# Rectifying no-hair theorems in Gauss-Bonnet theory

Alexandros Papageorgiou,* Chan Park,† and Miok Park‡
*Center for Theoretical Physics of the Universe, IBS, 34126 Daejeon, South Korea*

(Received 11 May 2022; accepted 3 October 2022; published 17 October 2022)

We revisit the no-hair theorems in Einstein-Scalar-Gauss-Bonnet theory with a general coupling function between the scalar and the Gauss-Bonnet term in four dimensional spacetime. We first resolve the conflict caused from the incomplete derivation of the old no-hair theorem by taking into account the surface term and restore its reliability. We also clarify that the novel no-hair theorem is always evaded for regular black hole solutions without any restrictions as long as the regularity conditions are satisfied.

DOI: 10.1103/PhysRevD.106.084024

## I. INTRODUCTION

The uniqueness theorems [1,2] made us believe that black holes might have no hair except for the mass, electromagnetic charge or angular momentum. This motivated the assertion of the no-hair theorem [3–5] which proved the nonexistence of black hole solutions with nontrivial scalar field for asymptotically flat spacetime. The cases for the massive vector or spinor field were also discussed respectively in [6,7]. However the desire to find new black hole solutions led to the discovery of several kinds of nontrivial field such as colored black holes [8] or Skyrmion hair black holes [9]. More recently, the evasion of the no-hair theorem has been shown for Einstein theory with a Gauss-Bonnet (GB) term which couples to massless scalar fields [10–17]. These results were subsequently extended to consider self-interactions [18,19] or a cosmological constant [20].

The theory of general relativity with higher derivatives is well motivated. The early study arose in quantum field theory by finding that higher derivative terms stabilize the divergent structure of gravity and helps to establish a renormalizable theory of gravity in the absence of matter fields [21,22]. Inspired by this result, application to cosmology was initially investigated in [23] and later more broad construction of modifying Einstein gravity has been widely studied in many works (see [24] and references therein) which have increasing interest due the emergence of novel ways to test the high curvature limit of GR via gravitational waves [25,26] and black hole shadows [27,28]. Furthermore, from the perspective of string theory, taking the low energy limit, gravity theory is reduced to Einstein theory with higher derivative terms whose coefficient is $\alpha'$, the inverse string tension, and associated with the dilaton coupling. Therefore the $\alpha'$ correction is considered as the stringy effect beyond Einstein gravity. Moreover, the swampland conjecture asserts the "no global symmetry" in quantum gravity regimes and this is supported by the no-hair theorem [29,30]. These circumstances draw attention to the existence of black hole solutions in higher derivative theories.

In particular, the Gauss-Bonnet theory has a special interest since it is topological in four dimensions. The no-hair theorems for Einstein-Scalar-Gauss-Bonnet theory (ESGB) are argued in [13] but their analysis was not complete both for the old no-hair theorem as well as the novel version of the theorem as follows.

First, the old no-hair theorem for ESGB theory in [10,13] showed the positive definite coupling $f(\varphi) > 0$ as a necessary requirement for the evasion of the old no-hair theorem. However the evasion of the old no-hair theorem was also found for $f(\varphi) < 0$ in [31], which is contrary to the previous studies. One might think that there is a privileged manner to validate the evasion of the old no-hair theorem, but then the theorem loses its universal power. This situation has caused suspicion on the reliability of the old no-hair theorem. Here we resolve this issue by giving a correct treatment of a surface term and restore the reliability of the old no-hair theorem. Indeed, this situation differs from the original work of Bekenstein [4], where the surface term vanishes because either the field is massive and therefore enjoys a Yukawa-like, exponential decay, or the field is massless and the theory is shift symmetric. In the latter case, the surface term is unphysical since it can always be canceled by a field shift which leaves the Lagrangian invariant. On the other hand, in ESGB theory with a massless scalar field, the surface term survives in

*papageo@ibs.re.kr
†iamparkchan@gmail.com
‡miokpark76@ibs.re.kr

general and it remains physical when the theory does not enjoy a shift symmetry. Thus the no-hair theorem should be understood on this basis.

