Hartree-Fock approach to dynamical mass generation in the generalized (2 + 1)-dimensional Thirring model

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The $(2+1)$-dimensional generalized massless Thirring model with four-component Fermi fields is investigated by the Hartree-Fock method. The Lagrangian of this model is constructed from two different four-fermion structures. One of them takes into account the vector × vector channel of fermion interaction with coupling constant G_v, the other, the scalar × scalar channel with coupling G_s. At some relation between bare couplings G_s and G_v, the Hartree-Fock equation for self-energy of fermions can be renormalized, and dynamical generation of the Dirac and Haldane fermion masses is possible. As a result, a phase portrait of the model consists of two nontrivial phases. In the first one the chiral symmetry is spontaneously broken due to dynamical appearance of the Dirac mass term, while in the second phase a spontaneous breaking of the spatial parity $\mathcal{P}$ is induced by a Haldane mass term. It is shown that in the particular case of a pure Thirring model, i.e., at $G_s = 0$, the ground state of the system is indeed a mixture of these phases. Moreover, it was found that dynamical generation of fermion masses is possible for any finite number of Fermi fields.

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I. INTRODUCTION

Over the past few decades, in quantum field theory, much attention has been paid to the study of various models in $(2+1)$-dimensional spacetime. This is due to the fact that in condensed matter physics there exist quite a few phenomena (quantum Hall effect, high-temperature superconductivity, physical processes in graphene, etc.) having a planar nature, and for their effective description it is convenient to use relativistic $(2+1)$-dimensional (D) models with four-fermion interaction. Among them is the Gross-Neveu model [1–7], Thirring model [8–21] etc. Note that both models are renormalizable, at least in the framework of the large-N technique [22–24] (N is the number of fermion fields). At the same time, various nonperturbative approaches (such as $1/N$ expansion method, Gaussian effective potential, optimized expansion techniques [22,25,26], etc.) to the study of the massless $(2+1)$-D Gross-Neveu model predict its qualitatively identical properties (dynamic generation of fermion masses, spontaneous chiral symmetry breaking, etc.). However, using different methods for studying the massless $(2+1)$-D Thirring model built on the basis of a *four-component* reducible spinor representation for fermion fields leads to conflicting results. Indeed, in a number of papers (see, for example, [9,10]), this model was investigated by the $1/N$ expansion method, where it was shown that only the Dirac fermion mass that breaks chiral symmetry can be generated (in this case, the spatial parity $\mathcal{P}$ remains unbroken). In contrast, an application of other research methods [12,13] to the same Thirring model gives the opposite result, because in these papers, the possibility of dynamical generation of a $\mathcal{P}$-odd (but chirally invariant) Haldane fermion mass was established. In addition, in the literature there is also a discrepancy in the predictions of the number of fermion fields N, with which dynamic mass generation is possible. Thus, for example, in the first of papers [9,10] this effect is predicted for any finite value of N, and in the second one only for $N < N_c = 128/3\pi^2$, and so on. [More details about the inconsistency of the results of the study of the $(2+1)$-D Thirring model by various nonperturbative methods can be found, for example, in Refs. [17–20].]

The discrepancy in the results points to the need for further and more thorough study of the $(2+1)$-dimensional Thirring model both within the framework of well-known methods, as well as by attracting new approaches.

To this end, in the present paper we use the so-called Hartree-Fock (HF) approach to investigate the possibility of dynamical fermion mass generation in the (generalized) $(2+1)$-dimensional Thirring model with four-component spinors. Earlier in Refs. [27,28] it was used to study the $(2+1)$-D Gross-Neveu model. Wherein it turned out that in the region of large N the HF method predicts the same properties as the nonperturbative $1/N$-expansion method widely used to study this model. In the region of small N, where the $1/N$-expansion method is not applicable, the HF approach predicts the existence of other nontrivial phases of the three-dimensional Gross-Neveu model, including the spontaneously non-Hermitian phase of the model [28]. The essence of the HF method consists, firstly, in using the Cornwall-Jackiw-Tomboulis (CJT) effective action for composite operators $\Gamma(S)$ [29] in field theory models with four-fermion interaction (here S denotes the full fermion propagator satisfying the stationarity equation $\delta\Gamma/\delta S = 0$), and, secondly, that $\Gamma(S)$ is considered in the first order in the coupling constants. The resulting stationary equation takes the form of the well-known Hartree–Fock equation for a fermion self-energy operator [30,31]. (This is the reason why we call this approach the HF method.) It should be especially noted that when studying the Thirring model by the HF method, auxiliary vector fields are not used at all, as it is usually practiced in most of the earlier approaches to the model, and due to which the mechanism of fermion mass generation in this model is more similar to the one found in $(2+1)$-D quantum electrodynamics (see, e.g., in Refs. [8,10]).

In our work, based on the HF approach, we explore the properties of not only the pure massless $(2+1)$-D Thirring model (with single vector $\times$ vector coupling G_v) composed of N reducible four-component spinors, but also a more general model, invariant under the same continuous symmetry group, in which the Lagrangian contains an additional scalar $\times$ scalar fermion interaction term with coupling constant G_s. We show that, depending on the relationship between G_v and G_s, the ground state of the generalized massless $(2+1)$-D Thirring model corresponds to either a chirally broken phase or phase in which fermions have a parity $\mathcal{P}$ violating mass. In contrast, at $G_s = 0$ in the ground state of the pure $(2+1)$-dimensional massless Thirring model these phases can coexist. Moreover, it is clear from our HF consideration that dynamical generation of fermion mass is allowed to occur at any finite value of N.

The paper is organized as follows. In Sec. II A we present the N-flavor massless $(2+1)$-D generalized Thirring model symmetric under discrete chiral and spatial $\mathcal{P}$ reflections. Here it is also shown that model is invariant under continuous $U(2N)$ group, and two different fermion mass terms, Dirac and Haldane, are defined. In Sec. II B the CJT effective action $\Gamma(S)$ of the composite bilocal and bifermion operator $\bar{\psi}(x)\psi(y)$ is constructed, which is actually the functional of the full fermion propagator $S(x,y)$. Then, the unrenormalized expression for $\Gamma(S)$ is obtained up to a first order in the bare coupling constants $G_{s,v}$ (it is the so-called Hartree-Fock approximation). Based on this expression, we show in Sec. III that for some well-defined behavior of the bare coupling constants $G_{s,v}(\Lambda)$ vs cutoff parameter Λ, there exist two different renormalized, i.e., without ultraviolet divergences, solutions of the stationary Hartree-Fock equation for the propagator. One of them corresponds to a phase in which the Haldane fermion mass term arises dynamically, and parity $\mathcal{P}$ is spontaneously broken down. Another solution of the HF equation corresponds to a chiral symmetry breaking phase with dynamically emerging Dirac mass term. Finally, in Sec. IV we use the renormalization group formalism and show that in the plane of dimensionless coupling constants there is at least one ultraviolet-stable fixed point of the model. Appendix A contains some information about two- and four-dimensional spinor representations of the $SO(2,1)$ group, whereas Appendix B gives all details of calculating the effective action $\Gamma(S)$ in the HF approximation.

II. (2+1)-DIMENSIONAL GENERALIZED THIRRING MODEL AND HARTREE-FOCK APPROACH

A. The model, its symmetries, and so on

The Lagrangian of the generalized massless and N-flavored $(2+1)$-D Thirring model under consideration has the following form (see, e.g., Refs. [16,32]):

$$L = \overline{\Psi}_k\gamma^\nu i\partial_\nu\Psi_k - \frac{G_v}{2N}(\overline{\Psi}_k\gamma^\mu\Psi_k)(\overline{\Psi}_l\gamma_\mu\Psi_l) + \frac{G_s}{2N}(\overline{\Psi}_k\tau\Psi_k)^2, \tag{1}$$

where for each $k = 1, \ldots, N$ the field $\Psi_k \equiv \Psi_k(t,x,y)$ is a (reducible) four-component Dirac spinor [its spinor indices are omitted in Eq. (1)], γ^ν ($\nu = 0,1,2$) are 4×4 matrices acting in the four-dimensional spinor space (the algebra of these γ matrices and their particular representation used in the present paper is given in Appendix A, where in addition the matrices γ^3, γ^5, and $\tau = -i\gamma^3\gamma^5$ are also introduced), and the summation over repeated k, l and μ, ν indices is assumed in Eq. (1) and below. The bare coupling constants G_v and G_s have a dimension of $[\text{mass}]^{-1}$. As discussed, e.g., in Refs. [16,32], at $N = 2$ the model (1) provides a fairly good description of the low-energy physics of graphene in the continuum limit. But we consider the N-flavor variant of the model in order to compare its phase structure obtained in the framework of the HF effective approach with the results of the large-N investigation [9,10].

