Connection between cosmological time and the constants of nature

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We examine in greater detail the proposal that time is the conjugate of the constants of nature. Fundamentally distinct times are associated with different constants, a situation often found in "relational time" settings. We show how in regions dominated by a single constant the Hamiltonian constraint can be reframed as a Schrödinger equation in the corresponding time, solved in the connection representation by outgoing-only monochromatic plane waves moving in a "space" that generalizes the Chern-Simons functional (valid for the equation of state $w = -1$) for other w. We pay special attention to the issues of unitarity and the measure employed for the inner product. Normalizable superpositions can be built, including solitons, "light-rays" and coherent/squeezed states saturating a Heisenberg uncertainty relation between constants and their times. A healthy classical limit is obtained for factorizable coherent states, both in monofluid and multifluid situations. For the latter, we show how to deal with transition regions, where one is passing on the baton from one time to another, and investigate the fate of the subdominant clock. For this purpose minisuperspace is best seen as a dispersive medium, with packets moving with a group speed distinct from the phase speed. We show that the motion of the packets' peaks reproduces the classical limit even during the transition periods, and for subdominant clocks once the transition is over. Deviations from the coherent/semiclassical limit are expected in these cases, however. The fact that we have recently transitioned from a decelerating to an accelerating Universe renders this proposal potentially testable, as explored elsewhere.

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I. INTRODUCTION

The problem of time in general relativity [1–4] and the mystery of the origin and value of the constants of nature [5] are two well-known conundrums embedded in the foundations of physics. In [6] we suggested that they could be inextricably intertwined. *A priori* this should certainly be the case. Physical time concerns relational *change*. In contrast, the constants of nature, if true to their name, are the hallmarks of *immutability*. We could therefore expect that time and the constants are conjugate dynamical variables, or at least complementary in a quantum sense. Should we, therefore, promote the constants to observables, with their complementaries providing physical definitions of time?

Naturally, a great many questions pour in. Foremost, given the plethora of constants (some more fundamental than others [7]), we have to settle on whether we should contend with different physical times, or if, instead, a select constant is the progenitor of a single time. In [6] we suggested following the first route. A clock is built with what is at hand. Depending on the constant(s) dominating the dynamics, different phase space regions should employ different times. Thus, we are led to times variously conjugate to the cosmological constant, Λ, the gravitational constant, G_N, or even the speed of light, c. Within such a pragmatic approach we need to know how to pass the baton from one clock to another. This adjustment of clocks should be seen as a physical feature of our world.

Our proposal may sound radically new, but in fact it is rooted in well-known literature. In the context of Λ it follows directly from findings in unimodular gravity [8] (in particular in the formulation of Henneaux and Teitleboim [9]), where the conjugate of Λ is identified with a time variable (unimodular time [9–11], a 4D version of Misner's volume time [12]). The proposal in [6] amounts to extending these ideas and applying them to constants other than Λ. A further point of contact is the concept of Chern-Simons time [13], related to York time [14], to be reinterpreted here.

The plan of this paper is as follows. In Sec. II we start by reviewing the origins of our proposal: the fully diffeomorphism invariant formulation of unimodular gravity [8,9], i.e. the formulation [9] which, contrary to its name, does *not* restrict the theory to unimodular diffeomorphisms. Our construction can be seen as a generalization

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of the prescription found in [9], targeting constants other than Λ. A reduction to minisuperspace is then presented in Sec. III, recovering the formalism in [6]. We quantize the problem and present general solutions for a single perfect fluid.

In Sec. IV we make further connections with previous literature by illustrating this procedure in the case of pure Λ. We show that in the connection representation the monochromatic plane waves move in a time (proportional to the time defined in [9]) conjugate to $1/\Lambda$ (seen as a "frequency"), and in a space which is the Chern-Simons (CS) functional (so that the spatial part of the wave is the real CS state [15]). But crucially, we can now superpose the monochromatic plane waves into normalizable solutions, as we show in Sec. V, where we identify solitons, light rays, and coherent/squeezed states. In Sec. VI we present a straightforward generalization to universes dominated by radiation and fluids with generic equations of state. In each of these the constant of choice is different, so that the chosen time is different. Having a quantum time variable and the ability to find peaked wave packets is instrumental in finding the correct classical limit. We explain how this is achieved by coherent states in Sec. VII, once classical cosmology is rephrased in the connection representation and with constant times.

The rest of the paper is spent formalizing how to change the clock in multifluid situations, as already sketched in [6] and reviewed in Sec. VIII. In Sec. IX we show how minisuperspace can be seen as a dispersive medium, with wave numbers that can be position and frequency dependent. By examining the associated group speed we can then find the equations of motion of the peak of suitable wave functions. Using this technique (and assuming that the wave function remains peaked) we prove in Sec. X that the correct semiclassical limit is still obtained in crossover regions. We also illuminate the fate of the minority clock once the handover of clocks is completed. In Sec. XI we formalize the reasons why a clock should indeed be built with "what is at hand," exposing the limitations of minority clocks.

In a concluding section we summarize our findings and discuss their ultimate implications.

II. DECONSTANTIZATION

The construction in this paper can be seen as a generalization of the fully diffeomorphism invariant formulation of unimodular gravity [8,9] (i.e. the formulation which does *not* restrict to unimodular diffeomorphisms). The unimodular theory of gravity is renown for converting the cosmological constant from a fixed parameter in the Lagrangian into an integration constant, owing its constancy to an equation of motion, i.e. demoting it to an on-shell constant only. To this process we will call "deconstantization."

In the formulation of [9], which we now review, one adds to a "base theory"[1] with action S_0 a new term:

$$S_0 \to S = S_0 - \int d^4x \Lambda \partial_\mu T^\mu_U = S_0 + \int d^4x (\partial_\mu \Lambda) T^\mu_U \quad (1)$$

(the two equivalent up to a boundary term). Λ is a scalar and T^μ_U is a vector *density*, so that the added term is indeed diffeomorphism invariant (note that for a density $\nabla_\mu T^\mu_U = \partial_\mu T^\mu_U$, and that the integrand has the correct weight for the integral to be a scalar). Upon a $3+1$ split, T^0_U becomes the canonical conjugate of Λ, and it was pointed out in [9–11] that T^0_U can be used to craft a definition of time. The density T^μ_U does not appear in S_0; therefore,

$$\frac{\delta S}{\delta T^\mu_U} = 0 \Rightarrow \partial_\mu \Lambda = -\frac{\delta S_0}{\delta T^\mu_U} = 0, \quad (2)$$

i.e. the promised on-shell constancy of Λ. The other equation of motion is

$$\frac{\delta S}{\delta \Lambda} = 0 \Rightarrow \partial_\mu T^\mu_U = \frac{\delta S_0}{\delta \Lambda} = -\frac{\sqrt{-g}}{8\pi G_N}. \quad (3)$$

As suggested in [9], the gauge invariance $T^\mu \to T^\mu + \epsilon^\mu$ (with $\partial_\mu \epsilon^\mu = 0$) implies that we should only consider as physical the zero mode of T^0, and none of its other components (this point is irrelevant in a minisuperspace reduction). Given (3) we find that on shell this is nothing but unimodular time [9–11] (a 4D version of Misner's volume time [12]). The second equation of motion, Eq. (3), should then be seen as the "time formula" of the theory.

The same prescription could be applied to any other supposed constant of nature appearing in S_0 (leading to the proposal in [6], as we will show). For a vector of D constants $\boldsymbol{\alpha}$ we should take

$$S_0 \to S = S_0 - \int d^4x \boldsymbol{\alpha} \cdot \partial_\mu \boldsymbol{T}^\mu_{\boldsymbol{\alpha}}, \quad (4)$$

where the dot denotes the Euclidean inner product in D dimensional space. As with Λ above, we obtain two extra equations of motion:

$$\frac{\delta S}{\delta \boldsymbol{T}^\mu_{\boldsymbol{\alpha}}} = 0 \Rightarrow \partial_\mu \boldsymbol{\alpha} = 0, \quad (5)$$

$$\frac{\delta S}{\delta \boldsymbol{\alpha}} = 0 \Rightarrow \partial_\mu \boldsymbol{T}^\mu_{\boldsymbol{\alpha}} = \frac{\delta S_0}{\delta \boldsymbol{\alpha}}. \quad (6)$$

[1]Which here will be just the standard action of general relativity plus a cosmological constant or whatever matter content; but this could be applied to any other "base theory."

These are the on-shell constancy of $\boldsymbol{\alpha}$ and generalized time formulas (several examples of which will be studied in Sec. VII A). We may either take one constant at a time (with alternative options), or consider a multiconstant setting with concurrent multiple definitions of time. This is not unusual in relational time formulations (see for example [16] for the implications this has for the singularity problem). We explain later how such concurrent times become a function of each other classically (Sec. VII A) and semiclassically (Sec. X), despite being intrinsically different, off-shell, and quantum mechanically.

