Supergravity *p*-branes with scalar charge

I. Bogush[1,*] and D. Gal'tsov[1,2,†]

[1]*Faculty of Physics, Moscow State University, 119899, Moscow, Russia*
[2]*Kazan Federal University, 420008 Kazan, Russia*

Standard dilatonic supergravity *p*-branes have scalar charges that are not independent parameters but are determined by the brane tension and Page charges. This feature can be traced to the no-hair theorem in the four-dimensional Einstein-scalar gravity, implying that more general solutions with independent scalar charges can have naked singularities. Since singular branes are also of interest as tentative classical counterparts of unstable tachyonic branes and/or brane-antibrane systems, it is worth investigating branes with independent scalar charges in more detail. Here, we study singular branes associated with the Fisher-Janis-Newman-Winicour solution of four-dimensional gravity. In the case of codimension three, we also construct singular branes endowed with a Zipoy-Voorhees-type oblateness parameter. It is expected that such branes will not be supersymmetric in the string theory. We demonstrate this in the special case of NS5-branes of type II theory. We analyze geodesics and test scalar perturbations of new solutions focusing on possible quantum healing of classical singularities.

I. INTRODUCTION

Supergravity dilatonic *p*-branes are solutions of the Einstein equations with dilaton and antisymmetric forms [1–3]. As is well-known, a single electrically or magnetically charged asymptotically flat brane with isotropic transverse space satisfying the cosmic censorship condition is determined by three parameters: tension, Page charge, and asymptotic dilaton value [4]. The scalar dilaton charge is also present, but it is of a secondary nature and is not a free parameter in accordance with the well-known no-hair theorem in the case $p = 0$. Black dilatonic branes have two horizons, the internal one being singular. So their extremal limit has a singular even horizon, though still exhibits supersymmetry. It turns out, however, that geodesic distance to the singularity is infinite, so the extremal horizon is not observed as naked singularity [5].

More general solutions were found by direct integration of Einstein equations (see, e.g., [6]), but these were shown to contain naked singularities [4]. Singular branes with extra parameters can also be of interest as tentative classical counterparts of tachyonic branes or brane-antibrane systems [7–11]. However, the general structure of singular branes and their relationship to singular solutions of general relativity do not seem to have been systematically investigated. As a step in this direction, we build and explore singular branes related to the famous Fisher-Janis-Newman-Winicour (FJNW) solution of Einstein's four-dimensional

gravity minimally coupled to a massless scalar field. Recall that the FJNW solution, first found by Fisher in 1948 [12], was rediscovered in various forms in Refs. [13–16], the equivalence of which is shown in [17,18]. This solution has a strong curvature singularity on the (would be) event horizon [19–21]. It can be extended to higher dimensions [22] and generalized to rotating solutions [23–25].

Recently, FJNW solution has become popular in four dimensions, as modeling deviations from the standard paradigm of the black hole physics in scalar-tensor theories [26–29]. Geodesics, charged particle trajectories, accretion disks in FJNW, and associated backgrounds were studied [30–34]. Some predictions were formulated for observations that should be taken into account in astrophysics in search of new physics. Additional interest in FJNW stems from the fact that its counterpart in modified gravity models such as Horndesky or DHOST may provide a nonsingular solution to these theories [35,36].

The FJNW solution with a certain value of the scalar charge can be recognized as the solution of the Einstein-Maxwell-dilaton (EMD) theory in the limit of a vanishing electromagnetic field. Mathematically, this connection can be attributed to the existence of a three-dimensional sigma model, the target space of which contains a subspace corresponding to Einstein's minimal scalar theory. This fact can also be used as a generating tool. The isometry transformations of the target space can be used to generate charged black holes with singular horizon starting with FJNW seed in four dimensions [23].

A similar generation technique has been proposed [37] for multidimensional gravity-antisymmetric form actions,

*igbogush@gmail.com
†galtsov@phys.msu.ru

and here we use it to generate p-brane solutions with a scalar charge as an independent parameter. It is achieved via the application of the generalized Harrison transformations [38] to a Fisher-related seed, which has no electric and/or suitable uncharged seed. The new solution opens a way to contact the standard BPS dilatonic branes from a new perspective. For a particular value of the dilaton charge, the new family reduces to the standard class of regular p-branes. This family contains, in particular, the NS5-brane from type IIA/B supergravities. By explicitly checking the Killing spinor equations, we prove that the extended family contains no new supersymmetric solutions, except for the well-known extremal dilaton class.

In the particular case of three-dimensional transverse space, we also derive more general p-brane solutions endowed, apart from an independent scalar charge, with an additional oblateness parameter of Zipoy-Voorhees type [39,40]. This is possible due to some special symmetry of the Weyl-class solutions with the scalar charge.

We discuss behavior of geodesics near singularities, identifying conditions when the latter are reached in finite and/or infinite time. We also consider behavior of the test scalar field near singularities, investigating whether they can be regarded as unobservable in quantum theory.

The plan of the paper is as follows. In Sec. II, we briefly recall the derivation of the Harrison transformation for branes. This technique is then applied to construct Fisher branes in arbitrary spacetime dimensions in Sec. III. In the same section, we classify new solutions according to their singularity structure, discuss their supersymmetry, and show how to add Zipoy-Voorhees deformation to the solution. The next Secs. IV and V are devoted to the study of geodesics and the test scalar field near the singularities.

II. HARRISON TRANSFORMATIONS FOR BRANES

Here, we briefly recall the generating technique for p-branes [37]. Consider Einstein gravity with dilaton and antisymmetric form in D-dimensional spacetime,

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} \left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{\exp(-\alpha\phi)}{2(n+1)!} F_{(n+1)}^2 \right),$$

$$(2.1)$$

where G_{MN} is a D-dimensional spacetime metric, $F_{(n+1)} = dA_{(n)}$ is an antisymmetric $n+1$-form, ϕ is a dilaton field, and α is the dilaton coupling constant, whose value is prescribed in a concrete theory. The antisymmetric form is squared with the weight one $F^2 = F_{M_1...M_{n+1}} F^{M_1...M_{n+1}}$. The corresponding equations of motion are

$$R_{MN} - \frac{1}{2} G_{MN} R = e^{-\alpha\phi} T_{MN}^{(F)} + T_{MN}^{(\phi)}, \qquad (2.2a)$$

$$\partial_M(e^{-\alpha\phi} \sqrt{-G} F_{(n+1)}^{MM_1...M_n}) = 0, \qquad (2.2b)$$

$$\frac{1}{\sqrt{-G}} \partial_M(\sqrt{-G} G^{MN} \partial_N \phi) + \frac{\alpha}{2(n+1)!} e^{-\alpha\phi} F_{(n+1)}^2 = 0,$$

$$(2.2c)$$

where the energy-momentum tensors for the antisymmetric form and the dilaton read

$$T_{MN}^{(F)} = \frac{1}{2n!} (F_{MM_1...M_n} F_N^{M_1...M_n}) - \frac{1}{4(n+1)!} G_{MN} F_{(n+1)}^2,$$

$$(2.3a)$$

$$T_{MN}^{(\phi)} = \frac{1}{2} \left(\partial_M \phi \partial_N \phi - \frac{1}{2} G_{MN} \partial_L \phi \partial^L \phi \right). \qquad (2.3b)$$

We assume that the brane is translation invariant in space and time, which implies the existence of d commuting Killing vectors, one of which is timelike. In adapted coordinates, the metric of the D-dimensional spacetime can be written in the form,

$$ds^2 = g_{\mu\nu}(x)dy^\mu dy^\nu + (\sqrt{-g})^{-2/s} h_{\alpha\beta}(x)dx^\alpha dx^\beta, \quad (2.4)$$

where $g_{\mu\nu}$ and $h_{\alpha\beta}$ are metrics on the d-dimensional brane world volume and the $s+2$-dimensional transverse space, respectively, so that $D = d + s + 2$. Both metrics depend on the transverse coordinates only, and their indices vary as $\mu, \nu = 0, ..., d-1$ and $\alpha, \beta = 1, ..., s+2$. We consider separately electric and magnetic branes. In the first case, $n = d$ and the form field $F_{(n+1)} = dA_{(n)}$ is generated by the potential,

$$A_{01...d-1} = v(x). \qquad (2.5)$$

In the magnetic case $n = s$, and the form field reads

$$F^{\alpha_1...\alpha_{s+1}} = \frac{e^{\alpha\phi}}{\sqrt{-G}} \epsilon^{\alpha_1...\alpha_{s+1}\beta} \partial_\beta u(x). \qquad (2.6)$$

Here, both the electric $v(x)$ and the magnetic $u(x)$ potentials depend on the transverse coordinates as well.

It is convenient to rescale the world-volume metric as $g_{\mu\nu} = f^{2/d} \tilde{g}_{\mu\nu}$, where $f = \sqrt{-g}$, $\det \tilde{g}_{\mu\nu} = -1$. In terms of new variables, the equations of motion reduce to equations of the $s+2$-dimensional sigma model,

$$S = \frac{1}{2\kappa_D^2} \int d^{s+2} x \sqrt{h} \left[R(h) - h^{\alpha\beta} \left(\frac{1}{2} \partial_\alpha \phi \partial_\beta \phi \right. \right.$$

$$+ \frac{s+d}{sd} \partial_\alpha(\ln f) \partial_\beta(\ln f) + \frac{1}{4} \tilde{g}^{\mu\lambda} \partial_\alpha \tilde{g}_{\lambda\nu} \tilde{g}^{\nu\sigma} \partial_\beta \tilde{g}_{\sigma\mu}$$

$$\left. \left. - \frac{1}{2} e^{-\psi} \partial_\alpha v \partial_\beta v \right) \right], \qquad (2.7)$$

where $R(h)$ is the Ricci scalar of the metric $h_{\alpha\beta}$ and $\psi = \alpha\phi + 2\ln f$. The matrix $\tilde{g}_{\mu\nu}$ decouples from the rest of variables, interacting with them only through the gravitational field $h_{\alpha\beta}$. Since $\tilde{g}_{\mu\nu}$ is a symmetric matrix with (minus) unit determinant, this part of variables parametrizes a coset $SL(d, R)/SO(1, d-1)$. Therefore, the metric on the world volume of the p-brane is to a high extent independent of the other σ-model variables, which only influence its determinant. In the magnetic case, one has to replace v by u and change the sign of the dilaton.

