


Bach equation and the matching of spacetimes in conformal cyclic cosmology models

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We consider the problem of matching two spacetimes, the previous and present aeons, in the conformal cyclic cosmology model. The common boundary between them inherits two sets of constraints—one for each solution of the Einstein field equations extended to the conformal boundaries. The previous aeon is assumed to be an asymptotically de Sitter spacetime, so the standard conformal formulation of the Einstein field equations suffice to derive the constraints on the future null infinity. For the future aeon, which is supposed to evolve from an initial singularity, they are obtained with the use of the Bach-type equation. This equation is regular at the past conformal infinity for conformally flat and conformally Einstein spacetimes, so we will mostly focus on them here. An example of the electrovacuum spacetime which does not fall into this class and has a regular conformal Bach tensor will be discussed in the Appendix.

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I. INTRODUCTION

The conformal cyclic cosmology (CCC) is an alternative cosmological model, introduced by Penrose [1]. It is based on the common agreement that our Universe evolves from an initial singularity, and that this evolution is influenced by the presence of a positive cosmological constant. Hence, its end state is assumed to approach an asymptotically de Sitter spacetime. This solution of the Einstein field equation has the remarkable property that its future conformal boundary is a spacelike hypersurface, which allows one to perform matching with the spacelike past conformal boundary of the other spacetime. Such gluing procedure can be viewed as a transition between two distinct solutions of the Einstein field equations. This is the bedrock of the CCC model: postulating that the history of our Universe consists of such building blocks, aeons, which are joined together by their spacelike conformal boundaries to form an infinite cycle. Because the aeons evolve from the big bang singularity into asymptotically de Sitter spacetimes, the CCC provides a natural explanation for the Weyl curvature hypothesis [2]. One of the other physical implications of this model is the postulated existence of the ringlike structures in the cosmic microwave background [3–5].

There have been many approaches to the construction of a viable model of the transition between two aeons in the CCC—see e.g., [6–9]. However, all of them assume that the conformal metric is given either in an exact form or in terms of a power series. The present article serves as a step toward the general case. In order to carry out our analysis, we make use of the conformal Einstein field equations and the Bach-type equation to study the constraints on the common boundary between aeons. The assumption that the conformal version of the latter is regular on the conformal boundary is a very restrictive condition. As will be seen, a natural way to satisfy it is to consider conformally Einstein spacetimes. This will be the main focus of this work. However, a class of electrovacuum spacetimes that are not conformally flat and have a regular Bach equation will also be discussed in the Appendix as an example of a more general scenario.

The structure of this article is as follows. In the next section, we will discuss the details of the transition between the two aeons of the CCC model. Section III provides a discussion of the constraints induced on the conformal boundary of the previous aeon by Friedrich’s conformal Einstein field equations [10,11]. After that, in Sec. IV, the Anderson-Fefferman-Graham equation [12,13] (which in the current case reduces to the equation with Bach tensor) and a nonlinear wave equation for the conformal factor will be used to describe the constraints on the past null infinity of the present aeon. The analysis of the big bang

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singularities that uses the Bach equation is viable only for a certain class of spacetimes—most notably the conformally flat ones. However, based on an example of the electrovacuum solution of the Einstein field equations discussed in the Appendix, we will make the assumption that the Bach equation is regular everywhere to analyze the present aeon. Lastly, in Sec. V the two sets of constraints obtained in Secs. III and IV will be evaluated on the common boundary between the aeons to obtain certain simplifications.

This article relies on the conformal formulation of the Einstein field equation, where asymptotically de Sitter spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is assumed to have a conformal extension (\mathcal{M}, g_{ab}) in the form of compact manifold \mathcal{M} with the boundary $\partial\mathcal{M}$ and the metric $g_{ab} = \Omega^2 \hat{g}_{ab}$. The function Ω is positive in the interior of \mathcal{M} and vanishes on $\partial\mathcal{M}$. In the analysis of the conformal field equations it is convenient to introduce the rescaled Weyl and Cotton tensors of \hat{g}_{ab} in the following way:

$$d^a{}_{bcd} := \frac{1}{\Omega} C^a{}_{bcd}, \quad Q_{abc} := \frac{1}{\Omega} \hat{A}_{abc}. \quad (1.1)$$

The spacetime with the initial (big bang) singularity will be assumed to satisfy the following regular Bach equation:

$$\bar{B}_{ab} = S_{ab}, \quad (1.2)$$

where \bar{B}_{ab} is the Bach tensor of the conformal metric and S_{ab} is the source term. With this preparation, we can formulate the main theorem of this article.

Theorem.—Let $(\hat{\mathcal{M}}, \hat{g}_{ab})$ be an asymptotically de Sitter spacetime with the cosmological constant $\hat{\Lambda}$ and the future conformal boundary \mathcal{I}^+ where the Cotton tensor vanishes. Assume that $(\check{\mathcal{M}}, \check{g}_{ab})$ is a spacetime with the big bang singularity located on the past conformal boundary \mathcal{I}^- , which is a solution of the regular conformal Bach equation. Then, if $(\hat{\mathcal{M}}, \hat{g}_{ab})$ and $(\check{\mathcal{M}}, \check{g}_{ab})$ are subsequent aeons in the conformal cyclic cosmology scenario the constraints

$$\begin{aligned} D^b e_{ab} &\doteq -Q_{\perp\perp a}^\top \doteq \sqrt{\frac{3}{\hat{\Lambda}}} S_{a\perp}^\top, & Q_{\perp bc}^\top &\doteq 0, \\ b_{bca} &\doteq \sqrt{\frac{3}{\hat{\Lambda}}} A_{abc}^{(3)}, & S_{\perp\perp} &\doteq 0 \end{aligned} \quad (1.3)$$

have to be satisfied on the common conformal boundary between them. The quantities e_{ab} and b_{abc} correspond to the electric and magnetic part of the rescaled Weyl tensor d_{abcd} and $A_{abc}^{(3)}$ is the Cotton tensor of the conformal boundary.

As an application of the main theorem we derive constraints relating the initial values of the matter fields in the present aeon with the same fields from the previous aeon if both spacetimes are assumed to be electrovacuum. This provides a partial answer to one of the questions posed by Tod in [14].

A. Notation and conventions

We will work with the four-dimensional spacetimes and use abstract index notation throughout the paper. The signature of the spacetime metrics will be $(-, +, +, +)$. The convention for the Riemann tensor R_{abcd} is as follows:

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = R_{abcd} v^d. \quad (1.4)$$

It can be decomposed into the Weyl tensor C_{abcd} and the Schouten tensor in the following way:

$$R_{abcd} = C_{abcd} + 2(g_{c[a} P_{b]d} + g_{d[b} P_{a]c}), \quad (1.5)$$

where

$$P_{ab} := \frac{1}{2} R_{ab} - \frac{R}{12} g_{ab}, \quad J := P_a{}^a = \frac{R}{6} \quad (1.6)$$

is the trace-corrected Ricci tensor—also called the Schouten tensor. The (anti)symmetrization brackets are defined as

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \quad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}) \quad (1.7)$$

with the obvious generalizations to more indices.