Together, all four-component spinor fields Ψ_k ($k = 1, \ldots, N$) form a fundamental multiplet of the $U(N)$ group, so the invariance of the Lagrangian (1) with respect to this group is obvious [and in the following the $U(N)$ symmetry of the model remains unbroken]. It is not so obvious that, in reality, the continuous symmetry group of the three-dimensional generalized Thirring model is wider and is $U(2N)$. This fact can be easily established if we rewrite the expression (1) in terms of two-component spinors. Namely, for each fixed $k = 1, \ldots, N$ we set $\Psi_k^t = (\psi_{2k-1}^t, \psi_{2k}^t)$, where the superscript t means the transposition operation, and ψ_{2k-1} and ψ_{2k} are two-component spinors (see Appendix A). Then we have

$$\begin{aligned} L_0 \equiv \overline{\Psi}_k \gamma^\nu i\partial_\nu \Psi_k &= \overline{\psi}_1 \tilde{\gamma}^\nu i\partial_\nu \psi_1 + \overline{\psi}_2 \tilde{\gamma}^\nu i\partial_\nu \psi_2 + \cdots + \overline{\psi}_{2N} \tilde{\gamma}^\nu i\partial_\nu \psi_{2N}, \\ \overline{\Psi}_k \gamma^\nu \Psi_k &= \overline{\psi}_1 \tilde{\gamma}^\nu \psi_1 + \overline{\psi}_2 \tilde{\gamma}^\nu \psi_2 + \cdots + \overline{\psi}_{2N} \tilde{\gamma}^\nu \psi_{2N}, \\ \overline{\Psi}_k \tau \Psi_k &= \overline{\psi}_1 \psi_1 + \overline{\psi}_2 \psi_2 + \cdots + \overline{\psi}_{2N} \psi_{2N}, \end{aligned} \tag{2}$$

where $\tilde{\gamma}^\nu$ are 2×2 matrices (see Appendix A). Assuming formally that the set of all two-component spinors ψ_{2k-1} and ψ_{2k} ($k = 1, \ldots, N$) forms a fundamental representation of the $U(2N)$ group, it is easy to see that both structures (2) and the entire Lagrangian (1) are invariant under this group.

More important for us is that the Lagrangian (1) is invariant under three discrete transformations; two of them are the so-called chiral transformations Γ^5 and Γ^3,

$$\begin{aligned} \Gamma^5:\ \Psi_k(t,x,y) &\to \gamma^5 \Psi_k(t,x,y); \qquad \overline{\Psi}_k(t,x,y) \to -\overline{\Psi}_k(t,x,y)\gamma^5, \\ \Gamma^3:\ \Psi_k(t,x,y) &\to \gamma^3 \Psi_k(t,x,y); \qquad \overline{\Psi}_k(t,x,y) \to -\overline{\Psi}_k(t,x,y)\gamma^3. \end{aligned} \tag{3}$$

The next one is the space reflection, or parity, transformation $\mathcal{P}$ under which $(t,x,y) \to (t,-x,y)$ and[1]

$$\begin{aligned} \mathcal{P}:\ \Psi_k(t,x,y) &\to \gamma^5\gamma^1 \Psi_k(t,-x,y); \\ \overline{\Psi}_k(t,x,y) &\to \overline{\Psi}_k(t,-x,y)\gamma^5\gamma^1. \end{aligned} \tag{4}$$

Due to the symmetry of the model (1) with respect to each of the discrete Γ^5, Γ^3, and $\mathcal{P}$ transformations, different mass terms are prohibited to appear perturbatively in this Lagrangian. Indeed, the most popular Dirac mass term has the form $m_D \overline{\Psi}_k \Psi_k = m_D(\overline{\psi}_{2k-1}\psi_{2k-1} - \overline{\psi}_{2k}\psi_{2k})$, but it breaks both $U(2N)$ and chiral Γ^5 and Γ^3 symmetries of the model, although it is $\mathcal{P}$ even. There is another well-known fermionic mass term that is often discussed in the literature. This is a mass term of the form $m_H \overline{\Psi}_k \tau \Psi_k = m_H(\overline{\psi}_{2k-1}\psi_{2k-1} + \overline{\psi}_{2k}\psi_{2k})$ (recall, here the 4×4 matrix τ is defined in Appendix A) and sometimes it is referred to as the Haldane mass term (see, e.g., Refs. [4,21]).[2] But nonzero Haldane mass m_H breaks the parity $\mathcal{P}$ invariance of the model [although it is $U(2N)$ invariant and chirally Γ^5 and Γ^3 symmetric]. So both Dirac and Haldane mass terms cannot appear in the model (1) when it is studied by the usual perturbative technique. However, within a framework of nonperturbative approximations (for example, in the $1/N$ expansion, etc.), fermion mass can arise dynamically, thereby breaking the original symmetry in a spontaneous way.

In our paper, we continue the investigation of $(2+1)$-D models with four-fermion interactions by the so-called HF method, which was started in our papers [27,28]. This time we use it to explore the possibility of dynamical mass generation within the framework of the generalized Thirring model (1). Note that theoretical ground of the HF method is the effective Cornwall-Jakiw-Tomboulis action for composite operators [29], which also provides a systematic way to go beyond the HF approximation.

B. From CJT to Hartree-Fock approach

Let us define $Z(K)$, the generating functional of the Green's functions of bilocal fermion-antifermion composite operators $\sum_{k=1}^N \bar{\Psi}_k^\alpha(x)\Psi_{k\beta}(y)$ in the framework of the $(2+1)$-D Thirring model (1) (the corresponding technique for theories with four-fermion interaction is elaborated in details, e.g., in Ref. [37])

[1] In $2+1$ dimensions, parity corresponds to inverting only one spatial axis [1,33], since the inversion of both axes is equivalent to rotating the entire space by π.

[2] The appearance of the Haldane mass term is related to the parity anomaly in $(2+1)$ dimensions, to generation of the Chern-Simons topological mass of gauge fields [34,35], as well as to the integer quantum Hall effect in planar condensed matter systems without external magnetic field, etc. [36].

$$Z(K) \equiv \exp(iNW(K)) = \int \mathcal{D}\bar{\Psi}_k \mathcal{D}\Psi_k \exp\left(i\left[I(\overline{\Psi},\Psi) + \int d^3x d^3y \bar{\Psi}_k^\alpha(x) K_\alpha^\beta(x,y)\Psi_{k\beta}(y)\right]\right), \tag{5}$$

where $\alpha, \beta = 1, 2, 3, 4$ are spinor indices, $K_\alpha^\beta(x,y)$ is a bilocal source of the fermion bilinear composite field $\bar{\Psi}_k^\alpha(x)\Psi_{k\beta}(y)$ (recall that in all expressions the summation over repeated indices is assumed).[3] Moreover, $I(\bar{\Psi},\Psi) = \int L d^3x$, where L is the Lagrangian (1) of the model under consideration. Hence,

$$I(\overline{\Psi},\Psi) = \int d^3x d^3y \overline{\Psi}_k^\alpha(x) D_\alpha^\beta(x,y)\Psi_{k\beta}(y) + I_{\rm int}(\overline{\Psi}_k^\alpha \Psi_{k\beta}), \qquad D_\alpha^\beta(x,y) = (\gamma^\nu)_\alpha^\beta i\partial_\nu \delta^3(x-y), \qquad I_{\rm int} = I_v + I_s,$$

$$\begin{aligned} I_v &= -\frac{G_v}{2N}\int d^3x (\overline{\Psi}_k \gamma^\mu \Psi_k)(\overline{\Psi}_l \gamma_\mu \Psi_l) \\ &= -\frac{G_v}{2N}\int d^3x d^3t d^3u d^3v \delta^3(x-t)\delta^3(t-u)\delta^3(u-v)\overline{\Psi}_k^\alpha(x)(\gamma^\mu)_\alpha^\beta \Psi_{k\beta}(t)\overline{\Psi}_l^\rho(u)(\gamma_\mu)_\rho^\xi \Psi_{l\xi}(v), \\ I_s &= \frac{G_s}{2N}\int d^3x (\bar{\Psi}_k \tau \Psi_k)(\bar{\Psi}_l \tau \Psi_l) \\ &= \frac{G_s}{2N}\int d^3x d^3t d^3u d^3v \delta^3(x-t)\delta^3(t-u)\delta^3(u-v)\bar{\Psi}_k^\alpha(x)(\tau)_\alpha^\beta \Psi_{k\beta}(t)\bar{\Psi}_l^\rho(u)(\tau)_\rho^\xi \Psi_{l\xi}(v). \end{aligned} \tag{6}$$

Note that in Eq. (6) and similar expressions below, $\delta^3(x-y)$ denotes the three-dimensional Dirac delta function. There is an alternative expression for $Z(K)$,

$$\begin{aligned} Z(K) &= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)\int \mathcal{D}\bar{\Psi}_k \mathcal{D}\Psi_k \exp\left(i\int d^3x d^3y \overline{\Psi}_k(x)[D(x,y)+K(x,y)]\Psi_k(y)\right) \\ &= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)[\det(D(x,y)+K(x,y))]^N \\ &= \exp\left(iI_{\rm int}\left(-i\frac{\delta}{\delta K}\right)\right)\exp[N{\rm Tr}\ \ln(D(x,y)+K(x,y))], \end{aligned} \tag{7}$$

where instead of each bilinear form $\overline{\Psi}_k^\alpha(s)\Psi_{k\beta}(t)$ appearing in $I_{\rm int}$ of Eq. (6) we use a variational derivative $-i\delta/\delta K_\alpha^\beta(s,t)$. Moreover, the Tr operation in Eq. (7) means the trace both over spacetime and spinor coordinates. The effective action (or CJT effective action) of the composite bilocal and bispinor operator $\overline{\Psi}_k^\alpha(x)\Psi_{k\beta}(y)$ is defined as a functional $\Gamma(S)$ of the full fermion propagator $S_\beta^\alpha(x,y)$ by a Legendre transformation of the functional $W(K)$ entering in Eq. (5),

$$\Gamma(S) = W(K) - \int d^3x d^3y S_\beta^\alpha(x,y) K_\alpha^\beta(y,x), \tag{8}$$

where

$$S_\beta^\alpha(x,y) = \frac{\delta W(K)}{\delta K_\alpha^\beta(y,x)}. \tag{9}$$

[3]We denote a matrix element of an arbitrary matrix (operator) $\hat{A}$ acting in the four-dimensional spinor space by the symbol A_β^α, where the upper (lower) index $\alpha(\beta)$ is the column (row) number of the matrix $\hat{A}$. In particular, the matrix elements of any γ^μ matrix is denoted by $(\gamma^\mu)_\beta^\alpha$.