As result, we will get $s + 2$-dimensional sigma-models on the transverse space with the metric $h_{\alpha\beta}$ realized by the target space variables $f, \phi, \tilde{g}^{\mu\nu}$, and v or u, respectively, for electric $(+)$ and magnetic $(-)$ cases [37]. The line elements of the corresponding target spaces are

$$dl_e^2 = Ad\xi_+^2 + Bd\psi_+^2 - \frac{1}{2}e^{-\psi_+}dv^2$$
$$+ \frac{1}{4}\text{tr}[\tilde{g}^{-1}d\tilde{g}\tilde{g}^{-1}d\tilde{g}], \qquad (2.8a)$$

$$dl_m^2 = Ad\xi_-^2 + Bd\psi_-^2 - \frac{1}{2}e^{-\psi_-}du^2$$
$$+ \frac{1}{4}\text{tr}[\tilde{g}^{-1}d\tilde{g}\tilde{g}^{-1}d\tilde{g}], \qquad (2.8b)$$

where $\xi_\pm, \psi_\pm$ are the following functions of ϕ and f:

$$\xi_\pm = \pm sd\phi - \alpha(s+d)\ln f, \quad \psi_\pm = \pm\alpha\phi + 2\ln f, \qquad (2.9)$$

and A, B are constants,

$$A = \frac{1}{sd\Delta}, \qquad B = \frac{s+d}{2\Delta}, \qquad \Delta = \alpha^2(s+d) + 2sd.$$

There is an electromagnetic duality between the electric (2.8a) and magnetic (2.8b) sigma-models,

$$\phi \to -\phi, \qquad u \leftrightarrow v, \qquad (2.10)$$

which translates an electric solution into a magnetic and vice versa. Keeping this in mind, let us consider the electric case (2.8a).

The isometry group of the target space $SL(d, R)/SO(1, d-1) \times SL(2, R)/SO(1, 1) \times R$ consists of the transformations of the matrix $\tilde{g}$, translations along ξ_+, and the nontrivial transformations in the subspace (ψ_+, v) relevant for the generating technique. Similarly to the four-dimensional Einstein-Maxwell model, one can define the Ernst potentials,

$$\Phi = \frac{v}{2\sqrt{2B}}, \qquad \mathcal{E} = \exp\psi_+ - \frac{v^2}{8B}. \qquad (2.11)$$

A nontrivial isometry transformation, which preserves an asymptotic behavior of the Ernst potentials $\mathcal{E} \to 1$, $\Phi \to 0$, is

$$\Phi = \frac{\Phi^{(0)} + c(c\Phi^{(0)} + \mathcal{E}^{(0)} - 1)}{1 - 2c\Phi^{(0)} - c^2\mathcal{E}^{(0)}},$$
$$\mathcal{E} = \frac{\mathcal{E}^{(0)} + 2c\Phi^{(0)} - c^2}{1 - 2c\Phi^{(0)} - c^2\mathcal{E}^{(0)}}, \qquad (2.12)$$

where the index $^{(0)}$ stands for the seed solution and c is a real transformation parameter. If the seed solution is uncharged $v_0 = 0$, the transformation (2.12) can be simplified as

$$\Phi = c\frac{\mathcal{E}^{(0)} - 1}{1 - c^2\mathcal{E}^{(0)}}, \qquad \mathcal{E} = \frac{\mathcal{E}^{(0)} - c^2}{1 - c^2\mathcal{E}^{(0)}}. \qquad (2.13)$$

The magnetic isometry transformations of the target space (2.8b) can be obtained by replacing v and ψ_+ with u and ψ_- in (2.11). These transformations represent the generalized Harrison map for the model (2.1).

III. BRANES WITH INDEPENDENT SCALAR CHARGE

To generate a charged brane with an independent scalar charge, we will start with the seed solution of the model (2.1) with the trivial antisymmetric form. This is related to the FJNW solution in higher dimensions.

A. FJNW solution in D-dimensional spacetime

Four-dimensional FJNW solution was generalized to arbitrary dimensions by Xanthopoulos and Zannias (XZ) in Ref. [41]. For the theory with the action,

$$S = \frac{1}{2\kappa_D^2}\int d^{\tilde{D}}x\sqrt{-g}\left(R - \frac{1}{2}(\nabla\phi)^2\right), \qquad (3.1)$$

the solution reads

$$ds^2 = -f_1^\sigma dt^2 + f_1^{\frac{1-\sigma}{\tilde{D}-3}-1}dr^2 + r^2 f_1^{\frac{1-\sigma}{\tilde{D}-3}}d\Omega_{(\tilde{D}-2)}^2,$$
$$\phi = \sqrt{\frac{\tilde{D}-2}{2(\tilde{D}-3)}}\frac{\Sigma\sigma}{M}\ln f_1, \qquad f_1 = 1 - \left(\frac{r_0}{r}\right)^{\tilde{D}-3},$$
$$r_0^{\tilde{D}-3} = \frac{2\kappa_D^2 M}{(\tilde{D}-2)\sigma S_{\tilde{D}-2}}, \qquad \sigma = \frac{M}{\sqrt{M^2 + \Sigma^2}}, \qquad (3.2)$$

where the mass M and the scalar charge Σ are independent parameters, $S_n = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ is the surface area of a unit n-sphere, $d\Omega_{(n)}^2$ is a line element of the unit n-sphere. One can directly check that the equations of motion for the model (3.1) correspond to the equations of motion (2.2) with vanishing antisymmetric form $F_{(n+1)} = 0$. Thus, the FJNW solution satisfies the equations of motion for the model (2.1). The horizon at $r = r_0$ is always singular for $\Sigma \neq 0$ [21]. Zero scalar charge $\Sigma = 0$ brings us back to

the standard Schwarzschild-Tangherlini solution in $\tilde{D}$ dimensions.

B. Generation of the charged FJNW-brane

To get the charged FJNW-branes, one constructs the seed solution complementing the FJNW metric (3.2) with a flat subspace parametrized by the coordinates $y_1, \ldots, y_{d-1}$. The subspace metric in the sector $t, y_1, \ldots, y_{d-1}$ and the space-like section of the metric (3.2) are identified with $g_{\mu\nu}^{(0)}$ and $h_{\alpha\beta}^{(0)}$ of the previous section, respectively. This gives us the seed potentials in the form $\psi_+^{(0)} = \sigma U \ln f_1$ and $\xi_+^{(0)} = \xi_+ = \sigma V \ln f_1$, where

$$U = \sqrt{\frac{s+1}{2s}\frac{\alpha \Sigma_0}{M_0} + 1},$$

$$V = sd\sqrt{\frac{s+1}{2s}\frac{\Sigma_0}{M_0}} - \frac{1}{2}\alpha(s+d). \qquad (3.3)$$

Substituting $\psi_+^{(0)}$ and $\xi_+^{(0)}$ in (2.11), we will get the seed Ernst potentials $\mathcal{E}^{(0)} = f_1^{\sigma U}$, $\Phi^{(0)} = 0$. Then the transformation (2.13) leads to the new Ernst potentials,

$$\mathcal{E} = \frac{f_1^{\sigma U} - c^2}{1 - c^2 f_1^{\sigma U}}, \qquad \Phi = c\frac{f_1^{\sigma U} - 1}{1 - c^2 f_1^{\sigma U}}. \qquad (3.4)$$

Using (2.9) and (2.11), one finds new v, ϕ and f for the generated solution. Finally, this leads to the metric of the electric p-brane,

$$\begin{aligned} ds^2 &= f_2^{4s/\Delta}(-f_1^\sigma dt^2 + dy_1^2 + \cdots + dy_{d-1}^2) \\ &\quad + f_2^{-4d/\Delta}f_1^{(1-\sigma)/s}(dr^2/f_1 + r^2 d\Omega_{s+1}^2), \\ \phi &= \frac{\sigma(U-1)}{\alpha}\ln f_1 + 4B\alpha \ln f_2, \end{aligned}$$

$$A_{01\ldots d-1} = 2c\sqrt{2B}(f_1^{\sigma U} f_2 - 1), \qquad (3.5)$$

where

$$f_1 = 1 - \frac{r_0^s}{r^s}, \qquad f_2 = \frac{1-c^2}{1-c^2 f_1^{\sigma U}},$$

$$r_0^s = \frac{2\kappa_D^2 M_0}{\sigma(s+1)S_{s+1}}, \qquad \sigma = \frac{M_0}{\sqrt{M_0^2 + \Sigma_0^2}}.$$

This new solution is physically meaningful only for integers s, $d \geq 1$. The independent parameters are the seed mass and scalar charge M_0, Σ_0, as well as the Harrison transformation parameter c. The solution (3.2) can be restored setting $c = 0$, $d = 1$.

The magnetic brane can be obtained via electric-magnetic duality (2.10). According to (2.6), the antisymmetric form will read

$$F_{\alpha_1\ldots\alpha_{s+1}} = \frac{c}{1-c^2}\frac{s}{s+1}\frac{4\kappa_D^2 M_0 U\sqrt{2B}}{S_{s+1}}\sqrt{\Omega_{(s+1)}}\epsilon_{\alpha_1\ldots\alpha_{s+1}}, \qquad (3.6)$$

where $\Omega_{(n)}$ is the determinant of the metric tensor of the unit n-sphere, $\epsilon_{\alpha_1\ldots\alpha_{s+1}} = \pm 1$.

Important examples of the generated family are F1 and NS5 branes in supergravities of type II. The fundamental string F1 is a solution with an electrically charged 3-form with $d = 2$, $s = 6$, $\alpha = 1$. Its dual one is a five-brane soliton NS5 with magnetic charge and $d = 6$, $s = 2$, $\alpha = 1$.

C. Charges

The ADM mass $\mathcal{M}$ and the electric Page charge $\mathcal{Q}$ per unit area of the FJNW brane can be presented as the surface integrals according to Eqs. (2.8) and (4.3) in Ref. [42],[1]

$$\mathcal{M} = M_0(1 + \delta_M), \qquad \delta_M \equiv \frac{4sU}{\Delta}\frac{c^2}{1-c^2}\left(d - \frac{s(s-1)}{s+1}\right), \qquad (3.7a)$$

$$\mathcal{Q} = \frac{1}{\kappa_D^2}\int_{S_\infty} e^{-\alpha\phi} * F^{(e)} = (-1)^d\frac{s}{s+1}\frac{c}{1-c^2}4\sqrt{2B}M_0 U, \qquad (3.7b)$$

where S_∞ is a hypersurface of an infinitely distant $s+1$-sphere (also, see Ref. [43] for details). In the magnetic case, the charge corresponding to the undualized form integral,

$$\mathcal{P} = \frac{1}{\kappa_D^2}\int_{S_\infty} F^{(m)}, \qquad (3.8)$$

formally coincides with the expression for the electric charge (3.7b) up to sign.