Let \mathcal{S} be a codimension-one spacelike hypersurface with the induced metric h_{ab} and the unit normal vector n^a . The projection to hypersurface tensors will be denoted by a superscript \top , whereas contraction with n^a by \perp , e.g.,

$$T_{a\perp}^\top = h_a{}^b T_{bc} n^c. \quad (1.8)$$

The Weyl tensor can be decomposed into its electric and magnetic part with respect to an unit vector u^a as follows:

$$E_{ab} := u^c u^d C_{acbd}, \quad H_{ab} := \frac{1}{2} u^c u^d \eta_{ackl} C^{kl}{}_{bd}, \quad (1.9)$$

where η_{abcd} is the covariant Levi-Civita tensor. Lastly, the Cotton and Bach tensors are defined as

$$\begin{aligned} A_{abc} &:= 2\nabla_{[b} P_{c]a}, \\ B_{ab} &:= -\nabla^c A_{abc} + P^{dc} C_{dacb}. \end{aligned} \quad (1.10)$$

Moreover, the Bianchi identity implies

$$A_{abc} = \nabla^d C_{dabc}. \quad (1.11)$$

II. CONFORMAL CYCLIC COSMOLOGIES

In the following, let $(\hat{\mathcal{M}}, \hat{g}_{ab})$ and $(\check{\mathcal{M}}, \check{g}_{ab})$ denote two solutions of the Einstein field equations to be called, respectively, the previous and present aeons. One has that

$$\begin{aligned}\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} + \hat{\Lambda}\hat{g}_{ab} &= \hat{T}_{ab}, \\ \check{R}_{ab} - \frac{1}{2}\check{R}\check{g}_{ab} + \check{\Lambda}\check{g}_{ab} &= \check{T}_{ab},\end{aligned}\quad (2.1)$$

where $\hat{\Lambda}$ (respectively $\check{\Lambda}$) are the positive cosmological constants and \hat{T}_{ab} (respectively \check{T}_{ab}) are the energy-momentum tensors describing the matter content of the corresponding aeon. We will assume that $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is an asymptotically de Sitter spacetime and that $(\check{\mathcal{M}}, \check{g}_{ab})$ evolves from an initial singularity. Those statements can be made precise with the assumption that both spacetimes admit a conformal compactification. Hence, in the sequel we will use the notion of conformal extensions of both aeons, (\mathcal{M}, g_{ab}) and $(\bar{\mathcal{M}}, \bar{g}_{ab})$, such that:

- (i) \mathcal{M} (respectively $\bar{\mathcal{M}}$) are compact manifolds with the boundaries $\partial\mathcal{M}$ (respectively $\partial\bar{\mathcal{M}}$) such that

$$\mathcal{M} = \hat{\mathcal{M}} \cup \partial\mathcal{M}, \quad \bar{\mathcal{M}} = \check{\mathcal{M}} \cup \partial\bar{\mathcal{M}}. \quad (2.2)$$

- (ii) The metrics g_{ab} (respectively \bar{g}_{ab}) are regular everywhere and there exist positive functions Ω and ω such that

$$g_{ab} = \Omega^2 \hat{g}_{ab}, \quad \bar{g}_{ab} = \frac{1}{\omega^2} \check{g}_{ab}. \quad (2.3)$$

That is, the zero locus $\mathcal{Z}(\Omega)$ corresponds to the future null infinity (future conformal boundary) \mathcal{I}^+ of $(\hat{\mathcal{M}}, \hat{g}_{ab})$ and $\mathcal{Z}(\omega)$ to the past conformal boundary \mathcal{I}^- of $(\check{\mathcal{M}}, \check{g}_{ab})$.

It is important to note here that we are not assuming that both spacetimes have the same conformal extension. If that were the case, then the reciprocal hypothesis could be used to determine the metric from the present aeon out of the metric from the past one, given that the conformal factor could be prescribed uniquely, or vice versa [1,7]. Instead, we will work with the two solutions of the Einstein field equations which independently satisfy the required conformal properties and study the matching conditions along their conformal boundary afterwards.

III. THE ASYMPTOTICALLY DE SITTER AEON: FRIEDRICH'S CONFORMAL FIELD EQUATIONS AND CONSTRAINTS

The objective of this section is to consider asymptotically de Sitter-like spacetimes $(\hat{\mathcal{M}}, \hat{g}_{ab})$ which admit conformal extensions (\mathcal{M}, g_{ab}) with

$$g_{ab} = \Omega^2 \hat{g}_{ab} \quad (3.1)$$

and derive the constraints on the future conformal boundary \mathcal{I}^+ ($\mathcal{Z}(\Omega)$). This is a standard setting for Friedrich's conformal Einstein field equations [10,11] and we will

briefly describe this approach with nonvanishing energy-momentum tensor—see [15] and the references therein for the discussion of trace-free matter models.

A. Conformal Einstein field equations

By the conformal Einstein field equations it is understood a conformal representation of the Einstein field equations—that is, they provide a set of equations which are formally regular at the conformal boundary (where the conformal factor Ω vanishes) of an asymptotically de Sitter spacetime, which imply, away from the conformal boundary, a solution to the Einstein field equations. There exist in the literature several candidates for suitable conformal field equations—see e.g., [15]. In this article we focus on Friedrich's conformal Einstein field equations [16]. These equations, which in the following we just simply call *the conformal Einstein equations*, have been instrumental in the study of the nonlinear stability of de Sitter-like and Minkowski-like spacetimes [17]. Although the conformal Einstein equations have mostly been considered in the vacuum setting, the formalism can be extended to the nonvacuum case. Indeed, these equations have been used to study the stability of a number of cosmological models (e.g., scalar field, dust, radiation) [18–20]. As it will be seen in the following, these equations allow the formulation of a regular asymptotic initial value problem at the conformal boundary.

