Covariant classification of conformal Killing vectors of locally conformally flat *n*-manifolds with an application to Kerr–de Sitter spacetimes

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We obtain a coordinate independent algorithm to determine the class of conformal Killing vectors of a locally conformally flat *n*-metric γ of signature (r, s) modulo conformal transformations of γ. This is done in terms of endomorphisms in the pseudo-orthogonal Lie algebra $\mathfrak{o}(r + 1, s + 1)$ up to conjugation of the group $O(r + 1, s + 1)$. The explicit classification is worked out in full for the Riemannian γ case ($r = 0$, $s = n$). As an application of this result, we prove that the set of five-dimensional, $(\Lambda > 0)$-vacuum, algebraically special metrics with nondegenerate optical matrix, analyzed in [G. Bernardi de Freitas *et al.*, Commun. Math. Phys. **340**, 291 (2015).] is in one-to-one correspondence with the metrics in the Kerr–de Sitter-like class. This class [M. Mars *et al.*, Classical Quantum Gravity **34**, 095010 (2017).; M. Mars and C. Peón-Nieto, Phys. Rev. D **105**, 044027 (2022).] exists in all dimensions, and its defining properties involve only properties at $\mathcal{I}$. The equivalence between two seemingly unrelated classes of metrics points towards interesting connections between the algebraically special type of the bulk spacetime and the conformal geometry at null infinity.

I. INTRODUCTION

Conformal invariance plays a fundamental role in many physical theories that include critical phenomena, conformal field theories, or electromagnetism, among many others. Conformal Killing vectors (CKV) in (a locally conformally) flat space, being the infinitesimal generators of (local) conformal transformations, are therefore also of great importance. The conformal group induces a natural equivalence relation between CKVs in flat space. Two such CKVs are said to be equivalent if there is a (local) conformal transformation that maps one to another. Conformal invariance in a theory means that the relevant object is the conformal class of CKVs instead of individual CKV representatives in the class.

As summarized in more detail below [1] (also [2–4]), the classification of conformal classes of CKVs in a flat *n*-dimensional space [of signature (r, s)] is equivalent to the algebraic classification of skew-symmetric endomorphisms in an $(n + 2)$-dimensional flat space of signature $(r + 1, s + 1)$ [i.e., elements of the Lie algebra $\mathfrak{o}(r + 1, s + 1)$] up to pseudo-orthogonal $O(r + 1, s + 1)$ transformations. The latter classification is worked out in detail in [4] by means of only elementary linear algebra

methods, although it is worth to remark that, in a more algebraic language, this is a particular case of classification of semisimple adjoint orbits, which is a well-known problem in Lie theory for which there actually exists a general framework (e.g., [5,6]).

For any classification result of the adjoint orbits of $\mathfrak{o}(r + 1, s + 1)$ to become of practical use to our case at hand, one first needs to find a map between the algebras of CKVs and $\mathfrak{o}(r + 1, s + 1)$. Such a map is easily constructed in Cartesian coordinates [1] (also [2–4]), but a general coordinate independent approach appears to be missing in the literature. This is specially relevant in physical contexts where general covariance is a key ingredient (e.g., general relativity), as it is often the case that the quantities are expressed in coordinate systems that are convenient for the problem at hand, and hence, *a priori* unrelated to any Cartesian description. The main objective of this paper is to provide a simple, algorithmic, and coordinate independent classification scheme for conformal classes of CKVs in locally conformally flat manifolds of arbitrary signature. Our main result is given in Theorem 2.5.

As already said, this result can be of interest in any physical problem (in a locally conformally flat space) where conformal invariance and diffeormorphism invariance play a crucial role. An example of paramount importance is the study of the asymptotic properties of

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spacetimes. A precise definition of the asymptotic region of a spacetime $(\tilde{M}, \tilde{g})$ can be given in terms of conformal scalings, as long as the metric satisfies the property of being *conformally extendable*. We say that a metric $\tilde{g}$ is conformally extendable if there exists a metric $g = \Omega^2 \tilde{g}$ for a (sufficiently) smooth positive function Ω on $\tilde{M}$, such that g admits a (sufficiently) smooth extension to $M := \tilde{M} \cup \partial \tilde{M}$, where $\partial \tilde{M} = \{\Omega = 0\}$ is usually called *null infinity* and denoted $\mathcal{I}$. A milestone result in general relativity by H. Friedrich [7,8] proves, using the conformal properties of the spacetime, that there exists a well-posed Cauchy problem with data on $\mathcal{I}$ for the $(\Lambda > 0)$-vacuum case in four dimensions [9]. Similar results also hold in higher dimensions [10–13], based on a different formalism. More details on these asymptotic Cauchy problems are given in Sec. III. Just like in the ordinary initial value problem of the Einstein equations [14], the presence of Killing vectors (KV) in the spacetime constraint the initial data of the asymptotic Cauchy problem with $\Lambda > 0$ [15,16]. For a KV X of $\tilde{g}$, it is a general fact that X extends to $\mathcal{I}$ as a tangent vector ξ, which is necessarily a conformal Killing vector of the induced metric γ at $\mathcal{I}$. In four spacetime dimensions, a Killing initial data (KID) equation was obtained in [16] for the asymptotic Cauchy problem of the $(\Lambda > 0)$-vacuum equations. This gives a necessary and sufficient condition that ξ and the initial data must satisfy in order for the spacetime metric to admit a KV X, which coincides with ξ at $\mathcal{I}$. The KID equation was extended to higher dimensions in [15], where it was also proved that, when the data are restricted to be analytic, it gives necessary and sufficient conditions for ξ to extend to a spacetime KV.

The fact that the asymptotic Cauchy problem for the $(\Lambda > 0)$-vacuum equations is formulated in terms of conformal metrics, leaves a conformal gauge freedom in the initial data (cf. Sec. III). Moreover, as a covariant theory, the data are also equivalent under diffeomorphisms of $\mathcal{I}$. This has the interesting consequence that the set of conformal diffeomorphisms (or conformal isometries) of $\mathcal{I}$ is a symmetry at $\mathcal{I}$ in the sense above. More specifically, the conformal group $\text{Conf}(\mathcal{I})$ is the set of diffeomorphisms $\varphi: \mathcal{I} \to \mathcal{I}$ satisfying $\varphi^\star(\gamma) = \omega^2 \gamma$ for some smooth positive function ω of $\mathcal{I}$. We show in Sec. III that if ξ satisfies the KID equation for certain data, then the vector fields of the form $\varphi_\star(\xi)$ for each $\varphi \in \text{Conf}(\mathcal{I})$ also satisfy the KID equation and correspond to the same symmetry of $\tilde{g}$. Thus, the symmetries of $\tilde{g}$ are in correspondence with the conformal classes of CKVs $[\xi]$ instead of with specific representatives in the class. As a consequence, whenever the geometry at $\mathcal{I}$ is locally conformally flat, the algorithmic classification result achieved in Theorem 2.5 is of direct applicability.

In the second part of the paper, we apply this theorem to establish the equivalence between two *a priori* unrelated families of $(\Lambda > 0)$-vacuum solutions of the Einstein field equations in five dimensions. The first class consists of the algebraically special spacetimes with nondegenerate optical matrix, classified in [17]. The second is the class of so-called Kerr–de Sitter-like class of metrics with conformally flat $\mathcal{I}$. This class is a natural generalization of Kerr–de Sitter, first defined in four dimensions in [2,18] and later generalized to arbitrary dimensions in [19] in the locally conformally flat $\mathcal{I}$ case. The characterizing property of this class is that its asymptotic data are constructed canonically from a CKV at $\mathcal{I}$. In fact, there is exactly one Kerr–de Sitter-like metric associated to each conformal class of CKVs of the metric at $\mathcal{I}$. Therefore, the space of conformal classes $[\xi]$ provides a good representation of the moduli space of metrics in the class. The explicit form of the metrics is the Kerr–de Sitter class in all dimensions was obtained in [19] via a rather unexpected equivalence (in all dimensions) between this class and the family of general $(\Lambda > 0)$-vacuum spacetimes of Kerr-Schild type and satisfying a natural fall-off condition at infinity. The fact that this family (in five dimensions) is also equivalent to the family obtained in [17] indicates that there might be interesting and unexpected connections for $(\Lambda > 0)$-vacuum spacetimes (in arbitrary dimension higher than four) between being algebraically special and being conformally extendable and having very special properties at null infinity. We emphasize that the family studied in [17] made *no a priori* assumption on the asymptotic properties of the spacetime.

Moreover, it is worth to emphasize that the metrics in [17] were heuristically found to be either Kerr–de Sitter or a limit thereof. This same fact holds for the Kerr–de Sitter-like class [19], but it can be seen as a natural consequence of the topological structure of the space of conformal classes of CKVs ξ, as well as the well-posedness of the asymptotic Cauchy problem. This perspective strengthens the uniqueness result of Kerr–de Sitter as understood in [17], in the sense that it proves that no further limits can be obtained from Kerr–de Sitter without, at least, substantially modifying the asymptotic properties.

The plan of the paper is as follows. In Sec. II, we prove our main result, Theorem 2.5, which provides a method for a coordinate independent classification of conformal classes of CKVs of any locally conformally flat n-manifold (Σ, γ) of signature (r, s). We do this in terms of the (simpler) algebraic classification of skew-symmetric endomorphisms of an $(n + 2)$-dimensional flat manifold $(\mathbb{R}^{r+1,s+1}, \eta)$ up to isometries of η, where η is a flat metric of signature $(r + 1, s + 1)$. The latter amounts to the classification of the Lie algebra $\mathfrak{o}(r + 1, s + 1)$ up to adjoint action of the Lie group $O(r + 1, s + 1)$, i.e., the equivalence classes $[F] = \{F' \in \mathfrak{o}(r + 1, s + 1) | F' = \Lambda \cdot F \cdot \Lambda^{-1}, \ \forall \ \Lambda \in O(r + 1, s + 1)\}$, where the dot stands for the usual matrix multiplication. A key result for Theorem 2.5 is (cf. Proposition 2.1) that we find a way to assign an element $F \in \mathfrak{o}(r + 1, s + 1)$ to any CKV ξ of γ based solely on pointwise properties of ξ (and its

derivatives). In addition, for later use, we give in Sec. II A the explicit definition of a set of quantities which uniquely determines a class $[F]$ in the Riemannian γ case (i.e., $r = 0$, $s = n$).

In Sec. III, we first review (in Sec. III A) some general facts on the asymptotic Cauchy problem of the $(\Lambda > 0)$-vacuum field equations in all dimensions. We then establish the equivalence of a class of CKVs at $\mathcal{I}$ satisfying the KID equation with a unique KV of the physical spacetime $\tilde{g}$. This result had already been proven for asymptotic data in the Kerr–de Sitter-like class in [2,15], and here we show it holds in general. Section III is concluded with Sec. III B, where we revisit the definition of the Kerr–de Sitter-like class of spacetimes in all dimensions and their main properties.

In Sec. IV, we apply Theorem 2.5 to establish the equivalence between the set of five dimensional algebraically special spacetimes with nondegenerate optical matrix, classified in [17], with the Kerr–de Sitter-like class of spacetimes. We start by calculating the asymptotic initial data of the metrics in [17]. This easily proves that such spacetimes are contained in the Kerr–de Sitter-like class. The nontrivial part is to verify that the metrics given in [17] exhaust the whole space of conformal classes of locally conformally flat four dimensional Riemannian metrics. The proof relies strongly on the results of Sec. II, because the coordinates in which the metrics in [17] are given are adapted to spacetime null congruences, and have therefore nothing to do with (conformally) Cartesian coordinates at infinity. In fact, it is hard to find an explicitly flat representative of the metric at $\mathcal{I}$.

We finish this paper with some observations in Sec. V. We emphasize that the application in Sec. IV goes beyond simply providing an explicit example where our main theorem can be applied. The application is useful to gain insight in the classification higher dimensional algebraically special spacetimes and point out several possible future results.

