Gauge and scalar fields on $\mathbb{CP}^2$: A gauge-invariant analysis. I. The effective action from chiral scalars

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A parametrization of gauge fields on complex projective spaces of arbitrary dimension is given as a generalization of the real two-dimensional case. Gauge transformations act homogeneously on the fields, facilitating a manifestly gauge-invariant analysis. Specializing to four dimensions, we consider the nature of the effective action due to chiral scalars interacting with the gauge fields. The key qualitatively significant terms include a possible gauge-invariant mass term and a finite four-dimensional Wess-Zumino-Witten (WZW) action. We comment on relating the mass term to lattice simulations as well as Schwinger-Dyson analyses and also on relating the WZW action to the instanton liquid picture of QCD.

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I. INTRODUCTION

The gauge-invariant analyses of the low-energy or long-distance properties of non-Abelian gauge theories remains a challenging problem even after decades of work. Large-scale numerical simulations have produced important insights as well as quantitative estimates of physically relevant observables, but the analytic understanding of the problem is far from satisfactory. Perhaps the most revelatory aspect of this state of affairs is concerning the foundational ingredient needed for the quantum theory, namely, the volume element for the gauge-orbit space. This space is the set of all gauge potentials ($\mathcal{A}$) modulo the set of all gauge transformations which are fixed to be identity at one point on the spacetime manifold ($\mathcal{G}_*$) [1]. Thus it is this space of gauge-invariant field configurations over which the functional integration for such theories has to be carried out to define the quantum theory; i.e., the volume element of this gauge-orbit space $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$ provides the measure of integration. There is still no satisfactory and explicit formula for this in the continuum four-dimensional theory. One can use gauge fixing and the Faddeev-Popov procedure to construct this volume element for a local section of $\mathcal{A}$ viewed as a $\mathcal{G}_*$ bundle over $\mathcal{C}$, or, equivalently, one may use the BRST (Becchi-Rouet-Stora-Tyutin) procedure. However, nonperturbative questions are generally beyond the reach of this procedure, although it may be adequate for the perturbative calculations.

In contrast to this, for gauge fields in two dimensions, the volume element for $\mathcal{C}$ can be calculated exactly in terms of a Wess-Zumino-Witten (WZW) action [2]. Although there are no propagating degrees of freedom for gauge fields in two dimensions, the result is relevant for the Chern-Simons-WZW relationship [3] and in the solution of Yang-Mills theory on Riemann surfaces [4]. This result may be taken as applying to the fields on a spatial slice in $(2+1)$ dimensions, and one can thus seek to utilize it in a Hamiltonian approach to $(2+1)$-dimensional Yang-Mills theories. Such an analysis has led to a formula for the string tension and also provided insights into the mass gap [5,6], including supersymmetric cases [7]. The expression for the string tension agrees very well with estimates via lattice simulations [8] and, more recently, estimates of the Casimir energy have provided independent verification of the mass gap (or the mass defined by the propagator) [9].

The calculation of the volume element of $\mathcal{C}$ in two dimensions was made possible by a parametrization of the gauge fields which relied on the fact that the two-dimensional space could be considered as a complex manifold. For $\mathbb{R}^4$, there is no unique complex structure, since there are many ways to pair the coordinates to form complex ones. One could consider a twistor space version which would include the set of all local complex structures. Calculations for the gauge theory would also require an infrared cutoff, so a compact space of finite volume is a better alternative. The simplest case of such a space would be the complex projective plane $\mathbb{CP}^2$, which is a complex

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Kähler manifold. The standard metric for this space is the Fubini-Study metric which is given in local coordinates z^a, $\bar{z}^{\bar{a}}$, $a = 1, 2$, $\bar{a} = 1, 2$, as

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z}/r^2)} - \frac{\bar{z} \cdot dz z \cdot d\bar{z}}{r^2(1 + z \cdot \bar{z}/r^2)^2} = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}, \quad (1)$$

where we have also included a scale parameter r for the coordinates. The volume of $\mathbb{CP}^2$ with this metric is $\pi^2 r^4/2$, so r can serve as an infrared cutoff. As $r \to \infty$, the metric becomes that of flat space (although there are some global issues which will not be important for us). Thus this space has a complex structure (which can help with the parametrization of the fields) and a finite volume, with a well-defined limit to the flat case. Indeed, a parametrization of the gauge potentials, as a generalization of the parametrization in two dimensions, was given in Ref. [10], where some preliminary results regarding the volume element of $\mathcal{C}$ were also given. Admittedly, the group of isometries for the space is $SU(3)$, rather than the 4d Euclidean group. However, this should not be an issue for many questions of interest. Recall that one can obtain insights into the physics by studying lattice gauge theories even though the lattice breaks the Euclidean invariance, recovering it only in the continuum limit. The analog for our use of $\mathbb{CP}^2$ would be the large r limit. Also, there are many other instances in which scenarios with reduced isometries can give insight into the physics of a problem, the Casimir effect being a classic example.

A closely related issue is the nature of quantum corrections (to the gauge field dynamics) due to matter fields. The calculation of such corrections in a manifestly gauge-invariant way using the parametrization mentioned above can give insights into the renormalization structure of the gauge theory and hence to some questions of physical interest. We propose to take up a more detailed analysis of the volume element for $\mathcal{C}$ and the nature of quantum corrections due to a scalar field on $\mathbb{CP}^2$. The present article will be devoted to the general framework and the corrections due to the scalar field, with more details on the volume element for $\mathcal{C}$ to be given in a follow-up paper [11].

There are two physical aspects of non-Abelian gauge theories for which our analysis can lead to useful insights. The first is about a possible mass term for gluons. There has been growing evidence, based on analytical and numerical studies, that the gluon acquires a "mass" [12–14]. Given these results, one can ask if we can find any evidence for a mass term in a manifestly gauge-invariant and analytic approach. This is what our analysis addresses. One of the terms we find is indeed a mass term consistent with gauge invariance and all the isometries of the underlying space $\mathbb{CP}^2$.

The second consideration is about the instanton liquid picture [15,16]. Analytical investigations as well as lattice simulations have shown that the infrared behavior of correlation functions for gluons, and for hadrons, is dominated by a dense collection of instantons, the so-called instanton liquid. Again, in a manifestly gauge-invariant analysis, one can ask if there are indications of an instanton liquid. Indeed, we find that one of the terms in the effective action is a four-dimensional WZW action whose critical points are anti-self-dual instantons [17,18]. We will comment on these issues in more detail later.

The organization of this paper is as follows. In Sec. II, we give the general parametrization of the fields, discussing in turn the nature of scalars, vectors and gauge potentials on $\mathbb{CP}^2$. We also introduce the action for the scalar field. Section III is devoted to the calculation of the quantum corrections due to the scalar field. After giving the general framework, we calculate the scalar field propagator for $\mathbb{CP}^2$ and discuss regularization issues. The leading terms among the quantum corrections are then obtained. These include a WZW action with a finite coefficient, a quadratically divergent mass term, and the expected log divergence of the wave function renormalization for the gauge fields. In Sec. IV, we discuss the physical implications. Several computational details are given in Appendixes, three of them, so as to avoid clutter and keep an uninterrupted flow of the general arguments in the text. Appendix A gives the parametrization of the gauge fields and the calculation of the scalar propagator on $\mathbb{CP}^k$, for arbitrary k, even though $k = 2$ is what is used in the text of the paper. In Appendix B, we give the calculation of a current relevant for the identification of the WZW term. Some of the subtleties regarding the WZW term are discussed in detail. The ultraviolet divergences are calculated in Appendix C.

II. PARAMETRIZATION OF FIELDS

As stated in the introduction, the manifold $\mathbb{CP}^2$ allows for a parametrization of the fields with a clear separation of the gauge-invariant degrees of freedom. This is most conveniently done in terms of the coset structure of the space as $\mathbb{CP}^2 = SU(3)/U(2)$. The manifold may be coordinatized in terms of a group element $g \in SU(3)$, with $U(2) \subset SU(3)$ as the local isotropy group and the coset directions corresponding to the translational degrees of freedom. This shows that functions on $\mathbb{CP}^2$ correspond to functions on $SU(3)$ which are invariant under $U(2)$, while vectors, tensors, etc., transform as specific nontrivial representations of $U(2)$.

Turning to more specific details, the defining fundamental representation is taken as a 3×3 unitary matrix g of unit determinant. It can be parametrized as $g = \exp(it_a \varphi^a)$, where t_a form a basis for traceless Hermitian 3×3 matrices, with $\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$, and φ^a are the coordinates for $SU(3)$. Following the familiar nomenclature from the quark model, we shall refer to the $SU(2)$ part of the $U(2)$ subgroup as isospin (denoted by I) and the $U(1)$ part of $U(2)$ as hypercharge (denoted by Y). The subgroup $SU(2)$ corresponds to the directions $a = 1, 2, 3$, with the generators t_1,

t_2, and t_3 for its Lie algebra; the hypercharge corresponds to $2t_8/\sqrt{3}$. On g, we can define left (L_a) and right (R_a) translation operators by

$$L_a g = t_a g, \qquad R_a g = g t_a. \tag{2}$$

The translation operators (or derivative operators) on $\mathbb{CP}^2$ can then be defined as

$$R_{\pm 1} = R_4 \pm i R_5, \qquad R_{\pm 2} = R_6 \pm i R_7. \tag{3}$$

These are the appropriate complex components; we shall denote them by R_i, $R_{\bar{i}}$, $i, \bar{i} = 1, 2$. The matrices corresponding to these combinations have all elements equal to zero, except for the $(i3)$ and $(3i)$ elements, which are equal to 1 for R_i and $R_{\bar{i}}$, respectively. The curvatures for $\mathbb{CP}^2$ take values in the Lie algebra of $U(2)$, with the operators R_α, $\alpha = 1, 2, 3$, and R_8 defining the analog of spin. Explicitly, R_a can be realized as differential operators:

$$g^{-1} dg = -i t_a E^a_i d\varphi^i, \qquad R_a = i (E^{-1})^i_a \frac{\partial}{\partial \varphi^i}. \tag{4}$$

A basis for functions on $SU(3)$ is given by the finite-dimensional unitary representation matrices for $SU(3)$, denoted by $\mathcal{D}^{(s)}_{AB}(g) = \langle s, A | \hat{g} | s, B \rangle$ (and often referred to as the Wigner functions). The action of R_a on these functions is given by

$$R_a \langle s, A | \hat{g} | s, B \rangle = \langle s, A | \hat{g} T_a | s, B \rangle = \langle s, A | \hat{g} | s, C \rangle (T_a)_{CB}, \tag{5}$$

where T_a are the matrix representatives of t_a in the representation labeled by s. Functions, vectors and tensors on $\mathbb{CP}^2$ have the mode expansion

$$F(g) = \sum_{s,A} C^{(s)}_A D^{(s)}_{A,w}(g) = \sum_{s,A} C^{(s)}_A \langle s, A | \hat{g} | s, w \rangle, \tag{6}$$

where the states on the right, namely, $|s, w\rangle$, must be so chosen as to give the correct transformation property for $F(g)$ under $U(2) \in SU(3)$.

A. Functions on $\mathbb{CP}^2$

Functions on $\mathbb{CP}^2$ must be invariant under $U(2)$, so we need states $|s, w\rangle$ with $Y = 0$ and $I = 0$. A state $|\{a_i\}, \{b_i\}\rangle$ which carries a general $SU(3)$ representation is of the form $T^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots b_q}$, $a_i, b_j = 1, 2, 3$, which we refer to as a (p, q)-type representation. These are totally symmetric in all the upper indices a_i's and totally symmetric in all the lower indices b_j's with the trace (or any contraction between any choice of upper and lower indices) vanishing. The $SU(3)$ action on $T^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots b_q}$ is given by

$$T^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots b_q} \to (g^{* a_1 a'_1} g^{* a_2 a'_2} \cdots)(g_{b_1 b'_1} g_{b_2 b'_2} \cdots) T^{a'_1 a'_2 \ldots a'_p}_{b'_1 b'_2 \ldots b'_q}. \tag{7}$$

Notice that the isospin subgroup acts on indices taking values 1 and 2, while the value of hypercharge is given as

$$Y = \begin{cases} -\frac{1}{3} & a_i = 1, 2, \\ \frac{2}{3} & a_i = 3, \end{cases} \qquad Y = \begin{cases} \frac{1}{3} & b_i = 1, 2, \\ -\frac{2}{3} & b_i = 3. \end{cases} \tag{8}$$

The choice of all indices equal to 3 with $p = q$ corresponds to the $U(2)$-invariant choice. Thus, for functions on $\mathbb{CP}^2$, we need representations of the (p, p) type with the mode expansion given by

$$f(g) = \sum_{s,A} C^{(p,p)}_A \langle s, A | \hat{g} | 0 \rangle,$$
$$|0\rangle \equiv |(p, p), w\rangle = |333\ldots, 333\ldots\rangle. \tag{9}$$

For brevity, we will denote the $U(2)$-invariant state as $|0\rangle$.

B. Vectors on $\mathbb{CP}^2$

The translation operators $R_{+i} = R_i$ and $R_{-i} = R_{\bar{i}}$ transform as doublets of $SU(2)$ and carry hypercharge $Y = 1, -1$, respectively. Thus vectors on $\mathbb{CP}^2$ must have a similar transformation property. This can be obtained for representations of the (p, p) type with $|s, w\rangle$ of the form $|33\ldots, i33\ldots\rangle$ and $|i33\ldots, 33\ldots\rangle$, corresponding to $Y = 1$ and $Y = -1$, respectively. These can be obtained from the invariant state $|33\ldots, 33\ldots\rangle$ by the application of R_i and $R_{\bar{i}}$, respectively. The corresponding vectors are the gradients of functions on $\mathbb{CP}^2$.

