Maximum size of black holes in our accelerating Universe

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In accordance with current models of the accelerating Universe as a spacetime with a positive cosmological constant, new results about a cosmological upper bound for the area of stable marginally outer trapped surfaces are found taking into account angular momentum, gravitational waves, and matter. Compared to previous results which take into account only some of the aforementioned variables, the bound is found to be tighter, giving a concrete limit to the size of black holes especially relevant in the early Universe.

I. INTRODUCTION

According to observations of our Universe, there are two accelerating expansion phases, the current phase and the very early stage. Such a Universe is approximately described as a spacetime with a positive cosmological constant. Considering the Einstein equation with a positive cosmological constant, the simplest solution is the de Sitter spacetime. Therefore, we can obtain useful insight into our Universe by examining the characteristics of asymptotically de Sitter spacetimes. One interesting result is the presence of an upper bound for the area of event horizons. This is consistent with the intuitive feature that black holes should be smaller than the cosmological horizon, as we can see for the Schwarzschild-de Sitter solution. There are several works showing the existence of a cosmological upper bound for the area of an apparent horizon or so, in general, spacetimes/axisymmetric spacetimes [1–8] (see Sec. IV.4 in Ref. [9] for a review). In some of these, the contribution from charge and angular momentum was also considered.

In this paper, taking into account the contribution from every relevant variable we can think of, with the exception of charge, we reexamine the cosmological bound for the area of a stable marginally outer trapped surface (stable MOTS) [10] in realistic spacetimes with a positive cosmological constant. The analysis is quasilocal; that is, the asymptotic behavior of spacetime is not required to be imposed. We can justify the omission of contribution from charge by the fact that it is unlikely that charged black holes exist in our neutral Universe [11,12], so this simplifies our setup.

This paper is organized as follows. In Sec. II, we present the setup and some basic concepts. In Sec. III, we derive the cosmological upper bounds for the sizes of black holes in spacetimes with a cosmological constant. Finally, we give a summary and discussion. The derivation of the key equation of the main result is presented separately in the Appendix.

II. SETUP AND BASICS

Let us consider a compact 2-surface (S, h_{ab}) in a three-dimensional spacelike hypersurface (Σ, q_{ab}) in a four-dimensional spacetime (M, g_{ab}). Denoting the future-directed unit normal vector to Σ in M by n^a and the outward-directed spacelike unit normal vector to S in Σ by r^a, the metric g_{ab} is decomposed as

$$g_{ab} = h_{ab} + r_a r_b - n_a n_b = q_{ab} - n_a n_b, \qquad (1)$$

and one can define the extrinsic curvatures of Σ in M and S in Σ as

$$K_{ab} = q_a^c q_b^d \nabla_c n_d \qquad (2)$$

and

$$k_{ab} = h_a^c h_b^d \nabla_c r_d, \qquad (3)$$

respectively, where ∇_a is the covariant derivative with respect to g_{ab}. We decompose K_{ab} as

$$K_{ab} = K_{(r)} r_a r_b + \kappa_{ab} + v_a r_b + v_b r_a, \qquad (4)$$

where

$$K_{(r)} := K_{ab} r^a r^b, \qquad (5)$$

$$\kappa_{ab} := h_a^c h_b^d K_{cd}, \qquad (6)$$

and

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$$v_a := h_a^c r^b K_{bc}. \qquad (7)$$

After some manipulation, we obtain the derivative of $\theta_+ = \kappa + k$ along the r^a direction [13,14] (see Appendix for the full derivation),

$$r^a \nabla_a \theta_+ = -\frac{1}{2}\theta_{+ab}\theta_+^{ab} - \frac{1}{2}\theta_+^2 + \theta_+(K_{(r)} + \kappa) + \frac{1}{2}{}^{(2)}R$$
$$+ \mathcal{D}_a V^a - V_a V^a - G_{ab}k^a n^b, \qquad (8)$$

where G_{ab} is the four-dimensional Einstein tensor,

$$\theta_{+ab} := \kappa_{ab} + k_{ab}, \qquad (9)$$

$$V_a := v_a - \mathcal{D}_a \ln \varphi, \qquad (10)$$

$$k^a := n^a + r^a, \qquad (11)$$

and $\mathcal{D}_a$ is the covariant derivative with respect to h_{ab}. φ is the lapse function for the radial direction so that $q_b^c r^a \nabla_a r^b = -\mathcal{D}^c \ln \varphi$. Note that k^a is a future-directed null vector, and V^a is spacelike vector tangent to S. Since $\theta_+ = h^{ab}\nabla_a k_b = h^{ab}\theta_{+ab}$ holds, θ_+ is the outgoing null expansion rate associated with the null vector k^a.

