as a superfluid.

The holographic semiclassical equation for holographic superconductors has been investigated previously [16]. The paper studies the issue by constructing a single vortex solution numerically (see, e.g., Refs. [17–21] for holographic vortices). However, it is desirable to show the Meissner effect analytically. Our results are summarized as follows:

(1) We first consider the case where the condensate is approximately constant and add a magnetic field perturbatively (Sec. III). The boundary current has the supercurrent as well as a contribution from the normal component which exists even in the pure Maxwell theory. The contribution can be interpreted as the bound current, and it changes the magnetic permeability (magnetic constant) from the vacuum value $\mu_0 = e^2$ to μ_m. The magnetic penetration length λ and the Ginzburg-Landau (GL) parameter κ have a nontrivial e dependence from the magnetic permeability. When $e \ll 1$, the result reduces to the standard GL result, but it deviates as one increases e. In the $e \to \infty$ limit, λ remains finite, and the extreme type I limit ($\lambda \to 0$) cannot be reached.

(2) One often imposes the Neumann boundary condition $\langle \mathcal{J}^i \rangle = 0$ in literature. This corresponds to the $e \to \infty$ limit because $\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle$. The nontrivial $e \to \infty$ limit explains why one can obtain a type II superconductor rather than the extreme type I superconductor under the Neumann boundary condition.

(3) In Sec. IV, we consider the case where the magnetic field is near the upper critical magnetic field H_{c2}. We obtain the holographic vortex lattice and show that the magnetic field decreases by the amount $|\psi|^2$, where ψ is the condensate: this also implies the Meissner effect.

(4) We focus on the $p = 2$ case, but the analysis of the $p = 3$ case is similar. For $p = 3$, an analytic solution is available [22,23], so one is able to obtain the GL parameter explicitly (Sec. V). Whether the holographic superconductor is type I or type II depends on e as well as the temperature. Also, we determine the dual GL theory.

II. PRELIMINARIES

We consider the bulk 4-dimensional s-wave holographic superconductor:

$$S_{\text{bulk}} = \int d^4x \sqrt{-g}(R - 2\Lambda) + S_m, \tag{2.1a}$$

$$S_m = -\frac{1}{g^2} \int d^4x \sqrt{-g} \left\{ \frac{1}{4} F_{MN}^2 + |D_M \Psi|^2 + m^2 |\Psi|^2 \right\}, \tag{2.1b}$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M, \tag{2.2a}$$

$$D_M = \nabla_M - iA_M, \tag{2.2b}$$

$$\Lambda = -\frac{3}{L^2}. \tag{2.2c}$$

Below we take the probe limit $g \gg 1$ where the backreaction of the matter fields onto the geometry is ignored. Then, the background metric is given by the Schwarzschild-AdS$_4$ (SAdS$_4$) black hole:

$$ds_4^2 = r^2(-f dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2 f} \tag{2.3a}$$

$$= \left(\frac{r_0}{u} \right)^2 (-f dt^2 + dx^2 + dy^2) + \frac{du^2}{u^2 f}, \tag{2.3b}$$

$$f = 1 - \left(\frac{r_0}{r} \right)^3 = 1 - u^3, \tag{2.3c}$$

where $u := r_0/r$. For simplicity, we set the AdS radius $L = 1$ and the horizon radius $r_0 = 1$. The Hawking temperature is given by $2\pi T = 3r_0/(2L^2)$. The bulk equations of motion are given by

$$0 = D^2 \Psi - m^2 \Psi, \tag{2.4a}$$

$$0 = \nabla_N F^{MN} - J^M, \tag{2.4b}$$

$$J_M = -i\{\Psi^\dagger D_M \Psi - \Psi(D_M \Psi)^\dagger\} \qquad (2.4c)$$

$$= 2\Im(\Psi^\dagger D_M \Psi). \qquad (2.4d)$$

In the $A_u = 0$ gauge, the static bulk equations become

$$0 = (-f\partial_u^2 - \Delta + 2|\varphi|^2)A_t, \qquad (2.5a)$$

$$0 = \{-\partial_u(f\partial_u) - \Delta + 2|\varphi|^2\}A_i - 2\Im(\varphi^\dagger \partial_i \varphi) + \partial_i(\vec{\partial}\cdot\vec{A}), \qquad (2.5b)$$

$$0 = \left\{-\partial_u(f\partial_u) + V - \frac{A_t^2}{f} - \delta^{ij}D_i D_j\right\}\varphi, \qquad (2.5c)$$

$$0 = \partial_u(\vec{\partial}\cdot\vec{A}) - 2\Im(\varphi^\dagger \partial_u \varphi), \qquad (2.5d)$$

where $\Delta := \partial_x^2 + \partial_y^2$, $(\vec{\partial}\cdot\vec{A}) := \delta^{ij}\partial_i A_j$, and

$$\Psi =: u\varphi, \qquad (2.6a)$$

$$V := \frac{m^2 + 2f - uf'}{u^2}. \qquad (2.6b)$$

In the $A_u = 0$ gauge, the asymptotic behaviors of matter fields are given by

$$A_\mu \sim \mathcal{A}_\mu + A_\mu^{(+)}u, \qquad (2.7a)$$

$$\Psi \sim \Psi^{(-)}u^{\Delta_-} + \Psi^{(+)}u^{\Delta_+}, \qquad (2.7b)$$

$$\Delta_\pm := \frac{3}{2} \pm \nu, \qquad \nu = \sqrt{\frac{9}{4} + m^2}. \qquad (2.7c)$$

$\mathcal{A}_t = \mu$ is the chemical potential, and $A_t^{(+)}$ represents the charge density $\langle\rho\rangle$. Similarly, $\mathcal{A}_i$ is the vector potential, and $A_i^{(+)}$ represents the current density $\langle\mathcal{J}_i\rangle$. $\Psi^{(+)}$ represents the order parameter $\langle\mathcal{O}\rangle$, and $\Psi^{(-)}$ is the external source for the order parameter.[2] According to the standard AdS/CFT dictionary,

$$\langle\mathcal{J}_\mu\rangle = \frac{1}{g^2}F_{u\mu}|_{u=0}, \qquad (2.8a)$$

$$\langle\mathcal{O}\rangle = \frac{1}{g^2}2\nu\Psi^{(+)}, \qquad (2.8b)$$

where one needs a standard counterterm action for the scalar field but the counterterm action for the Maxwell field makes no contribution for $p = 2$. Although we take the probe limit, we set $g = 1$ below for simplicity.

---

[2]For simplicity, we do not consider the "alternative quantization" where the role of $\Psi^{(-)}$ and $\Psi^{(+)}$ is exchanged [24].

We impose the mixed boundary condition:

$$\partial_j \mathcal{F}^{ij} = e^2\langle\mathcal{J}^i\rangle. \qquad (2.9)$$

Note that the vacuum magnetic permeability μ_0 and the vacuum electric permittivity ϵ_0 are given by $\mu_0 = 1/\epsilon_0 = e^2$.

III. SMALL MAGNETIC FIELD

We would like to know whether a magnetic field can enter the holographic superconductor. Below the critical temperature, a uniform condensate $\varphi_0 = \varphi_0(u)$ is a solution, and we apply a magnetic field there perturbatively. For simplicity, we consider $A_y = A_y(x, u)$ and $B = F_{xy} = \partial_x A_y$. We make the Fourier transformation:

$$A_y = \int \frac{dq}{(2\pi)}e^{iqx}\tilde{A}_y. \qquad (3.1)$$

Then, the bulk Maxwell equation becomes

$$0 = \{-\partial_u(f\partial_u) + q^2 + 2|\varphi_0|^2\}\tilde{A}_y. \qquad (3.2)$$

A. Dirichlet boundary condition

First, let us start with the standard Dirichlet boundary condition. In this case, there should be no Meissner effect, and the magnetic field can enter the superconductor however small the magnetic field is. One can formally integrate Eq. (3.2) as

$$f\partial_u\tilde{A}_y = -\int_u^1 du'(q^2 + 2|\varphi_0|^2)\tilde{A}_y(u'). \qquad (3.3)$$

Note that the right-hand side has a zero of degree 1 at the horizon $u = 1$. Further integrating the equation from the AdS boundary gives

$$\tilde{A}_y = \tilde{\mathcal{A}}_y - \int_0^u \frac{du'}{f(u')}\int_{u'}^1 du''(q^2 + 2|\varphi_0|^2)\tilde{A}_y(u'') \qquad (3.4a)$$

$$= \tilde{\mathcal{A}}_y\left\{1 - \int_0^u \frac{du'}{f(u')}\int_{u'}^1 du''(q^2 + 2|\varphi_0|^2) + \cdots\right\} \qquad (3.4b)$$

$$= \tilde{\mathcal{A}}_y\left\{1 - q^2\int_0^u \frac{du'}{1 + u' + u'^2} - \int_0^u \frac{du'}{f(u')}\int_{u'}^1 du''2|\varphi_0|^2 + \cdots\right\}. \qquad (3.4c)$$

We do not evaluate the integral explicitly, but one can obtain the regular solution in principle. The u' integral involves $1/f(u')$, which may give a logarithmic divergence

at $u' = 1$, but the u'' integral has a zero there, so the solution is regular.