One can also obtain the required states from representations of the $(p + 3, p)$ type with $|s, w\rangle = |i33\ldots, 33\ldots\rangle$, with $i = 1, 2$, corresponding to $Y = 1$, and from $(p, p + 3)$ type with $|s, w\rangle = |33\ldots, i33\ldots\rangle$ with $Y = -1$. Thus a vector on $\mathbb{CP}^2$ may be parametrized as

$$A_i = -R_i f - \eta_{i\bar{i}} \epsilon^{\bar{i}\bar{j}} \sum_{s,A} C^{(s)}_A \langle s, A | \hat{g} | \bar{j} 33\ldots, 33\ldots\rangle,$$
$$\bar{A}_{\bar{i}} = -R_{\bar{i}} \bar{f} - \eta_{\bar{i}i} \epsilon^{ij} \sum_{s^*,A} C^{(s^*)}_A \langle s^*, A | \hat{g} | 33\ldots, j33\ldots\rangle, \tag{10}$$

where, on the right-hand side, s indicates representations of the $(p + 3, p)$ type and s^* indicates the $(p, p + 3)$ type. The first terms on the right-hand side correspond to gradients of a function. The $(p + 3, p)$-type state $|\bar{j}33\ldots, 33\ldots\rangle$ can be obtained from the $SU(2)$-invariant states, with all indices equal to 3, by the application of $R_{\bar{j}}$ operators.[1] Specifically, we can write

[1]As defined in (7), the state $|j33\ldots, 33\ldots\rangle$ transforms under $SU(2)$ as the conjugate of the standard doublet representation (emphasized by using $\bar{j}$) the extra factor of $\eta_{i\bar{i}} \epsilon^{\bar{i}\bar{j}}$ converts the transformation to the usual doublet form.

$$\eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}|\bar{j}33\ldots,33\ldots\rangle = \eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}R_{\bar{j}}|33\ldots,33\ldots\rangle, \quad (11)$$

where $\eta_{i\bar{i}} = \delta_{i\bar{i}}$ (which is the metric for $\mathbb{CP}^2$ in the tangent frame) and $\epsilon^{\bar{i}\bar{j}}$ is the Levi-Civita tensor. The $SU(2)$-invariant state on the right-hand side has $Y = 2$, so the corresponding term in (10) may be written as $\eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}R_{\bar{j}}\chi$, where χ has $Y = 2$. [We may regard $\epsilon^{\bar{i}\bar{j}}\chi$ as a rank-2 tensor of the antiholomorphic type, so that the relevant term in (10) is the divergence of an antisymmetric tensor.] Similar statements hold for conjugates in the second line of (10), so that we can write the general parametrization as

$$\begin{aligned} A_i &= -R_i f - \eta_{i\bar{i}}\epsilon^{\bar{i}\bar{j}}R_{\bar{j}}\chi, \\ \bar{A}_{\bar{i}} &= -R_{\bar{i}}\bar{f} - \eta_{\bar{i}i}\epsilon^{ij}R_j\bar{\chi}. \end{aligned} \quad (12)$$

These can be written in terms of the standard covariant derivatives on $\mathbb{CP}^2$. R_i and $R_{\bar{i}}$ correspond to the tangent frame, with

$$R_i = (e^{-1})_i^m\nabla_m, \qquad R_{\bar{i}} = -(e^{-1})_{\bar{i}}^{\bar{m}}\bar{\nabla}_{\bar{m}}. \quad (13)$$

Here ∇'s include the spin connection as needed for χ and $\bar{\chi}$. e^i_m is the frame field for the metric on $\mathbb{CP}^2$, i.e., $\eta_{i\bar{i}}e^i_m e^{\bar{i}}_{\bar{m}} = g_{m\bar{m}}$. The explicit formulas for the frame field and its inverse for the Fubini-Study metric (1) are

$$\begin{aligned} e^a_m &= \frac{\delta^a_m}{\sqrt{1+\bar{z}\cdot z}} - \frac{\eta_{m\bar{m}}\bar{z}^{\bar{m}}z^a}{(1+\bar{z}\cdot z)(1+\sqrt{1+\bar{z}\cdot z})}, \\ (e^{-1})^m_a &= \sqrt{1+\bar{z}\cdot z}\left[\delta^m_a + \frac{\eta_{a\bar{a}}\bar{z}^{\bar{a}}z^m}{1+\sqrt{1+\bar{z}\cdot z}}\right]. \end{aligned} \quad (14)$$

In a coordinate basis the parametrization (12) takes the form

$$\begin{aligned} A_k &= -\nabla_k f + g_{k\bar{k}}\epsilon^{\bar{k}\bar{m}}\bar{\nabla}_{\bar{m}}\chi, \\ \bar{A}_{\bar{k}} &= \bar{\nabla}_{\bar{k}}\bar{f} - g_{\bar{k}k}\epsilon^{km}\nabla_m\bar{\chi}. \end{aligned} \quad (15)$$

In this paper, we choose the components of A to be related by $(A_i)^\dagger = -\bar{A}_{\bar{i}}$. This is in conformity with the use of anti-Hermitian components for the gauge fields, which is what is conventionally done for non-Abelian fields.

If we scale $z \to z/r$ and consider large values of r, $\mathbb{CP}^2$ reduces to a flat space, but with a complex structure since we still retain complex combinations of the real coordinates. In this case, (15) still retains its form:

$$A_k = -\partial_k f + \eta_{k\bar{k}}\epsilon^{\bar{k}\bar{m}}\bar{\partial}_{\bar{m}}\chi, \quad \bar{A}_{\bar{k}} = \bar{\partial}_{\bar{k}}\bar{f} - \eta_{\bar{k}k}\epsilon^{km}\partial_m\bar{\chi}. \quad (16)$$

An *a priori* and direct demonstration that this provides a complete and unique (see later) parametrization of the fields in the flat space limit is difficult without the group theoretic arguments which were used for $\mathbb{CP}^2$.

C. Gauge fields on $\mathbb{CP}^2$

We can use (15) as the parametrization for Abelian gauge fields (vector potentials) on $\mathbb{CP}^2$. In this case, $\chi, \bar{\chi}$ and the real part of f correspond to gauge-invariant degrees of freedom, while the imaginary part of f is the gauge parameter.

In generalizing to the non-Abelian case, we first note that the product of two functions on $\mathbb{CP}^2$ is still a function since it remains invariant under $U(2)$. So we can compose functions. Likewise the product of functions with χ or $\bar{\chi}$ retain the same $U(2)$ transformations as χ and $\bar{\chi}$. We can now write the generalization of (15) to the non-Abelian gauge fields as

$$\begin{aligned} A_i &= -\nabla_i M M^{-1} + g_{i\bar{i}}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}}, \\ \bar{A}_{\bar{i}} &= M^{\dagger -1}\bar{\nabla}_{\bar{i}}M^\dagger - g_{\bar{i}i}D_j\phi^{\dagger ij}. \end{aligned} \quad (17)$$

Here M and $M^\dagger$ are complex matrices which are group elements in the complexification of the gauge group. We will take the gauge group to be $SU(N)$ for simplicity. (This is easily generalized to any Lie group.) In this case, M and $M^\dagger$ are complex $N\times N$ matrices which may be viewed as elements of $SL(N,\mathbb{C})$. Further, $\phi^{\bar{i}\bar{j}} = \epsilon^{\bar{i}\bar{j}}\phi$ and $\phi^{\dagger ij} = \epsilon^{ij}\phi^\dagger$, where $\phi^{\bar{i}\bar{j}}$ and $\phi^{\dagger ij}$ are tensors valued in the Lie algebra of the gauge group $SU(N)$, in agreement with A_i and $\bar{A}_{\bar{i}}$ being Lie-algebra valued. Since ϕ is complex, we may also view it as an element of the Lie algebra of $SL(N,\mathbb{C})$, with $\phi^\dagger$ as its conjugate. The derivatives D_j and $\bar{D}_{\bar{j}}$ are defined by

$$\begin{aligned} D_j\Phi &= \nabla_j\Phi + [-\nabla_j M M^{-1}, \Phi], \\ \bar{D}_{\bar{j}}\Phi &= \bar{\nabla}_{\bar{j}}\Phi + [M^{\dagger -1}\bar{\nabla}_{\bar{j}}M^\dagger, \Phi] \end{aligned} \quad (18)$$

acting on a field Φ which transforms under the adjoint representation of the gauge group, $\Phi \to U\Phi U^\dagger$, where $U \in SU(N)$ is the gauge transformation. The potentials in (17) transform as connections with $M \to UM$, $M^\dagger \to M^\dagger U^\dagger$, $(\phi, \phi^\dagger) \to U(\phi, \phi^\dagger)U^\dagger$. The use of just $-\nabla_j M M^{-1}$, $M^{\dagger -1}\bar{\nabla}_{\bar{j}}M^\dagger$ in defining D_j and $\bar{D}_{\bar{j}}$ suffices to ensure that $D_j\Phi$ and $\bar{D}_{\bar{j}}\Phi$ transform covariantly under gauge transformations.[2] These derivatives are also Levi-Civita covariant.

There is another useful way to write the parametrization (17). Toward this we first note the identities

[2]An important point about gauge fields is that the space of connections is an affine space, so that one can reach any point in this space from any other point by a straight line. In other words, if $A^{(1)}$ and $A^{(2)}$ denote two potentials, then $A^{(1)}\tau + A^{(2)}(1-\tau)$, $0 \le \tau \le 1$ transforms like a connection for all τ. Therefore one can use a specific connection as a starting point and obtain every other connection by adding something that transforms covariantly. We may view $(-\nabla_i M M^{-1}, M^{\dagger -1}\bar{\nabla}_{\bar{i}}M^\dagger)$ as the starting connection and $(g_{i\bar{i}}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}}, -g_{\bar{i}i}D_j\phi^{\dagger ij})$ as what is added. In particular we can construct covariant derivatives for $(\phi^{\bar{i}\bar{j}}, \phi^{\dagger ij})$ using the starting connection $(-\nabla_i M M^{-1}, M^{\dagger -1}\bar{\nabla}_{\bar{i}}M^\dagger)$.

$$\begin{aligned}\bar{D}_{\bar{j}}\phi^{\bar{i}\bar{j}} &= \bar{\nabla}_{\bar{j}}\phi^{\bar{i}\bar{j}} + [M^{\dagger-1}\bar{\nabla}_{\bar{j}}M^\dagger, \phi^{\bar{i}\bar{j}}] \\ &= M[\bar{\nabla}_{\bar{j}}(M^{-1}\phi M)^{\bar{i}\bar{j}} + [H^{-1}\bar{\nabla}_{\bar{j}}H, (M^{-1}\phi M)^{\bar{i}\bar{j}}]]M^{-1} \\ &= M(\bar{\mathcal{D}}_{\bar{j}}(M^{-1}\phi M)^{\bar{i}\bar{j}})M^{-1} = M(\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}})M^{-1}, \\ D_j\phi^{\dagger ij} &= M^{\dagger-1}(\mathcal{D}_j\chi^{\dagger ij})M^\dagger, \end{aligned} \tag{19}$$

where $\chi^{\bar{i}\bar{j}} = \epsilon^{\bar{i}\bar{j}}(M^{-1}\phi M)$ and $\chi^{\dagger ij} = \epsilon^{ij}(M^\dagger\phi^\dagger M^{\dagger-1})$. Further, H is given as $H = M^\dagger M$ and $\bar{\mathcal{D}}_{\bar{j}}$ and $\mathcal{D}_j$ are defined with the connections $H^{-1}\bar{\nabla}_{\bar{j}}H$ and $-\nabla_j HH^{-1}$:

$$\begin{aligned}\bar{\mathcal{D}}_{\bar{j}}\Phi &= \bar{\nabla}_{\bar{j}}\Phi + [H^{-1}\bar{\nabla}_{\bar{j}}H, \Phi], \\ \mathcal{D}_j\Phi &= \nabla_j\Phi + [-\nabla_j HH^{-1}, \Phi].\end{aligned} \tag{20}$$

Using these identities, we can write (17) as

$$\begin{aligned}A_i &= -\nabla_i MM^{-1} + M(g_{i\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}})M^{-1}, \\ \bar{A}_{\bar{i}} &= M^{\dagger-1}\bar{\nabla}_{\bar{i}}M^\dagger + M^{\dagger-1}(-g_{\bar{i}i}\mathcal{D}_j\chi^{\dagger ij})M^\dagger.\end{aligned} \tag{21}$$

These equations can be reexpressed as

$$\begin{aligned}A_i &= -\nabla_i MM^{-1} - Ma_iM^{-1}, \\ \bar{A}_{\bar{i}} &= M^{\dagger-1}\bar{\nabla}_{\bar{i}}M^\dagger + M^{\dagger-1}\bar{a}_{\bar{i}}M^\dagger, \\ a_i &= -g_{i\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}}, \qquad \bar{a}_{\bar{i}} = -g_{\bar{i}i}\mathcal{D}_j\chi^{\dagger ij} = a_i{}^\dagger.\end{aligned} \tag{22}$$

It is easy to see that a_i and $\bar{a}_{\bar{i}}$ obey the following conditions:

$$g^{\bar{k}i}\bar{\mathcal{D}}_{\bar{k}}a_i = -\bar{\mathcal{D}}_{\bar{i}}\bar{\mathcal{D}}_{\bar{j}}\chi^{\bar{i}\bar{j}} = 0, \qquad g^{k\bar{i}}\mathcal{D}_k\bar{a}_{\bar{i}} = 0. \tag{23}$$

The gauge-invariant degrees of freedom are now easily identified as $H = M^\dagger M$ and $\chi = M^{-1}\phi M$, $\chi^\dagger = M^\dagger\phi^\dagger M^{\dagger-1}$. Equivalently, they may be taken as $H = M^\dagger M$ and a_i, $\bar{a}_{\bar{i}}$, where the latter are subject to the conditions (23). Yet another equivalent choice would be $\chi' = M^\dagger\phi M^{\dagger-1}$, $\chi'^\dagger = M^{-1}\phi^\dagger M$ and $H = M^\dagger M$. These fields constitute the coordinates for the space of gauge-invariant configurations, i.e., coordinates for the gauge-orbit space $\mathcal{C}$.

D. Uniqueness of the parametrization of fields

We now comment on the uniqueness of the parametrization of fields we have introduced. It is useful to consider the Abelian case first. The analysis based on group theory shows that the only representations of $SU(3)$ which contain a state transforming as a vector are of the (p, p) type (for which we take a derivative) and of the $(p+3, p)$ or $(p, p+3)$ types. This means that any vector can be parametrized as given in (12).[3]

Conversely, given A_i, we notice that

$$\eta^{i\bar{i}}R_{\bar{i}}A_i = -\eta^{i\bar{i}}R_{\bar{i}}R_i f - \epsilon^{\bar{i}\bar{j}}R_{\bar{i}}R_{\bar{j}}\chi = -\eta^{i\bar{i}}R_{\bar{i}}R_i f, \tag{24}$$

because $[R_{\bar{1}}, R_{\bar{2}}] = 0$. Since $\eta^{i\bar{i}}R_{\bar{i}}R_i$ is invertible (in fact the Green's function for this will be given later in this paper), we can find f in terms of derivatives of A_i. Once we have f, we can rewrite (12) as

$$\epsilon^{ij}A_j = -\epsilon^{ij}R_j f + \eta^{\bar{j}i}R_{\bar{j}}\chi, \tag{25}$$

which leads to

$$\epsilon^{ij}R_iA_j = \eta^{i\bar{i}}R_iR_{\bar{i}}\chi. \tag{26}$$

We can now invert this to obtain χ in terms of $\epsilon^{ij}R_iA_j$. (The Green's function for this case, namely, with $Y = \pm 2$, is given in [11].) Thus given A_i (and its conjugate), we can determine f and χ (and their conjugates). They will, of course, be nonlocal in terms of A's as expected. These arguments show the uniqueness of the parametrization for the Abelian case.

Going to the non-Abelian case, we note that the term $R_i f$ is of the form of an infinitesimal (complex) gauge transformation. Taking $\theta = f$ to be Lie-algebra valued, $-R_i f = -R_i MM^{-1}$ for $M \approx 1 + \theta$. We can "integrate" this to a finite transformation to the form $-R_i MM^{-1}$, $M = e^\theta$. The multiplication of functions on $\mathbb{CP}^2$ with functions is still a function, so there is no difficulty in doing this.

The remaining term in A_i should be Lie-algebra valued and transform homogeneously under gauge transformations, so we are led to the form (17), where we have to (gauge) covariantize the derivative acting on χ, as in (18), or in group theory language, $\mathcal{R}_{\bar{i}}\Phi = R_{\bar{i}}\Phi + [M^{\dagger-1}R_{\bar{i}}M, \Phi]$. In terms of these derivatives, (17) reads

$$\begin{aligned}A_i &= -R_iMM^{-1} - \eta_{i\bar{i}}\mathcal{R}_{\bar{j}}\phi^{\bar{i}\bar{j}}, \\ \bar{A}_{\bar{i}} &= -M^{\dagger-1}R_{\bar{i}}M^\dagger - \eta_{\bar{i}i}\mathcal{R}_j\phi^{\dagger ij}.\end{aligned} \tag{27}$$

The key point is that the covariant derivative $\mathcal{R}_{\bar{i}}$ has no curvature, i.e., $[\mathcal{R}_{\bar{i}}, \mathcal{R}_{\bar{j}}] = 0$. Therefore, we get

$$\eta^{\bar{i}i}\mathcal{R}_{\bar{i}}A_i = -\eta^{\bar{i}i}\mathcal{R}_{\bar{i}}(R_iMM^{-1}). \tag{28}$$

We can solve this iteratively in powers of A by starting with $M \approx 1 + \theta$, since $\eta^{\bar{i}i}R_{\bar{i}}R_i$ is invertible. Again once we have M (and $M^\dagger$ as its conjugate) we can use

$$\epsilon^{ij}R_iA_j + \epsilon^{ij}R_iMM^{-1}R_jMM^{-1} = -\epsilon^{ij}\eta_{j\bar{j}}R_i\mathcal{R}_{\bar{k}}\phi^{\bar{j}\bar{k}}. \tag{29}$$

The leading term on the right hand side is $-\epsilon^{ij}\eta_{j\bar{j}}\epsilon^{\bar{j}\bar{k}}R_iR_{\bar{k}}\chi = \eta^{i\bar{k}}R_iR_{\bar{k}}\chi$ and since $\eta^{i\bar{k}}R_iR_{\bar{k}}$ is invertible, again, we can, at least in principle, calculate χ in terms

[3] Notice that this may also be viewed as a holomorphic version of the Hodge decomposition for one-forms in terms of an exact form, a coexact form and a harmonic form. There is no "harmonic term" for us, since the Betti number b_1 of $\mathbb{CP}^2$ is zero.

of A_i as well. We have thus shown how we can go from A_i, $\bar{A}_{\bar{i}}$ to M, $M^\dagger$, χ, $\chi^\dagger$ and vice versa, showing uniqueness of the parametrization.

