Graded extension of Thomas-Whitehead gravity

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Thomas-Whitehead (TW) gravity was recently introduced as a projective gauge theory of gravity over a d-dimensional manifold that embeds reparametrization invariance into the action functional for gravitation through the use of the Thomas-Whitehead connection. The projective invariance in this d-dimensional theory enjoys an intimate relationship with the Virasoro coadjoint elements found in string theory as one of the components of the connection, $\mathcal{D}_{ab}$, is directly related to the coadjoint elements of the Virasoro algebra. TW gravity exploits projective Gauss-Bonnet terms in the action functional which allows the theory to collapse to Einstein's theory of general relativity in the limit that $\mathcal{D}_{ab}$ vanishes. In this paper we develop the graded extension of TW gravity, super-TW gravity, in the framework of a DeWitt supermanifold. We construct the Lagrangian for super-TW gravity, give a detailed derivation of the classical field equations, and discuss the graded extension of the projective connection as a prelude to a future understanding of TW-supergravity (which has manifest supersymmetry) and its relationship to the super-Virasoro algebra.

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I. INTRODUCTION

It is known that the method of coadjoint orbits [1,2] of the semidirect product of Kac-Moody algebras and the Virasoro algebras [3–6] leads to the two-dimensional Wess-Zumino-Witten (WZW) action [7,8] and the Polyakov action [9,10] respectively. One may arrive at this by integrating the Kirillov two-form [1,2] over any coadjoint orbit as prescribed in [11–13] which produces these *geometric* actions for their respective groups. One finds that the coadjoint elements have been promoted to fields in the geometric actions and the central extension to a coupling constant. The geometric action interprets the elements of the coadjoint representation of the Virasoro algebra as a background field coupling to the Polyakov metric. For the Virasoro group this background field has been called the diffeomorphism field $\mathcal{D}_{\mu\nu}$ and is akin to the Yang-Mills connection, A_μ, that related the coadjoint elements of the Kac-Moody group. One can extend these geometric actions by adding dynamics to A_μ through the addition of the Yang-Mills action in the WZW case, and similarly by adding the Thomas-Whitehead (TW) action [14–16] to the Polyakov action to give dynamics to the $\mathcal{D}_{\mu\nu}$. This reconciles the coadjoint elements of both the Kac-Moody algebra and the Virasoro algebra with geometric connections in higher dimensions. In [15] a detailed overview of Thomas-Whitehead gravity in a general setting is discussed. This includes a review of the relationship between the projective structure from Sturm-Liouville theory and the two cocycles of the Virasoro algebra as observed by Kirillov [2,17] as well as a derivation of the spin connection on the Thomas cone [18], the Dirac equation and the Dirac Lagrangian for spin $\frac{1}{2}$ spinors (fermions).

In [6,19,20], the authors applied the method of coadjoint orbits to the super-Virasoro algebra and later extended [21,22] in the context of studying superstring theories. This recovered the supersymmetric extension of Polyakov's action. A natural question to ask is what the supersymmetric extension of TW gravity is. In this paper, we study the preliminary question by generalizing the theory of Thomas-Whitehead gravity to a supermanifold with n ordinary coordinates and m Grassmann coordinates. A highly detailed discussion of the calculations can be found in [23].

In Sec. II we briefly review the theory of TW gravity and in Sec. III we briefly describe supervector spaces and supermanifolds following the approach of [21,24]. The consequences of the DeWitt topology and its relation to the theory of supersymmetry is reviewed from the perspective

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of [25,26]. Other excellent references on the theory of supermanifolds include [26–29]. In Sec. IV we generalize the TW connection to the graded manifolds by following the approach in [15]. The theory of TW connections has been studied in the graded setting before in [30–32].

II. REVIEW OF THOMAS-WHITEHEAD GRAVITY

In this section we review the TW gravity as developed in [14–16]. Projective geometry, as a theory of gravity, has been around for nearly a century as a strategy to incorporate the ambiguity of geodesics in relation to connections due to projective transformations [33–35]. The Thomas-Whitehead gravitational action [14] (named after mathematicians Tracy Thomas and J. H. C. Whitehead) uses the covariant derivative and the fibration of the Thomas cone from these early investigators to tie projective geometry to string theory and higher dimensional gravity through projective Gauss-Bonnet terms on the manifold. The projective connection and the metric are treated as independent in the spirit of the Palatini formalism [36]. This allows the field equations to collapse naturally to the Einstein-Hilbert field equations when the diffeomorphism field vanishes and when the fundamental projective invariant is evaluated on the affine connection compatible with the Einstein metric. In this way projective geometry can influence the Riemannian geometry by acting as sources in the energy-momentum tensor. This provides an avenue for geometric explanations of dark energy, dark matter, and other physical phenomena.

To proceed we are given a connection $\Gamma^a{}_{bc}$ on a Riemannian manifold. One can define the fundamental projective invariant as

$$\Pi^a{}_{bc} \equiv \Gamma^a{}_{bc} - \frac{1}{m+1}(\Gamma^d{}_{dc}\delta^a{}_b + \Gamma^d{}_{db}\delta^a{}_c), \tag{1}$$

which is invariant under projective transformations

$$\hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \delta^a{}_b v_c + \delta^a{}_c v_b. \tag{2}$$

Let J be the Jacobian of the coordinate transformation $x^a \to y^a$. Then we have the following identities:

$$\partial_c \log(\det(J)) = -J^a{}_b \partial_c (J^{-1})^b{}_a = -J^a{}_b \partial_c J_a{}^b, \tag{3}$$

$$J^h{}_f \frac{\partial^2 x^f}{\partial y^h \partial y^c} = -\partial_m \log(\det(J))(J^{-1})^m{}_c. \tag{4}$$

With this, the coordinate transformation law of the fundamental projective invariant is

$$\bar{\Pi}^a{}_{bc} = J^a{}_f\left(\Pi^f{}_{de}(J^{-1})^e{}_c(J^{-1})^d{}_b + \frac{\partial^2 x^f}{\partial y^b \partial y^c}\right) + \frac{1}{m+1}\frac{\partial}{\partial x^m}\log(\det(J))((J^{-1})^m{}_c\delta^a{}_b + (J^{-1})^m{}_b\delta^a{}_c). \tag{5}$$

From Eq. (5) it is apparent that Π itself is not a connection due to the extra terms arising in the transformation law. To construct a connection realizing projective invariance we adopt the approach of Thomas [34,35] and consider a connection not on M but instead on the volume bundle VM, which is now called the *Thomas cone*.

Given an m-manifold M, a volume form can be constructed from a smooth nonvanishing function $v\colon M \to \mathbb{R}_+$ and considering the m-form

$$|v(x)| dx^1 \wedge \ldots \wedge dx^m, \tag{6}$$

which is a generic volume form on M. The Thomas cone arises by interpreting the volume form as a a section of the volume bundle VM, where we take the absolute value of v to absolve the ambiguity of choice of orientation [34,35,37]. VM is then defined as the collection of all such sections, and is an $\mathbb{R}^+$ line bundle over M. As a manifold, VM is one dimension higher than M.

We use λ as the fiber coordinate on the Thomas cone, so the coordinates on VM are $(x^0, x^1, \ldots, x^{m-1}, \lambda)$, where $0 < \lambda < \infty$. In this section Greek letters (excluding λ) denote coordinates on VM while Latin letters range 0 to $m-1$ and denote coordinates on M.

The Thomas-Whitehead connection $\tilde{\Gamma}^\alpha{}_{\beta\gamma}$ lives on VM, and is both projectively invariant and houses Π as a component. The TW connection on VM can be decomposed as

$$\tilde{\Gamma}^\alpha{}_{\beta\gamma} = \begin{cases} \tilde{\Gamma}^a{}_{bc} = \Pi^a{}_{bc} \\ \tilde{\Gamma}^\lambda{}_{bc} = \lambda \mathcal{D}_{bc} \\ \tilde{\Gamma}^a{}_{b\lambda} = \tilde{\Gamma}^a{}_{\lambda b} = \frac{1}{\lambda}\delta^a{}_b \\ \tilde{\Gamma}^\lambda{}_{b\lambda} = \tilde{\Gamma}^\lambda{}_{\lambda b} = \tilde{\Gamma}^\lambda{}_{\lambda\lambda} = 0 \end{cases}, \tag{7}$$

where $\mathcal{D}_{bc}$ is a (nontensorial) rank 2 object on M. In general this need not be related to the Ricci tensor and when related to the Virasoro algebra in the literature, it is known as the diffeomorphism (diff) field [38]. Demonstrating this component of the TW connection appears in the geometric action of the diffeomorphism group of S^1 was the essence of [14]. For $\tilde{\Gamma}$ to be a connection on VM it must transform as

$$\tilde{\Gamma}^\alpha{}_{\beta\gamma} \to \frac{\partial y^\alpha}{\partial x^\delta}\frac{\partial x^\epsilon}{\partial y^\beta}\frac{\partial x^\eta}{\partial y^\gamma}\tilde{\Gamma}^\delta{}_{\epsilon\eta} + \frac{\partial y^\alpha}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial y^\gamma \partial y^\beta}, \tag{8}$$

under coordinate transformations on VM. For this to happen the transformation law for the diffeomorphism field must be [15,34,35]

$$\hat{\mathcal{D}}_{bc} = \left(\mathcal{D}_{ef} - \frac{\partial}{\partial x^e} j_f + \Pi^d{}_{ef} j_d - j_e j_f\right) \frac{\partial x^e}{\partial y^b} \frac{\partial x^f}{\partial y^c}, \quad (9)$$

where $j_a = \partial_a \log J^{-\frac{1}{m+1}}$. We emphasize that $\mathcal{D}_{bc}$ lives exclusively on M and is required to ensure that $\tilde{\Gamma}$ transform as a connection over VM.

III. A SUPERQUICK REVIEW OF SUPERMANIFOLDS

This section is based primarily on the geometric treatments in DeWitt's book [24] and Rogers's book [26], as well as the [25], where a Rogers supermanifold is defined. A review of a more algebraic approach is offered in [39].

A. Construction of supernumbers

Let θ^a denote the generators of an algebra subject to the relation

$$\theta^a \theta^b = -\theta^b \theta^a, \quad (10)$$

where $a, b = 1, \ldots, N$. This algebra is the N-dimensional Grassmann algebra Λ_N. As a vector space over $\mathbb{C}$, $\Lambda_N(\mathbb{C})$ is 2^N-dimensional with a basis given by

$$1, \theta^a, \theta^a\theta^b, \theta^a\theta^b\theta^c, \ldots, \theta^1\theta^2\ldots\theta^{N-1}\theta^N. \quad (11)$$

Throughout we restrict our attention to the field of complex numbers and denote $\Lambda_N(\mathbb{C})$ by Λ_N. Taking $N \to \infty$ we obtain the infinite-dimensional Grassmann algebra Λ_∞.

A supernumber $z \in \Lambda_\infty$ is a sum

$$z = \sum_{n=0}^{\infty} \frac{1}{n!} c_{a_1\ldots a_n} \theta^{a_n} \ldots \theta^{a_1}, \quad (12)$$

where $c_{a_1\ldots a_n} \in \mathbb{C}$. A useful decomposition of supernumbers is given by the splitting

$$z = u + v, \quad (13)$$

$$u = \sum_{n=0}^{\infty} \frac{1}{(2n)!} c_{a_1\ldots a_{2n}} \theta^{a_{2n}} \ldots \theta^{a_1}, \quad (14)$$

$$v = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1\ldots a_{2n+1}} \theta^{a_{2n+1}} \ldots \theta^{a_1}, \quad (15)$$

where u and v are the even and odd parts of z and are called c-numbers and a-numbers, respectively. The set of c-numbers and a-numbers are denoted by $\mathbb{C}_c$ and $\mathbb{C}_a$, respectively, and are 2^{N-1}-dimensional vector spaces over $\mathbb{C}$. $\mathbb{C}_c$ is a subalgebra of Λ_∞, while $\mathbb{C}_a$ is not as it is not closed under multiplication.