Second, the novel no-hair theorem involves analyzing the asymptotic behavior of the energy-momentum tensor near the horizon and at infinity under the assumption of a regular black hole solution and determines the possibility that the energy-momentum tensor smoothly matches both asymptotic limits. The application to the ESGB theory in [13] shows that imposing the regularity condition on the horizon is necessary to guarantee the evasion of the novel no-hair theorem and to further ensure the evasion the authors require the derivative of the energy-momentum tensor $(T_r{}^r)'$ to be negative near the horizon. We revisit this and find that the later condition plays no role in determining whether physically acceptable solutions exist.

In this paper, we revisit the no-hair theorem in ESGB theory studied in [13] and revise their argument with respect to the old no-hair theorem as well as extend the analysis of the novel no-hair theorem. In Sec. III we point out the omission of the surface term for the old no-hair theorem in [13]. In Sec. IV we show that in the case of the novel theorem the energy-momentum tensor can be regular without any further constraints as long as the condition for regularity of the scalar field is satisfied. Then we demonstrate our argument with numerical solutions in Sec. V. Thus we clarify the conditions for the no-hair theorem to hold and conclude that the no-hair theorem is more easily evaded than previously studied.

## II. EINSTEIN-SCALAR-GAUSS-BONNET THEORY

We start with the gravity action as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\nabla_\alpha\varphi\nabla^\alpha\varphi + f(\varphi)\mathcal{G}\right], \quad (1)$$

where we set $2\kappa^2 = 16\pi G = 1$ and $\mathcal{G}$ denotes the GB term that is written as

$$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \quad (2)$$

and which leads to the Einstein equations as follows:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} = \kappa^2\left[\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - (g_{\rho\mu}g_{\lambda\nu} + g_{\lambda\mu}g_{\rho\nu})\eta^{\kappa\lambda\alpha\beta}\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\nabla_\gamma\nabla_\kappa f\right], \quad (3)$$

where $\tilde{R}^{\rho\gamma}{}_{\alpha\beta} = \eta^{\rho\gamma\sigma\tau}R_{\sigma\tau\alpha\beta} = \frac{\epsilon^{\rho\gamma\sigma\tau}}{\sqrt{-g}}R_{\sigma\tau\alpha\beta}$ and the scalar field equation is

$$\nabla^2\varphi + \dot{f}\mathcal{G} = 0, \quad (4)$$

where "$\dot{}$" indicates the variation with respect to the scalar field $\varphi$. Employing the following metric ansatz

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2d\Omega_2, \quad (5)$$

the equations of motions are written as

$$\frac{rB' + B - 1}{r^2B} + \frac{\kappa^2}{2}\varphi'^2 + \frac{4\kappa^2}{r^2B} \times [(1 - 3B)B'\dot{f}\varphi' - 2(B - 1)B(\ddot{f}\varphi'^2 + \dot{f}\varphi')] = 0, \quad (6)$$

$$\frac{A'}{Ar} + \frac{B - 1}{Br^2} - \frac{\kappa^2}{2}\varphi'^2 + \frac{4\kappa^2(1 - 3B)A'\varphi'\dot{f}}{Ar^2} = 0, \quad (7)$$

$$\frac{A'(2A - rA')}{4A^2r} + \frac{B'(rA' + 2A)}{4ABr} + \frac{A''}{2A} + \frac{\kappa^2}{2}\varphi'^2 - \frac{2\kappa^2}{Ar}\left[\dot{f}\varphi'\left(2BA'' + 3A'B' - \frac{BA'^2}{A}\right) + 2BA'(\ddot{f}\varphi'^2 + \dot{f}\varphi'')\right] = 0, \quad (8)$$

$$\varphi'' + \frac{1}{2}\varphi'\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) + \frac{2\dot{f}}{Ar^2}\left[\frac{(3B - 1)A'B'}{B} - \frac{(B - 1)}{A}(A'^2 - 2AA'')\right] = 0, \quad (9)$$

where "$'$" indicates the variation with respect to the radial coordinate $r$.