III. COSMOLOGICAL UPPER BOUND FOR AREA OF STABLE MOTS

Following Ref. [10], the stable marginally outer trapped surface (stable MOTS) is defined as a compact 2-surface S satisfying[1]

$$\theta_+|_S = 0 \quad \text{and} \quad r^a \nabla_a \theta_+|_S \geq 0. \qquad (12)$$

Then, taking the integral of Eq. (8) over S and using the Einstein equation $G_{ab} = 8\pi T_{ab} - \Lambda g_{ab}$ with a positive cosmological constant Λ, we have

$$\frac{1}{2}\int_S {}^{(2)}R dA \geq \Lambda A + \int_S [V_a V^a + 8\pi(\rho_+ + \rho_{+gw})]dA, \qquad (13)$$

where A is the area of the stable MOTS S, $\rho_+ := T_{ab}k^a n^b = T_{ab}n^a n^b + T_{ab}r^a n^b$, and ρ_{+gw} is defined by

$$8\pi\rho_{+gw} := \frac{1}{2}\tilde{\theta}_{+ab}\tilde{\theta}_+^{ab} = \frac{1}{2}(\tilde{\kappa}_{ab}\tilde{\kappa}^{ab} + \tilde{k}_{ab}\tilde{k}^{ab}) + \tilde{\kappa}_{ab}\tilde{k}^{ab}. \qquad (14)$$

The tilde stands for the traceless part, i.e., $\tilde{\theta}_{+ab} = \theta_{+ab} - (1/2)\theta_+ h_{ab}$. Note that ρ_{+gw} is positive definite. The last expression in the right-hand side of Eq. (14) can be

seen as a summation of the energy density and flux of gravitational waves as well as in ρ_+. Now, assuming that the dominant energy condition holds, that is, $\rho_+ \geq 0$, one can see that the right-hand side of Eq. (13) is non-negative. From the Gauss-Bonnet theorem, this shows us that the topology of a stable MOTS is a 2-sphere,[2] that is, $\int_S {}^{(2)}R dA = 8\pi$. Thus, Eq. (13) gives us

$$A \leq \frac{4\pi}{\Lambda} - \frac{1}{\Lambda}\int_S [V_a V^a + 8\pi(\rho_+ + \rho_{+gw})]dA. \qquad (15)$$

This can be regarded as a generalization of the result obtained by Gibbons for stable minimal surfaces [4]. A minor arrangement gives us

$$\int_S [V^a V_a + 8\pi(\rho_+ + \rho_{+gw})]dA \leq 4\pi - \Lambda A. \qquad (16)$$

Bearing the cosmological setup in mind, one may define the surface-averaged total density $\bar{\rho}_{+tot}$ as

$$\bar{\rho}_{+tot} = \frac{1}{A}\int_S \rho_{+tot} dA, \qquad (17)$$

where $\rho_{+tot} := \rho_+ + \rho_{+gw}$. Then, Eq. (15) is simplified as

$$A \leq \frac{4\pi}{\Lambda_+} - \frac{1}{\Lambda_+}\int_S V^a V_a dA, \qquad (18)$$

where

$$\Lambda_+ := \Lambda + 8\pi\bar{\rho}_{+tot}. \qquad (19)$$

From the definition, V^a includes the contribution from the angular momentum. Indeed, v^a in V^a is approximately expressed as $v^a = h^{ab}r^c K_{bc} = -h^{ab}n^c\nabla_b r_c \sim \partial_r g_{t\varphi}$, and nontrivial $g_{t\varphi}$ appears when spacetimes have a rotation, where t, r, and φ are the time, radial, and angular coordinates. Thus, one can see how the size of stable MOTS is limited by the presence of gravitational waves, matter, and angular momentum.[3]

As an application to the event horizon of a slowly rotating Kerr–de Sitter spacetime, we can evaluate the second term in the right-hand side of Eq. (18). The metric of the Kerr–de Sitter spacetime is

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[1]In Refs. [2,15], the concept of a future trapping horizon was proposed instead, with the direction of the derivative in the second condition taken to be past-directed outgoing null.