The real counterparts of $\mathbb{C}_c$ and $\mathbb{C}_a$ are denoted by $\mathbb{R}_c$ and $\mathbb{R}_a$, respectively and are introduced by defining complex conjugation in the following fashion:

$$(z_1 + z_2)^* = z_1^* + z_2^*, \quad (16)$$

$$(z_1 z_2)^* = z_2^* z_1^*, \quad (17)$$

$$\theta^{a*} = \theta^a, \quad (18)$$

where $z_i \in \Lambda_\infty$ and θ^a are generators of Λ_∞. A supernumber z is real if $z^* = z$ and imaginary if $z^* = -z$.

B. Supervectors and supermatrices

Here we present the most salient features of supervectors and supermatrices we will encounter. The usual properties of vector spaces persist in the graded setting, except scalar multiplication is now distinct when acting on the left and right. This can be appreciated via the decomposition

$$X = U + V, \quad (19)$$

$$\alpha U = U\alpha, \quad (20)$$

$$\alpha V = -V\alpha, \quad (21)$$

where $\alpha \in \mathbb{C}_a$ and U, V are called the even and odd parts of X, respectively. If the odd (even) part of a supervector is not present the supervector is said to be c-type (a-type). Supernumbers and supervectors of a definite type are called pure. When both the supernumber and the supervector are pure this becomes $\alpha X = (-1)^{\alpha X} X\alpha$, with the power of -1 reflecting the parity of the vectors, where the "X" in the exponent of -1 is the parity of the pure vector X and "α" is the parity of the pure supernumber α. For example, if X is c-type then $(-1)^X = 1$, while if X is a-type then $(-1)^X = -1$. If X is not pure then $(-1)^X$ will depend on the component of X in question. Let $\{{}_i\mathbf{e}\}$ denote a discrete set of basis supervectors and their duals $\{\mathbf{e}^\mathbf{i}\}$ such that ${}_i\mathbf{e} \cdot \mathbf{e}^j = {}_i\delta^j$. Then a supervector may be denoted as $X = X^i{}_i\mathbf{e}$. Here we also introduce left and right derivations, ${}_a e = \frac{\bar{\partial}}{\partial x^a}$ and $e_a = \frac{\vec{\partial}}{\partial x^a}$. Therefore $X^i{}_i\mathbf{e} = X^i \frac{\vec{\partial}}{\partial x^i} = U - V$, while $X^i \mathbf{e}_i = X^i \frac{\bar{\partial}}{\partial x^i} = U + V$. If the object of interest is c-type (a-type), then its associated symbol equals 0 (1). For pure supervectors, one convenience is to write ${}^iX \equiv (-1)^{Xi} X^i$, so in particular for c-type supervectors ${}^iX = X^i$. The advantage of writing $X = X^i{}_i\mathbf{e}$ lies in the minimization of parity factors.

Let ${}_jK^i$ be a matrix of supernumbers. The components of a supervector transform by $\bar{X}^i = X^j{}_jK^i$, as summation is carried out only with adjacent indices. For K to preserve the parity of X it must be a block matrix of the form

$$K = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad (22)$$

where A and B are comprised of c-numbers and C and D are comprised of a-numbers. Such matrices are c-type precisely because they preserve the type of the supervector they act on and thus introduce no parity of their own. For c-type supermatrices we can form the product, supertranspose, and supertrace, respectively, as

$$\begin{aligned} {}_i(KL)^j &= {}_iK^k{}_kL^j \quad \text{[product rule]}, \\ {}^iK^{\sim}{}_j &= (-1)^{j(i+j)}{}_jK^i = (-1)^{j(i+1)}{}_jK^i \quad \text{[supertranspose]}, \\ \mathrm{str}(K) &= K_i{}^i = (-1)^i{}_iK^i \quad \text{[supertrace]}, \end{aligned} \tag{23}$$

where $\sim$ denotes the supertranspose operation. The superdeterminant is defined for any supermatrix and enjoys the usual multiplication laws. If K has the index structure ${}_iK^j$ or ${}^iK_j$ then

$$\mathrm{sdet}(K^{\sim}) = \mathrm{sdet}(K), \tag{24}$$

while if K has index structure ${}_iK_j$ or ${}^iK^j$ then

$$\mathrm{sdet}(M^{\sim}) = (-1)^n \mathrm{sdet}(M). \tag{25}$$

Let A, B, C, and D be $m\times m$, $n\times n$, $m\times n$, $n\times m$, $m\times n$, and $n\times m$ matrices with entries comprised of c, c, a, and a numbers, respectively. Then

$$\mathrm{sdet}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det(A - CB^{-1}D)(\det B)^{-1}, \tag{26}$$

where sdet is defined only if B is invertible. The superdeterminant of a c-type matrix is a real c-number. See [24] for a treatment of the superdeterminant for a-type matrices.

IV. THE SUPER THOMAS-WHITEHEAD CONNECTION

A. Super coordinate and projective transformations

In the graded setting we consider the coordinate transformation $x^a \to y^a$ with the super-Jacobian and its inverse

$$\begin{aligned} {}^aJ_b &= (-1)^{b(a+b)}{}_{b,}y^a = (-1)^{b(a+b)}\frac{\vec{\partial}}{\partial x^b}y^a, \\ {}^a(J^{-1})_b &= x^a\frac{\overleftarrow{\partial}}{\partial y^b} = (-1)^{b(a+b)}{}_{b,}x^a = (-1)^{b(a+b)}\frac{\vec{\partial}}{\partial y^b}x^a, \end{aligned} \tag{27}$$

which satisfies analogs of the usual identities

$${}^a(J^{-1})_b{}^bJ_c = {}^a\delta_c = x^a\frac{\overleftarrow{\partial}}{\partial x^c}, \tag{28}$$

$${}_cJ^b{}_b(J^{-1})^a = {}_c\delta^a = \frac{\vec{\partial}}{\partial x^c}x^a, \tag{29}$$

$$\text{and} \quad x^a{}_{,bc} = {}^a(J^{-1})_b\frac{\overleftarrow{\partial}}{\partial x^d}{}^d(J^{-1})_c. \tag{30}$$

There are two graded analogs of the Jacobi formula, one for left and one for right derivatives, respectively [24]:

$$\begin{aligned} &\frac{\vec{\partial}}{\partial x^a}\ln(\mathrm{sdet}(M)) = \mathrm{str}((M_ae)M^{-1}) = \mathrm{str}(M^{-1}(M_ae)), \\ &\ln(\mathrm{sdet}(M))\frac{\overleftarrow{\partial}}{\partial x^a} = (-1)^{ii}M_j\frac{\overleftarrow{\partial}}{\partial x^a}{}^j(M^{-1})_i \\ &\qquad = (-1)^{ii}(M^{-1})_j{}^jM_i\frac{\overleftarrow{\partial}}{\partial x^a}. \end{aligned} \tag{31}$$

The relationship between left and right derivatives can be used to show that the supertrace is invariant under cyclic permutations. A superprojective transformation is analogous to the ungraded case and is given by the relation

$$\hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + {}^a\delta_bv_c + {}^a\delta_c(-1)^{bc}v_b, \tag{32}$$

where v_a are the components of a c-type 1-form.

B. The superfundamental projective invariant

The fundamental projective invariant, Π, can be promoted to a graded setting by replacing the trace with the supertrace and adding necessary parity factors [30]. The resulting geometrical object is the superfundamental projective invariant

$$\Pi^a{}_{bc} \equiv \Gamma^a{}_{bc} - D({}^a\delta_b(-1)^e\Gamma^e{}_{ec} + {}^a\delta_c(-1)^{e+bc}\Gamma^e{}_{eb}), \tag{33}$$

where we set $D \equiv (m-n+1)^{-1}$ for future convenience. The coordinate transformation law of the superfundamental projective invariant is

$$\begin{aligned} \bar{\Pi}^a{}_{bc} &= {}^aJ_d((-1)^{g(f+b)}\Pi^d{}_{fg}{}^f(J^{-1})_b{}^g(J^{-1})_c + x^d{}_{,bc}) \\ &\quad + D\bigg({}^a\delta_b\ln(J)\frac{\overleftarrow{\partial}}{\partial x^f}{}^f(J^{-1})_c \\ &\quad + {}^a\delta_c(-1)^{bc}\ln(J)\frac{\overleftarrow{\partial}}{\partial x^f}{}^f(J^{-1})_b\bigg). \end{aligned} \tag{34}$$

C. The TW connection

The coefficients of the TW connection in the graded setting are

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \begin{cases} \tilde{\Gamma}^{a}{}_{bc} = \Pi^{a}{}_{bc} = (-1)^{bc}\Pi^{a}{}_{cb} \\ \tilde{\Gamma}^{\lambda}{}_{bc} = \lambda\mathcal{D}_{bc} = (-1)^{bc}\lambda\mathcal{D}_{cb} \\ \tilde{\Gamma}^{a}{}_{b}\lambda = \tilde{\Gamma}^{a}{}_{\lambda}b = \frac{1}{\lambda}\delta^{a}{}_{b} \\ \tilde{\Gamma}^{\lambda}{}_{b\lambda} = \tilde{\Gamma}^{\lambda}{}_{\lambda b} = \tilde{\Gamma}^{\lambda}{}_{\lambda\lambda} = 0 \end{cases}, \tag{35}$$

where any components not listed vanish. The measure transforms as

$$d^{m,n}x \to d^{m,n}\bar{x} = Jd^{m,n}x, \tag{36}$$

under a coordinate transformation $x^i \to \bar{x}^i(x)$, where $J \equiv \mathrm{sdet}(\bar{x}^i{}_{,j})$ is the super-Jacobian or Berezinian, and where the number of a-type coordinates must be even for the metric to be nonsingular. Latin indices range over the supermanifold coordinates while Greek indices (except λ) range over all coordinates, λ, x^a.

As before, we check the connection coefficients recover the transformation law for Π. The graded extensions of the identities from before are

$$j_a \equiv \log(J^{-D})\frac{\overleftarrow{\partial}}{\partial x^a} = \log(J^{-D})_{,a} = (\lambda J^{-D})\frac{\overleftarrow{\partial}}{\partial x^a}\frac{1}{(\lambda J^{-D})}, \tag{37}$$

$$\lambda\frac{\overleftarrow{\partial}}{\partial y^a}\frac{1}{\lambda} = \log(J^{D})\frac{\overleftarrow{\partial}}{\partial x^g}{}^{g}(J^{-1})_a = -j_g{}^{g}(J^{-1})_a, \tag{38}$$

$${}^{\lambda}(J^{-1})_a = -\lambda j_g{}^{g}(J^{-1})_a, \quad \text{and} \tag{39}$$

$${}^{\lambda}J_a = \lambda J^{-D}j_a. \tag{40}$$

With this, the transformation law for the super-TW connection is

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} \to {}^{\alpha}J_{\delta}x^{\delta}{}_{,\beta\gamma} + (-1)^{\eta(\epsilon+\beta)\alpha}J_{\delta}\tilde{\Gamma}^{\delta}{}_{\epsilon\eta}{}^{\epsilon}(J^{-1})_{\beta}{}^{\eta}(J^{-1})_{\gamma}. \tag{41}$$

The coordinate transformation law for the superdiffeomorphism field is

$$\mathcal{D}_{bc} \to (-1)^{f(b+e)}(\mathcal{D}_{ef} - j_{e,f} - j_e j_f + j_d\Pi^{d}{}_{ef})^{e}(J^{-1})_b{}^{f}(J^{-1})_c. \tag{42}$$