The first term represents the magnetic induction $\tilde{\mathcal{B}} = iq\tilde{\mathcal{A}}_y$. From the above result, a $\tilde{\mathcal{B}} \neq 0$ solution exists even when $\varphi_0 \neq 0$, and the magnetic induction does not decrease. This implies $H_{c1} = 0$ for the holographic superconductor under the Dirichlet boundary condition.

The remaining terms represent the current. The current is given by

$$\langle \tilde{\mathcal{J}}_y \rangle = \partial_u \tilde{A}_y |_{u=0} \tag{3.5a}$$

$$= \tilde{\mathcal{A}}_y \left(-q^2 - 2 \int_0^1 du |\varphi_0|^2 + \cdots \right). \tag{3.5b}$$

This is the London equation with added normal component.
  (1) The second term represents the supercurrent. Thus, the supercurrent itself exits, but there is no Ampère law $\nabla \times \mathcal{B} = e^2 \mathcal{J}$ on the boundary, so there is no Meissner effect.
  (2) The first term exists even in the pure Maxwell theory with $\varphi_0 = 0$. The term is interpreted as the bound current which produces a diamagnetic current as we see below.

B. Holographic semiclassical equation

We now change the boundary condition and impose the holographic semiclassical equation:

$$\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle. \tag{3.6}$$

However, the holographic semiclassical equation gives

$$q^2 \tilde{\mathcal{A}}_y = -e^2 (q^2 + 2I) \tilde{\mathcal{A}}_y \rightarrow \tilde{\mathcal{A}}_y = 0, \tag{3.7a}$$

$$I := \int_0^1 du |\varphi_0|^2. \tag{3.7b}$$

Namely, an inhomogeneous magnetic field is not allowed. A similar result holds in the GL theory.

In order to obtain a nontrivial solution, one must add an external source:

$$\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle + e^2 \mathcal{J}^i_{\text{ext}}. \tag{3.8}$$

First, let us consider the first term in Eq. (3.5), the bound current part. The semiclassical equation is rewritten as

$$q^2 \tilde{\mathcal{A}}_y = -e^2 q^2 \tilde{\mathcal{A}}_y + e^2 \tilde{\mathcal{J}}^{\text{ext}}_y, \tag{3.9a}$$

$$\rightarrow q^2 \tilde{\mathcal{A}}_y = \frac{e^2}{1 + e^2} \tilde{\mathcal{J}}^{\text{ext}}_y =: \mu_m \tilde{\mathcal{J}}^{\text{ext}}_y, \tag{3.9b}$$

$$\rightarrow \mu_m = \frac{e^2}{1 + e^2}. \tag{3.9c}$$

The left-hand side of Eq. (3.9b) is $\nabla \times \mathcal{B} = \nabla(\nabla \cdot \mathcal{A}) - \nabla^2 \mathcal{A} \rightarrow q^2 \tilde{\mathcal{A}}$, so the equation describes the Ampère law $\nabla \times \mathcal{B} = \mu_m \mathcal{J}$. Thus, the net effect of the bound current is to shift the magnetic permeability from the vacuum value $\mu_0 = e^2$ to μ_m.[3] The magnetic permeability μ_m and the magnetic susceptibility χ_m are related by

$$\mu_m = \mu_0 (1 + \chi_m), \tag{3.10a}$$

$$\rightarrow \chi_m = \frac{\mu_m}{\mu_0} - 1 = -\frac{e^2}{1 + e^2} < 0. \tag{3.10b}$$

If $\chi_m < 0$, a material is diamagnetic. If $\chi_m > 0$, a material is paramagnetic. In this case, the bound current produces a diamagnetic current.

We now include the supercurrent, and the semiclassical equation becomes

$$q^2 \tilde{\mathcal{A}}_y = \mu_m (-2I\tilde{\mathcal{A}}_y + \tilde{\mathcal{J}}^{\text{ext}}_y), \tag{3.11a}$$

$$\rightarrow \tilde{\mathcal{A}}_y \propto \frac{1}{q^2 + 2\mu_m I}. \tag{3.11b}$$

When $I \neq 0$,

$$\mathcal{A}_y \propto e^{-x/\lambda}, \tag{3.12a}$$

$$\lambda^2 = \frac{1}{2\mu_m I} = \frac{1 + e^2}{2e^2 I}, \tag{3.12b}$$

which implies the Meissner effect with magnetic penetration length λ.

In the standard GL theory (Appendix A),

$$\lambda^2_{\text{GL}} = \frac{1}{2e^2 |\psi|^2}, \tag{3.13}$$

and a superconductor is classified by the GL parameter κ_{GL}:

$$\kappa^2_{\text{GL}} = \frac{\lambda^2}{\xi^2} = \frac{b}{2e^2}, \tag{3.14}$$

where ξ is the correlation length of the order parameter. In the GL theory, a superconductor is type I when $\kappa^2_{\text{GL}} < 1/2$, and a superconductor is type II when $\kappa^2_{\text{GL}} > 1/2$.
  (1) At weak coupling $e \ll 1$, the e dependence coincides with the GL theory. But the result deviates it as

---

[3]The magnetic permeability has been discussed in holographic optics [25], but it differs from our definition. One can show that we essentially use the Landau-Lifshitz definition [26].

one increases e because of the nontrivial magnetic permeability μ_m.

(2) In particular, in the $e \to \infty$ limit, $\lambda_{GL} \to 0$, so $\kappa_{GL} \to 0$. Namely, it is the extreme type I limit and shows the strong Meissner effect. However, our holographic result shows that λ remains finite and it implies that the extreme type I limit cannot be reached.[4]

Restoring the horizon radius r_0 gives

$$\mu_m = \frac{e^2}{1 + e^2/r_0}, \tag{3.15a}$$

$$\chi_m = -\frac{e^2/r_0}{1 + e^2/r_0}, \tag{3.15b}$$

$$\lambda^2 = \frac{1}{2\mu_m r_0 I} = \frac{1 + e^2/r_0}{2e^2 I}\frac{1}{r_0}. \tag{3.15c}$$

Note that e^2 has scaling dimension 1 in $(2 + 1)$ dimensions and the horizon radius (or temperature) has scaling dimension 1.

C. Single vortex

In a type II superconductor, the magnetic field can enter the superconductors keeping the superconducting state. The magnetic field enters by forming vortices. As one increases the magnetic field, the magnetic field begins to penetrate into the superconductor, and vortices appear at the lower critical magnetic field H_{c1}.

Far from the vortex, the condensate is approximately constant and the magnetic field is small. We consider this region and obtain the magnetic field. We take the polar coordinate $d\vec{x}_2^2 = dr^2 + r^2 d\phi^2$. The $A_\phi = A_\phi(u, r)$ equation becomes

$$0 = \partial_u(f\partial_u A_\phi) + r\partial_r\left(\frac{1}{r}\partial_r A_\phi\right) - 2|\varphi_0|^2 A_\phi. \tag{3.16}$$

Using the ansatz $A_\phi = U(u)R(r)$, one obtains

$$\frac{1}{U}\partial_u(f\partial_u U) - 2|\varphi_0|^2 = -\frac{r}{R}\partial_r\left(\frac{1}{r}\partial_r R\right)$$

$$= -\frac{1}{\lambda^2}, \tag{3.17}$$

where λ is the separation constant. The R equation is the standard equation for the vortex, so

$$R \propto \sqrt{r}e^{-r/\lambda}, \qquad (r \to \infty). \tag{3.18}$$

---

[4]The correlation length ξ is obtained by solving the bulk scalar equation of motion, so it has no e dependence as we see explicitly in Sec. V.