If we assume the existence of a regular black hole, we require the following boundary conditions near the horizon

$$A(r) \sim A_h\epsilon, \qquad B(r) \sim B_h\epsilon, \qquad \varphi(r) \sim \varphi_h + \varphi_{h,1}\epsilon, \quad (10)$$

where $\epsilon = r - r_h$ is the expansion parameter and $\varphi_h$ is a finite value near the black hole horizon. First, as was pointed out in several works before [10,13], in order to ensure that the scalar field and its derivatives are finite one requires the following constraint, valid on the horizon

$$\varphi_{h,1} = -\frac{r_h}{4\dot{f}_h}\left(1 \mp \sqrt{1 - \frac{96}{r_h^4}\dot{f}_h^2}\right), \quad (11)$$

$$B_h = \frac{2}{r_h}\left(1 \pm \sqrt{1 - \frac{96}{r_h^4}\dot{f}_h^2}\right)^{-1}, \quad (12)$$

where $\dot{f}_h = \dot{f}(\varphi_h)$. We found that the numerical solutions are generated only for the minus sign in front of root in (11) with the plus sign in (12) and will just consider this case hereafter. To avoid $\varphi''(r_h)$ being divergent the inside of the root should not be zero, namely

$$\dot{f}_h^2 < \frac{r_h^4}{96}. \quad (13)$$

These regularity conditions (11)–(13) ensure that the solutions correspond to a regular black hole spacetime. We also write the near horizon expansion of the Riemann scalar invariant

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \sim \frac{64\left(2 \pm \sqrt{1-\frac{96}{r_h^4}\dot{f}_h^2} - \frac{24}{r_h^4}\dot{f}_h^2\right)\left(1-\frac{24}{r_h^4}\dot{f}_h^2\right)}{r_h^4\left(1 \pm \sqrt{1-\frac{96}{r_h^4}\dot{f}_h^2}\right)^4}, \tag{14}$$

which is finite unless the size of the horizon becomes zero $r_h \to 0$. This ensures that the spacetime is not a naked singularity but a regular black hole.

At infinity, the asymptotic flatness requires that the metric and scalar field are found to be

$$A(r) \sim 1 + \frac{A_1}{r}, \qquad B(r) \sim 1 + \frac{A_1}{r}, \qquad \varphi(r) \sim \varphi_\infty + \frac{\varphi_1}{r}, \tag{15}$$

where $\varphi_\infty$ takes a finite value that is physical if the theory does not enjoy a shift symmetry on $\varphi$. Here $\varphi_1$ is deeply related to scalar charge [32], which is defined by

$$Q = -\frac{1}{4\pi}\int_{S^2} \mathrm{d}^2\Sigma^\mu \nabla_\mu \varphi \tag{16}$$

at infinity. Plugging the expansion of the metric into the GB term yields

$$\mathcal{G} \sim \frac{48}{r_h^4}\left(1 \pm \sqrt{1-\frac{96}{r_h^4}\dot{f}_h^2}\right)^{-2} + \mathcal{O}(\epsilon), \quad (r \to r_h), \tag{17}$$

$$\mathcal{G} \sim \frac{12A_1^2}{r^6} + \mathcal{O}(r^{-7}), \quad (r \to \infty), \tag{18}$$

which are positive at both asymptotics. Using these expansions, we examine the existence of black hole solutions.