In accordance with the dilaton asymptotics for $r \to \infty$,

$$\phi \approx -\frac{1}{r^s}\frac{2\kappa_D^2}{(s+1)S_{s+1}}\sqrt{\frac{s+1}{2s}}\Sigma_0(1 + \delta_\Sigma),$$

$$\delta_\Sigma = 4B\alpha^2 \frac{U}{U-1}\cdot\frac{c^2}{1-c^2}, \qquad (3.9)$$

it is reasonable to define the dilaton charge $\mathcal{D}$ of the brane as follows:

---

[1]We have redefined the Page charge by a multiplicative constant in order to get rid of an extra κ_D in the following expressions and due to different weight of the Maxwell form in the action from Eq. (2.1).

$$\mathcal{D} = -\sqrt{\frac{2s}{s+1} \frac{\Sigma_0(1+\delta_\Sigma)}{\alpha}}$$

$$= -\left(1 + \frac{4B\alpha^2 c^2}{1-c^2}\right)\sqrt{\frac{2s}{s+1}\frac{\Sigma_0}{\alpha}} - \frac{8Bc^2}{1-c^2}\frac{s}{s+1}M_0. \quad (3.10)$$

Note that the dilaton charge is not associated with Noether symmetry, so its choice is rather the matter of convenience [44].

In Sec. III D, it will be shown that the necessary condition for the horizon to be regular is the seed scalar charge equal to zero: $\Sigma_0 = 0$. This condition does not mean a zero dilaton charge $\mathcal{D} = 0$, since in this case $U = 1$, and the product $\Sigma_0 \delta_\Sigma$ is finite. Rather, in this limit we obtain a constraint,

$$\mathcal{Q}^2 = \mathcal{D}\left(\mathfrak{B}\mathcal{D} - \frac{4s}{1+s}\mathcal{M}\right), \quad (3.11)$$

for the electric brane and

$$\mathcal{P}^2 = \mathcal{D}\left(\mathfrak{B}\mathcal{D} + \frac{4s}{1+s}\mathcal{M}\right), \quad (3.12)$$

for the magnetic one (which can be obtained replacing $\mathcal{D} \to -\mathcal{D}$, $\mathcal{Q} \to \mathcal{P}$), where

$$\mathfrak{B} = \frac{1}{2B} - \frac{4s(ds+d-s^2+s)}{(s+1)(d+s)}. \quad (3.13)$$

For the parameter values $d = s = 1$, $\alpha = \sqrt{3}$ ($\mathfrak{B} = 2$), the model is the four-dimensional $\alpha = \sqrt{3}$ Einstein-Maxwell-dilaton (EMD) theory, and the conditions (3.11), (3.12) coincide with the charge constraint found by Rasheed [45]. The Eq. (3.11) has two roots with respect to $\mathcal{D}$

$$\mathcal{D}_\pm = \frac{2\mathcal{M}s \pm \sqrt{4\mathcal{M}^2 s^2 + \mathfrak{B}(s+1)^2\mathcal{Q}^2}}{\mathfrak{B}(s+1)}. \quad (3.14)$$

If $\mathcal{Q} = 0$, the regularity condition (3.11) should coincide with the regularity condition of the seed solution, namely $\Sigma_0 = \mathcal{D} = 0$. Therefore, the root $\mathcal{D}_+$ is unphysical and does not correspond to the constraint $\Sigma_0 = 0$. Similarly, the plus sign in the solution for Eq. (3.12) does not correspond to regular magnetic solutions.

Another interesting limit is $U \to 0$, $c^2 \to 1$. In this case, the expressions for charges (3.7a), (3.7b), (3.10), and the solution itself (3.5) have a nontrivial limiting form with new metric function,

$$f_2 = (\zeta \ln f_1 + 1)^{-1}, \quad (3.15)$$

where ζ is a new independent parameter. For $\zeta > 0$, the function f_2 diverges at $r_\zeta = r_0(1 - e^{-1/\zeta})^{-1/s} > r_0$ as

$1/(r - r_\zeta)$. For $\zeta < 0$, the function f_2 does not have roots and tends to zero at r_0 as $1/\ln f_1$.

Similarly, the limit $M_0, \Sigma_0 \to 0, c^2 \to 1$ leaves the charges finite. Let us introduce an infinitesimal parameter ϵ, such that the quantities M_0, Σ_0, c depend on it as $M_0 = \xi\epsilon, \Sigma_0 = \zeta\epsilon, c = \pm(1-\epsilon)$, where ζ and ξ are some constants. Then, the limit $\epsilon \to 0$ leads to redefinition,

$$f_1 = 1, \qquad f_2 = \frac{r^s}{\rho^s + r^s}, \quad (3.16)$$

and the new charges will be

$$\mathcal{M} = \Phi\left(2B + \frac{s(d-s)}{\Delta}\right), \quad \mathcal{D} = -2B\Phi, \quad \mathcal{Q} = \pm\Phi\sqrt{2B}, \quad (3.17)$$

where

$$\rho^s = \frac{\kappa_D^2 \Phi}{sS_{s+1}}, \qquad \Phi = \frac{2s}{s+1}\xi + \sqrt{\frac{2s}{s+1}}\alpha\zeta.$$

In the magnetic case, the solution can be found from (3.17) again replacing $\mathcal{D} \to -\mathcal{D}, \mathcal{Q} \to \mathcal{P}$. It can be verified that the charge expressions satisfy identically the condition (3.14). These solutions were found previously in Ref. [46]. The charges (3.17) satisfy the following constraints:

$$\mathcal{M}^2 = \mathcal{Q}^2 \frac{(s+d+sd-s^2)^2}{(s+d)\Delta}, \quad (3.18a)$$

$$\mathcal{D}^2 = 2B\mathcal{Q}^2, \quad (3.18b)$$

which can be combined into

$$\mathcal{M}^2 + \left(\frac{1}{2B} - \left(\frac{s+d+sd-s^2}{s+d}\right)^2\right)\mathcal{D}^2 - \mathcal{Q}^2 = 0, \quad (3.19)$$

resembling the no-force condition [47,48]. For $d = s = 1$ the mass constraint (3.18a) coincides with the known result $\mathcal{M}^2 = \mathcal{Q}^2/(1+\alpha^2)$ for BPS-solutions in the EMD model [45,49–51].

The limit $c \to \infty$ leads to the function redefinition,

$$f_2 = f_1^{-\sigma U}, \quad (3.20)$$

while the antisymmetric form becomes trivial. This solution is a particular case of the wider family, which can be obtained from the seed solution with $SO(2)$ transformation considered in Sec. III F.

The first limiting solution (3.15) does not differ qualitatively from the general family, while the second limiting solution (3.20) is a special case of another wider family of uncharged solutions that requires a separate study. In what follows, we will discuss physical property of the general

solution (3.5) and the second limiting solution (3.16), denoting them as S_G and S_E, respectively.

D. Singularities

To perform an analysis of spacetime structure, we calculate the Ricci scalar, contracting indices in Eq. (2.2a), and substituting the general solution for electric/magnetic forms and the scalar field into the energy-momentum tensor,

$$
\begin{aligned}
R &= \frac{1}{2}\sigma^2 f_2^{4d/\Delta} f_1^{(\sigma-s-1)/s} f_1'^2 (R_F + R_\phi),\\
R_F &= \frac{4c^2 U^2(d-s)f_2^2 f_1^{\sigma U}}{(1-c^2)^2 \Delta},\\
R_\phi &= \left(\frac{\Sigma_0}{M_0}\sqrt{\frac{s+1}{2s}} + \frac{2\alpha c^2 U(d+s)f_2 f_1^{\sigma U}}{(1-c^2)\Delta} \right)^2. \quad (3.21)
\end{aligned}
$$

For the general family S_G, the function f_2 is regular at $r = r_0$; therefore, this factor does not lead to the curvature singularity. But for $\Sigma_0 \neq 0$, the Ricci scalar (3.21) contains a diverging factor proportional to $f_1^{(\sigma-s-1)/s}$ due to the negative exponent. For $\Sigma_0 = 0$, $M_0 > 0$, the surface $r = r_0$ is regular and represents an event horizon if $|c| < 1$.

The function f_2 diverges once its denominator is zero: $f_1^{\sigma U} = c^{-2}$. The function f_1 is bounded from above by a value 1; thus, this is possible for $c^2 > 1$ only. From Eq. (3.21), it follows that this leads to the existence of one more singularity at $r_0(1 - c^{-2/\sigma U})^{-s} > r_0$.

In the limiting case S_E, the scalar curvature becomes

$$
R = \left(\frac{2s\rho^s}{r^{s+1}\Delta} \right)^2 f_2^{4d/\Delta+2} d(\Delta - s(d+s)) \sim r^{4ds/\Delta-2}. \quad (3.22)
$$

To find out whether the point $r = 0$ is singular, we look at the extrema of the exponent $-2 + 4ds/\Delta$ with respect to α^2. This expression takes the lowest value -2 for $\alpha^2 \to \infty$ and the largest value 0 for $\alpha^2 = 0$. So, the family S_E is singular for $\alpha \neq 0$. For $\Delta - s(d+s) = 0$ [i.e., $\alpha^2 = s(s-d)/(s+d)$, $s \geq d$] the metric tensor is Ricci flat. However, the direct calculation of the scalars $R_{MN}R^{MN}$ and $R_{MNLK}R^{MNLK}$ lead to the asymptotic $\sim r^{4(d-s)/(d+s)}$ as $r \to 0$, so they diverge if $s \neq d$ (the case $s = d$ brings us back to $\alpha = 0$).