Thus, λ is the magnetic penetration length. The U equation gives

$$f\partial_u U = \int_u^1 du'(1/\lambda^2 - 2|\varphi_0|^2)U, \tag{3.19a}$$

$$U = \mathcal{U}\left\{1 + \int_0^u \frac{du'}{f(u')}\int_{u'}^1 du''(1/\lambda^2 - 2|\varphi_0|^2) + \cdots\right\}. \tag{3.19b}$$

The current is given by

$$\langle\mathcal{J}^\phi\rangle = \frac{1}{r^2}\partial_u A_\phi\Big|_{u=0} \tag{3.20a}$$

$$= \frac{R\mathcal{U}}{r^2}\int_0^1 du(1/\lambda^2 - 2|\varphi_0|^2) + \cdots. \tag{3.20b}$$

Imposing the semiclassical equation, one gets

$$0 = -\nabla_j\mathcal{F}^{\phi j} + e^2\langle\mathcal{J}^\phi\rangle \tag{3.21a}$$

$$= \frac{1}{r}\partial_r\left(\frac{1}{r}\partial_r A_\phi\right)\Big|_{u=0} + e^2\langle\mathcal{J}^\phi\rangle \tag{3.21b}$$

$$= \frac{U}{r}\partial_r\left(\frac{1}{r}\partial_r R\right) + e^2\frac{R}{r^2}\partial_u U\Big|_{u=0} \tag{3.21c}$$

$$\propto 1/\lambda^2 + e^2(1/\lambda^2 - 2I), \tag{3.21d}$$

where $I = \int_0^1 du|\varphi_0|^2$. Thus,

$$\lambda^2 = \frac{1 + e^2}{2e^2 I}. \tag{3.22}$$

Again, λ remains finite in the $e \to \infty$ limit. This limit corresponds to the Neumann boundary condition $\langle\mathcal{J}^i\rangle = 0$ because $\partial_j\mathcal{F}^{ij} = e^2\langle\mathcal{J}^i\rangle$. In fact, Eq. (3.20b) gives $\lambda^2 = 1/(2I)$ under the Neumann boundary condition.

IV. NEAR UPPER CRITICAL MAGNETIC FIELD

We discuss a single vortex in the previous section. As one increases the magnetic field further, more and more vortices are created, and the vortices form a lattice which is called the vortex lattice. Eventually, the superconducting state is completely broken at the upper critical magnetic field H_{c2}. Such holographic vortices have been investigated, and we follow Ref. [19] for the construction of the holographic vortex lattice.

The vortex lattice produces a supercurrent. However, in the standard holographic superconductor, there is no Maxwell equation on the AdS boundary, so the magnetic field can enter the superconductor not only at vortex cores.

We impose the holographic semiclassical equation and show the Meissner effect.

Near the upper critical magnetic field, the scalar field remains small, and one can expand matter fields as a series in ϵ, where ϵ is the deviation parameter from the critical point:

$$\varphi(\vec{x}, u) = \epsilon\varphi^{(1)} + \cdots, \tag{4.1a}$$

$$A_t(\vec{x}, u) = A_t^{(0)} + \epsilon^2 A_t^{(2)} + \cdots, \tag{4.1b}$$

$$A_i(\vec{x}, u) = A_i^{(0)} + \epsilon^2 A_i^{(2)} + \cdots. \tag{4.1c}$$

A. Zeroth order

At zeroth order, Eq. (2.5) becomes

$$0 = \mathcal{L}_t A_t^{(0)}, \tag{4.2a}$$

$$0 = \mathcal{L}_V A_i^{(0)} + \partial_i(\vec{\partial} \cdot \vec{A}^{(0)}), \tag{4.2b}$$

$$0 = \partial_u(\vec{\partial} \cdot \vec{A}^{(0)}), \tag{4.2c}$$

where

$$\mathcal{L}_t = -f\partial_u^2 - \Delta, \tag{4.3a}$$

$$\mathcal{L}_V = -\partial_u(f\partial_u) - \Delta, \tag{4.3b}$$

so the Maxwell equation gives[5]

$$A_t^{(0)} = \mu(1 - u), \tag{4.4a}$$

$$A_x^{(0)} = 0, \tag{4.4b}$$

$$A_y^{(0)} = Hx. \tag{4.4c}$$

B. First order

At first order, the bulk scalar equation becomes

$$0 = \left\{ -\partial_u(f\partial_u) + V - \frac{\mu^2(1-u)^2}{f} - \partial_x^2 - (\partial_y - iHx)^2 \right\}\varphi^{(1)}. \tag{4.5}$$

Using the ansatz

$$\varphi^{(1)} = e^{iqy}\chi_q(x)\rho(u), \tag{4.6}$$

---

[5]It is not clear if one should impose the semiclassical equation for these zero mode solutions. The zero modes satisfy $\partial_\nu \mathcal{F}^{\mu\nu} = 0$, so they are not induced by currents. For definiteness, we impose semiclassical equations only on nonzero modes in this paper.

one obtains

$$\left\{ -\partial_u(f\partial_u) + V - \frac{\mu^2(1-u)^2}{f} \right\}\rho = -E\rho, \tag{4.7a}$$

$$\left\{ -\partial_x^2 + H^2\left(x - \frac{q}{H}\right)^2 \right\}\chi_q = E\chi_q, \tag{4.7b}$$

where E is a separation constant. The regular bounded solution is given by Hermite function H_n as

$$\chi_q = e^{-z^2/2}H_n(z), \qquad z := \sqrt{H}\left(x - \frac{q}{H}\right), \tag{4.8}$$

with the eigenvalue

$$E = (2n + 1)H. \tag{4.9}$$

Below we set $n = 0$, so

$$\chi_q = \exp\left\{ -\frac{H}{2}\left(x - \frac{q}{H}\right)^2 \right\}. \tag{4.10}$$

What we obtained is the "droplet solution," but superpositions of the droplet solution give rise to a vortex lattice solution where a single vortex is arranged periodically. So, consider the general solution

$$\varphi^{(1)} = \rho_0(u)\Sigma(x, y), \tag{4.11a}$$

$$\Sigma(x, y) = \int_{-\infty}^{\infty} dq\, C(q)e^{iqy}\chi_q(x). \tag{4.11b}$$

Here, ρ_0 is the solution of Eq. (4.7a) with $E = H$. One can obtain the vortex lattice solution by choosing $C(q)$ appropriately. As discussed in Ref. [19], the most favorable solution thermodynamically is the triangular lattice for standard holographic superconductors.

The first order solution (4.11a) satisfies

$$(\partial_y - iA_y^{(0)})\varphi^{(1)} = i(\partial_x - iA_x^{(0)})\varphi^{(1)}, \tag{4.12}$$

so

$$2\Im[(\varphi^{(1)})^\dagger D_x^{(0)}\varphi^{(1)}] = -\partial_y|\varphi^{(1)}|^2, \tag{4.13a}$$

$$2\Im[(\varphi^{(1)})^\dagger D_y^{(0)}\varphi^{(1)}] = \partial_x|\varphi^{(1)}|^2, \tag{4.13b}$$

or

$$2\Im[(\varphi^{(1)})^\dagger D_i^{(0)}\varphi^{(1)}] = -\epsilon_i^j\partial_j|\varphi^{(1)}|^2, \tag{4.14}$$

where $\epsilon_{xy} = 1$.

C. Second order

The construction so far has been discussed in Ref. [19]. Let us proceed to the second order solution. We now solve the $A_i^{(2)}$ equation and obtain the current $\langle \mathcal{J}_i \rangle$. We then impose the holographic semiclassical equation $\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle$ and show the Meissner effect.

The Maxwell equation at second order is given by

$$0 = \mathcal{L}_V A_i^{(2)} + \epsilon_i^j \partial_j |\varphi^{(1)}|^2 + \partial_i (\vec{\partial} \cdot \vec{A}^{(2)}), \quad (4.15a)$$

$$0 = \partial_u (\vec{\partial} \cdot \vec{A}^{(2)}), \quad (4.15b)$$

where we use Eq. (4.14). From Eq. (4.15b), $(\vec{\partial} \cdot \vec{A}^{(2)})$ does not depend on u. Thus, one can choose $\vec{\partial} \cdot \vec{A}^{(2)} = 0$ by the gauge transformation which does not depend on u so that one can keep the $A_u = 0$ gauge. In momentum space,

$$0 = \tilde{\mathcal{L}}_V \tilde{A}_i^{(2)} + i\epsilon_i^j q_j |\widetilde{\varphi^{(1)}}|^2, \quad (4.16a)$$

$$\tilde{\mathcal{L}}_V = -\partial_u (f \partial_u) + q^2. \quad (4.16b)$$

Note that $|\widetilde{\varphi^{(1)}}|^2$ is the Fourier transformation of $|\varphi^{(1)}|^2$ and is not $|\tilde{\varphi}^{(1)}|^2$.

The second order solution can be constructed exactly, but it can be shown that it is a nonlocal function in the boundary direction [19]. This is because holographic results correspond to all orders in effective theory expansion. The GL theory takes only the first few terms in the expansion. In fact, at short wavelength, the London equation is replaced by a nonlocal expression known as the Pippard equation. In order to show the Meissner effect, it is enough to take the long-wavelength $q \to 0$ limit.