Taking into account the relation (5), it is clear that $S(x,y)$ is the full fermion propagator at $K(x,y) = 0$. Hence, in order to construct the CJT effective action $\Gamma(S)$ of Eq. (8), it is necessary to solve Eq. (9) with respect to K and then to use the obtained expression for K (in fact, it is a functional of S) in Eq. (8). It follows from the definition (8)–(9) that

$$\begin{aligned} \frac{\delta\Gamma(S)}{\delta S_\beta^\alpha(x,y)} &= \int d^3u d^3v \frac{\delta W(K)}{\delta K_\nu^\mu(u,v)}\frac{\delta K_\nu^\mu(u,v)}{\delta S_\beta^\alpha(x,y)} - K_\alpha^\beta(y,x) \\ &\quad - \int d^3u d^3v S_\mu^\nu(v,u)\frac{\delta K_\nu^\mu(u,v)}{\delta S_\beta^\alpha(x,y)}. \end{aligned} \tag{10}$$

[In Eq. (10) and below, the Greek letters α, β, μ, ν, etc., also denote the spinor indices, i.e., $\alpha, \ldots \nu, \ldots = 1, \ldots, 4$.] Now, due to the relation (9), it is easy to see that the first term in Eq. (10) cancels there the last term, so

$$\frac{\delta\Gamma(S)}{\delta S_\beta^\alpha(x,y)} = -K_\alpha^\beta(y,x). \tag{11}$$

Hence, in the true theory, in which bilocal sources $K_\alpha^\beta(y,x)$ are zero, the full fermion propagator is a solution of the following stationary equation:

$$\frac{\delta\Gamma(S)}{\delta S^{\alpha}_{\beta}(x,y)} = 0. \qquad (12)$$

Note that in the nonperturbative CJT approach the stationary/gap equation (12) for fermion propagator $S^{\beta}_{\alpha}(x,y)$ is indeed a Schwinger-Dyson equation [37]. Further, in order to simplify the calculations and obtain more detailed information about the phase structure of the model, we calculate both the effective action (8) and the gap equation (12) up to a first order in the couplings G_v and G_s.

We call such an approach to studying the properties of any model with four-fermion interactions [including the generalized Thirring model (1)] the Hartree-Fock method (a more detailed justification for this name is given at the end of this section).

In this case (see Appendix B)

$$\Gamma(S) = i\mathrm{Tr}\,\ln(iS) + \int d^3x d^3y S^{\alpha}_{\beta}(x,y) D^{\beta}_{\alpha}(y,x) - \frac{G_v}{2}\int d^3s\,\mathrm{tr}[\gamma^{\rho}S(s,s)]\mathrm{tr}[\gamma_{\rho}S(s,s)] \\ + \frac{G_v}{2N}\int d^3s\,\mathrm{tr}[\gamma^{\rho}S(s,s)\gamma_{\rho}S(s,s)] + \frac{G_s}{2}\int d^3s(\mathrm{tr}[\tau S(s,s)])^2 - \frac{G_s}{2N}\int d^3s\,\mathrm{tr}[\tau S(s,s)\tau S(s,s)]. \qquad (13)$$

Notice that in Eq. (13) the symbol tr means the trace of an operator over spinor indices only, but the symbol Tr is still the trace operation both over spacetime coordinates and spinor indices. Moreover, the expression for operator $D(x,y)$ is presented in Eq. (6). The stationary equation (12) for the CJT effective action (13) looks like[4]

$$-i[S^{-1}]^{\beta}_{\alpha}(x,y) - D^{\beta}_{\alpha}(x,y) = G_s\tau^{\beta}_{\alpha}\mathrm{tr}[\tau S(x,y)]\delta^3(x-y) - G_v(\gamma^{\rho})^{\beta}_{\alpha}\mathrm{tr}[\gamma_{\rho}S(x,y)]\delta^3(x-y) \\ - \frac{G_s}{N}[\tau S(x,y)\tau]^{\beta}_{\alpha}\delta^3(x-y) + \frac{G_v}{N}[\gamma^{\rho}S(x,y)\gamma_{\rho}]^{\beta}_{\alpha}\delta^3(x-y). \qquad (14)$$

Now suppose that $S(x,y)$ is a translationary invariant operator. Then

$$S^{\beta}_{\alpha}(x,y) \equiv S^{\beta}_{\alpha}(z) = \int\frac{d^3p}{(2\pi)^3}\overline{S^{\beta}_{\alpha}}(p)e^{-ipz}, \qquad \overline{S^{\beta}_{\alpha}}(p) = \int d^3z S^{\beta}_{\alpha}(z)e^{ipz}, \\ (S^{-1})^{\beta}_{\alpha}(x,y) \equiv (S^{-1})^{\beta}_{\alpha}(z) = \int\frac{d^3p}{(2\pi)^3}\overline{(S^{-1})^{\beta}_{\alpha}}(p)e^{-ipz}, \qquad (15)$$

where $z = x-y$ and $\overline{S^{\beta}_{\alpha}}(p)$ is a Fourier transformation of $S^{\beta}_{\alpha}(z)$. After Fourier transformation Eq. (14) takes the form

$$-i\overline{(S^{-1})^{\beta}_{\alpha}}(p) - (\hat{p})^{\beta}_{\alpha} = G_s\tau^{\beta}_{\alpha}\int\frac{d^3q}{(2\pi)^3}\mathrm{tr}[\tau\bar{S}(q)] - G_v(\gamma^{\rho})^{\beta}_{\alpha}\int\frac{d^3q}{(2\pi)^3}\mathrm{tr}[\gamma_{\rho}\bar{S}(q)] \\ - \frac{G_s}{N}\int\frac{d^3q}{(2\pi)^3}[\tau\bar{S}(q)\tau]^{\beta}_{\alpha} + \frac{G_v}{N}\int\frac{d^3q}{(2\pi)^3}[\gamma^{\rho}\bar{S}(q)\gamma_{\rho}]^{\beta}_{\alpha}, \qquad (16)$$

where $\hat{p} = p_{\nu}\gamma^{\nu}$. It is clear from Eq. (16) that in the framework of the four-fermion model (1) the Schwinger-Dyson equation for fermion propagator $\bar{S}(p)$ reads in the first order in $G_{s,v}$ like the Hartree-Fock equation for its self-energy operator $\Sigma(p)$ (the last quantity is nothing but the expression on the left side of this equation). As a result, we will henceforth refer to Eq. (16) as the Hartree-Fock equation. In particular, the first two terms on the right-hand side of Eq. (16) are the so-called Hartree contribution, whereas the last two terms there are the Fock contribution to fermion self-energy (for details, see, e.g., Sec. 4.3.1 in Ref. [30] or Sec. II C in Ref. [31]).

Finally, note that both the CJT (or HF) effective action (13) and its stationary equations (14)–(16), in which $G_{s,v}$ are bare coupling constants, contain ultraviolet (UV) divergences and need to be renormalized. In the next sections, using a rather general ansatz for propagator $\bar{S}(p)$, we find the corresponding modes of the coupling constants $G_{s,v}$ behavior vs cutoff parameter Λ, such that there occurs a renormalization of the gap Hartree-Fock equation (16), and it is possible to obtain its finite solution in the limit $\Lambda \to \infty$.

[4]The first term on the left-hand side of Eq. (14) can be easily obtained using Eq. (B2) from Appendix B.