The outermost singularity/horizon surface for S_G, $c^2 < 1$ is $r = r_0$. Solutions with positive (negative) seed mass M_0 have an infinite red shift $g_{tt} \to 0$ (blue shift $g_{tt} \to \infty$) at $r = r_0$. In Sec. IV, we will show that the point $r = r_0$ is approached by the radial geodesic in a finite time of the remote observer for any $\sigma \neq 1$. The outermost interesting surface for S_G, $c^2 > 1$ is the surface $f_2^{-1} = 0$. As $g_{tt} \to \infty$, such solutions are naked singularities with an infinite blue shift. The family S_G cannot be extremal in the sense $g_{tt}'(r_0) \neq 0$ or $g^{rr\prime}(r_0) \neq 0$, which can be verified directly.

According to this analysis, the generic family S_G is a naked singularity for $\sigma \neq 1$ but may have some black hole properties (redshift). The class S_E represent extreme solutions, that are regular only for $\alpha = 0$, representing extreme Reissner-Nordström singly charged solutions.

E. Supersymmetry

The fact that singularities of spacetime may be compatible with supersymmetry is known from the example of domain walls [52]. Therefore, it makes sense to check the possible supersymmetry of our singular branes from first principles. It can be seen that the solution subfamily S_E contains the known family of the BPS saturating solutions for any d, s, α, representing p-branes of the type IIA/IIB superstring theories [53–55]. However, the question may arise whether the general solution S_G contains other supersymmetric solutions.

We will study supersymmetry in the case of NS5-brane $(d = 6, \ s = 2, \ \alpha = 1)$ in type IIA supergravity. The supersymmetry conditions have the simplest form in the string frame $G_{MN}^{(s)} = e^{-\phi/2}G_{MN}$,

$$
\begin{aligned}
ds_{(s)}^2 &= f_1^{-\sigma(U-1)/2}(-f_1^\sigma dt^2 + dy_1^2 + \cdots + dy_5^2)\\
&\quad + f_2^{-1} f_1^{-(1+\sigma U)/2}(dr^2 + r^2 f_1 d\Omega_3^2),\\
\phi &= -\sigma(U-1)\ln f_1 - \frac{1}{2}\ln f_2. \quad (3.23)
\end{aligned}
$$

We will choose the three-sphere metric as follows:

$$
\begin{aligned}
d\Omega_3^2 &= d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2),\\
\sqrt{\Omega_{(3)}} &= \sin^2\psi \sin\theta, \quad (3.24)
\end{aligned}
$$

so the magnetic three-form from the Neveu-Schwarz sector H_{MNP} reads

$$
H_{\psi\theta\varphi} = p\sin^2\psi\sin\theta, \qquad p = \frac{c}{1-c^2}\frac{2\kappa_D^2 M_0 U}{3\pi^2}, \quad (3.25)
$$

where we introduced the constant p for convenience.

The variations of gravitino and dilatino for type IIA supergravity with truncated Ramond-Ramond fields are [56,57]

$$
\delta\psi_M = \left(\nabla_M - \frac{1}{8}H_{MNP}\Gamma^{NP}\Gamma_{11} \right)\epsilon, \quad (3.26a)
$$

$$
\delta\lambda = -\frac{1}{3}\left(\Gamma^\mu(\partial_\mu\phi)\Gamma_{11} - \frac{1}{12}H_{MNP}\Gamma^{MNP} \right)\epsilon, \quad (3.26b)
$$

where

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{MAB}\Gamma^{AB}, \qquad \Gamma^{M_1\ldots M_n} = \Gamma^{[M_1}\ldots\Gamma^{M_n]},$$

$$\Gamma_{11} = \Gamma_0\Gamma_1\ldots\Gamma_9, \tag{3.27}$$

ω_{MAB} is spin connection, and we use the convention $\{\Gamma_M,\Gamma_N\} = 2G_{MN}^{(s)}$.

Since the Killing spinor does not depend on the coordinates along the orbits of the commuting Killing vectors y^μ, so the condition $\delta\psi_\mu = 0$ reduces to $\omega_\mu{}^{\bar\nu\bar r}\Gamma_{\bar\nu\bar r}\epsilon = 0$, where the flat indices with respect to the ten-dimensional metric $G_{MN}^{(s)}$ are denoted by bars. As gamma matrices are not degenerate, this requires the spin connection to be zero, which is possible if $G_{\mu\nu}^{(s)}$ is constant (for $\mu,\nu = 0,\ldots 5$ only). This is not possible for the general solution S_G unless the function f_1 is constant. The function f_1 is constant for the subfamily S_E only with $f_1 = 1$ and $f_2 = r^s/(r^s + \rho^s)$, and there is no other solutions in S_G satisfying this condition. As we have mentioned, this solution is known to have Killing spinors. For the completeness of the analysis, we will obtain the resulting Killing spinors. The component $\delta\psi_r = \partial_r\epsilon$ is zero if the Killing spinor ϵ depends on the angles ψ, θ, and φ only. The remaining gravitino equations on the three-sphere are solved by the spinor,

$$\epsilon = \exp\left(-\Gamma^{\bar\psi}\Gamma^{\bar r}\psi/2\right)\exp\left(-\Gamma^{\bar\theta}\Gamma^{\bar\psi}\theta/2\right)\exp\left(-\Gamma^{\bar\psi}\Gamma^{\bar\theta}\varphi/2\right)\epsilon_0, \tag{3.28}$$

where ϵ_0 is a constant spinor and the exponents can be expanded as

$$e^{\Gamma^M\Gamma^N x} = \cos x + \Gamma^M\Gamma^N \sin x, \qquad M \neq N \neq t. \tag{3.29}$$

The remaining dilatino equation (3.26b) for S_E solutions is

$$(1 \pm \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5)\epsilon = 0, \tag{3.30}$$

where the sign depends on the sign of the magnetic charge. This condition breaks a half of supersymmetry.

F. FJNW-branes with oblateness parameter

Static uncharged solutions in general relativity with minimally coupled scalar field, which is the truncation of the theory (2.1) with $F = 0$, can be reduced to a simple sigma model. This opens a way to add the minimal massless scalar field via the transformations.

To construct an appropriate sigma model, we can truncate the previously derived ones (2.8a) or (2.8b) setting zero electric v or magnetic u potentials. This gives the target space metric,

$$dl_0^2 = \frac{s+d}{sd}(d\ln\sqrt{-g})^2 + \frac{1}{2}d\phi^2. \tag{3.31}$$

It has an $SO(2)$ symmetry, corresponding to the transformation used in [21],

$$\begin{cases} \ln\sqrt{-g} = \cos\beta\cdot\ln\sqrt{-g^{(0)}} - \sqrt{\frac{sd}{2(s+d)}}\sin\beta\cdot\phi^{(0)} \\ \phi = \sqrt{\frac{2(s+d)}{sd}}\sin\beta\cdot\ln\sqrt{-g^{(0)}} + \cos\beta\cdot\phi^{(0)} \end{cases}, \tag{3.32}$$

where β is a transformation parameter.

Starting with the four-dimensional vacuum axially symmetric Zipoy-Voorhees solution [39,40], the p-branes with $s = 1$ can be supplied with the deformation (oblateness) parameter δ. One chooses the seed solution,

$$ds^2 = -f_1(x)^\delta dt^2 + f_1(x)^{-\delta}ds_{(3)}^2, \qquad f_1(x) = \frac{x-1}{x+1},$$

$$ds_{(3)}^2 = k^2\left[\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2-1}\left(dx^2 + \frac{x^2-1}{1-y^2}dy^2\right)\right.$$
$$\left. + (x^2-1)(1-y^2)d\varphi^2\right], \tag{3.33}$$

where k and δ are some constants, x and y are oblate spheroidal coordinates, related to spherical coordinates as $x = \frac{r}{k} - 1$, $y = \cos\theta$. Substituting $d = s = 1$, $|g^{(0)}| = f^\delta$ and $\phi^{(0)} = 0$ in $SO(2)$-transformations (3.32) and redefining β through the charges, we will get a new solution,

$$ds^2 = -f_1^{\delta\sigma}dt^2 + f_1^{-\delta\sigma}ds_{(3)}^2, \qquad \phi = \frac{\Sigma}{k}\ln f_1, \tag{3.34}$$

where σ has the former expression, $k = \frac{M}{\sigma\delta}$. The obtained solution combines FJNW and Zipoy-Voorhees four-dimensional solutions. Next, we can extend the solution to $D = d + 3$ dimensions and apply the transformations (2.13). We will get solution in the form of the previously found FJNW brane (3.5) with the replacement $f_1^\sigma \to f_1^{\sigma\delta}$ and using the new form of $ds_{(3)}$,

$$ds^2 = f_2^{4/\Delta}(-f_1^{\sigma\delta}dt^2 + dy_1^2 + \cdots + dy_{d-1}^2)$$
$$+ f_2^{-4d/\Delta}f_1^{-\sigma\delta}ds_{(3)}^2,$$

$$\phi(r) = \frac{\Sigma}{k}\ln f_1 + 4B\alpha\ln f_2, \quad f_1(r) = \frac{x-1}{x+1},$$

$$f_2(r) = \frac{1-c^2}{1-c^2 f_1^{\sigma\delta U}}, \quad \sigma = \frac{M}{\sqrt{M^2+\Sigma^2}}, \quad U = 1 + \frac{\Sigma\alpha}{M}, \tag{3.35}$$

with the electric potential,

$$A_{01\ldots d-1} = \frac{2\sqrt{2B}}{c}(f_2 - 1),$$

or the magnetic form,

$$F_{y\varphi} = \frac{4c\sqrt{2B}}{1-c^2}(M + \alpha\Sigma).$$

This is a solution with both an independent scalar charge and an arbitrary oblateness parameter δ (see Ref. [23] for more details of the replacement $\sigma \to \sigma\delta$).