There is another feature of the parametrization (17) or (22) which will be important later. Notice that $(M, a_i, M^\dagger, \bar{a}_{\bar{i}})$ and $(M\bar{V}(\bar{x}), \bar{V}^{-1}(\bar{x})a_i\bar{V}(\bar{x}), V(x)M^\dagger, V(x)\bar{a}_{\bar{i}}V^{-1}(x))$ lead to the same gauge potentials, where $V(x)$ is an $SL(N,\mathbb{C})$ matrix with elements which are holomorphic functions, a holomorphic matrix for short, and $\bar{V}(\bar{x})$ is an antiholomorphic matrix. On $\mathbb{CP}^2$, there are no globally defined holomorphic or antiholomorphic functions, except for a constant. Thus globally, we have no such possibility of $M \to M\bar{V}, M^\dagger \to VM^\dagger$ and there are no additional degrees of freedom which could arise from this.

[We may note however that matrices V (or $\bar{V}$) defined as (anti)holomorphic in *local* neighborhoods can be useful to write nonsingular expressions for fields, in the same way that gauge transformations on intersections of coordinate patches can be used as transition functions for gauge fields which are specified patchwise. The use of V (or $\bar{V}$) as transition functions does not introduce additional functional degrees of freedom; they also do not show up in A_i and $\bar{A}_{\bar{i}}$. The metric and the expression for the volume element we calculate are also insensitive to V and $\bar{V}$ since our regularization preserves the correct transformation properties under these (anti)holomorphic transformations. A two-dimensional example of how the local use of V and $\bar{V}$ can be useful is given in [5].]

E. A scalar field on $\mathbb{CP}^2$

We now consider a massless scalar field multiplet Φ on $\mathbb{CP}^2$, with components Φ^α which transform under gauge transformations according to some representation of the gauge group $SU(N)$:

$$\Phi^\alpha \to \Phi'^\alpha = U^\alpha{}_\beta \Phi^\beta, \tag{30}$$

$U^\alpha{}_\beta$ being the representation matrices corresponding to U in the specific representation. The corresponding covariant derivatives are $(\nabla_i + A_i)\Phi$ and $(\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}})\Phi$. Before we write down the action, it is useful to discuss the volume element for $\mathbb{CP}^2$. Local complex coordinates $z^i, \bar{z}^{\bar{i}}$, $i = 1, 2$, can be introduced by taking the 3×3 matrix g to be such that

$$g_{13} = \frac{z_1}{\sqrt{1+\bar{z}\cdot z}}, \quad g_{23} = \frac{z_2}{\sqrt{1+\bar{z}\cdot z}}, \quad g_{33} = \frac{1}{\sqrt{1+\bar{z}\cdot z}}. \tag{31}$$

The metric, which is the restriction to $\mathbb{CP}^2$ of the Cartan-Killing metric on $SU(3)$, is given by the Fubini-Study metric

$$ds^2 = \left[\frac{d\bar{z}\cdot dz}{1+\bar{z}\cdot z} - \frac{z\cdot d\bar{z}\,\bar{z}\cdot dz}{(1+\bar{z}\cdot z)^2}\right] = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}. \tag{32}$$

We will use volume elements normalized so that the total volume is 1. It is then given by

$$d\mu = \frac{2}{\pi^2}\frac{d^4x}{(1+\bar{z}\cdot z)^3} = \frac{2}{\pi^2}(\det g) d^4x, \tag{33}$$

where $z^1 = x^1 - ix^2$ and $z^2 = x^3 - ix^4$. The use of this volume element is equivalent to using the Haar measure on $SU(3)$, again normalized to unity. We will consider a massless scalar with an action of the form

$$\begin{aligned} S_1 &= \int d\mu g^{\bar{i}i}[(\nabla_i + A_i)\Phi]^\dagger[(\nabla_i + A_i)\Phi] \\ &= \int d\mu g^{\bar{i}i}[\bar{\nabla}_{\bar{i}}\Phi^\dagger - \Phi^\dagger \bar{A}_{\bar{i}}][(\nabla_i + A_i)\Phi]. \end{aligned} \tag{34}$$

Notice that, upon carrying out an integration by parts, the action can be written as

$$S_1 = \int d\mu g^{\bar{i}i}\Phi^\dagger[-(\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}})(\nabla_i + A_i)]\Phi. \tag{35}$$

The relevant kinetic energy operator is thus $-g^{\bar{i}i}(\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}})(\nabla_i + A_i)$. One can also consider an action of the form

$$S_2 = \int d\mu g^{i\bar{i}}\Phi^\dagger[-(\nabla_i + A_i)(\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}})]\Phi. \tag{36}$$

This differs from the previous one by a term of the form $\Phi^\dagger g^{i\bar{i}} F_{i\bar{i}}\Phi$, where F is the field strength for the gauge field. Notice that either action is completely consistent with the isometries of the space $\mathbb{CP}^2$, so there is no *a priori* reason to favor one or the other, or any linear combination of the two. These actions essentially correspond to chiral scalars. One can also consider a nonchiral action of the form

$$\begin{aligned} S_3 = \frac{1}{2}\int d\mu g^{i\bar{i}}\Phi^\dagger[-((\nabla_i + A_i)(\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}}) \\ + (\bar{\nabla}_{\bar{i}} + \bar{A}_{\bar{i}})(\nabla_i + A_i))]\Phi. \end{aligned} \tag{37}$$

In this case the kinetic operator is the Laplace operator (suitably gauge covariantized) on the manifold $\mathbb{CP}^2$. S_3 is obviously $\frac{1}{2}(S_1 + S_2)$.

To briefly summarize, this section introduced general parametrizations for fields on $\mathbb{CP}^2$. The result for scalar fields is given in Eq. (9) and for non-Abelian gauge fields in Eqs. (17) or (22). The scalar field actions are given in (35)–(37). Our aim now is to calculate some of the quantum corrections due to the scalar field action, say, S_1, and interpret the physical implications of the results. This is discussed in the next section.

III. QUANTUM CORRECTIONS DUE TO THE SCALAR FIELD

Turning to the quantum corrections, we first note that the action S_1 can be written as

$$\begin{aligned} S_1 &= \int d\mu g^{\bar{i}i}[\Phi^\dagger(-\bar{D}\cdot D)\Phi - \Phi^\dagger M^{\dagger -1}\bar{a}M^\dagger \cdot D\Phi \\ &\quad + \Phi^\dagger \bar{D}\cdot MaM^{-1}\Phi + \Phi^\dagger M^{\dagger -1}\bar{a}M^\dagger MaM^{-1}\Phi] \\ &= \int d\mu g^{\bar{i}i}\Phi^\dagger(-\bar{D}\cdot D)\Phi + S_{\rm int}, \\ S_{\rm int} &= \int d\mu g^{\bar{i}i}[-\Phi^\dagger M^{\dagger -1}\bar{a}M^\dagger\cdot D\Phi + \Phi^\dagger MaM^{-1}\cdot\bar{D}\Phi \\ &\quad + \Phi^\dagger M^{\dagger -1}\bar{a}M^\dagger MaM^{-1}\Phi], \end{aligned} \tag{38}$$

where in getting $S_{\rm int}$ we used the transformation property of a in (23), $\bar{D}\cdot(MaM^{-1}) = M(\bar{\mathcal{D}}\cdot a)M^{-1} = 0$.

Denoting the [$(M, M^\dagger)$-dependent] propagator as

$$\langle\Phi(x)\Phi^\dagger(y)\rangle = \left(\frac{1}{(-\bar{D}\cdot D)}\right)_{x,y} = \mathcal{G}(x,y), \tag{39}$$

the effective action resulting from integrating out the scalar fields is $\Gamma = \Gamma_1 + \Gamma_2$, given by

$$\begin{aligned} e^{-\Gamma} &= \int [d\Phi d\Phi^\dagger] e^{-S_1}, \\ \Gamma_1 &= \mathrm{Tr}\,\log(-\bar{D}\cdot D), \\ \Gamma_2 &= \mathrm{Tr}\,\log[1 + M^{\dagger -1}\bar{a}M^\dagger\cdot(-D\mathcal{G}(x,y)) \\ &\quad + MaM^{-1}\cdot\bar{D}\mathcal{G}(x,y) \\ &\quad + M^{\dagger -1}\bar{a}M^\dagger MaM^{-1}\mathcal{G}(x,y)]. \end{aligned} \tag{40}$$

The first term Γ_1 will generate terms which only depend on $M, M^\dagger$. The second term can be expanded in powers of $(MaM^{-1}, M^{\dagger -1}\bar{a}M^\dagger)$; it will correspond to one-loop diagrams with $(MaM^{-1}, M^{\dagger -1}\bar{a}M^\dagger)$ at the vertices and with $\mathcal{G}$ as the propagator. Even though we have a massless field, there will be no infrared divergences in the calculation of Γ since $\mathbb{CP}^2$ is a compact manifold of finite volume. If we rescale the coordinates $z_i, \bar{z}_i \to z_i/r, \bar{z}_i/r$ to introduce appropriately dimensionful coordinates, $1/r$ will serve as the infrared cutoff. In the diagrammatic expansion of Γ, the first few terms will, however, be potentially ultraviolet divergent. These will include terms in Γ_1 and the expansion of Γ_2 to the quartic order in $(MaM^{-1}, M^{\dagger -1}\bar{a}M^\dagger)$. Our aim is to focus on these, evaluating them in a way fully consistent with gauge invariance and all the isometries of $\mathbb{CP}^2$. We will see that the first term Γ_1 generates a WZW action for $H = M^\dagger M$, with a finite coefficient with potential UV divergences canceling out. Such a term can have significant implications for physics since its critical points are instantons. There will also be other terms in Γ_1 which combine with terms from Γ_2. The leading terms of Γ_2 will be quadratic and logarithmic UV divergences.

While it is straightforward to use the standard diagrammatic expansion for Γ_2, for the evaluation of Γ_1 it is easier to consider its variation in $M, M^\dagger$. We find

$$\delta\Gamma_1 = \int \mathrm{Tr}[\delta(M^{\dagger -1}\bar{\nabla}M^\dagger)\langle\hat{J}\rangle + \delta(\nabla M M^{-1})\langle\hat{J}^\dagger\rangle], \tag{41}$$

$$\begin{aligned} \langle\hat{J}(x)\rangle &= -\langle D\Phi(x)\Phi^\dagger(y)\rangle_{y\to x} = -D_x\mathcal{G}(x,y)]_{y\to x}, \\ \langle\hat{J}^\dagger(x)\rangle &= -\langle\Phi(x)(D\Phi)^\dagger(y)\rangle_{y\to x} \\ &= (-\bar{\nabla}_y\mathcal{G}(x,y) + \mathcal{G}(x,y)M^{\dagger -1}\bar{\nabla}_x M^\dagger)]_{y\to x}. \end{aligned} \tag{42}$$

The limit $y \to x$ has to be taken with the properly regularized version of the propagator $\mathcal{G}(x,y)$. The problem is thus reduced to the evaluation of the expectation values of the currents as shown in (42). The currents are functions of $M, M^\dagger$ and obey a very useful condition related to the complex version of gauge transformations. The covariant derivatives for $(hM, M^\dagger h^{-1})$, where h is a complex matrix, are given by

$$D|_{hM} = hDh^{-1}, \qquad \bar{D}|_{M^\dagger h^{-1}} = h\bar{D}h^{-1}. \tag{43}$$

As a result, we find

$$\langle\hat{J}(hM, M^\dagger h^{-1})\rangle = h\langle\hat{J}(M, M^\dagger)\rangle h^{-1}. \tag{44}$$

This property will be very useful for the evaluation of Γ_1. A relation similar to (44) is what is used for the analogous calculation in two dimensions. We will be using a regularized version of this equation as discussed in Sec. III B.

A. The propagator for the scalar field

The free scalar field Φ has the mode expansion

$$\Phi = \sum_{p,A} C_A^{(p,p)}\sqrt{(p+1)^3}\,D_{A,0}^{(p,p)}(g). \tag{45}$$

Here $\sqrt{(p+1)^3}D_{A,0}^{(p,p)}(g)$ are the normalized eigenfunctions of $\eta^{\bar{i}i}R_{\bar{i}}R_i = -g^{\bar{i}i}\bar{\nabla}_{\bar{i}}\nabla_i$. (Our approach here will be somewhat similar to what was done for the case of $\mathbb{CP}^1 = S^2$ in [19].) Notice that, since $|0\rangle$ is invariant under $U(2)$ transformations,

$$\begin{aligned} \eta^{\bar{i}i}R_{\bar{i}}R_i D_{A,0}^{(p,p)}(g) &= \langle s, A\hat{g}\eta^{\bar{i}i}T_{\bar{i}}T_i|0\rangle \\ &= \langle s, A|\hat{g}(\eta^{\bar{i}i}T_{\bar{i}}T_i + T_1^2 + T_2^2 + T_3^2 + T_8^2)|0\rangle \\ &= \langle s, A|\hat{g}T_aT_a|0\rangle \\ &= p(p+2)D_{A,0}^{(p,p)}(g), \end{aligned} \tag{46}$$

where we have used the fact that the quadratic Casimir operator $T_a T_a$ has the eigenvalue $p(p+2)$ for the (p,p) representations. The propagator is thus given by

$$\begin{aligned} G(g,g') &= \sum_{p,A} \frac{(p+1)^3}{p(p+2)} D^{(p,p)}_{A,0}(g) D^{*(p,p)}_{A,0}(g') \\ &= \sum_{p=1} \frac{(p+1)^3}{p(p+2)} D^{(p,p)}_{0,0}(g'^\dagger g) \\ &= \sum_{p=1} \frac{(p+1)^3}{p(p+2)} \langle 0|g'^\dagger g|0\rangle, \end{aligned} \tag{47}$$

where we have used the group property to combine the two eigenfunctions to obtain $D^{(p,p)}_{0,0}(g'^\dagger g)$. The summation starts at $p=1$ because the eigenfunction for $p=0$ is a zero mode for $R_{\bar{i}}R_i$ and must be excluded from the sum. The propagator thus obeys the equation

$$\begin{aligned} \eta^{\bar{i}i} R_{\bar{i}} R_i G(g,g') &= \sum_{p=1} (p+1)^3 \langle 0|g'^\dagger g|0\rangle \\ &= \delta(g,g') - 1, \end{aligned} \tag{48}$$

where $\delta(g,g')$ is the Dirac delta function on $\mathbb{CP}^2$, normalized with the volume element (33). Explicitly, $\delta(g,g') - 1 = \sum_{p=1}^{\infty} (p+1)^3 D^{(p,p)}_{0,0}(g'^\dagger g)$. Notice that a (left) translation of g, g' by the same $SU(3)$ transformation h, i.e., $g \to hg$, $g' \to hg'$, leaves the propagator invariant. This is the expression of the translational invariance of $G(g,g')$.