II. COVARIANT CLASSIFICATION OF CKVS OF LOCALLY CONFORMALLY FLAT METRICS

We start with a well-known result in conformal geometry, which we prove for completeness. Recall that the Schouten tensor of a metric g of dimension $n \geq 3$ is defined as

$$L_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{\mathrm{Scal}_g}{2(n-1)}g\right),$$

and we denote the gradient, the Hessian, and its trace (the rough Laplacian) by grad_g, Hess_g, and Δ_g, respectively. Scalar product with g is denoted either by $g(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle_g$.

Lemma 2.1. Let (M, g) be a semi-Riemannian manifold of dimension n and Ω a smooth positive function in M.

Define $\bar{g} = \frac{1}{\Omega^2}g$. Then the respective Schouten tensors are related by

$$L_{\bar{g}} = L_g + \frac{1}{\Omega}\mathrm{Hess}_g\Omega - \frac{1}{2\Omega^2}|\mathrm{grad}_g\Omega|_g^2 g. \tag{1}$$

Proof.—The relationship between the Ricci tensors of g and $\bar{g}$ is well-known be (e.g., [20])

$$\mathrm{Ric}_{\bar{g}} = \mathrm{Ric}_g + \frac{n-2}{\Omega}\mathrm{Hess}_g\Omega + \left(\frac{1}{\Omega}\Delta_g\Omega - \frac{n-1}{\Omega^2}|\mathrm{grad}_g\Omega|_g^2\right)g.$$

Taking trace with respect to $\bar{g}$ and inserting in the expression for $L_{\bar{g}}$ the result follows at once. ∎

The following result relates the Hessians of scalar functions with respect to g and with respect to $\bar{g}$. The Levi-Civita covariant derivatives of g, $\bar{g}$ are denoted ∇, $\bar{\nabla}$, respectively. Indices in objects constructed using geometric quantities associated to g or to $\bar{g}$ and raised and lowered with its corresponding metric. Capital latin indices take values in $1, \ldots, n$.

Lemma 2.2. Let f be a smooth function on M and g, $\bar{g}$ and Ω as before Then,

$$\mathrm{Hess}_{\bar{g}}\left(\frac{f}{\Omega}\right) = \frac{1}{\Omega}\left(\mathrm{Hess}_g f + f(L_g - L_{\bar{g}})\right.$$
$$\left. + \left(\frac{f|\mathrm{grad}_g\Omega|_g^2}{2\Omega^2} - \frac{1}{\Omega}\langle\mathrm{grad}_g f, \mathrm{grad}_g\Omega\rangle_g\right)g\right).$$

Proof.—The difference tensor $\bar{\nabla} - \nabla$ is

$$S^A{}_{CD} = -\delta^A_C\frac{\nabla_D\Omega}{\Omega} - \delta^A_D\frac{\nabla_C\Omega}{\Omega} + \frac{\nabla^A\Omega}{\Omega}g_{CD}. \tag{2}$$

For a covector s_A, we therefore have

$$\bar{\nabla}_A s_B = \nabla_A s_B + \frac{1}{\Omega}(s_A\nabla_B\Omega + s_B\nabla_A\Omega - s_C\nabla^C\Omega g_{AB}).$$

Applying this to $s_B = \bar{\nabla}_B(\Omega^{-1}f) = \nabla_B(\Omega^{-1}f)$ and expanding the products in the right-hand side yields

$$\bar{\nabla}_A\bar{\nabla}_B\left(\frac{f}{\Omega}\right) = \frac{1}{\Omega}\left(\nabla_A\nabla_B f\right.$$
$$\left. + f\left(-\frac{1}{\Omega}\nabla_A\nabla_B\Omega + \frac{1}{\Omega^2}\nabla^C\Omega\nabla_C\Omega g_{AB}\right)\right.$$
$$\left. - \frac{1}{\Omega}\nabla^C f\nabla_C\Omega g_{AB}\right).$$

Replacing the Hessian of Ω with Eq. (1), the result follows. ∎

Consider a metric g_E of signature (r, s), which is locally flat on a manifold M and let $\mathcal{D}$ be the corresponding Levi-Civita derivative. Let $p \in M$ and U_p a neighborhood of p

where g_E is flat. Since the general solution of equation $\mathcal{D}_A \mathcal{D}_B f = 0$, $f(p) = 0$ on U_p is a linear combination of Cartesian coordinates centered at p, there exist precisely $n := r + s$ linearly independent functions $\{Y^C\}$ satisfying

$$\mathcal{D}_A \mathcal{D}_B Y^C = 0, \qquad Y^A|_p = 0. \tag{3}$$

We do not restrict $\{Y^A\}$ to be orthogonal. More specifically, let

$$h^{AB} := \mathcal{D}_C Y^A \mathcal{D}^C Y^B. \tag{4}$$

It is immediate from (3) that h^{AB} are constant on U_p. $\{Y^A\}$ being linearly independent and vanishing at p, it is immediate that they define a coordinate system on U_p. It follows that h^{AB} is invertible and has signature (r, s). We let h_{AB} be its inverse and introduce the functions,

$$Y^0 := 1, \qquad Y^{n+1} := \frac{1}{2} h_{AB} Y^A Y^B. \tag{5}$$

The following lemma provides a number of properties of the set $\{Y^\alpha\} := \{Y^0, Y^A, Y^{n+1}\}$.

Lemma 2.3. With the setup and definitions above, the following properties hold in U_p:

(i) The functions $\{Y^\alpha\}$ are linearly independent and satisfy

$$\text{Hess}_{g_E} Y^\alpha = \delta^\alpha_{n+1} g_E. \tag{6}$$

Moreover, the matrix,

$$Q^{\alpha\beta} := \mathcal{D}_C Y^\alpha \mathcal{D}^C Y^\beta - \delta^\alpha_{n+1} Y^\beta - \delta^\beta_{n+1} Y^\alpha,$$

is constant on U_p, nondegenerate and of signature $(r+1, s+1)$.

(ii) The general solution of

$$\text{Hess}_{g_E} f = c g_E, \qquad c \in \mathbb{R},$$

is a linear combination $f = c_\alpha Y^\alpha$ with $c_{n+1} = c$ and c_α for $\alpha \neq n+1$ arbitrary.

(iii) For each α, β, the vector field,

$$\zeta^{\alpha\beta} := Y^\alpha \text{grad}_{g_E} Y^\beta - Y^\beta \text{grad}_{g_E} Y^\alpha, \qquad \alpha < \beta \tag{7}$$

is a conformal Killing vector of g_E satisfying

$$\pounds_{\zeta^{\alpha\beta}} g_E = 2(Y^\alpha \delta^\beta_{n+1} - Y^\beta \delta^\alpha_{n+1}) g_E.$$

(iv) The set $\mathcal{B} := \{\zeta^{\alpha\beta}, \alpha < \beta\}$ is linearly independent and spans the conformal Killing algebra of (U_p, g_E).

Proof.—Firstly, it is trivial that $\text{Hess}_{g_E} Y^0 = 0$ and $\text{Hess}_{g_E} Y^A = 0$ holds. From definition (5) and since $\text{Hess}_{g_E} Y^A = 0$, we get

$$\mathcal{D}_C \mathcal{D}_D Y^{n+1} = h_{AB} \mathcal{D}_C Y^A \mathcal{D}_D Y^B.$$

Fix any point $q \in U_p$ and define the square matrix $A^A{}_B := \mathcal{D}_B Y^A|_q$. Using matrix notation $(A)^A{}_B = A^A_B$ where the upper index denotes row and the lower index column, we may write (4) as (t is the transpose)

$$(h^\sharp) = (A)^t (g^\sharp|_q)(A),$$

where $(g^\sharp_q)$ and $(h^\sharp)$ are the symmetric matrices with coefficients $(g^\sharp)^{AB} = g^{AB}_E|_q$ and $(h^\sharp)^{AB} = h^{AB}$. Since $(h^\sharp)$ is invertible, so it is (A) and

$$(h) = (A)^{-1}(g|_q)((A)^{-1})^t \Leftrightarrow (A)(h)(A)^t = (g|_q),$$

where $(g|_q)$ is the matrix with components $(g|_q)_{AB} = (g_E)_{AB}|_q$. In index notation, $h_{AB} \mathcal{D}_C Y^A \mathcal{D}_D Y^B = (g_E)_{CD}$ at all points in U_p. Thus, $\text{Hess}_{g_E} Y^{n+1} = g_E$ as claimed. The constancy of $Q^{\alpha\beta}$ follows from (6) because

$$\mathcal{D}_D Q^{\alpha\beta} = (\mathcal{D}_D \mathcal{D}_C Y^\alpha) \mathcal{D}^C Y^\beta + \mathcal{D}_C Y^\alpha \mathcal{D}_D \mathcal{D}^C Y^\beta$$
$$- \delta^\alpha_{n+1} \mathcal{D}_D Y^\beta - \delta^\beta_{n+1} \mathcal{D}_D Y^\alpha = 0.$$

Evaluating at p and using that $Y^A|_p = Y^{n+1}|_p = 0$ as well as (4) yields

$$Q^{\alpha\beta} = \begin{cases} -1 & \text{if } \alpha = 0, \beta = n+1 \\ -1 & \text{if } \alpha = n+1, \beta = 0 \\ h^{AB} & \text{if } \alpha = A, \beta = B \end{cases}$$

and zero otherwise because

$$Q^{An+1} = \mathcal{D}_C Y^A \mathcal{D}^C Y^{n+1} - Y^A$$
$$= \mathcal{D}_C Y^A (h_{BD} Y^B \mathcal{D}^C Y^D) - Y^A$$
$$= h^{AD} h_{BD} Y^B - Y^A = 0,$$

and the cases $\alpha = A, \beta = 0$ also vanish trivially. Since h^{AB} is of signature (r, s), it follows at once that $Q^{\alpha\beta}$ is of signature $(r+1, s+1)$ (at p and hence, everywhere) and, in particular, nondegenerate. This proves item (i).

For item (ii), let f be a function satisfying $\text{Hess}_g f = c g_E$ with c a constant. Define the constant $f_p := f(p)$ and the function $f_0 := f - f_p Y^0 - c Y^{n+1}$. It is clear that $f_0(p) = 0$ and $\text{Hess}_{g_E} f_0 = 0$. Thus, f_0 is a linear combination of $\{Y^A\}$. Therefore, $f = c_\alpha Y^\alpha$ with $c_0 = f_p$ and $c_{n+1} = c$. Note that the constant c_0 can be arbitrarily chosen since for any constant c' the function $f + c'$ also solves $\text{Hess}_g f = c g_E$.

For item (iii), the covector associated to $\zeta^{\alpha\beta}$ is

$$\zeta_D^{\alpha\beta} = Y^\alpha \mathcal{D}_D Y^\beta - Y^\beta \mathcal{D}_D Y^\alpha$$

$$\Rightarrow \mathcal{D}_C \zeta_D^{\alpha\beta} = \mathcal{D}_C Y^\alpha \mathcal{D}_D Y^\beta + Y^\alpha \delta_{n+1}^\beta (g_E)_{CD}$$
$$- \mathcal{D}_C Y^\beta \mathcal{D}_D Y^\alpha - Y^\beta \delta_{n+1}^\alpha (g_E)_{CD} \qquad (8)$$

so

$$\mathcal{D}_C \zeta_D^{\alpha\beta} + \mathcal{D}_D \zeta_C^{\alpha\beta} = 2(Y^\alpha \delta_{n+1}^\beta - Y^\beta \delta_{n+1}^\alpha)(g_E)_{CD}$$

which establishes (iii).