III. POSSIBILITY FOR DYNAMICAL GENERATION OF THE DIRAC AND HALDANE MASSES

Let us study on the basis of the HF equation (16) the possibility for dynamical generation of the Hermitian mass term $\overline{\Psi}_k(m_D + m_H\tau)\Psi_k$ in the massless (2 + 1)-D Thirring model (1). It means that we should find the solution $\bar{S}(p)$ of this equation, which looks like

$$\bar{S}(p) = -i(\hat{p} + m_D + m_H\tau)^{-1} = -i\begin{pmatrix} \tilde{p} + m_D + m_H, & 0 \\ 0, & -\tilde{p} + m_D - m_H \end{pmatrix}^{-1}$$
$$= -i\begin{pmatrix} (\tilde{p} + m_D + m_H)^{-1}, & 0 \\ 0, & (-\tilde{p} + m_D - m_H)^{-1} \end{pmatrix} = -i\begin{pmatrix} \frac{\tilde{p}-m_D-m_H}{p^2-(m_D+m_H)^2}, & 0 \\ 0, & \frac{-\tilde{p}-m_D+m_H}{p^2-(m_D-m_H)^2} \end{pmatrix}, \tag{17}$$

where m_D and m_H are finite unknown quantities, and in Eq. (17) the 4×4 matrix $\bar{S}(p)$ is presented in the form of a 2×2 matrix each element of which is, in tern, a 2×2 matrix. Moreover, there $\tilde{p} = \tilde{\gamma}^\nu p_\nu$, where $\tilde{\gamma}^\nu$ ($\nu = 0, 1, 2$) are 2×2 Dirac gamma matrices (see Appendix A). It is evident that in this case $\overline{S^{-1}}(p) = i(\hat{p} + m_D + \tau m_H)$. Using Eq. (17) in the HF gap equation (16), we obtain for the quantities m_D and m_H the following *unrenormalized* system of gap equations

$$m_D = \left(\frac{3iG_v}{2N} - \frac{iG_s}{2N}\right)\int\frac{d^3p}{(2\pi)^3}\left\{\frac{m_D + m_H}{p^2 - (m_D + m_H)^2} + \frac{m_D - m_H}{p^2 - (m_D - m_H)^2}\right\},$$
$$m_H = \left(2iG_s - \frac{iG_s}{2N} + \frac{3iG_v}{2N}\right)\int\frac{d^3p}{(2\pi)^3}\left\{\frac{m_D + m_H}{p^2 - (m_D + m_H)^2} - \frac{m_D - m_H}{p^2 - (m_D - m_H)^2}\right\}. \tag{18}$$

Performing in the integrals of Eq. (18) a Wick rotation, $p_0 \to ip_3$, and then using in the obtained three-dimensional Euclidean integration space the spherical coordinate system, $p_3 = p\cos\theta$, $p_1 = p\sin\theta\cos\phi$, $p_2 = p\sin\theta\sin\phi$, we have (after integration over angles, $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$, and cutting off the region of integration of the variable p, $0 \le p \le \Lambda$) the following *regularized* gap system:

$$m_D = \left(\frac{3G_v}{2N} - \frac{G_s}{2N}\right)\int_0^\Lambda\frac{p^2dp}{2\pi^2}\left\{\frac{m_D + m_H}{p^2 + (m_D + m_H)^2} + \frac{m_D - m_H}{p^2 + (m_D - m_H)^2}\right\},$$
$$m_H = \left(2G_s - \frac{G_s}{2N} + \frac{3G_v}{2N}\right)\int_0^\Lambda\frac{p^2dp}{2\pi^2}\left\{\frac{m_D + m_H}{p^2 + (m_D + m_H)^2} - \frac{m_D - m_H}{p^2 + (m_D - m_H)^2}\right\}. \tag{19}$$

Since

$$\int_0^\Lambda\frac{p^2}{p^2 + M^2}dp = \Lambda - \frac{\pi}{2}|M| + M\mathcal{O}\left(\frac{M}{\Lambda}\right), \tag{20}$$

the Eqs. (19) can be presented in the following asymptotic forms:

$$\frac{m_D}{A} = 2m_D\Lambda - \frac{\pi}{2}[(m_D + m_H)|m_D + m_H| + (m_D - m_H)|m_D - m_H|] + m_D\mathcal{O}\left(\frac{m_D}{\Lambda}\right),$$
$$\frac{m_H}{B} = 2m_H\Lambda - \frac{\pi}{2}[(m_D + m_H)|m_D + m_H| - (m_D - m_H)|m_D - m_H|] + m_H\mathcal{O}\left(\frac{m_H}{\Lambda}\right), \tag{21}$$

where

$$A = \frac{3G_v}{4N\pi^2} - \frac{G_s}{4N\pi^2}, \qquad B = \frac{G_s}{\pi^2} - \frac{G_s}{4N\pi^2} + \frac{3G_v}{4N\pi^2}. \tag{22}$$

To remove the UV divergences from Eqs. (21), we suppose that bare quantities $A \equiv A(\Lambda)$ and $B \equiv B(\Lambda)$ are such that

$$\frac{1}{A(\Lambda)} = 2\Lambda + \frac{\pi}{2} g_A + g_A \mathcal{O}\left(\frac{g_A}{\Lambda}\right),$$
$$\frac{1}{B(\Lambda)} = 2\Lambda + \frac{\pi}{2} g_B + g_B \mathcal{O}\left(\frac{g_B}{\Lambda}\right), \qquad (23)$$

where g_A and g_B are some finite Λ-independent and renormalization group invariant quantities with dimension of mass. In this case, at $\Lambda \to \infty$ the system of stationary equations (21) acquire the following *renormalized* form:

$$m_D g_A + (m_D + m_H)|m_D + m_H| + (m_D - m_H)|m_D - m_H| = 0,$$
$$m_H g_B + (m_D + m_H)|m_D + m_H| - (m_D - m_H)|m_D - m_H| = 0. \qquad (24)$$

Moreover, it is clear from Eq. (23) that at sufficiently large values of Λ

$$A(\Lambda) = \frac{1}{2\Lambda}\left(1 - \frac{\pi}{4\Lambda} g_A + \cdots\right),$$
$$B(\Lambda) = \frac{1}{2\Lambda}\left(1 - \frac{\pi}{4\Lambda} g_B + \cdots\right). \qquad (25)$$

So, taking into account the relations (22), we have for the bare constants $G_{s,v}$ the following asymptotic expansions at $\Lambda \to \infty$:

$$G_s \equiv G_s(\Lambda) = \pi^2(B - A) = \frac{\pi^3}{8\Lambda^2}(g_A - g_B) + \cdots,$$
$$G_v \equiv G_v(\Lambda) = \frac{\pi^2}{3}(B - A) + \frac{4\pi^2 N}{3} A$$
$$= \frac{2\pi^2 N}{3\Lambda} - \frac{\pi^3}{24\Lambda^2}[(4N - 1)g_A + g_B] + \cdots. \qquad (26)$$

As a rule, the stationary equation (12) has several solutions. To find which one is more preferable, it is necessary to consider the so-called CJT effective potential $V(S)$ of the model which is determined on the basis of the CJT effective action (8) by the following relation [29]:

$$V(S) \int d^3x \equiv -\Gamma(S)|_{\text{transl.-inv.}\, S(x,y)}, \qquad (27)$$

and that solution S of the stationarity equation (12), on which the effective potential $V(S)$ takes the least value, will correspond to the true fermion propagator $S(x, y)$ of the model. To find CJT effective potential $V(S)$ in the Hartree-Fock approximation, we should use in Eq. (27) the expressions (13) and (17) for CJT effective action $\Gamma(S)$ and for the full fermion propagator $S(x, y)$, respectively. But in this case the obtained expression for $V(S)$ contains UV divergences. However, they are eliminated if bare couplings $G_{s,v}$ are constrained by relations (26). As a result, in the Hartree-Fock approximation we obtain for the CJT effective potential $V(S) \equiv V(m_H, m_D)$ the following *renormalized* expression

$$V(m_D, m_H) = \frac{1}{12\pi}(3g_A m_D^2 + 3g_B m_H^2 + 2|m_D + m_H|^3 + 2|m_D - m_H|^3) \qquad (28)$$

(this expression is valid up toan unessential m_D, m_H-independent infinite constant). Note that the HF gap equations (24) are also the stationary equations for the effective potential (28). Now it is clear that the form of the global minimum point (GMP) of the function $V(m_D, m_H)$ determines the phase structure of the generalized Thirring model (1) when coupling constants $G_{s,v}$ are constrained by the conditions (26).

Let us study the behavior of the GMP of the function $V(m_D, m_H)$ (28) vs finite couplings $g_{A,B}$. First, note that this function is symmetric under the transformations $m_D \to -m_D$ and/or $m_H \to -m_H$. So, for simplicity, it is enough to look for its GMP only in the region $m_D, m_H \geq 0$. Second, it is evident that at $g_A, g_B \geq 0$ the GMP of $V(m_D, m_H)$ lies at the point $(m_D = 0, m_H = 0)$, which means that no fermion masses are generated in this region, and symmetry remains intact. In other regions for g_A and g_B, it is easy to find the following form of the GMP of the function $V(m_D, m_H)$ (28):

The region $g_B < 0$, $g_A > g_B$.—In this case the system of gap equations (24) has a nontrivial solution of the form $(m_D = 0, m_H = -g_B/2)$, which corresponds to the free-energy density equal to $V(m_D = 0, m_H = -g_B/2) = \frac{1}{48\pi} g_B^3 < 0$, and this quantity is smaller than the value of the CJT effective potential (28) at another, trivial solution $(m_D = 0, m_H = 0)$ of the gap equations (24). So, in the region under consideration only the Haldane mass term can be generated dynamically, and parity $\mathcal{P}$ (4) is broken spontaneously.

The region $g_A < 0$, $g_B > g_A$.—In this case the GMP of $V(m_D, m_H)$ is arranged at the point $(m_D = -g_A/2$, $m_H = 0)$. Hence, in this g_A, g_B region only the Dirac mass is allowed to be generated and the phase with spontaneous breaking both of the chiral [see in Eq. (3)] and $U(2N)$ symmetries is realized (parity $\mathcal{P}$ is conserved). The density of free energy in this phase is equal to $V(m_D = -g_A/2$, $m_H = 0) = \frac{1}{48\pi} g_A^3 < 0$.