IV. GEODESICS

Consider the geodesic equation,

$$\frac{d}{d\tau}(G_{MN}\dot{X}^N) - \frac{1}{2}\frac{dG_{PQ}}{dX^M}\dot{X}^P\dot{X}^Q = 0. \tag{4.1}$$

The metric (3.5) can be written in the form,

$$ds^2 = -a(r)dt^2 + b(r)d\boldsymbol{y}^2 + v(r)dr^2 + w(r)d\Omega_{s+1}^2, \tag{4.2}$$

where

$$a(r) = f_2^{4s/\Delta}f_1^\sigma, \quad b(r) = f_2^{4s/\Delta},$$
$$v(r) = f_2^{-4d/\Delta}f_1^{(1-s-\sigma)/s}, \quad w(r) = f_2^{-4d/\Delta}f_1^{(1-\sigma)/s}r^2. \tag{4.3}$$

By virtue of the spherical symmetry, we can choose geodesics lying in the equatorial plane of $s + 1$-sphere along the coordinate φ. Constants of motion for t, y^i and φ follows from Eq. (4.1),

$$\dot{t} = Ea^{-1}, \qquad \dot{y}^i = k^i b^{-1}, \qquad \dot{\varphi} = Lw^{-1}. \tag{4.4}$$

Let us choose the affine parameter τ so that $\dot{X}_M\dot{X}^M = -\epsilon$, where $\epsilon = -1, 0, 1$ for spacelike, lightlike, and timelike geodesics, respectively. Keeping in mind the constants of motion (4.4), the radial equation reads

$$\dot{r}^2 = \frac{1}{av}(E^2 - V_{\text{eff}}), \tag{4.5}$$

$$V_{\text{eff}}(r) = \frac{a}{b}k^2 + \frac{a}{w}L^2 + a\epsilon, \tag{4.6}$$

where k^2 is $k^i k^j \delta_{ij}$. In terms of the time of an infinitely distant observer, the radial equation has the form,

$$\left(\frac{dr}{dt}\right)^2 = \frac{a}{E^2 v}(E^2 - V_{\text{eff}}). \tag{4.7}$$

The proper time and the distant observer time of motion from r_1 to r_2, are

$$\Delta\tau = \pm\int_{r_1}^{r_2}\frac{\sqrt{av}\,dr}{\sqrt{E^2 - V_{\text{eff}}}},$$
$$\Delta t = \pm\int_{r_1}^{r_2}\sqrt{\frac{v}{a}}\frac{dr}{\sqrt{1 - V_{\text{eff}}/E^2}}. \tag{4.8}$$

Geodesics can reach the spherical surface S with radius r_S if the effective potential V_{eff} does not tend to $+\infty$ at this surface. For null and timelike worldlines, the effective potential is a positive function in the exterior region of the solution. Hence, it is enough to require V_{eff} to be bounded. If every term of the effective potential behaves as $\sim(r - r_S)^a$ near S, where a is some constant, then the necessary and sufficient condition is $a \geq 0$ for every term in the effective potential. If the geodesic curve intersects the surface S, then the required time interval in some frame of reference can be finite or infinite. For a distant observer, the geodesic reaches the surface S in a finite time interval Δt if and only if the quantity $\sqrt{v/a}$ is integrable near this surface. Therefore, if we want the time interval to be finite, the function $\sqrt{v/a}$ cannot diverge as $(r - r_S)^{-1}$ or faster. Similarly, for the proper time $\Delta\tau$, we have a condition that the function $\sqrt{va}$ cannot diverge as $(r - r_S)^{-1}$ or faster. According to the results of Table I, Table II gives the corresponding condition for the functions a, a/b, a/w, $\sqrt{va}$ and $\sqrt{v/a}$.

Behavior of geodesics in the background of solutions S_G with $c^2 < 1$ does not drastically differs from the motion in uncharged FJNW-branes. Depending on the value of σ, the effective potential can be either bounded or divergent in the singularity. For $\sigma \geq 1/(1 + s)$, the effective potential is always bounded for any geodesics; for $0 \leq \sigma < 1/(1 + s)$ the potential is bounded for geodesics with zero angular momentum $L = 0$ only; in the case $\sigma < 0$, the singularity r_0 is reachable for null radial geodesics only ($\epsilon = k = L = 0$).

TABLE I. Behavior of a, a/b, a/w, and $\sqrt{av}$, $\sqrt{v/a}$ near the surface S of the radius r_S up to a multiplicative constant, $x = r - r_S$.

Quantity	General expression	$S_G, c^2 < 1$	$S_G, c^2 > 1$	S_E
r_S		r_0	$r_0(1 - c^{-2/\sigma U})^{-s}$	0
a	$f_2^{4s/\Delta}f_1^\sigma$	x^σ	$x^{-4s/\Delta}$	$x^{4s^2/\Delta}$
a/b	f_1^σ	x^σ	x^0	1
a/w	$f_2^{4(s+d)/\Delta}f_1^{\sigma-\frac{1-\sigma}{s}}r^{-2}$	$x^{\sigma-\frac{1-\sigma}{s}}$	$x^{-4(s+d)/\Delta}$	$x^{-2+4s(s+d)/\Delta}$
$\sqrt{av}$	$f_2^{2(s-d)/\Delta}f_1^{\frac{(1-s)(1-\sigma)}{2s}}$	$x^{\frac{(1-s)(1-\sigma)}{2s}}$	$x^{2(d-s)/\Delta}$	$x^{2s(s-d)/\Delta}$
$\sqrt{v/a}$	$f_2^{-2(s+d)/\Delta}f_1^{\frac{-(s+1)\sigma+1-s}{2s}}$	$x^{\frac{-(s+1)\sigma+1-s}{2s}}$	$x^{2(s+d)/\Delta}$	$x^{-2s(s+d)/\Delta}$

TABLE II.   Regularity condition of various terms in effective potential (4.6) and conditions of finiteness of the required time for geodesics to traverse the surface S of the radius r_S in terms of the proper time τ and the distant observer time t for different solutions; $e_\epsilon, e_k, e_L, e_\tau, e_t$ are the exponents of the leading term in the expansion of the functions $a, a/b, a/w, \sqrt{av}, \sqrt{v/a}$ near the surface S, respectively.

Solution	r_S	$e_\epsilon \geq 0$	$e_k \geq 0$	$e_L \geq 0$	$e_\tau > -1$	$e_t > -1$
$S_G, c^2 < 1$	r_0	$\sigma \geq 0$	$\sigma \geq 0$	$\sigma \geq \frac{1}{1+s}$	Always	$\sigma \neq 1$
$S_G, c^2 > 1$	$r_0(1 - c^{-2/\sigma U})^{-s}$	Never	Always	Never	Always	Always
S_E	0	Always	Always	$\alpha^2 \leq \frac{2s^2}{s+d}$	Always	$\alpha^2 > \frac{2s^2}{s+d}$

Geodesics always traverse the surface r_0 in a finite proper time $\Delta\tau$. The distant observer can observe this in a finite time Δt only in the singular case $\sigma \neq 1$.

Solutions S_G with $c^2 > 1$ have another outermost singularity, which has different properties for geodesics. Timelike geodesics never reach this singularity due to an unbounded growth of the effective potential. Null geodesics can reach the singularity only if they are purely radial, $L = 0$. From the point of view of the distant observer, null geodesics can reach the singularity in a finite time Δt and a finite $\Delta\tau$.

The effective potential of geodesics in the background of S_E with $\alpha^2 \leq \alpha_{\text{crit}}^2$, $\alpha_{\text{crit}}^2 = 2s^2/(s + d)$ is bounded. Geodesics traverse the surface $r = 0$ in a finite $\Delta\tau$, but an infinite Δt. For the other case $\alpha^2 > \alpha_{\text{crit}}^2$, the effective potential with a nonzero angular momentum L unboundedly grows at the singularity, and geodesics with zero angular momentum $L = 0$ reaches the singularity with finite Δt and $\Delta\tau$. In the case of the NS5-brane, the dilaton coupling constant has a critical value $\alpha = \alpha_{\text{crit}} = 1$.

V. TEST SCALAR FIELD

The physical effects of the field theory that can manifest themselves in a singular spacetime are interesting both from the point of view of the search for new physics [58,59] and from a theoretical point of view. First of all, one has to construct the self-adjoint wave operators, which is a nontrivial task in singular curved spacetimes, since the usual axioms of the quantum field theory break there [60]. This is an important question, since it is believed that quantum effects are of decisive importance near singularities.

A related issue is the possible quantum unobservability of classical singularities. It was argued that if there is a unique self-adjoint extension of the wave operator in a singular spacetime, then classical singularity is quantum mechanically unobservable by the corresponding field [60,61]. This issue was investigated in a number of subsequent works. The situation simplifies in the static spaces. There are two simple ways to check the essential self-adjointness of symmetric operators. The first is the calculation of the defect indices (von Neumann). The second one uses Weyl's theorem: the wave operator is not self-adjoint if both local solutions in a neighborhood of

the singular submanifold satisfy the criterion of square integrability *limit circle*). Indeed, in this case, it is necessary to make additional assumptions about boundary conditions on which the evolution of the solution will depend. If only one solution is locally square integrable (*limit point*), the evolution of the waves will be uniquely determined, so the wave packet will not feel the presence of the singularity. In this case, in the vicinity of the singularity, as a rule, a repulsive barrier arises. The idea was formulated by Wald [60] and more specifically implemented by Horowitz and Marolph [61] in the case of the scalar field. Later work in this direction includes Refs. [62–64] and others. The scalar theory was generalized to the Maxwell and Dirac fields [64]. The possibility of quantum healing of geodesically incomplete manifolds has been tested for a number of static and conformally static singular solutions of Einstein theory in four and three dimensions (see the recent review [65]).

A closely related problem is that of stability of singular spacetimes. Stability of a solution is related to the sign of the imaginary part of the frequencies of normal modes defined with appropriate boundary conditions in singularities. It may happen that different choice of boundary conditions in the case where both local solutions are square integrable, may lead to different conclusions about stability [66,67]. Physically this means that the full specification of the singular spacetime must include prescription of boundary conditions for perturbations in singularities. This issue requires further study which is beyond the scope of the present work. But the crucial question is the number of local solutions near the singularities which are square integrable.