With the parametrization of g as in (31), $D^{(p,p)}_{0,0}(g'^\dagger g)$ is a polynomial of degree p in $\xi = (g'^\dagger g)_{33} (g'^\dagger g)^{*33} = (1+s)^{-1}$, where

$$\begin{aligned} s = \sigma^2_{z,y} &= \frac{1}{(g'^\dagger_y g_z)_{33} (g^\dagger_z g'_y)_{33}} - 1 = \frac{(1+\bar{z}\cdot z)(1+\bar{y}\cdot y)}{(1+\bar{z}\cdot y)(1+\bar{y}\cdot z)} - 1 \\ &= \frac{(\bar{z}-\bar{y})\cdot(z-y) + \bar{z}\cdot z\bar{y}\cdot y - \bar{z}\cdot y\bar{y}\cdot z}{(1+\bar{z}\cdot y)(1+\bar{y}\cdot z)}. \end{aligned} \tag{49}$$

By construction this is invariant under translations on $\mathbb{CP}^2$ since it only involves $(g'^\dagger g)_{33}$ and its conjugate. Further it is symmetric between the two points and vanishes when $g' = g$ and also reduces to $\bar{z}\cdot z$ when $y=0$, i.e., for $g'=1$. Therefore it is the appropriate generalization of $\bar{z}\cdot z$ to the square of the distance between two points.

It is possible to write down an expression for $D^{(p,p)}_{0,0}(g)$, but we do not display this expression here since it is still difficult to evaluate the sum in (47) in a closed form, except for certain special values of s. Instead, we will obtain the propagator by solving the differential equation (48). This can be done by writing the operator $R_i R_{\bar{i}}$ in terms of s. $\mathbb{CP}^2$ is a Kähler manifold with the Kähler potential

$$K = \log(1 + \bar{z}\cdot z), \tag{50}$$

so that the metric tensor can be written as $g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K$. The metric so obtained is the Fubini-Study metric given in (32). Explicitly, this metric tensor and its inverse are, respectively,

$$\begin{aligned} g_{a\bar{a}} &= \left[\frac{\eta_{a\bar{a}}}{(1+\bar{z}\cdot z)} - \eta_{a\bar{m}}\eta_{\bar{a}m} \frac{\bar{z}^{\bar{m}} z^m}{(1+\bar{z}\cdot z)^2}\right], \\ g^{a\bar{a}} &= (1+\bar{z}\cdot z)(\eta^{a\bar{a}} + z^a \bar{z}^{\bar{a}}). \end{aligned} \tag{51}$$

Because of the Kähler property, we also have $\partial_a (g^{a\bar{a}} \det g) = 0 = \partial_{\bar{a}} (g^{\bar{a}a} \det g)$, so that the operator of interest for us is given as

$$\begin{aligned} \eta^{\bar{i}i} R_{\bar{i},z} R_{i,z} G(z,y) &= -\frac{1}{\det g} \partial_{\bar{a}} (g^{\bar{a}a} \det g \partial_a G) \\ &= -g^{\bar{a}a} \bar{\partial}_{\bar{a}} \partial_a G \\ &= -(1+z\cdot\bar{z})(\bar{\partial}\cdot\partial + \bar{z}\cdot\bar{\partial} z\cdot\partial) G(s(z,y)) \\ &= -[s(1+s)^2 G'' + (2+s)(1+s) G'], \end{aligned} \tag{52}$$

where, in the fourth line, we have expressed the operator as it acts on a function of $s = \sigma^2_{z,y}$ as in (49), and the prime denotes the derivative with respect to s. We now consider points with nonzero separations $\sigma^2_{z,y}$, hence nonzero s, so that the δ function in (48) has no support. Equation (48) then becomes

$$s(1+s)^2 W' + (2+s)(1+s) W = 1, \qquad W = G'. \tag{53}$$

This is a first-order inhomogeneous differential equation for W and can be solved using an integrating factor. A further integration then leads to the following expression for G:

$$G = -\left(C_1 - \frac{1}{2}\right)\frac{1}{s} + C_1 \log s - \frac{1}{2}\log\left(\frac{s}{1+s}\right) + C_0, \tag{54}$$

where C_0 and C_1 are arbitrary constants. For short separations, $s \ll 1$, the first term on the right dominates. In this limit, we see from (52) that $R_{\bar{i}}R_i$ is well approximated by $-\bar{\partial}\cdot\partial$. To get the δ function upon the action of this operator, we need

$$G \approx \frac{1}{2|z-y|^2}. \tag{55}$$

[Recall that, for $\mathbb{R}^4$, $-\nabla^2$ acting on $1/(4\pi^2(x-x')^2)$ leads to the δ function. Here we have $-\bar{\partial}\cdot\partial = -\nabla^2/4$, so this removes a factor of 4. Further, we need a factor $\pi^2/2$, since we are using the δ function which integrates to 1, with the volume normalized to 1 rather than $\pi^2/2$. This leads to the result (55).] In this limit the separation s approaches that of flat space, namely, $|z-y|^2$. We conclude from (55) that $C_1 = 0$.

To determine C_0, we notice that, since the constant mode (or the zero mode) has been subtracted out in (48), the propagator is orthogonal to the zero mode. Thus we have the condition

$$\int d\mu G(g) = 0. \tag{56}$$

Carrying out the integration, we find $C_0 = -\frac{3}{4}$. The propagator for the massless scalar field on $\mathbb{CP}^2$ is thus obtained as

$$G(z,y) = \frac{1}{2s} - \frac{1}{2}\log\left(\frac{s}{1+s}\right) - \frac{3}{4}, \qquad s = \sigma^2_{z,y}. \tag{57}$$

More details on these calculations are given in Appendix A, where the propagator for a massless scalar on $\mathbb{CP}^k$, for any k, has been derived.

It is also useful to write this in terms of homogeneous coordinates for $\mathbb{CP}^2$. Let $Z = (Z^1, Z^2, Z^3)$ and $Y = (Y^1, Y^2, Y^3)$ be the homogeneous coordinates corresponding to the points we labeled by z^i and y^i. Then

$$s = \sigma^2_{z,y} = \frac{\bar{Z}\cdot Z \bar{Y}\cdot Y}{\bar{Z}\cdot Y \bar{Y}\cdot Z} - 1 \equiv \sigma^2(Z,Y). \tag{58}$$

Notice that this is invariant under the scaling $Z \to \lambda Z$, $Y \to \lambda' Y$, $\lambda, \lambda' \in \mathbb{C} - \{0\}$, so that it is defined on the projective space rather than $\mathbb{C}^3$. Scaling out Z^3 and Y^3, in a particular coordinate patch with $Z^3, Y^3 \neq 0$, we can write

$$\begin{aligned} Z &= Z^3(z^1, z^2, 1) = Z^3\sqrt{1+\bar{z}\cdot z}(g_{13}, g_{23}, g_{33}), \\ Y &= Y^3(y^1, y^2, 1) = Y^3\sqrt{1+\bar{y}\cdot y}(g'_{13}, g'_{23}, g'_{33}), \end{aligned} \tag{59}$$

where $z^i = Z^i/Z^3$ and $y^i = Y^i/Y^3$. We then see that s in (58) reduces to s in (49). Thus $G(s)$ in (57) with s given in homogeneous coordinates as in (58) and (59) gives a globally valid expression for the propagator for the scalar field.

B. Regularizations

With $\sigma^2_{z,y}$ as defined in (58), we have a globally valid expression for the propagator on $\mathbb{CP}^2$. We will now use this to define an ultraviolet regularization via point splitting, which is fully covariant, i.e., gauge covariant and consistent with all the isometries of $\mathbb{CP}^2$. Toward this, consider moving from y to a nearby point with coordinates y', which we express in terms of the homogeneous coordinates as

$$Y \to Y' = Y + \alpha\left(\frac{W\bar{Y}\cdot Y}{\bar{Y}\cdot W} - Y\right), \tag{60}$$

where α is a small complex number and W parametrizes the shift of coordinates. Notice that the added term has the same scaling behavior as Y. We then find

$$1+\sigma^2(Z,Y') = \frac{(1+\sigma^2(Z,Y))(1+\alpha\bar{\alpha}\sigma^2(Y,W))}{[1+\alpha(\frac{\bar{Z}\cdot W\bar{Y}\cdot Y}{\bar{Y}\cdot W\bar{Z}\cdot Y}-1)][1+\bar{\alpha}(\frac{\bar{W}\cdot Z\bar{Y}\cdot Y}{\bar{W}\cdot Y\bar{Y}\cdot Z}-1)]}. \tag{61}$$

The strategy is to use $G(\sigma^2(Z,Y'))$ as the regularized version of $G(\sigma^2(Z,Y))$, where we take Y' to be different from Y by a small amount proportional to $|\alpha| \sim \sqrt{\epsilon}$. ϵ will serve as the regularization parameter. Thus

$$G_{\text{Reg}}(Z,Y) = G(\sigma^2(Z,Y')), \tag{62}$$

where we will include an angular averaging over the displacement due to point splitting.

We will be calculating the effective action in a derivative expansion, so for most of the terms, it will turn out that we can take one of the points z, y to be at the origin, by virtue of translational invariance. A transformation which implements this will be given in Appendix B. But for now, if we take $y = 0$, i.e., $Y = (0,0,1)$, we find

$$\begin{aligned} 1+\sigma^2(Z,Y') &= \frac{(1+\bar{z}\cdot z)(1+\alpha\bar{\alpha}\bar{w}\cdot w)}{(1+\alpha\bar{z}\cdot w)(1+\bar{\alpha}\bar{w}\cdot z)} \\ &= \frac{\bar{Z}\cdot Z\bar{\tilde{W}}\cdot\tilde{W}}{\bar{Z}\cdot\tilde{W}\,\bar{\tilde{W}}\cdot Z}, \end{aligned} \tag{63}$$

where $\tilde{W} = (\alpha W_1, \alpha W_2, W_3) = W_3(\alpha w_1, \alpha w_2, 1)$. Further it is useful to make a change of variables from Z to Z' such that

$$\frac{Z'}{Z'_3\bar{\tilde{W}}_3} = \frac{Z}{\bar{\tilde{W}}\cdot Z}. \tag{64}$$

Notice that this transformation is covariant under independent scalings of Z, $\tilde{W}$, and Z'. Equation (64) is equivalent to

$$Z' = \lambda Z, \qquad \lambda = \frac{Z'_3\bar{\tilde{W}}_3}{\bar{\tilde{W}}\cdot Z}. \tag{65}$$

The key point is that, because of the homogeneity property, the Fubini-Study metric is unchanged under the change of variables in (64) or (65):

$$ds^2(Z',\bar{Z}') = ds^2(Z,\bar{Z}). \tag{66}$$

Correspondingly, the inverse metric, the volume measure, etc., can be taken to be defined by Z'. Equation (63) can then be written as

$$\begin{aligned} 1+\sigma^2(Z,Y') &= (1+\bar{z}'\cdot z')(1+\bar{\tilde{w}}\cdot\tilde{w}) \\ &\equiv 1+s(1+\epsilon)+\epsilon, \end{aligned} \tag{67}$$

where $s = \bar{z}' \cdot z'$ and $\bar{\tilde{w}} \cdot \tilde{w} = \epsilon$. Thus, in terms of the coordinates defined by Z', the effect of regularization is to replace $G(s)$ by $G(s(1+\epsilon)+\epsilon)$:

$$G_{\rm Reg}(s) = \frac{1}{2(s(1+\epsilon)+\epsilon)} - \frac{1}{2}\log\left(\frac{s(1+\epsilon)+\epsilon}{1+s(1+\epsilon)+\epsilon}\right) - \frac{3}{4}. \tag{68}$$

(The proper dimensions for ϵ and s can be restored by the scaling $z^i \to z^i/r$ and $\tilde{w}^i \to \tilde{w}^i/r$.)

This procedure provides a regularization which is covariant respecting the isometries of $\mathbb{CP}^2$, since s in (58) is an invariant quantity. Equation (68) may be viewed as a covariant point splitting and provides a uniform way to carry out calculations.

We now turn to the issue of gauge invariance. So far we have discussed the free propagator. In the presence of gauge fields, the propagator is $\mathcal{G}(x,y) = (-\bar{D}\cdot D)^{-1}_{x,y}$. For most of the calculations we do, this will be expanded in powers of the gauge field as

$$\mathcal{G}(x,y) = G(x,y) + \int_{y_1} G(x,y_1)\mathbb{V}_{y_1}G(y_1,y) + \int_{y_1,y_2} G(x,y_1)\mathbb{V}_{y_1}G(y_1,y_2)\mathbb{V}_{y_2}G(y_2,y) + \cdots, \tag{69}$$

where $\mathbb{V} = \bar{A}\cdot\partial + A\cdot\bar{\partial} + (\bar{\partial}\cdot A) + \bar{A}\cdot A$. In calculating currents such as $\langle \hat{J}(x)\rangle = -D_x\mathcal{G}(x,y)]_{y\to x}$ in (42), we must ensure that the point splitting is gauge covariant as well. The point splitting amounts to writing $G_{\rm Reg}(x,y) = G(x,y')$. Since $\mathcal{G}_{\rm Reg}(x,y)$ must transform as $\mathcal{G}_{\rm Reg}(x,y) \to U(x)\mathcal{G}_{\rm Reg}(x,y)U^\dagger(y)$ under the gauge transformation $M \to UM, M^\dagger \to M^\dagger U^\dagger$, we see that a gauge-invariant point splitting is given by

$$\begin{aligned}\mathcal{G}_{\rm Reg}(x,y) &= \mathcal{G}(x,y')\mathcal{P}\exp\left(-\int_y^{y'}(M^{\dagger-1}\bar{\nabla}M^\dagger - \nabla M M^{-1})\right)\\ &= \left[G_{\rm Reg}(x,y) + \int_{y_1} G(x,y_1)\mathbb{V}(y_1)G_{\rm Reg}(y_1,y) + \cdots\right]\\ &\quad\times \mathcal{P}\exp\left(-\int_y^{y+\delta y}(M^{\dagger-1}\bar{\nabla}M^\dagger - \nabla M M^{-1})\right).\end{aligned} \tag{70}$$

Here $G_{\rm Reg}$ is as in (62) and $y' = y+\delta y$, with $\delta y^a \delta\bar{y}^{\bar{a}} \to \epsilon\eta^{a\bar{a}}$ in taking the small ϵ limit in a symmetric way. Notice that, because the path-ordered exponential involves the integral of one-forms, we can use local coordinates y, y' in (70).

In principle, we can now calculate Γ according to (40)–(42), using the expression given above. But before doing that, we discuss some issues regarding the infrared side of calculations with (62). As mentioned earlier, on $\mathbb{CP}^2$, we do not expect infrared divergences. Nevertheless, there is a subtlety we need to address. Here we will consider only the first few terms in the expansion of Γ, which are potentially ultraviolet divergent, to understand the nature of counterterms which might be needed. The key point is that we cannot carry out an exact calculation of all of the one-loop contributions. We can evaluate the first few terms in a diagrammatic expansion (and the WZW term which is rather special). So we need some control over the diagrams with higher numbers of vertices. Thus we need to develop an expansion scheme where the diagrams with more and more vertices are parametrically smaller. To see how this can be implemented, a comparison with flat space is useful. Basically, we are saying that the diagrammatic expansion of Γ will contain two types of terms. The first few diagrams, which are potentially ultraviolet divergent, do not have infrared divergences even in flat space. If we evaluate them in flat space with an infrared cutoff, they will be insensitive to this or at worst have a marginal (logarithmic) dependence. The remaining terms in Γ, corresponding to higher numbers of vertices, will be infrared divergent in flat space. Such contributions, if we evaluate them with an infrared cutoff λ, will be proportional to inverse powers of λ. We can use an analogous procedure for $\mathbb{CP}^2$, evaluating the corresponding diagrams with an infrared cutoff λ. Since at short distances the propagator on $\mathbb{CP}^2$ approaches the flat space version, these will carry inverse powers of λ as well. The relevant parameter will then be λr^2, where r is the $\mathbb{CP}^2$ radius, and for $\lambda r^2 \gg 1$, these terms are parametrically small. As we shall see in the next section the dominant term for $\lambda r^2 \gg 1$ is a WZW term, which is also UV finite. The other dominant contributions are from the potentially ultraviolet-divergent terms. The calculation of the effective action along these lines is very much in the spirit of Wilsonian renormalization.

