# Second-order energy-momentum tensor of a scalar field

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We investigate the second-order effective energy-momentum tensor (2EMT) constructed by the quadratic terms of the linear scalar cosmological perturbations while the universe is dominated by a scalar field. We show that 2EMT is gauge dependent. We then study 2EMT in three (longitudinal, spatially flat, and comoving) gauge conditions in the slow-roll stage of inflation. We find that 2EMT exhibits an effective fluid of $w = -1/3$ on superhorizon scales in all of those gauge conditions.

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## I. INTRODUCTION

An inflationary universe is a successful scenario so far, according to the observation of the cosmic microwave background (CMB). Cosmological perturbations produced during the inflationary period explain the CMB data successfully. As the precision of the observational technique improves, the higher-order perturbations beyond the linear order have attracted more attention. Thus, it is increasingly more important to understand the higher-order nature of cosmological perturbations to test the paradigm of inflation and to discriminate among different models of inflation. The second-order cosmological perturbations during inflation were investigated in Refs. [1,2], in particular, the second-order effective energy-momentum tensor (2EMT) constructed by the quadratic terms of the linear perturbations was considered. This has been studied as a way of a backreaction of perturbations on the background. The gauge invariance of 2EMT was also discussed.

Recently 2EMT of cosmological perturbations produced by a perfect fluid was investigated in the Friedmann universe [3]. The gauge dependence of 2EMT was presented in general. The 2EMT in three gauge conditions (longitudinal, spatially flat, and comoving gauges) was investigated in the matter- and radiation-dominated epochs. No convergence of 2EMT was observed, i.e., each gauge condition gives a different 2EMT in the given epoch.

The story of the gauge invariance of the Einstein's equation can be discussed as below. One can expand the Einstein's equation by order as (let $8\pi G = 1$ in this section)

$$G_{\mu\nu} = T_{\mu\nu}$$
$$\Rightarrow G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \cdots. \quad (1)$$

The equality is satisfied order by order, $G^{(n)}_{\mu\nu} = T^{(n)}_{\mu\nu}$. Let us consider the first-order equation, $G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu}$. The Einstein tensor $G^{(1)}_{\mu\nu}$ and the energy-momentum tensor $T^{(1)}_{\mu\nu}$ are not gauge invariant by themselves. By adding terms on both sides of the first-order equation, one can construct the equation in a gauge-invariant form,

$$\bar{G}^{(1)}_{\mu\nu} = \bar{T}^{(1)}_{\mu\nu}, \quad (2)$$

where the gauge-invariant quantities are given by

$$\bar{G}^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu} + \delta G^{(1)}_{\mu\nu}, \qquad \bar{T}^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} + \delta T^{(1)}_{\mu\nu}. \quad (3)$$

Here, $\delta G^{(1)}_{\mu\nu}$ and $\delta T^{(1)}_{\mu\nu}$ are the terms added to make gauge-invariant quantities.

Similarly, the second-order equation can be put into the gauge-invariant form, $\bar{G}^{(2)}_{\mu\nu} = \bar{T}^{(2)}_{\mu\nu}$. However, each quantity is composed in a more complicated way,

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$$\bar{G}^{(2)}_{\mu\nu} = G^{(2)}_{\mu\nu} + \delta G^{(2)}_{\mu\nu} = G^{(1)}_{\mu\nu}[g^{(2)}] + \underline{G^{(2)}_{\mu\nu}[g^{(1)}]} + \delta G^{(1)}_{\mu\nu}[g^{(2)}] + \delta G^{(2)}_{\mu\nu}[g^{(1)}], \quad (4)$$

$$\bar{T}^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu} + \delta T^{(2)}_{\mu\nu} = T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \underline{T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}]} + \delta T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \delta T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}], \quad (5)$$

where $g^{(n)}$ and $\psi^{(n)}$ represent the metric and the matter perturbations of $n$th order. The second-order quantity consists of two parts; one is linear order of the second-order perturbations (e.g., $G^{(1)}_{\mu\nu}[g^{(2)}]$), and the other is quadratic order of the linear perturbations (e.g., $G^{(2)}_{\mu\nu}[g^{(1)}]$).

2EMT is constructed by the underlined terms in Eqs. (4) and (5),

$$T^{(2,\mathrm{eff})}_{\mu\nu} \equiv T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}]. \quad (6)$$

Considering $\bar{G}^{(2)}_{\mu\nu}$ and $\bar{T}^{(2)}_{\mu\nu}$ are gauge invariant individually, it is difficult to expect $T^{(2,\mathrm{eff})}_{\mu\nu}$ in which only a collection of parts of them are gauge invariant.

In Ref. [3], we derived 2EMT of a barotropic fluid and showed that it is indeed gauge dependent. In this work, we study 2EMT of a minimally coupled canonical scalar field. We will derive 2EMT and show its gauge dependence. We shall try to impose three gauge conditions (longitudinal, spatially flat, and comoving gauges), and investigate the convergence of 2EMT in the slow-roll background during inflation. Very interestingly, we find that 2EMT in all gauge choices converges to an effective fluid in the long-wavelength limit. This convergence is absent in the barotropic fluid in the Friemann universe studied in our previous work [3].

## II. EINSTEIN'S EQUATION AND 2EMT OF SCALAR FIELD

In this section, we derive 2EMT starting from linear perturbations. We consider the general metric

$$ds^2 = a^2(\eta)[-(1+2A)d\eta^2 - 2B_i d\eta dx^i + (\delta_{ij} + 2C_{ij})dx^i dx^j], \quad (7)$$

where the metric perturbations $A$, $B_i$, and $C_{ij}$ are to be expanded in all orders. The Einstein tensor $G_{\mu\nu}$ up to second order was presented in our previous work [3], which we will not duplicate here. In this section, we present the energy-momentum tensor of a minimally coupled canonical scalar field $\phi$.

### A. Energy-momentum tensor

The energy-momentum tensor of a minimally coupled scalar field $\phi$ is given by the perturbed matter action $S_\phi$ with respect to the metric $g^{\mu\nu}$,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} - V(\phi)\right], \quad (8)$$

where $V(\phi)$ is the potential of $\phi$. We split $\phi$ into the unperturbed background $\phi_0$ and the perturbation $\delta\phi$,

$$\phi = \phi_0 + \delta\phi. \quad (9)$$

Then each component of the energy-momentum tensor is readily found as, up to second order in perturbations,

$$T_{00} = \frac{1}{2}(\phi_0')^2 + a^2V(\phi_0) + \phi_0'\delta\phi' + 2AV(\phi_0) + a^2V_\phi(\phi_0)\delta\phi + \frac{1}{2}(\delta\phi')^2 + \frac{1}{2}(\nabla\delta\phi)^2 + \frac{1}{2}a^2V_{\phi\phi}(\phi_0)\delta\phi^2 + 2a^2AV_\phi(\phi_0)\delta\phi + \frac{1}{2}B_kB_k(\phi_0')^2 - \phi_0'B_k\delta\phi_{,k}, \quad (10)$$

$$T_{0i} = \phi_0'\delta\phi_{,i} - \frac{1}{2}B_i(\phi_0')^2 + a^2B_iV(\phi_0) + \delta\phi'\delta\phi_{,i} - B_i\phi_0'\delta\phi' + AB_i(\phi_0')^2 + a^2B_iV_\phi(\phi_0)\delta\phi, \quad (11)$$

$$T_{ij} = \delta_{ij}\left[\frac{1}{2}(\phi_0')^2 - a^2V(\phi_0)\right] + \delta_{ij}[\phi_0'\delta\phi' - A(\phi_0')^2 - a^2V'(\phi_0)\delta\phi] + C_{ij}[(\phi_0')^2 - 2a^2V(\phi_0)] + \delta\phi_{,i}\delta\phi_{,j} + 2C_{ij}[\phi_0'\delta\phi' - A(\phi_0')^2 - a^2V_\phi(\phi_0)\delta\phi] + \delta_{ij}\left[\frac{1}{2}(\delta\phi')^2 - \frac{1}{2}(\nabla\delta\phi)^2 + \left(2A^2 - \frac{1}{2}B_kB_k\right)(\phi_0')^2 - 2A\phi_0'\delta\phi' + B_k\phi_0'\delta\phi_{,k} - \frac{1}{2}a^2V_{\phi\phi}(\phi_0)\delta\phi^2\right], \quad (12)$$

where a prime denotes a derivative with respect to the conformal time $\eta$, $V_\phi = \partial V/\partial\phi|_0$ and $V_{\phi\phi} = \partial^2V/\partial\phi^2|_0$. From now on, the potential is always evaluated with respect to the background value, i.e., $V = V(\phi_0)$ and so on.