To investigate the stability and quantum healing of singularities, one has to deal with perturbations in supergravity/string theory backgrounds, which are driven by Lichnerowitz type operators. But here we will not delve into these rather complicated issues, confining ourselves to a simpler analysis of the Klein-Gordon operator following [60–65]. Using the definitions (4.3), the Klein-Gordon equation $(\Box - \mu^2)\phi = 0$ can be written in the form,

$$-a^{-1}\partial_t^2\phi + b^{-1}\partial_i^2\phi + \frac{1}{R}\partial_r(Rv^{-1}\partial_r\phi) + \frac{1}{w}\Delta_S^{s+1}\phi - \mu^2\phi = 0,$$

(5.1)

where Δ_S^{s+1} is the Laplace-Beltrami operator of a unit $s + 1$-sphere, and

$$R(r) = \sqrt{-G/\Omega_{(s+1)}} = \sqrt{ab^{d-1}vw^{s+1}} = f_2^{-4d/\Delta} f_1^{\frac{1-\sigma}{s}} r^{s+1}.$$

Substituting the ansatz $\phi = \exp\{-i\omega t + ik_i y^i\}\Phi(r)Y_l^K(\theta)$, where K is a set of quantum numbers of the $s + 1$-sphere harmonics Y_l^K, except for the quantum number of the total angular moment l. Taking into account the eigenvalues of the harmonics $\Delta_S^{s+1} Y_l^K = -l(l + s)Y_l^K$ [the eigenvalue l is degenerate with the degree $(2l + s)(l + s - 1)!/l!s!$ [68]], we will get

$$\frac{1}{R}\partial_r(Rv^{-1}\partial_r\Phi) + \left(\frac{\omega^2}{a} - \frac{k^2}{b} - \frac{l(l+s)}{w} - \mu^2\right)\Phi = 0. \quad (5.2)$$

Let us make a coordinate transformation $\partial_r = W(\varrho)\partial_\varrho$ and substitution $\Phi = P(r)\chi(r)$ with some arbitrary functions W and P. We would like to find functions W, P, such that the equation takes the Schrödinger form,

$$\chi'' + (\omega^2 - V_{\text{eff}})\chi = 0, \quad (5.3)$$

where prime denotes the derivative with respect to ϱ. To achieve this goal, we must put

$$W^2 = \frac{v}{a},$$

$$P = \sqrt{\frac{v}{WR}} = f_2^{\frac{s+d}{\Delta}} f_1^{-(1-\sigma)(s+1)/4s} r^{-(s+1)/2}. \quad (5.4)$$

Substituting (5.4) in Eq. (5.2), we will get the effective potential V_{eff} for the test scalar field,

$$V_{\text{eff}} = k^2\frac{a}{b} + l(l + s)\frac{a}{w} + a\mu^2 + V_0, \quad (5.5)$$

$$V_0 = (\ln P)'^2 - (\ln P)''. \quad (5.6)$$

The first three terms of V_{eff} coincide with the effective potential for geodesics up to the change of constants of motion to the quantum numbers. Note that L^2 integrability of the solutions does not depend on the choice of the functions W and P, if we choose the correct integration measure in the corresponding Sturm-Liouville problem: R/adr for (5.2), and $d\varrho$ for (5.3).

Consider the limit $r \to +\infty$. Then, the measure asymptotically tends to $r^{s+1}dr$, so the L^2-integrable scalar field mode should tend to zero faster than $r^{-1-s/2}$. At the same time, the solution of Eq. (5.2) has an asymptotic form,

$$\Phi \approx r^{-s/2}(C_1 J_{l+s/2}(\kappa r) + C_2 Y_{l+s/2}(\kappa r)),$$
$$\kappa^2 = \omega^2 - k^2 - \mu^2.$$

For $\kappa^2 > 0$, the solutions are wavelike, and they are not L^2 integrable, which is evident from the asymptotic behavior of the oscillation amplitude $r^{-(s+1)/2}$. The case $\kappa^2 = 0$ leads to the solution of the form $\Phi \approx C_1 r^l + C_2 r^{-l-s}$, where the only square integrable mode is the second term for $s > 2$ or $l > 0$. In the case $\kappa^2 < 0$, one mode exponentially diverges and another exponentially decays. So only half of them are L^2 integrable. Summing up, the L^2 integrability implies the inequality $\omega^2 < k^2 + \mu^2$ (which can be not strict for $s > 2$ or $l > 0$).

Consider the solution near the outermost singularity (which is of the Fisher type for $S_G, c^2 < 1$). The asymptotic behavior of the function $W = \sqrt{v/a}$ has been classified in Table I. Generally, $W \sim x^m$, where m is some constant, depending on the parameters of the theory. The new coordinate can be expressed in terms of the old one asymptotically as follows: $\varrho \sim x^{m+1}$ (or $\rho \sim \ln x$ for $m = -1$) up to a multiplicative constant and an arbitrary additive constant. The new radial coordinate, which brings the equation to the Schrödinger form, strongly depends on whether the geodesics can achieve the singularity with a finite time interval of the distant observer or not. Near the singularity the function P behaves as x^n up to a coefficient, where n is some constant. From Eq. (5.6), for $m \neq -1$, it follows that $V_0 \approx (\nu^2 - 1)\varrho^{-2}/4$, where

$$\nu = \frac{2n + m + 1}{m + 1}. \quad (5.7)$$

Let us find the form of the solution in the vicinity of the singularity for each metric subfamily.

A. Solution S_G with $c^2 < 1$

The exponent $m = -1 + (1 + s)(1 - \sigma)/2s$ achieves its minimum for $\sigma = 1$ equal to -1. The case $\sigma = 1$ is similar to the test scalar field case in the background of the regular Schwarzschild or Reissner-Nordström solutions [69,70], so we will not consider it here. For all other cases, we have $m > -1$, so near the singularity $\varrho \to 0$. To avoid terms in V_{eff} more singular than ϱ^{-2} from V_0, it is necessary to require the following inequality:

$$\frac{\sigma}{m + 1} \geq -2, \qquad \left(\sigma - \frac{1 - \sigma}{s}\right)/(m + 1) \geq -2.$$

Both conditions strictly hold after the substitution of m, so the most singular term of the effective potential is contained in V_0. Function P has an asymptotic $x^{-\frac{m+1}{2}}$, where we find $n = -(m + 1)/2$ and $\nu = 0$. The solution of the Eq. (5.3) with such an effective potential near $r = r_0$ has the form,

$$\Phi \approx P\varrho^{1/2}(C_1 + C_2 \ln \varrho) \approx C_1' + C_2' \ln x. \quad (5.8)$$

Both modes are square integrable near the singularity (limit circle). Thus, in this case the singularity is not healed. The test scalar field is regular if $C'_2 = 0$ [71].

B. Solution S_G with $c^2 > 1$

With similar calculations, we have $m = 2(s + d)/\Delta > 0$, $n = -m/2$, and $\varrho \to 0$ near the singularity. Conditions for the other terms in V_{eff} to be less singular than V_0, are

$$(m + 1)\Delta - 2s > 0, \qquad (m + 1)\Delta - 2(d + s) > 0,$$

and they hold for any parameters. Substituting the values of m and n in (5.7), we get $\nu = 1/(m + 1)$. The solution of Eq. (5.3) with such an effective potential near the singularity is

$$\Phi \approx P\sqrt{\varrho}(C_1\varrho^{+\nu/2} + C_2\varrho^{-\nu/2}) \approx C_1 x + C_2. \qquad (5.9)$$

Both modes are regular and square integrable near the horizon. The singularity is not healed either.

C. Solution S_E with $\alpha^2 \neq \alpha^2_{\text{crit}} = 2s^2/(s + d)$

Such solutions in a theory with $\alpha = 0$ are regular extreme Reissner-Nordström black holes [72] and will not be considered here; we will assume $\alpha \neq 0$ only. From the asymptotics of W and P, we find

$$m = -2s(s + d)/\Delta \neq -1, \qquad n = -\frac{m + s + 1}{2}, \qquad (5.10)$$

$$\varrho \approx \frac{r^{m+1}}{\rho^m(m + 1)}. \qquad (5.11)$$

Let us keep only the leading terms in the expansion for each term in the effective potential,

$$V_{\text{eff}} \approx k^2 + q\left(\frac{r^{m+1}}{\rho^m(m + 1)}\right)^{-2} + \mu^2\left(\frac{r}{\rho}\right)^{4s^2/\Delta}, \qquad (5.12)$$

where

$$q = \frac{l(l + s)}{(m + 1)^2} + \frac{\nu^2 - 1}{4} = \left(\frac{l + s/2}{m + 1}\right)^2 - \frac{1}{4}. \qquad (5.13)$$

Solutions will be square integrable near the singularity if $R|\Phi|^2/a$ decays faster than r^{-1}. Taking into consideration Eq. (3.16), an integrable solution Φ must decay faster than $r^{-s/2-1+2s(d+s)/\Delta}$ near $r = 0$. We will use this condition further to analyze cases $\alpha^2 > \alpha^2_{\text{crit}}$ and $\alpha^2 < \alpha^2_{\text{crit}}$ separately.

For $m > -1$ ($\alpha^2 > \alpha^2_{\text{crit}}$), the leading term in Eq. (5.3) comes from the second term of V_{eff}, i.e., $q\varrho^{-2}$. In this case, the solution near the singularity is

$$\Phi \approx P\sqrt{\varrho}\left(C_1\varrho^{\sqrt{q+1/4}} + C_2\varrho^{-\sqrt{q+1/4}}\right)$$
$$\approx C'_1 r^l + C'_2 r^{-l-s}. \qquad (5.14)$$

The first mode r^l satisfies the square integrability condition if $l > 2s(d + s)/\Delta - 1 - s/2$. As we consider the case $\alpha^2 > \alpha^2_{\text{crit}}$, we can substitute $\alpha^2 = \alpha^2_{\text{crit}} + x_\alpha$, $x_\alpha > 0$ obtaining a new constraint $l > -1 - s/2 + 2s/(2s + x_\alpha)$. This inequality always holds, and the mode r^l is always integrable. The second mode r^{-l-s} is integrable if $l < -s/2 + x_\alpha/(2s + x_\alpha)$, which is possible only for $l = 0$, $s = 1$, $\alpha^2 > 2(d + 2)/(d + 1)$. This mode, however should be excluded from the spectrum on the same ground as the corresponding mode in the nonrelativistic quantum mechanics (appearance of the delta function under the action of the full Laplacian). Therefore, this case corresponds to the limit point.

For the case $m < -1$ ($\alpha^2 < \alpha^2_{\text{crit}}$) with $k^2 \neq \omega^2$, the coordinate ϱ tends to infinity, so the equation for χ has the form,

$$\chi'' + \kappa^2\chi \approx 0, \qquad \kappa^2 = \omega^2 - k^2.$$

The solution for this equation asymptotically has the form,

$$\Phi \approx r^n(C_1 \exp(i\kappa\varrho) + C_2 \exp(-i\kappa\varrho)). \qquad (5.15)$$

For $\kappa^2 > 0$, the solution near the singularity oscillates infinitely fast, it diverges due to $n < 0$ and is not integrable. For $\kappa^2 < 0$, one of two modes exponentially diverges, and another exponentially decays, being square integrable near the horizon. However, for $\kappa^2 = 0$, the leading term is $q\varrho^{-2}$. Thus, this case is similar to the previous one $\alpha^2 > \alpha^2_{\text{crit}}$ with the solution (5.14) except the fact that $x_\alpha < 0$ now. As the rhs of the inequality for the square integrability is a monotonic function of α^2, we can draw conclusions from the corner cases $\alpha^2 = 0, \alpha^2_{\text{crit}}$. One can find that the mode r^l is integrable except the case $d = 1$ with $\alpha^2 \leq 2s^2/(s + 1)(s + 2)$, and the mode r^{-l-s} is always nonintegrable.