Obviously a function of a constant is also a constant, so there is a *classical* ambiguity in this construction. It leads to theories related by a canonical transformation:

$$\boldsymbol{\alpha} \to \boldsymbol{\beta}(\boldsymbol{\alpha}). \tag{7}$$

As with any such canonical transformation, the conjugates transform according to

$$\boldsymbol{T}^0_{\boldsymbol{\beta}} = \frac{\delta \boldsymbol{\alpha}}{\delta \boldsymbol{\beta}} \boldsymbol{T}^0_{\boldsymbol{\alpha}}. \tag{8}$$

More generally, on-shell, we have that

$$\partial_\mu \boldsymbol{T}^\mu_{\boldsymbol{\beta}} = \frac{\delta S_0}{\delta \boldsymbol{\beta}} = \frac{\delta \boldsymbol{\alpha}}{\delta \boldsymbol{\beta}} \frac{\delta S_0}{\delta \boldsymbol{\alpha}} = \frac{\delta \boldsymbol{\alpha}}{\delta \boldsymbol{\beta}} \partial_\mu \boldsymbol{T}^\mu_{\boldsymbol{\alpha}} \tag{9}$$

so that, using $\partial_\mu \boldsymbol{\alpha} = \partial_\mu \boldsymbol{\beta} = 0$, we have

$$\boldsymbol{T}^\mu_{\boldsymbol{\beta}} = \frac{\delta \boldsymbol{\alpha}}{\delta \boldsymbol{\beta}} \boldsymbol{T}^\mu_{\boldsymbol{\alpha}}. \tag{10}$$

All theories generated by an arbitrary choice of $\boldsymbol{\beta}(\boldsymbol{\alpha})$ are classically (or "on-shell") equivalent between themselves (and to GR). Yet their quantum mechanics is very different. For example, a state can only be coherent and factorizable for one of the choices of $\boldsymbol{\beta}(\boldsymbol{\alpha})$. The inner product we will propose is also not invariant under such transformations. This will be essential in deriving the simplest quantum theory later.

We note that several works found in the previous literature can be expressed within this framework. The solution to the cosmological constant problem going by the name of "sequester" [17,42], in its local formulation [18], is an example of one of these theories. The "fluxes" defined in [18] are nothing but the age of the Universe according to two possible times, associated with deconstantized parameters. Indeed the stabilized *observed* cosmological constant in the sequester becomes the ratio of two such ages. In this construction one takes the basis of constants:

$$\boldsymbol{\alpha} = \left(\frac{1}{16\pi G_N}, \rho_0 = \frac{\Lambda}{8\pi G_N} \right) \tag{11}$$

(where G_N is Newton's gravitational constant, we recall), i.e. the Planck mass squared and the bare vacuum energy ρ_0, respectively. This leads to times' formulas:

$$\partial_\mu \boldsymbol{T}^\mu_{\boldsymbol{\alpha}} = \sqrt{-g}(-R, 1), \tag{12}$$

(where R is the Ricci scalar) that is, unimodular time, and a version thereof weighted by the Ricci scalar, let us call it Ricci time. This leads to two ages of the Universe (considering a past and future boundary defined in [42]), $\Delta T_{\alpha 1}$ the Ricci time age, and $\Delta T_{\alpha 2}$ the volume time age. The space-time average of the Ricci scalar can be written as a ratio between these two ages:

$$\langle R \rangle = -\frac{\Delta T_1}{\Delta T_2}. \tag{13}$$

After some manipulations [18] it is then proved that the observed stabilized cosmological constant is given precisely by

$$\Lambda_{\rm obs} = \frac{1}{4} \langle R \rangle = -\frac{\Delta T_1}{4 \Delta T_2}. \tag{14}$$

In a future publication we will investigate the connection between the results in this paper and the sequester. Other similar prescriptions targeting the gravitational coupling and the Planck constant were considered in [19,20].

III. REDUCTION TO MINISUPERSPACE

It is straightforward to prove that these theories reduce to [6] in minisuperspace. We take for base action the Einstein-Cartan action reduced to homogeneity and isotropy (e.g. [21,22])

$$S_0 = 6\kappa V_c \int dt \left(\dot{b} a^2 - N a \left[-(b^2 + k) + \sum_i \frac{m_i}{a^{1+3w_i}} \right] \right), \tag{15}$$

where the last term describes a set of generic perfect fluids with equation of state w_i (to be initially examined one at a time). Here $\kappa = 1/(16\pi G_N)$, $k = 0, \pm 1$ is the normalized spatial curvature, a is the expansion factor (the only metric variable), and the connection variable b is the off-shell version of the Hubble parameter (since $b = \dot{a}$ on-shell, if there is no torsion). The Lagrange multiplier N is the lapse function and $V_c = \int d^3x$ is the comoving volume of the region under study (which could be the whole manifold, should this be compact).

The summation term can accommodate a large number of models, but their details will not be relevant here. For the cosmological constant we have $m_i = \Lambda/3$ and $w_i = -1$. For dust and radiation we have $w_i = 0, 1/3$, and we can set $m_i = C_i 8\pi G_{N0}/3$, where C_i are conserved quantities, such as those defined in [16], and the gravitational coupling is kept fixed $G_N = G_{N0}$. But we can also set $m_i = C_i 8\pi G_N/3$, and deconstantize the gravitational coupling, G_N, instead.

None of these details (leading to alternative theories) will be relevant to the solutions to be found here.

Hence, the Poisson bracket associated with the base action is

$$\{b, a^2\} = \frac{1}{6\kappa V_c}, \tag{16}$$

leading to commutator:

$$[\hat{b}, \hat{a^2}] = i\frac{l_P^2}{3V_c} = i\mathfrak{h}, \tag{17}$$

where $l_P = \sqrt{8\pi G_N \hbar}$ is the reduced Planck length, implying an effective "Planck's constant"[2]:

$$\mathfrak{h} = \frac{l_P^2}{3V_c}. \tag{18}$$

In the b representation (17) can be implemented by

$$\hat{a}^2 = -i\frac{l_P^2}{3V_c}\frac{\partial}{\partial b} = i\mathfrak{h}\frac{\partial}{\partial b}. \tag{19}$$

We now focus on the case of a single fluid with equation of state w (or an epoch where a fluid dominates all the others). From S_0, a Hamiltonian H_0 can be derived,

$$H_0 = 6\kappa V_c N a\left(-(b^2 + k) + \frac{m}{a^{1+3w}}\right), \tag{20}$$

which leads to the standard Wheeler-DeWitt (WDW) equation,

$$H_0\psi_s(b, m) = 0 \tag{21}$$

with suitable ordering [namely that implied by (20)]. A possible way to solve (21) is to solve instead $\mathcal{H}_0\psi_s(b,m) = 0$ with

$$\mathcal{H}_0 = h_\alpha(b)a^2 - \alpha = 0, \tag{22}$$

where:

$$h_\alpha(b) = (b^2 + k)^{\frac{2}{1+3w}}, \tag{23}$$

$$\alpha = m^{\frac{2}{1+3w}}, \tag{24}$$

that is, to solve

$$\left[-i\mathfrak{h}h_\alpha(b)\frac{\partial}{\partial b} - \alpha\right]\psi_s(b;\alpha) = 0. \tag{25}$$

This is a standard way to get a solution in the connection representation in the case of Λ (as we review in the next section), and generalizes trivially for radiation, and for other fluids (Sec. VI) (in multifluid cases some subtleties may arise; see Sec. X and [24]). It leads to the (real) Chern-Simons state and its adaptations. Setting

$$X_\alpha(b) = \int \frac{db}{h_\alpha(b)} \tag{26}$$

the WDW Eq. (25) becomes a plane-wave equation in X_α,

$$\left(-i\frac{l_P^2}{3V_c}\frac{\partial}{\partial X_\alpha} - \alpha\right)\psi_s = 0, \tag{27}$$

with solution

$$\psi_s(b;\alpha) = \mathcal{N}\exp\left[i\frac{3V_c}{l_P^2}\alpha X_\alpha(b)\right]. \tag{28}$$

Notice that we use the subscript α to index X_α not because it is a function of α, but because the function $X(b)$ depends on the α targeted.