The parity of $\mathcal{D}_{ab}$ is $(-1)^{a+b}$. Under an infinitesimal coordinate transformation $x^a \to x^a - \delta\epsilon^a$,

$$\begin{aligned}\mathcal{D}_{bc}(x) \to \mathcal{D}_{bc}(x) + \delta(&\mathcal{D}_{bc,i}\epsilon^i + \mathcal{D}_{bf}(x)\epsilon^f{}_{,c} \\ &+ (-1)^{c(b+e)}\mathcal{D}_{ec}(x)\epsilon^e{}_{,b} - D(-1)^i\epsilon^i{}_{,ibc} \\ &+ D(-1)^i\epsilon^i{}_{,id}\Pi^d{}_{bc}(x)),\end{aligned}$$

where the coefficient of δ is the super Lie derivative with respect to the vector field ϵ. Setting the fermionic dimension to zero and the bosonic dimension to one and noting that the superdiffeomorphism has a single component, we recover the following reduction:

$$\mathcal{D} \to \mathcal{D} + \delta\left(\mathcal{D}'\epsilon + 2\mathcal{D}\epsilon' - \frac{1}{2}\epsilon'''\right). \tag{43}$$

After a redefinition of $\mathcal{D}$, we recover the coordinate transformation on a coadjoint Virasoro element [14,15]. It is sometimes convenient to express Π in terms of a member in the equivalence class of connections and subtracting out the trace of this member which we denote by α. In general, our torsionless affine connection Γ will not be compatible with the metric. In terms of this member, we have

$$\alpha_a \equiv -D(-1)^e\Gamma^e{}_{ea}, \tag{44}$$

$$\Pi^a{}_{bc} = \Gamma^a{}_{bc} + \delta^a{}_b\alpha_c + (-1)^{bc}\delta^a{}_c\alpha_b. \tag{45}$$

We may then express the projective Ricci symbol in terms of this connection and its trace and write

$$\begin{aligned}\mathcal{R}_{bd} = &(-1)^{c(b+c+d)}\Gamma^c{}_{bd,c} - (-1)^{d(b+c)+c}\Gamma^c{}_{df}\Gamma^f{}_{cb} \\ &+ (-1)^{db}(\alpha_{d,b} - \alpha_f\Gamma^f{}_{db}) + \alpha_{b,d} - \alpha_f\Gamma^f{}_{bd} \\ &+ (m-n-1)\alpha_b\alpha_d.\end{aligned} \tag{46}$$

If Γ is Levi-Civita then $\alpha_a = -D^{-1}\log(g^{1/2})_{,a}$ and

$$(-1)^{db}\alpha_{d,b} = \alpha_{b,d}. \tag{47}$$

The superprojective Ricci symbol transforms as the superdiffeomorphism field up to a constant. This provides us with an alternative way to deduce the parity of $\mathcal{D}$. Furthermore, we may define a tensor on M defined as

$$\mathcal{P}_{ab} = \mathcal{D}_{ab} - \alpha_{a,b} + \alpha_f\Gamma^f{}_{ab} + \alpha_a\alpha_b. \tag{48}$$

$\mathcal{P}_{ab}$ is a rank-2c-type tensor with parity $(-1)^{a+b}$, and is known to differential geometers in the ungraded setting as the projective Schouten tensor. If our connection is Levi-Civita then $\mathcal{P}$ is supersymmetric, i.e., $\mathcal{P}_{ab} = (-1)^{ab}\mathcal{P}_{ba}$. By inserting a parity factor and contracting over the first and third indices of the superprojective Riemann curvature symbol, we obtain the superprojective Ricci symbol from before except that $\mathcal{P}$ and $\mathcal{D}$ are now present,

$$\begin{aligned}\mathcal{R}_{bd} &= (-1)^{c(b+c)}\mathcal{R}^c{}_{bcd} \\ &= R_{bd} + (m-n-1)(\mathcal{P}_{bd} - \mathcal{D}_{bd}) \\ &\quad + (-1)^{bd}\alpha_{d,b} - \alpha_{b,d}.\end{aligned} \tag{49}$$

Rearranging the above, we have the following form for the superdiffeomorphism field:

$$\mathcal{D}_{bd} = -\frac{1}{m-n-1}\mathcal{R}_{bd} + \left(\frac{1}{m-n-1}(R_{bd} + (-1)^{bd}\alpha_{d,b} - \alpha_{b,d}) + \mathcal{P}_{bd}\right). \tag{50}$$

One can always shift the superdiffeomorphism field by any symmetric rank-2 tensor on M.

V. THE SUPER-THOMAS-WHITEHEAD CURVATURE TENSOR

It is natural to form a curvature tensor from the super-TW connection, which we call the super-TW curvature tensor or the superprojective Riemann curvature tensor,

$$\mathcal{K}^{\alpha}{}_{\beta\gamma\delta} = -\tilde{\Gamma}^{\alpha}{}_{\beta\gamma,\delta} + (-1)^{\gamma\delta}\tilde{\Gamma}^{\alpha}{}_{\beta\delta,\gamma} - (-1)^{\delta(\epsilon+\beta+\gamma)}\tilde{\Gamma}^{\alpha}{}_{\epsilon\delta}\tilde{\Gamma}^{\epsilon}{}_{\beta\gamma} + (-1)^{\gamma(\epsilon+\beta)}\tilde{\Gamma}^{\alpha}{}_{\epsilon\gamma}\tilde{\Gamma}^{\epsilon}{}_{\beta\delta}. \tag{51}$$

Before discussing how $K^{\alpha}{}_{\beta\gamma\delta}$ transforms, we recall how a tensor of rank (1,3) transforms under a change of basis,

$$\bar{T}^{a_1}{}_{a_2a_3a_4} = (-1)^{\Delta_4(a+b,b)}T^{b_1}{}_{b_2b_3b_4b_1}(L^{-1\sim})^{a_1b_2}L_{a_2}{}^{b_3}L_{a_3}{}^{b_4}L_{a_4}, \tag{52}$$

where Δ for a tensor of rank (r,s) is defined as $(q = r+s)$

$$\Delta_q(a,b) \equiv \sum_{\substack{t,u=1\\ t<u}}^{n} a_t b_u. \tag{53}$$

Hence, $\mathcal{K}^{\alpha}{}_{\beta\gamma\delta}$ transforms as

$$\mathcal{K}^{\alpha_1}{}_{\alpha_2\alpha_3\alpha_4} \to (-1)^{\Delta_4(\alpha+\beta,\beta)}\mathcal{K}^{\beta_1}{}_{\beta_2\beta_3\beta_4\beta_1}(L^{-1\sim})^{\alpha_1\beta_2}L_{\alpha_2}{}^{\beta_3}L_{\alpha_3}{}^{\beta_4}L_{\alpha_4}. \tag{54}$$

The nonvanishing components of $\mathcal{K}^{\alpha}{}_{\beta\gamma\delta}$ are

$$\mathcal{K}^{\lambda}{}_{abc} = \lambda((-1)^{bc}\mathcal{D}_{ac,b} - \mathcal{D}_{ab,c} + (-1)^{b(a+d)}\mathcal{D}_{db}\Pi^{d}{}_{ac} - (-1)^{c(a+b+d)}\mathcal{D}_{dc}\Pi^{d}{}_{ab}), \tag{55}$$

$$\begin{aligned}\mathcal{K}^{a}{}_{bcd} &= \mathcal{R}^{a}{}_{bcd} + (-1)^{bc}\delta^{a}{}_{c}\mathcal{D}_{bd} - (-1)^{d(b+c)}\delta^{a}{}_{d}\mathcal{D}_{bc}\\ &= R^{a}{}_{bcd} - \delta^{a}{}_{b}((-1)^{cd}\mathcal{P}_{dc} - \mathcal{P}_{cd}) + (-1)^{bc}\delta^{a}{}_{c}\mathcal{P}_{bd} - (-1)^{d(b+c)}\delta^{a}{}_{d}\mathcal{P}_{bc},\end{aligned}$$

where $K^{a}{}_{bcd}$ is called the superprojective Riemann curvature tensor on M. For convenience we also introduce the tensor

$$\check{\mathcal{K}}_{abc} = \frac{1}{\lambda}\mathcal{K}^{\lambda}{}_{abc}. \tag{56}$$

Under a superprojective transformation $\mathcal{P}$ transforms as

$$\mathcal{P}_{ab} \to \mathcal{P}_{ab} + v_{a;b} - v_a v_b. \tag{57}$$

This transformation law arises from the definition of $\mathcal{P}_{bc}$ in terms of $\mathcal{D}_{bc}$, as $\mathcal{D}_{bc}$ is invariant under projective transformations. Rewriting the other components of the super-Thomas-Whitehead curvature tensor, we have

$$\check{\mathcal{K}}_{abc} = (-1)^{cb}\mathcal{P}_{ac,b} - \mathcal{P}_{ab,c} + (-1)^{cb}\alpha_a\mathcal{P}_{cb} - \alpha_a\mathcal{P}_{bc} + (-1)^{b(a+d)}\mathcal{P}_{db}\Gamma^{d}{}_{ac} - (-1)^{c(a+b+d)}\mathcal{P}_{dc}\Gamma^{d}{}_{ab} + \mathcal{P}_{ab}\alpha_c - (-1)^{bc}\mathcal{P}_{ac}\alpha_b - \alpha_f R^{f}{}_{abc}. \tag{58}$$

Similarly, the super-TW Ricci tensor is formed by contracting the first and third indices of the super-TW curvature tensor

$$\mathcal{K}_{bd} = R_{bd} + (m-n)\mathcal{P}_{bd} - (-1)^{db}\mathcal{P}_{db}. \tag{59}$$

We can write the super-TW Ricci tensor in terms of the connection by taking the trace

$$R_{jl} = (-1)^{k(j+1)}R^{k}{}_{jkl}$$

and the super-TW Ricci scalar as

$$\mathcal{K} = \mathcal{K}_{ab}g^{ba} = R + (m-n-1)\mathcal{P}. \tag{60}$$

VI. THE SUPER-THOMAS-WHITEHEAD ACTION

The TW action [14–16] is constructed from the sum of the projective Einstein-Hilbert and projective Gauss-Bonnet Lagrangians. In this section we construct the analogous super-TW action as the sum of the superprojective Einstein-Hilbert Lagrangian, $\mathcal{L}_{\text{SPEH}}$, and the superprojective Gauss-Bonnet Lagrangian, $\mathcal{L}_{\text{SPGB}}$. In the TW action, g_{ab}, Π^{a}_{bc}, and $\mathcal{D}_{ab}$ are all independent field degrees of freedom, where the metric serves to build coordinate invariant objects.

In the super-TW action, $\mathcal{L}_{\text{SPEH}}$ generates the super Einstein-Hilbert term and couples the metric on M and the superdiffeomorphism field. The tensor $\check{\mathcal{K}}_{abc}$ contains $\mathcal{D}$, Π, and derivatives on $\mathcal{D}$. The square of $\check{\mathcal{K}}_{abc}$ sources dynamics of $\mathcal{D}$ that arise in the projective Gauss-Bonnet action. In the limit where Π is compatible with the metric and the diffeomorphism field vanishes the TW action collapses to the Einstein-Hilbert action. This follows from the fact that the Gauss-Bonnet action is a topological invariant in four dimensions. Furthermore, it is known that in any dimension, $\mathcal{L}_{\text{SPGB}}$ has only second-order derivatives of the metric [40,41], which keeps the field equations from developing higher time derivatives. Finally, we emphasize that the TW Lagrangian over VM is invariant under both supercoordinate and superprojective transformations.