As mentioned above, the strategy behind the conformal Einstein field equations is to find a system that is regular at the conformal boundary \mathcal{I}^+ and is equivalent to the usual Einstein field equation under suitable conditions. Let

$$s := \frac{1}{24}R\Omega + \frac{1}{4}\square\Omega \quad (3.2)$$

where the curvature quantities and derivative operator is associated with the conformal metric g_{ab} . It is also useful to define the rescaled version of the tensors corresponding to the de Sitter-like (physical) metric \hat{g}_{ab} ,

$$d^a{}_{bcd} := \frac{1}{\Omega}C^a{}_{bcd}, \quad Q_{abc} := \frac{1}{\Omega}\hat{A}_{abc}. \quad (3.3)$$

The conformal Einstein field equations are given by the following system:

$$\nabla_a \nabla_b \Omega = -\Omega P_{ab} + s g_{ab} + \frac{\Omega}{2} \overset{\circ}{T}_{ab}, \quad (3.4a)$$

$$\nabla_a s = -P_a{}^b \nabla_b \Omega + \frac{1}{6} \nabla^c (\overset{\circ}{T}_{ac} \Omega), \quad (3.4b)$$

$$\nabla_d d^d{}_{abc} = Q_{abc}, \quad (3.4c)$$

$$2\nabla_{[b} P_{c]a} = d^d{}_{abc} \nabla_d \Omega + \Omega Q_{abc}, \quad (3.4d)$$

$$\hat{\Lambda} = 6\Omega s - 3\nabla_c \Omega \nabla^c \Omega + \frac{\hat{T}}{4}, \quad (3.4e)$$

where $\overset{\circ}{T}_{ab}$ denotes the trace-free part of the physical energy-momentum-tensor

$$\overset{\circ}{T}_{ab} = \hat{T}_{ab} - \frac{\hat{T}}{4} \hat{g}_{ab}, \quad (3.5)$$

further complemented by the conservation equation for \hat{T}_{ab} and its conformal transformation rule

$$T_{ab} = \frac{1}{\Omega^q} \hat{T}_{ab} \quad (3.6)$$

where T_{ab} is the regular unphysical energy-momentum tensor. Because the spacetime is asymptotically de Sitter we will restrict ourselves to $q \geq 0$.

A solution to the system (3.4) has the form of a collection of fields

$$(\Omega, g_{ab}, s, P_{ab}, d^a{}_{bcd}, T_{ab}) \quad (3.7)$$

over \mathcal{M} .

Remark 1. As $\nabla_a \Omega$ is orthogonal to level set of Ω , Eq. (3.4e) is equivalent to the statement that $\mathcal{Z}(\Omega)$ is a spacelike hypersurface—as long as $\hat{T} = 0$ there.

Remark 2. If the trace \hat{T} vanishes, then for $q = 2$ the conservation equation $\hat{\nabla}_b \hat{T}_a{}^b = 0$ is conformally invariant and regular everywhere.

B. Constraints

Let \mathcal{S} be a spacelike hypersurface of \mathcal{M} with unit normal vector n^a , $n^a n_a = -1$. The unphysical metric g_{ab} induces a metric h_{ab} on \mathcal{S} , given by

$$h_{ab} := g_{ab} + n_a n_b. \quad (3.8)$$

The extrinsic curvature of \mathcal{S} will be denoted by K_{ab} and is defined as

$$K_{ab} := \nabla_a n_b + n_a a_b, \quad (3.9)$$

where $a_b = \nabla_n n_b$ is the acceleration of n^a . Let $\sigma := \nabla_{\perp} \Omega$, so that one has the decomposition

$$\nabla_a \Omega = D_a \Omega - n_a \sigma, \quad (3.10)$$

where D_a is the Levi-Civita connection of h_{ab} .

Before discussing the constraints induced by the conformal field equations on \mathcal{S} , it is useful to consider decompositions of the relevant tensors with respect to n^a . We have

$$\overset{\circ}{T}_{ab} = \overset{\circ}{T}_{ab}^{\top} - 2n_{(a} \overset{\circ}{T}_{b)\perp}^{\top} + n_a n_b \overset{\circ}{T}_{\perp\perp}. \quad (3.11)$$

Similarly,

$$P_{ab} = P_{ab}^{\top} - 2n_{(a} P_{b)\perp}^{\top} + n_a n_b P_{\perp\perp}. \quad (3.12)$$

The decomposition of the rescaled Weyl tensor d_{abcd} is given by

$$\begin{aligned} d_{abcd} = & d_{abcd}^{\top} - n_d b_{abc} + n_c b_{abd} - n_b b_{cda} \\ & + n_a b_{cdb} + n_a n_c e_{bd} + n_b n_d e_{ac} \\ & - n_b n_c e_{ad} - n_a n_d e_{bc}, \end{aligned} \quad (3.13)$$

where $e_{ab} := d_{a\perp b\perp}^{\top}$ is its electric part and $b_{abc} := d_{abc\perp}^{\top}$ defines its magnetic part. Because of the symmetries of d_{abcd} its fully projected part can be decomposed further as

$$d_{abcd}^{\top} = e_{ac} h_{bd} + e_{bd} h_{ac} - e_{ad} h_{bc} - e_{bc} h_{ad}. \quad (3.14)$$

The constraints induced on \mathcal{S} by (3.4) can be obtained by considering intrinsic and normal-intrinsic components of this equations. They read

$$D_a D_b \Omega = \sigma K_{ab} - \Omega P_{ab}^{\top} + s h_{ab} + \frac{\Omega}{2} \overset{\circ}{T}_{ab}^{\top}, \quad (3.15a)$$

$$D_a \sigma = K_a{}^b D_b \Omega - \Omega P_{a\perp}^{\top} + \frac{\Omega}{2} \overset{\circ}{T}_{a\perp}^{\top}, \quad (3.15b)$$

$$\begin{aligned} D_a s = & -P_{ab}^{\top} D^b \Omega + P_{a\perp}^{\top} \sigma + \frac{1}{6} \overset{\circ}{T}_{ab}^{\top} D^b \Omega \\ & - \frac{1}{6} \overset{\circ}{T}_{a\perp}^{\top} \sigma + \frac{\Omega}{6} (D^b \overset{\circ}{T}_{ab}^{\top} + \overset{\circ}{T}_{ab}^{\top} a^b \\ & - K_a{}^b \overset{\circ}{T}_{b\perp}^{\top} - K \overset{\circ}{T}_{a\perp}^{\top} - h_a{}^b \nabla_{\perp} \overset{\circ}{T}_{b\perp}^{\top} \\ & + \overset{\circ}{T}_{\perp\perp} a_a), \end{aligned} \quad (3.15c)$$

$$D_b e^b{}_a = -b_{bac} K^{bc} - \overset{\circ}{Q}_{\perp\perp a}, \quad (3.15d)$$

$$D^a b_{bca} = 2e_{a[b} K^a{}_{c]} + \overset{\circ}{Q}_{\perp bc}, \quad (3.15e)$$

$$\begin{aligned} 2D_{[b} P_{c]a}^{\top} = & -2P_{\perp[b}^{\top} K_{c]a} + \Omega \overset{\circ}{Q}_{abc}^{\top} + \sigma b_{bca} \\ & - 2h_{a[b} e_{c]d} D^d \Omega - 2e_{a[b} D_{c]} \Omega, \end{aligned} \quad (3.15f)$$