Having established the base theory (no extension has been applied yet), we now subject the theory to prescription (1), targeting α (suitably normalized by 6κ for convenience). Hence, in minisuperspace,

$$S_0 \to S = S_0 + 6\kappa V_c \int dt\dot{\alpha}T_\alpha. \tag{29}$$

This addition does not change the Hamiltonian constraint, but it does introduces a new pair of canonical variables,

$$\{\alpha, T_\alpha\} = \frac{1}{6\kappa V_c}, \tag{30}$$

so that

$$[\alpha, T_\alpha] = i\mathfrak{h}. \tag{31}$$

It has the virtue of converting the WDW equation (25) into a Schrödinger equation

$$\left[-i\mathfrak{h}h_\alpha(b)\frac{\partial}{\partial b} - i\mathfrak{h}\frac{\partial}{\partial T_\alpha}\right]\psi(b, T_\alpha) = 0, \tag{32}$$

with the wave function now depending on time T_α. Its monochromatic solutions are

$$\psi(b, T_\alpha) = \psi_s(b;\alpha)\exp\left[-\frac{i}{\mathfrak{h}}\alpha T_\alpha\right] \tag{33}$$

[2]Throughout the paper we will use the shorthand $\mathfrak{h}$ or not depending on convenience, and comparison with previous literature. Its interpretation as the actual Planck constant [23] will not be relevant here.

with the "spatial" ψ_s satisfying the original (25). The full monochromatic solutions are therefore plane waves in X_α moving at fixed speed (set to 1 by the 6κ normalization of α),

$$\psi(b, T_\alpha) = \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2}\alpha(X_\alpha(b) - T_\alpha)\right], \quad (34)$$

with the choice

$$\mathcal{N} = \frac{1}{\sqrt{2\pi\mathfrak{h}}} \quad (35)$$

to be justified later. Notice that $X_\alpha(b)$ is like a linearizing variable in doubly special relativity: the speed of propagation is variable if measured in terms of the more physically available b, rather than X_α. The most general solution is a superposition of monochromatic solutions:

$$\psi(b, T_\alpha) = \int \frac{d\alpha}{\sqrt{2\pi\mathfrak{h}}}\mathcal{A}(\alpha) \exp\left[\frac{i}{\mathfrak{h}}\alpha(X_\alpha(b) - T_\alpha)\right]. \quad (36)$$

Note that the free Schrödinger equation (32) is in fact a wave equation accepting only retarded waves:

$$\left(\frac{\partial}{\partial X_\alpha} + \frac{\partial}{\partial T_\alpha}\right)\psi = 0. \quad (37)$$

Its associated conserved current is

$$j^0 = j^1 = |\psi|^2 \quad (38)$$

i.e. all the waves are outgoing (or retarded time) solutions. The general solution takes the form

$$\psi(b) = F(T_\alpha - X_\alpha), \quad (39)$$

where F can be any function. At once a definition of probability is suggested, but we defer the matter to Sec. V D.

IV. PURE LAMBDA AND A REINTERPRETATION OF CHERN-SIMONS TIME

We first illustrate these principles with the cosmological constant Λ, showing that the implications are a twist on both unimodular gravity [8,9] (specifically the time variable defined in [9]), and the concept of Chern-Simons time [13]. Indeed our proposal leads to a hybrid between these works, with a significant reinterpretation of Chern-Simons "time."

We start by reviewing some standard results. For a pure Λ we have in minisuperspace (ignoring torsion [21,25])

$$H = 6\kappa V_c N a\left(-(b^2 + k) + \frac{\Lambda}{3}a^2\right). \quad (40)$$

A direct solution to the quantum Hamiltonian constraint in the b representation

$$\left[-(b^2 + k) - i\frac{\Lambda l_P^2}{9V_c}\frac{\partial}{\partial b}\right]\psi = 0 \quad (41)$$

is given by the (real) Chern-Simons state [15,26,27] reduced to minisuperspace [21,22,25]:

$$\psi_{CS} = \mathcal{N} \exp\left[i\frac{9V_c}{\Lambda l_P^2}\left(\frac{b^3}{3} + bk\right)\right]. \quad (42)$$

As is well known, this is a pure phase, which is the product of a "frequency" proportional to $1/\Lambda$, and the "Chern-Simons time" as proposed by Smolin and Soo [13].

Something similar can be obtained from our construction. We can put the Hamiltonian constraint associated with (40) in the form

$$\mathcal{H}_0 = \frac{1}{b^2 + k}a^2 - \frac{3}{\Lambda} = 0. \quad (43)$$

By doing this the Hamiltonian constraint acquires the form (22) with

$$h_\alpha(b) = \frac{1}{b^2 + k}, \quad (44)$$

$$\alpha = \phi = \frac{3}{\Lambda}. \quad (45)$$

A Schrödinger equation (32) follows, with a time variable T_ϕ identified with p_ϕ, normalized such that

$$[\phi, T_\phi] = i\frac{l_P^2}{3V_c} \equiv i\mathfrak{h} \quad (46)$$

[cf. (29) and (31)]. Its monochromatic solutions are

$$\psi = \psi_s(b; \phi) \exp\left[-i\frac{3V_c}{l_P^2}\phi T_\phi\right] \quad (47)$$

with the "spatial" factor of the wave function satisfying

$$\left[-i\frac{l_P^2}{3V_c}h_\alpha(b)\frac{\partial}{\partial b} - \frac{3}{\Lambda}\right]\psi_s = 0. \quad (48)$$

This is the point of bringing the Hamiltonian to form (22) and choosing the ordering we chose. As in (34), the ψ_s are plane waves:

$$\psi_s(b; \phi) = \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2}\phi X_\phi(b)\right] \quad (49)$$

in "spatial" variable:

$$X_\phi(b) = \int \frac{db}{h_\alpha(b)} = \frac{b^3}{3} + kb. \tag{50}$$

This is nothing but the Chern-Simons functional (in minisuperspace) and (49) is the Chern-Simons state (42).

However, our interpretation of Chern-Simons time is different from that of Smolin and Soo. The full monochromatic solution is

$$\psi(b, T_\phi) = \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2}\phi(T_\phi - X_\phi(b))\right] \tag{51}$$

with $T_\phi \equiv p_\phi$. Hence the (unitary) time evolution happens in terms of a time which is *not* the Chern-Simons functional, but the momentum conjugate to $1/\Lambda$ (up to a conventional proportionality constant). Here $X_i = \Im(Y_{CS})$ is not a time, but a spatial variable. Time, instead, is the conjugate of $\alpha = 3/\Lambda$. The waves, however, move at constant speed (set to one by the conventional proportionality factors) in terms of X_ϕ and T_ϕ. This is true of the phase speed and also of the group speed if we construct wave packets, as we shall do in the next section. Hence the spatial X_ϕ and the time T_ϕ can be loosely confused if ψ is peaked. Its peak moves along the outgoing "light-ray" $T_\phi = X_\phi$, and hence confusing the two may be harmless for some purposes.

V. MORE GENERAL STATES

By demoting Λ to a circumstantial constant we gain more than a time variable in the quantum theory: we enlarge the space of solutions. Instead of being restricted to (51) we can now admit the most general superposition of these monochromatic plane waves:

$$\psi(b) = \int \frac{d\phi}{\sqrt{2\pi\mathfrak{h}}}\mathcal{A}(\phi)\exp\left[\frac{i}{\mathfrak{h}}\phi(X_\phi(b) - T_\phi)\right], \tag{52}$$

with the probability for the cosmological constant given from

$$P(\phi) = |\mathcal{A}(\phi)|^2 \tag{53}$$

with measure $d\phi$ (fast-forward to the end of this section for more details; also see [28] for alternatives).

Everything we state in this section about general states for Λ, parametrized by ϕ, works for any other α, with suitable modifications (i.e. $\phi \to \alpha$ in all relevant quantities).

A. Extreme cases

Two limiting cases are of interest. At one extreme we may have a completely undetermined ϕ:

$$\mathcal{A}(\phi) = \epsilon \tag{54}$$

leading to

$$\psi = \sqrt{2\pi\mathfrak{h}}\epsilon\delta(T_\phi - X_\phi). \tag{55}$$

This is the conformal constraint present in the parity-even branch of the quasitopological theories of [21,25,28], where Λ is allowed to vary by virtue of multiplying a Gauss-Bonnet topological term. In such theories $1/\Lambda$ has a conjugate momentum which is forced to equal the Chern-Simons functional by a primary constraint. Here we see that this constraint is interpreted as a "light ray" in minisuperspace: of the many waves generally acceptable only a delta function ray is possible in this theory. Time T_ϕ is fully fixed by the position X_ϕ along this ray. Whereas in standard relativity any state (52) is a solution, in these quasitopological theories one is forced to have an infinitely sharp clock, with total delocalization in "constant" ϕ.