D. Solution S_E with $\alpha^2 = \alpha^2_{\text{crit}} = 2s^2/(s + d)$

In this case, we have $m = -1$, $n = -s/2$, $\varrho \approx \rho \ln r \to -\infty$. Substituting the background solution in Eq. (5.6), we get an exact expression for V_0,

$$V_0 = \frac{1}{4}(r^s + \rho^s)^{-\frac{2(s+1)}{s}}((s^2 - 1)r^{2s} + 2s(s + 1)r^s\rho^s + s^2\rho^{2s}). \qquad (5.16)$$

In the singular point, the expression (5.16) tends to the finite positive value $V_0|_{r=0} = (s/2\rho)^2$. With account for the other terms, the effective potential tends to the value $V_{\text{eff}} \approx k^2 + (l + s/2)^2/\rho^2$. In this case, the equation has

TABLE III. The number of square integrable modes and regular modes for each class of solutions and a certain interval of ω^2. If the case contains some special case, this special case is given in the last column, and the number of modes for the special case is given in the brackets.

Solution	Condition on ω^2	No. of square int. modes	No. of regular modes	Special case
$S_G, c^2 < 1$	Any	2	1	
$S_G, c^2 > 1$	Any	2	2	
$S_E, \alpha^2 > \alpha_{\text{crit}}^2$	Any	1 (2)	1 (1)	$l = 0, s = 1, \alpha^2 > 2\frac{d+2}{d+1}$
$S_E, \alpha^2 < \alpha_{\text{crit}}^2$	$\omega^2 > k^2$	0	0	
	$\omega^2 = k^2$	0 (1)	0 (1)	$d = 1, \alpha^2 \leq \frac{2s^2}{(s+1)(s+2)}$
	$\omega^2 < k^2$	1	1	
$S_E, \alpha^2 = \alpha_{\text{crit}}^2$	$\omega^2 - k^2 \geq \left(\frac{l+s/2}{\rho}\right)^2$	0	0	
	$\frac{l(l+s)}{\rho^2} < \omega^2 - k^2 < \left(\frac{l+s/2}{\rho}\right)^2$	1	0	
	$\omega^2 - k^2 \leq \frac{l(l+s)}{\rho^2}$	1	1	

the form (5.15) up to the replacement of $\kappa^2 \to \omega^2 - k^2 - (l + s/2)^2/\rho^2$ and the dependence $\varrho(r)$. For $\kappa^2 > 0$, modes remain singular and nonintegrable. For $\kappa^2 < 0$, the solution can be simplified

$$\Phi \approx C_1 r^{-|\kappa|\rho - s/2} + C_2 r^{|\kappa|\rho - s/2}.$$

The first mode is singular and always nonintegrable. The other mode is regular if $|\kappa|\rho \geq s/2$ [which can be rewritten $\omega^2 \leq k^2 + l(l + s)/\rho^2$], and it is always integrable. In the case $\kappa^2 = 0$, we have to choose the next leading term in the effective potential with the lowest exponent $V_{\text{eff}} \approx b_1 r^{a_1}$. For the massive scalar field, the effective potential asymptotically behaves as $V_{\text{eff}} \approx \mu^2 (r/\rho)^{a_1}$, where $a_1 = 2s/(d + s)$, and for the massless case it is $V_{\text{eff}} \approx s^{-1}(s^2 - 2ls - 2l^2)\rho^{-s-2} r^s$ with $a_1 = s$ (note, the expression $s^2 - 2ls - 2l^2$ cannot be zero for integer l, s). Then the solution is

$$\chi'' - b_1 \exp\left(a_1 \varrho/\rho\right)\chi \approx 0,$$

$$\Phi \approx r^{-s/2}\left(C_1 I_0\left(\frac{2\sqrt{b_1}\rho}{a_1} r^{a_1/2}\right) + C_2 K_0\left(\frac{2\sqrt{b_1}\rho}{a_1} r^{a_1/2}\right)\right)$$

$$\approx C_1' r^{-s/2} + C_2' r^{-s/2} \ln r,$$

where I_0, K_0 are modified Bessel functions of the second kind. Solutions we obtained diverge, and they are not integrable for any C_1, C_2. As there are at most one integrable mode for each set of quantum numbers, the singularity is healed.

The summary of the square integrability and regularity of the modes in the background of the different classes of the solution is given in Table III. The general class is always unhealed since the number of square integrable modes is two. In the case of the Fisher-like singularity ($S_G, c^2 < 1$), one of these modes is integrable. Contrary, in the solution with a singularity of the Reissner-Nordström type ($S_G, c^2 > 1$), both modes are regular. In the solution S_E,

there is at most one square integrable mode, except for one special case. This case is $l = 0$, for the solution $S_E, \alpha^2 > 2\frac{d+2}{d+1} > \alpha_{\text{crit}}^2$ with $s = 1$. However, similarly to quantum mechanics, this mode can be excluded from the spectrum as a solution for the pointlike (δ-function distribution) source. If we exclude this mode, all S_E solutions contain a quantum unobservable singularity; otherwise, the singularity is observable only in the special case. All square integrable modes are regular except this special $l = 0$ case, and one more case with a critical coupling constant $\alpha^2 = \alpha_{\text{crit}}^2$ within the certain interval of the frequency $\frac{l(l+s)}{\rho^2} + k^2 < \omega^2 < \left(\frac{l+s/2}{\rho}\right)^2 + k^2$.

VI. CONCLUSIONS

In this article, we have constructed charged p-branes with primary scalar charge in Einstein gravity with dilaton and antisymmetric forms using Harrison transformations adapted for branes applied to an extended Fisher solution. The obtained branes, spherically symmetric in the transverse space, generically have naked singularities, except for those obtained from the regular subfamily of the seed solutions and Harrison transformation parameter $c^2 < 1$.

Using some limiting procedure in the space of parameters, we also found a special solution, denoted as S_E, satisfying a constraint on the physical charges which reduces to the "no-force" condition in EMD theory with $\alpha^2 = 3$. Exploring supersymmetry of the full new family of solutions corresponding to NS5 branes, we found that supersymmetry holds only in the limiting case S_E, which coincides with the standard BPS NS5-branes.

We investigated the geodesic motion in the vicinity of the outermost singularity of the obtained solutions. In the general case with $c^2 < 1$ with Fisher singularity, the parameter σ plays the key role in the behavior of the geodesics. In the general case with $c^2 > 1$, the behavior

does not differ appreciably for different theory parameters and charges of a solution. The behavior of motion near the singularity for S_E family depends on the relation between the coupling constant α and its critical value $\alpha_{\text{crit}}^2 = 2s^2/(s+d)$. If $\alpha^2 \leq \alpha_{\text{crit}}^2$; geodesics near the singularity behave similarly to the motion near a black hole surface: the effective potential is always bounded, a particle achieves the singularity with an infinite time with respect to a distant observer. Otherwise, if $\alpha^2 > \alpha_{\text{crit}}^2$, the effective potential for geodesics with nonzero angular moment diverges, but radial geodesics achieve the singularity with a finite time of a distant observer.

We considered a massive test scalar field in the background of the obtained solutions. By transforming the dynamical variable and the radial coordinate, one can represent the Klein-Gordon equation in the form of the Schrödinger equation with some effective potential. We investigated the possibility to heal the singularity in a quantum sense, based on the analysis of square integrability of the test field. For the solution family S_E, it was shown that the behavior of the test scalar field depends on whether the coupling constant α get over the critical value α_{crit} or not. For S_E, at most one mode is square integrable near the singularity (and so the singularity is healed), except the case $s = 1$, $\alpha^2 > 2(d+2)/(d+1) > \alpha_{\text{crit}}^2$ for the mode

$l = 0$. But this latter mode should be excluded from the spectrum of perturbations as a solution for a pointlike source, similarly to the nonrelativistic quantum mechanics.

For a general family with arbitrary c^2, all modes are square integrable near the singularity. Thus, the choice of boundary conditions remains an important issue of the stability of the solution with respect to test disturbances of the scalar field in the general case.

Finally, in the case of codimension three, we constructed the branes equipped with Zipoy-Voorhees oblateness parameter, which do not have spherical symmetry in transverse space. These solutions do not seem to be known before.

ACKNOWLEDGMENTS

[1] M. J. Duff and J. X. Lu, Black and super p-branes in diverse dimensions, Nucl. Phys. **B416**, 301 (1994).

[2] H. Lu and C. N. Pope, P-brane solitons in maximal super-gravities, Nucl. Phys. **B465**, 127 (1996).

[3] K. S. Stelle, Lectures on supergravity p-branes, in Trieste 1996, High Energy Physics and Cosmology, pp. 287–339, arXiv:hep-th/9701088.

[4] D. V. Gal'tsov, J. P. S. Lemos, and G. Clement, Supergravity p-branes revisited: Extra parameters, uniqueness, and topological censorship, Phys. Rev. D **70**, 024011 (2004).

[5] M. J. Duff, R. R. Khuri, and J. X. Lu, String and five-brane solitons: Singular or non-singular?, Nucl. Phys. **B377**, 281 (1992).

[6] B. Zhou and C. J. Zhu, The complete black brane solutions in D-dimensional coupled gravity system, arXiv:hep-th/9905146.

[7] J. X. Lu and S. Roy, Supergravity approach to tachyon condensation on the brane—anti-brane system, Phys. Lett. B **599**, 313 (2004).

[8] J. X. Lu and S. Roy, Static, non-SUSY p-branes in diverse dimensions, J. High Energy Phys. 02 (2005) 001.

[9] S. Kobayashi, T. Asakawa, and S. Matsuura, Open string tachyon in supergravity solution, Mod. Phys. Lett. A **20**, 1119 (2005).