The infrared cutoff can be included by using a simple integral representation for the propagator. We write

$$G_{\rm Reg}(x,y) = \frac{1}{r^2}\int_{\lambda r^2}^\infty dt\left[e^{-t\sigma^2_{x,y'}/r^2}\left(\frac{1}{2}+\frac{1}{2t}(1-e^{-t})\right) - \frac{3}{4}e^{-t}\right]. \tag{71}$$

We have introduced r^2 via the scaling of coordinates. The infrared cutoff λ appears as the lower limit of the integration over t. When λ is set to zero, we clearly reproduce (62). This result, combined with (69), can be used for calculating the effective action.

C. The WZW action

We now discuss the explicit calculations, starting with the evaluation of Γ_1. This requires, according to (41) and (42), the current $\langle \hat{J}(x)\rangle$. Notice that according to (44) we can evaluate this by choosing $h = M^\dagger$, so that

$$\langle \hat{J}(M, M^\dagger)\rangle = M^{\dagger -1}\langle \hat{J}(H,1)\rangle M^\dagger. \quad (72)$$

For $\langle \hat{J}(H,1)\rangle$, the relevant propagator (obtained by $M \to H, M^\dagger \to 1$) is

$$\mathcal{G}_{\rm Reg}(x,y) = \langle x| \frac{1}{(-\bar{\nabla}\cdot\mathcal{D})}|y'\rangle \mathcal{P}\exp\left(\int_y^{y'} \nabla H H^{-1}\right). \quad (73)$$

The current is then $\langle \hat{J}(H,1)\rangle = -\mathcal{D}_x \mathcal{G}_{\rm Reg}(x,y)$ with $y \to x$. We can expand the expression (73) in powers of $\nabla H H^{-1}$. This leads to the result

$$\langle \hat{J}_a(H,1)\rangle = -\frac{\pi}{2} C \nabla_a H H^{-1} + \cdots, \quad (74)$$

$$C = \frac{1}{\pi r^2}\left[1 - \log 2 + \frac{3}{2}e^{-\lambda r^2} + \frac{\lambda r^2}{4}\right] + \frac{1}{\pi r^2}\left[(E_1(\lambda r^2) - E_1(2\lambda r^2)) - \frac{1}{2}e^{-\lambda r^2}(1 - e^{-\lambda r^2})\right] + \frac{1}{\pi r^2}\left[\frac{(1-e^{-\lambda r^2})^2}{4\lambda r^2} + \lambda r^2 (e^{\lambda r^2} - 1)E_1(2\lambda r^2)\right]. \quad (75)$$

Here E_1 denotes the exponential integral

$$E_1(w) = \int_1^\infty \frac{dt}{t} e^{-wt} = e^{-w}\int_0^\infty dt \frac{e^{-t}}{w+t}. \quad (76)$$

(The details of this calculation are given in Appendix B. There are additional terms which involve more powers of gradients of H as indicated by the ellipsis. Some of these terms will contribute to the $\log \epsilon$ terms; see below.) Going back to (41), we can now write the variation of Γ_1 (with respect to $M^\dagger$) as

$$\begin{aligned}\delta\Gamma_1 &= \int g^{\bar{a}a}{\rm Tr}\left[\delta(M^{\dagger -1}\bar{\nabla}_{\bar{a}}M^\dagger)M^{\dagger -1}\left(-\frac{\pi}{2}C\nabla_a H H^{-1}\right)M^\dagger\right] + \cdots \\ &= -\frac{\pi}{2}C\int g^{\bar{a}a}{\rm Tr}[\bar{\nabla}_{\bar{a}}(\delta M^\dagger M^{\dagger -1})\nabla_a H H^{-1}] + \cdots. \quad (77)\end{aligned}$$

We can now identify the part of Γ_1 corresponding to (77). The four-dimensional WZW action is given by [17,18]

$$\begin{aligned}S_{\rm wzw}(H) &= \frac{1}{2\pi}\int \frac{\pi^2}{2} d\mu g^{a\bar{a}}{\rm Tr}(\nabla_a H \bar{\nabla}_{\bar{a}} H^{-1}) - \frac{i}{24\pi}\int \omega \wedge {\rm Tr}(H^{-1}dH)^3 \\ &= \frac{\pi}{4}\int d\mu g^{a\bar{a}}{\rm Tr}(\nabla_a H\bar{\nabla}_{\bar{a}}H^{-1}) - \frac{i}{24\pi}\int \omega\wedge {\rm Tr}(H^{-1}dH)^3, \quad (78)\end{aligned}$$

where ω is the Kähler two-form on $\mathbb{CP}^2$ given in local coordinates as

$$\omega = i g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}} \quad (79)$$

with $g_{a\bar{a}}$ given by the Fubini-Study metric (32). The last term in (78) is, as usual, over a five-manifold which has $\mathbb{CP}^2$ as the boundary. [The extra factor of $\pi^2/2$ in (78) compared to the standard normalizations used for this action is due to the fact that we normalized the volume to 1. Also, we use a slightly different convention for the normalization of ω, compared to [10].] It is easily verified by direct computation that $S_{\rm wzw}$ obeys the 4d version of the Polyakov-Wiegmann identity [20], namely,

$$S_{\rm wzw}(NH) = S_{\rm wzw}(N) + S_{\rm wzw}(H) - \frac{\pi}{2}\int g^{\bar{a}a}{\rm Tr}[(N^{-1}\bar{\nabla}_{\bar{a}}N)\nabla_a H H^{-1}]. \quad (80)$$

Introducing a left variation of $M^\dagger$ by $N \approx 1 + \delta M^\dagger M^{\dagger -1}$, we find

$$\delta S_{\rm wzw}(H) = -\frac{\pi}{2}\int d\mu g^{a\bar{a}}{\rm Tr}[\bar{\nabla}_{\bar{a}}(\delta M^\dagger M^{\dagger -1})\nabla_a H H^{-1}]. \quad (81)$$

Comparing with (77), we see that we can write

$$\Gamma_1 = C S_{\rm wzw}(H) + \cdots. \quad (82)$$

The coefficient C is as given in (75) and is finite. It is useful to simplify it for limiting values of λr^2. For small λr^2, we can use the expansion $E_1(w) \approx -\gamma - \log w + \cdots$ to obtain

$$C \approx \frac{1}{\pi r^2}\left[\frac{5}{2} - \frac{5\lambda r^2}{2}\right], \qquad \lambda r^2 \ll 1. \quad (83)$$

This shows that, as $\lambda \to 0$, we still get a finite result with no infrared divergence, consistent with the expectations for a compact space of finite volume. For $\lambda r^2 \gg 1$, which is the case of interest to us in view of the discussion at the end of Sec. III B,

$$C \approx \frac{1}{\pi r^2}\left[1 - \log 2 + \frac{\lambda r^2}{4}\right], \qquad \lambda r^2 \gg 1. \quad (84)$$

A number of remarks are in order at this point. First of all, we have only evaluated $\langle \hat{J}(M, M^\dagger)\rangle$. The term in $\delta\Gamma_1$, Eq. (41), where we vary M can be obtained via Hermitian conjugation of the term resulting from $\langle \hat{J}(M, M^\dagger)\rangle$. We can then verify, via the identity (80), that the WZW term of Γ_1 is consistent with the variation of M as well.

A second point is the following. The result (82) was obtained by choosing $h = M^\dagger$ and using (72) for the current. We can then ask the question whether we obtain the same result if we use the identity (44) with $h = M^{-1}$, thus setting $M \to 1$, $M^\dagger \to H$. In this case $D \to \nabla$, and the relevant propagator is $(-\bar{\mathcal{D}} \cdot \nabla)^{-1}$. In this case, it is not possible to obtain the result (74) in any expansion of $(-\bar{\mathcal{D}} \cdot \nabla)^{-1}$ in powers of $H^{-1}\bar{\nabla} H$ to any finite order. A resummation of an infinite series of terms is necessary. With the resummation, we do get the same result. The situation is similar to what happens in two dimensions. (A more detailed explanation is given in Appendix B.)

Finally, we note that the leading term of $S_{\rm wzw}$ is negative definite. Thus, in $e^{-\Gamma}$, which is to be used for the subsequent integration over the gauge fields, it has the "wrong" sign, leading to divergent functional integrals. What this means of course is that higher terms in gradient of H are not negligible in regimes where integration over H starts diverging (which can happen when the gradients of H become large). Also, there is a similar WZW term which arises in the calculation of the functional measure for the gauge fields, which is analyzed in the follow-up paper [11]. It turns out that the coefficient of the combined WZW terms has the appropriate sign to ensure convergence, at least for some number of chiral scalar fields of low-dimensional representations.

D. The mass term

We now turn to terms in Γ_2, Eq. (40). First, we notice that similarly as for $\langle \hat{J}(x) \rangle$ we can factorize out $M^\dagger$ and $M^{\dagger -1}$ in the trace and send $M \to H$, $M^\dagger \to 1$. This gives us

$$\Gamma_2 = \mathrm{Tr}\,\log\left[1 + (-\bar{a} \cdot \mathcal{D} + HaH^{-1} \cdot \bar{\nabla} + \bar{a} HaH^{-1})\mathcal{G}(x,y)\right], \tag{85}$$

where $\mathcal{G} = 1/(-\bar{\nabla} \cdot \mathcal{D})$. The UV-divergent terms can then be calculated by first expanding Γ_2 and then further using the expansion of $\mathcal{G}$ in terms of $\nabla H H^{-1}$. The first set of terms will have one power of $\bar{a}$ or HaH^{-1}. Notice that the coefficient of $\bar{a}$ in (85) is $(-\mathcal{D}\mathcal{G})$ which is the current we have already discussed. We may therefore expect a *finite* term of the form $\mathrm{Tr}[\bar{a} \nabla H H^{-1}]$. Unlike the case for $S_{\rm wzw}(H)$, this term is not invariant under $M^\dagger \to V M^\dagger$, so it is sensitive to the ambiguity of how $M^\dagger$ is defined. Recall that for the contribution to Γ_1 the terms with higher powers of $\nabla H H^{-1}$, or higher number of derivatives, do not contribute to the leading term with the minimal number of derivatives, i.e., $S_{\rm wzw}(H)$. However, for $\bar{a}$, the situation is less clear, since we have a tensor $\chi^{\dagger ij}$. The commutator of derivatives on this gives a term with no derivatives, albeit at the cost of a power of $1/r^2$ due to the curvature. Presumably some combination of such terms will combine with $\mathrm{Tr}[\bar{a} \nabla H H^{-1}]$ to produce a result insensitive to the ambiguity $M^\dagger \to V M^\dagger$. So calculating finite terms is rather involved requiring the careful accounting of powers of $1/r^2$. We do not carry this out here. Instead we will focus on the potentially ultraviolet-divergent terms. [The only finite term with significant physical implications is $S_{\rm wzw}(H)$, which we have already discussed.]

The next set of terms will be of the quadratic order. It is straightforward to work this out as

$$\Gamma = \frac{1}{4\epsilon} \int d\mu g^{a\bar{a}} \mathrm{Tr}(\bar{a}_{\bar{a}} H a_a H^{-1}) + \mathcal{O}(\log \epsilon). \tag{86}$$

The leading-order ultraviolet-divergent term is thus a mass term for the fields a and $\bar{a}$.

It is useful to contrast this with the situation in flat space. Consider a scalar field Φ (in flat space) coupled to A_μ. For the sake of the argument, we will take the field Φ to be massive to avoid any issues of infrared divergences. Then the quantum corrections due to Φ can also, naively, lead to a mass term for the gauge field, namely, a term of the form $\int d^4x A^2$, due to vacuum polarization effects. However, usually we reject such a term by requiring that any term we generate via quantum corrections should be gauge invariant and preserve the isometries of the underlying space. In flat space, the latter condition is equivalent to requiring invariance under Poincaré symmetry, or the corresponding Euclidean symmetry of 4d rotations and translations. The mass term $\int d^4x A^2$ does not pass this test and hence can be avoided in any regulator (such as dimensional regularization) which preserves the required invariances. Notice also that since Φ has a mass, we can expand diagrams with higher numbers of vertices in powers of the inverse mass and the terms so generated will be local. As a result, a nonlocal mass term of the form

$$\Gamma_{\rm mass} \sim \int d^4x \mathrm{Tr}\left[F^{\mu\nu}\left(\frac{1}{-D_\alpha D^\alpha}\right)F_{\mu\nu}\right], \tag{87}$$

which we may think of as $\int d^4x A^2$ completed by an infinite series of nonlocal terms to obtain the required invariance, is also not possible.

However, if we relax the invariance conditions, the Ward-Takahashi identities for the gauge symmetry do allow for mass terms. A classic well-known case is at finite temperature. If we use the Matsubara formalism, the relevant spacetime is $\mathbb{R}^3 \times S^1$, which has less isometries than $\mathbb{R}^4$. In this case, we get gauge-invariant screening masses for gauge fields, compatible with the Ward-Takahashi identities. The situation with the present case of $\mathbb{CP}^2$ is similar. *The mass term on $\mathbb{CP}^2$ given in (86) is gauge invariant and is fully consistent with the isometries of the underlying space. Thus there is no a priori reason to reject it*. The divergence also implies that it is a short-distance effect and not eliminated at large values of r. So the correct way to handle this is to define a renormalized theory where such a term has a coefficient renormalized to a finite value.

E. The log-divergent terms

Calculating further terms in the expansions of Γ_1 and Γ_2 we find the following logarithmically divergent terms:

$$\Gamma_{\log\epsilon}=\frac{\log\epsilon}{24}\int \mathrm{Tr}[(g^{a\bar{a}}(\bar{\nabla}_{\bar{a}}(\nabla_a HH^{-1})+[\bar{a}_{\bar{a}},Ha_aH^{-1}]))^2 -2g^{a\bar{a}}g^{b\bar{b}}(\bar{\nabla}_{\bar{a}}(\nabla_b HH^{-1})[\bar{a}_{\bar{b}},Ha_aH^{-1}] +\bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(Ha_bH^{-1}))-2g^{a\bar{a}}g^{b\bar{b}}(\mathcal{D}_a(Ha_bH^{-1})[\bar{a}_{\bar{a}},\bar{a}_{\bar{b}}] -\bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}[Ha_aH^{-1},Ha_bH^{-1}]) +g^{a\bar{a}}g^{b\bar{b}}[Ha_aH^{-1},Ha_bH^{-1}][\bar{a}_{\bar{a}},\bar{a}_{\bar{b}}]]. \quad (88)$$

The term which is independent of a and $\bar{a}$ is from Γ_1 due to the terms in $\langle\hat{J}\rangle$ with higher powers of ∇HH^{-1} and derivatives.

Even though (88) is a rather complicated looking expression, it simplifies neatly when written in terms of the field strength tensors. The calculation is straightforward and $\Gamma_{\log\epsilon}$ reduces to the covariant form

$$\Gamma_{\log\epsilon}=\frac{\log\epsilon}{24}\int \mathrm{Tr}g^{a\bar{a}}g^{b\bar{b}}[2F_{ab}F_{\bar{a}\bar{b}}-F_{a\bar{b}}F_{\bar{a}b}] =\frac{\log\epsilon}{384}\int \mathrm{Tr}g^{\mu\lambda}g^{\nu\delta}F_{\mu\nu}F_{\lambda\delta}+\frac{\log\epsilon}{16}\int \mathrm{Tr}(g^{a\bar{a}}F_{a\bar{a}})^2, \quad (89)$$

where the field strength tensors are, as usual, defined as

$$F_{ab}=[\nabla_a+A_a,\nabla_b+A_b],$$
$$F_{\bar{a}\bar{b}}=[\bar{\nabla}_{\bar{a}}+\bar{A}_{\bar{a}},\bar{\nabla}_{\bar{b}}+\bar{A}_{\bar{b}}],$$
$$F_{a\bar{b}}=[\nabla_a+A_a,\bar{\nabla}_{\bar{b}}+\bar{A}_{\bar{b}}]. \quad (90)$$

The details of the calculations, of both (86) and (88), are given in Appendix C. Notice that the first term in $\Gamma_{\log\epsilon}$, in the second line of (89), is proportional to the familiar action for gauge fields. The second term is allowed for a complex manifold such as $\mathbb{CP}^2$, since $g^{a\bar{a}}F_{a\bar{a}}$ does not have to vanish. The appearance of this term is linked to the chiral nature of the action S_1 for the scalar field in (35), with a kinetic operator $-\bar{D}\cdot D$. One can verify that a nonchiral kinetic energy term $-\frac{1}{2}(D\cdot\bar{D}+\bar{D}\cdot D)$ does not produce the last term in (89). Further, writing

$$-\bar{D}\cdot D=-\frac{1}{2}[(D\cdot\bar{D}+\bar{D}\cdot D)-(D\cdot\bar{D}-\bar{D}\cdot D)] =-\frac{1}{2}[(D\cdot\bar{D}+\bar{D}\cdot D)-g^{a\bar{a}}F_{a\bar{a}}], \quad (91)$$

we can trace the origin of the last term in (89) to the $\frac{1}{2}g^{\bar{a}a}F_{\bar{a}a}$ term in (91).