At the opposite extreme we may completely fix ϕ:

$$\mathcal{A}(\phi) = \delta(\phi - \phi_0) \tag{56}$$

leading to

$$\psi = \mathcal{N}\exp\left[-i\frac{3V_c\phi_0}{l_P^2}(T_\phi - X_\phi(b))\right]. \tag{57}$$

This is the Chern-Simons state in the usual Einstein-Cartan theory, where Lambda is fully fixed. It implies a uniform distribution in X_ϕ. The crests of this infinite plane wave still move at the speed of light, but its "location" does not, because it is not localized. Hence time effectively disappears, since the wave function is fully smeared in X and T. This is an example of a more general fact: *infinitely sharp constants are failed clocks*. They imply complete delocalization in time. This clarifies and reinterprets the discussion on a timelike tower of turtles in [29]. Obviously these states are not strictly speaking normalizable.

B. Coherent squeezed states

In between these two extremes we can build coherent/squeezed states centred at ϕ_0:

$$\mathcal{A}(\phi) = \sqrt{\mathbf{N}(\phi_0, \sigma_\phi)} = \frac{\exp\left[-\frac{(\phi-\phi_0)^2}{4\sigma_\phi^2}\right]}{(2\pi\sigma_\phi^2)^{1/4}} \tag{58}$$

(where $\mathbf{N}$ denotes a normal distribution). Evaluating the integral we get

$$\begin{aligned}\psi(b, T) &= \mathcal{N}' \exp\left[\frac{-\sigma_\phi^2(X_\phi - T_\phi)^2 + i\phi_0(X_\phi - T_\phi)}{\mathfrak{h}^2}\right] \\ &= \mathcal{N}'\psi(b, T_\phi; \phi_0)\exp\left[-\frac{\sigma_\phi^2(X_\phi - T_\phi)^2}{\mathfrak{h}^2}\right].\end{aligned} \tag{59}$$

The last expression relates the infinite norm Chern-Simons state for ϕ_0 to the finite norm wave packet built around a

fixed Λ. We see that it is dressed by a Gaussian, which regularizes it. This is just a Gaussian distribution in $X - T$, with variance:

$$\sigma_T^2 = \sigma_X^2 = \frac{\mathfrak{h}^2}{4\sigma_\phi^2}. \tag{60}$$

A Heisenberg uncertainty principle can therefore be established, with: $\sigma_X = \sigma_T$ and

$$\sigma_T \sigma_\phi \geq \frac{l_P^2}{6V_c}. \tag{61}$$

For a coherent squeezed state this inequality is saturated.

Note that here (as in [30]) there is an ambiguity in defining zero squeezing. Requiring $\sigma_T^2 = \sigma_\phi^2 = l_P^2/(6V_c)$ would be dimensionally wrong. This is nothing but the ambiguity in defining coherent nonsqueezed states for free particles [31].[3] It is well known that, unlike for a harmonic oscillator or EM radiation, coherent states for a free particle lack a natural scale with which to define dimensionless quadratures [31,32] and the squeezing parameter. One may ignore this and simply look at squeezed-coherent states as a general class of solutions, or else introduce a scale in the problem (as was done in [30]).

C. Solitons

Note also that we do not need to use monochromatic waves to build states. As already explained, any function of the form

$$\psi = F(X_\phi - T_\phi) \tag{62}$$

would work, as we saw in the discussion leading to (39). Namely F can be just a Gaussian, without the plane wave factor, as for (59):

$$\psi(b, T) = \mathcal{N}'' \exp\left[\frac{(X_\phi - T_\phi)^2}{4\sigma^2}\right]. \tag{63}$$

Such "solitons" could be interesting. We stress that unlike coherent states, such solitons are not well localized in ϕ: for that we need the internal beats of a plane wave, for which this F would be the envelope.

D. Normalizability and measure

All of these solutions are normalizable with the "naive" inner product. No longer do we need to blame the trivial inner product for the nonnormalizability of the monochromatic solutions. Of course monochromatic solutions are strictly speaking non-normalizable by themselves; their superpositions, on the other hand, are normalizable in the standard sense.

Specifically, mimicking the procedure in [33] for the simpler current (38), we can infer the inner product

$$\langle \psi_1 | \psi_2 \rangle = \int dX_\phi \psi_1^\star(b, T_\phi) \psi_2(b, T_\phi) \tag{64}$$

with unitarity

$$\frac{\partial}{\partial T_\phi} \langle \psi_1 | \psi_2 \rangle = 0 \tag{65}$$

enforced by current conservation

$$\frac{\partial}{\partial T_\phi} \langle \psi_1 | \psi_2 \rangle = \int dX_\phi \frac{\partial}{\partial X_\phi} (\psi_1^\star(b, T_\phi) \psi_2(b, T_\phi)) = 0. \tag{66}$$

(This vanishes only with suitable boundary conditions, with subtleties like the ones highlighted in [16], e.g. singularities, etc.) We can also swap X_ϕ and T_ϕ in this definition,

$$\langle \psi_1 | \psi_2 \rangle = \int dT_\phi \psi_1^\star(b, T_\phi) \psi_2(b, T_\phi) \tag{67}$$

with the two definitions equivalent and amounting to

$$\langle \psi_1 | \psi_2 \rangle = \int d\phi \mathcal{A}_1^\star(\phi) \mathcal{A}_2(\phi), \tag{68}$$

in view of (52). The normalizability condition $|\langle \psi | \psi \rangle| = 1$ therefore supports (53) identifying the probability of ϕ.

We stress that this argument is valid for each dominating fluid i, adopting its associated α, X_α, and T_α. The argument for unitarity is far more complicated in the transition regions for multifluids, or for the minority clock, i.e. clocks corresponding to subdominant components, as we shall see later in this paper.

VI. RADIATION AND GENERAL PERFECT FLUIDS

Our approach can be used to find the equivalent of the Chern-Simons state for universes filled with a single fluid with an equations of state more general than $w = -1$.

A. The example of radiation

A radiation dominated universe ($w = 1/3$) is a particularly simple case. Then the Hamiltonian is

$$H = 6\kappa V_c \frac{N}{a} (-(b^2 + k)a^2 + m) \tag{69}$$

already in the form (22), up to a redefinition of the lapse function

[3] Our deconstantized constants can be seen as the momentum of a free abstract particle, moving with uniform speed in an abstract "space" which is our time variable.

$$\tilde{N} = 6\kappa \frac{N}{a} V_c \tag{70}$$

even off shell. Time, therefore, is the conjugate momentum to m and this can be identified with conformal time (more on this later). The monochromatic solutions to the time-dependent Schrödinger equations are

$$\psi = \psi_s(b;m) \exp\left[-\frac{i}{\mathfrak{h}} m p_m\right] \tag{71}$$

with

$$\left[i\mathfrak{h}(b^2+k)\frac{\partial}{\partial b} + m\right]\psi_s = 0 \tag{72}$$

so that we have plane waves in terms of

$$\begin{aligned} X_r(b) = \int \tfrac{db}{b^2+k} &= \tfrac{1}{\sqrt{k}} \arctan\left[\tfrac{b}{\sqrt{k}}\right] && \text{if } k > 0 \\ &= -\tfrac{1}{b} && \text{if } k = 0 \\ &= -\tfrac{1}{\sqrt{|k|}} \operatorname{argtanh}\left[\tfrac{b}{\sqrt{|k|}}\right] && \text{if } k < 0 \end{aligned}$$

to be seen as the equivalent of the Chern-Simons functional for a radiation dominated universe. The plane wave solutions at a generic time therefore are

$$\psi(b, T_r; m) = \mathcal{N} \exp\left[\frac{i}{\mathfrak{h}} m (X_r - T_r)\right] \tag{73}$$

(with $T_r = p_m$). These solutions will form the basis for the solution of the singularity problem proposed in [34].

B. One exception: Milne or curvature domination

The general solution (23) breaks down for $w = -1/3$, an equation of state degenerate with spatial curvature k (or kc^2, to put it suggestively). Backtracking to (20) we see that the problem is that we lose the spatial differential operator contained in a^2. The spatial solution then becomes $\psi_s = \delta(b^2 - m)$, where m can include, or indeed be just $-kc^2$. The monochromatic solution is

$$\psi(b, T_m; m) = \mathcal{N}\delta(b^2 - m) e^{-\frac{i}{\mathfrak{h}} m T_m} \tag{74}$$

with $T_m = p_m$ as usual. Hence in this case there is no time evolution, since for any superposition we have

$$\begin{aligned} \psi(b, T_m) &= \int dm \mathcal{A}(m) \delta(b^2 - m) e^{-\frac{i}{\mathfrak{h}} m T_m} \\ &= \mathcal{A}(b^2) e^{-\frac{i}{\mathfrak{h}} b^2 T_m} \end{aligned} \tag{75}$$

so that $|\psi|^2 = |\mathcal{A}(b^2)|^2$. The reason why this happens will be made clear in the next section.