## III. OLD NO-HAIR THEOREM

As studied in [10,13], let us start with the scalar field equation multiplied by the coupling function $f(\varphi)$ and then take an integration over four dimensional spacetime

$$\int_{\mathcal{V}} \mathrm{d}^4x\sqrt{-g}f[\nabla^2\varphi + \dot{f}\mathcal{G}] = 0 \tag{19}$$

$$= -\int_{\mathcal{V}} \mathrm{d}^4x\sqrt{-g}\dot{f}(\partial^\mu\varphi\partial_\mu\varphi - f\mathcal{G}) + \int_{\partial\mathcal{V}} \mathrm{d}^3x\sqrt{-h}fn^\mu\partial_\mu\varphi, \tag{20}$$

where $h$ is the induced metric on a hypersurface defined by a normal vector $n^\mu$. Since we assume that the static scalar field depends on the radial coordinate, only the $\mu = r$ component makes a non-trivial contribution. Since the GB effect vanishes approaching infinity as seen in Eq. (18), we expect that the scalar field behaves like a massless particle there and so their falloff would be slow enough for the surface term to survive. To explicitly show this, we plug our metric ansatz into the general expression above. Factoring out the time and angular integrations, it reduces to

$$\int_{r_h}^\infty \mathrm{d}r\sqrt{\frac{A}{B}}r^2\dot{f}(\partial^\mu\varphi\partial_\mu\varphi - f\mathcal{G}) - \left(\sqrt{\frac{A}{B}}r^2g^{rr}f\partial_r\varphi\right)\Big|_{r\to\infty} = 0, \tag{21}$$

where the surface term, that is the second line above, vanishes at the horizon since $g^{rr} \to 0$ as $r \to r_h$. Substituting the asymptotic expansion (15) to the surface term, we see that it remains finite and approaches $f(\varphi_\infty)\varphi_1$. The presence of a surface term makes it nontrivial to prove the nonexistence of black hole solutions for a general coupling function $f(\varphi)$. But for the special cases such as in the absence of the surface term by $f(\varphi_\infty) = 0$ or $\varphi_1 = 0$, the no-hair theorem still holds when $f(\varphi) < 0$, but is evaded again if $f(\varphi) > 0$ as previously studied in [13]. Nevertheless, the case of $f(\varphi_\infty) = 0$ cannot be achieved for a general coupling. For the cases of $f(\varphi) = \alpha e^{\gamma\varphi(r)}$ or $f(\varphi) = \alpha\varphi(r)^{-n}$ with positive $n$, the regular scalar field solution cannot make $f(\varphi_\infty)$ to vanish and hence the surface term is always present unless $\varphi_1 = 0$. On the other hand, for the cases of $f(\varphi) = \alpha\varphi(r)^n$ or $f(\varphi) = \alpha(1 - e^{\gamma\varphi^2})$, the surface term will disappear if $\varphi_\infty = 0$ and so the no-hair theorem still holds for $f(\varphi) < 0$, while if $\varphi_\infty \neq 0$ the surface term survives. For the later case, black hole solutions would exist for either positive or negative sign of a coupling function. Thus, black hole solutions exist regardless of the sign of $f(\varphi)$ in these cases. This fact will be numerically demonstrated with the explicit coupling functions in Sec. V.

## IV. NOVEL NO-HAIR THEOREM

The novel no-hair theorem was formulated for scalar fields that are minimally coupled to gravity in [33]. They assume the positivity of the energy density $\mathcal{E} = -T_t{}^t > 0$ and illustrate the asymptotic behaviors of $T_r{}^r$ and $(T_r{}^r)'$ at horizon and infinity. They then show the impossibility of smoothly matching these asymptotic conditions by making explicit use of the Einstein equations. The same methodology was applied to the nonminimally coupled ESGB theory, in [10,13], which showed that the smooth matching of the energy-momentum tensor at both asymptotics may be achieved if $T_r{}^r > 0$ and $(T_r{}^r)' < 0$ close to the horizon. We find that the second assumption is not required to smoothly connect the energy-momentum tensor at both asymptotics. Our argument takes the following form.