The first task is to endow our supermanifold M with a metric g, and then promote g to a metric G on VM. Let us recall the structure of a c-type matrix k that acts on an (m,n)-dimensional supervector space:

$$k = \begin{pmatrix} A_{m\times m} & C_{m\times n} \\ D_{n\times m} & B_{n\times n} \end{pmatrix}, \tag{61}$$

where A, B, C, and D are of type c, c, a, and a, respectively. For the Thomas cone we need to increase the dimension of the bosonic sector by one and arrange the decomposition of the matrix in order to showcase both the pure and mixed subsectors

$$K = \begin{pmatrix} A^1_{1\times 1} & A^2_{1\times m} & C^1_{1\times n} \\ A^3_{m\times 1} & A^4_{m\times m} & C^2_{m\times n} \\ D^1_{n\times 1} & D^2_{n\times m} & B_{n\times n} \end{pmatrix}. \tag{62}$$

Even though we have changed the bosonic dimension and offered a refined decomposition, the components of A^i, B^j, C^k, and D^l are still of type c, c, a, and a, respectively. We promote the metric over VM [14,15] to the graded setting

$$\begin{aligned} {}_\mu G_\nu &= \begin{pmatrix} -\frac{\lambda_0^2}{\lambda^2} & -\frac{\lambda_0^2}{\lambda} g_b & -\frac{\lambda_0^2}{\lambda} g_B \\ -\frac{\lambda_0^2}{\lambda} {}_a g & {}_a g_b - \lambda_0^2 {}_a g g_b & {}_a g_B - \lambda_0^2 {}_a g g_B \\ -\frac{\lambda_0^2}{\lambda} {}_A g & {}_A g_b - \lambda_0^2 {}_A g g_b & {}_A g_B - \lambda_0^2 {}_A g g_B \end{pmatrix} \\ &= \begin{pmatrix} {}_\lambda G_\lambda & {}_\lambda G_b & {}_\lambda G_B \\ {}_a G_\lambda & {}_a G_b & {}_a G_B \\ {}_A G_\lambda & {}_A G_b & {}_A G_B \end{pmatrix}, \end{aligned} \tag{63}$$

where a and b range over the even coordinates (except λ), A and B range over the odd coordinates, μ and ν range over all coordinates, and $g_a \equiv -D^{-1}\log(g^{1/2})_{,a} = (-1)^a({}_a g)$. If we choose the Levi-Civita connection, then $g_a = \alpha_a$. λ_0 is introduced in order to render the components of the metric dimensionless and g_a has units of inverse length.

Considering the case $M = \mathbb{R}^m_c \times \mathbb{R}^n_a$, the metric simplifies considerably as $g_{aA} = g_{Aa} = 0$ and $g_a = g_A = 0$. The canonical form of the metric on $\mathbb{R}^2_c \times \mathbb{R}^2_a$ is [24]

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} {}_a\eta_b & 0 \\ 0 & {}_A\eta_B \end{pmatrix}, \tag{64}$$

implying the metric on the volume bundle of $\mathbb{R}^m_c \times \mathbb{R}^n_a$ is

$${}_\mu G_\nu = \begin{pmatrix} -\frac{\lambda_0^2}{\lambda^2} & 0 & 0 \\ 0 & {}_a\eta_b & 0 \\ 0 & 0 & {}_A\eta_B \end{pmatrix}. \tag{65}$$

We have the relationship between the metric on M and on VM

$$\mathrm{sdet}({}_\mu G_\nu) = -\frac{\lambda_0^2}{\lambda^2}\mathrm{sdet}({}_{\dot\mu} g_{\dot\nu}), \tag{66}$$

where $\dot\mu$, $\dot\nu$ range over all coordinates except λ. G satisfies $G^{\mu\nu} = (-1)^{\mu\nu} G^{\nu\mu}$ and transforms on VM as

$$G_{\mu\nu} \to (-1)^{\sigma(\rho+\mu)} G_{\rho\sigma}{}^\rho(L^{-1})_\mu{}^\sigma(L^{-1})_\nu, \tag{67}$$

where the metric with both of its indices to the bottom right takes the form

$$G_{\mu\nu} = \begin{pmatrix} -\frac{\lambda_0^2}{\lambda^2} & 0 & 0 \\ 0 & g_{ab} & 0 \\ 0 & 0 & g_{AB} \end{pmatrix} = \begin{pmatrix} -\frac{\lambda_0^2}{\lambda^2} & 0 \\ 0 & g_{\dot\mu\dot\nu} \end{pmatrix} (-1)^\mu {}_\mu G_\nu. \tag{68}$$

This metric is symmetric and invariant under superprojective transformations by construction. The inverse is

$$G^{\mu\nu} = \begin{pmatrix} -\frac{\lambda^2}{\lambda_0^2} & 0 & 0 \\ 0 & g^{ab} & 0 \\ 0 & 0 & g^{AB} \end{pmatrix} = \begin{pmatrix} -\frac{\lambda^2}{\lambda_0^2} & 0 \\ 0 & g^{\dot\mu\dot\nu} \end{pmatrix} = {}^\mu G^\nu. \tag{69}$$

Now that we have a metric on VM we are ready to construct an action. We revert back to our old convention where Greek and Latin indices range over the coordinates of VM and M, respectively. This change causes the metric on VM to take the shape

$${}_\mu G_\nu = \begin{pmatrix} {}_\lambda G_\lambda & {}_\lambda G_b \\ {}_a G_\lambda & {}_a G_b \end{pmatrix} = \begin{pmatrix} -\lambda^{-2} & -\lambda^{-1} g_b \\ -\lambda^{-1}{}_a g & {}_a g_b - {}_a g g_b \end{pmatrix}, \tag{70}$$

with inverse given by

$${}^\nu G^\rho = \begin{pmatrix} \lambda^2(g_m{}^m g^n{}_n g - 1) & -\lambda g_m{}^m g^c \\ -\lambda^b g^m{}_m g & {}^b g^c \end{pmatrix}, \tag{71}$$

Our next task is to construct the square of the superprojective Riemann curvature tensor, which has 12 parity terms. Shifting the metrics to the left will result in many more parity terms:

$$\mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}_{\alpha}{}^{\beta\gamma\delta} = (-1)^{\rho(\mu+\nu+\sigma+\alpha+\beta)+\sigma(\mu+\nu+\alpha+\beta+\gamma)+\nu(\mu+\alpha)} \times \mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\nu\rho\sigma\mu}G_{\alpha}G^{\nu\beta}G^{\rho\gamma}G^{\sigma\delta}. \quad (72)$$

The next summand we need is the square of the superprojective Ricci tensor, given by

$$\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta} = (-1)^{(\nu+\beta)\delta}\mathcal{K}_{\beta\delta}\mathcal{K}_{\nu\sigma}G^{\sigma\delta}G^{\nu\beta}, \quad (73)$$

which can be written in terms of its ancestor, the superprojective Riemann curvature tensor, as

$$\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta} = (-1)^{\gamma(\nu+\sigma+\beta+\gamma)+\alpha(\nu+\sigma)+\rho(\nu+\rho)+\sigma(\nu+\beta)} \times \mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\nu\rho\sigma}\delta^{\rho}{}_{\mu}\delta^{\gamma}{}_{\alpha}G^{\nu\beta}G^{\sigma\delta}. \quad (74)$$

Finally, we need the square of the superprojective Ricci scalar

$$\mathcal{K}^{2} = (-1)^{\gamma(\beta+\gamma)+\rho(\nu+\rho)+(\nu+\sigma)(\alpha+\gamma)}\mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\nu\rho\sigma}\delta^{\rho}{}_{\mu}\delta^{\gamma}{}_{\alpha}G^{\sigma\nu}G^{\delta\beta}. \quad (75)$$

If we change the order of the tensors in this expression more parity factors will arise, so our convention in expressing candidates for $\mathcal{L}$ will begin with products of K, followed by δ and then G. Bringing everything together, we have the superprojective Einstein-Hilbert Lagrangian and the superprojective Gauss-Bonnet Lagrangian:

$$\mathcal{L}_{\mathrm{SPEH}} = \mathcal{K}, \quad (76)$$

$$\mathcal{L}_{\mathrm{SPGB}} = \mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}_{\alpha}{}^{\beta\gamma\delta} - 4\mathcal{K}_{\beta\delta}\mathcal{K}^{\beta\delta} + \mathcal{K}^{2} = \mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\nu\rho\sigma}\mathcal{C}^{\sigma\delta\rho\gamma\nu\beta}{}_{\mu\alpha}, \quad (77)$$

where $\mathcal{C}$ is the superprojective Gauss-Bonnet tensor on VM

$$\mathcal{C}^{\sigma\delta\rho\gamma\nu\beta}{}_{\mu\alpha} = ((-1)^{\delta(\mu+\nu+\rho+\alpha+\beta+\gamma)+\gamma(\mu+\nu+\alpha+\beta)+\beta(\mu+\alpha)+\mu}G^{\sigma\delta}G^{\rho\gamma}G^{\nu\beta}G_{\mu\alpha} - 4(-1)^{\gamma(\nu+\sigma+\beta+\gamma)+\alpha(\nu+\sigma)+\rho(\nu+\rho)+\sigma(\nu+\beta)}\delta^{\rho}{}_{\mu}\delta^{\gamma}{}_{\alpha}G^{\nu\beta}G^{\sigma\delta} + (-1)^{\gamma(\beta+\gamma)+\rho(\nu+\rho)+(\nu+\sigma)(\alpha+\gamma)}\delta^{\rho}{}_{\mu}\delta^{\gamma}{}_{\alpha}G^{\sigma\nu}G^{\delta\beta}). \quad (78)$$

The motivation for the projective Einstein-Hilbert action arises directly from general relativity. The projective Gauss-Bonnet term is added as it contains second-order derivatives on $\mathcal{D}$, thus rendering $\mathcal{D}$ dynamical, while maintaining second-order differential equations for the metric, which recovers the Einstein field equations in a certain limit. The possible relation between the projective Gauss-Bonnet term and topological properties of supermanifolds is under investigation.

We define the super-Gauss-Bonnet symbol $\mathcal{G}$ on M and the super-Gauss-Bonnet tensor $\mathcal{B}$ on M as

$$\mathcal{G}^{hdgcfb}{}_{ea} \equiv (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}g^{hd}g^{gc}g^{fb}(g_{ea} - g_{e}g_{a}) - 4(-1)^{c(f+h+b+c)+a(f+h)+g(f+g)+h(f+b)}\delta^{g}{}_{e}\delta^{c}{}_{a}g^{fb}g^{hd} + (-1)^{c(b+c)+g(f+g)+(f+h)(a+c)}\delta^{g}{}_{e}\delta^{c}{}_{a}g^{hf}g^{db}, \quad (79)$$

$$\mathcal{B}^{hdgcfb}{}_{ea} \equiv \mathcal{G}^{hdgcfb}{}_{ea} \quad (80)$$

$$= B_{1}g^{hd}g^{gc}g^{fb}g_{ea} - B_{2}\delta^{g}{}_{e}\delta^{c}{}_{a}g^{fb}g^{hd} + B_{3}\delta^{g}{}_{e}\delta^{c}{}_{a}g^{hf}g^{db}, \quad (81)$$

where we have introduced

$$B_{1} \equiv (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}, \quad (82)$$

$$B_{2} \equiv 4(-1)^{c(f+h+b+c)+a(f+h)+g(f+g)+h(f+b)}, \quad (83)$$

$$B_{3} \equiv (-1)^{c(b+c)+g(f+g)+(f+h)(a+c)}. \quad (84)$$

We are now ready to expand the superprojective Gauss-Bonnet Lagrangian as

$$\mathcal{L}_{\mathrm{SPCGB}} = \mathcal{K}^{\alpha}{}_{\beta\gamma\delta}\mathcal{K}^{\mu}{}_{\nu\rho\sigma}\mathcal{C}^{\sigma\delta\rho\gamma\nu\beta}{}_{\mu\alpha} = \mathcal{K}^{a}{}_{bcd}\mathcal{K}^{e}{}_{fgh}\mathcal{G}^{hdgcfb}{}_{ea} + \mathcal{K}^{\lambda}{}_{bcd}\mathcal{K}^{\lambda}{}_{fgh}\mathcal{C}^{hdgcfb}{}_{\lambda\lambda} + \mathcal{K}^{\lambda}{}_{bcd}\mathcal{K}^{e}{}_{fgh}\mathcal{C}^{hdgcfb}{}_{e\lambda} + \mathcal{K}^{a}{}_{bcd}\mathcal{K}^{\lambda}{}_{fgh}\mathcal{C}^{hdgcfb}{}_{\lambda a}. \quad (85)$$