For the last item, let $F_{\alpha\beta}$, for $\alpha < \beta$, be any set of constants and define $F_{\beta\alpha} := -F_{\alpha\beta}$. The most general linear combination of elements in $\mathcal{B}$ is

$$\zeta := \sum_{\alpha<\beta} F_{\alpha\beta} \zeta^{\alpha\beta} = \frac{1}{2} F_{\alpha\beta} \zeta^{\alpha\beta} = F_{\alpha\beta} Y^\alpha \text{grad}_{g_E} Y^\beta.$$

From item (iii), this vector satisfies

$$\pounds_\zeta g_E = 2F_{\alpha\beta} Y^\alpha \delta_{n+1}^\beta g_E$$
$$= 2F_{\alpha n+1} Y^\alpha g_E$$
$$= 2(F_{0n+1} Y^0 + 2F_{An+1} Y^A) g_E.$$

The functions $\{Y^0, Y^A\}$ are linearly independent, so $\zeta = 0$ implies $F_{\alpha n+1} = 0$ and then $\zeta = 0$ reduces to

$$\zeta_D = F_{0B} \mathcal{D}_D Y^B + F_{AB} Y^A \mathcal{D}_D Y^B = 0.$$

Evaluating at p (where Y^A vanishes) and using that $(A)^A{}_D = \mathcal{D}_D Y^A|_p$ is invertible, we find $F_{0B} = 0$. Finally, from (8)

$$\mathcal{D}_C \zeta_D|_p = F_{AB} A^A{}_C A^B{}_D = 0$$

from which $F_{AB} = 0$ and the only vanishing linear combination in $\mathcal{B}$ is the zero vector. Finally, the number of independent constants $F_{\alpha\beta}$ equals to $\sum_{\beta=1}^{n+1} \sum_{\alpha=0}^{\beta-1} 1 = \sum_{\beta=1}^{n+1} \beta = (n+1)(n+2)/2$, which is the dimension of the conformal Killing algebra of locally conformally flat n-metrics (e.g., [1]). ∎

Let now g be a locally conformally flat metric and ξ a conformal Killing vector of g. This means that at any point $p \in M$, there exists a neighborhood U_p of p and a flat metric g_E on U_p conformal to g. We restrict ourselves to U_p in everything that follows and denote the covariant derivative with respect to g as ∇ and the covariant derivative with respect to g_E as $\mathcal{D}$.

Let $\Omega: U_p \to \mathbb{R}$ be the smooth positive function satisfying $g_E = \Omega^{-2} g$. By Lemma 2.1 and $L_{g_E} = 0$, this function satisfies the equation,

$$\text{Hess}_g \Omega = \frac{1}{2\Omega} |\text{grad}_g \Omega|_g^2 g - \Omega L_g. \qquad (9)$$

We next show that we may assume that the function Ω satisfies, in addition, $\Omega_p = 1$ and $\nabla_A \Omega|_p = 0$. The underlying reason is the freedom to conformally rescale a flat metric in such a way that it remains flat. We seek for a smooth function $\omega: U_p \to \mathbb{R}$, positive near p such that $\overline{g_E} = \omega^{-2} g_E$ is also flat. Since the curvature of g_E is zero, the curvature of $\overline{g_E}$ will be zero if and only if $L_{\overline{g_E}} = 0$ (indeed, this is immediate in dimension $n = 3$ and as a consequence of the conformal invariance of the Weyl tensor in higher dimension). From Lemma 2.1, the metric $\bar{g}_E$ has $L_{\bar{g}_E} = 0$ if and only if ω satisfies the partial differential equation,

$$\text{Hess}_{g_E} \omega = \frac{1}{2\omega} |\text{grad}_{g_E} \omega|_{g_E}^2 g_E. \qquad (10)$$

As a consequence of the flatness of g_E, the divergence of the above equation gives

$$\mathcal{D}^A \nabla_A \mathcal{D}_B \omega = \mathcal{D}_B \mathcal{D}^A \mathcal{D}_A \omega = \mathcal{D}_B \left(\frac{1}{2\omega} |\text{grad}_{g_E} \omega|_{g_E}^2 \right)$$

$$\Rightarrow \mathcal{D}^A \mathcal{D}_A \omega = \frac{1}{2\omega} |\text{grad}_{g_E} \omega|_{g_E}^2 + K \qquad (11)$$

for a constant K. On the other hand, the trace of (10) is

$$\mathcal{D}^A \mathcal{D}_A \omega = \frac{n}{2\omega} |\text{grad}_{g_E} \omega|_{g_E}^2,$$

which comparing with (11) yields that $(2\omega)^{-1} |\text{grad}_{g_E} \omega|^2$ is constant on U_p. Denoting this constant by c, the set of equations to be solved is

$$\text{Hess}_{g_E} \omega = c g_E, \qquad (|\text{grad}_{g_E} \omega|_{g_E}^2 - 2c\omega)|_p = 0. \qquad (12)$$

Let $\{Y^\alpha\}$ be defined as before, i.e., $Y^0 = 1$,

$$\mathcal{D}_C \mathcal{D}_D Y^A = 0, \qquad Y^A|_p = 0, \qquad Y^{n+1} := \frac{1}{2} h_{AB} Y^A Y^B$$

with h_{AB} the inverse of h^{AB} defined in (4). By item (ii) in Lemma 2.3, the general solution of the first equation in (12) is $\omega = a_0 + a_A Y^A + c Y^{n+1}$ where a_0, a_A are arbitrary constants. Since $Y^A|_p = Y^{n+1}|_p = \mathcal{D}_A Y^{n+1}|_p = 0$, we get

$$0 = (\mathcal{D}_C \omega \mathcal{D}^C \omega - 2c\omega)|_p$$
$$= (a_A a_B \mathcal{D}_C Y^A \mathcal{D}^C Y^B - c)|_p$$
$$= a_A a_B h^{AB} - 2c.$$

Thus, the general solution of (12) is

$$\omega = a_0 Y^0 + a_A Y^A + \frac{1}{2} a_A a_B h^{AB} Y^{n+1}.$$

Given any values $\omega_0, \omega_A \in \mathbb{R}$, there exists a unique solution ω satisfying

$$\omega|_p = \omega_0, \qquad \mathcal{D}_C \omega|_p = \omega_C,$$

because the algebraic problem,

$$\omega|_p = (a_0 Y^0 + a_A Y^A)|_p = a_0 = \omega_0,$$
$$\mathcal{D}_C \omega|_p = a_A \mathcal{D}_C Y^A|_p = a_A A^A{}_C = \omega_C$$

always admits a unique solution $\{a_0, a_A\}$. Now, define $\bar{\Omega} := \omega \Omega$ with

$$\omega_0 = (\Omega|_p)^{-1}, \qquad \omega_A = -\left(\frac{1}{\Omega^2} \nabla_A \Omega\right)\bigg|_p.$$

It is immediate that $\bar{\Omega}|_p = 1$, $\nabla_C \bar{\Omega}|_p = 0$ and that $\bar{g}_E = \omega^{-2} g_E = \omega^{-2} \Omega^{-2} g = \bar{\Omega}^{-2} g$ is flat (in a suitable connected neighborhood of p where ω remains positive). Clearly $\bar{\Omega}$ also satisfies (9). Dropping the overlines, we have shown:

Lemma 2.4. For any locally conformally flat manifold (M, g) and point $p \in M$, there exists a unique choice of conformal factor Ω which satisfies that $g_E := \Omega^2 g$ is a flat metric in a neighborhood of p and

$$\Omega|_p = 1, \qquad \nabla_A \Omega|_p = 0. \tag{13}$$

From now on, *we make the choice of conformal factor as in Lemma 2.4.* Since g_E^{AB} [the inverse of $(g_E)_{AB}$] is given by $g_E^{AB} = \Omega^2 g^{AB}$, the $(n+1)(n+2)/2$ vector fields $\zeta^{\alpha\beta}$ introduced in (7) can also be written in the form,

$$\zeta^{\alpha\beta} = \Omega^2 (Y^\alpha \text{grad}_g Y^\beta - Y^\beta \text{grad}_g Y^\alpha), \qquad \alpha < \beta.$$

We have shown in Lemma 2.3 that these vector fields span the conformal Killing algebra of g_E in U_p and hence, also the conformal Killing algebra of g on the same domain. We intend to compute coefficients of the decomposition,

$$\xi = \sum_{\alpha < \beta} F_{\alpha\beta} \zeta^{\alpha\beta} = F_{\alpha\beta} \Omega^2 Y^\alpha \text{grad}_g Y^\beta, \qquad F_{\beta\alpha} = -F_{\alpha\beta}.$$

The strategy to do that is the well-known fact (see, e.g., [21]) that two local conformal Killing vectors ξ_1 and ξ_2 on a semi-Riemannian manifold (M, g), i.e., vector fields defined on a common open nonempty connected neighborhood $U \subset M$ and satisfying

$$\pounds_{\xi_1} g = 2\Psi_1 g, \qquad \pounds_{\xi_2} g = 2\Psi_2 g$$

are the same on U if and only if, at some point $p \in U$, it holds

$$(\xi_1)_A|_p = (\xi_2)_A|_p,$$
$$(\nabla_{[A}(\xi_1)_{B]})|_p = (\nabla_{[A}(\xi_2)_{B]})|_p,$$
$$\Psi_1|_p = \Psi_2|_p,$$
$$\nabla_A \Psi_1|_p = \nabla_A \Psi_2|_p, \tag{14}$$

where the indices between brackets are antisymmetrized. Assume that we are given a conformal Killing vector ξ on (M, g), so we can compute the function Ψ_ξ defined by $\pounds_\xi g = 2\Psi_\xi g$. We therefore may regard the following quantities as known (p is, as before, any chosen point in M),

$$\hat{\xi}_A := \xi_A|_p, \qquad \hat{\omega}_{AB} := (d\boldsymbol{\xi})_{AB}|_p = (\nabla_A \xi_B - \nabla_B \xi_A)|_p,$$
$$\hat{\Psi} := \Psi_\xi|_p, \qquad \hat{\Psi}_A := \nabla_A \Psi_\xi|_p,$$

where $\boldsymbol{\xi} := g(\xi, \cdot)$. Let ζ_F be defined by

$$\zeta_F = \frac{1}{2} F_{\alpha\beta} \zeta^{\alpha\beta} = F_{\alpha\beta} \Omega^2 Y^\alpha \text{grad}_g Y^\beta = F_{\alpha\beta} Y^\alpha \text{grad}_{g_E} Y^\beta,$$

where $F_{\alpha\beta} = F_{[\alpha\beta]}$ are arbitrary constants. By item (iii) in Lemma 2.3, we have

$$\pounds_{\zeta_F} g = \pounds_{\zeta_F}(\Omega^2 g_E)) = \Omega^2 \left(2 \frac{\zeta_F(\Omega)}{\Omega} g_E + \pounds_{\zeta_F} g_E\right)$$

$$= \Omega^2 \left(2 \frac{\zeta_F(\Omega)}{\Omega} g_E + 2 F_{\alpha\beta} Y^\alpha \delta_{n+1}^\beta g_E\right)$$

$$= 2 \left(\frac{\zeta_F(\Omega)}{\Omega} + F_{\alpha\beta} Y^\alpha \delta_{n+1}^\beta\right) g =: 2\Psi_F g, \tag{15}$$

with the last equality defining Ψ_F. Let us compute the differential of this function,

$$\nabla_C \Psi_F = \nabla_C \left(\frac{1}{\Omega} \zeta_F^D \nabla_D \Omega\right) + F_{\alpha\beta} \nabla_C Y^\alpha \delta_{n+1}^\beta$$

$$= -\frac{1}{\Omega^2} \zeta_F^D \nabla_D \Omega \nabla_C \Omega + \frac{1}{\Omega} \zeta_F^D \nabla_C \nabla_D \Omega$$

$$+ \frac{1}{\Omega}(\nabla_C \zeta_F^D) \nabla_D \Omega + F_{\alpha\beta} \mathcal{D}_C Y^\alpha \delta_{n+1}^\beta$$

$$= -\frac{1}{\Omega^2} \zeta_F^D \nabla_D \Omega \nabla_C \Omega + \zeta_F^D \left(\frac{\nabla_A \Omega \nabla^A \Omega}{2\Omega^2} g_{CD} - (L_g)_{CD}\right)$$

$$+ \frac{1}{\Omega}(\nabla_C \zeta_F^D) \nabla_D \Omega + F_{\alpha\beta} \mathcal{D}_C Y^\alpha \delta_{n+1}^\beta,$$

where in the third equality, we inserted (9). We elaborate the third term using the difference tensor $S = \mathcal{D} - \nabla$, explicitly given in (2),

$$\nabla_C \zeta_F^D = \mathcal{D}_C \zeta_F^D - S^D{}_{AC} \zeta_F^A$$

$$= \mathcal{D}_C \zeta_F^D + \zeta_F^D \frac{\nabla_C \Omega}{\Omega} + \delta_C^D \frac{1}{\Omega} \zeta_F^A \nabla_A \Omega - \frac{\nabla^D \Omega}{\Omega} g_{AC} \zeta_F^A,$$

and insert above to get

$$\nabla_C \Psi_F + \zeta_F^D L_{gCD} = \frac{1}{\Omega^2} \zeta_F^D \nabla_D \Omega \nabla_C \Omega - \frac{\nabla_A \Omega \nabla^A \Omega}{2\Omega^2} \zeta_{FC}$$
$$+ \frac{1}{\Omega} (\mathcal{D}_C \zeta_F^D) \nabla_D \Omega + F_{\alpha\beta} \mathcal{D}_C Y^\alpha \delta_{n+1}^\beta. \tag{16}$$

We may now determine the coefficients $F_{\alpha\beta}$ in terms of (14).