$$2D_{[a} P_{b]\perp}^{\top} = 2K_{[a}{}^c P_{b]c}^{\top} + b_{abc} D^c \Omega + \Omega \overset{\circ}{Q}_{\perp ab}, \quad (3.15g)$$

$$\hat{\Lambda} = 6\Omega s - 3D_a \Omega D^a \Omega + 3\sigma^2 + \frac{\hat{T}}{4}, \quad (3.15h)$$

where $K := K_a{}^a$. Any other projections are either trivial or can be expressed by the linear combination of (3.15). This system of equations is supplemented by

$$P_{ab}^{(3)} = \Omega e_{ab} + P_{ab}^\top - K \left(K_{ab} - \frac{1}{4} h_{ab} K \right) + K_{ac} K_b^c - \frac{1}{4} h_{ab} K_{cd} K^{cd}, \quad (3.16)$$

where $P_{ab}^{(3)}$ is the Schouten tensor of h_{ab} , and

$$2D_{[a}K_{b]c} = \Omega b_{abc} + 2h_{c[a}P_{b]\perp}^\top. \quad (3.17)$$

These two equations are a consequence of the Gauss-Codazzi and Codazzi-Mainardi relations.

C. Constraints on $\mathcal{Z}(\Omega)$

Now, suppose that \mathcal{S} is the future conformal boundary of the asymptotically de Sitter spacetime—i.e., $\mathcal{S} = \mathcal{Z}(\Omega)$. The constraint equations (3.15) reduce greatly in that case and read

$$D_a \sigma \doteq 0, \quad \hat{\Lambda} \doteq 3\sigma^2, \quad (3.18a)$$

$$sh_{ab} \doteq -\sigma K_{ab}, \quad (3.18b)$$

$$D_a s \doteq \sigma \left(P_{a\perp}^\top - \frac{1}{6} T_{a\perp}^\top \right), \quad (3.18c)$$

$$P_{ab}^{(3)} \doteq P_{ab}^\top - K \left(K_{ab} - \frac{1}{4} h_{ab} K \right) + K_{ac} K_b^c - \frac{1}{4} h_{ab} K_{cd} K^{cd}, \quad (3.18d)$$

$$D_{[a}K_{b]c} \doteq h_{c[a}P_{b]\perp}^\top, \quad (3.18e)$$

$$2D_{[a}P_{b]\perp}^\top \doteq 2K_{[a}^c P_{b]c}^\top, \quad (3.18f)$$

$$2D_{[b}P_{c]a}^\top \doteq \sigma b_{bca} - 2P_{\perp[b}^\top K_{c]a}, \quad (3.18g)$$

$$D^b e_{ab} \doteq -Q_{\perp\perp a}^\top - K^{bc} b_{bac}, \quad (3.18h)$$

$$D^a b_{bca} \doteq Q_{\perp bc}^\top - 2K_{[b}^a e_{c]a}, \quad (3.18i)$$

where \doteq denotes the equality on the zero set of Ω . Equations (3.18) yield $\sigma \doteq \sqrt{\hat{\Lambda}/3}$. Because this function (the normal derivative of Ω) is constant on $\mathcal{Z}(\Omega)$ it is convenient to define

$$\kappa := \frac{s}{\sigma}. \quad (3.19)$$

Then, from (3.18b) and (3.18c)

$$K_{ab} \doteq -\kappa h_{ab}, \quad P_{a\perp}^\top \doteq D_a \kappa + \frac{1}{6} T_{a\perp}^\top, \quad (3.20)$$

e.g., $\mathcal{Z}(\Omega)$ is an umbilic hypersurface. The Gauss-Codazzi constraint (3.18d) may serve as a definition of the projected part of the unphysical Schouten tensor. More precisely, one has

$$P_{ab}^\top \doteq P_{ab}^{(3)} + \frac{1}{2} \kappa^2 h_{ab}. \quad (3.21)$$

It can be readily verified that the Codazzi-Mainardi relation constraint (3.18e) yields $T_{a\perp}^\top \doteq 0$ in the current case, which leads to (3.18f) being identically satisfied.

The constraint (3.18g) may now be viewed as a definition of the magnetic part of the rescaled Weyl tensor, i.e.,

$$b_{bca} \doteq \frac{1}{\sigma} 2D_{[b}P_{c]a}^{(3)} = \frac{1}{\sigma} A_{abc}^{(3)}, \quad (3.22)$$

where $A_{abc}^{(3)}$ is the unphysical three-dimensional Cotton tensor. The last two constraints, (3.18h) and (3.18i), reduce to

$$D^b e_{ab} \doteq -Q_{\perp\perp a}^\top, \quad Q_{\perp ab}^\top \doteq 0, \quad (3.23)$$

where the divergence-free nature of $A_{abc}^{(3)}$ has been used.

This section can be summarized as follows:

Theorem 1.—Let $(\hat{\mathcal{M}}, \hat{g}_{ab})$ be an asymptotically de Sitter spacetime with conformal extension (\mathcal{M}, g_{ab}) where $g_{ab} = \Omega^2 \hat{g}_{ab}$. Then, the constraints induced on the future conformal boundary \mathcal{I}^+ (zero locus of Ω) are as follows:

$$\begin{aligned} \nabla_\perp \Omega &\doteq \sqrt{\frac{\hat{\Lambda}}{3}}, & K_{ab} &\doteq -\sqrt{\frac{3}{\hat{\Lambda}}} s h_{ab} \\ P_{a\perp}^\top &\doteq \sqrt{\frac{3}{\hat{\Lambda}}} D_a s, & P_{ab}^\top &\doteq P_{ab}^{(3)} + \frac{3}{2\hat{\Lambda}} s^2 h_{ab}, \\ b_{abc} &\doteq \sqrt{\frac{3}{\hat{\Lambda}}} A_{cab}^{(3)}, & D^b e_{ab} &\doteq -Q_{\perp\perp a}^\top, \\ Q_{\perp ab}^\top &\doteq 0, \end{aligned} \quad (3.24)$$

where the unhatted quantities correspond to the conformal metric g_{ab} and the superscript (3) indicates quantities intrinsic to \mathcal{I}^+ . Moreover, $s := \frac{1}{24} R\Omega + \frac{1}{4} \square\Omega$, e_{ab} and b_{abc} correspond to the electric and magnetic part of the rescaled Weyl tensor $\Omega^{-1} C_{abcd}$ and $Q_{abc} := \Omega^{-1} \hat{A}_{abc}$ is the rescaled Cotton tensor.

Remark 3. Following the postulates of the CCC, the conformal boundary is the natural hypersurface to perform matching between a past aeon and a future one.