One could use the coordinate u, but it is simpler to use the tortoise coordinate u_*:

$$ds^2 = \frac{1}{u^2}\left(-f dt^2 + \frac{du^2}{f}\right) + \cdots \quad (4.17a)$$

$$= \frac{f}{u^2}(-dt^2 + du_*^2) + \cdots, \quad (4.17b)$$

$$du_* := \frac{du}{f}. \quad (4.17c)$$

Here, we take $u_* : 0 \to \infty$, and $u_* \to \infty$ corresponds to the horizon. Then,

$$0 = \mathcal{L}^* \tilde{A}_i^{(2)} + g_i, \quad (4.18a)$$

$$\mathcal{L}^* = -\partial_*^2 + q^2 f, \quad (4.18b)$$

$$g_i = i\epsilon_i^j q_j f |\widetilde{\varphi^{(1)}}|^2. \quad (4.18c)$$

Using the bulk Green's function, the solution is formally written as

$$\tilde{A}_i^{(2)} = a_i - \int_0^\infty du_*' G(u_*, u_*') g_i(u_*'), \quad (4.19a)$$

$$\mathcal{L}^* G(u_*, u_*') = \delta(u_* - u_*'). \quad (4.19b)$$

We impose the boundary conditions (1) $G(u_* = 0, u_*') = 0$, and (2) $\partial_* G|_{u_* \to \infty} = 0$. The first term a_i is the homogeneous solution:

$$(-\partial_*^2 + q^2 f) a_i = 0. \quad (4.20)$$

We impose the boundary conditions (1) regular at the horizon and (2) $a_i = \tilde{A}_i^{(2)}$ at $u = 0$.

One can construct the homogeneous solution by the q expansion:

$$a_i = F_0 + q^2 F_2 + \cdots. \quad (4.21)$$

The solution which satisfies the boundary conditions is

$$a_i = \tilde{A}_i^{(2)}\left(1 - q^2 \int_0^u \frac{du'}{1 + u' + u'^2}\right) + O(q^4) \quad (4.22a)$$

$$\sim \tilde{A}_i^{(2)}(1 - q^2 u + \cdots), \quad (u \to 0). \quad (4.22b)$$

The function g_i is $O(q)$, and it is enough to construct the Green's function at $q = 0$:

$$-\partial_*^2 G = \delta(u_* - u_*'). \quad (4.23)$$

Such a Green's function is obtained from two homogeneous solutions. The homogeneous solutions are

$$A_b = u_*, \quad (4.24a)$$

$$A_h = 1, \quad (4.24b)$$

$$W := A_h \partial_* A_b - (\partial_* A_h) A_b = 1. \quad (4.24c)$$

The solution A_b satisfies the boundary condition at the AdS boundary and A_h satisfies the boundary condition at the horizon. Then, the Green's function is given by

$$G(u_*, u_*') = \begin{cases} A_h(u_*) A_b(u_*') = u_*' & (u_*' < u_* < \infty) \\ A_h(u_*') A_b(u_*) = u_* & (0 < u_* < u_*') \end{cases}.$$

Thus,

$$\tilde{A}_i^{(2)} = a_i - u_* \int_{u_*}^{\infty} du'_* g_i(u'_*) - \int_0^{u_*} du'_* u'_* g_i(u'_*) + O(q^3).$$

$$(4.25)$$

The current is given by

$$\langle \tilde{\mathcal{J}}_i \rangle = \partial_u \tilde{A}_i^{(2)}|_{u=0} = \partial_* \tilde{A}_i^{(2)}|_{u=0} \tag{4.26a}$$

$$= \partial_* a_i - \int_0^{\infty} du'_* g_i(u'_*) \tag{4.26b}$$

$$\sim -q^2 \tilde{A}_i^{(2)} - i\epsilon_i^j q_j \int_0^1 du |\widetilde{\varphi^{(1)}}|^2 + O(q^3) \tag{4.26c}$$

$$= \tilde{\mathcal{J}}_i^n + \tilde{\mathcal{J}}_i^s. \tag{4.26d}$$

The second term of Eq. (4.26c) is the supercurrent. Once again, the supercurrent itself exists even under the Dirichlet boundary condition, but there is no Meissner effect. The first term of Eq. (4.26c) exists even for the pure Maxwell theory, and it is interpreted as the bound current.

D. Holographic semiclassical equation

We now impose the semiclassical equation as the boundary condition:

$$\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle. \tag{4.27}$$

In momentum space, $\partial_j \mathcal{F}^{ij} = -\Delta \mathcal{A}_i \to q^2 \tilde{\mathcal{A}}_i$ in the gauge $\partial_i \mathcal{A}^i = 0$. Thus, the holographic semiclassical equation becomes

$$q^2 \tilde{\mathcal{A}}_i^{(2)} = e^2 \tilde{\mathcal{J}}_i^n + e^2 \tilde{\mathcal{J}}_i^s, \tag{4.28a}$$

$$\to q^2 \tilde{\mathcal{A}}_i^{(2)} = \frac{e^2}{1+e^2} \tilde{\mathcal{J}}_i^s =: \mu_m \tilde{\mathcal{J}}_i^s. \tag{4.28b}$$

So,

$$\tilde{\mathcal{A}}_i^{(2)} = \frac{\mu_m}{q^2} \tilde{\mathcal{J}}_i^s \tag{4.29a}$$

$$= -i \frac{\mu_m}{q^2} \epsilon_i^j q_j \int_0^1 du |\widetilde{\varphi^{(1)}}|^2. \tag{4.29b}$$

$\tilde{\mathcal{B}}$ is then obtained as

$$\tilde{\mathcal{B}}^{(2)} = i\epsilon^{ij} q_i \tilde{\mathcal{A}}_j^{(2)} = -\mu_m \int_0^1 du |\widetilde{\varphi^{(1)}}|^2. \tag{4.30}$$

Going back to the real space,

$$\mathcal{B}^{(2)} = -\mu_m \int_0^1 du |\varphi^{(1)}|^2. \tag{4.31}$$

By adding the zeroth order solution,

$$\mathcal{B} = H - \epsilon^2 \mu_m \int_0^1 du |\varphi^{(1)}|^2 \tag{4.32}$$

with $H := \mathcal{B}_{\infty}$.

Finally, let us rewrite the result in terms of the operator expectation value $\langle \mathcal{O} \rangle$. Recall

$$\Psi = u\varphi, \tag{4.33a}$$

$$\varphi = \epsilon \varphi^{(1)} + \cdots = \epsilon \rho_0(u)\Sigma + \cdots, \tag{4.33b}$$

$$\rho_0 \sim \rho_0^{(-)} u^{\Delta_- - 1} + \rho_0^{(+)} u^{\Delta_+ - 1}, \tag{4.33c}$$

so

$$\langle \mathcal{O} \rangle = 2\nu \Psi^{(+)} = 2\nu \epsilon \rho_0^{(+)} \Sigma. \tag{4.34}$$

Then,

$$\mathcal{B} = H - \epsilon^2 \mu_m |\Sigma|^2 \int_0^1 du |\rho_0|^2 \tag{4.35a}$$

$$= H - \frac{\mu_m}{(2\nu)^2} |\langle \mathcal{O} \rangle|^2 \int_0^1 du |\rho_0/\rho_0^{(+)}|^2, \tag{4.35b}$$

$$\mu_m = \frac{e^2}{1+e^2}. \tag{4.35c}$$

Just like in the GL theory (A37), the magnetic induction $\mathcal{B}$ reduces by the amount $|\langle \mathcal{O} \rangle|^2$ which implies the Meissner effect.

(1) At weak coupling $e \ll 1$,

$$\mathcal{B} \sim H - e^2 |\langle \mathcal{O} \rangle|^2 \tag{4.36}$$

apart from numerical factors, and the e dependence coincides with the GL theory. But the result deviates it as one increases e.

(2) In particular, in the $e \to \infty$ limit,

$$\mathcal{B} \to H - \frac{1}{(2\nu)^2} |\langle \mathcal{O} \rangle|^2 \int_0^1 du |\rho_0/\rho_0^{(+)}|^2. \tag{4.37}$$

Unlike in the GL theory, there is a nontrivial $e \to \infty$ limit.

We discuss vortex lattices, but the analysis itself does not tell whether the holographic superconductor is type I or type II. Using the GL parameter κ, H_{c2} and thermodynamic critical magnetic field H_c are related by

$$H_{c2} = \sqrt{2}\kappa H_c. \tag{4.38}$$

When $\kappa^2 > 1/2$, $H_{c2} > H_c$, and the superconductor is type II. When $\kappa^2 < 1/2$, $H_{c2} < H_c$, and the superconductor

is type I. The existence of a vortex solution itself does not imply that the superconductor is type II. Let us lower the magnetic field. For a type I superconductor, the material can "supercool"; namely it can remain the normal state even for $H < H_c$. Then, at $H = H_{c2}$, nucleation occurs, and the vortex lattice forms. In order to determine that our holographic superconductor is type I or II, one needs to determine κ.