### B. Einstein's equation

We can construct order by order the Einstein's equation by equating each component of the Einstein tensor and the corresponding component of the energy-momentum tensor (10)–(12).

#### 1. Background equations

The background equations, which contain no perturbations but only background variables, are given by the 00 and $ij$ components,

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left[\frac{1}{2}(\phi_0{}')^2 + a^2 V\right], \tag{13}$$

$$2\mathcal{H}' + \mathcal{H}^2 = 8\pi G\left[-\frac{1}{2}(\phi_0{}')^2 + a^2 V\right], \tag{14}$$

where $\mathcal{H} \equiv (da/d\eta)/a$, and for later use, $H \equiv (da/dt)/a$ with $dt = ad\eta$. The equation for $\phi_0$ can be found from the energy-momentum tensor conservation,

$$\phi_0'' + 2\mathcal{H}\phi_0' + a^2 V_\phi = 0. \tag{15}$$

#### 2. Linearized equations

Using the background equations (13)–(15), we find the most convenient form of the first-order equations,

$$\begin{aligned} &2(\mathcal{H}' + 2\mathcal{H}^2)A - 2\mathcal{H}B_{k,k} - 2\mathcal{H}C_{kk}' + \Delta C_{kk} - C_{kl,kl} \\ &\quad = 8\pi G[(\phi_0'\delta\phi)' + 2\mathcal{H}\phi_0'\delta\phi - 2\phi_0'\delta\phi'], \end{aligned} \tag{16}$$

$$2\mathcal{H}A_{,i} + B_{[i,k]k} + C_{ik,k}' - C_{kk,i}' = 8\pi G\phi_0'\delta\phi_{,i}, \tag{17}$$

$$\begin{aligned} &\delta_{ij}[2\mathcal{H}A' + 2(\mathcal{H}' + 2\mathcal{H}^2)A + \Delta A - B_{k,k}' - 2\mathcal{H}B_{k,k} \\ &\quad - C_{kk}'' - 2\mathcal{H}C_{kk}' + \Delta C_{kk} - C_{kl,kl}] \\ &\quad - A_{,ij} + B_{(i,j)}' + 2\mathcal{H}B_{(i,j)} + C_{ij}'' + 2\mathcal{H}C_{ij}' \\ &\quad - \Delta C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} \\ &= 8\pi G\delta_{ij}[(\phi_0'\delta\phi)' + 2\mathcal{H}\phi_0'\delta\phi]. \end{aligned} \tag{18}$$

The equation of motion for the scalar-field perturbation $\delta\phi$ is given by

$$\begin{aligned} &\delta\phi'' + 2\mathcal{H}\delta\phi' - \Delta\delta\phi + a^2 V_{\phi\phi}\delta\phi - (A' - B_{k,k} - C_{kk}')\phi_0' \\ &\quad + 2a^2 V_\phi A = 0. \end{aligned} \tag{19}$$

#### 3. Gauge-invariant variables

Now, we consider only scalar components of the metric perturbations such that they are written in terms of four functions,

$$A = \alpha, \qquad B_i = \beta_{,i}, \qquad C_{ij} = -\psi\delta_{ij} + E_{,ij}. \tag{20}$$

These functions transform under the infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ as

$$\alpha \to \alpha - \xi^{0\prime} - \mathcal{H}\xi^0, \tag{21}$$

$$\beta \to \beta - \xi^0 + \xi', \tag{22}$$

$$\psi \to \psi + \mathcal{H}\xi^0, \tag{23}$$

$$E \to E - \xi. \tag{24}$$

Then, it can be readily shown that the following variables are gauge invariant up to first order,

$$\overline{\delta\phi} = \delta\phi - \phi_0' Q, \tag{25}$$

$$\Phi = \alpha - Q' - \mathcal{H}Q, \tag{26}$$

$$\Psi = \psi + \mathcal{H}Q, \tag{27}$$

where $Q$ represents a time translation defined by

$$Q = \beta + E'. \tag{28}$$

Now, Eqs. (16)–(18) enable us to express $\overline{\delta\phi}$ and $\overline{\delta\phi}'$ in terms of the gauge-invariant variable $\Psi = \Phi$,

$$\overline{\delta\phi} = \frac{\Psi' + \mathcal{H}\Psi}{4\pi G\phi_0'}, \tag{29}$$

$$\overline{\delta\phi}' = \frac{\Delta\Psi - K\Psi' - L\Psi}{4\pi G\phi_0'}, \tag{30}$$

where $K$ and $L$ depend only on the background variables,

$$K = 3\mathcal{H} + a^2\frac{V_\phi}{\phi_0'}, \tag{31}$$

$$L = \mathcal{H}' + 2\mathcal{H}^2 + a^2\mathcal{H}\frac{V_\phi}{\phi_0'}. \tag{32}$$

Then, the linear equations (16)–(18) can be written solely in terms of $\Psi$ as

$$\Psi'' - \Delta\Psi + 2K\Psi' + 2L\Psi = 0. \tag{33}$$

### C. 2EMT

Since we have obtained the Einstein and the energy-momentum tensors up to second order, and the Einstein's equation up to linear order, we now can construct the 2EMT in a similar manner as we did in our previous work [3]. As we discussed in the Introduction, the second-order Einstein's equation is rearranged as

$$G_{\mu\nu}^{(1)}[g^{(2)}] = 8\pi G T_{\mu\nu}^{(1)}[g^{(2)}, \delta\phi^{(2)}] + 8\pi G T_{\mu\nu}^{(2,\mathrm{eff})} \quad \text{with}$$

$$T_{\mu\nu}^{(2,\mathrm{eff})} \equiv T_{\mu\nu}^{(2)}[g^{(1)}, \delta\phi^{(1)}] - \frac{G_{\mu\nu}^{(2)}[g^{(1)}]}{8\pi G}, \tag{34}$$

where the two quadratic terms make the 2EMT that describes the *backreaction* of the linear perturbations.