F. Summary of Sec. III

It will be useful to have a short summary of this rather long section. We set up the expansion scheme for calculating the effective action Γ obtained by integrating out the scalar fields Φ and $\Phi^\dagger$. The propagator for the scalar field and its regularized form were given in (57) and (71), respectively. The result for Γ can be summarized as

$$\Gamma=\int \mathrm{Tr}\left[\frac{1}{4\epsilon}g^{a\bar{a}}(\bar{a}_{\bar{a}}Ha_aH^{-1})+\frac{\log\epsilon}{384}g^{\mu\lambda}g^{\nu\delta}F_{\mu\nu}F_{\lambda\delta} +\frac{\log\epsilon}{16}(g^{a\bar{a}}F_{a\bar{a}})^2\right]+CS_{\mathrm{wzw}}(H)+\text{finite terms}, \quad (92)$$

where the coefficient C is given in (75). These are the leading terms in the following sense. The first three terms give the potential ultraviolet-divergent terms, corresponding to a mass term and the wave function renormalization of the gauge field. There is one finite term, which is rather special, which we have singled out. This is the WZW action for H; it is the finite term with the minimal number of derivatives on the H field. The terms which we have not calculated are finite terms with higher numbers of derivatives on H or involving powers of a_a and $\bar{a}_{\bar{a}}$.

IV. SUMMARY AND PHYSICAL IMPLICATIONS

As mentioned in the introduction, the manifold $\mathbb{CP}^2$ has many features making it attractive for analyzing the dynamics of gauge fields. With this in mind, we have worked out the parametrization of the gauge potentials on $\mathbb{CP}^2$, very much along the lines of a similar parametrization used in two dimensions. This allowed for a simple separation of the gauge-invariant degrees of freedom, making it possible to perform calculations in a manifestly gauge-invariant way. We have also obtained the form of the (chiral) scalar field propagator on $\mathbb{CP}^2$ and worked out the leading terms in the effective action obtained by integrating out the scalar fields. The result is summarized in (92).

We now turn to the physical implications of the results we have obtained. We start with considerations regarding the mass term with the quadratically divergent coefficient in (92). This term is manifestly consistent with gauge invariance and, also, it preserves all the isometries of $\mathbb{CP}^2$. Therefore, we have no reason to reject a possible mass term. Further, the ultraviolet singularities in a field theory are only sensitive to local geometry, so they are essentially the same as in flat space. The appearance of this term with a divergent coefficient therefore shows that it will survive to the large r limit. (The fact that we have reduced isometries even in the large r limit is important for this, unlike the situation in $\mathbb{R}^4$ where a mass term can be ruled out on grounds of invariance.) The existence of the mass term implies that we have to include a counterterm

$$S_{\text{mass}} = \mu^2 \int g^{a\bar{a}} \text{Tr}(\bar{a}_{\bar{a}} H a_a H^{-1}) \tag{93}$$

in the action for the gauge fields. We can then use μ^2 to absorb the divergence and define a renormalized mass $\mu^2_{\text{Ren}} = \mu^2 + (1/4\epsilon)$. The natural question is then: What value should we assign to μ^2_{Ren}? Recall that the four-dimensional non-Abelian gauge theory is not defined until we pick a dimensionful parameter which sets the basic scale for the theory. So one option is to regard μ^2_{Ren} as providing this dimensional transmutation. In this case, other renormalization effects will include this parameter as an infrared cutoff for the transverse modes. Thus the usual dimensional parameter Λ_{QCD} will be a function of this parameter μ^2_{Ren}. Equivalently, we may take the dimensional parameter to be the usual Λ_{QCD} defined via the one-loop renormalization of the coupling constant and regard μ^2_{Ren} as determined by the theory in terms of Λ_{QCD}. The full effective action by construction includes quantum effects in the sense that it determines the quantum dynamics via its equations of motion. These are essentially the Schwinger-Dyson equations of the theory. So we can think of μ^2_{Ren} as determined via the Schwinger-Dyson equations, if we have already chosen the dimensional parameter as Λ_{QCD}.

The idea of a soft gluon mass[4] goes back to the 1980s [12], but it is only more recently that systematic attempts have been made to develop this to the level of quantitative predictions [13,14]. There has been considerable evidence based on lattice simulations, where one sees clearly that the gluon propagator in the Landau gauge saturates to a finite nonzero value at low momenta [21]. These lattice results do require an explanation. On the analytical side, there has been a lot of effort in calculating the gluon self-energy via Schwinger-Dyson equations and showing that it is nonzero at zero momentum; for a review, see [14]. The propagator, by construction, is gauge dependent, but the result for the mass is gauge invariant since BRST Ward-Takahashi identities are preserved. Our analysis, which is manifestly gauge invariant, thus provides an understanding for the possible genesis of a mass term as expected from the lattice data and is in conformity with the analyses done using the Schwinger-Dyson equations.

Regarding the log-divergent terms, there is not much to say, except that it contributes to the field (or wave function) renormalization and eventually gets folded into the running of the coupling constant. Turning to the WZW action, we first note that if $\mu^2_{\text{Ren}} \neq 0$, then the modes corresponding to a_a and $\bar{a}_{\bar{a}}$ are not relevant at low energies and the theory is controlled primarily by $S_{\text{wzw}}(H)$. This term comes with a finite coefficient even if we take $\lambda \to 0$, i.e., there are no infrared divergences, the result being

$$\Gamma_{\text{wzw}} = \frac{5}{2\pi r^2} S_{\text{wzw}}(H). \tag{94}$$

The kinematic regime of interest to us is however for $\lambda r^2 \gg 1$ with the coefficient of $S_{\text{wzw}}(H)$ equal to C as in (84); in this regime, terms in the effective action with vertices of higher mass dimension are parametrically subdominant, as explained at the end of Sec. III B. This provides a consistent argument for the theory being controlled by $S_{\text{wzw}}(H)$. As we noted before, the coefficient in (94) or (84) has the wrong sign, contributing a growing exponential for the subsequent integration over the gauge fields. But this is only for the contribution due to scalar matter fields; as we will see in [11], there is a WZW term which arises in the measure for the gauge fields as well; it is of the right sign and convergence for the integration over the gauge fields is obtained, at least for some number of massless chiral scalar fields of low-dimensional representations. In this case, the low-energy dynamics will be dominated by the critical points, i.e., solutions of the equations of motion, of $S_{\text{wzw}}(H)$. The critical points are anti-self-dual instantons and related to holomorphic vector bundles. Essentially, M and $M^\dagger$ define the holomorphic frames for the bundle. This action has a long history. Originally, Donaldson considered this action in the context of anti-self-dual instantons [17]. The same arises in attempting to generalize the WZW theory to four dimensions and relating it to the Kähler-Chern-Simons theory [18], similar to the WZW-CS relation in two and three dimensions [3]. As shown in [18] and elaborated in [22,23], this action also leads to a holomorphically factorized current algebra, very similar to the situation in two dimensions. Such theories have also been found in higher-dimensional quantum Hall systems [24] and are also realized as the target space dynamics of (world-sheet) $N = 2$ heterotic superstrings [25].

As mentioned in the introduction, the correlation functions for gauge fields seem to be dominated by instantons at low energies. A number of numerical simulations starting with the work of the MIT group have shown clear evidence for this; see, for example, [15]. On flat $\mathbb{R}^4$, it is difficult to isolate a part of the effective action as pertaining just to the instantons, so it is difficult to see how instanton dominance can emerge. By considering a complex manifold such as $\mathbb{CP}^2$ and obtaining S_{wzw} by integrating out fields, we obtain some analytical evidence pointing to an instanton liquid picture. Further discussion of these matters will be taken up in the second part of this work.

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[4] If the self-energy $\Sigma(p)$ as a function of the momentum p has the property that $\Sigma(0) = \mu^2 \neq 0$ and $\Sigma(p) \to 0$ as $|p| \to \infty$, it is referred to as a "soft" mass.

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APPENDIX A: GAUGE FIELDS AND THE SCALAR PROPAGATOR ON $\mathbb{CP}^k$

In this Appendix, we consider the generalization of our parametrization of gauge fields to complex projective spaces of arbitrary dimension, i.e., to $\mathbb{CP}^k$. We will also discuss the propagator for a scalar field on such spaces. While we do not carry out explicit calculations of effective action or gauge-invariant measures (for arbitrary k), this analysis does serve to illustrate that there is a uniform way to treat all $\mathbb{CP}^k$.

Regarding the parametrization of the gauge fields, we can proceed in a way similar to what we did for $\mathbb{CP}^2$ by noting that $\mathbb{CP}^k$ is the group coset space $SU(k+1)/U(k)$. Thus functions, vectors, etc., on this space may be realized in terms of the Wigner function $D^{(s)}_{A,B}(g) = \langle s, A|\hat{g}|s, B\rangle$ which is the representative of an $SU(k+1)$ element g in a general irreducible representation. We designate the representation by s, and A and B label the states within the representation. For the defining fundamental representation, g is a $(k+1)\times(k+1)$ unitary matrix of unit determinant. The generators of the group in this representation will be denoted by $\{t_a\}$, as we did for $SU(3)$. The subalgebra $\underline{U(k)}$ is embedded in the standard way in the algebra of $\overline{SU(k+1)}$, as the upper left block in the fundamental representation, while the $U(1)$ generator, which is the analog of the hypercharge, is given by

$$Y = \sqrt{\frac{2k}{k+1}}\, t_{k^2+2k} = \frac{1}{k+1}\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -k \end{bmatrix}. \tag{A1}$$

The right translation operators are defined as in (2), with R_i and $R_{\bar{i}}$ given by the coset generators.

A function on $\mathbb{CP}^k$ must be invariant under $U(k) \in SU(k+1)$, so it can be expanded as

$$F(g) = \sum_{s,A} C^{(s)}_A \langle s, A|\hat{g}|s, w\rangle \equiv \sum_{s,A} C^{(s)}_A D^{(s)}_{A,0}(g), \tag{A2}$$

where $C^{(s)}_A$ are arbitrary coefficients characterizing the function and the state $|s, w\rangle \equiv |0\rangle$ is invariant under $U(k)$. As in the case of $SU(3)$, we can think of the carrier space of $SU(k+1)$ representations in the tensor notation as

$$T^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots b_q} \equiv |a_1, a_2, \ldots a_p; b_1, b_2, \ldots, b_q\rangle, \tag{A3}$$

where each index can take values 1 to $k+1$. But unlike the case of $SU(3)$, in general, we do not have symmetry under permutation of the a's or the b's. To obtain a $U(k)$-invariant state within such a representation, which is to be identified as $|s, w\rangle$ in (A2), we will need $p = q$; the invariant state would then correspond to the choice of $a_i = k+1$, $b_i = k+1$; i.e.,

$$\begin{aligned} |s, w\rangle &\equiv |0\rangle \\ &= |k+1, k+1, \ldots, k+1; \\ &\quad k+1, k+1, k+1, \ldots, k+1\rangle. \end{aligned} \tag{A4}$$

This will also mean that the representations of interest for functions on $\mathbb{CP}^k$ are of the $s = (p, p)$ type and are symmetric in all a's and symmetric in all b's.

The components of the gauge potential must transform in the same way as R_i and $R_{\bar{i}}$. These operators transform as the fundamental and antifundamental representations of $SU(k)$ and have $Y = 1$ and -1, respectively. Thus, the gauge potentials can be expanded as in (A2), but with $|s, w\rangle$ having $Y = \pm 1$ and transforming as fundamental and antifundamental of $SU(k)$. Since functions are $U(k)$ invariant, derivatives of functions will have these properties and will qualify as components of the gauge potential. As before, these will describe the gradient part (i.e., the pure gauge part and the H part) of the vector potentials. There are other choices for $|s, w\rangle$ as well. For example, a state

$$|s, i\rangle = |k+1, k+1, \ldots, k+1; i, k+1, k+1, \ldots, k+1\rangle \tag{A5}$$

with all a's and b's being set to $k+1$, except for b_1, which is identified as the index i taking values 1 to k, satisfies the requirements, with $Y = 1$. If b_1 is symmetric with the other b's, then this will be obtained by acting on the state given in (A4). This is the gradient part we have already mentioned. But we can also have states where b_1 is antisymmetric with all the other b's, which are themselves symmetric among themselves. Such a state cannot be written in terms of the action of R_i on a highest weight state of the form (A4). We also have a corresponding state in the conjugate representation given by

$$|s, \bar{i}\rangle = |i, k+1, \ldots, k+1; k+1, k+1, k+1, \ldots, k+1\rangle \tag{A6}$$

This will have $Y = -1$. If we are considering an Abelian gauge potential on $\mathbb{CP}^k$, we can now write it as

$$\begin{aligned} A_i &= -R_i f + \sum_{s,A} a^s_A \langle s, A|\hat{g}|s, i\rangle = -R_i f + a_i, \\ \bar{A}_{\bar{i}} &= -R_{\bar{i}} \bar{f} + \sum_{s,A} a^{s*}_A \langle s, A|\hat{g}|s, \bar{i}\rangle = -R_{\bar{i}} \bar{f} + \bar{a}_{\bar{i}}, \end{aligned} \tag{A7}$$

where f is a complex function with a mode expansion

$$f = \sum_{s,A} \lambda_A \langle s, A|\hat{g}|s, w\rangle. \tag{A8}$$

Since the product of a $U(k)$-invariant state like $|s, w\rangle$ with another $U(k)$-invariant state will contain only $U(k)$-invariant states when it is reduced to irreducible components, we see that products of functions also qualify as functions. Thus, for a non-Abelian theory we can generalize (A7) as

$$A_i = -\nabla_i M M^{-1} - M a_i M^{-1},$$
$$\bar{A}_{\bar{i}} = M^{\dagger -1}\bar{\nabla}_{\bar{i}} M^{\dagger} + M^{\dagger -1}\bar{a}_{\bar{i}} M^{\dagger}, \tag{A9}$$

where M is a complex matrix taking values in the complexified gauge group and a_i and $\bar{a}_i$ are given by

$$a_i = \sum_{s,A} a_A^s \langle s, A|\hat{g}|s, i\rangle \tag{A10}$$

with $\bar{a}_i = a_i^\dagger$. [In (A9), we have also changed from R_i and $R_{\bar{i}}$ to the gradient operators, as in (13).]

Again, it is useful (and important) to count the number of polarizations shown in the parametrization (A7) or (A10). For a $U(1)$ gauge field on a complex k-dimensional space such as $\mathbb{CP}^k$, we need k complex components or k independent functions to begin with. In f and $\bar{f}$, we have one complex function. The remaining terms in (A7), namely, a_i and $\bar{a}_{\bar{i}}$, give k complex (or $2k$ real) components. But there is a constraint, just as in the case of $\mathbb{CP}^2$, since the state $|s, i\rangle$ is a highest weight state. The action of a raising operator on it, whereby the index i is changed to $k+1$, vanishes because the index b_1 was taken to be antisymmetric under exchange with any of the other b's. This means that we have the condition $\eta^{i\bar{i}}\bar{R}_{\bar{i}} a_i = 0$. Thus, effectively, we have $k-1$ independent (complex) functions in a_i, so that with the f and $\bar{f}$, we have a total of k complex functions as needed.