V. BULK 5 DIMENSIONS

The analysis of the bulk 5-dimensional holographic superconductor is similar, but an analytic solution is available for a particular value of m^2 [22,23], and one can obtain λ, the correlation length ξ, and the GL parameter κ explicitly.

A. The GL parameter

We again consider a small magnetic field. In this case,

$$\tilde{A}_y = \tilde{\mathcal{A}}_y \left\{ 1 - \int_0^u \frac{u' du'}{f(u')} \int_{u'}^1 du'' \frac{1}{u''} \left(\frac{q^2}{r_0^2} + 2|\varphi_0|^2 \right) + \cdots \right\}.$$
(5.1)

For SAdS$_5$, $r_0 = \pi T$. For $p = 3$, one must add a counterterm action

$$S_{CT} = - \int d^4 x \frac{1}{4g^2} \sqrt{-\gamma} \gamma^{\mu\nu} \gamma^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \times \ln(u/r_0), \quad (5.2)$$

where $\gamma_{\mu\nu}$ is the $(p+1)$-dimensional boundary metric. The current is then given by

$$\langle \tilde{\mathcal{J}}_y \rangle = \frac{r_0^2}{u} \partial_u \tilde{A}_y - \partial_\nu (\sqrt{-\gamma} F^{y\nu}) \times \ln(u/r_0)|_{u=0} \quad (5.3a)$$

$$= \tilde{\mathcal{A}}_y \left(q^2 \ln r_0 - 2r_0^2 \int_0^1 du \frac{1}{u} |\varphi_0|^2 + \cdots \right). \quad (5.3b)$$

Again, the first term of Eq. (5.3b) is a bound current, and the second term is the supercurrent.

We again impose the holographic semiclassical equation with an external source:

$$q^2 \tilde{\mathcal{A}}_y = e^2 (q^2 c_1 - 2r_0^2 I) \tilde{\mathcal{A}}_y + e^2 \tilde{\mathcal{J}}_y^{ext}, \quad (5.4a)$$

$$c_1 = \ln r_0, \quad (5.4b)$$

$$I = \int_0^1 du \frac{1}{u} |\varphi_0|^2. \quad (5.4c)$$

The bound current part is rewritten as

$$(1 - c_1 e^2) q^2 \tilde{\mathcal{A}}_y = e^2 \tilde{\mathcal{J}}_y^{ext}, \quad (5.5a)$$

$$\to q^2 \tilde{\mathcal{A}}_y = \frac{e^2}{1 - c_1 e^2} \tilde{\mathcal{J}}_y^{ext} =: \mu_m \tilde{\mathcal{J}}_y^{ext}, \quad (5.5b)$$

$$\to \mu_m = \frac{e^2}{1 - c_1 e^2} = \frac{e^2}{1 - e^2 \ln(\pi T)}. \quad (5.5c)$$

The magnetic susceptibility χ_m is given by

$$\chi_m = \frac{c_1 e^2}{1 - c_1 e^2} = \frac{e^2 \ln(\pi T)}{1 - e^2 \ln(\pi T)}. \quad (5.6)$$

At $T = 0$, $\chi_m < 0$ or diamagnetic. As one increases temperature, $\chi_m > 0$ or paramagnetic. Then, χ_m diverges at $e^2 \ln(\pi T) = 1$, and $\chi_m < 0$ at high temperatures. Then,

$$\tilde{\mathcal{A}}_y \propto \frac{1}{q^2 + 2\mu_m r_0^2 I}, \quad (5.7a)$$

$$\mathcal{A}_y \propto e^{-x/\lambda}, \quad (5.7b)$$

$$\lambda^2 = \frac{1}{2\mu_m I} \frac{1}{r_0^2}, \quad (5.7c)$$

which implies the Meissner effect.

When $(p, \Delta) = (3, 2)$, there exits a simple analytic solution at the critical point [22]. The scalar solution is parametrized by a dimensionless parameter μ/T, and $\mu/T = 2\pi$ is the critical point. We fix T and vary μ. The solution is

$$\varphi_0 = \sqrt{\frac{24}{r_0} (\mu - \mu_c)} \frac{u}{1 + u^2}. \quad (5.8)$$

See Appendix B for details. The factor $(\mu - \mu_c)^{1/2}$ shows the mean-field behavior with critical exponent $\beta = 1/2$. The solution is a special case of a one-parameter family of holographic Lifshitz superconductors [23]. Then, one can evaluate I explicitly:

$$I = \int_0^1 du \frac{1}{u} |\varphi_0|^2 = \frac{6}{r_0} (\mu - \mu_c), \quad (5.9a)$$

$$\lambda^2 = \frac{1}{2\mu_m I} \frac{1}{r_0^2} \quad (5.9b)$$

$$= \frac{1 - e^2 \ln(\pi T)}{e^2} \frac{1}{12(\mu - \mu_c)} \frac{1}{\pi T}. \quad (5.9c)$$

On the other hand,

$$\xi^2 = \frac{1}{2(\mu - \mu_c)} \frac{1}{\pi T}, \quad (5.10)$$

so

$$\kappa^2 = \frac{\lambda^2}{\xi^2} = \frac{1 - e^2 \ln(\pi T)}{6e^2}. \tag{5.11}$$

The factor $1/6$ was found previously [23].

(1) Focus on $e^2 \ln(\pi T) < 1$ where μ_m and λ^2 are positive. First, consider a fixed T. At weak coupling $e \ll 1$, the e dependence coincides with the GL theory. But the result deviates it as one increases e.

(2) In particular, in the $e \to \infty$ limit, $\lambda_{\rm GL} \to 0$, so $\kappa_{\rm GL} \to 0$. But in the holographic superconductor, λ remains finite and the extreme type I limit cannot be reached [when $e^2 \ln(\pi T) < 1$].

(3) In general, whether the holographic superconductor is type I or II depends on T as well (Fig. 1). As $T \to 0$, $\kappa \to \infty$, so it is the extreme type II. As one increases T, κ decreases. A similar result holds in many superconducting materials including high-T_c materials [27].

The boundary between type I or II is given by $\kappa^2 = 1/2$, so

$$T = \frac{1}{\pi} \exp\left(\frac{1 - 3e^2}{e^2}\right) \tag{5.12a}$$

$$\to \frac{1}{\pi} e^{-3} \sim 0.016, \qquad (e \to \infty). \tag{5.12b}$$
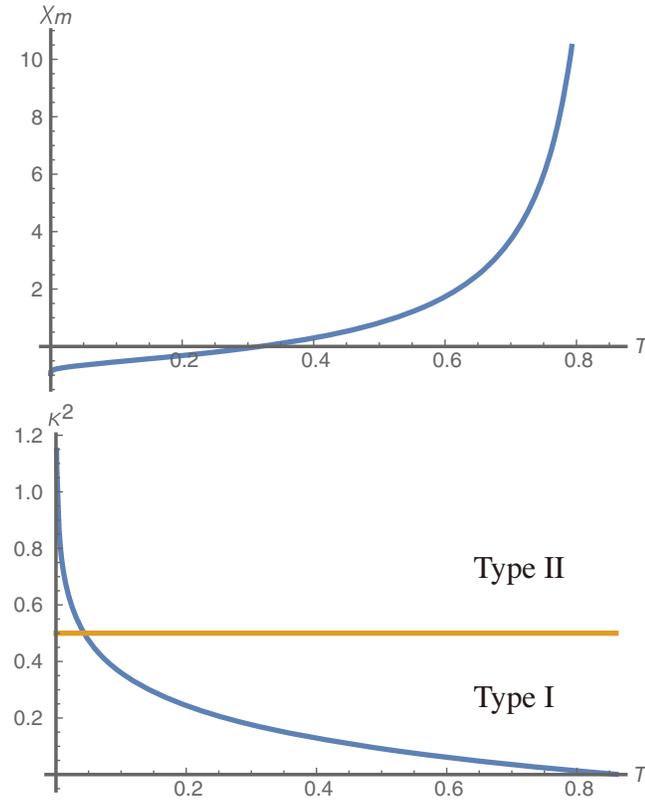

FIG. 1. The magnetic susceptibility χ_m (top) and the GL parameter κ^2 (bottom) for $e^2 \ln(\pi T) < 1$ with $e = 1$.

B. The dual GL theory

From the results we obtained, one is able to determine the dual GL theory. Writing $\psi = \langle \mathcal{O} \rangle$, the GL theory is given by

$$F = \int d^3x \left\{ c|D_i\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{4\mu_m}F_{ij}^2 \right.$$

$$\left. - (\psi J^\dagger + \psi^\dagger J) \right\}, \tag{5.13}$$

where J is the source of the order parameter. From the GL theory, one obtains the following (see Appendix A for the details):

(1) The spontaneous condensate: $|\psi_0|^2 = -a/b$.
(2) The penetration length: $\lambda^2 = 1/(2c\mu_m|\psi_0|^2)$.
(3) The correlation length: $\xi^2 = -c/a$.