VII. THE CLASSICAL LIMIT

Given that "time evolution" is the most obvious feature of classical cosmology, it is obvious that any quantum cosmology scheme lacking a "time" will have trouble connecting with the classical world. Reciprocally, the discovery of a quantum time should be used in the first instance to make sure that the classical limit is sound, before exploring possible quantum departures/corrections.

In this section we first present the format in which the classical results are best presented so that they can be recovered by appropriately (semi)classical wave functions, within our scheme. We then prove that coherent states reproduce the classical limit.

A. The classical "time formula"

We first find a classical expression for our physical times T_α as a function of the nonphysical coordinate time t associated with lapse N. From the second Hamilton equation [using (20), (30), and (24)] we can derive the time formula:

$$\begin{aligned} \dot{T}_\alpha = \dot{p}_\alpha = \{p_\alpha, H\} &= -\frac{1+3w}{2} N a^{-3w} m^{\frac{3w-1}{1+3w}} \\ &= -\frac{1+3w}{2} N a^{-3w} \alpha^{\frac{3w-1}{2}}. \end{aligned} \tag{76}$$

Note that we have used the original Hamiltonian, and not $\mathcal{H}_0$, to work out the relation between T_α and time. We can now set $N = 1$ to derive the relation between T_α and cosmological proper time t:

$$\frac{dT_\alpha}{dt} = -\frac{1+3w}{2} a^{-3w} m^{\frac{3w-1}{1+3w}} \tag{77}$$

or else set

$$N = N_\alpha = -\frac{2}{1+3w} a^{3w} m^{\frac{1-3w}{1+3w}} \tag{78}$$

to ensure we are using a time coordinate coincident with the physical time T.

Within this scheme (but note [35]), we highlight the following facts:

(i) Radiation is unique in that its time does not depend on m, so when this goes to zero its time is still well defined.

(ii) Specifically, radiation time is minus conformal time, $T_r = -\eta$, since

$$\dot{T}_r = -N/a. \tag{79}$$

This is in agreement with [16].

(iii) Dust time is proportional to minus proper cosmological time, with

$$T_m = -\frac{t}{2m}. \tag{80}$$

(iv) Our Lambda time is proportional to unimodular time [8]

$$\dot{T}_\phi = N\frac{a^3}{\phi^2} = N\frac{\Lambda^2}{9}a^3. \tag{81}$$

A canonical transformation relates the two: this is responsible for linearizing the dispersion relations. Unimodular time is related to Misner's volume time [12]; indeed it can be seen as a 4D version, where time is measured by the 4-volume to the past of an observer.

(v) The only degenerate case in (76) is $w = -1/3$, but this case is exceptional, as already discussed in Sec. VI B. In this case we should not transform from m to α, so that $\dot{T}_m = -Na$.

The sign in the time-formula (76) is important and we note that it changes from $w > -1/3$ to $w < -1/3$. This is a key feature of our formalism and we will comment further on this below.

B. The classical trajectory: A connection space picture

All classical descriptions are equivalent, so we may select whichever makes better contact with our quantum theory. In our case we pick a description which is unusual in the following ways:

(i) Instead of the expansion factor a, we take for dependent variable the minisuperspace connection variable b. Recall that when torsion is zero, on shell, this is the comoving inverse Hubble length $\dot{a} = b$. Quantum mechanically b is an independent and complementary variable to the metric (or rather, the densitized inverse triad a^2).

(ii) Instead of using some coordinate time t as an independent variable, we use the physical time(s) T_α. These are classically (on shell) a function of t, as just calculated in (76). Fundamentally, and quantum mechanically, there can be many T, but in the classical limit they all become functions of t (so that there is only one time classically, but there are several quantum mechanical times).

Hence, the classical description we are aiming for takes the form $b = b(T)$, possibly in the parametric form,

$$b = b(t), \tag{82}$$

$$T_\alpha = T(t), \tag{83}$$

rather than the textbook $a = a(t)$.

Then, we can show that the classical trajectory for a single fluid system is given by

$$\dot{X}_\alpha = \dot{T}_\alpha. \tag{84}$$

Indeed the full content of the classical equations can be obtained from the first Friedman equation (equivalent to the Hamiltonian constraint $H = 0$),

$$b^2 + k = \frac{m}{a^{1+3w}} \tag{85}$$

which should be assumed throughout, as well as the two dynamical Hamilton equations:

$$\dot{a} = \{a, H\} = Nb, \tag{86}$$

$$\dot{b} = \{b, H\} = -\frac{1+3w}{2}Na\frac{m}{a^{3(1+w)}} \tag{87}$$

[where we have used (85) in the second equation]. Together, these two dynamical equations imply the Raychaudhuri (second Friedman) equation (for $N = 1$):

$$\ddot{a} = -\frac{1+3w}{2}a\frac{m}{a^{3(1+w)}}. \tag{88}$$

It is easy to see that (84) implies

$$\frac{\dot{b}}{h_\alpha(b)} = -\frac{1+3w}{2}a^{-3w}m^{\frac{3w-1}{1+3w}} \tag{89}$$

which upon some manipulations reproduces the dynamical equation (87) [assuming the constraint (85) throughout].

Using this unconventional description [i.e. (84)] may take some getting used to, even though it is classically equivalent to the $a = a(t)$ description. Points of note include the following:

(i) Expanding and contracting universes correspond to $b > 0$ and $b < 0$, with $b = 0$ representing a static universe (and its vicinity the loitering model).

(ii) Hence, the ekpyrotic [36], or any such similar "bouncing" models, will see b go through zero. Tunneling between branches with different signs may also be possible.

(iii) For a given matter content, b can either only increase or only decrease in parameter time t; the first if $w < -1/3$, the second if $w > -1/3$. For $w = -1/3$ (or for the Milne universe, for example) b does not change (this starts shedding light on the anomaly found in Sec. VI B).

(iv) Hence, the equivalent of a "bounce" in b space is a universe undergoing a transition from decelerated to accelerated expansion, such as we have seemingly undergone recently. At the end of inflation the

reverse happens, the equivalent of a "turnaround" in b space.

The fact that in this picture we have recently emerged from a b bounce must have quantum mechanical implications; quantum reflection always leaves its traces. The matter will be studied in more detail in [24].

C. Parenthesis on the "arrow of time"

In view of what we said, the issue of the arrow of "time" merits a parenthesis. In the metric representation flipping the time arrow interconverts expanding (increasing a) and contracting (decreasing a) universes. There are always two branches of solutions, as required by time-reversal symmetry.

In the $b = b(T)$ description, face value, there is only one solution, for which b must increase with T_α,

$$\frac{db}{dT_\alpha} > 0, \tag{90}$$

as implied by (84). This is reflected in the quantum mechanical solutions [cf. (39)]; there can only be outgoing waves. Thus, there is a sense in which there is only one arrow of time in the connection representation and using the times T_α. A Feynman absorber is not needed to set the physical arrow of time.

This fact is actually an expression of the horizon (and ultimately also the flatness) problem, as well as its standard solution, as we now show. Let us first assume expanding universes ($b > 0$). Then, the horizon problem is that for $w > -1/3$ the comoving Hubble length $1/b$ increases in time, whereas its solution follows from that it decreases if $w < -1/3$. In our description this is expressed by (90) and the fact that, using t as the auxiliary arbitrary time arrow, the sign in the time-formula (76) changes from $w > -1/3$ to $w < -1/3$. Converting this into $b(T_\alpha)$ we then get that (90) is a statement of the horizon problem for $w > -1/3$ and its solution for $w < -1/3$.

The actual arrow of t does not matter, because it cancels in its double effect on $b = \dot{a}$ and on $\dot{b}$ [note the invariance of the Raychaudhuri equation (88) under time reversal]. Solutions to the horizon problem based on a contracting phase (e.g. the ekpyrotic scenario [36]) can be understood in our description from (90) being independent of the sign of b; however, the statement of the problem and its solution involves $|b|$, not b. So the criteria for the problem and solution are reversed for models in a contracting phase ($b < 0$), and this is still expressed by (90).

D. Coherent states and the classical limit

Having a quantum time variable and a larger space of solutions (as described in Sec. V) are the two reasons why contact with the classical limit is possible. Monochromatic waves, such as those solving the fixed constant theory, imply a uniform distribution [in $X(b)$], hardly a prediction, but they are also not immediately physical. One needs *both* a time variable *and* the ability to superpose plane waves into normalized peaked distributions to recover something minimally physical.

In fact having a peak is not enough. For example, since $\dot{X}_\alpha = \dot{T}_\alpha$ represents the classical trajectory (as just discussed), one might think that the light ray, $\psi \propto \delta(X_\alpha - T_\alpha)$, described in Sec. V A would be perfectly classical. But such a state would have a totally undefined α, and so the $T_\alpha = T_\alpha(t)$ part of the argument could not be true [note that α generally appears in the right-hand side of (76)].