IV. THE BIG BANG SINGULARITY: THE PRESENT AEON

Having described the constraints on the future conformal boundary of the asymptotically de Sitter (previous) aeon we

will move to the formulation of the conformally regular version of the Einstein field equations for the present aeon. As will be seen, the approach described in the previous section will not work due to the requirement of an initial singularity.

Recall that $(\check{\mathcal{M}}, \check{g}_{ab})$ is a solution of the Einstein field equations with the energy-momentum tensor \check{T}_{ab} and positive cosmological constant $\check{\Lambda}$. Moreover, we assume the existence of a conformal extension of this spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ such that

$$\bar{g}_{ab} = \frac{1}{\omega^2} \check{g}_{ab} \quad (4.1)$$

and $\mathcal{Z}(\omega)$ corresponds to the initial (big bang) singularity. Since the physical energy-momentum tensor is singular at $\mathcal{Z}(\omega)$ let

$$\check{T}_{ab} = \omega^{-v} \bar{T}_{ab} \Rightarrow \check{T} = \omega^{-2-v} \bar{T}, \quad (4.2)$$

where $v \geq 0$ and \bar{T}_{ab} (the unphysical energy-momentum tensor) is regular everywhere. The Einstein field equations can be written as

$$\check{P}_{ab} = \frac{1}{2} \check{T}_{ab} + \frac{1}{6} \check{g}_{ab} (\check{\Lambda} - \check{T}). \quad (4.3)$$

In terms of conformal quantities this last expressions implies

$$\begin{aligned} \omega^2 \bar{P}_{ab} + 2 \bar{\nabla}_a \omega \bar{\nabla}_b \omega - \frac{1}{2} \bar{g}_{ab} \bar{\nabla}_c \omega \bar{\nabla}^c \omega - \omega \bar{\nabla}_a \bar{\nabla}_b \omega \\ = \frac{1}{2} \omega^{2-v} \left(\bar{T}_{ab} - \frac{1}{3} \bar{g}_{ab} \bar{T} \right) + \frac{1}{6} \bar{g}_{ab} \omega^4 \check{\Lambda}. \end{aligned} \quad (4.4)$$

Remark 4. Equation (4.4) can be used to derive regularity conditions on the unphysical energy-momentum tensor \bar{T}_{ab} at $\mathcal{Z}(\omega)$. After taking its trace one obtains

$$\omega^{2-v} \bar{T} \doteq 0. \quad (4.5)$$

Similarly, if $\bar{n}_a := \bar{\nabla}_a \omega$ is a normal vector to the level set of ω , then

$$2 \bar{n}_a \bar{n}_b - \frac{1}{2} \bar{g}_{ab} \bar{n}_c \bar{n}^c \doteq \frac{1}{2} \omega^{2-v} \left(\bar{T}_{ab} - \frac{1}{3} \bar{g}_{ab} \bar{T} \right) \Big|_{\omega=0}. \quad (4.6)$$

If $\mathcal{Z}(\omega)$ is a spacelike hypersurface, then after contracting the equality above with the metric \bar{h}_{ab} [intrinsic to $\mathcal{Z}(\omega)$] one gets

$$\omega^{2-v} \bar{T}_{\perp\perp} \Big|_{\omega=0} \geq 0. \quad (4.7)$$

Because of the trace-free character of the ω^{-2} terms in Eq. (4.3) written in terms of conformal quantities, a similar

regularization procedure as Friedrich's conformal Einstein field equations cannot be employed here to obtain a system that is regular on $\mathcal{Z}(\omega)$. However, in the sequel we will argue that if the physical energy-momentum tensor of the matter is restricted in a certain way, then an approach that uses the Bach tensor can be employed to produce a suitable set of regular conformal equations.

A. The Bach equation

The Bach tensor \check{B}_{ab} can be related to the energy-momentum tensor \check{T}_{ab} with the use of Einstein field equations. We have

$$\begin{aligned} \check{B}_{ab} = \frac{1}{2} \check{\square} \check{T}_{ab} - \frac{1}{6} \check{g}_{ab} \check{\square} \check{T} + \frac{1}{6} \check{\nabla}_a \check{\nabla}_b \check{T} - \check{T}_a{}^c \check{T}_{bc} \\ + \frac{1}{6} \check{g}_{ab} \check{T} (\check{\Lambda} - \check{T}) + \check{C}_{abcd} \check{T}^{cd} + \frac{1}{4} \check{g}_{ab} \check{T}_{cd} \check{T}^{cd} \\ - \frac{2}{3} \check{T}_{ab} (\check{\Lambda} - \check{T}). \end{aligned} \quad (4.8)$$

It can be verified that for $\check{g}_{ab} \mapsto \bar{g}_{ab} = \omega^{-2} \check{g}_{ab}$ the Bach tensor transforms as

$$\check{B}_{ab} \mapsto \bar{B}_{ab} = \omega^2 \check{B}_{ab}, \quad (4.9)$$

so the initial step in obtaining the conformal field equations which describe an initial singularity spacetime is to verify whether the right-hand side of (4.8) multiplied by ω^2 and written in terms of the unphysical energy-momentum tensor is regular on $\mathcal{Z}(\omega)$. Unfortunately, a simple calculation reveals that this is not the case for generic \bar{T}_{ab} . Nevertheless, driven by an example concerning certain class of electrovacuum spacetimes (as discussed in the Appendix) where the source term in the Bach equation is regular everywhere, we will proceed with the analysis of the constraints assuming that the conformal equation has the form

$$\bar{B}_{ab} = S_{ab}, \quad (4.10)$$

where S_{ab} is a (regular) source term. It should be stressed, however, that although Eq. (4.10) bears close resemblance to the equation appearing in the conformally invariant theories of gravity (see [21] for a recent developments), here we take the point of view that it is a consequence of the Einstein field equations.

Remark 5. The source term S_{ab} vanishes for conformally flat and conformally Einstein spacetimes.

The Bach tensor is equivalent, in dimension 4, to the Fefferman-Graham obstruction tensor—see [22]. Hence, an approach based on the equation $B_{ab} = 0$ has been used to generalize Friedrich's approach to higher-dimensional asymptotically de Sitter spacetimes [12,13]—see also [23]. In our analysis the conformal extension of the present aeon

is a solution of the Bach equation with a source term, so a similar approach can be used to show the well-posedness of this system. The relation between its solution and the physical big bang singularity spacetime can be achieved with the use of the nonlinear wave-type equation for the conformal factor ω .

B. The regularized Bach equation constraints on the conformal boundary \mathcal{I}^-

Following the discussion in the previous subsection we will assume that the conformal version of the Einstein field equations of $(\check{\mathcal{M}}, \check{g}_{ab})$ are given by

$$\begin{aligned} \bar{B}_{ab} &= S_{ab}, \\ \check{\square}\omega &= \bar{J}\omega - \frac{2}{3}\omega^3\check{\Lambda} + \frac{1}{6}\bar{T}\omega^{1-v}, \end{aligned} \quad (4.11)$$

where S_{ab} is a regular source term which depends on the matter content of the spacetime and the wave-type equation for the conformal factor ω arises from taking a trace of (4.4).