Also, $\kappa^2 = b/(2\mu_m c^2)$.

Our holographic results are as follows:

(1) According to the standard AdS/CFT dictionary, $\langle \mathcal{O} \rangle = -\Psi^{(+)}$. The spontaneous condensate is $|\psi|^2 = 24\epsilon_\mu$, where $\epsilon_\mu := \mu - \mu_c$.
(2) The penetration length: $\lambda^2 = 2/(\mu_m|\psi|^2)$.
(3) The correlation length: $\xi^2 = 1/(2\epsilon_\mu)$.

(For simplicity, we set $r_0 = 1$.) Comparing these results fixes all GL parameters:

$$F = \int d^3x \left\{ \frac{1}{4}|D_i\psi|^2 - \frac{\epsilon_\mu}{2}|\psi|^2 + \frac{1}{96}|\psi|^4 + \frac{1}{4\mu_m}F_{ij}^2 \right.$$

$$\left. - (\psi J^\dagger + \psi^\dagger J) \right\}. \tag{5.14}$$

Just like in the GL theory, this free energy should be regarded as leading terms. For example, we do not include the $O(\psi^6)$ term and higher, and the numerical coefficients are leading ones. Note that the order parameter does not have the canonical normalization. Rather, the normalization is chosen from the GKP-Witten relation. Namely, we fix the normalization of ψ so that the source term is given by $(\psi J^\dagger + \psi^\dagger J)$ with $J = \Psi^{(-)}$.

VI. DISCUSSION

(1) Reference [16] studies the holographic semiclassical equation by constructing a single vortex solution numerically. However, there is a puzzle in previous analysis. For the $p = 2$ holographic superconductor, they consider the Neumann boundary condition $\langle \mathcal{J}^i \rangle = 0$. This is equivalent to the $e \to \infty$ limit because $\partial_j \mathcal{F}^{ij} = e^2 \langle \mathcal{J}^i \rangle$. Then, one expects the extreme type I superconductor, but they obtain a type II superconductor. This was explained in terms of S-duality [28].

Our analysis gives an alternative interpretation. This is because the holographic magnetic penetration

length has a nontrivial $e \to \infty$ limit. A similar analysis can be done for the $p = 3$ case. Thus, the Neumann boundary condition should be possible for the $p = 3$ case as well.

(2) In the above analysis, we focus on $e^2 \ln(\pi T) < 1$, but it is interesting to consider $e^2 \ln(\pi T) > 1$, where $\mu_m < 0$. Using ϵ, μ_m, one can classify a material as follows:

   (a) In the vacuum, $\epsilon, \mu_m > 0$, and the speed of light c is given by $c^2 = 1/(\epsilon \mu_m)$.
   (b) For metals, $\epsilon < 0$, $\mu_m > 0$, and it implies that the material is not transparent to light.
   (c) When $\epsilon < 0$, $\mu_m < 0$, the material is transparent to light. Such a material is called a metamaterial and shows the negative refractive index [25].
   (d) When $\epsilon > 0$, $\mu_m < 0$, the material is not transparent to light again.

The $\mathcal{N} = 4$ plasma has $\mu_m < 0$ for $e^2 \ln(\pi T) > 1$. Also, one can show that $\epsilon > 0$ for the plasma. Thus, the plasma corresponds to (d) and is not transparent to light. It is interesting to study the implications for the quark-gluon plasma. On the other hand, $\lambda^2 < 0$ in this case, so it implies that the Meissner effect does not occur. Namely, the magnetic field can enter the material. We are unaware if such an effect is discussed in superconductor literature.

(3) One limitation of our analysis is that we take the probe limit $g \gg 1$. g appears in the combination e^2/g^2, and the $e \to \infty$ limit really means the $e/g \to \infty$ limit. It is certainly interesting to take the back-reaction into account, but it is difficult to study the system analytically.

(4) Finally, we apply the holographic semiclassical equation to the Meissner effect, but it is interesting to explore the other backreaction problems.

ACKNOWLEDGMENTS

APPENDIX A: GINZBURG-LANDAU THEORY

The GL theory is given by

$$F = \int d^p x \left\{ c|D_i \psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{4\mu_m} F_{ij}^2 \right\}, \quad \text{(A1a)}$$

$$D_i := \partial_i - iA_i, \quad \text{(A1b)}$$

where μ_m is the magnetic permeability. In the standard GL theory, $\mu_m = e^2$. Namely, we slightly generalize the GL theory where the material has a magnetization (not due to supercurrent) and there exists a bound current. We take

$$a = a_0(T - T_c) + \cdots, \qquad (a_0 > 0), \quad \text{(A2a)}$$

$$b = b_0 + \cdots, \qquad (b_0 > 0). \quad \text{(A2b)}$$

The equations of motion are given by

$$0 = -cD^2\psi + a\psi + b\psi|\psi|^2, \quad \text{(A3a)}$$

$$0 = \partial_j F^{ij} - \mu_m j^i, \quad \text{(A3b)}$$

$$\begin{aligned} j_i &= -\frac{\delta F_\psi}{\delta A^i} \\ &= -ic\{\psi^\dagger D_i \psi - \psi D_i \psi^\dagger\} = 2c\Im[\psi^\dagger D_i \psi]. \quad \text{(A3c)} \end{aligned}$$

Below $T < T_c$, a homogeneous spontaneous condensate is a solution:

$$|\psi_0|^2 = -\frac{a}{b} \propto T_c - T. \quad \text{(A4)}$$

There are two characteristic scales for a superconductor:

(1) One is the correlation length of the order parameter. This comes from the order parameter mass and is given by

$$\xi^2 = \frac{c}{|a|}. \quad \text{(A5)}$$

(2) The other is the magnetic penetration length. This comes from the gauge field mass and is given by

$$\lambda^2 = \frac{1}{2c\mu_m|\psi_0|^2} = \frac{1}{2c\mu_m}\frac{b}{|a|}. \quad \text{(A6)}$$

Then, a superconductor is characterized by a dimensionless parameter κ, the GL parameter:

$$\kappa^2 = \frac{\lambda^2}{\xi^2} = \frac{b}{2\mu_m c^2}. \quad \text{(A7)}$$

A superconductor is classified by the value of κ:

   (i) type I: $\kappa^2 < 1/2$ or $\xi > \sqrt{2}\lambda$.
   (ii) type II: $\kappa^2 > 1/2$ or $\xi < \sqrt{2}\lambda$.

The factor $1/2$ is determined from the free energy analysis below. In a type I superconductor, the penetration length is shorter than the correlation length, and the magnetic field cannot enter the superconductor. As one increases the magnetic field, eventually superconductivity is broken. In a type II superconductor, the penetration length is longer than the correlation length, and the magnetic field can enter the superconductor keeping the superconducting state. The magnetic field enters by forming vortices.

Below $T < T_c$, a homogeneous condensate is a solution. Then, apply a small magnetic field. For simplicity, consider a 2-dimensional superconductor in the (x, y) plane, and apply a magnetic field in the z direction: $A_y = A_y(x)$ and

$B = F_{xy}$. We also add an external source j_y^{ext}. The Maxwell equation becomes

$$q^2 \tilde{A}_y = \mu_m(-2c|\psi_0|^2 \tilde{A}_y + j_y^{\text{ext}}), \qquad \text{(A8a)}$$

$$\to \tilde{A}_y \propto \frac{1}{q^2 + 2c\mu_m|\psi_0|^2}. \qquad \text{(A8b)}$$

The inverse Fourier transformation gives

$$A_y \propto e^{-x/\lambda}. \qquad \text{(A9)}$$

1. Single vortex

Far from the vortex, $\psi = \psi_0$ is approximately constant. In the cylindrical coordinate $ds^2 = dr^2 + r^2 d\phi^2 + \cdots$, the $A_\phi = A_\phi(r)$ equation becomes

$$0 = \frac{1}{r}\left(\frac{1}{r}A_\phi'\right)' - \frac{1}{\lambda^2}\frac{A_\phi}{r^2}. \qquad \text{(A10)}$$

Then, the solution is given by the modified Bessel function K_1:

$$A_\phi = \frac{r}{\lambda}K_1(r/\lambda) \to \sqrt{\frac{\pi r}{2\lambda}}e^{-r/\lambda}, \qquad \text{(A11)}$$

where we used the asymptotic formula

$$K_1(z) \to \sqrt{\frac{\pi}{2z}}e^{-z}, \qquad (z \to \infty). \qquad \text{(A12)}$$