Using the results in the previous section, we can write $T_{\mu\nu}^{(2,\mathrm{eff})}$ in terms of the gauge-invariant variable $\Psi$ and the gauge-dependent variables $Q$ and $E$. After taking the spatial average denoted by bra-ket notations, $\tau_{\mu\nu} \equiv \langle T_{\mu\nu}^{(2,\mathrm{eff})} \rangle$, we obtain

$$
\begin{aligned}
8\pi G\tau_{00} = {} & \frac{1}{\mathcal{H}' - \mathcal{H}^2}\left\{ -\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle - 2(K+\mathcal{H})\langle\nabla\Psi'\cdot\nabla\Psi\rangle - [3(\mathcal{H}' - \mathcal{H}^2) + K^2 + a^2 V_{\phi\phi}]\langle(\Psi')^2\rangle \right. \\
& + (9\mathcal{H}' - 10\mathcal{H}^2 - 2L)\langle(\nabla\Psi)^2\rangle + 2\left[6\mathcal{H}(\mathcal{H}' - \mathcal{H}^2) - KL + 2(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} - a^2\mathcal{H}V_{\phi\phi}\right]\langle\Psi\Psi'\rangle \\
& \left. - \left[L^2 - 4\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + a^2\mathcal{H}^2 V_{\phi\phi}\right]\langle\Psi^2\rangle \right\} \\
& + 2\left\langle \left\{ a^2\frac{V_\phi}{\phi_0'}Q' + \left[3\mathcal{H}' + (4\mathcal{H}+K)a^2\frac{V_\phi}{\phi_0'} + a^2 V_{\phi\phi}\right]Q - \Delta Q + \Delta E' - 2\mathcal{H}\Delta E \right\}\Psi' \right\rangle \\
& + 2\left\langle \left\{ -\left(L + 6\mathcal{H}^2 - 2a^2\mathcal{H}\frac{V_\phi}{\phi_0'}\right)Q' + \left[2\mathcal{H}(L - 3\mathcal{H}') + (L - 2\mathcal{H}' + 4\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + \mathcal{H}a^2 V_{\phi\phi}\right]Q \right. \right. \\
& \left. \left. - \Delta Q' - (K - \mathcal{H})\Delta Q + 2\mathcal{H}\Delta E' + \Delta^2 E \right\}\Psi \right\rangle - (\mathcal{H}' + 2\mathcal{H}^2)\langle Q'^2\rangle - 2\left[\mathcal{H}(\mathcal{H}' - 4\mathcal{H}^2) + (\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'}\right]\langle Q'Q\rangle \\
& - \left\{ (\mathcal{H}' - \mathcal{H}^2)\left[4\mathcal{H}^2 + 8\mathcal{H}a^2\frac{V_\phi}{\phi_0'} + a^4\left(\frac{V_\phi}{\phi_0'}\right)^2 + a^2 V_{\phi\phi}\right] + 3\mathcal{H}'(\mathcal{H}' - 4\mathcal{H}^2)\right\}\langle Q^2\rangle \\
& - 2\mathcal{H}\langle\nabla Q'\cdot\nabla Q\rangle + \langle[4\mathcal{H}^2 Q + 2\mathcal{H}\Delta E - (\mathcal{H}' + 2\mathcal{H}^2)E']\Delta E'\rangle + 4\mathcal{H}\langle(\mathcal{H}Q' + \mathcal{H}'Q)\Delta E\rangle, \qquad (35)
\end{aligned}
$$

$$
8\pi G\tau_{0i} = 0, \qquad (36)
$$

$$
\begin{aligned}
8\pi G\tau_{ij} = {} & \delta_{ij}\left[\frac{1}{\mathcal{H}' - \mathcal{H}^2}\left\{\frac{1}{3}\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2\left(\frac{\mathcal{H}}{3} - K\right)\langle\nabla\Psi'\cdot\nabla\Psi\rangle + (\mathcal{H}' - \mathcal{H}^2 - K^2 + a^2 V_{\phi\phi})\langle(\Psi')^2\rangle \right.\right. \\
& + \left[\frac{1}{3}(11\mathcal{H}' - 10\mathcal{H}^2) - 2L\right]\langle(\nabla\Psi)^2\rangle + 2[\mathcal{H}a^2 V_{\phi\phi} - KL + 2(\mathcal{H}' - \mathcal{H}^2)(K + 2\mathcal{H})]\langle\Psi\Psi'\rangle \\
& \left.\left. + [4(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}' + 2\mathcal{H}^2 + L) - L^2 + \mathcal{H}^2 a^2 V_{\phi\phi}]\langle\Psi^2\rangle \right\} \right. \\
& + 2\left\langle \left[ Q'' + 3(3\mathcal{H} - K)Q' + (\mathcal{H}' + 12\mathcal{H}^2 + K^2 - 7\mathcal{H}K - a^2 V_{\phi\phi})Q + \frac{1}{3}\Delta E' - \frac{4}{3}(K - \mathcal{H})\Delta E \right]\Psi' \right\rangle \\
& + 2\left\langle \left\{ 4\mathcal{H}Q'' + (6\mathcal{H}' + 10\mathcal{H}^2 - 3L)Q' + \Delta Q' - (K - 3\mathcal{H})\Delta Q \right. \right. \\
& \left. \left. - [2\mathcal{H}'' + a^2\mathcal{H}V_{\phi\phi} - \mathcal{H}K^2 + (5\mathcal{H}' + 2\mathcal{H}^2)K - 11\mathcal{H}'\mathcal{H} - 9\mathcal{H}^3]Q - \frac{1}{3}\Delta E'' - \frac{2}{3}\mathcal{H}\Delta E' - \frac{4}{3}L\Delta E + \frac{2}{3}\Delta^2 E \right\}\Psi \right\rangle \\
& + \frac{2}{3}\langle\nabla Q''\cdot\nabla Q\rangle + \frac{2}{3}\langle(\nabla Q')^2\rangle + \frac{4\mathcal{H}}{3}\langle\nabla Q'\cdot\nabla Q\rangle + 2\mathcal{H}\langle Q''Q'\rangle - 2\mathcal{H}'\langle Q''Q\rangle - (\mathcal{H}' - 2\mathcal{H}^2)\langle Q'^2\rangle \\
& - 2[2\mathcal{H}'' + 3\mathcal{H}'\mathcal{H} + (\mathcal{H}' - \mathcal{H}^2)(K + \mathcal{H})]\langle Q'Q\rangle \\
& - \left\{ 4\mathcal{H}\mathcal{H}'' + 3\mathcal{H}'^2 + 4\mathcal{H}'\mathcal{H}^2 + (\mathcal{H}' - \mathcal{H}^2)\left[4\mathcal{H}^2 + a^2 V_{\phi\phi} + a^4\left(\frac{V_\phi}{\phi_0'}\right)^2\right]\right\}\langle Q^2\rangle \\
& - \frac{2}{3}\langle(2\mathcal{H}Q - 3\mathcal{H}E' + \Delta E)\Delta E''\rangle - \frac{1}{3}\langle[8\mathcal{H}Q' + 8(\mathcal{H}' + \mathcal{H}^2)Q - 3(\mathcal{H}' + 2\mathcal{H}^2)E' + 2\Delta E' + 4\mathcal{H}\Delta E]\Delta E'\rangle \\
& \left. - \frac{4}{3}\langle\{\mathcal{H}Q'' + 2(\mathcal{H}' + \mathcal{H}^2)Q' + 2[3\mathcal{H}\mathcal{H}' - \mathcal{H}^3 - (\mathcal{H}' - \mathcal{H}^2)K]Q\}\Delta E\rangle\right]. \qquad (37)
\end{aligned}
$$

As we can see, 2EMT does not depend only on the gauge-invariant variable $\Psi$ but also on the gauge-dependent variables $Q$ and $E$. Therefore, we conclude that the 2EMT for a scalar field is *gauge dependent* like for a perfect fluid in Ref. [3].

## III. 2EMT IN DIFFERENT GAUGES

In the previous section, we have seen that $\tau_{\mu\nu}$ for a scalar field is gauge dependent. Still, we may expect that even under different gauge conditions $\tau_{\mu\nu}$ behaves similarly in certain wavelength limits so that $\tau_{\mu\nu}$ describes effectively a well-behaved fluid independent of gauge choices. In this section, we examine this possibility under three gauge conditions: longitudinal, spatially flat, and comoving gauges. We consider the long- and short-wavelength limits of $\tau_{\mu\nu}$ in each gauge. Here, the long-wavelength limit means the physical wavelengths $\lambda_{\rm phys}$ of the perturbation modes are much larger than the Hubble scale $H^{-1}$, $\lambda_{\rm phys} \gg H^{-1}$, or in terms of the (comoving) momentum,

$$k \ll \mathcal{H}. \tag{38}$$

In the short-wavelength limit, conversely $k \gg \mathcal{H}$.