The constraints on the initial hypersurface implied by the Bach equation can be obtained from the normal-normal and normal-intrinsic components of Eq. (4.10). More precisely, one has that

$$\bar{B}_{\perp\perp} = S_{\perp\perp}, \quad \bar{B}_{a\perp}^{\perp} = S_{a\perp}^{\perp}. \quad (4.12)$$

In a gauge where $\bar{\nabla}_{\perp}\bar{n}_a = 0$ on $\mathcal{Z}(\omega)$ we have (see e.g., [24])

$$\begin{aligned} \bar{B}_{\perp\perp} &\doteq \bar{D}^a\bar{D}^b E_{ab} - \bar{D}^c(C_{cab\perp}^{\perp}\bar{K}^{ab}) + \bar{P}_{ab}^{\perp}E^{ab} + \bar{K}^{ab}\bar{A}_{a\perp b}^{\perp}, \\ \bar{B}_{a\perp}^{\perp} &\doteq -\bar{D}^b\bar{A}_{a\perp b}^{\perp} - (\bar{D}^b E_{cb})\bar{K}_a^c \\ &\quad + \bar{K}_a^d\bar{K}^{bc}C_{\perp bcd} + \bar{P}^{\perp cd}C_{dac\perp}^{\perp} + \bar{P}_{\perp b}^{\perp}E_a^b \end{aligned} \quad (4.13)$$

where $\bar{h}_{ab} := \bar{g}_{ab} + \bar{n}_a\bar{n}_b$ and \bar{K}_{ab} are the first two fundamental forms of $\mathcal{Z}(\omega)$ and \bar{D}_a the Levi-Civita connection of \bar{h}_{ab} . Moreover, $E_{ab} = C_{a\perp b\perp}$ is the electric part of the Weyl tensor. To avoid introducing new notation symbol \doteq will denote equality on $\mathcal{Z}(\omega)$ here.

The boundary conditions for the nonlinear equation for the conformal factor ω consist of prescribing $\omega = 0$ on the past conformal boundary \mathcal{I}^- and the condition that its gradient has a negative unit length there (which corresponds to \mathcal{I}^- being a spacelike hypersurface).

The discussion in the previous paragraphs can be summarized in the following:

Theorem 2.—Let $(\check{\mathcal{M}}, \check{g}_{ab})$ be a spacetime with conformal extension $(\bar{\mathcal{M}}, \bar{g}_{ab})$ described by the conformal field equations

$$\begin{aligned} \bar{B}_{ab} &= S_{ab}, \\ \check{\square}\omega &= \bar{J}\omega - \frac{2}{3}\omega^3\check{\Lambda} + \frac{1}{6}\bar{T}\omega^{1-v}, \end{aligned} \quad (4.14)$$

where $\bar{g}_{ab} = \omega^{-2}\check{g}_{ab}$ and S_{ab} is a regular tensor which depends on the matter content on the spacetime. Moreover, the initial conditions for the conformal factor ω are as follows:

$$\omega = 0, \quad \bar{g}^{ab}\bar{\nabla}_a\omega\bar{\nabla}_b\omega = -1 \text{ on } \mathcal{I}^-(\check{\mathcal{M}}), \quad (4.15)$$

i.e., $\mathcal{I}^-(\check{\mathcal{M}}) \equiv \mathcal{Z}(\omega)$ and this hypersurface is spacelike. Then, the constraints induced on the past conformal boundary $\mathcal{I}^-(\check{\mathcal{M}})$ by the first equation are

$$\begin{aligned} S_{\perp\perp} &= \bar{D}^a\bar{D}^b E_{ab} - \bar{D}^c(C_{cab\perp}^{\perp}\bar{K}^{ab}) + \bar{P}_{ab}^{\perp}E^{ab} + \bar{K}^{ab}\bar{A}_{a\perp b}^{\perp}, \\ S_{a\perp}^{\perp} &= -\bar{D}^b\bar{A}_{a\perp b}^{\perp} - (\bar{D}^b E_{cb})\bar{K}_a^c \\ &\quad + \bar{K}_a^d\bar{K}^{bc}C_{\perp bcd} + \bar{P}^{\perp cd}C_{dac\perp}^{\perp} + \bar{P}_{\perp b}^{\perp}E_a^b, \end{aligned} \quad (4.16)$$

where $E_{ab} := C_{\perp a\perp b}$ is the electric part of the Weyl tensor.

V. MATCHING CONDITIONS FOR THE CONFORMAL BOUNDARY BETWEEN AEONS

In the previous sections, we discussed two sets of constraints on conformal boundaries. The first one holds on the future conformal boundary \mathcal{I}^+ of the previous aeon $(\hat{\mathcal{M}}, \hat{g}_{ab})$ (asymptotically de Sitter spacetime) and the second on the past conformal boundary \mathcal{I}^- (the big bang singularity) of the present aeon $(\check{\mathcal{M}}, \check{g}_{ab})$. In the CCC scenario, those two hypersurfaces are identified and form a common boundary between the aeons. In the sequel, we will study the consequences of this assumption, i.e., use the constraints induced on the future conformal boundary of the previous aeon to simplify the Bach equation constraints on the past conformal boundary of the present aeon.

The constraints coming from the conformal Einstein field equations on the future conformal boundary of the previous aeon imply that \mathcal{I}^+ is an umbilic hypersurface and the projections of P_{ab} can be expressed in terms of $P_{ab}^{(3)}$, h_{ab} , and $D_a s$ as in (3.24). If we use this information in the Bach equation constraints (4.13), then

$$\begin{aligned} S_{\perp\perp} &\doteq \bar{D}^a\bar{D}^b E_{ab} + \bar{P}_{ab}^{(3)}E^{ab}, \\ S_{a\perp}^{\perp} &\doteq -\bar{D}^b\bar{A}_{a\perp b}^{\perp} + \sqrt{\frac{3}{\Lambda}}s\bar{D}^b E_{ab} + \sqrt{\frac{3}{\Lambda}}E_a^b\bar{D}_b s, \end{aligned} \quad (5.1)$$

where the fact that $C_{abc\perp}^{\perp}$ vanishes on the umbilic hypersurface has been used.