The semiclassical limit is only recovered for the coherent states $\psi(b, T_\alpha)$ described in Sec. V B. For these, the second Hamilton equation (76) is true not only on average (an expression of Ehrenfest's theorem), but with minimal and balanced uncertainties in the complementary α and T_α appearing on the two sides of (76). Both sides of the argument implying that $\dot{X}_\alpha = \dot{T}_\alpha$ represents the classical trajectory can now be reproduced, and so we have a truly semiclassical state.

We can also understand the result in Sec. VI B. We do not have propagating waves in this case. However, the classical equation of motion is $\dot{b} = 0$, i.e. the universe is static in b, as already explained. Any coherent state in m therefore reproduces this result (as well as the time formula in terms of m).

VIII. MULTITIME

Naturally, we end up with the usual problem in quantum gravity: either there is no time, or, if we succeed in defining one, we are left with a multitude of choices. How do we deal with multiple times and multiple fluids, even in situations where there are epochs where one fluid dominates? The proposal in [6] was to accept this multitude of times, with the adjustment of clocks across different "time zones" to be seen as a physical feature of our world. For the rest of this paper we will examine further how the handover between clocks can be made seamless for some states.

Let $\boldsymbol{\alpha}$ be a vector with dimension D representing the whole set of relevant constants and $\boldsymbol{T}$ their conjugates. The D components of $\boldsymbol{T}$ are *a priori* independent variables, so we have a plethora of times instead of a single one.[4] Hence the "Schrödinger" equation is a partial differential equation in multiple times obtained from taking the Hamiltonian following from (15) and applying the replacement:

$$H\left[b, a^2; \boldsymbol{\alpha} \to i\frac{l_P^2}{3V_c}\frac{\partial}{\partial \boldsymbol{T}}\right]\psi = 0 \tag{91}$$

(in whatever representation, a^2, or b as chosen here). Its general solutions are

[4]We used "*a priori*" here because "*a posteriori*", i.e. on-shell or semiclassically, all the T become a function of each other, as we will see later, so that this is not to be confused with classical theories with extra time dimensions.

$$\psi(b,\boldsymbol{T}) = \int d\boldsymbol{\alpha}\, \mathcal{A}(\boldsymbol{\alpha})\, \exp\left[-i\frac{3V_c}{l_P^2}\boldsymbol{\alpha}\boldsymbol{T}\right]\psi_s(b;\boldsymbol{\alpha}), \tag{92}$$

where $\psi_s(b;\boldsymbol{\alpha})$ solves the WDW equation with constant $\boldsymbol{\alpha}$. We fix

$$|\psi_s|^2 = |\mathcal{N}_D|^2 = \frac{1}{(2\pi\hbar)^D} \tag{93}$$

to streamline the algebra.

As outlined in [6], if the Hamiltonian divides phase space into regions dominated by a single constant (or a single fluid), the readjustment of quantum clocks across such regions is seamless if we assume coherent states in all α_i *and* factorization:

$$\mathcal{A}(\boldsymbol{\alpha}) = \prod_i \sqrt{\mathbf{N}(\alpha_{0i},\sigma_i)}\,. \tag{94}$$

Then, as explained in [6], the $\psi_s(b;\boldsymbol{\alpha})$ is a piecewise plane wave in the $X_i(b)$ associated with each dominant α_i. Inserting into (92), each of these pieces gets grouped into a factor with the phase associated with the corresponding T_i (hence, the approximate single-time Schrödinger equation), producing a wave packet describing the correct classical limit (as described before).

In the next sections we will make this argument more explicit, while addressing details such as what happens in the transition regions between single fluid domination, or what happens to the minority component clocks in each phase where one fluid dominates. In order to address these questions in detail, and properly deal with multi-fluid situations, we need to first introduce a new tool for generating solutions in minisuperspace.

IX. MINISUPERSPACE AS A DISPERSIVE MEDIUM

The arguments in Sec. III for linearizing variables α, X_α, and T_α are straightforward to apply only when one fluid dominates the universe. The real world, however, is more complicated. For example, at the crossover between two epochs dominated by different fluids, we will find an X variable which is a function not only of b but also of α [as can be seen from the prescription leading to (26), and will be found explicitly in the next section]. In such cases it is more fruitful to revert to the original variable b [instead of any function $X(b)$] and regard minisuperspace as a dispersive medium. From this point of view the variables α, X_α, and T_α, when they exist, are the "linearizing variables" of the dispersive medium, to use the terminology of [37]. They can and should be used where they exist, but more generally we should face the dispersive nature of minisuperspace head on.

In the general case we can still define times $\boldsymbol{T}$ for the various $\boldsymbol{\alpha}$, impose the monochromatic ansatz (92), and find the spatial solutions ψ_s, which in general will not be plane waves in any $X(b)$ variable independent of α. The monochromatic solutions can still be superposed into peaked wave packets, as in (92). However it is important to realize that, as with any other dispersive medium, the envelope of such packets moves with a group speed that should not be confused with the phase speed.

Specifically, writing

$$\psi_s(b,\boldsymbol{\alpha}) = \mathcal{N}_D\, \exp\left[i\frac{3V_c}{l_P^2}P(b,\boldsymbol{\alpha})\right] \tag{95}$$

we identify dispersion relations:

$$\boldsymbol{\alpha}\cdot\boldsymbol{T} - P(b,\boldsymbol{\alpha}) = 0. \tag{96}$$

Assuming that the amplitude $\mathcal{A}(\boldsymbol{\alpha})$ is factorizable and sufficiently peaked around $\boldsymbol{\alpha}_0$, we can expand

$$P(b,\boldsymbol{\alpha}) = P(b;\boldsymbol{\alpha_0}) + \sum_i \left.\frac{\partial P}{\partial\alpha_i}\right|_{\boldsymbol{\alpha_0}}(\alpha_i - \alpha_{i0}) + \cdots \tag{97}$$

to find that the wave function factorizes as

$$\psi \approx \mathcal{N}_D e^{i\frac{3V_c}{l_P^2}(P(b;\boldsymbol{\alpha_0})-\boldsymbol{\alpha_0}\cdot\boldsymbol{T})}\prod_i \psi_i(b,T_i). \tag{98}$$

The first factor is the monochromatic (generally nonplane) wave centered on $\boldsymbol{\alpha_0}$. The other factors describe envelopes of the form

$$\psi_i(b,T_i) = \int d\alpha_i \mathcal{A}(\alpha_i) e^{-i\frac{3V_c}{l_P^2}(\alpha_i-\alpha_{i0})(T_i-\frac{\partial P}{\partial\alpha_i})} \tag{99}$$

which therefore move according to

$$T_i = \left.\frac{\partial P(b)}{\partial\alpha_i}\right|_{\boldsymbol{\alpha_0}}. \tag{100}$$

We can also dot this equation, to find the group speed on $\{b, T_i\}$ space:

$$c_g = \left.\frac{db}{dT}\right|_{\text{peak}} = \left.\frac{\dot b}{\dot T}\right|_{\text{peak}} = \frac{1}{\frac{\partial^2 P}{\partial\alpha\partial b}}. \tag{101}$$

The motion of these envelopes (and so of the peak of the distribution) should agree with the classical equations of motion. We will show in the rest of this paper that indeed it does so, for coherent states, in a number of nontrivial situations (such as for mixtures of fluids during transition periods when none of them dominates, or for the subdominant clock).

This obviously generalizes the construction for single fluids, for which a variable $X(b)$ can be found such that $P = \alpha X(b)$ for some α. Then, with a a suitable choice of α (and canonical T_α) we can always make the first term in the dispersion relations αT_α (for example, in the case of Lambda by $\Lambda \to \phi = 3/\Lambda$, $T_\Lambda \to T_\phi = -T_\Lambda/\phi^2$). They are "linearizing" variables because $c_{\rm lin} = \dot{X}/\dot{T} = 1$.

X. DEALING WITH CROSSOVER REGIONS

As it happens we are sitting right on a bounce in b. How do we deal with such transitions? In this section we show that the correct semiclassical limit is still obtained *assuming the wave function remains sharply peaked*. What actually happens to the wave function is left to future work [24]. We also investigate the fate of the minority clock (i.e. the radiation and Lambda clocks in the Lambda and radiation epochs) once the handover of clocks is completed. We will use as a working model a mixture of radiation and Lambda, because the algebra is clearer, but generalizations to the more relevant case of dust and Lambda behave in the same way.