Proposition 2.1. Let (M, g) be a locally conformally flat semi-Riemannian manifold of arbitrary signature (r, s) and dimension $n \geq 3$. Fix any point p in M and a sufficiently small simply connected neighborhood of p. Define Ω in U_p as the unique solution of (9) satisfying (13) and let $g_E := \Omega^{-2} g$. This metric is flat in U_p, and we may introduce the functions $\{Y^\alpha\}$ and the conformal Killing vectors $\zeta^{\alpha\beta}$ on U_p as described above.

Let ξ be any conformal Killing vector on (U_p, g). Define Ψ_ξ by $\pounds_\xi g = 2\Psi_\xi g$ and introduce the quantities,

$$\hat{\xi}_C := \xi_C|_p, \qquad \hat{\omega}_{CD} := \frac{1}{2}(d\xi)_{CD}|_p \qquad \hat{\Psi} := \Psi_\xi|_p,$$
$$\hat{W}_C := \nabla_C \Psi_\xi + \xi^D (L_g)_{CD}|_p,$$

where L_g is the Schouten tensor of g. Then ξ admits the decomposition,

$$\xi = \frac{1}{2} F_{\alpha\beta} \zeta^{\alpha\beta} = F_{\alpha\beta} \Omega Y^\alpha \mathrm{grad}_g Y^\beta,$$

with $F_{\alpha\beta} = F_{[\alpha\beta]}$ given by

$$F_{0B} = B^C{}_B \hat{\xi}_C, \qquad F_{AB} = \hat{\omega}_{CD} B^C{}_A B^D{}_B,$$
$$F_{An+1} = \hat{W}_C B^C{}_A, \qquad F_{0n+1} = \hat{\Psi},$$

where $B^A{}_B$ is the inverse of $A^A{}_C := \nabla_C Y^A|_p$. i.e.,

$$A^A{}_C B^C{}_B = \delta^A{}_B.$$

Proof.—Since $\zeta_{FA} = \Omega^2 F_{\alpha\beta} Y^\alpha \nabla_A Y^\beta$, we need to impose (recall that $\mathcal{D}_C Y^{n+1}|_p = \mathcal{D}_C \Omega|_p = 0$)

$$\hat{\xi}_C = \zeta_{FC}|_p = F_{\alpha\beta} \delta_0^\alpha \mathcal{D}_C Y^\beta|_p = F_{0B} A^B{}_C,$$
$$2\hat{\omega}_{CD} = (d\zeta_F)_{CD}|_p = 2\nabla_{[C}(\Omega^2 F_{\alpha\beta} Y^\alpha \nabla_{D]} Y^\beta)|_p$$
$$= 2F_{\alpha\beta} \nabla_{[C} Y^\alpha \nabla_{D]} Y^\beta|_p = 2F_{\alpha\beta} \mathcal{D}_{[C} Y^\alpha \mathcal{D}_{D]} Y^\beta|_p$$
$$= 2F_{AB} A^B{}_C A^B{}_D,$$
$$\hat{\Psi} = \Psi_F|_p = \left(\frac{1}{\Omega} \xi_F(\Omega) + F_{\alpha\beta} Y^\alpha \delta_{n+1}^\beta\right)\Big|_p = F_{0n+1},$$
$$\hat{W}_C = F_{\alpha\beta} \mathcal{D}_C Y^\alpha \delta_{n+1}^\beta|_p = F_{An+1} A^A{}_C,$$

where in the third equality we used Ψ_F as given in (15), and in the last equality, we applied (16). The result follows at once from these expressions. ∎

Remark 2.1. The definition of $\{Y^A\}$ allows one to choose <u>any</u> invertible matrix $A^A{}_C$ for its construction. However, if one wants $\{Y^A\}$ to define a Cartesian coordinate system on U_p, then $A^A{}_B$ must be chosen so that

$$g^{CD}|_p A^A{}_C A^B{}_D = \eta^{AB}, \tag{17}$$

where $\eta^{AB} = \mathrm{diag}\{\overbrace{-1, \ldots, -1}^{r}, \overbrace{1, \cdots 1}^{s}\}$. Note that the left-hand side of (17) is by definition $h^{AB}|_p$ and that h^{AB}, which gives the entries of g^{AB} in coordinates $\{Y^A\}$ [see (4)], is constant on U_p.

With this proposition at hand, we can relate the coefficients $F_{\alpha\beta}$ with the determination of the conformal class to which ξ belongs. For simplicity of presentation, let us assume that (M, g) is compact and simply connected and let $\mathrm{Conf}(M, g)$ be the conformal group, i.e., the collection of all conformal diffeomorphism of (M, g), namely transformations $\varphi: M \to M$ satisfying $\varphi^\star(g) = \theta^2 g$, for some smooth positive function $\theta: M \to \mathbb{R}$. The vector field $\varphi_\star^{-1}(\xi)$ defines a conformal Killing vector of (M, g), and by definition, the conformal class is the collection of all such conformal vector fields. It is known (see [1,22]) that the conformal Killing algebra of (M, g) is isomorphic (as a vector space) to the set of skew-symmetric endomorphisms (V, η), where $V \simeq \mathbb{R}^{n+2}$ and η is a metric of signature $(r + 1, s + 1)$. The map is also an anti-isomorphism of Lie algebras. Note that the skew symmetry here is defined with respect to the interior product $\langle \cdot, \cdot \rangle$ defined by η, namely, an endomorphism F is skew symmetric if for every pair of vectors $u, v \in \mathbb{R}$ it satisfies $\langle F(u), v \rangle = -\langle u, F(v) \rangle$.

Let us call the conformal Killing algebra of (M, g) by $\mathrm{CKill}(M, g)$, the vector space (V, η) as $\mathbb{R}^{(r+1,s+1)}$, the set of skew-symmetric endomorphisms of $\mathbb{R}^{(r+1,s+1)}$ by $\mathrm{SkewEnd}(\mathbb{R}^{(r+1,s+1)})$ and denote the isomorphism above by

$$\Psi: \mathrm{CKill}(M, g) \to \mathrm{SkewEnd}(\mathbb{R}^{(r+1,s+1)}).$$

It turns out that the conformal class of ξ is

$$[\xi] = \{\Psi^{-1}(F_\Lambda) \quad \text{for all}$$
$$F_\Lambda = \Lambda^{-1} \Psi(\xi)\Lambda, \qquad \Lambda \in O(r + 1, s + 1)\},$$

where the orthogonal group $O(r + 1, s + 1)$ acts on $\mathbb{R}^{(r+1,s+1)}$ in a natural way.

The construction of the map Ψ can be done in several ways and is a consequence of the fact that (M, g) can be isometrically embedded in the null cone of the origin in $\mathbb{R}^{r+1,s+1}$. The explicit representation of Ψ relies on a choice of a flat representative g_E in the conformal class $[g]$ of g in a sufficiently small neighborhood U_p of p and a choice of

Cartesian coordinates X^A in U_p. To g_E and $\{X^A\}$, one associates an orthonormal basis $B := \{e_\alpha\}$ of $\mathbb{R}^{r+1,s+1}$ with e_0 timelike. The most general conformal Killing vector $\xi \in \mathrm{CKill}(M, g)$ restricted to U_p can be written in terms of constants $\{b_A, \nu, a_A, \omega_{AB} = \omega_{[AB]}\}$ as

$$\xi = \left(b^A + \nu X^A + (a_B X^B) X^A - \frac{1}{2}(X_B X^B) a^A - \omega^A_{\ B} X^B \right) \partial_{X^A},$$

(18)

where indices are raised and lowered with η^{AB} (cf. remark 2.1) and its inverse. Then, the endomorphism $\Psi(\xi) \in \mathrm{SkewEnd}(\mathbb{R}^{r+1,s+1})$ expressed in the basis B, i.e., $(F_\xi)^\alpha_{\ \beta} e_\alpha = \Psi(\xi)(e_\beta)$ is given by $F_{\xi}{}^\alpha_{\ \beta} = \eta^{\alpha\mu} F_{\xi \mu \beta}$,

$$\eta_{\alpha\mu} = \eta^{\alpha\mu} = \mathrm{diag}\{-1, 1, \underbrace{-1, \ldots, -1}_{r}, \underbrace{1, \ldots, 1}_{s}\},$$

and the matrix $(F_{\xi})_{\mu\beta}$ is given by (the first index is row and the second column)

$$F_{\xi} = \begin{pmatrix} 0 & \nu & a^t - b^t/2 \\ -\nu & 0 & -a^t - b^t/2 \\ -a + b/2 & a + b/2 & -\omega \end{pmatrix},$$

where $a, b \in \mathbb{R}^n$ are column vectors with components a_A, b_A, respectively, t denotes the transpose and ω is the $n \times m$ skewsymmetric matrix with components ω_{AB}. We can now make connection to the previous decomposition. Assume for the moment that we take $A^A_{\ B}$ such that (17) holds. Then $\{Y^A\}$ defines a Cartesian coordinate system of the flat metric g_E in U_p. Moreover, since $Y^{n+1} = \frac{1}{2}\eta_{AB} Y^A Y^B$, it is straightforward to check that setting $X^A = Y^A$, the conformal Killing vector (18) can also be written as

$$\xi = F_{\alpha\beta} Y^\alpha \mathcal{D}^A Y^\beta \partial_{Y^A} = F_{\alpha\beta} Y^\alpha \mathrm{grad}_{g_E} Y^\beta,$$

with

$$F_{0A} = b_A, \qquad F_{0n+1} = \nu, \qquad F_{AB} = \omega_{AB} \qquad F_{C,n+1} = a_C.$$

In order to match the two constructions, we need to introduce a vector space of dimension $n + 2$ and a metric of signature $(r + 1, s + 1)$. Define $\hat{V} := \mathrm{span}\{Y^\alpha\}$ and endow this space with the scalar product Q defined by

$$Q(Y^\alpha, Y^\beta) := Q^{\alpha\beta},$$

where the constants $Q^{\alpha\beta}$ are defined in item (i) of Lemma 2.3. This metric is well-defined (i.e., independent of the

choice of functions Y^A) because under a $GL(n)$ transformation[1] M,

$$Y'^A = M^A_{\ B} Y^B,$$

we have $Y'^0 = Y^0$ (obvious) and $Y'^{n+1} = Y^{n+1}$ because

$$Y'^{n+1} = \frac{1}{2} h'_{AB} Y'^A Y'^B = \frac{1}{2} h'_{AB} M^A_{\ C} M^B_C Y^C Y^D$$

$$= \frac{1}{2} h_{CD} Y^A Y^D = Y^{n+1},$$

and the last equality follows from definition (4) (and its corresponding prime) together with

$$A'^A_B := \mathcal{D}_B Y'^A\big|_p = M^A_{\ C} A^C_{\ B}.$$

Hence, from the definition of $Q^{\alpha\beta}$, we have

$$Q'^{\alpha\beta} = Q^{\mu\nu} M^\alpha_{\ \mu} M^\beta_{\ \nu},$$

with $M^0_{\ 0} = M^{n+1}_{\ n+1} = 1$, $M^\alpha_{\ \beta} = M^A_{\ B}$ for $\alpha = A, \beta = B$, and the rest are zero.