A further simplification occurs after assuming that the physical Cotton tensor vanishes at \mathcal{I}^+ , e.g., $q \geq 1$ in the conformal transformation rule (3.6). Then $C_{abcd} \doteq 0$

(see e.g., [25] and Theorem 10.3 in [15]). Moreover, on the level set of Ω an equality $A_{abc} = \hat{A}_{abc} - \sigma d_{\perp abc}$ holds, so in that case

$$A_{ab\perp}^{\top} \doteq \sqrt{\frac{\hat{\Lambda}}{3}} e_{ab}. \quad (5.2)$$

Hence, the Bach equation constraints on the transition hypersurface between the aeons in the CCC reduce now to

$$\begin{aligned} S_{\perp\perp} &\doteq 0, \\ S_{a\perp}^{\top} &\doteq \sqrt{\frac{\hat{\Lambda}}{3}} \bar{D}^b e_{ab}. \end{aligned} \quad (5.3)$$

The main result of this paper can be now stated in the form of the full set of the matching conditions that are imposed on the matching hypersurface between the aeons in the CCC scenario with the regular Bach equation.

Theorem 3.—Let $(\hat{\mathcal{M}}, \hat{g}_{ab})$ be an asymptotically de Sitter spacetime with cosmological constant $\hat{\Lambda}$ and the future conformal boundary \mathcal{I}^+ where the Cotton tensor vanishes. Assume that $(\check{\mathcal{M}}, \check{g}_{ab})$ is a spacetime with big bang singularity located on the past conformal boundary \mathcal{I}^- , which is a solution of the regular conformal Bach equations. Then, if $(\hat{\mathcal{M}}, \hat{g}_{ab})$ and $(\check{\mathcal{M}}, \check{g}_{ab})$ are subsequent aeons in the conformal cyclic cosmology scenario the constraints

$$\begin{aligned} D^b e_{ab} &\doteq -Q_{\perp\perp a}^{\top} \doteq \sqrt{\frac{3}{\hat{\Lambda}}} S_{a\perp}^{\top}, & Q_{\perp bc}^{\top} &\doteq 0, \\ b_{bca} &\doteq \sqrt{\frac{3}{\hat{\Lambda}}} A_{abc}^{(3)}, & S_{\perp\perp} &\doteq 0 \end{aligned} \quad (5.4)$$

have to be satisfied on the common conformal boundary between $(\hat{\mathcal{M}}, \hat{g}_{ab})$ and $(\check{\mathcal{M}}, \check{g}_{ab})$. The quantities e_{ab} and b_{abc} correspond to the electric and magnetic part of the rescaled Weyl tensor d_{abcd} , Q_{abc} is the rescaled Cotton tensor, and $A_{abc}^{(3)}$ is the Cotton tensor of the conformal boundary. Tensor S_{ab} denotes a source term in the regular Bach equation.

Remark 6. The source term S_{ab} in the Bach equation is regular everywhere by assumption. Hence, the constraints on the common boundary between the aeons imply that the divergence of the electric part of the rescaled Weyl tensor e_{ab} and the $Q_{\perp\perp a}^{\top}$ component of the rescaled Cotton tensor are regular there.

A. Initial values of the (electro)magnetic fields

One of the questions posed in [14] is related with the initial value of the magnetic field at the conformal boundary of the present aeon $(\check{\mathcal{M}}, \check{g}_{ab})$. We will show that the constraints obtained in Theorem 3 can be naturally used

to partially determine this value from the matter content of the previous aeon.

Suppose that $(\check{\mathcal{M}}, \check{g}_{ab})$ is the electrovacuum spacetime which satisfies the regularized Bach equations. Then,

$$\bar{B}_{ab} = X_{ab} + \frac{2}{3} \check{\Lambda} \omega^2 \check{T}_{ab}, \quad (5.5)$$

(see the Appendix), where X_{ab} is a regular tensor on the conformal extension $(\check{\mathcal{M}}, \bar{g}_{ab})$. If we assume that $\bar{T}_{ab} = \omega^2 \check{T}_{ab}$, then the constraints (5.4) will be equivalent with

$$\frac{2}{3} \check{\Lambda} \bar{T}_{\perp\perp} + X_{\perp\perp} \doteq 0 \quad (5.6)$$

and

$$\frac{2}{3} \check{\Lambda} \bar{T}_{\perp a}^{\top} + X_{\perp a}^{\top} \doteq -\sqrt{\frac{\hat{\Lambda}}{3}} Q_{\perp\perp a}^{\top}. \quad (5.7)$$

Before analyzing Eq. (5.7) let us first focus our attention on (5.6). If we use the standard expression of the electromagnetic energy-momentum tensor in terms of the electric and magnetic field, $\check{\mathcal{E}}^a$ and $\check{\mathcal{B}}^a$ respectively, then (5.6) will read

$$\frac{2}{3} \check{\Lambda} (\bar{\mathcal{E}}_a \bar{\mathcal{E}}^a + \bar{\mathcal{B}}_a \bar{\mathcal{B}}^a) + X_{\perp\perp} \doteq 0 \quad (5.8)$$

where $\bar{\mathcal{E}}_a = \omega \check{\mathcal{E}}_a$ and $\bar{\mathcal{B}}_a = \omega \check{\mathcal{B}}_a$ are the unphysical electric and magnetic fields. Hence, this equation is a constraint on the energy density of the electromagnetic field on the conformal boundary.

To see how (5.7) can be viewed as a constraint on the initial values of $\check{\mathcal{E}}^a$ and $\check{\mathcal{B}}^a$ suppose that the energy-tensor of the previous aeon has the form of electromagnetic field, i.e.,

$$\hat{T}_{ab} = 2\hat{F}_{ac}\hat{F}_b{}^c - \frac{1}{2}\hat{g}_{ab}\hat{F}_{cd}\hat{F}^{cd}. \quad (5.9)$$

If we introduce its unphysical counterpart in a way which makes the conservation equation $\hat{\nabla}^b \hat{T}_a{}^b = 0$ conformally invariant (see Remark 2), i.e.,

$$T_{ab} = \frac{1}{\Omega^2} \hat{T}_{ab} \quad (5.10)$$

then (5.7) reads

$$\frac{2}{3} \check{\Lambda} \bar{T}_{\perp a}^{\top} + X_{\perp a}^{\top} \doteq -\frac{1}{3} \hat{\Lambda} T_{\perp a}^{\top} \quad (5.11)$$

or

$$-\frac{4}{3} \check{\Lambda} (\bar{\mathcal{E}} \times \bar{\mathcal{B}})_a + X_{\perp a}^{\top} \doteq \frac{2}{3} \hat{\Lambda} (\mathcal{E} \times \mathcal{B})_a \quad (5.12)$$

where \times denotes the vector product and $\mathcal{E}^a = \Omega^{-1}\hat{\mathcal{E}}^a$ and $\mathcal{B}^a = \Omega^{-1}\hat{\mathcal{B}}^a$ are the unphysical electric and magnetic fields in the previous aeon. Hence, we see that the initial value of the vector product of \mathcal{E}^a and \mathcal{B}^a is determined by an analogous quantity computed with the use of the unphysical fields in the previous aeon.