We tackle this Lagrangian one term at a time:

$$\mathcal{L}_{1} = \mathcal{K}^{a}{}_{bcd}\mathcal{K}^{e}{}_{fgh}(\mathcal{B}^{hdgcfb}{}_{ea} - (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}g^{hd}g^{gc}g^{fb}g_{e}g_{a}), \quad (86)$$

$$\mathcal{L}_{2} = -\check{\mathcal{K}}_{bcd}\check{\mathcal{K}}_{fgh}(-1)^{d(f+g+b+c)+c(f+b)}g^{hd}g^{gc}g^{fb}, \quad (87)$$

$$\mathcal{L}_3 = -\check{\mathcal{K}}_{bcd}\mathcal{K}^e{}_{fgh}(-1)^{d(e+f+g+b+c)+c(e+f+b)+e(b+e)}g^{hd}g^{gc}g^{fb}g_e, \tag{88}$$

$$\mathcal{L}_4 = -\mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}(-1)^{d(f+g+a+b+c)+c(f+a+b)+ba}g^{hd}g^{gc}g^{fb}g_a. \tag{89}$$

Let us rearrange $\mathcal{L}_3$ to share the same index structure as $\mathcal{L}_4$, and vice versa:

$$\mathcal{L}_3 = -\mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}g_a(-1)^{a+f(c+d)+g(d+f)+h(f+g)} = \tilde{\mathcal{L}}_4, \tag{90}$$

$$\tilde{\mathcal{L}}_4 + \mathcal{L}_4 = -\mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}g_a(-1)^{a+f(c+d)+g(d+f)+h(f+g)}((-1)^{a(a+b+c+d)+b(c+d)+cd+f(g+h)+gh}+1), \tag{91}$$

$$\mathcal{L}_4 = \mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}(-1)^{d(f+g+a+b+c)+c(f+a+b)+ba}g^{hd}g^{gc}g^{fb}(-g_a) = \tilde{\mathcal{L}}_3, \tag{92}$$

$$\tilde{\mathcal{L}}_3 + \mathcal{L}_3 = -\check{\mathcal{K}}_{bcd}\mathcal{K}^e{}_{fgh}g^{hd}g^{gc}g^{fb}g_e(-1)^{d(e+f+g+b+c)+c(e+f+b)+e(b+e)}((-1)^{e(e+f+g+h)+b(c+d)+cd+f(g+h)+gh}+1). \tag{93}$$

A short calculation shows that $(\tilde{\mathcal{L}}_3 + \mathcal{L}_3) = (\tilde{\mathcal{L}}_4 + \mathcal{L}_4)$. As they are equivalent we pick $(\tilde{\mathcal{L}}_4 + \mathcal{L}_4)$ to participate in our action because there are fewer parity factors. For what follows, we define four parity factors that will be useful:

$$P_0 \equiv (-1)^{c(b+c)}, \tag{94}$$

$$P_1 \equiv (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}, \tag{95}$$

$$P_2 \equiv (-1)^{d(f+g+b+c)+c(f+b)}, \tag{96}$$

$$P_3 \equiv (-1)^{a+f(c+d)+g(d+f)+h(f+g)} \times ((-1)^{a(a+b+c+d)+b(c+d)+cd+f(g+h)+gh}+1). \tag{97}$$

Reintroducing the scale λ_0 and introducing two coupling constants, α_0 and β_0, the final form of the super TW gravity action is

$$S_{STW} = \alpha_0\int \mathcal{K}^a{}_{bcd}\delta^c{}_a g^{db}P_0 G^{\frac{1}{2}}d\lambda d^{m,n}x + \beta_0\int \mathcal{K}^a{}_{bcd}\mathcal{K}^e{}_{fgh}(\mathcal{B}^{hdgcfb}{}_{ea} - \lambda_0^2 g^{hd}g^{gc}g^{fb}g_e g_a P_1)G^{\frac{1}{2}}d\lambda d^{m,n}x$$
$$- \beta_0\lambda_0^2\int \check{\mathcal{K}}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}P_2 G^{\frac{1}{2}}d\lambda d^{m,n}x - \beta_0\lambda_0^2\int \mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}g_a P_3 G^{\frac{1}{2}}d\lambda d^{m,n}x. \tag{98}$$

As stated earlier, setting $\mathcal{P}$ to zero recovers the Einstein-Hilbert action from the superprojective Riemann curvature tensor since

$$\mathcal{K}^a{}_{bcd} = R^a{}_{bcd}, \quad \text{and} \tag{99}$$

$$\check{\mathcal{K}}_{bcd} = -\alpha_a R^a{}_{bcd}. \tag{100}$$

Define the constant C as

$$C = \int_{\lambda_1}^{\lambda_2}\frac{\lambda_0}{\lambda}d\lambda = \lambda_0\log\left(\frac{\lambda_2}{\lambda_1}\right), \tag{101}$$

where $0 < \lambda_1 < \lambda_2 < \infty$. Recalling the relationship between G and g, we have the simplification

$$\int_{\lambda_1}^{\lambda_2} G^{\frac{1}{2}}d\lambda = Cg^{\frac{1}{2}}. \tag{102}$$

Setting n to zero and m to four, and letting $\mathcal{P}\to 0$, the super-TW action reduces to

$$S_{\rm TW} = \alpha_0 C\int Rg^{\frac{1}{2}}d^4x + \beta_0 C \times \int (R^a{}_{bcd}R_a{}^{bcd} - 4R_{bd}R^{bd} + R^2)g^{\frac{1}{2}}d^4x.$$

The first term is the Einstein-Hilbert action, while the second term is the Gauss-Bonnet action, which does not contribute to the Einstein field equations in four dimensions [40].

VII. THE FIELD EQUATIONS

The variation of the canonical measure function is given by

$$\delta g^{\frac{1}{2}} = \frac{1}{2}g^{-\frac{1}{2}}\delta g = \frac{1}{2}g^{\frac{1}{2}}(-1)^{ii}g^j\delta_j g_i. \tag{103}$$

The variation of the inverse metric arises from

$$0 = \delta({}^a\delta_c), \tag{104}$$

$$\delta g^{ad} = \delta g^{ab}{}_b g_c g^{cd} = -g^{ab}\delta_b g_c g^{cd} = -(-1)^{(b+c)(c+d)} g^{ab} g^{cd}\delta_b g_c. \tag{105}$$

Let M be a Riemannian supermanifold. If M is compact, then [24]

$$\int_M (-1)^i (g^{\frac{1}{2}} X^i)_{,i} d^{m,n}x = \int_M (-1)^i g^{\frac{1}{2}} X^i{}_{;i} d^{m,n}x = 0, \tag{106}$$

implying

$$(\delta\Gamma^j{}_{ji} g^{ki})_{;k} = (-1)^{k(i+k)}\delta\Gamma^j{}_{ji;k} g^{ki}, \tag{107}$$

$$-g^{\frac{1}{2}}(-1)^j\delta\Gamma^j{}_{ji;k}g^{ki} = -g^{\frac{1}{2}}(-1)^{j+k(i+k)}(\delta\Gamma^j{}_{ji}g^{ki})_{;k} = -g^{\frac{1}{2}}(-1)^k((-1)^j\delta\Gamma^j{}_{ji}g^{ik})_{;k} = -g^{\frac{1}{2}}(-1)^k X^k{}_{;k}, \tag{108}$$

$$X^k = (-1)^j\delta\Gamma^j{}_{ji}g^{ik}, \tag{109}$$

$$(\delta\Gamma^j{}_{ik}g^{ki})_{;j} = (-1)^{j(i+k)}\delta\Gamma^j{}_{ik;j}g^{ki}, \tag{110}$$

$$g^{\frac{1}{2}}(-1)^{j(i+j+k)}\delta\Gamma^j{}_{ik;j}g^{ki} = g^{\frac{1}{2}}(-1)^j(\delta\Gamma^j{}_{ik}g^{ki})_{;j} = g^{\frac{1}{2}}(-1)^j Y^j{}_{;j}, \tag{111}$$

$$Y^j = \delta\Gamma^j{}_{ik}g^{ki}. \tag{112}$$

We are now in a position to find field equations of the dynamical fields Π, $\mathcal{D}$, and g in the action. We express the action in terms of the superprojective Cotton-York symbol, reducing the action to a functional that is on M, viz,

$$S_{\mathrm{STW}} = \sum_{i=1}^{5} S_i, \tag{113}$$

$$S_1 = \alpha_0 C\int \mathcal{K}^a{}_{bcd}\delta^c{}_a g^{db}P_0 g^{\frac{1}{2}} d^{m,n}x, \tag{114}$$

$$S_2 = \beta_0 C\int \mathcal{K}^a{}_{bcd}\mathcal{K}^e{}_{fgh}\mathcal{B}^{hdgcfb}{}_{ea} g^{\frac{1}{2}} d^{m,n}x, \tag{115}$$

$$S_3 = -\beta_0 C\lambda_0^2\int \mathcal{K}^a{}_{bcd}\mathcal{K}^e{}_{fgh}g^{hd}g^{gc}g^{fb}g_e g_a P_1 g^{\frac{1}{2}} d^{m,n}x, \tag{116}$$

$$S_4 = -\beta_0 C\lambda_0^2\int \check{\mathcal{K}}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}P_2 g^{\frac{1}{2}} d^{m,n}x, \tag{117}$$

$$S_5 = -\beta_0 C\lambda_0^2\int \mathcal{K}^a{}_{bcd}\check{\mathcal{K}}_{fgh}g^{hd}g^{gc}g^{fb}g_a P_3 g^{\frac{1}{2}} d^{m,n}x, \tag{118}$$

$$\mathcal{B}^{hdgcfb}{}_{ea} = B_1 g^{hd}g^{gc}g^{fb}g_{ea} - B_2\delta^g{}_e\delta^c{}_a g^{fb}g^{hd} + B_3\delta^g{}_e\delta^c{}_a g^{hf}g^{db}. \tag{119}$$

Observe that the dependence on Π and $\mathcal{D}$ resides in the superprojective Cotton-York symbol and the superprojective Riemann curvature tensor, while the metric dependence resides only in the superdeterminant of the metric, the inverse metric, and the super-Gauss-Bonnet tensor. Before we proceed, we note that S_{STW} can be expressed differently with a particular combination [14,15] of the nontrivial coefficients of the superprojective Riemann curvature tensor on VM. This combination happens to be a tensor over M, known as the projective Cotton-York tensor. The ungraded version is

$$g_a K^a{}_{bcd} + \check{\mathcal{K}}_{bcd}. \tag{120}$$

In every variation below, we hide the constants, α_0, β_0, λ_0, and C.