A. Monochromatic solutions

Our working model has classical Hamiltonian,

$$H = Na\left(-(b^2 + k) + \frac{a^2}{\phi} + \frac{m}{a^2}\right), \tag{102}$$

spanning a two-dimensional constant space with

$$\boldsymbol{\alpha} = \left(\phi \equiv \frac{3}{\Lambda}, m\right). \tag{103}$$

Its multitime "Schrödinger" equation (91) has solutions of the form (92). One way to find the spatial ψ_s is to put $H = 0$ in the form (22) with $\alpha = \phi$, aware that $h_\alpha(b, \alpha)$ may then be a function of α too. To this end we solve the quadratic in a^2 equivalent to $H = 0$ to find

$$a^2 = \frac{g \pm \sqrt{g^2 - 4m/\phi}}{2/\phi} \tag{104}$$

with $g(b) = b^2 + k$. Since a^2 must be real (although not necessarily positive) we have

$$g^2 \geq g_0^2 = \frac{4m}{\phi} = \frac{4}{3}\Lambda m. \tag{105}$$

The plus branch contains Lambda domination when $g^2 \gg g_0^2$; the minus branch contains radiation domination, also with $g^2 \gg g_0^2$. The transition happens when $g^2 \approx g_0^2$ (with $g^2 > g_0^2$). Thus, we have a "bounce" in b space at $g = g_0$, i.e. a transition from decelerated expansion (decreasing b) to accelerated expansion (increasing b). The Hamiltonian constraint is therefore equivalent to two constraints of the required form:

$$\mathcal{H}_\pm = h_\pm(b; \phi, m)a^2 - \phi = 0 \tag{106}$$

with the important novelty that h (and so H_0) is "energy" dependent (dependent on the conjugate of time; i.e. the constants):

$$h_\pm = \frac{2}{g \pm \sqrt{g^2 - 4m/\phi}}. \tag{107}$$

This is of course irrelevant for the ψ_s, which is given by

$$\psi_{s\pm}(b; \phi, m) = \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2}\phi X_\pm(b; \phi, m)\right] \tag{108}$$

with

$$X_\pm(b; \phi, m) = \int db \frac{1}{2}\left(g \pm \sqrt{g^2 - 4m/\phi}\right). \tag{109}$$

We see that for $g^2 \gg m/\phi$ the $+/-$ branches have

$$X_+(b; \phi, m) \approx X_\phi = \frac{b^3}{3} + kb, \tag{110}$$

$$X_-(b; \phi, m) \approx \frac{m}{\phi} X_r \tag{111}$$

leading to the correct limits:

$$\psi_{s+}(b; \phi, m) \approx \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2}\phi X_\phi(b)\right], \tag{112}$$

$$\psi_{s-}(b; \phi, m) \approx \mathcal{N} \exp\left[i\frac{3V_c}{l_P^2} m X_r(b)\right]. \tag{113}$$

This illustrates with a concrete example the comments made just after Eq. (94): the $\psi_s(b; \boldsymbol{\alpha})$ is a piece-wise plane wave in the relevant α and X_α in each region of single fluid domination. To leading order it might seem that if $\mathcal{A}(\boldsymbol{\alpha})$ factorizes, then all the other times factorize, too, and stop describing the b evolution since they became b-independent phases. However this is not the case, as we now show by considering the next to leading order in the expansion.

B. What happens to the minority clock(s)?

Before addressing the handover region itself, we first examine in more detail what happens to the "minority" clock once the handover is finished. Expanding (109) to the next order we find

$$X_+(b) \approx X_\phi - \frac{m}{\phi} X_r + \cdots, \tag{114}$$

$$X_-(b) \approx \frac{m}{\phi} X_r + \frac{m^2}{\phi^2} \int \frac{db}{g^3} + \cdots. \tag{115}$$

1. The radiation clock in the Lambda epoch

Including the next order term in X_+ we find that deep in the Lambda epoch the monochromatic wave function is

$$\psi(b, \boldsymbol{T}; \boldsymbol{\alpha}) = \mathcal{N} \exp\left[-i \frac{3V_c}{l_P^2} (\phi(T_\phi - X_\phi) + m(T_r + X_r))\right].$$

Inserting into (92) we find for any factorizable amplitude

$$\psi(b, \boldsymbol{T}) = F_1(X_\phi - T_\phi) F_2(X_r + T_r). \tag{116}$$

In particular we could choose factorizable Gaussian amplitudes for ϕ and m leading to coherent F_1 and F_2 [of the form (59)]. As we will see this is more the exception than the rule.

We see that *both* factors reproduce the classical equations of motion. These amount to the first and second Friedmann equations:

$$b^2 + k = \frac{a^2}{\phi} + \frac{m}{a^2}, \tag{117}$$

$$\dot{b} = \frac{a}{\phi} - \frac{m}{a^3}. \tag{118}$$

In addition, the times formulas (replicated quantum mechanically by coherent factorizable states, just as before) are

$$\dot{T}_\phi = \frac{a^3}{\phi^2}, \tag{119}$$

$$\dot{T}_r = -\frac{1}{a}. \tag{120}$$

Evaluating

$$\dot{X}_\phi = \dot{b}(b^2 + k), \tag{121}$$

$$\dot{X}_r = \frac{\dot{b}}{b^2 + k} \tag{122}$$

we can then recover the monofluid equations of motion in the appropriate epochs with $\dot{X}_\alpha \approx \dot{T}_\alpha$, for $\alpha = \phi, m$. But some more algebra also reveals that deep in the Lambda era we can write the classical trajectory as

$$\dot{X}_r \approx -\dot{T}_r \tag{123}$$

and this is equivalent to $\dot{X}_\phi \approx \dot{T}_\phi$. Hence, the peak of both factors in (116) describes the classical trajectory.

This sheds light on what happens to our quantum "multitime" in semiclassical situations, given that classically only one time can exist. Quantum mechanically the two T_i are independent variables and fundamentally remain so, even for semiclassical states. There is never a constraint between the different T_i. What happens is that for peaked states the peak of the joint distribution maps out a trajectory of b in 2D space $\boldsymbol{T}$ (in this case $X_\phi(b) = T_\phi$ *and* $X_r(b) = -T_r$). This implies a constraint between the two times *at the peak of the joint distribution*, so classically only one time exists. The quantum fluctuations or these different times, on the other hand, would remain independent.

2. The Lambda clock in the radiation epoch

Deep in the radiation epoch we have instead

$$\psi(b, \boldsymbol{T}; \boldsymbol{\alpha}) = \mathcal{N} \exp\left[-\frac{i}{\mathfrak{h}}\left(m(T_r - X_r) + \phi\left(T_\phi - \frac{m^2}{\phi^2}\int \frac{db}{g^3}\right)\right)\right] \tag{124}$$

with the novelty that the factor associated with the subdominant component (Lambda) now depends on m as well. As a result, the wave packets never factorize into separate radiation and Lambda factors, even if the amplitudes $\mathcal{A}(\boldsymbol{\alpha})$ do. In addition the minority factor no longer is a plane wave in the original $X(b)$ and $\alpha = \phi$. Hence, even if the original amplitudes were a diagonal Gaussian, the wave functions will be very distorted. The simple arguments for unitarity for single fluids also break down for a minority clock in this situation. This will be discussed further in the next section.

Nonetheless, we can show that for a peaked second factor, the motion of the peak still reproduces the correct classical limit (the first one obviously does). Using (101) and

$$c_g^{-1} = \frac{\partial^2}{\partial\phi\partial b} \frac{m^2}{\phi} \int \frac{db}{g^3} = -\frac{m^2}{\phi^2 g^3} \tag{125}$$

we find that for the peak,

$$c_g = \left.\frac{\dot{b}}{\dot{T}_\phi}\right|_{\text{peak}} \tag{126}$$

implies

$$\dot{b} = -\frac{m}{a^3} \tag{127}$$

which is nothing but approximately (118) in the radiation epoch.

Notice that within the same approximations used in Sec. IX to derive (101), the wave function effectively factorizes as

$$\psi(b, \boldsymbol{T}; \boldsymbol{\alpha}) \approx \psi_1(T_r - X_r)\psi_2(b, T_\phi; m_0). \tag{128}$$

Hence to this order, the arguments at the end of Sec. X B 1 in support of a single classical time still apply.

C. Semiclassical limit in the transition region

We can also show that *if* the distribution remains peaked [24], then the peak follows the classical trajectory even during the b bounce, for both branches $\pm$. In this case the P function defined in Sec. IX is given by

$$P_\pm(b,m,\phi) = \phi X_\pm(b;m/\phi) = \phi \int db \frac{1}{2}\left(g \pm \sqrt{g^2 - 4m/\phi}\right). \quad (129)$$

Within the approximations of Sec. IX [see (98) in particular] the wave function must be approximately given by the monochromatic solution times the product of two envelopes $\psi_1(b,T_\phi)\psi_2(b,T_r)$. The latter move with group speeds

$$c_{g1} = \left.\frac{\dot b}{\dot T_\phi}\right|_{\text{peak}} = \frac{1}{\frac{\partial^2 P}{\partial\phi\partial b}}, \quad (130)$$

$$c_{g2} = \left.\frac{\dot b}{\dot T_r}\right|_{\text{peak}} = \frac{1}{\frac{\partial^2 P}{\partial m\partial b}}. \quad (131)$$

It is now a matter of algebra to show that in both branches ($\pm$) these are equivalent to the classical equation of motion (118) [with (117) assumed throughout].