In the case that $\{Y^A\}$ are Cartesian coordinates (i.e., when $h^{AB} = \eta^{AB}$), then we can construct an orthonormal basis of $\hat{V}$ by introducing

$$E^0 := Y^0 + \frac{1}{2} Y^{n+1}, \quad E^1 := Y^0 - \frac{1}{2} Y^{n+1}, \quad E^{A+1} := Y^A.$$

The endomorphism of $\hat{V}$ defined by

$$F_\xi(Y^\alpha) := Q^{\alpha\beta} F_{\beta\mu} Y^\mu$$

is identical to the endomorphism $\Psi(\xi)$ if we identify $V = \hat{V}$ and the basis vectors $e_\alpha = \eta_{\alpha\beta} E^\beta$.

The classification of the endomorphism $\Psi(\xi)$ up to the conjugacy class is obviously independent of the choice of basis. Thus, once we have established the equivalence of $\Psi(\xi)$ and F_ξ, we may use *any* basis $\{Y^\alpha\}$, not necessary orthogonal. From the point of view of the original space (M, g), a natural choice is the (nonorthogonal) basis defined by

$$\mathcal{D}_A Y^B\big|_p = \delta^B_{\ A}.$$

In such a basis, the expression of $F_{\alpha\beta}$ is simplest, while the expression of $Q^{\alpha\beta}$ is just

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[1] Recall that $\{Y^A\}$ spans the solution space of (3). Thus, any other set of linearly independent solutions $\{Y'^A\}$ must necessarily differ from $\{Y^A\}$ by a $GL(n)$ transformation.

$$Q^{0,n+1} = Q^{n+1,0} = -1, \qquad Q^{AB} = g^{AB}|_p.$$

We summarize this result in the following theorem:

Theorem 2.5. Let (M, g) be a locally conformally flat semi-Riemannian space of arbitrary signature (r, s) and dimension $n \geq 3$. Fix a point $p \in M$ and a local conformal Killing vector ξ defined in a sufficiently small open neighborhood U_p of p. Let $\Psi_\xi : U_p \to \mathbb{R}$ be defined by $\pounds_\xi g = 2\Psi_\xi$.

Then the conformal class of ξ is determined by the conjugacy class under $O(r+1, s+1)$ of the skew-symmetric endomorphism F_ξ on $(\mathbb{R}^{r+1,s+1}, Q)$ defined by $F_\xi(v^\alpha) = Q^{\alpha\beta} F_{\beta\mu} v^\mu$ where v^α is a basis of $\mathbb{R}^{r+1,s+1}$ with nonzero scalar products,

$$Q^{\alpha\beta} := Q(v^\alpha, v^\beta)$$
$$= \begin{cases} -1 & \text{if } \alpha = 0, \beta = n+1 \text{ or } \alpha = n+1, \beta = 0 \\ g^{AB}|_p & \text{if } \alpha = A, \beta = B \\ 0 & \text{rest of terms} \end{cases},$$

and the coefficients $F_{\beta\mu} = F_{[\beta\mu]}$ are given by

$$F_{0,A} = \xi_A|_p, \qquad F_{AB} = \nabla_{[A}\xi_{B]}|_p, \qquad F_{0,n+1} = \Psi_\xi|_p,$$
$$F_{A,n+1} = (\nabla_A \Psi_\xi + \xi^B(L_g)_{AB})|_p,$$

where L_g is the Schouten tensor of g and ∇ its covariant derivative.

A. The Riemannian case $(r = 0, s = n)$

The method of classification of CVKs employed in [15,19] requires to find explicitly a flat representative g_E in the class of locally conformally flat metrics and also Cartesian coordinates for g_E. However, this may be a very hard task. Theorem 2.5 improves the classification method in [15,19] as it allows one to obtain the conformal class of a CKV with *independence on the coordinates and the representative* of the class of conformally flat metrics. We shall provide an interesting application of this result in Sec. IV. For that, we now introduce the explicit classification of CKVs in conformally flat Riemannian metrics.

From the discussion above it follows that, for locally conformally flat Riemannian n-metrics, the classification of conformal classes of CKVs is equivalent to the classification of $\text{SkewEnd}(\mathbb{R}^{(1,n+1)})$ up to $O(1, n+1)$ transformations. In order to uniquely characterize the conjugacy class $[F] = \{F_\Lambda \in \text{SkewEnd}(\mathbb{R}^{(1,n+1)}) | F_\Lambda = \Lambda F \Lambda^{-1}, \Lambda \in O(1, n+1)\}$, one needs to find a sufficient number of $O(1, n+1)$-invariant quantities. A possibility [4] is to give the eigenvalues[2] of F^2 together with the causal

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[2]It is preferable to use the eigenvalues of F^2 because they are real, and they are in one-to-one correspondence with the (complex) eigenvalues of F.

character of $\ker F$ (see also [2] for an alternative classification in terms of the traces of even powers of F and its matrix rank). Observe [4] that, as a consequence of the skew symmetry of F, all eigenvalues of F^2 are at least of double multiplicity, and there is always a vanishing eigenvalue if n is odd. Hence, it turns out [4] to be sufficient to determine the roots of the following polynomial:

$$\mathcal{Q}_{F^2}(x) := (\mathcal{P}_{F^2}(-x))^{1/2} \quad \text{(if } n \text{ even)},$$
$$\mathcal{Q}_{F^2}(x) := \left(\frac{\mathcal{P}_{F^2}(-x)}{x}\right)^{1/2} \quad \text{(if } n \text{ odd)}, \tag{19}$$

where $\mathcal{P}_{F^2}(-x)$ refers to the characteristic polynomial of $-F^2$. Counting multiplicity, $\mathcal{Q}_{F^2}(x)$ has p roots, where p is the natural number related to the dimension n by

$$p := \left\lfloor \frac{n+1}{2} \right\rfloor - 1,$$

being $\lfloor x \rfloor \in \mathbb{Z}$ the floor function for all $x \in \mathbb{R}$, i.e., the largest integer which is equal or less than x. Then, the classification result of equivalence classes of F is given by the following Proposition:

Proposition 2.2. ([4]). Let $\text{Roots}(\mathcal{Q}_{F^2})$ denote the set of roots of $\mathcal{Q}_{F^2}(x)$ repeated as many times as their multiplicity and arranged as follows:
(a) If n odd, $\{\sigma; \mu_1^2, ..., \mu_p^2\} := \text{Roots}(\mathcal{Q}_{F^2})$ sorted by $\sigma \geq \mu_1^2 \geq \cdots \geq \mu_p^2$ if $\ker F$ is timelike, where in this case, necessarily $\sigma > 0$. Otherwise, $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq \sigma$.
(b) If n even, $\{-\mu_t^2, \mu_s^2; \mu_1^2, ..., \mu_p^2\} := \text{Roots}(\mathcal{Q}_{F^2})$ sorted by $\mu_1^2 \geq \cdots \geq \mu_p^2 \geq \mu_s^2 = -\mu_t^2 = 0$ if $\ker F$ is null. Otherwise, $\mu_s^2 \geq \mu_1^2 \geq \cdots \geq \mu_p^2 \geq 0 \geq -\mu_t^2$, where either μ_s^2 or μ_t^2 are nonzero.
Then the parameters $\{\sigma; \mu_1^2, ..., \mu_p^2\}$ for n odd and $\{-\mu_t^2, \mu_s^2; \mu_1^2, ..., \mu_p^2\}$ for n even determine uniquely the class of F up to $O(1, n+1)$ transformations and hence, also the class of $\xi := \Psi^{-1}(F)$ up to conformal transformations.

Remark 2.2. It holds in general that $\mathcal{Q}_{F^2}$ has at most one negative root [4]. When n is even and $\ker F$ is null, then $\mathcal{Q}_{F^2}$ has a root at least double at zero and no negative roots [4]. These are necessary facts that follow from Proposition 2.2.

Note that endomorphisms with equal roots of $\mathcal{Q}_{F^2}$ (hence, equal eigenvalues with same multiplicities) can belong to different conformal classes. The idea in [4] is to introduce an additional invariant, namely the causal character of $\ker F$, to remove this ambiguity by defining $\text{Roots}(\mathcal{Q}_{F^2})$, whose elements are sorted depending on $\ker F$. This gives a well-defined parametrization of the space of conformal classes, which will be key in Sec. IV.

III. THE ASYMPTOTIC CAUCHY PROBLEM AND THE KERR–DE SITTER-LIKE CLASS

In this section, we review the basics on the asymptotic Cauchy problem for $(\Lambda > 0)$-vacuum, $(n + 1)$-dimensional spacetimes and introduce the definition and properties of the Kerr–de Sitter-like class of spacetimes, in four dimensions [2,18] as well as its extension to higher dimensions [15,19]. The results in this section are not new, but they will be needed in Sec. IV. The following discussion is meant to summarize these results in order to make the paper self-contained.

A. Asymptotic Cauchy problem with $\Lambda > 0$

As we already mentioned in the introduction, in some situations, $(n + 1)$-dimensional, $(\Lambda > 0)$-vacuum spacetimes $(\tilde{M}, \tilde{g})$ admitting a conformal extension $(M, g := \Omega^2 \tilde{g})$ can be characterized in a neighborhood of $\mathcal{I}$ by asymptotic initial data (i.e., data prescribed at $\mathcal{I}$). This is true in general for $n = 3$ by the classical results by Friedrich [7–9]. In higher dimensions, the asymptotic characterization results stem from the Fefferman and Graham formalism [23,24], and hold in general for $(n + 1)$ even dimensions [10–12] or when the asymptotic initial data are analytic [13].

In what follows, we restrict to $(n + 1)$-dimensional (with $n \geq 3$), $(\Lambda > 0)$-vacuum metrics, which admit a locally conformally flat $\mathcal{I}$. In this case, the asymptotic initial data [15] are a conformally flat Riemannian n-manifold (Σ, γ), which prescribes the geometry of $\mathcal{I}$ via an isometric embedding $\iota : \Sigma \hookrightarrow \mathcal{I} \subset M$, together with a transverse and traceless tensor, or TT tensor, D which prescribes the electric part of the rescaled Weyl tensor. Namely,

$$D := \iota^\star (|\mathrm{grad}_g \Omega|^{-2} \Omega^{2-n} C(\mathrm{grad}_g \Omega, \cdot, \mathrm{grad}_g \Omega, \cdot)), \quad (20)$$

where C is the 4-covariant Weyl tensor of (M, g). The conformal flatness of $\mathcal{I}$ is not required in the $n = 3$ case for (20) to hold true, but it is indeed needed for $n > 3$ [15]. Actually, the (smooth) extendability of $\Omega^{2-n} C(\mathrm{grad}_g \Omega, \cdot, \mathrm{grad}_g \Omega, \cdot)$ to $\mathcal{I}$ is a nontrivial result (cf. [15,25]) for $n > 3$, which relies strongly on the assumption of local conformal flatness of $\mathcal{I}$. For $n = 3$, the extendability of $\Omega^{-1} C$ to a generic $\mathcal{I}$ is a consequence of the Weyl tensor vanishing identically in three dimensions.