We now turn to the derivation of the propagator for a scalar field on $\mathbb{CP}^k$. The mode expansion for such a field was given in (A2) and (A4). The propagator can be written in terms of a mode expansion as in (47), with the local coordinates of $\mathbb{CP}^2$ given in (31). These local coordinates are related to the homogeneous coordinates Z as in (59). More generally, on $\mathbb{CP}^k$, the required representations $D^{(p,p)}_{A,0}(g)$ are polynomials in $g_{a(k+1)}$ and the conjugate $g^{*a(k+1)}$. This implies that $D^{(p,p)}_{0,0}(g)$ is a function of $s = \eta_{a\bar{a}} z^a \bar{z}^{\bar{a}}$ [and, likewise, $D^{(p,p)}_{0,0}(g'^\dagger g)$ is a function of $s = \sigma^2(z, y)$ as defined in (49)]. The action of the operator $\eta^{\bar{i}i} R_{\bar{i}} R_i = -g^{\bar{i}i}\bar{\nabla}_{\bar{i}}\nabla_i$ on the $U(k)$-invariant state is given by

$$\eta^{\bar{i}i} R_{\bar{i}} R_i D^{(p,p)}_{A,0}(g) = p(p+k) D^{(p,p)}_{A,0}(g), \tag{A11}$$

where $p(p+k)$ is the eigenvalue of the quadratic Casimir operator for a (p,p) representation of $SU(k+1)$. Hence, as for $\mathbb{CP}^2$, the eigenfunction for $p = 0$ is a zero mode and it must be excluded from the mode expansion of the propagator. Thus the propagator obeys the equation

$$\eta^{\bar{i}i} R_{\bar{i}} R_i G(g, g') = \delta(g, g') - 1. \tag{A12}$$

Since $\mathbb{CP}^k$ is a Kähler manifold, the metric tensor and its inverse are given by Eq. (51). The normalized volume element is

$$d\mu = \frac{k!}{\pi^k} \frac{d^{2k}x}{(1 + \bar{z}\cdot z)^{k+1}} = \frac{k!}{\pi^k} (\det g) d^{2k}x. \tag{A13}$$

The operator of interest acting on G gives us

$$\eta^{\bar{i}i} R_{\bar{i}} R_i G = -g^{\bar{a}a}\bar{\partial}_{\bar{a}}\partial_a G$$
$$= -(1+s)[s(1+s)G'' + (k+s)G']. \tag{A14}$$

Following the propagator calculation for $\mathbb{CP}^2$, if we consider nonzero s, (A14) becomes

$$s(1+s)^2 W' + (k+s)(1+s)W = 1, \quad W = G'. \tag{A15}$$

Using a suitable integrating factor and performing an integration on W, we get the following equation for G:

$$G = -\left(C_1 - \frac{1}{k}\right)\sum_{n=1}^{k-1} \frac{1}{n} \mathcal{C}^{k-1}_n \frac{1}{s^n} + C_1 \log s$$
$$-\frac{1}{k}\log\left(\frac{s}{1+s}\right) + C_0,$$
$$\mathcal{C}^{k-1}_n = \frac{(k-1)!}{n!(k-n-1)!}, \tag{A16}$$

where the first term is present only for $k > 1$.

We fix the constant C_1 by looking at the short-distance expansion of the propagator. As $s \ll 1$, $G \to -(C_1 - \frac{1}{k})\frac{1}{k-1}\frac{1}{s^{k-1}}$ [for $k = 1$, $(C_1 - 1/k)\log s$]. In this limit $R_{\bar{i}} R_i$ can be approximated by the flat space operator $-\bar{\partial}\cdot\partial = -\nabla^2/4$. For $\mathbb{R}^{2k}$, the Green function for the operator $-\nabla^2$ is $(k-2)!/(4\pi^k (x-x')^{2(k-1)})$ (for $k = 1$, $-\frac{\log(x-x')^2}{4\pi}$). Removing a factor of 4 (since we are considering $-\nabla^2/4$) and multiplying by a factor of $\frac{\pi^k}{k!}$ from the volume normalization, we conclude that G should have the following short-distance limit:

$$G \approx \frac{1}{k(k-1)s^{k-1}}, \qquad k > 1,$$
$$G \approx -\log s, \qquad k = 1. \tag{A17}$$

This implies that $C_1 = 0$.

To find C_0 we notice that G is given by an expansion of modes with the eigenfunction for $p = 0$ removed. Hence it must be orthogonal to the $p = 0$ eigenfunction, which is a constant. The propagator must then obey the equation

$$0=\int d\mu G$$
$$=\int d\mu\left[\frac{1}{k}\sum_{n=1}^{k-1}\frac{1}{n}\mathcal{C}_n^{k-1}\frac{1}{s^n}-\frac{1}{k}\log\left(\frac{s}{1+s}\right)\right]+C_0. \quad \text{(A18)}$$

Solving the integral on the left we identify the constant C_0 as

$$C_0=-\frac{1}{k}\sum_{n=1}^{k}\frac{1}{n}. \quad \text{(A19)}$$

Thus, the massless scalar propagator for $\mathbb{CP}^k$ is

$$G=\frac{1}{k}\sum_{n=1}^{k-1}\frac{1}{n}\mathcal{C}_n^{k-1}\frac{1}{s^n}-\frac{1}{k}\log\left(\frac{s}{s+1}\right)-\frac{1}{k}\sum_{n=1}^{k}\frac{1}{n}. \quad \text{(A20)}$$

In particular, for $k=2$,

$$G=\frac{1}{2s}-\frac{1}{2}\log\left(\frac{s}{s+1}\right)-\frac{3}{4}, \quad \text{(A21)}$$

which is the same as our result in (57) for $\mathbb{CP}^2$.

APPENDIX B: CALCULATING THE EXPECTATION VALUE OF THE CURRENT $\langle\hat{J}\rangle$

In this Appendix we go over some of the details of the calculation of the result (74) for $\langle\hat{J}\rangle$. The terms we need come from the expansion of the propagator in (73) up to terms with one power of ∇HH^{-1}. The current is then given as

$$\langle\hat{J}\rangle=-\mathcal{D}_x\mathcal{G}_{\rm Reg}(x,y)|_{y\to x}$$
$$=\text{Term 1}+\text{Term 2}+\text{Term 3}+\cdots,$$
$$\text{Term 1}=-\nabla_{xa}G(x,y')\mathcal{P}\exp\left(\int_y^{y'}\nabla HH^{-1}\right)\Big|_{y\to x},$$
$$\text{Term 2}=(\nabla_aHH^{-1})_xG(x,x'),$$
$$\text{Term 3}=\int_z\nabla_{xa}G(x,z)g_z^{b\bar{b}}(\nabla_bHH^{-1})_z\bar{\nabla}_{z\bar{b}}G(z,x'). \quad \text{(B1)}$$

The primed homogenous coordinate is as in (52):

$$X'=X+\alpha\left(\frac{W\bar{X}\cdot X}{\bar{X}\cdot W}-X\right). \quad \text{(B2)}$$

For each term we do an angular average over α and W with the conditions that $\alpha\bar{\alpha}=\epsilon$ and $\sigma^2(x,w)=1$.

For Term 1 in (B1), on averaging over α and w, we get

$$\text{Term 1}=-(\nabla_bHH^{-1})_x\int_\alpha\delta(\alpha\bar{\alpha}-\epsilon)$$
$$\times\int_w\delta(\sigma^2(x,w)-1)\nabla_{ax}G(x,y')\Big|_{y\to x}(x'-x)^b. \quad \text{(B3)}$$

We can then make a coordinate transformation $w\to w'$ such that

$$w^a=x^a+(e_x^{-1})_b^a\frac{w'^b}{1-\bar{x}\cdot w'}, \quad \text{(B4)}$$

where e_x^{-1} are the (inverse) frame fields at x as given in (14). This sets $w'\cdot\bar{w}'=\sigma^2(x,w)$. In group theoretic terms we are using the translational invariance of the integral to change $g_{x'}^\dagger g_x$ to $g_{w'}^\dagger$, effectively setting $x\to 0$ and $w\to w'$ in the integral in (B3).

Using the following:

$$\sigma^2(x,x')=\alpha w'\cdot\bar{\alpha}\bar{w}',$$
$$\nabla_{xa}\sigma^2(x,y')|_{y\to x}=-(1+\alpha w'\cdot\bar{\alpha}\bar{w}')\eta_{a\bar{a}}(e_x)_{\bar{i}}^{\bar{a}}\bar{\alpha}\bar{w}'^{\bar{i}},$$
$$(x'-x)^b=(e_x^{-1})_i^b\frac{\alpha w'^i}{1-\bar{x}\cdot\alpha w'}, \quad \text{(B5)}$$

Eq. (B3) becomes

$$\text{Term 1}=(\nabla_aHH^{-1})_x\frac{\epsilon}{2r^2}\left(1+\frac{\epsilon}{r^2}\right)G'\left(\frac{\epsilon}{r^2}\right), \quad \text{(B6)}$$

where we have included the scaling $\epsilon\to\epsilon/r^2$ and $G'(s)=\frac{\partial G}{\partial s}$. The scalar propagator is given by

$$G(s)=\frac{1}{r^2}\int_{\lambda r^2}^\infty d\rho\left[e^{-\rho s}\left(\frac{1}{2}+\frac{1}{2\rho}(1-e^{-\rho})\right)-\frac{3}{4}e^{-\rho}\right] \quad \text{(B7)}$$

so that Term 1 is

$$\text{Term 1}=(\nabla_aHH^{-1})_x\left[-\frac{1}{4\epsilon}-\frac{1}{2r^2}\right]. \quad \text{(B8)}$$

Doing a similar coordinate transformation for w in Term 2 in (B1), the rescaled Term 2 becomes

$$\text{Term 2}=(\nabla_aHH^{-1})_xG\left(\frac{\epsilon}{r^2}\right)$$
$$=(\nabla_aHH^{-1})_x\left[\frac{1}{2\epsilon}-\frac{1}{2r^2}\log\left(\frac{\epsilon}{r^2}\right)-\frac{\lambda}{2}-\frac{3}{4r^2}e^{-\lambda r^2}\right.$$
$$\left.-\frac{1}{2r^2}(E_1(\lambda r^2)+\gamma+\log(\lambda r^2))\right]. \quad \text{(B9)}$$

Finally, for Term 3 in (B1), we can do two coordinate transformations: one, as above, $w\to w'$ such that

$\sigma^2(x,w) = w' \cdot \bar{w}'$, and another for $z \to z'$ such that $\sigma^2(z,x) = z' \cdot \bar{z}'$. These transformations effectively set $x \to 0$ in the integral. Furthermore, in Term 3, $\nabla_b HH^{-1}$ is at point z, but since we are focusing on terms without derivatives on $\nabla_b HH^{-1}$, we evaluate it at x. The term then becomes

$$\begin{aligned}\text{Term 3} = (\nabla_b HH^{-1})_x &\int_\alpha \delta(\alpha\bar{\alpha} - \epsilon) \int_{w'} \delta(|w'|^2 - 1)\\ &\times \int d\mu(z') G'(|z'|^2)(1+|z'|^2) G'(\sigma^2(z',w'))\\ &\times (1+\sigma^2(z',w'))(-\eta_{a\bar{a}}(e_x)^{\bar{a}}_{\bar{m}} \bar{z}'^{\bar{m}})\\ &\times (e_x^{-1})^b_m \left(\frac{z'^m}{1 - \bar{x}\cdot z'} - \frac{\alpha w'^m}{1-\bar{x}\cdot \alpha w'}\right)\\ &\times \frac{(1+|z'|^2)(1-\alpha w'\cdot\bar{x})}{(1+\alpha w'\cdot \bar{z}')(1-z'\cdot\bar{x})},\end{aligned} \tag{B10}$$

where

$$\begin{aligned}1+\sigma^2(z',w') &= \frac{(1+z'\cdot\bar{z}')(1+\alpha\bar{\alpha}w'\cdot\bar{w}')}{(1+\alpha\bar{z}'\cdot w')(1+\bar{\alpha}\bar{w}'\cdot z')}\\ &= \frac{\bar{Z}'\cdot Z'\bar{\tilde{W}}'\cdot\tilde{W}'}{\bar{Z}'\cdot\tilde{W}'\bar{\tilde{W}}'\cdot Z'}\end{aligned} \tag{B11}$$

and $\tilde{W}' = (\alpha W_1', \alpha W_2', W_3') = W_3'(\alpha w_1', \alpha w_2', 1)$. It is now useful to make a final change of coordinates from variables Z' to $\tilde{Z}$ given by

$$\frac{\tilde{Z}}{\tilde{Z}_3 \bar{\tilde{W}}_3'} = \frac{Z'}{\bar{\tilde{W}}'\cdot Z'}. \tag{B12}$$

Equation (B11) can then be written as

$$\begin{aligned}1+\sigma^2(z',w') &= (1+\bar{\tilde{z}}\cdot\tilde{z})(1+\alpha\bar{\alpha}w'\cdot\bar{w}')\\ &= (1+\bar{\tilde{z}}\cdot\tilde{z})(1+\epsilon)\end{aligned} \tag{B13}$$

upon angular averaging. Term 3 then simplifies to

$$\begin{aligned}\text{Term 3} &= -(\nabla_b HH^{-1})_x \int d\mu(\tilde{z})[G'(|\tilde{z}|^2)(1+\epsilon)\\ &\quad\times G'(|\tilde{z}|^2(1+\epsilon)+\epsilon)\\ &\quad\times (1+|\tilde{z}|^2)^3 \eta_{a\bar{a}}(e_x)^{\bar{a}}_{\bar{m}}(e_x^{-1})^b_m \bar{\tilde{z}}^{\bar{m}}\tilde{z}^m]\\ &= -(\nabla_a HH^{-1})_x \int_0^\infty ds s^2 G'(s)(1+\epsilon)\\ &\quad\times G'(s(1+\epsilon)+\epsilon),\end{aligned} \tag{B14}$$

where, in the final line, $s = |\tilde{z}|^2$. After rescaling and carrying out the integral this term becomes

$$\begin{aligned}\text{Term 3} = (\nabla_a HH^{-1})_x \Big[&-\frac{1}{4\epsilon} + \frac{1}{2r^2}\log\left(\frac{\epsilon}{r^2}\right) + \frac{3\lambda}{8}\\ &+ \frac{1}{2r^2}(E_1(2\lambda r^2) + \gamma + \log(2\lambda r^2))\\ &+ \frac{\lambda}{2}(1-e^{\lambda r^2})E_1(2\lambda r^2) + \frac{1}{4r^2}e^{-\lambda r^2}(1-e^{-\lambda r^2})\\ &- \frac{1}{8\lambda r^4}(1-e^{-\lambda r^2})^2\Big].\end{aligned} \tag{B15}$$

Combining expressions (B8), (B9), and (B15), we get

$$\langle \hat{J}_a \rangle = -\frac{\pi}{2} C \nabla_a HH^{-1} + \cdots \tag{B16}$$

with C as given in (75).

In arriving at (B16), we used (72) with $h = M^\dagger$, effectively eliminating $M^\dagger$ and replacing M by H. What is the result if we eliminate M? In this case, the relevant propagator is $(-\bar{\mathcal{D}}\cdot\nabla)^{-1}$ and we do not obtain (74) in any expansion of $(-\bar{\mathcal{D}}\cdot\nabla)^{-1}$ in powers of $H^{-1}\bar{\nabla}H$ to any finite order. A resummation is then needed but the final result is the same. We mentioned this point in Sec. III C, but here we go over the arguments in some more detail.