Indeed, for the Lambda wave packet factor we have

$$\frac{\partial^2 P}{\partial\phi\partial b} = \frac{1}{h} \pm \frac{m}{\phi}\frac{1}{\sqrt{g^2 - g_0^2}}. \quad (132)$$

Using

$$\pm\sqrt{g^2 - g_0^2} = \frac{2a^2}{\phi} - g = \frac{a^2}{\phi} - \frac{m}{a^2} \quad (133)$$

and $h = \phi/a^2$ we have

$$\frac{\partial^2 P}{\partial\phi\partial b} = \frac{a^4/\phi^2}{\frac{a^2}{\phi} - \frac{m}{a^2}} \quad (134)$$

implying that the peak moves along

$$\dot b = \frac{\dot T_\phi}{a^4/\phi^2}\left(\frac{a^2}{\phi} - \frac{m}{a^2}\right) = \frac{a}{\phi} - \frac{m}{a^3} \quad (135)$$

i.e. (118), as required.

Likewise, for the m wave packet factor we have

$$\frac{\partial^2 P}{\partial m\partial b} = \mp\frac{1}{\sqrt{g^2 - g_0^2}} = -\frac{1}{\frac{a^2}{\phi} - \frac{m}{a^2}} \quad (136)$$

leading to

$$\dot b = -\dot T_r\left(\frac{a^2}{\phi} - \frac{m}{a^2}\right) \quad (137)$$

or (118).

Hence the correct classical limit is always obtained, assuming the wave functions remain peaked. Whether this is a good approximation remains to be seen [24]. In addition there are other issues regarding the semiclassical limit, as we now explain.

XI. WHY SWAP CLOCKS?

In this section we explain better why "a clock [should be] crafted with what is at hand" as proposed in [6]. This is not just common sense: It affects the semiclassical limit. As we have just seen in detail, the probability peak's motion has the correct classical limit (assuming Ehrenfest's theorem) even for the minority clock, but this hides the fact that typically the state will not be coherent in such a setup, and so departures from the semiclassical regime are expected.

The Lambda clock in the radiation epoch is a good illustration of this. As we saw in Sec. X B 2 [cf. Eq. (124)], to leading order (in the saddle approximation of Sec. IX), the Lambda factor in the radiation epoch changes its dependence on b from X_ϕ to

$$X = \int \frac{db}{g^3} \quad (138)$$

and its α from ϕ to

$$\alpha = \frac{m_0}{\phi^2}. \quad (139)$$

In Sec. X B 2 we studied in detail the peak of the wave function, but the semiclassical limit requires also the arguments in Sec. VII for the correct representation of the time formula, and so we need more than a peaked distribution; we should have a coherent state in m_0/ϕ^2. But if we chose a Gaussian amplitude in ϕ, then this will not be Gaussian in m_0/ϕ^2, quite the contrary: strong distortions are expected. This is representative of what usually happens to minority clocks. Indeed, the radiation clock in the Lambda epoch (see Sec. X B 1) is the exception to this rule. It is a rare case where a coherent majority clock remains coherent in the subdominant phase.

We can also add the issue of unitarity and inner product to the discussion. We can define a conserved inner product as in Sec. V D, but it only leads to a simple conserved current and reexpression in terms of a measure in b in the dominant epoch. As we saw, in monofluid situations (and so using the dominant clock in a multifluid situation) there is a range of options for setting up the inner product and conserved current. These are all ultimately equivalent: we can either use Eq. (64), leading to a general expression,

$$d\mu(b) = dX_\alpha = \frac{db}{(b^2+k)^{\frac{2}{1+3w}}}, \tag{140}$$

or we can trade X_α for T_α leading to (67), or we can use (141) (with ϕ replaced by the applicable α). However, only

$$\langle\psi_1|\psi_2\rangle = \int d\alpha \mathcal{A}^\star(\alpha_1)\mathcal{A}(\alpha_2) \tag{141}$$

generalizes to multifluid situations. Bearing in mind that the general solution now is

$$\psi(b,T) = \int d\alpha\, \mathcal{A}(\alpha)\, \exp\left[-i\frac{3V_c}{l_P^2}\alpha T\right]\psi_s(b;\alpha), \tag{142}$$

where $\psi_s(b;\alpha)$ solves the WDW equation with constant α, it is obvious that (141) reduces to (64) and (67) for single fluids. For multifluids we recover (67) if and only if $\psi_s(b;\alpha)$ is a pure phase (so in cases where there is no b bounce), but not (64). For multitimes this is still more complicated. But whatever the case, the definition (141) remains a general time-independent definition for the inner product. However, it is only when the dominant clock and its X are used that it leads to a simple inner product in terms of b (see [24] for further discussion).

XII. CONCLUSIONS

In summary, in this paper we proposed an amplification of the standard theory allowing the constants of nature to be nonconstant off shell. Classically, the "constants" remain constant in the equations of motion, and so nothing changes, but quantum mechanically it all changes. Each constant generates a space of quantum states composed of superpositions of waves indexed by the value of the constant (these can be seen as monochromatic partial waves). The waves propagate in a fundamentally dispersive medium where the "space" is the connection, the "time" is the momentum conjugate to the constant, and the "energy" and "momentum" are functions of the constant. In some regions (or "epochs") we can find simple linearizing variables (X_α, α and T_α), in terms of which the partial waves are plane waves moving at fixed speed, conventionally set to 1. In such regions, for a given constant, our construction provides a good clock and rod, leading to a simple inner product and definition of unitarity, with coherent states providing the perfect definition of a semiclassical state. However, this construction is never global, and hence the need to change clocks at different "time zones" in our Universe.

Specifically, we showed that the dominant fluid always generates a good clock, but the "minority clock" is generally problematic in that the linearizing variables are not always available, and so the dispersive nature of the medium has to be faced. Even if approximate linearizing variables can be found, they will generally be different functions of the constant than in the dominant phase. If the wave function was coherent in the original variable, it will not be in the second. If the wave function is peaked, the peak follows the classical equations of motion; however the quantum nature of the system can never be erased. Unitarity can be defined in a universal way, but it becomes cumbersome when written in terms of the minority clock. Changing clocks is therefore advisable.

The interesting point remains that in transition regions we may expect anomalies, and so interesting phenomenology. To make matters more poignant, we happen to be loosely sitting on the fence separating matter and Lambda domination. Our clocks, therefore, have been likely nonideal at one point somewhere a few billion years ago. The fact that the transition from deceleration to acceleration is a quantum bounce, with its inevitable ringing, only compounds the issue (see [24] for a preliminary investigation). This was followed up in [38] in a more realistic setting and in connection with the Hubble tension anomaly, providing the perfect arena for testing this idea observationally.

Obviously many questions remain. Our predictions in [24,38] will be in the form of semiclassical corrections, but suppose the universe enters one of the nonsemiclassical states also considered in this paper. How would we see a universe going quantum? We are used to quantum systems as microscopical subsystems living inside larger classical macroscopical systems; but this is just the opposite. What if our local classical world were encased in a universe which on the very largest scales is behaving quantum mechanically? How would we see it? We are currently investigating this matter within the context of the decoherent histories approach [39,40].

Even ignoring this question, other interesting problems remain. Clock swapping is essential for keeping the classical description, but why would the universe choose to swap clocks? A selection principle seems to be at play, and this may shed light on other fine-tuning problems, such as the cosmological constant problem (e.g. [17–20,41–43]). Also, what implications are there for the constants of nature if they are not allowed to be infinitely sharp in a classical world? In our framework, an infinitely sharp constant implies total delocalization in the conjugate time. In this sense, a perfect constant is a failed clock.[5] Would this have implications, either for our local physics, or for our description of the large-scale universe (the two being essentially complementary)? Finally, one may ask what happens if more than one dominant clock is at play? Preliminary work suggests that they would get in each other's way regarding classicality [35], but how problematic is this? Is this, instead, another observational window of opportunity?

[5]One may also appeal to Borges's quote, "[Eternity can be defined] as the simultaneous and lucid possession of all the instants of time". With this definition, an exact constant of Nature is equivalent to eternity *in one of the many possible times* (viz. the one dual to the infinitely sharp constant).

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