Since we are interested in a universe dominated by a scalar field, i.e., the inflationary epoch representatively, we expand $\tau_{\mu\nu}$ in the power of the slow-roll parameters,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G\frac{\dot{\phi}_0^2}{H^2}, \tag{39}$$

$$\delta \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = \epsilon - \frac{\dot{\epsilon}}{2H\epsilon}. \tag{40}$$

With the Fourier-mode expansion

$$\Psi(\eta,\boldsymbol{x}) = \sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{41}$$

solving Eq. (33) in the long- and short-wavelength limits gives the following solutions:

$$\Psi_{\boldsymbol{k}}(\eta) = \begin{cases} A_1\epsilon & \text{(long wavelength)} \\ 4\pi G\dot{\phi}_0[c_1\sin(k\eta) + c_2\cos(k\eta)] & \text{(short wavelength)} \end{cases}, \tag{42}$$

where $A_1$ and $c_i$ are complex constants. Notice that since $\Psi_{\boldsymbol{k}}$ is approximately constant on long-wavelength scales and is rapidly oscillating on short-wavelength scales, their cross terms in computing $\tau_{\mu\nu}$ from Eqs. (35), (36), and (37) vanish. We thus only consider the cases when two $\Psi_{\boldsymbol{k}}$'s are both in the long- and short-wavelength limits separately. Also, we mention that when the scalar field $\phi$ is oscillating, it is completely equivalent to a pressureless matter component [4]. Then the analysis of the corresponding 2EMT can follow straightly our previous study [3].

### A. Longitudinal gauge

Let us take the longitudinal gauge by imposing

$$\beta = E = 0. \tag{43}$$

This in turn gives $Q = 0$. $\tau_{\mu\nu}$ in this gauge is found to be described solely by the gauge-invariant variable $\Psi$,

$$\begin{aligned}\tau_{00} = \frac{1}{8\pi G(\mathcal{H}' - \mathcal{H}^2)}\Bigg\{&-\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle - 2(K+\mathcal{H})\langle\nabla\Psi'\cdot\nabla\Psi\rangle - [3(\mathcal{H}' - \mathcal{H}^2) + K^2 + a^2V_{\phi\phi}]\langle(\Psi')^2\rangle \\ &+ (9\mathcal{H}' - 10\mathcal{H}^2 - 2L)\langle(\nabla\Psi)^2\rangle + 2\left[6\mathcal{H}(\mathcal{H}' - \mathcal{H}^2) - KL + 2(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} - a^2\mathcal{H}V_{\phi\phi}\right]\langle\Psi\Psi'\rangle \\ &- \left[L^2 - 4\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + a^2\mathcal{H}^2V_{\phi\phi}\right\}\langle\Psi^2\rangle\Bigg],\end{aligned} \tag{44}$$

$$\begin{aligned}\tau_{ij} = \frac{\delta_{ij}}{8\pi G(\mathcal{H}' - \mathcal{H}^2)}\Bigg\{&\frac{1}{3}\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2\left(\frac{\mathcal{H}}{3} - K\right)\langle\nabla\Psi'\cdot\nabla\Psi\rangle + (\mathcal{H}' - \mathcal{H}^2 - K^2 + a^2V_{\phi\phi})\langle(\Psi')^2\rangle \\ &+ \left[\frac{1}{3}(11\mathcal{H}' - 10\mathcal{H}^2) - 2L\right]\langle(\nabla\Psi)^2\rangle + 2[\mathcal{H}a^2V_{\phi\phi} - KL + 2(\mathcal{H}' - \mathcal{H}^2)(K + 2\mathcal{H})]\langle\Psi\Psi'\rangle \\ &+ [4(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}' + 2\mathcal{H}^2 + L) - L^2 + \mathcal{H}^2a^2V_{\phi\phi}]\langle\Psi^2\rangle\Bigg\}.\end{aligned} \tag{45}$$

#### 1. Long-wavelength limit

In the long-wavelength limit $k \ll \mathcal{H}$, we find $\tau_{\mu\nu}$ in the longitudinal gauge as

$$\tau_{00} \approx \frac{|A_1|^2 k^2}{8\pi G}\left[2\epsilon + 3\frac{\mathcal{H}^2}{k^2}(-3\epsilon^2 + \epsilon\delta)\right], \tag{46}$$

$$\tau_{ij} \approx \frac{|A_1|^2 k^2}{8\pi G}\left[-\frac{2}{3}\epsilon + 3\frac{\mathcal{H}^2}{k^2}(3\epsilon^2 - \epsilon\delta)\right]. \tag{47}$$

We note that the long-wavelength limit can be divided into two sublimits, depending on which term inside the square brackets dominates: (i) the ultralong-wavelength limit for which the second term dominates so that $\mathcal{H}/k \gtrsim 1/\sqrt{\epsilon}$, and (ii) the infralong-wavelength limit for which the first term dominates so that $\mathcal{H}/k \lesssim 1/\sqrt{\epsilon}$.

(i) Ultralong-wavelength limit: in this limit, $k \lesssim \sqrt{\epsilon}\mathcal{H}$ so that the second term, i.e., the $\mathcal{O}(\epsilon^2)$ term, in the 2EMT is dominant. The value of the equation-of-state parameter $w$ in this limit is given by

$$w \equiv \frac{\mathfrak{p}}{\varrho} \approx -1, \tag{48}$$

$$\varrho \approx -\frac{3H^2|A_1|^2(3\epsilon^2 - \epsilon\delta)}{8\pi G}, \tag{49}$$

where the effective energy density and pressure of the 2EMT are given by $\varrho = \tau_{00}/a^2$ and $\mathfrak{p} = \tau_{ii}/a^2$. Note that this result agrees with that in Ref. [2] in which the authors showed that $w \approx -1$ with $\varrho < 0$.

(ii) Infralong-wavelength limit: in this limit, $k \gtrsim \sqrt{\epsilon}\mathcal{H}$ so that the first term, i.e., the $\mathcal{O}(\epsilon)$ term, in the 2EMT is dominant. The value of $w$ in this limit is given by

$$w \approx -\frac{1}{3}, \tag{50}$$

$$\varrho \approx \frac{k^2|A_1|^2\epsilon}{4\pi G a^2} > 0. \tag{51}$$

It is interesting to note that the scale dividing the ultra- and infralong-wavelength limits coincides with the so-called decoupling limit, where we can neglect the perturbations in the metric and may only consider those in the matter sector. That is, the gravitational background is considered to be completely classical, and the Goldstone boson picture for the curvature perturbation becomes manifest [5]—only the scalar field is quantum mechanical in the classical gravitational background. Interestingly, in such a limit the equation of state of 2EMT is $w \approx -1/3$ in *all gauge choices* we examine in this section.