We first expand the energy momentum tensor and its derivative around the horizon as well as at infinity

$$T_r{}^r = T_t{}^t = -\frac{1}{rB'}T_\theta{}^\theta = -\frac{2B'\dot{f}\varphi'}{r^2} + \mathcal{O}(\epsilon), \quad (22)$$

$$(T_r{}^r)' = \frac{BA'}{A}\left[\frac{4(rB'+1)\dot{f}\varphi'}{r^3} - \frac{r\varphi'^2}{4(r+2\dot{f}\varphi')} - \frac{2(\ddot{f}\varphi'^2+\dot{f}\varphi'')}{r(r+2\dot{f}\varphi')}\right] + \mathcal{O}(\epsilon) \quad (23)$$

and

$$T_r{}^r = -T_t{}^t = -T_\theta{}^\theta = \frac{1}{4}\varphi'^2 + \mathcal{O}(r^{-5}), \quad (24)$$

$$(T_r{}^r)' = -\frac{1}{r}\varphi'^2 + \mathcal{O}(r^{-6}), \quad (25)$$

where the GB term is subdominant at large $r$. The product $\dot{f}\varphi'$ is always negative definite for regular black hole (11) and results in

$$T_r{}^r|_{r\to r_h} > 0, \qquad (T_r{}^r)'|_{r\to r_h}\text{: undetermined} \quad (26)$$

$$T_r{}^r|_{r\to\infty} > 0, \qquad (T_r{}^r)'|_{r\to\infty} < 0 \quad (27)$$

where $(T_r{}^r)'|_{r\to\infty}$ decays asymptotically as $r^{-5}$. As pointed out in [10,13], this indicates that the energy density is negative near the horizon, $\mathcal{E} = -T_t{}^t < 0$, which is opposed to the minimally coupled case in [33]. This is an effect crucially driven by the GB term. The authors considered the possible smooth matching of the energy-momentum tensor at both asymptotics by requiring the condition

$$V = \ddot{f}\varphi'^2 + \dot{f}\varphi'' = \partial_r(\dot{f}\varphi')|_{r_h} > 0, \quad (28)$$

which guarantees $(T_r{}^r)'|_{r\to r_h}$ to be negative in smoothly connecting the two asymptotic limits. However this is not the only way of matching. We first argue that the condition (28) is overly restrictive. To explicitly show this, we explore the parameter space of $(T_r{}^r)'|_{r\to r_h}$ by using the near horizon expansion which takes the form

$$(T_r{}^r)' = -\frac{\beta^2[68-41\beta^2+\sqrt{1-\beta^2}(68-9\beta^2)]}{4r_h^3\sqrt{1-\beta^2}(1+\sqrt{1-\beta^2})^4} - \frac{36\ddot{f}_h\beta^2}{r_h^5\sqrt{1-\beta^2}(1+\sqrt{1-\beta^2})^3} + \mathcal{O}(\epsilon), \quad (29)$$

where $\ddot{f}_h$ denotes $\ddot{f}(\varphi_h)$ and we used a new variable

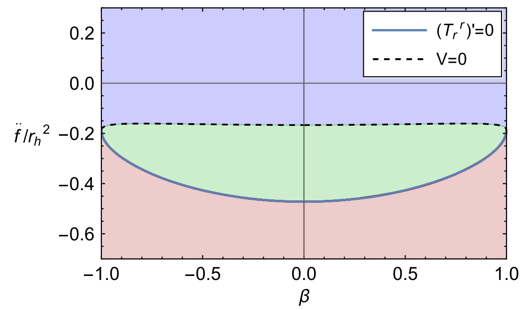


FIG. 1. Parameter space for $(T_r{}^r)'|_{r\to r_h}$. The blue ($V > 0$) and green ($V < 0$) regions satisfy $(T_r{}^r)'|_{r\to r_h} < 0$ while the red region $(T_r{}^r)'|_{r\to r_h} > 0$, and $\beta = \pm 1$ is excluded.