A. Field equations for Π

Varying S_1 with respect to the superfundamental projective invariant gives

$$\begin{aligned}\delta S_1 = \int \delta\Pi^x{}_{yz}(&\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_1{}^c{}_a{}^{db}{}_{,d}(-1)^{d(a+b+c+d)+c} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_1{}^c{}_a{}^{db}{}_{,c}(-1)^{c(a+b)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{c(b+c+f)}\\ &+\Pi^a{}_{fc}\delta^f{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{c(b+c+f)+(x+y+z)(a+c+f)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{d(b+c+f)+c}\\ &-\Pi^a{}_{fd}\delta^f{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{d(b+c+f)+c+(x+y+z)(a+d+f)}),\\ \mathcal{F}_1{}^c{}_a{}^{db} = (-1)^{bc}&\delta^c{}_a g^{db} g^{\frac{1}{2}},\end{aligned} \tag{121}$$

The variation of S_2 with respect to Π is

$$\begin{aligned}\delta S_2 = \int \delta\Pi^x{}_{yz}(&\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_2{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_2{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_2{}^{dcb}{}_a(-1)^{c(f+b)}\\ &+\delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_2{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_2{}^{dcb}{}_a(-1)^{d(b+c+f)}\\ &-\delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_2{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)}),\end{aligned} \tag{122}$$

$$\mathcal{F}_2{}^{dcb}{}_a = \mathcal{K}^e{}_{fgh}(\mathcal{B}^{hdgcfb}{}_{ea} + \mathcal{B}^{dhcgbf}{}_{ae}P_7)g^{\frac{1}{2}}, \tag{123}$$

$$P_7 = (-1)^{(a+b+c+d)(e+f+g+h)}. \tag{124}$$

The variation of S_3 with respect to Π is

$$\delta S_3 = -\int \delta\Pi^x{}_{yz}(\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_3{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_3{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_3{}^{dcb}{}_a(-1)^{c(f+b)}$$
$$+ \delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_3{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_3{}^{dcb}{}_a(-1)^{d(b+c+f)}$$
$$- \delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_3{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)}). \tag{125}$$

The variation of S_4 with respect to Π is

$$\delta S_4 = -\int \delta\Pi^x{}_{yz}(\delta^e{}_x\delta^y{}_b\delta^z{}_d\mathcal{D}_{ec}(-1)^{(e+c)(e+b+d)+c(b+e)} - \delta^e{}_x\delta^y{}_b\delta^z{}_c\mathcal{D}_{ed}(-1)^{d(b+c+e)+(e+d)(e+b+c)})\mathcal{F}_4{}^{dcb}{}_a, \tag{126}$$

$$\mathcal{F}_4{}^{dcb}{}_a = \mathcal{K}_{fgh}(g^{hd}g^{gc}g^{fb}P_2 + g^{dh}g^{cg}g^{bf}\tilde{P}_2P_8)g^{\frac{1}{2}}, \tag{127}$$

$$\tilde{P}_2 = (-1)^{h(f+g+b+c)+g(f+b)}, \tag{128}$$

$$P_8 = (-1)^{(b+c+d)(f+g+h)}. \tag{129}$$

Finally, the variation of S_5 with respect to Π is

$$\delta S_5 = -\int \delta\Pi^x{}_{yz}(\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_5{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_5{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_5{}^{dcb}{}_a(-1)^{c(f+b)}$$
$$+ \delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_5{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c+f)}$$
$$- \delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)} + \delta^e{}_x\delta^y{}_f\delta^z{}_h\mathcal{D}_{eg}\mathcal{F}_6{}^{hgf}(-1)^{(e+g)(e+f+h)+g(f+e)}$$
$$- \delta^e{}_x\delta^y{}_f\delta^z{}_g\mathcal{D}_{eh}\mathcal{F}_6{}^{hgf}(-1)^{h(f+g+e)+(e+h)(e+f+g)}), \tag{130}$$

$$\mathcal{F}_5{}^{dcb}{}_a = \mathcal{K}_{fgh}g^{hd}g^{gc}g^{fb}g_aP_3g^{\frac{1}{2}}, \tag{131}$$

$$\mathcal{F}_6{}^{hgf} = \mathcal{K}^a{}_{bcd}g^{hd}g^{gc}g^{fb}g_aP_3P_9g^{\frac{1}{2}}, \tag{132}$$

$$P_9 = (-1)^{(a+b+c+d)(f+g+h)}. \tag{133}$$

Adding up the variations gives the field equations for Π. The full field equations can be found in Appendix B [see Eq. (B1)].

B. Field equations for $\mathcal{D}_{bd}$

Again we start with S_1:

$$\delta S_1 = \int \delta\mathcal{D}_{xy}(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\mathcal{F}_7{}^c{}_a{}^{db},$$
$$\mathcal{F}_7{}^c{}_a{}^{db} = \delta^c{}_ag^{db}P_0g^{\frac{1}{2}}. \tag{134}$$

The variation of S_2 is

$$\delta S_2 = \int \delta\mathcal{D}_{xy}(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\mathcal{F}_2{}^{dcb}{}_a.$$

The variation of S_3 is

$$\delta S_3 = -\int \delta\mathcal{D}_{xy}(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\mathcal{F}_3{}^{dcb}{}_a.$$

The variation of S_4 is

$$\delta S_4 = -\int \delta\mathcal{D}_{xy}(-\delta^x{}_b\delta^y{}_d\mathcal{F}_4{}^{dcb}{}_{a,c}(-1)^{dc+c(a+b+c+d)}$$
$$+ \delta^x{}_b\delta^y{}_c\mathcal{F}_4{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)}$$
$$+ \delta^x{}_e\delta^y{}_c\Pi^e{}_{bd}\mathcal{F}_4{}^{dcb}{}_a(-1)^{c(b+e)}$$
$$- \delta^x{}_e\delta^y{}_d\Pi^e{}_{bc}\mathcal{F}_4{}^{dcb}{}_a(-1)^{d(b+c+e)}), \tag{135}$$

and lastly the variation of S_5 is

$$\delta S_5 = -\int \delta\mathcal{D}_{xy}(\delta^x{}_b\delta^y{}_d\delta^a{}_c\mathcal{F}_5{}^{dcb}{}_a(-1)^{bc+(a+c)(b+d)} - \delta^x{}_b\delta^y{}_c\delta^a{}_d\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c)+(b+c)(a+d)}$$
$$- \delta^x{}_f\delta^y{}_h\mathcal{F}_6{}^{hgf}{}_{,g}(-1)^{hg+g(f+g+h)} + \delta^x{}_f\delta^y{}_g\mathcal{F}_6{}^{hgf}{}_{,h}(-1)^{h(f+g+h)}$$
$$+ \delta^x{}_e\delta^y{}_g\Pi^e{}_{fh}\mathcal{F}_6{}^{hgf}(-1)^{g(f+e)} - \delta^x{}_e\delta^y{}_h\Pi^e{}_{fg}\mathcal{F}_6{}^{hgf}(-1)^{h(f+g+e)}). \tag{136}$$

The full field equations are again in Appendix B [see Eq. (B2)].

C. The field equations for g_{ab}

Let us begin by rearranging and relabeling the action to make the metric dependence explicit, while also keeping in mind the connection and metric are independent. We start by breaking up the action to give

$$S_1 = \int \sqrt{g}g^{db}\mathcal{F}_{8bd}, \tag{137}$$

$$\mathcal{F}_{8bd} = \mathcal{K}^a{}_{bcd}\delta^c{}_a P_0(-1)^{b+d}, \tag{138}$$

$$S_2 = \int \sqrt{g}\mathcal{B}^{hdgcfb}{}_{ea}\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh}, \tag{139}$$

$$\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh} = \mathcal{K}^a{}_{bcd}\mathcal{K}^e{}_{fgh}P_{10}, \tag{140}$$

$$P_{10} = (-1)^{a+b+c+d+e+f+g+h}, \tag{141}$$

$$S_3 = \int \sqrt{g}g^{hd}g^{gc}g^{fb}g_e g_a\mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh}, \tag{142}$$

$$\mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh} = -\mathcal{K}^a{}_{bcd}\mathcal{K}^e{}_{fgh}P_1P_{10}, \tag{143}$$

$$S_4 = \int \sqrt{g}g^{hd}g^{gc}g^{fb}\mathcal{F}_{11bcdfgh}, \tag{144}$$

$$\mathcal{F}_{11bcdfgh} = -\mathcal{K}_{bcd}\mathcal{K}_{fgh}P_2(-1)^{b+c+d+f+g+h}, \tag{145}$$

$$S_5 = \int \sqrt{g}g^{hd}g^{gc}g^{fb}g_a\mathcal{F}_{12}{}^a{}_{bcdfgh}, \tag{146}$$

$$\mathcal{F}_{12}{}^a{}_{bcdfgh} = -\mathcal{K}^a{}_{bcd}\mathcal{K}_{fgh}P_3(-1)^{a+b+c+d+f+g+h}. \tag{147}$$

We will vary with respect to g^{ab}. The family of variations that will be needed are

$$\delta g_c = \delta g^{ab}V_{1bac} + \delta g^{ab}{}_{,c}V_{2ba},$$
$$V_{1bac} = \frac{1}{2D}g_{ba,c}(-1)^{a+b}, \tag{148}$$

$$V_{2ba} = \frac{1}{2D}g_{ba}(-1)^{c(a+b)+a+b}, \tag{149}$$

$$\tilde{B}_2 = -(-1)^{(f+b+h+d)(g+e+c+a)}B_2, \tag{150}$$

$$\tilde{B}_3 = (-1)^{(h+f+d+b)(g+e+c+a)}B_3, \tag{151}$$

$$\delta\mathcal{B}^{hdgcfb}{}_{ea} = \delta g^{xy}[B_1(\delta^h{}_x\delta^d{}_y g^{gc}g^{fb}g_{ea} + \delta^g{}_x\delta^c{}_y g^{hd}g^{fb}g_{ea}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y g^{hd}g^{gc}g_{ea}(-1)^{(h+d+g+c)(f+b)}$$
$$+ V_{0exya}g^{hd}g^{gc}g^{fb}(-1)^{(h+d+g+c+f+b)(e+a)}) + \tilde{B}_2(\delta^f{}_x\delta^b{}_y g^{hd}\delta^g{}_e\delta^c{}_a + \delta^h{}_x\delta^d{}_y g^{fb}\delta^g{}_e\delta^c{}_a(-1)^{(f+b)(h+d)})$$
$$+ \tilde{B}_3(\delta^h{}_x\delta^f{}_y g^{db}\delta^g{}_e\delta^c{}_a + \delta^d{}_x\delta^b{}_y g^{hf}\delta^g{}_e\delta^c{}_a(-1)^{(h+f)(d+b)})],$$
$$\delta g_a = \delta g^{xy}V_{1yxa} + \delta g^{xy}{}_{,a}V_{2yx}. \tag{152}$$

Using the above identities, the variation of S_1 with respect to the metric yields

$$\delta S_1 = \int \delta g^{xy}\left[\left(-\frac{1}{2}\sqrt{g}_y g_x(-1)^x\right)g^{db}\mathcal{F}_{8bd} + \delta^d{}_x\delta^b{}_y\sqrt{g}\mathcal{F}_{8bd}\right]. \tag{153}$$

Similarly, the variation of S_2 with respect to the metric is

$$\delta S_2 = \int \delta g^{xy}\Big[\Big(-\frac{1}{2}\sqrt{g}_y g_x(-1)^x\Big)\mathcal{B}^{hdgcfb}{}_{ea}\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh} + (B_1(\delta^h{}_x\delta^d{}_y g^{gc}g^{fb}g_{ea} + \delta^g{}_x\delta^c{}_y g^{hd}g^{fb}g_{ea}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y g^{hd}g^{gc}g_{ea}(-1)^{(h+d+g+c)(f+b)} + V_{0exya}g^{hd}g^{gc}g^{fb}(-1)^{(h+d+g+c+f+b)(e+a)}) + \tilde{B}_2(\delta^f{}_x\delta^b{}_y g^{hd}\delta^g{}_e\delta^c{}_a + \delta^h{}_x\delta^d{}_y g^{fb}\delta^g{}_e\delta^c{}_a(-1)^{(f+b)(h+d)}) + \tilde{B}_3(\delta^h{}_x\delta^f{}_y g^{db}\delta^g{}_e\delta^c{}_a + \delta^d{}_x\delta^b{}_y g^{hf}\delta^g{}_e\delta^c{}_a(-1)^{(h+f)(d+b)}))\sqrt{g}\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh}\Big]. \qquad (154)$$