2. Critical magnetic field

The critical magnetic field H_c is defined by the condition that the homogeneous condensate is thermodynamically favorable compared with the normal state. It is convenient to write $\psi = \rho e^{i\theta}$ and use the gauge-invariant variable $\hat{A}_i$:

$$\hat{A}_i := A_i - \partial_i\theta. \qquad \text{(A13)}$$

The free energy becomes

$$F = \int d^p x \left\{ c(\partial_i\rho)^2 + (a + c\hat{A}_i^2)\rho^2 + \frac{b}{2}\rho^4 + \frac{1}{4\mu_m}F_{ij}^2 \right\}. \qquad \text{(A14)}$$

The equations of motion are given by

$$0 = -c\partial_i^2\rho + (a + c\hat{A}_i^2)\rho + b\rho^3, \qquad \text{(A15a)}$$

$$0 = \partial_j F^{ij} + 2c\mu_m\rho^2\hat{A}_i. \qquad \text{(A15b)}$$

The variation of F includes the term

$$\delta F = \cdots + \frac{1}{\mu_m}\int dS_i \, F^{ij}\delta\hat{A}_j. \qquad \text{(A16)}$$

Then, F is appropriate when one fixes $\hat{A}_i$ on the boundary but is not appropriate when one fixes the external magnetic field $F_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i$. In order to obtain H_c, one fixes the external magnetic field, so one should use the Gibbs free energy. We define the Gibbs free energy by

$$G = F - \frac{1}{\mu_m}\int dS_i \, F^{ij}\hat{A}_j \qquad \text{(A17a)}$$

$$= F - \frac{1}{\mu_m}\int d^p x \, \partial_i(F^{ij}\hat{A}_j). \qquad \text{(A17b)}$$

Then, the variation becomes

$$\delta G = \cdots - \frac{1}{\mu_m}\int dS_i \, \delta F^{ij}\hat{A}_j. \qquad \text{(A18)}$$

Using the Maxwell equation, the on-shell Gibbs free energy becomes

$$G = \int d^p x \left\{ c(\partial_i\rho)^2 + (a + c\hat{A}_i^2)\rho^2 + \frac{b}{2}\rho^4 + 2c\hat{A}_i^2\rho^2 - \frac{1}{4\mu_m}F_{ij}^2 \right\}. \qquad \text{(A19)}$$

In the superconducting phase, $\rho^2 = -a/b$ and $\hat{A}_i = 0$ (due to the Meissner effect), so

$$G_s = -\frac{a^2}{2b}V_p, \qquad \text{(A20)}$$

where V_p is the p-dimensional volume. In the normal phase, $\rho = 0$ and $F_{xy} = H$, so

$$G_n = -\frac{1}{2\mu_m}H^2 V_p. \qquad \text{(A21)}$$

When $G_s < G_n$, the superconducting phase is favorable, so

$$H < H_c = -a\sqrt{\frac{\mu_m}{b}}. \qquad \text{(A22)}$$

As we see below, $H_{c2} = -a/c$, so

$$H_{c2} = \frac{-a}{c} = \sqrt{2}\kappa H_c. \qquad \text{(A23)}$$

When $\kappa^2 < 1/2$, $H_{c2} < H_c$, and the superconductor is type I. When $\kappa^2 > 1/2$, $H_{c2} < H_c$, and the superconductor is type II.

3. Upper critical magnetic field

Near the upper critical magnetic field H_{c2}, ψ remains small, and one can expand matter fields as a power series:

$$\psi = \epsilon\psi^{(1)} + \cdots, \tag{A24a}$$

$$A_i = A_i^{(0)} + \epsilon^2 A_i^{(2)} + \cdots. \tag{A24b}$$

At zeroth order, the Maxwell equation is

$$0 = \partial_j F_{(0)}^{ij}, \tag{A25}$$

so one has a homogeneous magnetic field

$$A_y^{(0)} = Hx. \tag{A26}$$

At first order, the order parameter field obeys

$$0 = -c(\partial_i - iA_i^{(0)})^2\psi^{(1)} + a\psi^{(1)}. \tag{A27}$$

Using the ansatz

$$\psi^{(1)} = e^{iqy}\chi_q(x), \tag{A28}$$

the equation becomes

$$c\left\{-\partial_x^2 + H^2\left(x - \frac{q}{H}\right)^2\right\}\chi_q = -a\chi_q. \tag{A29}$$

This is the Landau problem, and the solution is given by the Hermite function H_n as

$$\chi_q = e^{-z^2/2}H_n(z), \qquad z := \sqrt{H}\left(x - \frac{q}{H}\right). \tag{A30}$$

The eigenvalue is given by

$$E_n = (2n+1)H = \frac{-a}{c}. \tag{A31}$$

H takes the maximum value when $n = 0$ or $H_{c2} = -a/c$. The general solution is written as

$$\psi^{(1)} = \int_{-\infty}^{\infty} dq\, C(q)e^{iqy}\chi_q(x). \tag{A32}$$

The first order solution (A32) satisfies

$$(\partial_y - iA_y^{(0)})\psi^{(1)} = i(\partial_x - iA_x^{(0)})\psi^{(1)}, \tag{A33}$$

so

$$J_x^{(2)} = 2c\Im[(\psi^{(1)})^\dagger D_x^{(0)}\psi^{(1)}] = -c\partial_y|\psi^{(1)}|^2, \tag{A34a}$$

$$J_y^{(2)} = c\partial_x|\psi^{(1)}|^2, \tag{A34b}$$

or

$$J_a^{(2)} = 2c\Im[(\psi^{(1)})^\dagger D_a^{(0)}\psi^{(1)}] = -c\epsilon_a^b\partial_b|\psi^{(1)}|^2, \tag{A35}$$

where the Latin indices a, b run though x and y, and $\epsilon_{xy} = 1$. Then, at second order,

$$0 = \partial^b F_{ab}^{(2)} - \mu_m J_a^{(2)} \tag{A36a}$$

$$= \epsilon_{ab}\partial^b(F_{xy}^{(2)} - c\mu_m|\psi^{(1)}|^2). \tag{A36b}$$

One can integrate the equation. Asymptotically, $|\psi^{(1)}| \to 0$, so $F_{xy} \to H$. Then,

$$F_{xy} = B = H - c\mu_m|\psi^{(1)}|^2. \tag{A37}$$

Thus, the magnetic induction B reduces by the amount $|\psi^{(1)}|^2$ which implies the Meissner effect. We discuss its holographic counterpart in Sec. IV.

APPENDIX B: ANALYTIC SOLUTION OF HOLOGRAPHIC SUPERCONDUCTOR

For the SAdS_{p+2} background, the Hawking temperature is $\pi T = (p+1)r_0/4$. In the gauge $A_u = 0$, the static bulk equations become

$$0 = -fu^{p-2}\partial_u\left(\frac{1}{u^{p-2}}\partial_u A_t\right) - \frac{\Delta}{r_0^2}A_t + 2|\varphi|^2 A_t, \tag{B1a}$$

$$0 = -u^{p-2}\partial_u\left(\frac{f}{u^{p-2}}\partial_u A_i\right) - \frac{\Delta}{r_0^2}A_i - 2\Im(\varphi^\dagger D_i\varphi)$$
$$+ \frac{1}{r_0^2}\partial_i(\vec{\partial}\cdot\vec{A}), \tag{B1b}$$

$$0 = -u^{p-2}\partial_u\left(\frac{f}{u^{p-2}}\partial_u\varphi\right) + \left(V - \frac{A_t^2}{r_0^2 f} - \frac{\delta^{ij}D_i D_j}{r_0^2}\right)\varphi, \tag{B1c}$$

$$0 = \frac{1}{r_0^2}\partial_u(\vec{\partial}\cdot\vec{A}) - 2\Im(\varphi^\dagger\partial_u\varphi), \tag{B1d}$$

where $\Delta := \delta^{ij}\partial_i\partial_j$, $(\vec{\partial}\cdot\vec{A}) := \delta^{ij}\partial_i A_j$, and

$$\Psi =: u\varphi, \tag{B2a}$$

$$f = 1 - u^{p+1}, \tag{B2b}$$

$$V := \frac{m^2 + pf - uf'}{u^2}. \tag{B2c}$$

For simplicity, we set $r_0 = 1$ below. The asymptotic behaviors of matter fields are given by

$$A_\mu \sim \mathcal{A}_\mu + A_\mu^{(+)} u^{p-1}, \tag{B3a}$$

$$\Psi \sim \Psi^{(-)} u^{\Delta_-} + \Psi^{(+)} u^{\Delta_+}, \tag{B3b}$$

$$\Delta_\pm := \frac{p+1}{2} \pm \nu, \qquad \nu = \sqrt{\frac{(p+1)^2}{4} + m^2}. \tag{B3c}$$

When the Breitenlohner-Freedman (BF) bound [29] is saturated or

$$m_{\mathrm{BF}}^2 = -\frac{(p+1)^2}{4}, \tag{B4}$$

the asymptotic behavior is replaced by

$$\Psi \sim \Psi^{(-)} u^{\Delta} \ln u + \Psi^{(+)} u^{\Delta}, \qquad \Delta := \frac{p+1}{2}. \tag{B5}$$

At high temperature, the equations of motion admit a solution

$$A_t = \mu(1 - u^{p-1}), \qquad A_i = 0, \qquad \Psi = 0. \tag{B6}$$

But the $\Psi = 0$ solution becomes unstable at the critical point and is replaced by a $\Psi \neq 0$ solution.