#### 2. Short-wavelength limit

In the short-wavelength limit $k \gg \mathcal{H}$, we find $\tau_{\mu\nu}$ as

$$\tau_{00} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[6 + \frac{\mathcal{H}^2}{k^2}(14\epsilon - \delta)\right], \tag{52}$$

$$\tau_{ij} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[\frac{10}{3} + \frac{\mathcal{H}^2}{k^2}\left(\frac{4\epsilon}{3} - \frac{5\delta}{3}\right)\right]. \tag{53}$$

The value of $w$ in this limit is given by

$$w \approx \frac{5}{9}, \tag{54}$$

$$\varrho \approx \frac{3k^4(|c_1|^2 + |c_2|^2)}{2a^4}. \tag{55}$$

### B. Spatially flat gauge

Let us take the spatially flat gauge,

$$\psi = E = 0, \tag{56}$$

so that the spatial metric is unperturbed. It is equivalent to set

$$Q = \frac{\Psi}{\mathcal{H}}. \tag{57}$$

$\tau_{\mu\nu}$ in this gauge is then found to be described solely by $\Psi$,

$$\begin{aligned}\tau_{00} = \frac{1}{8\pi G(\mathcal{H}' - \mathcal{H}^2)}\Big\{&-\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2(L - M)\langle\nabla\Psi'\cdot\nabla\Psi\rangle \\ &- \left[\left(K + \frac{3(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}}\right)^2 - \frac{4}{\mathcal{H}^2}(\mathcal{H}' - \mathcal{H}^2)(2\mathcal{H}' + \mathcal{H}^2) + a^2V_{\phi\phi}\right]\langle(\Psi')^2\rangle \\ &- \frac{1}{\mathcal{H}^2}[(\mathcal{H}' - 2\mathcal{H}^2)(\mathcal{H}' - 2\mathcal{H}^2 - 2L) + (\mathcal{H}' - 4\mathcal{H}^2)(\mathcal{H}' - \mathcal{H}^2)]\langle(\nabla\Psi)^2\rangle \\ &- \left[\frac{2L(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}^2}\left(3K - \frac{\mathcal{H}' + 11\mathcal{H}^2}{\mathcal{H}}\right) - \frac{2(\mathcal{H}' - 2\mathcal{H}^2)}{\mathcal{H}}a^2V_{\phi\phi}\right]\langle\Psi'\Psi\rangle \\ &- \frac{(\mathcal{H}' - 2\mathcal{H}^2)^2}{\mathcal{H}^2}\left[(L - \frac{2(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}})^2 + 12(\mathcal{H}' - \mathcal{H}^2) + a^2V_{\phi\phi}\right]\langle\Psi^2\rangle\Big\},\end{aligned} \tag{58}$$

$$\tau_{ij} = \frac{\delta_{ij}}{8\pi G\mathcal{H}^2(\mathcal{H}'-\mathcal{H}^2)}\left\{\frac{1}{3}(2\mathcal{H}'-\mathcal{H}^2)\langle(\nabla\Psi')^2\rangle - \frac{1}{3}(\mathcal{H}'+2\mathcal{H}^2)\langle(\Delta\Psi)^2\rangle\right.$$
$$-\frac{2}{3\mathcal{H}}[4\mathcal{H}'^2+3\mathcal{H}'\mathcal{H}^2-8\mathcal{H}^4+(2\mathcal{H}'+\mathcal{H}^2)\mathcal{H}K]\langle\nabla\Psi'\cdot\nabla\Psi\rangle$$
$$-[3(\mathcal{H}'-\mathcal{H}^2)(3\mathcal{H}'-7\mathcal{H}^2)+14\mathcal{H}(\mathcal{H}'-\mathcal{H}^2)K+\mathcal{H}^2K^2-\mathcal{H}^2a^2V_{\phi\phi}]\langle(\Psi')^2\rangle$$
$$+\frac{1}{3\mathcal{H}^2}(\mathcal{H}'-2\mathcal{H}^2)[2\mathcal{H}'^2+10\mathcal{H}'\mathcal{H}^2-13\mathcal{H}^4+2(2\mathcal{H}'+\mathcal{H}^2)L]\langle(\nabla\Psi)^2\rangle$$
$$+\left[\frac{2}{\mathcal{H}}(\mathcal{H}'-2\mathcal{H}^2)(L^2+12(\mathcal{H}'-\mathcal{H}^2)L-4(\mathcal{H}'+2\mathcal{H}^2)(\mathcal{H}'-\mathcal{H}^2))-2\mathcal{H}(\mathcal{H}'-\mathcal{H}^2)a^2V_{\phi\phi}\right]\langle\Psi'\Psi\rangle$$
$$\left.-\frac{(\mathcal{H}'-2\mathcal{H}^2)}{\mathcal{H}^2}[L^2+12(\mathcal{H}'-\mathcal{H}^2)L-\mathcal{H}^2a^2V_{\phi\phi}-4(\mathcal{H}'+2\mathcal{H}^2)(\mathcal{H}'-\mathcal{H}^2)]\langle\Psi^2\rangle\right\}. \quad (59)$$

#### 1. Long-wavelength limit

In the long-wavelength limit $k \ll \mathcal{H}$, we find $\tau_{\mu\nu}$ in the spatially flat gauge as

$$\tau_{00} \approx \frac{|A_1|^2k^2}{8\pi G}\left[\epsilon + 3\frac{\mathcal{H}^2}{k^2}(-3\epsilon^2+\epsilon\delta)\right], \quad (60)$$

$$\tau_{ij} \approx \frac{|A_1|^2k^2}{8\pi G}\left[-\frac{1}{3}\epsilon + 3\frac{\mathcal{H}^2}{k^2}(3\epsilon^2-\epsilon\delta)\right]. \quad (61)$$

Again, the long-wavelength limit is divided into the ultra- and infralong-wavelength limits.

(i) Ultralong-wavelength limit: in this limit, the result is exactly the same as that of the longitudinal gauge.
(ii) Infra long-wavelength limit: in this limit, the value of $w$ is the same as that in the longitudinal gauge, but the energy density is smaller by factor 2,

$$w \approx -\frac{1}{3}, \quad (62)$$

$$\varrho \approx \frac{k^2|A_1|^2\epsilon}{8\pi Ga^2}. \quad (63)$$

#### 2. Short-wavelength limit

In the short-wavelength limit $k \gg \mathcal{H}$, we find

$$\tau_{00} \approx (|c_1|^2+|c_2|^2)\frac{k^4}{4a^2}\left[2+\frac{\mathcal{H}^2}{k^2}(-4\epsilon+\delta)\right], \quad (64)$$

$$\tau_{ij} \approx (|c_1|^2+|c_2|^2)\frac{k^4}{4a^2}\left[\frac{2}{3}+\frac{\mathcal{H}^2}{k^2}\left(2\epsilon-\frac{7\delta}{3}\right)\right]. \quad (65)$$

And the value of $w$ in this limit is given by

$$w \approx \frac{1}{3}, \quad (66)$$

$$\varrho \approx \frac{k^4(|c_1|^2+|c_2|^2)}{2a^4}. \quad (67)$$

### C. Comoving gauge

Let us impose the comoving gauge condition in the scalar mode,

$$\delta\phi = E = 0, \quad (68)$$

for which the gauge-invariant variable $\Psi$ and the gauge-dependent variable $Q$ are given by

$$\Psi = \psi + \mathcal{H}\beta, \quad (69)$$

$$Q = \frac{\Psi'+\mathcal{H}\Psi}{\mathcal{H}'-\mathcal{H}^2}. \quad (70)$$

$\tau_{\mu\nu}$ in this gauge is found to be

$$\tau_{00} = \frac{1}{8\pi G(\mathcal{H}'-\mathcal{H}^2)^2}\left[-2\mathcal{H}\langle\Delta\Psi'\Delta\Psi\rangle - (4\mathcal{H}'-3\mathcal{H}^2)\langle(\Delta\Psi)^2\rangle\right.$$
$$+\frac{F_1}{\mathcal{H}'-\mathcal{H}^2}\langle(\nabla\Psi')^2\rangle + \frac{2F_2}{\mathcal{H}'-\mathcal{H}^2}\langle\nabla\Psi'\cdot\nabla\Psi\rangle + \frac{F_3}{\mathcal{H}'-\mathcal{H}^2}\langle(\nabla\Psi)^2\rangle$$
$$\left.-\frac{F_4}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle(\Psi')^2\rangle - \frac{2F_5}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle\Psi'\Psi\rangle - \frac{F_6}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle\Psi^2\rangle\right], \quad (71)$$