In terms of the finite g_A, g_B couplings, the phase portrait of the model is depicted in Fig. 1. Note that on the line

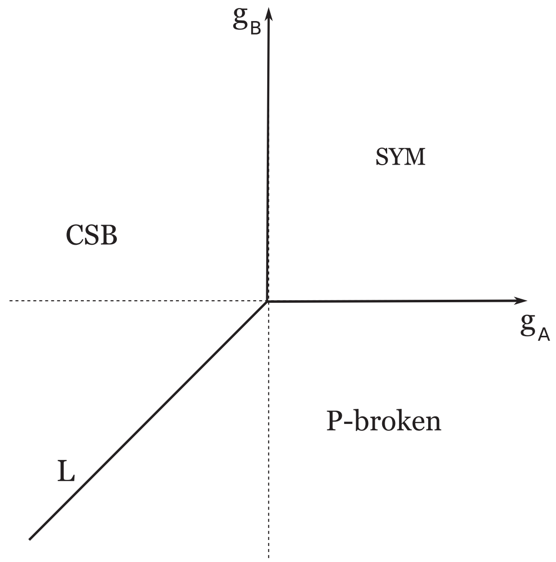

FIG. 1. (g_A, g_B) phase portrait of the generalized Thirring model. Here, we use the notations CSB and "P-broken" for chiral symmetry breaking and parity $\mathcal{P}$-breaking phases, respectively. The notation SYM means the symmetrical phase without mass generation. On the straight line L $g_A = g_B$.

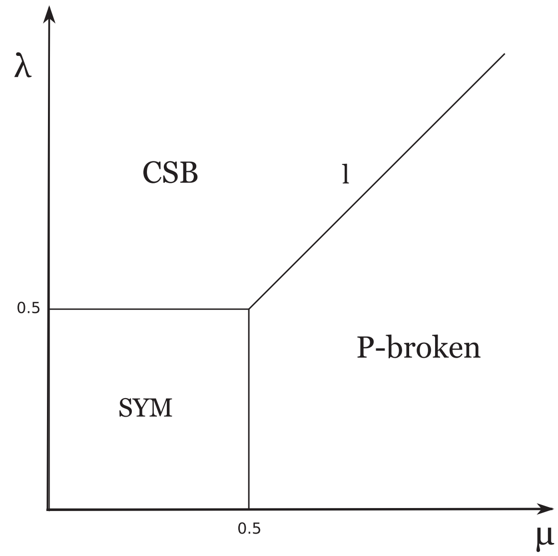

FIG. 2. Phase portrait of the model in terms of dimensionless bare couplings μ and λ defined in the text before Eq. (29). The point (0.5,0.5) of the (μ, λ) plane is the UV-fixed point. The line l is defined by equation $\lambda = \mu$. Other notations are introduced in Fig. 1.

$L = \{(g_A, g_B)\colon g_A = g_B, g_A < 0\}$ of this figure there is a first-order phase transition from chiral symmetry broken (CSB) (at $g_B > g_A$) to $\mathcal{P}$-broken phase (at $g_A > g_B$). On the line L we have an equality of the free-energy densities of the ground states of these phases. In other words, it means that at $g_A = g_B$ the two phases coexist. Moreover, it is clear from (26) that at $g_A = g_B$ we have $A = B$, i.e., $G_s = 0$. So in a massless $(2+1)$-D *pure* Thirring model (without G_s coupling) the ground state is indeed a mixture of CSB and $\mathcal{P}$-breaking phases, i.e., this state can be imagined as a space filled with one of the above phases, in which bubbles of another phase can exist.

IV. PHASE PORTRAIT IN TERMS OF DIMENSIONLESS BARE COUPLINGS

Finally, let us look at the properties of the model (1) from a renormalization group point of view, i.e., try to find a position of its UV-stable fixed point as well as depict its phase portrait, in contrast to the phase diagram of Fig. 1, in terms of some dimensionless parameters. To this aim, we should attract some dimensionless bare quantities and then find the zeros of the corresponding Callan-Simanzik β functions. In our case, it is most convenient to deal with the bare quantities A and B (22). We have shown that in the Hartree-Fock approximation the model is renormalizable if these couplings behave vs Λ as it is shown in Eqs. (23). Taking into account such a dependence of A and B on Λ, we can now determine the following dimensionless bare quantities $\lambda \equiv \lambda(\Lambda) = \Lambda A(\Lambda)$ and $\mu \equiv \mu(\Lambda) = \Lambda B(\Lambda)$, and the corresponding Callan-Simanzik β functions (for definition, see, e.g., Sec. 2.7 of Ref. [22])

$$\beta_A(\lambda) \equiv \Lambda \partial \lambda / \partial \Lambda = 2\lambda \left(\frac{1}{2} - \lambda\right),$$
$$\beta_B(\mu) \equiv \Lambda \partial \mu / \partial \Lambda = 2\mu \left(\frac{1}{2} - \mu\right). \tag{29}$$

Due to the structure (29) of these Callan-Simanzik β functions, it is clear that both $\lambda(\Lambda)$ and $\mu(\Lambda)$ tend to $1/2$ when $\Lambda \to \infty$. It means that in the (μ, λ) plane there exists a UV-stable fixed point with coordinates $(1/2, 1/2)$.[5] Then, taking into account the relations (25), it is also possible to establish that at sufficiently high values of Λ

$$\lambda - 1/2 = -\frac{\pi g_A}{8\Lambda} + \cdots, \quad \mu - 1/2 = -\frac{\pi g_B}{8\Lambda} + \cdots. \tag{30}$$

It follows from Eqs. (30) that at $\lambda < 1/2$ and $\mu < 1/2$ we have both $g_A > 0$ and $g_B > 0$. According to a phase portrait of Fig. 1, it corresponds to the symmetrical phase of the model. It means that in the region $\{(\mu, \lambda)\colon \lambda < 1/2, \mu < 1/2\}$ of the (μ, λ) plane the symmetric phase is arranged. In a similar way, using the relations (30) between dimensional g_A, g_B and dimensionless λ, μ couplings and taking into account the phase diagram of Fig. 1, one can draw the phase portrait of the model in terms of λ and μ in some vicinity of the UV-stable fixed point with coordinates $(1/2, 1/2)$ (see Fig. 2).

Alternatively, it is also possible to remake the phase portrait of Fig. 2 of the model in terms of other, more

[5]Note that this conclusion also follows directly from Eq. (25) when $\Lambda \to \infty$.

natural and physically acceptable dimensionless coupling constants, $g_s \equiv \Lambda G_s$ and $g_v \equiv \Lambda G_v$. Due to Eqs. (22), they are connected with λ and μ by the relations

$$4N\pi^2\lambda = -g_s + 3g_v, \quad 4N\pi^2\mu = (4N-1)g_s + 3g_v. \quad (31)$$

It is clear from Eq. (31) that the lines $\mu = \lambda$, $\lambda = 1/2$, and $\mu = 1/2$ of Fig. 2 transforms, respectively, to the lines $g_s = 0$, l_1, and l_2 of the (g_s, g_v) plane, where

$$l_1: g_v = \frac{1}{3}g_s + \frac{2N\pi^2}{3}, \quad l_2: g_v = -g_s\frac{4N-1}{3} + \frac{2N\pi^2}{3}. \quad (32)$$

These lines intersect in the UV-fixed point with coordinates $(g_s = 0, g_v = g_v^*)$, where $g_v^* = \frac{2N\pi^2}{3}$. So in Fig. 3 the (g_s, g_v)-phase portrait of the model is presented in some neighborhood of this UV-fixed point.

It follows from the phase diagram of Fig. 3 that in the framework of the HF approximation the initial symmetry of the generalized Thirring model (1) can be broken dynamically at an arbitrary fixed value of N. Namely, suppose that $g_v \gtrsim g_v^*$. Then at sufficiently small and positive values of g_s the $\mathcal{P}$-breaking phase is realized in the model and the Haldane fermion mass is dynamically generated. However, when g_s is small and negative, then fermions acquire dynamically the Dirac mass, and in this case both chiral and $U(2N)$ symmetries of the model are broken spontaneously. In the particular case when $g_s = 0$, but $g_v > g_v^*$, there is a coexistence of these phases [this situation is realized in the original $(2+1)$-D Thirring model with only one nonzero coupling G_v].