In addition, the characterization of spacetimes in terms of asymptotic initial data is independent of the conformal factor Ω. As a consequence the data have the following conformal equivalence:

$$(\Sigma, \gamma, D) \simeq (\Sigma, \omega^2 \gamma, \omega^{2-n} D), \qquad \forall \omega \in C^\infty(\Sigma, \mathbb{R}^+), \quad (21)$$

in the sense that any pair of data correspond to the same physical spacetime $(\tilde{M}, \tilde{g})$ if and only if they are related by (21).

Now, let (Σ, γ, D) be asymptotic data for $(\tilde{M}, \tilde{g})$. Then, for a CKV ξ of γ, the following KID equation:

$$\mathcal{L}_\xi D + \frac{n - 2}{n} (\mathrm{div}_\gamma \xi) D = 0 \quad (22)$$

is proven to be a necessary and sufficient condition for $\tilde{g}$ to admit a KV X such that $\xi = X|_{\mathcal{I}}$, in general if $n = 3$ [16] and assuming that (γ, D) are analytic for $n > 3$ [15] [there is no proof yet for the general $n > 3$ case, but we believe that (22) asymptotically characterizes symmetries in general.] It is a matter of direct computation to show that if $(\Sigma, \gamma, D; \xi)$ satisfies the KID equation, so does $(\Sigma, \omega^2 \gamma, \omega^{2-n} D; \xi)$. Then, for any element $\varphi \in \mathrm{Conf}(\Sigma, \gamma)$, the following equivalences are ready:

$$\begin{aligned} (\Sigma, \gamma, D; \xi) &\simeq (\Sigma, \varphi^\star(\gamma), \varphi^\star(D); \varphi_\star^{-1}(\xi)) \\ &= (\Sigma, \omega^2 \gamma, \omega^{2-n} D; \varphi_\star^{-1}(\xi)) \\ &\simeq (\Sigma, \gamma, D; \varphi_\star^{-1}(\xi)), \end{aligned} \quad (23)$$

the first one arising from the fact that φ is a diffeomorphism and the last one from (21). Therefore, a particular KV of the bulk spacetime is not associated to a single CKV ξ satisfying (22), but its whole conformal class $[\xi]$ of CKVs satisfying (22) [here it is crucial that each representative of $[\xi]$ is a solution of (22)].

B. The Kerr–de Sitter-like class

In four spacetime dimensions (i.e., $n = 3$), the vanishing of the Mars-Simon tensor for a particular KV [26,27] characterizes Kerr–de Sitter and related spacetimes [28]. The class of four-dimensional spacetimes, which are $(\Lambda > 0)$-vacuum, conformally extendable and admitting a KV X whose Mars-Simon tensor vanishes defines the so-called Kerr–de Sitter-like class of spacetimes. In the locally conformally flat $\mathcal{I}$ case[3] [2], the asymptotic data characterizing this class is a conformally flat Riemannian 3-manifold (Σ, γ) and a TT tensor D of the form,

$$D = \kappa D_\xi, \quad \text{where } D_\xi := \frac{1}{|\xi|_\gamma^5} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{3} \gamma \right),$$

κ is a real constant, ξ is a CKV of γ satisfying $X|_{\mathcal{I}} = \xi$ [thus also (22)] and $\boldsymbol{\xi} := \gamma(\xi, \cdot)$. Using the results discussed above on asymptotic characterization of $(n + 1)$-dimensional spacetimes, one can extend, by means of asymptotic data, the definition of the Kerr–de Sitter-like class in the conformally flat $\mathcal{I}$ case to all dimensions [19]. Namely:

Definition 3.1. The $(n + 1)$-dimensional Kerr–de Sitter-like class with conformally flat $\mathcal{I}$ is defined as the set of

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[3]The nonconformally flat $\mathcal{I}$ cases were also studied in [18], but the results are not needed for our purposes here.

($\Lambda > 0$)-vacuum spacetimes, which admit a conformally flat $\mathcal{I}$ and such that

$$D = \kappa D_\xi, \quad \text{where } D_\xi := \frac{1}{|\xi|_\gamma^{n+2}} \left(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \frac{|\xi|_\gamma^2}{n} \gamma \right), \quad (24)$$

κ is a real constant and ξ is a CKV of the (conformally flat) metric γ at $\mathcal{I}$.

We remark that this is not an *ad hoc* definition, but it naturally follows after checking [15] that the asymptotic data of the Gibbons *et al.* definition of the Kerr–de Sitter metrics [29] consist of a conformally flat manifold (Σ, γ) and a tensor D of the form (24), for a particular choice of ξ. Allowing ξ to be an arbitrary CKV keeps the traceless and transverse property of D_ξ, and thus, provides a natural generalization of the data, which in turn generalizes the definition of Kerr–de Sitter-like class to higher dimensions. Moreover, in all cases, ξ satisfies the KID equation (22).

From the data equivalence (21), it follows [19] that the datasets in the Kerr–de Sitter-like class satisfy the following equivalence property:

$$(\Sigma, \gamma, \kappa D_\xi) \simeq (\Sigma, \gamma, \kappa D_{\xi'}) \Leftrightarrow \xi' = \varphi(\xi),$$
$$\text{for some } \varphi \in \text{Conf}(\Sigma, \gamma). \quad (25)$$

This has more serious consequences than simply the fact that the conformal class $[\xi]$ characterizes a unique KV [cf. (23)]. Due to the role that $[\xi]$ plays in the construction of the data in the Kerr–de Sitter-like class, equivalence (25) actually means that the metrics in the Kerr–de Sitter-like class (with conformally flat $\mathcal{I}$) are in one-to-one correspondence with the conformal classes of CKVs of locally conformally flat metrics. Hence, the moduli space of metrics in this class is represented by the space of parameters in Proposition 2.2. Moreover, the quotient topology naturally defined in the space of conformal classes is inherited by the space of metrics in the Kerr–de Sitter-like class. The main result in [19] exploits this fact to provide an explicit reconstruction of all the metrics in the Kerr–de Sitter-like class as limits of Kerr–de Sitter or an analytic extension thereof. Moreover, they are proven to exhaust the set of Kerr-Schild metrics on a locally de Sitter background with an additional decay condition:

Theorem 3.1. [19] Let $(\tilde{\mathcal{M}}, \tilde{g})$ be a ($\Lambda > 0$)-vacuum $(n + 1)$-dimensional spacetime, such that $\tilde{g}$ admits the Kerr-Schild form on a locally de Sitter background, namely,

$$\tilde{g} = \tilde{g}_{dS} + \mathcal{H} \boldsymbol{k} \otimes \boldsymbol{k}, \quad (26)$$

where $\tilde{g}_{dS}$ is locally isometric to de Sitter, $\mathcal{H}$ is a smooth function on $\tilde{\mathcal{M}}$ and $\boldsymbol{k}$ a null one-form (with respect to both $\tilde{g}$ and $\tilde{g}_{dS}$). Additionally, assume that $\tilde{g}$ admits a smooth conformal extension such that $\Omega^2 \mathcal{H} \boldsymbol{k} \otimes \boldsymbol{k} = O(\Omega)$. Then

and only then $\tilde{g}$ belongs to the Kerr–de Sitter-like class with locally conformally flat $\mathcal{I}$.

IV. AN APPLICATION: CLASSIFICATION OF ALGEBRAICALLY SPECIAL ($\Lambda > 0$)-VACUUM SOLUTION IN FIVE DIMENSIONS

In [17], the problem of determining the most general algebraically special spacetime in five dimensions that solves the vacuum Einstein field equations,

$$R_{\alpha\beta} = 4\lambda g_{\alpha\beta}, \qquad \lambda \in \mathbb{R},$$

is addressed.[4] Algebraically special means that the spacetime (M, g) admits a multiple Weyl aligned null direction (WAND) ℓ. A multiple WAND is a nonidentically vanishing null vector field satisfying [31]

$$\ell^b \ell_{[e} C_{a]b[cd} \ell_{f]} = 0,$$

where C_{abcd} is the Weyl tensor of (M, g). Multiple WANDs are always geodesic, $\nabla_\ell \ell \propto \ell$. Admitting a multiple WAND is equivalent to the algebraic classification of the Weyl tensor, as extended by Coley *et al.* at d-dimensions [32], being of type II or more special.

The problem in [17] is solved under the additional hypothesis that the so-called optical matrix of ℓ is nondegenerate.[5] The optical matrix ρ encodes the kinematical properties of the congruence of null geodesics defined by ℓ. Geometrically, it is a (0, 2)-tensor defined on the quotient vector space at each point $p \in M$ of equivalence classes $\ell^\perp/\sim$, where two vectors $X, Y \in T_pM$ orthogonal to ℓ are related by $\sim$ if and only if $X - Y$ is proportional to ℓ. Its definition is

$$\rho(\bar{X}, \bar{Y}) := g(X, \nabla_Y \ell), \qquad X \in \bar{X}, Y \in \bar{Y},$$

and one checks at once that ρ is well-defined, i.e., independent of the choice of representative $X \in \bar{X}, Y \in \bar{Y}$.

Since our interest here lies in the case $\lambda > 0$, we quote the results in [17] restricted to this situation. Specifically, the main result in [17] states that under the above conditions and $\lambda > 0$ the most general solution of the Einstein field equations belongs to one of three families of metrics, classified according to the eigenvalues of ρ. Namely, the eigenvalues of ρ can be written in terms of r and χ, being the former the affine parameter along the null geodesics of ℓ and the latter a constant function along the same congruence of geodesics. The three cases arise depending on whether χ is not everywhere constant or, if constant, whether this constant is zero or not.

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[4]In [30], $\Lambda = 4\lambda$ is used instead. We prefer λ, which matches directly the notation used [15,19].

[5]The problem is also solved in the degenerate case in [30] and references therein. We restrict to the nondegenerate case.

Case 1 ($\chi \neq 0, d\chi \neq 0$). The functions $\{r, \chi\}$ are completed to local coordinates $\{r, u, \chi, x, y\}$ and in terms of real constants A_0, μ_0, C_0, E_0 such that

$$P(\chi) := C_0 - \frac{E_0^2}{\chi^2} - 2A_0\chi^2 - \lambda\chi^4$$

is positive in some interval $I \subset \mathbb{R}\setminus\{0\}$ and with χ taking values on I the metric is

$$\tilde{g}_1 = -2\boldsymbol{\sigma}^1 dr + \frac{r^2 + \chi^2}{P(\chi)} d\chi^2 + h_{ij}\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j,$$

where the one-forms $\boldsymbol{\sigma}^i$ are $\boldsymbol{\sigma}^1 := du + \chi^2 dy$, $\boldsymbol{\sigma}^2 := dx - \frac{E_0}{\chi^2}dy$, $\boldsymbol{\sigma}^3 := dy$ and h_{ij} is the matrix,

$$h_{ij} = \begin{pmatrix} -2H(r) & E_0 & -P(\chi) \\ E_0 & r^2\chi^2 & 0 \\ -P(\chi) & 0 & (r^2 + \chi^2)P(\chi) \end{pmatrix},$$

$$H(r) := A_0 - \frac{\lambda}{2}(r^2 - \chi^2) - \frac{\mu_0}{2(r^2 + \chi^2)}.$$

Case 2 ($\chi \neq 0, d\chi = 0$). Here χ acts as a parameter, and there exist local coordinates $\{r, u, x, \theta, \phi\}$ and real constants E_0, μ_0 such that, with $\boldsymbol{\tau} := du + \frac{2\chi}{F_0}\cos\theta d\phi$, the metric reads

$$\tilde{g}_2 = -2\boldsymbol{\tau}dr - G(r)\boldsymbol{\tau}^2 + r^2\left(dx + \frac{E_0}{\chi^3}\left(1 + \frac{\chi^2}{r^2}\right)\boldsymbol{\tau}\right)^2$$
$$+ \frac{r^2 + \chi^2}{F_0}(d\theta^2 + \sin^2\theta d\phi^2),$$

$$G(r) := \frac{E_0^2}{\chi^4}\left(1 + \frac{\chi^2}{r^2}\right) - \lambda(r^2 + \chi^2) - \frac{\mu_0}{r^2 + \chi^2},$$

$$F_0 := 4\left(\lambda\chi^2 + \frac{E_0^2}{\chi^4}\right).$$

Case 3 ($\chi = 0, d\chi = 0$). The metric is

$$\tilde{g}_3 = -\left(\kappa - \frac{\mu_0}{r^2} - \lambda r^2\right)du^2 - 2dudr + r^2h,$$

where $\mu_0 \in \mathbb{R}$ and h is a Riemannian three-dimensional metric of constant curvature $\kappa \in \{-1, 0, 1\}$.