[10] D. Gal'tsov, S. Klevtsov, D. Orlov, and G. Clement, More on general p-brane solutions, Int. J. Mod. Phys. A **21**, 3575 (2006).

[11] T. Asakawa, S. Kobayashi, and S. Matsuura, Excited D-branes and supergravity solutions, Int. J. Mod. Phys. A **21**, 1503 (2006).

[12] I. Z. Fisher, Scalar mesostatic field with regard for gravitational effects, Zh. Eksp. Teor. Fiz. **18**, 636 (1948); [English translation arXiv:gr-qc/9911008].

[13] A. I. Janis, E. T. Newman, and J. Winicour, Reality of the Schwarzschild Singularity, Phys. Rev. Lett. **20**, 878 (1968).

[14] J. Winicour, A. I. Janis, and E. T. Newman, Static, axially symmetric point horizons, Phys. Rev. **176**, 1507 (1968).

[15] M. Wyman, Static spherically symmetric scalar fields in general relativity, Phys. Rev. D **24**, 839 (1981).

[16] H. A. Buchdahl, Reciprocal static metrics and scalar fields in the general theory of relativity, Phys. Rev. **115**, 1325 (1959).

[17] K. S. Virbhadra, Janis-Newman-Winicour and Wyman solutions are the same, Int. J. Mod. Phys. A **12**, 4831 (1997).

[18] A. Bhadra and K. K. Nandi, On the equivalence of the Buchdahl and the Janis-Newman-Winnicour solutions, Int. J. Mod. Phys. A **16**, 4543 (2001).

[19] J. E. Chase, Event horizons in static scalar-vacuum space-times, Commun. Math. Phys. **19**, 276 (1970).

[20] K. S. Virbhadra, S. Jhingan, and P. S. Joshi, Nature of singularity in Einstein massless scalar theory, Int. J. Mod. Phys. D **06,** 357 (1997).

[21] S. Abdolrahimi and A. A. Shoom, Analysis of the Fisher solution, Phys. Rev. D **81,** 024035 (2010).

[22] L. Sokolowski and B. J. Carr, Can black holes in classical Kaluza-Klein theory have scalar hair?, Phys. Lett. B **176,** 334 (1986).

[23] I. Bogush and D. Gal'tsov, Generation of rotating solutions in Einstein-scalar gravity, Phys. Rev. D **102,** 124006 (2020).

[24] M. Astorino, Stationary axisymmetric spacetimes with a conformally coupled scalar field, Phys. Rev. D **91,** 064066 (2015).

[25] B. Chauvineau, New method to generate exact scalar-tensor solutions, Phys. Rev. D **100,** 024051 (2019).

[26] K. S. Virbhadra and G. F. R. Ellis, Gravitational lensing by naked singularities, Phys. Rev. D **65,** 103004 (2002).

[27] G. N. Gyulchev and S. S. Yazadjiev, Gravitational lensing by rotating naked singularities, Phys. Rev. D **78,** 083004 (2008).

[28] T. Crisford and J. E. Santos, Violating the Weak Cosmic Censorship Conjecture in Four-Dimensional Anti-de Sitter Space, Phys. Rev. Lett. **118,** 181101 (2017).

[29] J. P. DeAndrea and K. M. Alexander, Negative time delay in strongly naked singularity lensing, Phys. Rev. D **89,** 129904 (2014); **89,** 123012(E) (2014).

[30] S. Zhou, R. Zhang, J. Chen, and Y. Wang, Geodesic structure of Janis-Newman-Winicour spacetime, Int. J. Theor. Phys. **54,** 2905 (2015).

[31] A. N. Chowdhury, M. Patil, D. Malafarina, and P. S. Joshi, Circular geodesics and accretion disks in Janis-Newman-Winicour and Gamma metric, Phys. Rev. D **85,** 104031 (2012).

[32] G. Z. Babar, M. Jamil, and Y.-K. Lim, Dynamics of a charged particle around a weakly magnetized naked singularity, Int. J. Mod. Phys. D **25,** 1650024 (2016).

[33] G. Gyulchev, P. Nedkova, T. Vetsov, and S. Yazadjiev, Image of the Janis-Newman-Winicour naked singularity with a thin accretion disk, Phys. Rev. D **100,** 024055 (2019).

[34] S. Sau, I. Banerjee, and S. SenGupta, Imprints of the Janis-Newman-Winicour spacetime on observations related to shadow and accretion, Phys. Rev. D **102,** 064027 (2020).

[35] J. Ben Achour, H. Liu, and S. Mukohyama, Hairy black holes in DHOST theories: Exploring disformal transformation as a solution-generating method, J. Cosmol. Astropart. Phys. 02 (2020) 023.

[36] D. V. Gal'tsov, Conformal and kinetic couplings as two Jordan frames of the same theory: Conformal and kinetic couplings, Eur. Phys. J. C **80,** 443 (2020).

[37] D. V. Gal'tsov and O. A. Rytchkov, Generating branes via sigma models, Phys. Rev. D **58,** 122001 (1998).

[38] B. K. Harrison, New solutions of the Einstein-Maxwell equations from old, J. Math. Phys. (N.Y.) **9,** 1744 (1968).

[39] D. M. Zipoy, Topology of some spheroidal metrics, J. Math. Phys. (N.Y.) **7,** 1137 (1966).

[40] B. H. Voorhees, Static axially symmetric gravitational fields, Phys. Rev. D **2,** 2119 (1970).

[41] B. C. Xanthopoulos and T. Zannias, Einstein gravity coupled to a massless scalar field in arbitrary space-time dimensions, Phys. Rev. D **40,** 2564 (1989).

[42] J. X. Lu, ADM masses for black strings and p-branes, Phys. Lett. B **313,** 29 (1993).

[43] P. de Lange, D. R. Mayerson, and B. Vercnocke, Structure of six-dimensional microstate geometries, J. High Energy Phys. 09 (2015) 075.

[44] G. W. Gibbons, Antigravitating black hole solitons with scalar hair in $N = 4$ supergravity, Nucl. Phys. **B207,** 337 (1982).

[45] D. Rasheed, The rotating dyonic black holes of Kaluza-Klein theory, Nucl. Phys. **B454,** 379 (1995).

[46] H. Lu, C. N. Pope, E. Sezgin, and K. S. Stelle, Stainless super p-branes, Nucl. Phys. **B456,** 669 (1995).

[47] A. A. Tseytlin, 'No force' condition and BPS combinations of p-branes in eleven-dimensions and ten-dimensions, Nucl. Phys. **B487,** 141 (1997).

[48] J. Scherk, Antigravity: A crazy idea?, Phys. Lett. **88B,** 265 (1979).

[49] M. Nozawa, On the Bogomol'nyi bound in Einstein-Maxwell-dilaton gravity, Classical Quantum Gravity **28,** 175013 (2011).

[50] S. J. Poletti, J. Twamley, and D. L. Wiltshire, Dyonic dilaton black holes, Classical Quantum Gravity **12,** 1753 (1995); **12,** 2355(E) (1995).

[51] W. J. Geng, B. Giant, H. Lü, and C. N. Pope, Mass of dyonic black holes and entropy super-additivity, Classical Quantum Gravity **36,** 145003 (2019).

[52] R. Kallosh, Supersymmetry in singular spaces and domain walls, Int. J. Mod. Phys. A **16,** 683 (2001).

[53] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, Superstrings and solitons, Nucl. Phys. **B340,** 33 (1990).

[54] M. J. Duff, R. R. Khuri, and J. X. Lu, String solitons, Phys. Rep. **259,** 213 (1995).

[55] G. Clement, D. Gal'tsov, and C. Leygnac, Black branes on the linear dilaton background, Phys. Rev. D **71,** 084014 (2005).

[56] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, England, 2016).

[57] R. Kallosh, Democratic formulation of $D = 10$ supersymmetry, Theor. Math. Phys. **128,** 1193 (2001).

[58] P. Liao, J. Chen, H. Huang, and Y. Wang, Absorption and scattering of scalar wave by naked singularity, Gen. Relativ. Gravit. **46,** 1752 (2014).

[59] A. Chowdhury and N. Banerjee, Echoes from a singularity, Phys. Rev. D **102,** 124051 (2020).

[60] R. M. Wald, Dynamics in nonglobally hyperbolic, static spacetimes, J. Math. Phys. (N.Y.) **21,** 2802 (1980).

[61] G. T. Horowitz and D. Marolf, Quantum probes of space-time singularities, Phys. Rev. D **52,** 5670 (1995).

[62] A. Ishibashi and A. Hosoya, Who's afraid of naked singularities? Probing timelike singularities with finite energy waves, Phys. Rev. D **60,** 104028 (1999).

[63] A. Ishibashi and R. M. Wald, Dynamics in nonglobally hyperbolic static spacetimes. 2. General analysis of prescriptions for dynamics, Classical Quantum Gravity **20,** 3815 (2003).

[64] T. M. Helliwell, D. A. Konkowski, and V. Arndt, Quantum singularity in quasiregular spacetimes, as indicated by Klein-Gordon, Maxwell and Dirac fields, Gen. Relativ. Gravit. **35,** 79 (2003).

[65] D. A. Konkowski and T. M. Helliwell, Quantum healing of spacetime singularities: A review, Mod. Phys. Lett. A **33,** 1830002 (2018).

[66] G. W. Gibbons, S. A. Hartnoll, and A. Ishibashi, On the stability of naked singularities, Prog. Theor. Phys. **113,** 963 (2005).

[67] A. Sadhu and V. Suneeta, A naked singularity stable under scalar field perturbations, Int. J. Mod. Phys. D **22,** 1350015 (2013).

[68] J. P. Harnad and P. Winternitz, Hyperspherical harmonics, separation of variables and the Bethe ansatz, Lett. Math. Phys. **33,** 61 (1995).

[69] J. A. H. Futterman, F. A. Handler, and R. A. Matzner, *Scattering From Black Holes* (Cambridge University Press, Cambridge, England, 2012).

[70] C. Kehle and Y. Shlapentokh-Rothman, A scattering theory for linear waves on the interior of Reissner-Nordström black holes, Ann. Inst. Henri Poincaré **20,** 1583 (2019).

[71] K. A. Bronnikov, Scalar fields as sources for wormholes and regular black holes, Particles **1,** 56 (2018).

[72] S. Dain and G. Dotti, The wave equation on the extreme Reissner-Nordström black hole, Classical Quantum Gravity **30,** 055011 (2013).