[2]Even for cases with a nonpositive cosmological constant, Eq. (13) holds. Then, if the integration in the right-hand side of Eq. (13) is large enough that the right-hand side is positive, the topology of a stable MOTS is also shown to be a 2-sphere.

[3]Even for cases with a nonpositive cosmological constant, the result here is true if Λ_+ is positive.

$$ds^2 = -\frac{\zeta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\lambda}d\phi\right)^2 + \frac{\rho^2}{\zeta}dr^2 + \frac{\rho^2}{\chi}d\theta^2$$
$$+ \frac{\chi\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\lambda}d\phi\right)^2, \tag{20}$$

where $\zeta = (r^2+a^2)(1-\Lambda r^2/3) - 2mr$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\lambda = 1 + \Lambda a^2/3$, and $\chi = 1 + (\Lambda/3)a^2\cos^2\theta$. a is the Kerr parameter, and m is the mass parameter. The unit normal vector r^a to the horizon in the $t = \text{const}$ slice is

$$r_a = \frac{\rho}{\sqrt{\zeta}}(dr)_a, \tag{21}$$

and then, one can read φ as

$$\varphi = \frac{\rho}{\sqrt{\zeta}}. \tag{22}$$

At the leading order, the components of V_a are given by

$$V_\theta = -\partial_\theta \ln \varphi = \frac{a^2\cos\theta\sin\theta}{\rho^2} = O(a^2) \tag{23}$$

and

$$V_\phi = \frac{\sqrt{\zeta}}{\rho}K_{\phi r} = -\frac{3am}{r^2}\sin^2\theta + O(a^3). \tag{24}$$

Note that the θ component of V_a does not contribute to the evaluation at the leading order. Integrating over the event horizon, we have

$$\int_{S_H} V_a V^a dA = 24\pi\frac{(am)^2}{r_H^4} + O(a^4). \tag{25}$$

Notice that at the leading order there is no dependence on the cosmological constant. Although we can compute higher order terms of a, the expression would be rather complicated, and it would be difficult to discern a physical or geometrical meaning. Nevertheless, since $(r_H^4/24)\int_{S_H} V_a V^a dA$ gives us the angular momentum $(am)^2$ at the leading order, the above observation may prompt us to define the surface-averaged angular momentum as[4]

$$(8\pi\bar{J})^2 := \frac{A^2}{6\pi}\int_S V^a V_a dA. \tag{26}$$

Then, Eq. (18) gives us

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[4]In Ref. [16], $(8\pi\bar{J})^2 := (A^2/6\pi)\int_S v^a v_a dA$ was used. Note that v^a is in the integrand, not V^a.

$$A + \frac{6\pi(8\pi\bar{J})^2}{\Lambda_+ A^2} \le \frac{4\pi}{\Lambda_+}. \tag{27}$$

Due to the presence of the angular momentum, the area A has a lower bound too. A minor rearrangement of Eq. (27) gives us

$$(8\pi\bar{J})^2 \le \frac{2}{3}A^2\left(1 - \frac{\Lambda_+}{4\pi}A\right). \tag{28}$$

This is similar to the inequality for the Komar angular momentum J_K in axisymmetric spacetimes [6],

$$(8\pi J_K)^2 \le A^2\left(1 - \frac{\Lambda}{4\pi}A\right)\left(1 - \frac{\Lambda}{12\pi}A\right). \tag{29}$$

Note however that $\bar{J}$ does not coincide with the Komar angular momentum J_K, even for axisymmetric spacetimes. Indeed, one can show the following inequality [16,17]:

$$\bar{J}^2 \le \left(\frac{\mathcal{R}_A}{\mathcal{R}_\phi}\right)^4 J_K^2, \tag{30}$$

where $\mathcal{R}_\phi$ is defined by[5]

$$\mathcal{R}_\phi^4 := \frac{3}{8\pi}\int_S \phi_a\phi^a dA, \tag{31}$$

with ϕ^a the axisymmetric Killing vector. We used the fact that $\int_S V^a\phi_a dA = \int_S v^a\phi_a dA$. For the Kerr-de Sitter spacetime, $J_K = am/\lambda$ (for example, see Ref. [6]). Then, one can see that $J_K = \bar{J} + O(a^2)$ for small a cases.