In all cases, the metrics admit a smooth conformal compactification. Indeed, the metric $g_i = \frac{1}{r^2}\tilde{g}_i$ $i = 1, 2, 3$ followed by the change of variable $\Omega = 1/r$ yields a metric that is smooth in Ω and extends as a Lorentzian metric to $\mathcal{I} := \{\Omega = 0\}$. The corresponding metrics at $\mathcal{I}$, denoted by γ_i take the form,

$$\gamma_1 = \frac{d\chi^2}{P(\chi)} + \lambda(du + \chi^2 dy)^2 + \chi^2\left(dx - \frac{E_0}{\chi^2}dy\right)^2 + P(\chi)dy^2,$$

$$\gamma_2 = \lambda\boldsymbol{\tau}^2 + \left(dx + \frac{E_0}{\chi^3}\boldsymbol{\tau}\right)^2 + \frac{1}{F_0}(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\gamma_3 = \lambda du^2 + h.$$

By direct computation, one can check that the Weyl tensor of each one of these metrics is identically zero, so γ_i are locally conformally flat. As discussed in Sec. III, in the locally conformally flat $\mathcal{I}$ case the rescaled Weyl tensor is smoothly extendable to $\mathcal{I}$ and prescribes the TT tensor D in the asymptotic initial data [cf. (20)], after a suitable identification via the isometric embedding $\iota:(\Sigma, \gamma) \hookrightarrow (M, g)$ such that $\iota(\Sigma) = \mathcal{I}$.

In the present case, $\mathcal{I}$ is simply $\{\Omega = 0\}$ and the embedding is trivial in these adapted coordinates, so it is a matter of direct computation to determine the tensor D via formula (20). It turns out that in all three cases, this tensor takes the following form:

$$D_{AB} = \frac{4\mu_0\lambda^2}{|\xi|_\gamma^6}\left(\xi_A\xi_B - \frac{|\xi|_\gamma^2}{4}\gamma_{AB}\right)$$

and the vector ξ are given in each case by the following expressions:

(Case 1): $\xi_1 = \partial_u,$

(Case 2): $\xi_2 = \partial_u - \frac{E_0}{\chi^3}\partial_x,$

(Case 2): $\xi_3 = \partial_u.$

Moreover it is also a matter of direct computation to check that each vector field ξ_i is a conformal Killing vector of the corresponding metric γ_i. In other words: *all the three cases belong to the Kerr–de Sitter-like class* (cf. Definition 3.1). Now a natural question is to ask whether they are a subset within or they span the whole Kerr–de Sitter-like class. To address this question, we use the fact that the metrics in this class are in one-to-one correspondence [cf. (21)] with the conformal classes of the CKVs determining D. Hence, we must check that all possible conformal classes $\{[\xi_i]\}$ for the admissible values of the parameters in $\tilde{g}_i$ cover the space of conformal classes given in Proposition 2.2.

Observe that the parameter μ_0 always appears as a scaling constant in D_{AB}. This means that it can be set to[6] $\epsilon = \text{sign}(\mu_0)$ by suitably absorbing its norm into ξ. Namely, by defining $\xi' := |\mu_0|^{-1/2}\xi$, it follows

$$D_{AB} = \frac{4\mu_0\lambda^2}{|\xi|_\gamma^6}\left(\xi_A\xi_B - \frac{|\xi|_\gamma^2}{4}\gamma_{AB}\right)$$
$$= \frac{4\epsilon\lambda^2}{|\xi'|_\gamma^6}\left(\xi'_A\xi'_B - \frac{|\xi'|_\gamma^2}{4}\gamma_{AB}\right). \qquad (27)$$

This scale freedom will be relevant to prove that $\{[\xi_i]\}$ covers all possible conformal classes of four-dimensional conformally flat metrics.

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[6]By considering $\text{sign}(0) = 0$, we may include the case $\mu_0 = 0$, which corresponds to de Sitter-spacetime.

In order to determine the conformal class of ξ_i, we use the results of Sec. II. One simply needs to fix any point $p \in \mathcal{I}$, compute the quantities associated to ξ_i that appear in Theorem 2.5 and construct the endomorphism. The result, expressed in the basis v^α of Theorem 2.5 is

$$\text{Case 1: } F_{\xi_1} = \begin{pmatrix} 0 & 0 & -\lambda(A_0 + \frac{\lambda}{2}\chi_0^2) & E_0\lambda & \lambda(A_0\chi_0^2 - C_0 + \frac{\lambda}{2}\chi_0^4) & 0 \\ 0 & 0 & 0 & 0 & \lambda\chi_0 P(\chi_0) & 0 \\ -1 & \lambda\chi_0^3 P(\chi_0)^{-1} & 0 & 0 & 0 & -A_0 + \frac{\lambda}{2}\chi_0^2 \\ 0 & -E_0\lambda\chi_0^{-1} P(\chi_0)^{-1} & 0 & 0 & 0 & 0 \\ 0 & -\lambda\chi_0 P(\chi_0)^{-1} & 0 & 0 & 0 & -\lambda \\ 0 & 0 & -\lambda & 0 & -\lambda\chi_0^2 & 0 \end{pmatrix},$$

$$\text{Case 2: } F_{\xi_2} = \begin{pmatrix} 0 & \frac{\lambda F_0}{8} & \frac{\lambda E_0}{\chi} & 0 & \frac{1}{4}\lambda\chi\cos\theta_0 & 0 \\ -1 & 0 & 0 & -\frac{2\lambda\chi^2}{F_0}\cotan\theta_0 & 0 & \frac{1}{2}(\lambda\chi^2 - \frac{E_0^2}{\chi^4}) \\ \frac{E_0}{\chi^3} & 0 & 0 & 0 & 0 & \frac{E_0 F_0}{8\chi^3} \\ 0 & 0 & 0 & 0 & -\lambda\chi\sin\theta_0 & 0 \\ 0 & 0 & 0 & \frac{\lambda\chi}{\sin\theta_0} & 0 & 0 \\ 0 & -\lambda & 0 & 0 & -\frac{2\lambda\chi\cos\theta_0}{F_0} & 0 \end{pmatrix},$$

$$\text{Case 3: } F_{\xi_3} = \begin{pmatrix} 0 & -\frac{\lambda}{2}\kappa & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the point p has coordinates $\{\chi_0, u_0, x_0, y_0\}$ in case 1 and coordinates $\{u_0, x_0, \theta_0, \phi_0\}$ in case 2. (It is not necessary to give values to the coordinates of p in case 3 as they do not explicitly appear in the matrix.)

In order to simplify the notation we denote $\mathcal{Q}_i := \mathcal{Q}_{F_{\xi_i}}$ the polynomials (19) corresponding to the three cases. They are given (up to an irrelevant global sign) by

Case 1: $\mathcal{Q}_1 = z^3 + 2A_0\lambda z^2 - C_0\lambda^3 z + E_0^2\lambda^5$,

Case 2: $\mathcal{Q}_2 = \left(z + \frac{\lambda E_0^2}{\chi^4}\right)(z - \lambda^2\chi^2)^2$,

Case 3: $\mathcal{Q}_3 = z^2(z + \kappa\lambda)$.

Observe that these polynomials are independent of the point p. This is a necessary fact because the conformal class of the Killing vector is independent of the point, and hence the algebraic classification of F_ξ up to conjugation must also be independent of the point. This provides a nontrivial test both to the validity of Theorem 2.5 and for the calculations in this section.

We know from Proposition 2.2 (cf. [4]) that the classification of F_{ξ_i} is given by the set of parameters $\text{Roots}(\mathcal{Q}_i) = \{-\mu_t^2, \mu_s^2, \mu^2\}$, which are the roots of $\mathcal{Q}_1$ sorted in a way determined by the causal character of $\ker(F_{\xi_i})$. Thus, it is necessary to establish case by case the connection between the parameters defining the metrics $\tilde{g}_i$ and $\text{Roots}(\mathcal{Q}_i)$.

Case 1.
First, observe that

$$\mathcal{Q}_1(\tilde{\chi}^2) = -\lambda^3\tilde{\chi}^2 P(\lambda^{-1}\tilde{\chi}). \tag{28}$$

Hence, the roots of $\mathcal{Q}_1$ are determined by the roots of the polynomial $\chi^2 P(\chi)$. The parameters $\{A_0, C_0, E_0\}$ defining $P(\chi)$ are restricted by the condition of $P(\chi)$ being positive in some interval $I \in \mathbb{R}\backslash\{0\}$, which is clearly equivalent to imposing this same condition on the polynomial $\chi^2 P(\chi)$. Moreover, $\chi^2 P(\chi)$ is an even polynomial, which has negative dominant term, i.e., $\chi^2 P(\chi) \to \infty$ as $\chi \to \pm\infty$. If $E_0 = 0$, then $\chi^2 P(\chi)$ has a root at $\chi = 0$ which is at least double. So if $\chi^2 P(\chi)$ is to be positive on I, it either has one positive root (and its corresponding negative root) if $\{0\}$ is
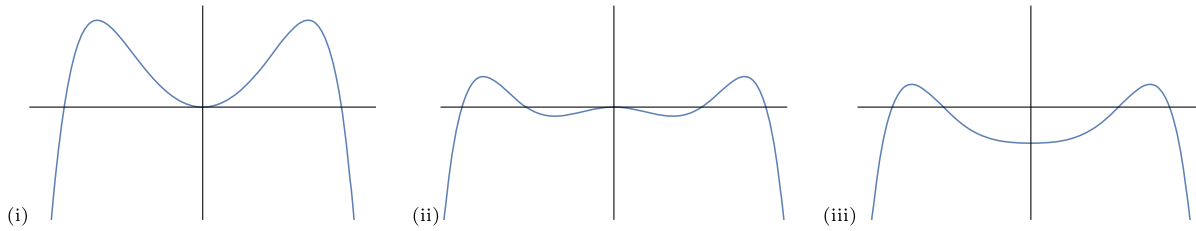
FIG. 1. Possible profiles of the polynomial $\chi^2 P(\chi)$: (i) $E_0 = 0$ with one real positive root, (ii) $E_0 = 0$ with two real positive roots, (iii) $E_0 \neq 0$.

in the closure of I, or two positive roots (and their corresponding negative roots) otherwise (cf. Fig. 1). If $E_0 \neq 0$, then $\chi^2 P(\chi)$ is negative at the origin, so if $\chi^2 P(\chi)$ is to be positive on I, it must have exactly two positive different roots (and their corresponding negative roots) (cf. Fig. 1).

All the above cases are covered by the polynomial decomposition,

$$\chi^2 P(\chi) = -\lambda(\chi^2 - a^2)(\chi^2 - b^2)(\chi^2 + c^2),$$

where a^2, b^2, c^2 are real constants satisfying

$$a^2 + b^2 - c^2 = 2\frac{A_0}{\lambda}, \qquad -a^2b^2 + a^2c^2 + b^2c^2 = \frac{C_0}{\lambda},$$

$$a^2b^2c^2 = \frac{E_0^2}{\lambda}, \tag{29}$$

with $a^2 \neq b^2$ [as otherwise the positivity of $\chi^2 P(\chi)$ cannot hold]. After swapping a and b if necessary we may assume $a^2 > b^2 \geq 0$. The argument can be read in the opposite direction: only by choosing any constants a, b, c such that $a \neq b$ and $a^2 > b^2 \geq 0$ and defining the parameters $\{A_0, C_0, E_0\}$ via (29), one obtains a function $P(\chi)$ which is positive in some interval $I \in \mathbb{R}\backslash\{0\}$.