First, we define $\mathcal{F}_{13}$ and $\mathcal{F}_{14}$ to clean up the variation below:

$$\mathcal{F}_{13}{}^{ae} = \sqrt{g}g^{hd}g^{gc}g^{fb}\mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh}(-1)^{(e+a)(h+d+g+c+f+b)}, \qquad (155)$$

$$\mathcal{F}_{14bcdfgh} = g_e g_a \mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh}. \qquad (156)$$

The variation of S_3 with respect to the metric is

$$\delta S_3 = \int \delta g^{xy}\Big[V_{1yxe}g_a\mathcal{F}_{13}{}^{ae} - (V_{2yx}g_a\mathcal{F}_{13}{}^{ae})_{,e} + V_{1yxa}g_e\mathcal{F}_{13}{}^{ae}(-1)^{ae} - (V_{2yx}g_e\mathcal{F}_{13}{}^{ae}(-1)^{ae})_{,a} + \Big(\Big(-\frac{1}{2}\sqrt{g}_y g_x(-1)^x\Big)g^{hd}g^{gc}g^{fb} + \delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb} + \delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big)\mathcal{F}_{14bcdfgh}\Big]. \qquad (157)$$

The variation of S_4 with respect to the metric is

$$\delta S_4 = \int \delta g^{xy}\Big[\Big(-\frac{1}{2}\sqrt{g}_y g_x(-1)^x\Big)g^{hd}g^{gc}g^{fb} + \delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb} + \delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big]\mathcal{F}_{11bcdfgh}. \qquad (158)$$

We define $\mathcal{F}_{15}$ and $\mathcal{F}_{16}$ in order to clean up the variation below:

$$\mathcal{F}_{15bcdfgh} = g_a\mathcal{F}_{12}{}^a{}_{bcdfgh} \qquad (159)$$

$$\mathcal{F}_{16}{}^a = \sqrt{g}g^{hd}g^{gc}g^{fb}\mathcal{F}_{12}{}^a{}_{bcdfgh}(-1)^{a(h+d+g+c+f+c)}. \qquad (160)$$

The variation of S_5 with respect to the metric is

$$\delta S_5 = \int \delta g^{xy}\Big[\Big(\Big(-\frac{1}{2}\sqrt{g}_y g_x(-1)^x\Big)g^{hd}g^{gc}g^{fb} + \delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb} + \delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big)\mathcal{F}_{15bcdfgh} + V_{1yxa}\mathcal{F}_{16}{}^a - (V_{2yx}\mathcal{F}_{16}{}^a)_{,a}(-1)^{a(a+x+y)}\Big]. \qquad (161)$$

As before, the equations of motion for g_{ab} are found in Appendix B [see Eq. (B3)]. From here one may define an energy-momentum tensor. In order to write the usual Einstein equations, one would be obliged to decompose the fundamental projective invariant into an affine connection and a traceless Palatini tensor. Then one could have the usual Riemannian geometric objects on the left-hand side of these field equations, while the Palatini field equations and the contributions from the diffeomorphism field would move to the right-hand side forming projective geometric sources.

VIII. SIMPLE COSMOLOGICAL MODEL

As a practical example we present the field equations in one of the simplest possible limits and recover de Sitter space as a solution. Assume that Π contains in its equivalence class the metric compatible connection, so that Π and g are no longer independent degrees of freedom. The Π field equation will be trivially satisfied in the spirit of Palatini formalism [36], so there are only two independent fields in the theory, $\mathcal{D}$ and g. Motivated by cosmological implications, we will also impose the condition $\mathcal{D}_{bc} = \Lambda g_{bc}$ where Λ is some constant, so $\mathcal{D}$ is playing the role of a cosmological constant. The field equation for $\mathcal{D}_{bc}$ reduces to

$$\begin{aligned}&\alpha_0(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)} - \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\delta^c{}_a g^{db}P_0\\&+\beta_0(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)} - \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\\&\times((-1)^{fg}\delta^e{}_g\Lambda g_{fh} - (-1)^{h(f+g)}\delta^e{}_h\Lambda g_{fg})(\mathcal{B}^{hdgcfb}{}_{ea} + \mathcal{B}^{dhcgbf}{}_{ae}P_7) = 0,\end{aligned} \tag{162}$$

which is satisfied whenever $\mathcal{D}_{bc} = \Lambda g_{bc}$, while the metric field equations become

$$\begin{aligned}&\alpha_0\left(-\frac{m-n-1}{2}{}_y g_x g^{db}\Lambda g_{bd}(-1)^x + \delta^d{}_x\delta^b{}_y(m-n-1)\Lambda g_{bd}\right) + \beta_0\left[-\frac{1}{2}{}_y g_x \mathcal{B}^{hdgcfb}{}_{ea}((-1)^{bc}\delta^a{}_c\Lambda g_{bd} - (-1)^{d(b+c)}\delta^a{}_d\Lambda g_{bc})\right.\\&\times((-1)^{fg}\delta^e{}_g\Lambda g_{fh} - (-1)^{h(f+g)}\delta^e{}_h\Lambda g_{fg})P_{10}(-1)^x + (B_1(\delta^h{}_x\delta^d{}_y g^{gc}g^{fb}g_{ea} + \delta^g{}_x\delta^c{}_y g^{hd}g^{fb}g_{ea}(-1)^{(h+d)(g+c)}\\&+\delta^f{}_x\delta^b{}_y g^{hd}g^{gc}g_{ea}(-1)^{(h+d+g+c)(f+b)} - g_{ex}g_{ya}g^{hd}g^{gc}g^{fb}(-1)^{(h+d+g+c+f+b)(e+a)+(x+y)(e+x)+y})\\&+\tilde{B}_2\delta^f{}_x\delta^b{}_y g^{hd}\delta^g{}_e\delta^c{}_a + \tilde{B}_2\delta^h{}_x\delta^d{}_y g^{fb}\delta^g{}_e\delta^c{}_a(-1)^{(f+b)(h+d)} + \tilde{B}_3\delta^h{}_x\delta^f{}_y g^{db}\delta^g{}_e\delta^c{}_a\\&\left.+\tilde{B}_3\delta^d{}_x\delta^b{}_y g^{hf}\delta^g{}_e\delta^c{}_a(-1)^{(h+f)(d+b)})((-1)^{bc}\delta^a{}_c\Lambda g_{bd} - (-1)^{d(b+c)}\delta^a{}_d\Lambda g_{bc})((-1)^{fg}\delta^e{}_g\Lambda g_{fh} - (-1)^{h(f+g)}\delta^e{}_h\Lambda g_{fg})P_{10}\right] = 0,\end{aligned}$$

which are proportional to the energy-momentum tensor. We recognize this latter equation as the graded extension [15,16] of the differential equation for (anti-)de Sitter space, so taking the aforementioned limits we are able to recover graded (anti-)de Sitter space as a solution to the graded TW field equations.

IX. CONCLUSION

In this paper, we have generalized TW gravity to a graded setting in the framework of a DeWitt supermanifold. The super-TW gravity action is invariant under superprojective transformations, yields second-order partial differential equations for the metric, the superfundamental projective invariant, and the superdiffeomorphism field. Our construction generated an infinitesimal coordinate transformation law for the superdiffeomorphism field, which recovered the coadjoint action on a coadjoint Virasoro element in a particular limit. Additionally, setting the number of fermionic coordinates to zero, the number of bosonic coordinates to four, and the tensorial relative to the superdiffeomorphism field to zero, the super-TW action simplified to the Einstein-Hilbert action. The super-TW action is the natural precursor to understanding a theory of projective supergravity with dynamical projective connections intimately connected to the super-Virasoro algebra. For instance, setting one of indices of the super Diffeomorphism field to a bosonic index and the other to a fermionic index, one would expect the appearance of a spin-3/2 Rarita-Schwinger field. Also, we expect a supersymmetric version of the super-TW action to make contact with the supersymmetric extension of the 2D Polyakov action [6,9,19,20]. The super-TW action described in this paper is not restricted to supersymmetric coordinates and can be used to investigate other superspace phenomena. Also, our analysis focused completely on tensors and does not address the study of spinors in superspaces and their coupling to the TW connection. Details on the investigation of fermions in TW gravity in the ungraded setting were discussed in detail in [15].

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APPENDIX A: SUMMARY OF CONVENTIONS

Parity factors:

$$P_0 \equiv (-1)^{c(b+c)}, \tag{A1}$$

$$P_1 \equiv (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}, \tag{A2}$$

$$\tilde{P}_1 \equiv (-1)^{h(e+f+g+a+b+c)+g(e+f+a+b)+f(e+a)+a}, \tag{A3}$$

$$P_2 \equiv (-1)^{d(f+g+b+c)+c(f+b)}, \tag{A4}$$

$$\tilde{P}_2 \equiv (-1)^{h(f+g+b+c)+g(f+b)}, \quad \text{(A5)}$$

$$P_3 \equiv (-1)^{a+f(c+d)+g(d+f)+h(f+g)} \times ((-1)^{a(a+b+c+d)+b(c+d)+cd+f(g+h)+gh} + 1), \quad \text{(A6)}$$

$$P_4 \equiv (-1)^{a+e(b+c+d+e)+f(c+d)+g(d+f)+h(f+g)} \times (1 - (-1)^{b(c+d)+cd+f(g+h)+gh}), \quad \text{(A7)}$$

$$P_5 \equiv (-1)^{f(c+d)+g(d+f)+h(f+g)}, \quad \text{(A8)}$$

$$P_6 \equiv (-1)^{b(c+d)+cd+f(g+h)+gh}, \quad \text{(A9)}$$

$$P_7 \equiv (-1)^{(a+b+c+d)(e+f+g+h)}, \quad \text{(A10)}$$

$$P_8 \equiv (-1)^{(b+c+d)(f+g+h)}, \quad \text{(A11)}$$

$$P_9 \equiv (-1)^{(a+b+c+d)(f+g+h)}, \quad \text{(A12)}$$

$$P_{10} \equiv (-1)^{a+b+c+d+e+f+g+h}, \quad \text{(A13)}$$

$$B_1 \equiv (-1)^{d(e+f+g+a+b+c)+c(e+f+a+b)+b(e+a)+e}, \quad \text{(A14)}$$

$$B_2 \equiv 4(-1)^{c(f+h+b+c)+a(f+h)+g(f+g)+h(f+b)}, \quad \text{(A15)}$$

$$\tilde{B}_2 \equiv -(-1)^{(f+b+h+d)(g+e+c+a)} B_2, \quad \text{(A16)}$$

$$B_3 \equiv (-1)^{c(b+c)+g(f+g)+(f+h)(a+c)}, \quad \text{(A17)}$$

$$\tilde{B}_3 \equiv (-1)^{(h+f+d+b)(g+e+c+a)} B_3. \quad \text{(A18)}$$

Tensors and symbols:

$$\mathcal{F}_1{}^c{}_a{}^{db} = (-1)^{bc} \delta^c{}_a g^{db} g^{\frac{1}{2}}, \quad \text{(A19)}$$

$$\mathcal{F}_2{}^{dcb}{}_a = \mathcal{K}^e{}_{fgh} (\mathcal{B}^{hdgcfb}{}_{ea} + \mathcal{B}^{dhcgfb}{}_{ae} P_7) g^{\frac{1}{2}}, \quad \text{(A20)}$$

$$\mathcal{F}_3{}^{dcb}{}_a = \mathcal{K}^e{}_{fgh} (g^{hd} g^{gc} g^{fb} g_e g_a P_1 + g^{dh} g^{cg} g^{bf} g_a g_e \tilde{P}_1 P_7) g^{\frac{1}{2}}, \quad \text{(A21)}$$

$$\mathcal{F}_4{}^{dcb}{}_a = \mathcal{K}_{fgh} (g^{hd} g^{gc} g^{fb} P_2 + g^{dh} g^{cg} g^{bf} \tilde{P}_2 P_8) g^{\frac{1}{2}}, \quad \text{(A22)}$$