When $(p, \Delta) = (3, 2)$, there exits a simple analytic solution at the critical point [22,23]. In other words, this is the case where the scalar mass saturates the BF bound. We briefly discuss the solution for completeness.

1. Low-temperature background

Consider the solution of the form

$$\Psi = \Psi(u), \qquad A_t = A_t(u), \qquad A_i = 0. \tag{B7}$$

Near the critical point, the scalar field remains small, and one can expand matter fields. Namely, we construct the low-temperature background perturbatively:

$$\Psi(u) = \epsilon \Psi^{(1)} + \epsilon^3 \Psi^{(3)} + \cdots, \tag{B8a}$$

$$A_t(u) = A_t^{(0)} + \epsilon^2 A_t^{(2)} + \cdots. \tag{B8b}$$

At zeroth order,

$$A_t^{(0)} = \mu(1 - u^2). \tag{B9}$$

At first order, there exists a simple solution:

$$\Psi^{(1)} = \frac{u^2}{1 + u^2} \quad \text{at } \mu_c = \Delta = 2. \tag{B10}$$

This is the solution only at the critical point. Also, this is the solution of the linear equation of motion, so the overall constant ϵ is undetermined. To resolve these issues, one needs to proceed to higher orders.

We impose the following boundary conditions:
(1) $\Psi^{(n)}$: no fast falloff ($n \geq 3$) and no slow falloff. The former means that $\mathcal{O}$ comes only from $\Psi^{(1)}$. The latter is the condition for a spontaneous condensate. At the horizon, we impose the regularity condition.
(2) $A_t^{(n)}$: $A_t^{(n)} = 0$ at the horizon.
Namely, we fix the fast falloff $\mathcal{O}$, but the chemical potential is corrected as

$$\mu = \mu_c + \epsilon^2 \delta\mu_2 + \cdots. \tag{B11}$$

At higher orders,

$$A_t^{(2)} = (1 - u^2)\delta\mu_2 - \frac{u^2(1 - u^2)}{4(1 + u^2)} \tag{B12a}$$

$$\sim \delta\mu_2 + \frac{1}{4}(-1 - 4\delta\mu_2)u^2 + \cdots, \tag{B12b}$$

$$\Psi^{(3)} = \frac{-2u^4 + u^2(1 + u^2)\ln(1 + u^2)}{24(1 + u^2)^2}, \tag{B13a}$$

$$\delta\mu_2 = \frac{1}{24}. \tag{B13b}$$

Here, $\delta\mu_2$ is determined from the boundary condition. Then, at $O(\epsilon^2)$, the chemical potential becomes

$$\mu = A_t|_{u=0} = 2 + \frac{1}{24}\epsilon^2 + \cdots, \tag{B14a}$$

$$\Psi \sim \epsilon u^2, \qquad (u \to 0). \tag{B14b}$$

This fixes the overall constant of the condensate ϵ as

$$\epsilon^2 = 24(\mu - \mu_c). \tag{B15}$$

2. Correlation length

At high temperatures, the background solution is given by Eq. (B6). Consider the linear perturbation from the background $\Psi = 0 + \delta\Psi$. We consider the perturbation of the form $e^{-i\omega t + iqx}$. When $\Psi = 0$, δA_t and δA_i decouple from the $\delta\Psi$ equation, and it is enough to consider the $\delta\Psi$ equation. We impose the boundary conditions (1) regular at the horizon and (2) no slow falloff. Namely, we obtain quasinormal modes. Set $\epsilon_\mu = \mu - \mu_c < 0$ and employ the (ϵ_μ, q) expansion:

$$\delta\Psi = \psi_c + \epsilon_\mu \psi_\epsilon + q^2 \psi_q + \cdots. \tag{B16}$$

The solution is given by

$$\delta\Psi \propto \frac{u^2}{4(1+u^2)}\left(-q^2\ln u + 4 - 2\epsilon_\mu \ln\frac{1+u^2}{u}\right) \quad \text{(B17a)}$$

$$\sim -\frac{1}{4}(q^2 - 2\epsilon_\mu)u^2\ln u + u^2. \quad \text{(B17b)}$$

Thus,

$$q^2 - 2\epsilon_\mu = q^2 + \xi^{-2} = 0 \rightarrow \xi^2 = \frac{-1}{2\epsilon_\mu}. \quad \text{(B18)}$$

[1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).

[2] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[3] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[4] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).

[5] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. A. Wiedemann, *Gauge/String Duality, Hot QCD and Heavy Ion Collisions* (Cambridge University Press, Cambridge, England, 2014).

[6] M. Natsuume, AdS/CFT duality user guide, Lect. Notes Phys. **903**, 1 (2015).

[7] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality: Foundations and Applications* (Cambridge University Press, Cambridge, England, 2015).

[8] J. Zaanen, Y. W. Sun, Y. Liu, and K. Schalm, *Holographic Duality in Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2015).

[9] S. A. Hartnoll, A. Lucas, and S. Sachdev, *Holographic Quantum Matter* (The MIT Press, Cambridge, MA, 2018).

[10] M. Baggioli, *Applied Holography: A Practical Mini-Course*, SpringerBriefs in Physics (Springer, New York, 2019).

[11] G. Compere and D. Marolf, Setting the boundary free in AdS/CFT, Classical Quantum Gravity **25**, 195014 (2008).

[12] C. Ecker, W. van der Schee, D. Mateos, and J. Casalderrey-Solana, Holographic evolution with dynamical boundary gravity, J. High Energy Phys. 03 (2022) 137.

[13] S. S. Gubser, Breaking an Abelian gauge symmetry near a black hole horizon, Phys. Rev. D **78**, 065034 (2008).

[14] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Building a Holographic Superconductor, Phys. Rev. Lett. **101**, 031601 (2008).

[15] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Holographic superconductors, J. High Energy Phys. 12 (2008) 015.

[16] O. Domenech, M. Montull, A. Pomarol, A. Salvio, and P. J. Silva, Emergent gauge fields in holographic superconductors, J. High Energy Phys. 08 (2010) 033.

[17] T. Albash and C. V. Johnson, Vortex and droplet engineering in holographic superconductors, Phys. Rev. D **80**, 126009 (2009).

[18] M. Montull, A. Pomarol, and P. J. Silva, The Holographic Superconductor Vortex, Phys. Rev. Lett. **103**, 091601 (2009).

[19] K. Maeda, M. Natsuume, and T. Okamura, Vortex lattice for a holographic superconductor, Phys. Rev. D **81**, 026002 (2010).

[20] V. Keranen, E. Keski-Vakkuri, S. Nowling, and K. P. Yogendran, Inhomogeneous structures in holographic superfluids: II. vortices, Phys. Rev. D **81**, 126012 (2010).

[21] Ó. J. C. Dias, G. T. Horowitz, N. Iqbal, and J. E. Santos, Vortices in holographic superfluids and superconductors as conformal defects, J. High Energy Phys. 04 (2014) 096.

[22] C. P. Herzog, An analytic holographic superconductor, Phys. Rev. D **81**, 126009 (2010).

[23] M. Natsuume and T. Okamura, Holographic Lifshitz superconductors: Analytic solution, Phys. Rev. D **97**, 066016 (2018).

[24] I. R. Klebanov and E. Witten, AdS/CFT correspondence and symmetry breaking, Nucl. Phys. **B556**, 89 (1999).

[25] A. Amariti, D. Forcella, A. Mariotti, and G. Policastro, Holographic optics and negative refractive index, J. High Energy Phys. 04 (2011) 036.

[26] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, New York, 1984).

[27] L. P. Gor'kov, Critical supercooling field in superconductivity theory, Sov. Phys. JETP **37**, 593 (1960).

[28] E. Witten, SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry, arXiv:hep-th/0307041.

[29] P. Breitenlohner and D. Z. Freedman, Positive energy in anti-de Sitter backgrounds and gauged extended supergravity, Phys. Lett. **115B**, 197 (1982).