Therefore, by (28), the roots of $\mathcal{Q}_1$ are always $\{\lambda a^2, \lambda b^2, -\lambda c^2\}$. According to Proposition 2.2, to determine their correspondence with $\{-\mu_t^2, \mu_s^2, \mu^2\}$, we must sort $\{\lambda a^2, \lambda b^2, -\lambda c^2\}$ depending on the causal character of $\ker F_{\xi_1}$. By Proposition 2.2, the sorting is such that $-\mu_t^2 = \mu_s^2 = 0$ and $\mu^2 = \lambda a^2$, whenever $\ker(F_{\xi_1})$ is null (in which case, by Remark 2.2, $\mathcal{Q}_1$ has an at least double root at zero and no negative roots) and $-\mu_t^2 = -\lambda c^2, \mu_s^2 = \lambda a^2, \mu^2 = \lambda b^2$ when $\ker(F_{\xi_1})$ is non-null.

If $E_0 \neq 0$, then zero is not a root of $\mathcal{Q}_1$. Being $\mathcal{Q}_1$ the square root of the characteristic polynomial of F^2 [see (19)], this implies that F_{ξ_1} has no zero eigenvalues. Thus, $\ker F_{\xi_1} = \{0\}$ and by Proposition 2.2, the conformal class of ξ_1 is given by

(1a) $-\mu_t^2 = -\lambda c^2$, $\mu_s^2 = \lambda a^2$ and $\mu^2 = \lambda b^2$ covering the region $\{\mu_s^2 > \mu^2 > 0 > -\mu_t^2\}$.

When $E_0 = 0$ it is straightforward to compute the kernel of F_{ξ_1}. The result is

$\ker(F_{\xi_1})$

$$= \text{span}\left\{u^1 := v^4, u^2 := v^1 + v^2 + \frac{1}{\chi_0}v^3 - \left(A_0 + \frac{\lambda}{2}\chi_0^2\right)v^6\right\},$$

so the kernel is two-dimensional in this case. The pull-back of the scalar product Q on this space is

$$Q(u^1, u^1) = \frac{1}{\chi_0^2}, \qquad Q(u^1, u^2) = 0, \qquad Q(u^2, u^2) = \frac{C_0}{\chi_0^2}.$$

This is timelike if $C_0 < 0$, null if $C_0 = 0$ and spacelike if $C_0 > 0$. By (29), $E_0 = 0$ requires the vanishing of b^2 and/or c^2, and by Proposition 2.2, the conformal class of ξ_1 is given by

(1b) If $c^2 = 0, b^2 \neq 0$, then $C_0 < 0$ and $-\mu_t^2 = 0, \mu_s^2 = \lambda a^2$, $\mu = \lambda b^2$ cover the region $\{\mu_s^2 > \mu^2 > 0 = -\mu_t^2\}$.

(1c) If $c^2 \neq 0, b^2 = 0$, then $C_0 > 0$ and $-\mu_t^2 = -\lambda c^2$, $\mu_s^2 = \lambda a^2, \mu = 0$ cover the region $\{\mu_s^2 > \mu^2 = 0 > -\mu_t^2\}$.

(1d) If $c^2 = 0, b^2 = 0$, then $C_0 = 0$ and $-\mu_t^2 = 0, \mu_s^2 = 0$, $\mu = \lambda a^2$ cover the region $\{\mu^2 > 0 = \mu_s^2 = -\mu_t^2\}$.

Case 2.

In this case, the roots of $\mathcal{Q}_2$ are immediately found to be $-\lambda E_0^2/\chi^4$ together with the double root $\lambda^2\chi^2$. If $E_0 \neq 0$, it again follows that $\ker F_{\xi_2} = \{0\}$; thus, the conformal class of ξ_2 is given by the parameters $-\mu_t^2 = -\lambda E_0^2/\chi^4$, $\mu_s^2 = \mu = \lambda^2\chi^2$ covering the region $\{\mu_s^2 = \mu^2 > 0 > -\mu_t^2\}$. When $E_0 = 0$ the kernel of F_{ξ_2} is

$$\ker(F_{\xi_2}) = \text{span}\left\{u^1 := v^3, u^2 := v^1 + \frac{\lambda\chi^2}{2}v^6\right\},$$

so the kernel is again two dimensional and Q restricted to this space is

$$Q(u^1, u^1) = 1, \qquad Q(u^1, u^2) = 0, \qquad Q(u^2, u^2) = -\frac{4}{\lambda\chi^2}.$$

Thus, $\ker(F_{\xi_2})$ is timelike, and the conformal class of ξ_2 is given by

(2) $-\mu_t^2 = 0$, $\mu_s^2 = \mu = \lambda^2\chi^2$ covering the region $\{\mu_s^2 = \mu^2 > 0 = -\mu_t^2\}$.

Case 3.

In this case, the roots of $\mathcal{Q}_3$ are also trivial. There is a double root at zero and another one at $-\lambda\kappa$. Note that from the scaling freedom of ξ [cf. (27)], F_{ξ_3} is also defined up to

a scaling factor $|\mu_0|^{1/2}$ and $F_{\xi_3}^2$ up to $|\mu_0|$. Then the root $-\lambda\kappa$ can be scaled by a nonzero positive factor $-\lambda|\mu_0|\kappa$, which will be relevant to cover the maximal space of parameters in the space of conformal classes. Observe that (in any of the three cases) there is no restriction on the value of μ_0.

The kernel of the endomorphism is

$$\ker(F_{\xi_3}) = \mathrm{span}\left\{u^1 := v^1 - \frac{\kappa}{2}v^6, u^2 := v^3, u^3 := v^4, u^4 := v^5\right\};$$

hence, four dimensional and Q restricted to this space is

$$Q(u^1, u^1) = \kappa, \qquad Q(u^2, u^2) = 1,$$
$$Q(u^3, u^3) = \frac{1}{\Sigma^2|_p}, \qquad Q(u^4, u^4) = \frac{1}{(\Sigma^2\sin^2\theta)|_p},$$

and the rest zero, and where we have written the constant curvature metric as

$$h = d\psi^2 + \Sigma^2(\psi)(d\theta^2 + \sin^2\theta d\phi^2).$$

The kernel is now spacelike if $\kappa > 0$, degenerate if $\kappa = 0$ and timelike if $\kappa < 0$. Thus, for each case, the conformal class of ξ_3 is

(3a) For $\kappa = 1$, $-\mu_t^2 = -\lambda|\mu_0|$, $\mu_s^2 = \mu = 0$ covering the region $\{\mu_s^2 = \mu^2 = 0 > -\mu_t^2\}$.

(3b) For $\kappa = 0$, $-\mu_t^2 = \mu_s^2 = \mu = 0$ covering a single point.

(3c) For $\kappa = -1$, $\mu_s^2 = \lambda|\mu_0|$, $-\mu_t^2 = \mu = 0$ covering the region $\{\mu_s^2 > \mu^2 = 0 = -\mu_t^2\}$.

Summarizing, the cases 1,2 and 3 correspond to CKVs ξ_1, ξ_2 and ξ_3 whose respective conformal classes cover the space of parameters,

$$\mathcal{A} = \{(-\mu_t^2, \mu_s^2, \mu^2) \in \mathbb{R}^3 | \mu_s^2 \geq \mu^2 \geq 0 \geq -\mu_t^2, \text{ with } \mu_t^2 \text{ or } \mu_s^2 \neq 0\}$$
$$\cup \{(-\mu_t^2, \mu_s^2, \mu^2) \in \mathbb{R}^3 | \mu^2 \geq 0 = \mu_s^2 = -\mu_t^2\}.$$

By Proposition (2.2), $\mathcal{A}$ corresponds to the entire space parameterizing the conformal classes of CKVs of conformally flat 4-metrics. Thus, we have proven that these metrics correspond exactly to all metrics in the five-dimensional Kerr–de Sitter-like class. This, combined with Theorem 3.1, yields the following result:

Theorem 4.1. In five spacetime dimensions, the following classes of $(\Lambda > 0)$-vacuum metrics are equivalent:

(1) The Kerr–de Sitter-like class (cf. Definition 3.1).

(2) The Kerr-Schild type metrics on a locally de Sitter background [cf. (26)] admitting a smooth conformal extension such that $\Omega^2 \mathcal{H} \boldsymbol{k} \otimes \boldsymbol{k} = O(\Omega)$.

(3) The algebraically special metrics with nondegenerate optical matrix.

V. DISCUSSION

We have obtained a method to determine the conformal class of an arbitrary CKV ξ of a locally conformally flat metric γ of any dimension and signature. Such method is based on pointwise properties of the CKVs, and it is independent on the coordinates and the representative of the class of metrics conformal to γ. This improves previously existing results (cf. [15,19]), which require to find an explicitly flat representative in Cartesian coordinates.

Our result is stated as a computationally neat algorithm in Theorem 2.5, which allows for a straightforward application in Sec. IV. Namely, we classify the asymptotic data of all five-dimensional, algebraically special, $(\Lambda > 0)$-vacuum spacetimes, whose optical matrix is nondegenerate (cf. [17]). Such asymptotic data are determined by the conformal class of a CKV of a conformally flat $\mathcal{I}$. Furthermore, we prove equivalence of this collection of spacetimes with the Kerr–de Sitter-like class as well as with the $(\Lambda > 0)$-vacuum Kerr-Schild spacetimes satisfying a natural asymptotic condition (cf. Theorem 3.1).

It is worth commenting that the results of Sec. IV, besides providing an application of Theorem 2.5, outline the way for potential future results. As pointed out in [17], in dimensions higher than four, there exist several results (cf. [33,34]) supporting the idea that the class of algebraically special solutions is more rigid than in four dimensions. One then wonders whether Theorem 3.1 also holds in any dimension higher than five. Surprisingly, the results in [17] do not rely on any asymptotic property of the spacetime, while in [19] it is central. Yet, the class of spacetimes studied [17] and in [19] (in five spacetime dimensions) happen to be equivalent, as we have shown in this paper. This hints a possible connection between asymptotic properties of spacetimes and the algebraic classification of the Weyl tensor. More precisely, the algebraically special condition with non-degenerate optical matrix implies conformal extendability with locally conformally flat $\mathcal{I}$. A better understanding of this aspect would be of substantial intrinsic interest and key for extending Theorem 3.1 to arbitrary dimensions.

It is also interesting to observe that the Weyl tensor C of the spacetime determines, in any dimension, the Weyl tensor[7] c of the metric γ induced at $\mathcal{I}$. Indeed, an asymptotic expansion of C, shows that its components fully tangent to $\mathcal{I}$ coincide with c to the leading order. The Weyl tensor contains a lot of information about the conformal class of metrics of dimension equal or higher than four,[8] so it would not be surprising that the

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[7]Note that c and D as defined in (20), are indeed independent objects at $\mathcal{I}$, however both obtainable from the spacetime Weyl tensor C at $\mathcal{I}$.

[8]The conformal class determines the Weyl tensor, but the opposite is not always true, cf. [35].

algebraically special condition on C imposes strong conditions on the conformal class of (four or higher dimensional) γ. However, in four spacetime dimensions, γ is three dimensional and thus, c vanishes identically, so it is not clear in this case how the algebraic type of C affects the conformal class of γ. The difference between the four and higher dimensional cases may be responsible for the lack of rigidity of algebraically special metrics in four dimensions, because recall (cf. Sec. III) that the conformal class of γ is one of the freely speciable data in the asymptotic Cauchy problem. This, however, does not rule out other possible relations between the algebraic type of the four-dimensional spacetime metrics and their asymptotic properties. One connection may arise from the fact that the other component of the asymptotic data is, in four dimensions, the electric part of the rescaled Weyl tensor (cf. Sec. III).

In addition, constraints on the conformal class of γ may also appear as a consequence of the relation between C and Cotton tensor of γ, which plays a similar role than the Weyl tensor for three-dimensional metrics. These potential connections are worth to investigate in the future.

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