$$\beta = \pm\frac{\sqrt{96}|\dot{f}_h|}{r_h^2}, \quad (30)$$

which ranges $-1 < \beta < 1$ and this will carry the same sign as $\alpha$ in our examples. We plot the boundary in which $(T_r{}^r)'|_{r\to r_h}$ changes sign in Fig. 1 as a solid line. Inspection of the figure allows one to visualize the various possibilities regarding the sign of $(T_r{}^r)'$ near the horizon but this does not mean that all parameter space yields black hole solutions. As seen in Fig. 1 the green region also makes $(T_r{}^r)'$ to be negative even with a negative value of $V$. We second point out that in spite of the undetermination of $(T_r{}^r)'$ the rest of the conditions in (26)–(27) do not prevent having $(T_r{}^r)'|_{r\to r_h} > 0$ to smoothly join the energy-momentum tensor from the horizon to infinity. Namely, either sign $(T_r{}^r)'$ is allowed to match $T_r{}^r$ at both asymptotic regions. In conclusion the whole region except the points with $\beta = \pm 1$ in Fig. 1 should be considered to generate black hole solutions.

In the next section we explore all three regions and numerically find solutions for regular black holes with scalar hair for all three cases.

## V. EXAMPLES

Here we verify the evasion of no-hair theorems for two specific coupling functions. In order to generate the numerical solution the expansion coefficients $A_h$, $\varphi_h$ are used as the parameters to give the initial conditions for Eqs. (6)–(9), but $A_h$ is determined by the boundary condition at infinity. Here we produce a family of solutions for varying $\varphi_h$ fixing $r_h = 1$.

### A. $f(\varphi) = \alpha e^{\gamma\varphi(r)}$

As addressed before, the surface term does not disappear in this case unless $\varphi_1 = 0$ and therefore the no-hair theorem is evaded for any values of $\alpha$ as long as the coupling function satisfies (11)–(12). For a given $\varphi_h$, the value of $\varphi_\infty/\varphi_h$ is depicted as a function of $\beta$ and the scalar functions $\varphi(r)$ for the cases $\beta = -0.71$, 0.136, and 0.5

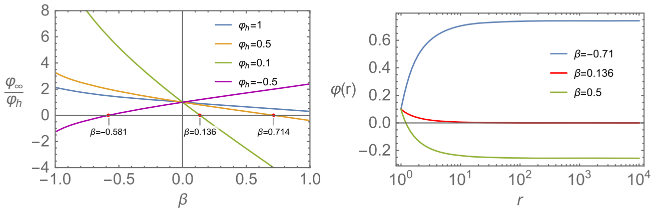


FIG. 2. For $f = \alpha e^{\gamma\varphi}$ (left) $\varphi_\infty/\varphi_h$ vs $\beta$ and (right) $\varphi(r)$ for different values of $\beta$ fixing $\varphi_h = 0.1$.

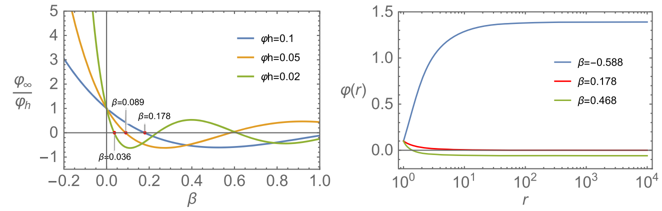


FIG. 4. For $f = \alpha\varphi^2$, (left) $\varphi_\infty/\varphi_h$ vs $\beta$ and (right) $\varphi(r)$ for different values of $\beta$ fixing $\varphi_h = 0.1$.