$$\tau_{ij} = \frac{\delta_{ij}}{8\pi G(\mathcal{H}' - \mathcal{H}^2)^2}\left[-\frac{2}{3}\langle(\Delta\Psi')^2\rangle - \frac{2}{3}\langle\Delta^2\Psi\Delta\Psi\rangle + \frac{2P_1}{3(\mathcal{H}' - \mathcal{H}^2)}\langle\Delta\Psi'\Delta\Psi\rangle - \frac{P_2}{3(\mathcal{H}' - \mathcal{H}^2)}\langle(\Delta\Psi)^2\rangle \right.$$
$$+ \frac{P_3}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle(\nabla\Psi')^2\rangle - \frac{2P_4}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle\nabla\Psi'\cdot\nabla\Psi\rangle + \frac{P_5}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle(\nabla\Psi)^2\rangle$$
$$\left. - \frac{P_6}{3(\mathcal{H}' - \mathcal{H}^2)^3}\langle(\Psi')^2\rangle + \frac{2P_7}{(\mathcal{H}' - \mathcal{H}^2)^3}\langle\Psi'\Psi\rangle - \frac{P_8}{3(\mathcal{H}' - \mathcal{H}^2)^3}\langle\Psi^2\rangle\right]. \tag{72}$$

The time-dependent coefficients $F_i$ and $P_i$ are given in the Appendix.

#### 1. Long-wavelength limit

In the long-wavelength limit $k \ll \mathcal{H}$, we find in the comoving gauge

$$\tau_{00} \approx \frac{|A_1|^2\mathcal{H}^2}{8\pi G}[9 + (49\epsilon - 36\delta) + 4(16\epsilon^2 - 25\epsilon\delta + 9\delta^2)], \tag{73}$$

$$\tau_{ij} \approx \frac{|A_1|^2\mathcal{H}^2}{8\pi G}[-3 + (5\epsilon + 12\delta) + 4(13\epsilon^2 - 8\epsilon\delta - 3\delta^2)]. \tag{74}$$

In the comoving gauge, unlike the other gauges we have examined previously, the long-wavelength limit is not subdivided into ultra- and infralong-wavelength limits. The value of $w$ in the long-wavelength limit is the same as the infralong-wavelength limit of the previous gauge choices, but with a different energy density,

$$w \approx -\frac{1}{3}, \tag{75}$$

$$\varrho \approx \frac{9\mathcal{H}^2|A_1|^2}{8\pi G a^2} = \frac{9H^2|A_1|^2}{8\pi G}. \tag{76}$$

Notice that this comoving gauge condition coincides with the uniform energy density condition $\delta\rho = 0$ in the limit of $k \to 0$, see e.g., [6]. However, here we work out only up to second order in perturbations, thus when higher-order contributions are taken into account our results may well change.

#### 2. Short-wavelength limit

In the short-wavelength limit $k \gg \mathcal{H}$, we find

$$\tau_{00} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[-\frac{1}{\epsilon} + \left(3 - \frac{2\delta}{\epsilon}\right) + \frac{\mathcal{H}^2}{k^2}(5\epsilon + 2\delta)\right], \tag{77}$$

$$\tau_{ij} \approx -(|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[\frac{11}{3\epsilon} + \left(1 + \frac{2\delta}{3\epsilon}\right) + \frac{2\delta^2}{\epsilon}\right]. \tag{78}$$

The value of $w$ in this limit is given by

$$w \approx \frac{11}{3}, \tag{79}$$

$$\varrho \approx -\frac{k^4(|c_1|^2 + |c_2|^2)}{4a^4\epsilon}. \tag{80}$$

## IV. CONCLUSIONS

We considered the cosmological perturbations in the universe dominated by a scalar field. Introducing the scalar modes of the metric perturbations and the matter field perturbation, we derived the Einstein's equation up to the second order, and constructed a 2EMT, which is composed of the quadratic terms of the linear perturbations, and is responsible for the so-called gravitational backreaction.

Like the first order, the second-order Einstein's equation may be put into a gauge-invariant form after adding terms on both sides of the equation. However, a 2EMT which is the collection of only quadratic terms of the equation is not guaranteed to have gauge invariance. Some works have been done in this context, e.g., in Refs. [7–11]. However, the gauge invariance of a 2EMT was doubted in Refs. [12–14].

In this work, we derived a 2EMT directly and considered its gauge dependence. As we observed in the Friedmann universe driven by perfect fluid [3], the 2EMT in the inflation period driven by a scalar field also exhibits gauge dependence; the 2EMT does not depend only on the gauge-invariant variable but also on the gauge-dependent variables.

Although the 2EMT is gauge dependent, we investigated it in certain wavelength limits under different gauge conditions in order to examine if its behavior converges. We examined in three conditions (longitudinal, spatially flat, and comoving gauges) in the slow-roll regime of inflation. The result showed that the 2EMT behaves as an effective fluid of $w = -1/3$ in all gauges in the long-wavelength limit. Specifically, it was observed in the *infralong-wavelength* limit, $\sqrt{\epsilon}\mathcal{H} \lesssim k \ll \mathcal{H}$. This behavior was not observed in the Friedmann universe dominated by fluid in Ref. [3].

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## APPENDIX: TIME-DEPENDENT COEFFICIENTS IN THE COMOVING GAUGE

Here, we list the time-dependent coefficients $F_i$ and $P_i$ of the 2EMT in the comoving gauge in Eqs. (71) and (72):

$$F_1 = 2\mathcal{H}\mathcal{H}'' + (\mathcal{H}')^2 + 4\mathcal{H}(K-2\mathcal{H})\mathcal{H}' - \mathcal{H}^3(4K-3\mathcal{H}), \tag{A1}$$

$$F_2 = -(2\mathcal{H}' - \mathcal{H}^2)\mathcal{H}'' - (4K-5\mathcal{H})(\mathcal{H}')^2 + 2\mathcal{H}(2K\mathcal{H} + L - 3\mathcal{H}^2)\mathcal{H}' - \mathcal{H}^3(2L-3\mathcal{H}^2), \tag{A2}$$

$$\begin{aligned} F_3 = {} & -4\mathcal{H}\mathcal{H}'\mathcal{H}'' + 13(\mathcal{H}')^3 - \left[8L - 2\left(K - \frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H} + 12\mathcal{H}^2\right](\mathcal{H}')^2 \\ & - \mathcal{H}^2\left(4\mathcal{H}K - 8L + \mathcal{H}^2 - 4\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H}' + 2\mathcal{H}^5\left(K + 4\mathcal{H} - \frac{a^2V_\phi}{\phi_0'}\right), \end{aligned} \tag{A3}$$

$$\begin{aligned} F_4 = {} & (\mathcal{H}' + 2\mathcal{H}^2)(\mathcal{H}'')^2 - 4[2\mathcal{H}(\mathcal{H}')^2 - K(\mathcal{H}')^2 - K\mathcal{H}^2\mathcal{H}' + \mathcal{H}^4(2K+\mathcal{H})]\mathcal{H}'' \\ & + \left[5K^2 - 16K\mathcal{H} + 7\mathcal{H}^2 - 2Ka^2V'(\phi_0)/\phi_0' + \left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3 \\ & - \left[3K^2 - 16\mathcal{H}K + 5\mathcal{H}^2 - 6Ka^2V'(\phi_0)/\phi_0' + 3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^2(\mathcal{H}')^2 \\ & - \left[9K^2 + 8\mathcal{H}K - 17\mathcal{H}^2 + 6Ka^2V'(\phi_0)/\phi_0' - 3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^4(\mathcal{H}') \\ & + \left[7K^2 + 8\mathcal{H}K - 7\mathcal{H}^2 + 2Ka^2V'(\phi_0)/\phi_0' - \left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^6, \end{aligned} \tag{A4}$$