$$\mathcal{F}_5{}^{dcb}{}_a = \mathcal{K}_{fgh} g^{hd} g^{gc} g^{fb} g_a P_3 g^{\frac{1}{2}}, \quad \text{(A23)}$$

$$\mathcal{F}_6{}^{hgf} = \mathcal{K}^a{}_{bcd} g^{hd} g^{gc} g^{fb} g_a P_3 P_9 g^{\frac{1}{2}}, \quad \text{(A24)}$$

$$\mathcal{F}_7{}^c{}_a{}^{db} = \delta^c{}_a g^{db} P_0 g^{\frac{1}{2}}, \quad \text{(A25)}$$

$$\mathcal{F}_{8bd} = \mathcal{K}^a{}_{bcd} \delta^c{}_a P_0 (-1)^{b+d}, \quad \text{(A26)}$$

$$\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh} = \mathcal{K}^a{}_{bcd} \mathcal{K}^e{}_{fgh} P_{10}, \quad \text{(A27)}$$

$$\mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh} = -\mathcal{K}^a{}_{bcd} \mathcal{K}^e{}_{fgh} P_1 P_{10}, \quad \text{(A28)}$$

$$\mathcal{F}_{11bcdfgh} = -\mathcal{K}_{bcd} \mathcal{K}_{fgh} P_2 (-1)^{b+c+d+f+g+h}, \quad \text{(A29)}$$

$$\mathcal{F}_{12}{}^a{}_{bcdfgh} = -\mathcal{K}^a{}_{bcd} \mathcal{K}_{fgh} P_3 (-1)^{a+b+c+d+f+g+h}, \quad \text{(A30)}$$

$$\mathcal{F}_{13}{}^{ae} = \sqrt{g} g^{hd} g^{gc} g^{fb} \mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh} (-1)^{(e+a)(h+d+g+c+f+b)}, \quad \text{(A31)}$$

$$\mathcal{F}_{14bcdfgh} = g_e g_a \mathcal{F}_{10}{}^a{}_{bcd}{}^e{}_{fgh}, \quad \text{(A32)}$$

$$\mathcal{F}_{15bcdfgh} = g_a \mathcal{F}_{12}{}^a{}_{bcdfgh}, \quad \text{(A33)}$$

$$\mathcal{F}_{16}{}^a = \sqrt{g} g^{hd} g^{gc} g^{fb} \mathcal{F}_{12}{}^a{}_{bcdfgh} (-1)^{a(h+d+g+c+f+c)}, \quad \text{(A34)}$$

$$V_{0dabc} = -g_{da} g_{bc} (-1)^{(a+b)(a+d)+b}, \quad \text{(A35)}$$

$$V_{1bac} = \frac{1}{2D} g_{ba,c} (-1)^{a+b}, \quad \text{(A36)}$$

$$V_{2ba} = \frac{1}{2D} g_{ba} (-1)^{c(a+b)+a+b}. \quad \text{(A37)}$$

APPENDIX B: FIELD EQUATIONS

1. Field equations for Π

$$\alpha_0 C(\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_1{}^c{}_a{}^{db}{}_{,d}(-1)^{d(a+b+c+d)+c} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_1{}^c{}_a{}^{db}{}_{,c}(-1)^{c(a+b)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{c(b+c+f)}$$
$$+ \Pi^a{}_{fc}\delta^f{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{c(b+c+f)+(x+y+z)(a+c+f)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{d(b+c+f)+c}$$
$$- \Pi^a{}_{fd}\delta^f{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_1{}^c{}_a{}^{db}(-1)^{d(b+c+f)+c+(x+y+z)(a+d+f)}) + \beta_0 C(\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_2{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)}$$
$$- \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_2{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_2{}^{dcb}{}_a(-1)^{c(f+b)} + \delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_2{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)}$$
$$- \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_2{}^{dcb}{}_a(-1)^{d(b+c+f)} - \delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_2{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)})$$
$$+ \beta_0 C\lambda_0^2(\delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_3{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_3{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)} + \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_3{}^{dcb}{}_a(-1)^{c(f+b)}$$
$$+ \delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_3{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_3{}^{dcb}{}_a(-1)^{d(b+c+f)}$$
$$- \delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_3{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)} + \delta^e{}_x\delta^y{}_b\delta^z{}_d\mathcal{D}_{ec}\mathcal{F}_4{}^{dcb}{}_a(-1)^{(e+c)(e+b+d)+c(b+e)}$$
$$- \delta^e{}_x\delta^y{}_b\delta^z{}_c\mathcal{D}_{ed}\mathcal{F}_4{}^{dcb}{}_a(-1)^{d(b+c+e)+(e+d)(e+b+c)} + \delta^a{}_x\delta^y{}_b\delta^z{}_c\mathcal{F}_5{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)} - \delta^a{}_x\delta^y{}_b\delta^z{}_d\mathcal{F}_5{}^{dcb}{}_{a,c}(-1)^{c(a+b+c)}$$
$$+ \delta^a{}_x\delta^y{}_f\delta^z{}_c\Pi^f{}_{bd}\mathcal{F}_5{}^{dcb}{}_a(-1)^{c(f+b)} - \delta^a{}_x\delta^y{}_f\delta^z{}_d\Pi^f{}_{bc}\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c+f)}$$
$$+ \delta^f{}_x\delta^y{}_b\delta^z{}_d\Pi^a{}_{fc}\mathcal{F}_5{}^{dcb}{}_a(-1)^{c(f+b)+(f+b+d)(a+f+c)} - \delta^f{}_x\delta^y{}_b\delta^z{}_c\Pi^a{}_{fd}\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c+f)+(a+f+d)(f+b+c)}$$
$$+ \delta^e{}_x\delta^y{}_f\delta^z{}_h\mathcal{D}_{eg}\mathcal{F}_6{}^{hgf}(-1)^{(e+g)(e+f+h)+g(f+e)} - \delta^e{}_x\delta^y{}_f\delta^z{}_g\mathcal{D}_{eh}\mathcal{F}_6{}^{hgf}(-1)^{h(f+g+e)+(e+h)(e+f+g)}) = 0. \tag{B1}$$

2. Field equations for $\mathcal{D}$

$$\alpha_0 C(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)} - \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\mathcal{F}_7{}^c{}_a{}^{db} + \beta_0 C(\delta^x{}_b\delta^y{}_d\delta^a{}_c(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d(-1)^{d(b+c)+(b+c)(a+d)})\mathcal{F}_2{}^{dcb}{}_a + \beta_0 C\lambda_0^2(\delta^x{}_b\delta^y{}_d\delta^a{}_c\mathcal{F}_3{}^{dcb}{}_a(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d\mathcal{F}_3{}^{dcb}{}_a(-1)^{d(b+c)+(b+c)(a+d)} - \delta^x{}_b\delta^y{}_d\mathcal{F}_4{}^{dcb}{}_{a,c}(-1)^{dc+c(a+b+c+d)} + \delta^x{}_b\delta^y{}_c\mathcal{F}_4{}^{dcb}{}_{a,d}(-1)^{d(a+b+c+d)}$$
$$+ \delta^x{}_e\delta^y{}_c\Pi^e{}_{bd}\mathcal{F}_4{}^{dcb}{}_a(-1)^{c(b+e)} - \delta^x{}_e\delta^y{}_d\Pi^e{}_{bc}\mathcal{F}_4{}^{dcb}{}_a(-1)^{d(b+c+e)} + \delta^x{}_b\delta^y{}_d\delta^a{}_c\mathcal{F}_5{}^{dcb}{}_a(-1)^{bc+(a+c)(b+d)}$$
$$- \delta^x{}_b\delta^y{}_c\delta^a{}_d\mathcal{F}_5{}^{dcb}{}_a(-1)^{d(b+c)+(b+c)(a+d)} - \delta^x{}_f\delta^y{}_h\mathcal{F}_6{}^{hgf}{}_{,g}(-1)^{hg+g(f+g+h)} + \delta^x{}_f\delta^y{}_g\mathcal{F}_6{}^{hgf}{}_{,h}(-1)^{h(f+g+h)}$$
$$+ \delta^x{}_e\delta^y{}_g\Pi^e{}_{fh}\mathcal{F}_6{}^{hgf}(-1)^{g(f+e)} - \delta^x{}_e\delta^y{}_h\Pi^e{}_{fg}\mathcal{F}_6{}^{hgf}(-1)^{h(f+g+e)}) = 0. \tag{B2}$$

3. Field equations for g

$$\alpha_0 C\left(-\frac{1}{2}\sqrt{g}_y g_x g^{db}\mathcal{F}_{8bd}(-1)^x + \delta^d{}_x\delta^b{}_y\sqrt{g}\mathcal{F}_{8bd}\right) + \beta_0 C\left[-\frac{1}{2}\sqrt{g}_y g_x \mathcal{B}^{hdgcfb}{}_{ea}\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh}(-1)^x\right.$$
$$+ (B_1(\delta^h{}_x\delta^d{}_y g^{gc}g^{fb}g_{ea} + \delta^g{}_x\delta^c{}_y g^{hd}g^{fb}g_{ea}(-1)^{(h+d)(g+c)} + \delta^f{}_x\delta^b{}_y g^{hd}g^{gc}g_{ea}(-1)^{(h+d+g+c)(f+b)}$$
$$+ V_{0exya}g^{hd}g^{gc}g^{fb}(-1)^{(h+d+g+c+f+b)(e+a)}) + \tilde{B}_2\delta^f{}_x\delta^b{}_y g^{hd}\delta^g{}_e\delta^c{}_a + \tilde{B}_2\delta^h{}_x\delta^d{}_y g^{fb}\delta^g{}_e\delta^c{}_a(-1)^{(f+b)(h+d)}$$
$$\left.+ \tilde{B}_3\delta^h{}_x\delta^f{}_y g^{db}\delta^g{}_e\delta^c{}_a + \tilde{B}_3\delta^d{}_x\delta^b{}_y g^{hf}\delta^g{}_e\delta^c{}_a(-1)^{(h+f)(d+b)})\sqrt{g}\mathcal{F}_9{}^a{}_{bcd}{}^e{}_{fgh}\right]$$

$$
\begin{aligned}
&-\beta_0 C\lambda_0^2\Big[V_{1yxe}g_a\mathcal{F}_{13}{}^{ae}-(V_{2yx}g_a\mathcal{F}_{13}{}^{ae})_{,e}+V_{1yxa}g_e\mathcal{F}_{13}{}^{ae}(-1)^{ae}-(V_{2yx}g_e\mathcal{F}_{13}{}^{ae}(-1)^{ae})_{,a}\\
&+\Big(-\frac{1}{2}\sqrt{g}_y g_x g^{hd}g^{gc}g^{fb}(-1)^x+\delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb}+\delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)}\\
&+\delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big)\mathcal{F}_{14bcdfgh}+\Big(-\frac{1}{2}\sqrt{g}_y g_x g^{hd}g^{gc}g^{fb}(-1)^x\\
&+\delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb}+\delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)}+\delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big)\mathcal{F}_{11bcdfgh}\\
&+\Big(-\frac{1}{2}\sqrt{g}_y g_x g^{hd}g^{gc}g^{fb}(-1)^x+\delta^h{}_x\delta^d{}_y\sqrt{g}g^{gc}g^{fb}+\delta^g{}_x\delta^c{}_y\sqrt{g}g^{hd}g^{fb}(-1)^{(h+d)(g+c)}\\
&+\delta^f{}_x\delta^b{}_y\sqrt{g}g^{hd}g^{gc}(-1)^{(f+b)(h+d+g+c)}\Big)\mathcal{F}_{15bcdfgh}+V_{1yxa}\mathcal{F}_{16}{}^a-(V_{2_{yx}}\mathcal{F}_{16}{}^a)_{,a}(-1)^{a(a+x+y)}\Big]=0. \qquad \text{(B3)}
\end{aligned}
$$

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