are displayed in Fig. 2. We fix $\gamma = 1$ and only focus on the positive case for $\gamma$ since the equations of motion are symmetric with $-\gamma$ under the change of the sign of the scalar field $\varphi(r)$. Since there is a shift symmetry of the Dilaton field in this case, a nonzero value of $\varphi_\infty$ can be shifted to zero under a rescaling of the radial coordinate $r$ [10]. To examine the evasion of the old no-hair theorem (21), we plot the bulk term and surface term, which are the first and second line in Eq. (21) respectively, as a function of $\beta$ in Fig. 3. This demonstrates that the value of the surface term is not zero but takes the opposite sign to the bulk term and therefore plays a crucial role for the evasion of the old no-hair theorem. We also investigated the energy-momentum tensor, but since our parameter choices ($\gamma = 1$) are firmly in the blue region of Fig. 1, $T_r{}^r$ is positive and monotonically deceasing and $(T_r{}^r)'$ is negative and monotonically increasing having positive $V$ in (28). We have found solutions for both the green as well as the red region of Fig. 1 by changing the value of $\gamma$, but observed that the behavior of the metric functions differs considerably from the Schwarzschild-like behavior [34]. We instead will explore the green and red regions of Fig. 1 for the novel no-hair theorem in the next example.

### B. $f(\varphi) = \alpha\varphi(r)^2$

In this case, the surface term can vanish when $\varphi_\infty = 0$ or $\varphi_1 = 0$ which reduces the bulk integration to be zero. This cannot be achieved for $f(\varphi) < 0$ or $\alpha < 0$ and so the no-hair theorem is expected to hold. This is numerically shown in Fig. 4, where the solutions show an ever increasing value of $\varphi_\infty$ for negative $\beta$ and therefore certainly there are no solutions that yield $\varphi_\infty = 0$ for negative values of $\beta$. Moreover, as shown in Fig. 5, the bulk integration values

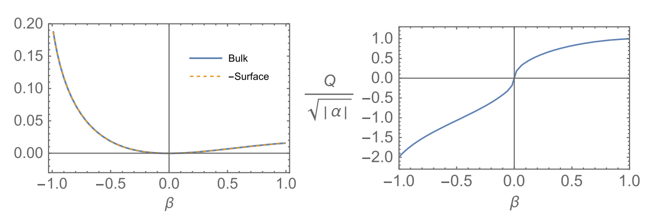


FIG. 3. Old no-hair theorem: For $f = \alpha e^{\gamma\varphi}$ with $\varphi_h = 0.1$ (left) plot of bulk and surface term (right) the scalar charge $Q/\sqrt{|\alpha|}$ vs $\beta$.

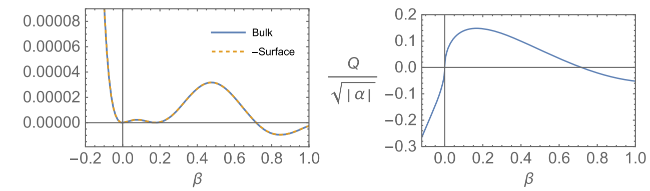


FIG. 5. Old no-hair theorem: For $f = \alpha\varphi^2$ with $\varphi_h = 0.1$ (left) plot of bulk and surface term (right) the scalar charge $Q/\sqrt{|\alpha|}$ vs $\beta$.

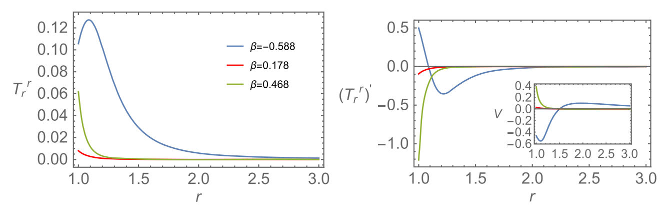


FIG. 6. Novel no-hair theorem: For $f = \alpha\varphi^2$ and $\varphi_h = 0.1$, (left) $T_r{}^r$ and (right) $(T_r{}^r)'$ and $V$.