$$\begin{aligned} F_5 = {} & (\mathcal{H}' + 2\mathcal{H}^2)\mathcal{H}(\mathcal{H}'')^2 - \left[(\mathcal{H}')^3 - \left(2\mathcal{H}K + L - 14\mathcal{H}^2 + \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)(\mathcal{H}')^2\right. \\ & \left. - \left(2\mathcal{H}K + 4L + 15\mathcal{H}^2 - 2\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H}^2\mathcal{H}' + \left(4\mathcal{H}K + 5L + 12\mathcal{H}^2 - \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H}^4\right]\mathcal{H}'' \\ & - 2(K-2\mathcal{H})(\mathcal{H}')^4 + \left[3KL - 18\mathcal{H}^2K - 7\mathcal{H}L + (\mathcal{H}K - L - \mathcal{H}^2)\frac{a^2V_\phi}{\phi_0'} + \mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3 \\ & + \left[-\mathcal{H}^2\frac{a^2V_\phi}{\phi_0'}\left(3\mathcal{H}K - 3L - \mathcal{H}^2 + 3\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) + \mathcal{H}^2(3KL + 50\mathcal{H}^2K + 7\mathcal{H}L - 68\mathcal{H}^3)\right](\mathcal{H}')^2 \\ & + \left[\mathcal{H}^4\frac{a^2V_\phi}{\phi_0'}\left(3\mathcal{H}K - 3L + \mathcal{H}^2 + 3\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) - 5\mathcal{H}^4(3KL + 10\mathcal{H}^2K + \mathcal{H}L - 14\mathcal{H}^3)\right]\mathcal{H}' \\ & + \left[-\frac{a^2V_\phi}{\phi_0'}\left(\mathcal{H}K - L + \mathcal{H}^2 + \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) + 9KL + 20\mathcal{H}^2K + 5\mathcal{H}L - 20\mathcal{H}^3\right]\mathcal{H}^6, \end{aligned} \tag{A5}$$

$$
\begin{aligned}
F_6 =& (\mathcal{H}' + 2\mathcal{H}^2)\mathcal{H}^2(\mathcal{H}'')^2 \\
&+ 2\mathcal{H}\left[-(\mathcal{H}')^3 + \left(L - 10\mathcal{H}^2 + \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)(\mathcal{H}')^2 + \left(4L + 15\mathcal{H}^2 - 2\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H}^2\mathcal{H}' - \mathcal{H}^4\left(5L + 10\mathcal{H}^2 - \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\right]\mathcal{H}'' \\
&+ (\mathcal{H}')^5 - \left(2L - 33\mathcal{H}^2 + 2\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)(\mathcal{H}')^4 + \left[\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\left(2L + 6\mathcal{H}^2 + \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) + L^2 - \mathcal{H}^2(42L + 65\mathcal{H}^2)\right](\mathcal{H}')^3 \\
&+ \left[-\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\left(6L + 10\mathcal{H}^2 + 3\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) + 9L^2 + 110L\mathcal{H}^2 + 39\mathcal{H}^4\right]\mathcal{H}^2(\mathcal{H}')^2 \\
&+ \left[\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\left(6L + 10\mathcal{H}^2 + 3\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) - 21L^2 - 110L\mathcal{H}^2 + 8\mathcal{H}^4\right]\mathcal{H}^4\mathcal{H}' \\
&+ \left[-\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\left(2L + 4\mathcal{H}^2 + \mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right) + 11L^2 + 44\mathcal{H}^2L - 4\mathcal{H}^4\right]\mathcal{H}^6,
\end{aligned} \tag{A6}
$$

$$
P_1 = 4\mathcal{H}'' + (6K - 11\mathcal{H})\mathcal{H}' - 3\mathcal{H}^2(2K - \mathcal{H}), \tag{A7}
$$

$$
P_2 = 4\mathcal{H}\mathcal{H}'' + 4(\mathcal{H}')^2 + (8\mathcal{H}K - 8L - 15\mathcal{H}^2)\mathcal{H}' - \mathcal{H}^2(8\mathcal{H}K - 8L - 3\mathcal{H}^2) \tag{A8}
$$

$$
\begin{aligned}
P_3 =& -2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 6(\mathcal{H}'')^2 + 2[(8K - 13\mathcal{H})\mathcal{H}' - \mathcal{H}^2(8K - \mathcal{H})]\mathcal{H}'' + 9(\mathcal{H}')^3 \\
&+ [8K(2K - 5\mathcal{H}) - 4L + 27\mathcal{H}^2](\mathcal{H}')^2 - [16K(2K - 3\mathcal{H}) - 8L + 17\mathcal{H}^2]\mathcal{H}^2\mathcal{H}' \\
&+ [8K(2K - \mathcal{H}) - 4L + 5\mathcal{H}^2]\mathcal{H}^4,
\end{aligned} \tag{A9}
$$

$$
\begin{aligned}
P_4 =& -\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 6\mathcal{H}(\mathcal{H}'')^2 - [3(\mathcal{H}')^2 - (22\mathcal{H}K - 8L - 19\mathcal{H}^2)\mathcal{H}' + 2\mathcal{H}^2(11\mathcal{H}K - 4L + \mathcal{H}^2)]\mathcal{H}'' \\
&- (8K - 29\mathcal{H})(\mathcal{H}')^3 + [4K(5K\mathcal{H} - 3L - 7\mathcal{H}^2) + \mathcal{H}(16L - 69\mathcal{H}^2)](\mathcal{H}')^2 \\
&- \mathcal{H}^2[4\mathcal{H}K(10K - 9\mathcal{H}) - 8L(3K - 2\mathcal{H}) - 107\mathcal{H}^3]\mathcal{H}' + \mathcal{H}^4[4K(5\mathcal{H}K - 3L) - 43\mathcal{H}^3],
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
P_5 =& 4\mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' - 18\mathcal{H}^2(\mathcal{H}'')^2 + \mathcal{H}[60(\mathcal{H}')^2 - 4(3\mathcal{H}K + 8L + 11\mathcal{H}^2)\mathcal{H}' \\
&+ 4\mathcal{H}^2(3\mathcal{H}K + 8L + 14\mathcal{H}^2)]\mathcal{H}'' - 23(\mathcal{H}')^4 + (60\mathcal{H}K + 4L - 131\mathcal{H}^2)(\mathcal{H}')^3 \\
&- [4\mathcal{H}K(10L + 39\mathcal{H}^2) - 4L(2L + 17\mathcal{H}^2) - 333\mathcal{H}^4](\mathcal{H}')^2 \\
&+ \mathcal{H}^2[4\mathcal{H}K(20L + 39\mathcal{H}^2) - 4L(4L + 21\mathcal{H}^2) - 357\mathcal{H}^4]\mathcal{H}' \\
&- 2\mathcal{H}^4[10\mathcal{H}K(2L + 3\mathcal{H}^2) - 2L(2L + 3\mathcal{H}^2) - 53\mathcal{H}^4],
\end{aligned} \tag{A11}
$$