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APPENDIX: ELECTROVACUUM SPACETIMES WITH REGULAR CONFORMAL BACH TENSOR

Following the discussion at the end of Sec. IV, we present an example of electrovacuum spacetimes that admit regular conformal Bach equation. This is based on the analysis of the Chevreton tensor given in [26]. As the equations simplify greatly with the use of spinorial formalism, we will employ it here. The signature of the metric will be changed to $(+, -, -, -)$, which is usual in this setting. The other spinorial conventions and notation follow from [15] with the exception that the complex conjugation will be denoted by $'$.

1. Bach tensor for source-free electromagnetic fields

Let \check{T}_{ab} be a source-free electromagnetic energy-momentum tensor. It can be expressed via Maxwell spinor $\check{\phi}_{AB}$ as

$$\check{T}_{ab} = \check{\phi}_{AB}\check{\phi}'_{A'B'}. \quad (\text{A1})$$

Since

$$\check{\phi}_A{}^B\check{\phi}_{BC} = \frac{1}{2}\check{\epsilon}_{AB}\check{\phi}_{CD}\check{\phi}^{CD}, \quad (\text{A2})$$

then

$$\check{T}_a{}^c\check{T}_{bc} - \frac{1}{4}\check{g}_{ab}\check{T}_{cd}\check{T}^{cd} = 0 \quad (\text{A3})$$

and the Bach tensor reads

$$\check{B}_{ab} = \frac{1}{2}\check{\square}\check{T}_{ab} + \check{C}_{abcd}\check{T}^{cd} - \frac{2}{3}\check{\Lambda}\check{T}_{ab}. \quad (\text{A4})$$

The Maxwell equations are equivalent to $\check{\nabla}'_{A'}\check{\phi}_{AB} = 0$, so

$$0 = \check{\nabla}_{CA'}\check{\nabla}'_{A'}\check{\phi}^A{}_B = -\frac{1}{2}\check{\square}\check{\phi}_{CA} + \check{\square}_{CA}\check{\phi}^A{}_B \quad (\text{A5})$$

or with the use of $\check{\square}_{CA}\check{\phi}^A{}_B = \Psi_{CBAD}\check{\phi}^{AD} + \frac{2}{3}\check{\Lambda}\check{\phi}_{BC}$,

$$\check{\square}\check{\phi}_{AB} = 2\Psi_{ABCD}\check{\phi}^{CD} + \frac{4}{3}\check{\Lambda}\check{\phi}_{AB}. \quad (\text{A6})$$

This relation can be used to express $\check{\square}\check{T}_{ab}$ in terms of lower order derivatives, i.e.,

$$\check{\square}(\check{\phi}_{AB}\check{\phi}'_{A'B'}) = 2\check{\nabla}_{CC'}\check{\phi}_{AB}\check{\nabla}^{CC'}\check{\phi}'_{A'B'} - 2\check{C}_{abcd}\check{T}^{cd} + \frac{8}{3}\check{\Lambda}\check{T}_{ab}. \quad (\text{A7})$$

Ultimately, the Bach tensor equals

$$\check{B}_{ab} = \check{\nabla}_{CC'}\check{\phi}_{AB}\check{\nabla}^{CC'}\check{\phi}'_{A'B'} + \frac{2}{3}\check{\Lambda}\check{\phi}_{AB}\check{\phi}'_{A'B'} \quad (\text{A8})$$

or, when written in terms of tensorial quantities (compare with [27]),

$$\check{B}_{ab} = \frac{1}{2}\check{g}_{ab}\check{\nabla}_c\check{F}_{de}\check{\nabla}^c\check{F}^{de} - 2\check{\nabla}_c\check{F}_{ad}\check{\nabla}^c\check{F}_b{}^d + \frac{2}{3}\check{\Lambda}\check{T}_{ab} \quad (\text{A9})$$

where

$$\check{F}_{ab} = \frac{1}{2}\check{\phi}_{AB}\epsilon_{A'B'} + \frac{1}{2}\check{\phi}'_{A'B'}\epsilon_{AB} \quad (\text{A10})$$

is the Maxwell tensor.

The first term on the right-hand side of (A8) will be singular when the equation is written in terms of unphysical tensors associated with $\bar{g}_{ab} = \omega^{-2}\check{g}_{ab}$ and $\bar{\phi}_{AB} = \omega\check{\phi}_{AB}$ —which makes Maxwell equations conformally invariant. Hence, in order for the conformal Bach equation to be regular, an assumption has to be made about the Maxwell spinor. The result in [26] showed that the simple condition $\check{\nabla}_{CC'}\check{\phi}_{AB}\check{\nabla}^{CC'}\check{\phi}'_{A'B'} = 0$ restricts the spacetime to the Petrov type **N** or **O** and vanishing cosmological constant, so in that case the Bach tensor vanishes.

However, if we consider the following equation

$$\check{\nabla}_{CC'}\check{\phi}_{AB}\check{\nabla}^{CC'}\check{\phi}'_{A'B'} = f\check{\phi}_{AB}\check{\phi}'_{A'B'} \quad (\text{A11})$$

where f is a regular function on the conformal extension \bar{M} , we get a nonsingular and nontrivial Bach equation

$$\bar{B}_{ab} = \left(f + \frac{2}{3}\check{\Lambda}\right)\bar{\phi}_{AB}\bar{\phi}'_{A'B'}. \quad (\text{A12})$$

We see that one possibility for (A11) to be satisfied is

$$\check{\nabla}_{CC'}\check{\phi}_{AB} = \check{\phi}_{(AB}\check{t}_{C)C'} \quad (\text{A13})$$

where \check{t} is one of the null tetrad vectors. The other valid approach would be to consider a generalization of condition (24) from [26], where the homogeneous case was considered ($f = 0$). For example,

$$\check{\nabla}_{CC'}\check{\phi}_{AB} = \eta o_A o_B o_C l_{C'} \quad (\text{A14})$$

where o_A, ι_A are the spin basis elements and η a complex function, could in principle be used to achieve the desired result. However, it can be verified that this assumption leads to a vanishing electromagnetic field if the cosmological constant is not zero. On the other hand, a more general approach with

$$\check{\nabla}_{CC'}\check{\phi}_{AB} = \eta o_A o_B o_C o_{C'} + \chi \iota_A \iota_B \iota_C \iota_{C'} \quad (\text{A15})$$

where χ is another complex function, still leads to the scenario where null vector $l^a = o^A o^{A'}$ is geodetic and shear-free, but does not correspond to the Weyl spinor of type N or O .

It should be stressed that the regularized Bach equation (A12) is only a necessary condition for an electrovacuum spacetime to fit into the CCC scenario described in Sec. V. It is not sufficient, because of the requirement that the present aeon has a big bang singularity.

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