It is clear from Eq. (32) that straight lines l_1 and l_2 intersect the g_s axis of Fig. 3 in the points g_s^* and g_s^{**},

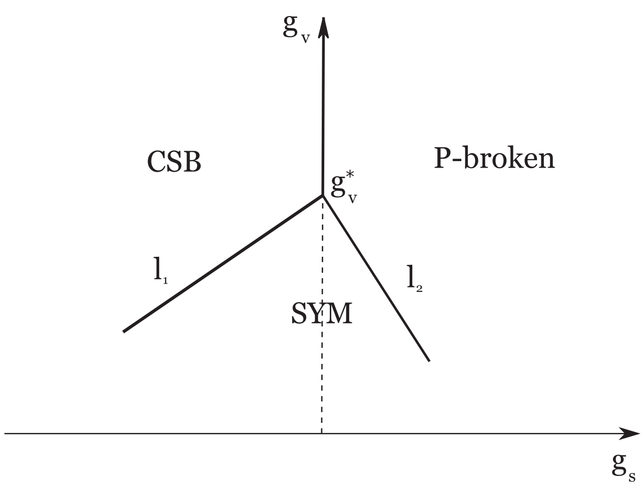

FIG. 3. Phase portrait of the model in terms of dimensionless bare couplings g_s and g_v defined in the text before Eq. (31). The point $(0, g_v^*)$ (where $g_v^* = 2N\pi^2/3$) of the (g_s, g_v) plane is the UV-fixed point. The lines l_1 and l_2 are defined in Eq. (32). Other notations are introduced in Fig. 1.

respectively, where $g_s^* = -2N\pi^2$ and $g_s^{**} = \frac{2N\pi^2}{4N-1}$. Hence if $N \to \infty$, then we have $g_v^* \to \infty$, $g_s^* \to -\infty$ and $g_s^{**} \to \pi^2/2$. As a result, we have in this case the expansion of the symmetrical phase over the whole region of the (g_s, g_v) plane, such that $g_s < \pi^2/2$ (of course, it contains the g_v axis). This fact corresponds to the absence of symmetry breaking effects in the generalized massless $(2+1)$-D Thirring model if it is studied by the HF method at $N \to \infty$ (and for sufficiently small values of $g_s < \pi^2/2$). The similar property of the pure $(2+1)$-D massless Thirring model is observed when it is investigated in the framework of the leading order of the large-N technique (see, e.g., Ref. [9]).

V. SUMMARY AND CONCLUSIONS

In this work, the phase structure of the massless $(2+1)$-D generalized Thirring model (1), in which fermions are four-component, is studied by the Hartree-Fock method. The method is based on the Cornwall-Jackiw-Tomboulis effective action for composite operators (see in Sec. II) calculated up to the first order in the coupling constants [29]. In our opinion, one of the advantages of this CJT approach is the possibility to study the phase structure of any four-fermionic quantum field theory model without introducing auxiliary scalar (as it is often done in the case of the Gross-Neveu models) or vector fields—in the case of Thirring model, etc.

Prior to this, the HF approach was not used when considering the properties of the pure Thirring model, i.e., when $G_s = 0$ in Eq. (1). At the same time, other approaches ($1/N$ expansion, variational method, etc.) gave contradictory information regarding the structure of the ground state ($\equiv$ of the vacuum) of the model. For example, some papers predict the dynamic generation of the Dirac mass $m_D\bar{\Psi}_k\Psi_k$ and appearance of a phase with broken chiral symmetry [9–11]. In others, the ground state is characterized by $\mathcal{P}$-parity violation and the appearance of Haldane mass $m_H\bar{\Psi}_k\tau\Psi_k$ [12,13] for fermions (about other inconsistences see, e.g., Refs. [17–20]).

Using the HF approach, we were able to show that there is no contradiction between the above mentioned results, since in fact the vacuum of the $(2+1)$-D pure Thirring model is really a mixed state in which these two phases coexist. In other words, in a part of the two-dimensional space, the Dirac mass is dynamically generated for fermions, and chiral symmetry is spontaneously broken in this region. At the same time, the region of this phase can border on areas of another phase, in which the Haldane mass is generated, and $\mathcal{P}$ parity is spontaneously broken. During the transition from one phase to another, a first-order phase transition occurs in the system (see the last paragraph of Sec. III). Moreover, it is clear that in the framework of the HF approach to the pure $(2+1)$-D Thirring model (1) the dynamical mass generation comes at any finite N only from the Fock term of Eq. (16). At $G_s = 0$ the Hartree term makes no contribution to the regularized HF equation (19). Since the Fock term is proportional to $1/N$, we see the absence of dynamical mass generation in the limit $N \to \infty$. The same result was obtained in the leading order of the $1/N$-expansion approach to this model [9].

Returning to the results obtained when considering the properties of the generalized Thirring model (1) by the HF method, we note that renormalized (i.e., finite) expressions both for the effective potential (28) and for the HF equation (16) itself can only be obtained for a well-defined behavior (26) of the bare coupling constants $G_s(\Lambda)$ and $G_v(\Lambda)$ vs cutoff parameter Λ, in which two finite (and renormalization group invariant) constants g_A and g_B appear. In this case, for arbitrary fixed values of g_A and g_B, only one of the phases is realized in the model, symmetric, $\mathcal{P}$-breaking, or a phase with chiral symmetry breaking (the last two phases are characterized by dynamic appearance of the Haldane or Dirac mass, respectively), and the (g_A, g_B)-phase portrait of the model is shown in Fig. 1.

Then, using the dimensionless couplings $g_s \equiv \Lambda G_s(\Lambda)$ and $g_v \equiv \Lambda G_v(\Lambda)$, we have shown that generalized Thirring model (1) is characterized by nontrivial UV stability. It means that in the (g_s, g_v) plane there exists a so-called UV-stable fixed point $(g_s = 0, g_v = g_v^*)$, where $g_v^* = \frac{2N\pi^2}{3}$, such that in the limit $\Lambda \to \infty$ we have $(g_s, g_v) \to (0, g_v^*)$. Phase portrait of the model in some neighborhood of this UV-stable fixed point is given in Fig. 3.

It follows from Fig. 3 that at each fixed N, when the UV-fixed point is finite, dynamical generation of fermion mass, Dirac or Haldane, is possible in the generalized Thirring model (see discussion at the end of Sec. IV). However, if $N \to \infty$, then the UV-fixed point tends to ∞ along the g_v axis, and for arbitrary fixed values of dimensionless couplings g_s and g_v (when g_s is a rather small) the point (g_s, g_v) lies in the region corresponding to symmetrical phase, i.e., the dynamical generation of any fermion mass is absent [similar to the results of Refs. [9,11] obtained in the pure $(2+1)$-D Thirring model].

Finally, two remarks are in order. First, in the recent study of the $(2+1)$-D Gross-Neveu model by HF method [27] just the Hartree term gives the main contribution to the dynamic generation of the fermion mass, i.e., to the effect that is also observed in the leading order of the large-N approximation [22] (the contribution of the Fock term in this case is not so significant). In contrast, the present investigation of the generalized $(2+1)$-D Thirring model (1) by the HF method shows that at $G_s \neq 0$ the Fock terms of the stationary equation (16) play a more important role in the dynamical generation of a fermion mass (and at $G_s = 0$, i.e., in the original Thirring model, the appearance of this effect is entirely due to the Fock terms). Since the Fock terms are proportional to $1/N$, it might seem that the HF approach to fermion self-energy is equivalent to its study in the framework of the first two orders of large-N expansion. Indeed, as it is discussed in Ref. [31], taking into account the Hartree terms in equations of the type (16) is equivalent to considering the properties of the fermion propagator in the leading order of the $1/N$ expansion. However, there are a lot of diagrams that are of $1/N$ order, but which lie outside the scope of the HF consideration and are not described by Fock terms [31]. Thus, it is because of the presence of the Fock terms that the difference between the HF and large-N methods appears.

Second, the HF method is a kind of the well-known mean-field approach widely used in both field theory and many-particle physics. And, of course, its scope is limited. For example, the HF approach to $(2+1)$-D Thirring model predicts dynamical symmetry breaking at any fixed finite value of N. However, if the model is investigated by other and more sophisticated nonperturbative methods (such as the functional renormalization group, the Dyson-Schwinger method, and, especially, the lattice approach, which is based on the first principles of quantum field theory), the acceptable values of $N < N_c$ are rather small (the value of N_c is discussed, e.g., in the recent review [19]). In such cases, in our opinion, it is possible to improve the results of the HF method by going beyond the mean-field approach using the next orders over couplings $G_{s,v}$ in the CJT effective action $\Gamma(S)$ (8).

ACKNOWLEDGMENTS

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APPENDIX A: ALGEBRA OF THE γ MATRICES IN THE CASE OF $SO(2,1)$ GROUP

The two-dimensional irreducible representation of the $(2+1)$-dimensional Lorentz group $SO(2,1)$ is realized by the following 2×2 $\tilde{\gamma}$ matrices:

$$\tilde{\gamma}^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\gamma}^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\tilde{\gamma}^2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A1}$$

acting on two-component Dirac spinors. They have the properties

$$\mathrm{Tr}(\tilde{\gamma}^\mu \tilde{\gamma}^\nu) = 2g^{\mu\nu}; \qquad [\tilde{\gamma}^\mu, \tilde{\gamma}^\nu] = -2i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha;$$
$$\tilde{\gamma}^\mu \tilde{\gamma}^\nu = -i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha + g^{\mu\nu}, \tag{A2}$$

where $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$, $\tilde{\gamma}_\alpha = g_{\alpha\beta}\tilde{\gamma}^\beta$, $\varepsilon^{012} = 1$. There is also the relation

$$\mathrm{Tr}(\tilde{\gamma}^\mu \tilde{\gamma}^\nu \tilde{\gamma}^\alpha) = -2i\varepsilon^{\mu\nu\alpha}. \tag{A3}$$