$$
\begin{aligned}
P_6 =& -6\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)[\mathcal{H}'' + 2(K - \mathcal{H})\mathcal{H}' - 2\mathcal{H}^2K]\mathcal{H}''' + 12\mathcal{H}(\mathcal{H}'')^3 - [9(\mathcal{H}')^2 - \mathcal{H}(48K - 57\mathcal{H})\mathcal{H}' + 6\mathcal{H}^3(8K + \mathcal{H})](\mathcal{H}'')^2 \\
&+ 12[-(3K - 7\mathcal{H})(\mathcal{H}')^3 + \mathcal{H}(6K^2 - 9\mathcal{H}K - L - 3\mathcal{H}^2)(\mathcal{H}')^2 \\
&- \mathcal{H}^3(12K^2 - 11\mathcal{H}K - 2L - 13\mathcal{H}^2)\mathcal{H}' + \mathcal{H}^5(6K^2 + \mathcal{H}K - L - 5\mathcal{H}^2)]\mathcal{H}'' \\
&+ 3\left[-13K^2 + 62\mathcal{H}K - 49\mathcal{H}^2 + \left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^4 \\
&+ 12\mathcal{H}\left[4K^3 - 2\mathcal{H}K^2 - 32\mathcal{H}^2K - 2KL + 2\mathcal{H}L + 29\mathcal{H}^3 - \mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3 \\
&- 6\mathcal{H}^3\left[24K^3 - 27\mathcal{H}K^2 - 90\mathcal{H}^2K - 12KL + 8\mathcal{H}L + 105\mathcal{H}^3 - 3\mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^2 \\
&+ 12\mathcal{H}^5\left[12K^3 - 8\mathcal{H}K^2 - 40\mathcal{H}^2K - 6KL + 2\mathcal{H}L + 36\mathcal{H}^3 - \mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}' \\
&- 3\mathcal{H}^7\left[16K^3 + \mathcal{H}K^2 - 46\mathcal{H}^2K - 8KL + 33\mathcal{H}^3 - \mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right],
\end{aligned} \tag{A12}
$$

$$
\begin{aligned}
P_7 = {} & [2\mathcal{H}^2(\mathcal{H}'-\mathcal{H}^2)\mathcal{H}''-4\mathcal{H}(\mathcal{H}')^3+2\mathcal{H}(\mathcal{H}K+L+4\mathcal{H}^2)(\mathcal{H}')^2-4\mathcal{H}^3(\mathcal{H}K+L+2\mathcal{H}^2)\mathcal{H}'+2\mathcal{H}^5(\mathcal{H}K+L+2\mathcal{H}^2)]\mathcal{H}''' \\
& -4\mathcal{H}^2(\mathcal{H}'')^3+[12\mathcal{H}(\mathcal{H}')^2-\mathcal{H}(8\mathcal{H}K+8L-\mathcal{H}^2)\mathcal{H}'+\mathcal{H}^3(8\mathcal{H}K+8L+11\mathcal{H}^2)](\mathcal{H}'')^2 \\
& +[-9(\mathcal{H}')^4+(25\mathcal{H}K+5L-36\mathcal{H}^2)(\mathcal{H}')^3-\mathcal{H}(4\mathcal{H}K^2+39\mathcal{H}^2K+20KL-23\mathcal{H}L-54\mathcal{H}^3)(\mathcal{H}')^2 \\
& +\mathcal{H}^3(8\mathcal{H}K^2+35\mathcal{H}^2K+40KL-29\mathcal{H}L-76\mathcal{H}^3)\mathcal{H}'-\mathcal{H}^5(4\mathcal{H}K^2+21\mathcal{H}^2K+20KL-\mathcal{H}L-19\mathcal{H}^3)]\mathcal{H}'' \\
& +\left(-19K+55\mathcal{H}+2\frac{a^2V_\phi}{\phi_0'}\right)(\mathcal{H}')^5+\left[20\mathcal{H}K^2-6\mathcal{H}^2K+9KL-29\mathcal{H}L-112\mathcal{H}^3-10\mathcal{H}^2\frac{a^2V_\phi}{\phi_0'}-\mathcal{H}\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^4 \\
& -2\mathcal{H}\left[36\mathcal{H}^2K^2-58\mathcal{H}^3K-2L^2+8K^2L-27\mathcal{H}^2L-8\mathcal{H}KL-45\mathcal{H}^4-10\mathcal{H}^3\frac{a^2V_\phi}{\phi_0'}-2\mathcal{H}^2\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3 \\
& +2\mathcal{H}^3\left[52\mathcal{H}^2K^2-103\mathcal{H}^3K-6L^2+24K^2L-38\mathcal{H}^2L-31\mathcal{H}KL+22\mathcal{H}^4-10\mathcal{H}^3\frac{a^2V_\phi}{\phi_0'}-3\mathcal{H}^2\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^2 \\
& -\mathcal{H}^5\left[72\mathcal{H}^2K^2-151\mathcal{H}^3K-12L^2+48K^2L-74\mathcal{H}^2L-40\mathcal{H}KL+61\mathcal{H}^4-10\mathcal{H}^3\frac{a^2V_\phi}{\phi_0'}-4\mathcal{H}^2\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}' \\
& +\mathcal{H}^7\left[20\mathcal{H}^2K^2-36\mathcal{H}^3K-4L^2+16K^2L-23\mathcal{H}^2L-3\mathcal{H}KL+16\mathcal{H}^4-2\mathcal{H}^3\frac{a^2V_\phi}{\phi_0'}-\mathcal{H}^2\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right], \qquad \text{(A13)}
\end{aligned}
$$

$$
\begin{aligned}
P_8 = {} & -6\mathcal{H}^2(\mathcal{H}'-\mathcal{H}^2)[\mathcal{H}\mathcal{H}''-4(\mathcal{H}')^2+2(L+3\mathcal{H}^2)\mathcal{H}'-2\mathcal{H}^2(L+2\mathcal{H}^2)]\mathcal{H}'''+12\mathcal{H}^3(\mathcal{H}'')^3 \\
& -3\mathcal{H}^2[21(\mathcal{H}')^2-(16L+17\mathcal{H}^2)\mathcal{H}'+4\mathcal{H}^2(4L+5\mathcal{H}^2)](\mathcal{H}'')^2 \\
& +6\mathcal{H}[13(\mathcal{H}')^4-(\mathcal{H}K+23L-6\mathcal{H}^2)(\mathcal{H}')^3+(8L^2+3\mathcal{H}^3K+4\mathcal{H}KL+33\mathcal{H}^2L-24\mathcal{H}^4)(\mathcal{H}')^2 \\
& -\mathcal{H}^2(16L^2+3\mathcal{H}^3K+8\mathcal{H}KL+29\mathcal{H}^2L-34\mathcal{H}^4)\mathcal{H}'+\mathcal{H}^4(8L^2+\mathcal{H}^3K+4\mathcal{H}KL+19\mathcal{H}^2L-5\mathcal{H}^4)]\mathcal{H}'' \\
& -45(\mathcal{H}')^6+6[6\mathcal{H}K+11L-22\mathcal{H}^2](\mathcal{H}')^5 \\
& -3\left[7L^2+2\mathcal{H}^2K^2+40\mathcal{H}^3K+36\mathcal{H}KL-52\mathcal{H}^2L-152\mathcal{H}^4-\mathcal{H}^2\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^4 \\
& +12\mathcal{H}\left[4KL^2+2\mathcal{H}^3K^2-4\mathcal{H}L^2-60\mathcal{H}^3L+32\mathcal{H}^2KL+11\mathcal{H}^4K-43\mathcal{H}^5-\mathcal{H}^3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3 \\
& -3\mathcal{H}^3\left[48KL^2+12\mathcal{H}^3K^2-58\mathcal{H}L^2-356\mathcal{H}^3L+184\mathcal{H}^2KL+12\mathcal{H}^4K-41\mathcal{H}^5-6\mathcal{H}^3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^2 \\
& +6\mathcal{H}^5\left[24KL^2+4\mathcal{H}^3K^2-20\mathcal{H}L^2-123\mathcal{H}^3L+64\mathcal{H}^2KL-4\mathcal{H}^4K+6\mathcal{H}^5-2\mathcal{H}^3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}' \\
& -3\mathcal{H}^7\left[16KL^2+2\mathcal{H}^3K^2-5\mathcal{H}L^2-56\mathcal{H}^3L+36\mathcal{H}^2KL-4\mathcal{H}^4K+6\mathcal{H}^5-\mathcal{H}^3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]. \qquad \text{(A14)}
\end{aligned}
$$


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