are growing in the negative $\beta$ region. In addition, when $\varphi_\infty \neq 0$ the surface term survives unless $\varphi_1 = 0$ and the equality (21) becomes nontrivial, which indicates the existence of black hole solutions. Thus the old no-hair theorem is expected to be evaded and this is verified by generating numerical solutions in Fig. 4. The comparison between the bulk and surface terms are then plotted in Fig. 5 and show that the bulk term is exactly the same as the surface term with the opposite sign. To explicitly demonstrate our argument for the novel no-hair theorem, the energy-momentum tensor $T_r{}^r$ and its derivative $(T_r{}^r)'$ are depicted in Fig. 6, which shows that $(T_r{}^r)'$ can take positive values with $V$ being negative, while smoothly connecting the two asymptotic regimes. This demonstrates that one may obtain solutions for any of the regions displayed in Fig. 1.

## VI. CONCLUSION

We investigated the no-hair theorems in ESGB theory previously studied in [13,14]. In the formulation of the old no-hair theorem, the surface term has so far been neglected. However, due to the nonminimal coupling, the asymptotic behaviors of the scalar fields are drastically altered and hence the surface term cannot be ignored in principle. Consequently, the evasion of the old no-hair theorem should be discussed taking the presence of the surface term into account. We provided the right criteria for the old no-hair theorem to hold or to be evaded. This explains the existence of the numerical solutions when the coupling function is negative, which was excluded in the previous studies [13,14], and this fact is demonstrated for the cases of $f = \alpha e^{\gamma\varphi}$ and $f = \alpha\varphi^2$. In the case of the novel no-hair theorem, one generally expects it to be evaded for non-minimal couplings since the original version explicitly assumed a minimal coupling to gravity. We confirm this fact by finding regular solutions numerically, which merely obey the regularity condition (11) on the horizon and we find that it is not necessary for the derivative of the energy-momentum tensor to be negative. Instead, $(T_r{}^r)'$ may admit either sign on the horizon while still yielding an acceptable solution. In summary, the novel no-hair theorem is evaded automatically in ESGB theory while the old no-hair theorem holds for a very specific choice of parameters and is, in fact, the only way to limit the possible black hole solutions with nontrivial scalar hair.

This study explicitly clarified the parameter regimes of new black hole solutions in ESGB theory and depending on the sign of the coupling function the black hole solutions will show the distinctive properties from ones of standard general relativity. Due to the recent illuminating development of the astrophysical observation, we expect that this deviation can be tested in various ways of detection. First, one of aims for the gravitational waves analysis is to test classical general relativity since Einstein's theory of gravity is considered as an effective theory. One idea to do so is to test the no-hair theorem [35]. Analyzing the data of gravitational waves in the ringdown phase from the LIGO can be compared to the waveform that is numerically generated with given mass and angular momentum in Einstein theory [25]. The agreement can prove the no-hair theorem which is predicted by the standard general relativity, but the disagreement will indicate that the black holes have more hair than just mass and angular momentum. In the latter case, it would be interesting to see if ESGB theory corrects the deviation. Second, these solutions can be used to obtain the direct image of the shadow of the black hole as observed by the Einstein Horizon Telescope. If we know the mass of the black hole with high accuracy for Sagittarius A* (Sgr A*) or Messier 87 * (M87*) we get a precise prediction for the shadow radius analytically. However, in theories such as ESGB theory this shows that the radius deviates from a Schwarzchild black hole for a given mass. Since these supermassive black holes have very low angular momenta the present work is a good approximation and therefore our results are directly applicable for constraining the theory using the observed black hole shadow [27,28].

## ACKNOWLEDGMENTS

We wish to thank Jinho Kim, Chang Sub Shin, and Sang Hui Im for useful discussion. We were supported by the Institute for Basic Science (Grants No. IBS-R018-D1, No. IBS-R018-Y1). We appreciate APCTP for its hospitality during completion of this work.

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