Note that the definition of chiral symmetry is slightly unusual in $(2+1)$ dimensions [spin is here a pseudoscalar rather than a (axial) vector]. The formal reason is simply that there exists no other 2×2 matrix anticommuting with the Dirac matrices $\tilde{\gamma}^\nu$ which would allow the introduction of a γ^5 matrix in the irreducible representation. The important concept of "chiral" symmetries and their breakdown by mass terms can nevertheless be realized also in the framework of $(2+1)$-dimensional quantum field theories by considering a four-component reducible representation for Dirac fields. In this case the Dirac spinors ψ have the following form:

$$\psi(x)=\begin{pmatrix}\tilde{\psi}_1(x)\\ \tilde{\psi}_2(x)\end{pmatrix}, \tag{A4}$$

with $\tilde{\psi}_1,\tilde{\psi}_2$ being two-component spinors. In the reducible four-dimensional spinor representation one deals with 4×4 γ matrices: $\gamma^\mu=\mathrm{diag}(\tilde{\gamma}^\mu,-\tilde{\gamma}^\mu)$, where $\tilde{\gamma}^\mu$ are given in (A1). (This particular reducible representation for γ matrices is used, e.g., in Ref. [33].) One can easily show that (μ, $\nu=0$, 1, 2):

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu)=4g^{\mu\nu};\qquad \gamma^\mu\gamma^\nu=\sigma^{\mu\nu}+g^{\mu\nu};$$
$$\sigma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]=\mathrm{diag}(-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha,-i\varepsilon^{\mu\nu\alpha}\tilde{\gamma}_\alpha). \tag{A5}$$

In addition to the Dirac matrices γ^μ ($\mu=0,1,2$) there exist two other matrices, γ^3 and γ^5, which anticommute with all γ^μ ($\mu=0$, 1, 2) and with themselves

$$\gamma^3=\begin{pmatrix}0, & I\\ I, & 0\end{pmatrix},\qquad \gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3=i\begin{pmatrix}0, & -I\\ I, & 0\end{pmatrix},$$
$$\tau=-i\gamma^3\gamma^5=\begin{pmatrix}I, & 0\\ 0, & -I\end{pmatrix} \tag{A6}$$

with I being the unit 2×2 matrix.

APPENDIX B: CALCULATION OF THE $\Gamma(S)$ UP TO A FIRST ORDER IN $G_{s,v}$: HARTREE-FOCK APPROXIMATION

1. The case $G_{s,v}=0$

In this case $\exp(iI_{\mathrm{int}}(-i\frac{\delta}{\delta K}))=1$, so we have from Eqs. (5)–(7)

$$\exp(iNW(K))=\exp[N\mathrm{Tr}\ \ln(D(x,y)+K(x,y))]$$
$$\Rightarrow W(K)=-i\mathrm{Tr}\ \ln(D(x,y)+K(x,y)). \tag{B1}$$

Now, using a well-known relation [see, e.g., Eq. (11.101) of Ref. [38]],

$$\frac{\partial}{\partial\alpha}\mathrm{Tr}\ \ln M(\alpha)=\mathrm{Tr}\left\{M^{-1}\frac{\partial M}{\partial\alpha}\right\}, \tag{B2}$$

where $M\equiv M(\alpha)$ is a matrix, we have from Eqs. (9) and (B1)

$$S^\alpha_\beta(x,y)=\frac{\delta W(K)}{\delta K^\beta_\alpha(y,x)}=-i\int d^3sd^3t\sum_{\mu\nu}[(D+K)^{-1}]^\mu_\nu(s,t)\frac{\delta K^\nu_\mu(t,s)}{\delta K^\beta_\alpha(y,x)}$$
$$=-i\int d^3sd^3t\sum_{\mu\nu}[(D+K)^{-1}]^\mu_\nu(s,t)\delta^3(t-y)\delta^3(s-x)\delta_{\nu\beta}\delta_{\mu\alpha}=-i[(D+K)^{-1}]^\alpha_\beta(x,y). \tag{B3}$$

Solving this equation with respect to K, we obtain

$$K=-iS^{-1}-D. \tag{B4}$$

Finally, after substituting the relation (B4) into Eq. (B1) and taking into account the definition (8) of the CJT effective action $\Gamma(S)$, we have (omitting independent S terms) for it the following expression at $G=0$:

$$\Gamma(S)=-i\mathrm{Tr}\ \ln(-iS^{-1})+\int d^3xd^3yS^\alpha_\beta(x,y)D^\beta_\alpha(y,x). \tag{B5}$$

Starting from the CJT effective action (B5), it is possible to obtain the stationary equation [see Eq. (12)] for the genuine spinor propagator S of the generalized $(2+1)$-D Thirring model at $G_{s,v}=0$. Taking into account the relation (B2), it can be presented in the following form:

$$0=i\int d^3sd^3t\sum_{\mu\nu}[S^{-1}]^\mu_\nu(s,t)\frac{\delta S^\nu_\mu(t,s)}{\delta S^\alpha_\beta(x,y)}+D^\beta_\alpha(y,x)$$
$$=i[S^{-1}]^\beta_\alpha(y,x)+D^\beta_\alpha(y,x), \tag{B6}$$

where a trivial relation $\frac{\delta S^\nu_\mu(t,s)}{\delta S^\alpha_\beta(x,y)}=\delta^3(t-x)\delta^3(s-y)\delta_{\nu\alpha}\delta_{\mu\beta}$ is taken into consideration. Hence, in the absence of interaction in the generalized Thirring model (1), i.e., at $G_{s,v}=0$, the stable and stationary form of the propagator is the following, $S=-iD^{-1}$, where D is presented in Eq. (6).

2. CJT effective action in the first order in coupling constants

In this case the functional $W(K)$ (7) looks like (here and below we use the definition $\Delta \equiv D + K$)

$$\exp(iNW(K)) = \left(1 + iI_v\left(-i\frac{\delta}{\delta K}\right) + iI_s\left(-i\frac{\delta}{\delta K}\right)\right)\exp(N\mathrm{Tr}\ \ln\Delta)$$
$$= \left\{1 + i\frac{G_v}{2N}\int d^3s d^3t d^3u d^3v \delta^3(s-t)\delta^3(t-u)\delta^3(u-v)(\gamma^\rho)^\beta_\alpha \frac{\delta}{\delta K^\beta_\alpha(s,t)}(\gamma_\rho)^\nu_\mu \frac{\delta}{\delta K^\nu_\mu(u,v)}\right.$$
$$\left. - i\frac{G_s}{2N}\int d^3s d^3t d^3u d^3v \delta^3(s-t)\delta^3(t-u)\delta^3(u-v)\tau^\beta_\alpha \frac{\delta}{\delta K^\beta_\alpha(s,t)}\tau^\nu_\mu \frac{\delta}{\delta K^\nu_\mu(u,v)}\right\}\exp(N\mathrm{Tr}\ \ln\Delta). \quad \text{(B7)}$$

In the following, two relations are needed,

$$\frac{\delta \mathrm{Tr}\ \ln\Delta}{\delta K^\nu_\mu(u,v)} = (\Delta^{-1})^\mu_\nu(v,u), \quad \text{(B8)}$$

which is a consequence of Eq. (B3) or Eq. (B2), and

$$\frac{\delta}{\delta K^\beta_\alpha(s,t)}(\Delta^{-1})^\mu_\nu(v,u) = -\int d^3v' d^3u' \sum_{\mu',\nu'}(\Delta^{-1})^\mu_{\mu'}(v,v')\frac{\delta\Delta^{\mu'}_{\nu'}(v',u')}{\delta K^\beta_\alpha(s,t)}(\Delta^{-1})^{\nu'}_\nu(u',u). \quad \text{(B9)}$$

Note that Eq. (B9) follows from a rather general formula (11.94) of Ref. [38]. Taking into account in Eq. (B9) that $\frac{\delta\Delta^{\mu'}_{\nu'}(v',u')}{\delta K^\beta_\alpha(s,t)} = \delta^3(v'-s)\delta^3(u'-t)\delta^{\mu'\beta}\delta_{\nu'\alpha}$, we have

$$\frac{\delta}{\delta K^\beta_\alpha(s,t)}(\Delta^{-1})^\mu_\nu(v,u) = -(\Delta^{-1})^\mu_\beta(v,s)(\Delta^{-1})^\alpha_\nu(t,u). \quad \text{(B10)}$$

Applying the relations (B8) and (B10) in Eq. (B7), we obtain

$$\exp(iNW(K)) = \left\{1 + i\frac{G_vN}{2}\int d^3s(\mathrm{tr}[\gamma^\rho\Delta^{-1}(s,s)]\mathrm{tr}[\gamma_\rho\Delta^{-1}(s,s)]) - i\frac{G_v}{2}\int d^3s\,\mathrm{tr}[\gamma^\rho\Delta^{-1}(s,s)\gamma_\rho\Delta^{-1}(s,s)]\right.$$
$$\left. - i\frac{G_sN}{2}\int d^3s(\mathrm{tr}[\tau\Delta^{-1}(s,s)])^2 + i\frac{G_s}{2}\int d^3s\,\mathrm{tr}[\tau\Delta^{-1}(s,s)\tau\Delta^{-1}(s,s)]\right\}\exp(N\mathrm{Tr}\ \ln\Delta), \quad \text{(B11)}$$

where tr means the trace operation only in the spinor space. It follows from Eq. (B11) that up to a first order in $G_{v,s}$

$$W(K) = -i\mathrm{Tr}\ \ln\Delta - \frac{G_s}{2}\int d^3s(\mathrm{tr}[\tau\Delta^{-1}(s,s)])^2 + \frac{G_s}{2N}\int d^3s\,\mathrm{tr}[\tau\Delta^{-1}(s,s)\tau\Delta^{-1}(s,s)]$$
$$+ \frac{G_v}{2}\int d^3s(\mathrm{tr}[\gamma^\rho\Delta^{-1}(s,s)]\mathrm{tr}[\gamma_\rho\Delta^{-1}(s,s)]) - \frac{G_v}{2N}\int d^3s\,\mathrm{tr}[\gamma^\rho\Delta^{-1}(s,s)\gamma_\rho\Delta^{-1}(s,s)]. \quad \text{(B12)}$$