We start by regarding $\langle \hat{J} \rangle$ as a function of M and $M^\dagger$. Then, rather than using (44), consider just setting $M^\dagger = 1$ in the expression for the current. This leads to $\langle \hat{J}(M,1)\rangle = -(\pi/2)C\nabla MM^{-1}$. This calculation is the same as in arriving at (74) except that we just have M now, not H as in the argument of the current. [We can view this as giving the functional derivative of Γ_1 at the point $(M,1)$ in the space of configurations $(M, M^\dagger)$. One may then seek to integrate functionally.] Naturally, the result $\langle \hat{J}\rangle = -(\pi/2)C\nabla MM^{-1}$ is not gauge covariant (since we set $M^\dagger = 1$), but we know $\langle \hat{J}\rangle$ should be. Clearly, this has to be obtained by $M^\dagger$-dependent correction terms. We may then ask: What $M^\dagger$-dependent terms can we add to ∇MM^{-1} to make it covariant? To eliminate the inhomogeneous term in ∇MM^{-1} from the gauge transformation $M \to UM$, we need a term of the form $-M^{\dagger-1}\nabla M^\dagger$. This gives $-\nabla MM^{-1} - M^{\dagger-1}\nabla M^\dagger$ and leads to the result (74). Notice that we have the *holomorphic* derivative of $M^\dagger$ in this expression. Since $M^\dagger$ comes with the antiholomorphic derivative in $\bar{D}$, various terms must combine to produce $M^\dagger$ from $M^{\dagger-1}\bar{\nabla}M^\dagger$ (and then the holomorphic derivative) which will require an infinite series. Effectively, the identity (44) is a way of carrying out this resummation. Another way to see the argument for the gauge-covariant expression for the current is the following. We consider the derivative $\bar{\nabla}_{\bar{a}}\langle \hat{J}_a\rangle$. With the result $\langle \hat{J}(M,1)\rangle = -(\pi/2)C\nabla MM^{-1} + \cdots$, this becomes

$$\bar{\nabla}_{\bar{a}}\langle \hat{J}_a(M,1)\rangle = (\pi/2)C[\bar{\nabla}_{\bar{a}}(-\nabla_a MM^{-1})] + \cdots. \tag{B17}$$

The term on the right-hand side is the first term in the field strength for the potentials,

$$\begin{aligned}\mathcal{F}_{\bar{a}a} &= \bar{\nabla}_{\bar{a}}(-\nabla_a MM^{-1}) - \nabla_a(M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}) \\ &\quad + [(M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}), (-\nabla_a MM^{-1})] \\ &= \bar{\mathcal{D}}_{\bar{a}}(-\nabla_a MM^{-1}) - \nabla_a(M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}). \end{aligned} \tag{B18}$$

Since this is the gauge-covariant version of $\bar{\nabla}_{\bar{a}}(-\nabla_a MM^{-1})$ with the minimal number of derivatives, we see that the gauge-covariant extension of (B17) is

$$\bar{\mathcal{D}}_{\bar{a}}\langle \hat{J}_a(M,M^{\dagger})\rangle = \frac{\pi}{2}C[\bar{\mathcal{D}}_{\bar{a}}(-\nabla_a MM^{-1}) - \nabla_a(M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger})] + \cdots. \tag{B19}$$

Notice further that we have the identity

$$\begin{aligned}&\nabla_b(M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}) - \bar{\nabla}_{\bar{a}}(M^{\dagger-1}\nabla_b M^{\dagger}) \\ &\quad + [M^{\dagger-1}\nabla_b M^{\dagger}, M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}] = 0.\end{aligned} \tag{B20}$$

Combining this with (B19) we get

$$\bar{\mathcal{D}}_{\bar{a}}\left[\langle \hat{J}_a(M,M^{\dagger})\rangle + \frac{\pi}{2}C(\nabla_a MM^{-1} + M^{\dagger-1}\nabla_a M^{\dagger})\right] + \cdots = 0, \tag{B21}$$

which has the solution

$$\begin{aligned}\langle \hat{J}_a(M,M^{\dagger})\rangle &= M^{\dagger-1}\left[-\frac{\pi}{2}C\nabla_a HH^{-1}\right]M^{\dagger} \\ &\quad + M^{\dagger-1}[V^{-1}\nabla_a V]M^{\dagger} + \cdots,\end{aligned} \tag{B22}$$

where V is a holomorphic matrix. This result leads us back to (74). The second term on the right -hand side is an ambiguity corresponding to the holomorphic ambiguity in defining M and $M^{\dagger}$ mentioned at the end of Sec. II C. It can be removed by redefining $M^{\dagger}$; the WZW action is insensitive to this. Effectively, in solving (B19) using (B20), we are carrying out a resummation.

The situation is exactly analogous to what happens in two dimensions. In calculating $\mathrm{Tr}\log\bar{D}$ in two dimensions, one uses the result

$$\begin{aligned}\left(\frac{1}{\bar{D}}\right)_{x,y} &= M^{\dagger-1}(x)\left[\frac{1}{\pi(x-y)}\right]_{x,y}M^{\dagger}(y) \\ &\approx \left[\frac{1}{\pi(x-y)}\right]_{x,y}[1+(y-x)M^{\dagger-1}\partial M^{\dagger}+\cdots].\end{aligned} \tag{B23}$$

If $\bar{D}^{-1}$ is expanded in powers of $M^{\dagger-1}\bar{\partial}M^{\dagger}$, clearly one needs to resum an infinite series to get the holomorphic derivative on the right-hand side.

APPENDIX C: CALCULATING THE UV-DIVERGENT TERMS

In this Appendix we will go over some of the calculations leading to the UV-divergent terms in (86) and (88). First, we will find the divergent terms in Γ_1 by calculating the expectation value of the current as in the appendix above. As we have seen in Appendix B, Γ_1 has at most log-divergent terms. Such terms can be calculated in the large r^2 limit treating the space as effectively flat.[5] Following Eq. (42),

$$\langle \hat{J}(H,1)\rangle(x) = [-\mathcal{D}_x\mathcal{G}(x,y)]_{y\to x}, \tag{C1}$$

where $\mathcal{D}$ has the connection $-\nabla HH^{-1}$ and $\mathcal{G}(x,y) = (-\bar{\nabla}\cdot\mathcal{D})^{-1}_{x,y}$.

Expanding the propagator $\mathcal{G}$ in powers of ∇HH^{-1},

$$\begin{aligned}\langle \hat{J}(H,1)\rangle(x) &= -\mathcal{D}_x\left(G(x,y') + \int_z G(x,z)\mathbb{X}_z G(z,y') + \cdots\right) \\ &\quad \times\mathcal{P}\exp\left(\int_y^{y'}\nabla HH^{-1}\right)\bigg|_{y\to x} \\ &= \langle\hat{J}\rangle^{(1)} + \langle\hat{J}\rangle^{(2)} + \cdots,\end{aligned} \tag{C2}$$

where $-\bar{\nabla}\cdot\mathcal{D} = -\bar{\nabla}\cdot\nabla - \mathbb{X}$, or explicitly, $\mathbb{X} = -\bar{\nabla}(\nabla HH^{-1}) - \nabla HH^{-1}\cdot\bar{\nabla}$. The UV-divergent terms arise from the first three terms of the expansion.

For log-divergent terms we are only interested in the flat space part of the propagator $G(x,y) = \frac{1}{2|x-y|^2}$. Performing similar coordinate transformations as in Appendix B we introduce the regulator in the following way:

$$G_{\mathrm{Reg}}(x,y) \to \frac{1}{2(|x-y|^2+\epsilon)}. \tag{C3}$$

Using results from Appendix B and performing calculations for the log terms,

$$\begin{aligned}\langle\hat{J}_a\rangle^{(1)} &= \left(\frac{1}{4\epsilon} - \frac{\log\epsilon}{2\mathrm{r}^2}\right)\nabla_a HH^{-1}, \\ \langle\hat{J}_a\rangle^{(2)} &= -\left(\frac{1}{4\epsilon} - \frac{\log\epsilon}{2\mathrm{r}^2}\right)\nabla_a HH^{-1} \\ &\quad + \frac{\log\epsilon}{12}(-2\nabla_a\bar{\nabla}(\nabla HH^{-1}) + 3\nabla_a HH^{-1}\bar{\nabla}(\nabla HH^{-1}) \\ &\quad + \nabla\cdot\bar{\nabla}(\nabla_a HH^{-1})), \\ \langle\hat{J}_a\rangle^{(3)} &= \frac{\log\epsilon}{12}(-\nabla_a HH^{-1}\bar{\nabla}(\nabla HH^{-1}) \\ &\quad - 2\bar{\nabla}(\nabla HH^{-1})\nabla_a HH^{-1} \\ &\quad + g^{b\bar{b}}[-\nabla_b HH^{-1}, \bar{\nabla}_{\bar{b}}(\nabla_a HH^{-1})]).\end{aligned} \tag{C4}$$

[5] By dimensional analysis, terms that are at most log divergent are of the form of monomials of fields and their derivatives of scaling dimension 4, integrated over all space. So we can calculate them in the flat space limit and then promote the metric and volume element to the curved space ones to obtain the covariant expressions.

Gathering these terms together we find

$$\langle \hat{J}_a \rangle (H,1) = -\frac{\log \epsilon}{12} \mathcal{D}_a \bar{\nabla}(\nabla H H^{-1}). \tag{C5}$$

Looking back at $\delta\Gamma_1$ in Eq. (41), and given that $\langle \hat{J}(M, M^\dagger) \rangle = M^{\dagger -1} \langle \hat{J}(H,1) \rangle M^\dagger$, we find that

$$\begin{aligned} \delta\Gamma_1 &= \int \mathrm{Tr}[\delta(M^{\dagger-1}\bar{\nabla}M^\dagger)\langle \hat{J}(M,M^\dagger)\rangle + \mathrm{H.c.}] \\ &= \int \mathrm{Tr}[\bar{\nabla}(\delta M^\dagger M^{\dagger -1})\langle \hat{J}(H,1)\rangle + \mathrm{H.c.}] \\ &= \frac{\log\epsilon}{12} \int \mathrm{Tr}[\delta(\bar{\nabla}(\nabla H H^{-1}))\bar{\nabla}(\nabla H H^{-1})]. \end{aligned} \tag{C6}$$

This identifies Γ_1 as

$$\Gamma_1 = \frac{\log \epsilon}{24} \int \mathrm{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2. \tag{C7}$$

To get the divergent terms in Γ_2 the calculation scheme is similar as for Γ_1. For mass terms we use calculations as in Appendix B; for log terms we simplify calculations by treating the space as flat. Starting from Eq. (85),

$$\begin{aligned} \Gamma_2 &= \mathrm{Tr}\log[1 + (-\bar{a}\cdot\mathcal{D} + HaH^{-1}\cdot\bar{\nabla} + \bar{a}HaH^{-1})_x \mathcal{G}(x,y)] \\ &= \mathrm{Tr}\log[1 + \mathbb{Y}_x \mathcal{G}(x,y)] \\ &= \int_x \mathrm{Tr}\mathbb{Y}_x \mathcal{G}(x,y') W(y',y)\Big|_{y\to x} \\ &\quad - \frac{1}{2}\int_{x,z} \mathrm{Tr}\mathbb{Y}_x \mathcal{G}(x,z)\mathbb{Y}_z\mathcal{G}(z,x')W(x',x)\cdots \\ &= \Gamma_2^{(1)} + \Gamma_2^{(2)} + \cdots, \end{aligned} \tag{C8}$$

where $\mathbb{Y} = -\bar{a}\cdot(\nabla - \nabla H H^{-1}) + HaH^{-1}\cdot\bar{\nabla} + \bar{a}HaH^{-1}$ and $W(y',y) = \mathcal{P}\exp(\int_y^{y'} \nabla H H^{-1})$.

As above, we expand the propagator $\mathcal{G}$ in terms of $\nabla H H^{-1}$ to find the following UV-divergent terms:

$$\begin{aligned} \Gamma_2^{(1)} &= \left(\frac{1}{2\epsilon} - \frac{\log\epsilon}{2\mathrm{r}^2}\right)\int \mathrm{Tr}\bar{a}HaH^{-1} + \log\epsilon \int \mathrm{Tr}\frac{1}{4}\bar{\nabla}(\nabla H H^{-1})\bar{a}HaH^{-1}, \\ \Gamma_2^{(2)} &= \left(-\frac{1}{4\epsilon} + \frac{\log\epsilon}{2\mathrm{r}^2}\right)\int \mathrm{Tr}\bar{a}HaH^{-1} \\ &\quad + \log\epsilon \int \mathrm{Tr}\Bigg[-\frac{1}{6}\bar{\nabla}(\nabla H H^{-1})\left(\bar{a}HaH^{-1} + \frac{1}{2}HaH^{-1}\bar{a}\right) + \frac{1}{4}(\bar{a}HaH^{-1})^2 \\ &\quad - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}(\nabla_a(Ha_bH^{-1}) + [-\nabla_a H H^{-1}, Ha_bH^{-1}]) - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}}(\nabla_b H H^{-1})[\bar{a}_{\bar{b}}, Ha_aH^{-1}]\Bigg], \\ \Gamma_2^{(3)} &= \log\epsilon\int \mathrm{Tr}\Bigg[-\frac{1}{4}(\bar{a}HaH^{-1})^2 - \frac{1}{4}\bar{a}\cdot HaH^{-1}HaH^{-1}\cdot\bar{a} - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}[-Ha_aH^{-1}, Ha_bH^{-1}] \\ &\quad - \frac{1}{12} g^{a\bar{a}} g^{b\bar{b}}[\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}](\nabla_a(Ha_bH^{-1}) + [-\nabla_a H H^{-1}, Ha_bH^{-1}])\Bigg], \\ \Gamma_2^{(4)} &= \log\epsilon\int \mathrm{Tr}\left[\frac{1}{24}(\bar{a}HaH^{-1})^2 + \frac{1}{24}(HaH^{-1}\bar{a})^2 + \frac{1}{6}\bar{a}\cdot HaH^{-1}HaH^{-1}\cdot\bar{a} + \frac{1}{24} g^{a\bar{a}} g^{b\bar{b}}[\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}][Ha_aH^{-1}, Ha_bH^{-1}]\right]. \end{aligned} \tag{C9}$$

Combining the four terms together we get

$$\begin{aligned} \Gamma_2 &= \frac{1}{4\epsilon}\int \mathrm{Tr}\bar{a}HaH^{-1} + \frac{\log\epsilon}{24}\int \mathrm{Tr}[2\bar{\nabla}(\nabla H H^{-1})[\bar{a}, HaH^{-1}] + [\bar{a}, HaH^{-1}]^2 \\ &\quad - 2g^{a\bar{a}} g^{b\bar{b}}(\bar{\nabla}_{\bar{a}}(\nabla_b H H^{-1})[\bar{a}_{\bar{b}}, Ha_aH^{-1}] + \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(Ha_bH^{-1})) \\ &\quad - 2g^{a\bar{a}} g^{b\bar{b}}(\mathcal{D}_a(Ha_bH^{-1})[\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}] - \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}[Ha_aH^{-1}, Ha_bH^{-1}]) + g^{a\bar{a}} g^{b\bar{b}}[Ha_aH^{-1}, Ha_bH^{-1}][\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}]]. \end{aligned} \tag{C10}$$

Combining Γ_1 and Γ_2 from (C7) and (C10) above, the ultraviolet-divergent terms are

$$\begin{aligned} \Gamma_{\mathrm{div}} &= \frac{1}{4\epsilon}\int \mathrm{Tr}\bar{a}HaH^{-1} + \frac{\log\epsilon}{24}\int \mathrm{Tr}[(\bar{\nabla}\cdot(\nabla H H^{-1}) + [\bar{a}, HaH^{-1}])^2 \\ &\quad - 2g^{a\bar{a}} g^{b\bar{b}}(\bar{\nabla}_{\bar{a}}(\nabla_b H H^{-1})[\bar{a}_{\bar{b}}, Ha_aH^{-1}] + \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}\mathcal{D}_a(Ha_bH^{-1})) \\ &\quad - 2g^{a\bar{a}} g^{b\bar{b}}(\mathcal{D}_a(Ha_bH^{-1})[\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}] - \bar{\nabla}_{\bar{a}}\bar{a}_{\bar{b}}[Ha_aH^{-1}, Ha_bH^{-1}]) + g^{a\bar{a}} g^{b\bar{b}}[Ha_aH^{-1}, Ha_bH^{-1}][\bar{a}_{\bar{a}}, \bar{a}_{\bar{b}}]], \end{aligned} \tag{C11}$$

which is the result in (86) and (88).

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