To find the effective action $\Gamma(S)$ in the first order of G_v and G_s (i.e. in the HF approximation), we must use in Eq. (8), as well as in Eq. (9), the expression (B12) for $W(K)$. In particular, it follows from Eqs. (9) and (B12) that

$$S^\alpha_\beta(x,y) \equiv \frac{\delta W(K)}{\delta K^\beta_\alpha(y,x)} = -i(\Delta^{-1})^\alpha_\beta(x,y) + G_s\int d^3s[\Delta^{-1}(x,s)\tau\Delta^{-1}(s,y)]^\alpha_\beta \mathrm{tr}[\tau\Delta^{-1}(s,s)]$$
$$- \frac{G_s}{N}\int d^3s[\Delta^{-1}(x,s)\tau\Delta^{-1}(s,s)\tau\Delta^{-1}(s,y)]^\alpha_\beta$$
$$- G_v\int d^3s[\Delta^{-1}(x,s)\gamma^\rho\Delta^{-1}(s,y)]^\alpha_\beta \mathrm{tr}[\gamma_\rho\Delta^{-1}(s,s)] + \frac{G_v}{N}\int d^3s[\Delta^{-1}(x,s)\gamma^\rho\Delta^{-1}(s,s)\gamma_\rho\Delta^{-1}(s,y)]^\alpha_\beta, \quad \text{(B13)}$$

where the relation (B10) was applied. Now, the next problem is to express the bilocal source K as a function(al) of S with the help of Eq. (B13). We will use the perturbation approach over the coupling constants $G_{s,v}$, i.e., will suppose that the solution of Eq. (B13) has the form

$$K(S) = K_0 + \delta K, \tag{B14}$$

where $\delta K \sim G_{s,v}$ and K_0 is the solution of Eq. (B13) at $G_{s,v} = 0$, and it is given in Eq. (B4), i.e., $K_0 = -iS^{-1} - D$. Recall that Δ^{-1} in Eq. (B13) is indeed a functional of K, i.e., $\Delta^{-1} \equiv \Delta^{-1}(K)$. So, let us expand this quantity in a Taylor series around K_0 up to a first order in a small perturbation δK of Eq. (B14),

$$(\Delta^{-1}(K))^\alpha_\beta(x,y) = (\Delta^{-1}(K_0))^\alpha_\beta(x,y) + \int d^3u d^3v \delta K^\nu_\mu(u,v) \frac{\delta(\Delta^{-1}(K))^\alpha_\beta(x,y)}{\delta K^\nu_\mu(u,v)}\bigg|_{K=K_0} + \cdots. \tag{B15}$$

Taking into account in Eq. (B15) the derivative rule (B10) as well as the trivial relation $(\Delta^{-1}(K_0))^\alpha_\beta(x,y) = iS^\alpha_\beta(x,y)$, we obtain

$$(\Delta^{-1}(K))^\alpha_\beta(x,y) = iS^\alpha_\beta(x,y) + \int d^3u d^3v S^\alpha_\nu(x,u)\delta K^\nu_\mu(u,v) S^\mu_\beta(v,y) + \cdots. \tag{B16}$$

After a substitution of the relation (B16) instead of a first term in the right-hand side of Eq. (B13) and replacing all Δ^{-1} in other terms of Eq. (B13) by iS, we find the following equation on the quantity δK:

$$\begin{aligned}\int d^3u d^3v S^\alpha_\nu(x,u)\delta K^\nu_\mu(u,v) S^\mu_\beta(v,y) = &-G_s \int d^3s [S(x,s)\tau S(s,y)]^\alpha_\beta \mathrm{tr}[\tau S(s,s)] + \frac{G_s}{N}\int d^3s [S(x,s)\tau S(s,s)\tau S(s,y)]^\alpha_\beta \\ &+ G_v \int d^3s [S(x,s)\gamma^\rho S(s,y)]^\alpha_\beta \mathrm{tr}[\gamma_\rho S(s,s)] - \frac{G_v}{N}\int d^3s [S(x,s)\gamma^\rho S(s,s)\gamma_\rho S(s,y)]^\alpha_\beta.\end{aligned} \tag{B17}$$

Its solution with respect to δK has the following form:

$$\begin{aligned}\delta K^\alpha_\beta(x,y) = &-G_s \tau^\alpha_\beta \mathrm{tr}[\tau S(x,x)]\delta^3(x-y) + \frac{G_s}{N}[\tau S(x,x)\tau]^\alpha_\beta \delta^3(x-y) + G_v (\gamma^\rho)^\alpha_\beta \mathrm{tr}[\gamma_\rho S(x,x)]\delta^3(x-y) \\ &- \frac{G_v}{N}[\gamma^\rho S(x,x)\gamma_\rho]^\alpha_\beta \delta^3(x-y).\end{aligned} \tag{B18}$$

Bearing in mind this expression for δK as well as that $K_0 = -iS^{-1} - D$, we obtain, up to a first order in $G_{s,v}$, the solution $K(S)$ (B14) of Eq. (B13),

$$\begin{aligned}K^\alpha_\beta(x,y) = &-i(S^{-1})^\alpha_\beta(x,y) - D^\alpha_\beta(x,y) - G_s \tau^\alpha_\beta \mathrm{tr}[\tau S(x,x)]\delta^3(x-y) + \frac{G_s}{N}[\tau S(x,x)\tau]^\alpha_\beta \delta^3(x-y) \\ &+ G_v (\gamma^\rho)^\alpha_\beta \mathrm{tr}[\gamma_\rho S(x,x)]\delta^3(x-y) - \frac{G_v}{N}[\gamma^\rho S(x,x)\gamma_\rho]^\alpha_\beta \delta^3(x-y).\end{aligned} \tag{B19}$$

It follows from Eq. (B19) that

$$\begin{aligned}\Delta(K)^\alpha_\beta(x,y) \equiv K^\alpha_\beta(x,y) + D^\alpha_\beta(x,y) = &-i(S^{-1})^\alpha_\beta(x,y) - G_s \tau^\alpha_\beta \mathrm{tr}[\tau S(x,x)]\delta^3(x-y) + \frac{G_s}{N}[\tau S(x,x)\tau]^\alpha_\beta \delta^3(x-y) \\ &+ G_v (\gamma^\rho)^\alpha_\beta \mathrm{tr}[\gamma_\rho S(x,x)]\delta^3(x-y) - \frac{G_v}{N}[\gamma^\rho S(x,x)\gamma_\rho]^\alpha_\beta \delta^3(x-y).\end{aligned} \tag{B20}$$

Now, it is clear from Eqs. (B19) and (B20) that (also up to a first order in $G_{s,v}$)

$$-i\mathrm{Tr}\,\ln\Delta = -i\mathrm{Tr}\,\ln(-iS^{-1}) - G_s\int d^3s[\mathrm{tr}(\tau S(s,s))]^2 + \frac{G_s}{N}\int d^3s\,\mathrm{tr}[\tau S(s,s)\tau S(s,s)]$$
$$+ G_v\int d^3s\,\mathrm{tr}[\gamma^\rho S(s,s)]\mathrm{tr}[\gamma_\rho S(s,s)] - \frac{G_v}{N}\int d^3s\,\mathrm{tr}[S(s,s)\gamma^\rho S(s,s)\gamma_\rho],$$
$$-\int d^3x d^3y S^\alpha_\beta(x,y)K^\beta_\alpha(y,x) = \int d^3x d^3y S^\alpha_\beta(x,y)D^\beta_\alpha(y,x) + G_s\int d^3s[\mathrm{tr}(\tau S(s,s))]^2$$
$$-\frac{G_s}{N}\int d^3s\,\mathrm{tr}[\tau S(s,s)\tau S(s,s)] - G_v\int d^3s\,\mathrm{tr}[\gamma^\rho S(s,s)]\mathrm{tr}[\gamma_\rho S(s,s)]$$
$$+\frac{G_v}{N}\int d^3s\,\mathrm{tr}[S(s,s)\gamma^\rho S(s,s)\gamma_\rho] \tag{B21}$$

(the last equation is valid up to an unessential and S-independent infinite constant). Finally, replacing all Δ^{-1} functions in the last four terms of Eq. (B12) for $W(K)$ by iS, and taking into account the relations (B21), we obtain in the first order in $G_{s,v}$ for the CJT effective action $\Gamma(S)$ (8) the expression (13) [where we also took into account the trivial relation, $\ln(-iS^{-1}) = -\ln(iS)$].

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