IV. SUMMARY AND DISCUSSION

When one thinks of gravitational collapse of matter spreading over the cosmological horizon, a part of the matter of the outside region of the cosmological horizon does not collapse due to the cosmological expansion. Then, one can expect that the size of a black hole is smaller than that of the Universe, and we aim to show this. In particular, we derive a new inequality for the area of stable MOTS, such that the size of a black hole is limited by the presence of matter, gravitational waves, and angular momentum. A main application of this consequence is to give certain considerations to the population and features of primordial black holes formed in the very early stage of our Universe. When one considers standard cosmology, the surface-averaged density (plus flux) $\bar{\rho}_+ = (1/A)\int_S T_{ab}n^a k^b dA$ is composed of a homogeneous isotropic part and

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[5]Considering the inverse mean curvature flow for the spacelike hypersurface Σ, one can show that $\mathcal{R}_\phi \ge \mathcal{R}_A$ for the oblate cases and $\mathcal{R}_\phi \le \mathcal{R}_A$ for the prolate cases [16]. Only for the spherically symmetric case $\mathcal{R}_\phi = \mathcal{R}_A$ holds.

cosmological perturbations, and the latter imply only tiny effects at the second order for the inequality. However, in nonlinear events such as black hole formation, the argument would be drastically changed. Note that even if $\Lambda = 0$, we would still have a cosmological upper bound for stable MOTS due to the presence of a nonzero $\bar{\rho}_+$ in the expanding Universe, so that $\Lambda_+ \simeq 8\pi\bar{\rho}_+$ in standard cosmology as expected (see Ref. [8] for a similar argument).

One may be interested in higher-dimensional cases. For the n-dimensional cases, Eq. (13) becomes

$$\frac{1}{2}\int_S {}^{(n-1)}R\,dA \geq \Lambda A + \int_S [V_a V^a + 8\pi(\rho_+ + \rho_{+\mathrm{gw}})]dA. \tag{32}$$

This inequality does hold regardless of dimensions and the sign of the cosmological constant. For $n = 3$, we can evaluate the left-hand side from the Gauss Bonnet theorem, that is, $\int_S {}^{(n-1)}RdA = 8\pi(1 - g)$, where g is the number of genus. However, for $n \geq 4$, the Gauss-Bonnet theorem does not work, and thus, the value and sign of the left-hand side cannot be fixed. It gives us an obstacle to show the upper bound for the area of the stable MOTS in higher dimensions.

There are a few remaining issues. Equations (15) and (16) do not depend on the area-averaged quantities, which we introduce as a simplification. In reality, the exact physical meaning of the surface integral of $V^a V_a$ is unclear, although it can be definitely said that it includes the contribution from the angular momentum or a three-dimensional vector-type quantity. Furthermore, it is interesting to look at its variance for the evolution sequence of stable MOTSs. In the final phase of black hole formation, spacetime approaches a vacuum and stationary state near the black hole, and then, the stable MOTS coincides with the cross section of the event horizon. This means that the second part of the integrand in Eq. (15) decreases with evolution, but the first term does not and will be dominate in the integrand because of the existence of a possible angular momentum. In future studies, it would be nice to address these features.

APPENDIX: SKETCH OF DERIVATION OF EQ. (8)

Here, we briefly describe the derivation of Eq. (8).

From the Codazzi equation, $q_a^b R_{bc} n^c = D_b K_a^b - D_a K$, we have

$$\begin{aligned}
R_{ab} n^a r^b &= r^a D_b K_a^b - r^a D_a K \\
&= -k_{ab}\kappa^{ab} + K_{(r)}k + \mathcal{D}_a v^a + 2v^a \mathcal{D}_a \ln\varphi - r^a D_a \kappa,
\end{aligned} \tag{A1}$$

where D_a is the covariant derivative with respect to q_{ab}. From the definition of the three-dimensional Riemann tensor, ${}^{(3)}R_{ab}r^a r^b$ gives us

$$\begin{aligned}
r^a D_a k &= -k_{ab}k^{ab} - {}^{(3)}R_{ab}r^a r^b - \varphi^{-1}\mathcal{D}^2\varphi \\
&= -\frac{1}{2}k_{ab}k^{ab} - \frac{1}{2}k^2 + \frac{1}{2}{}^{(2)}R - \frac{1}{2}{}^{(3)}R - \varphi^{-1}\mathcal{D}^2\varphi,
\end{aligned} \tag{A2}$$

where, in the second line, we used the double traced Gauss equation for S in Σ.

Using the double trace of the Gauss equation for Σ in M, we have

$$G_{ab} n^a n^b = \frac{1}{2}({}^{(3)}R - K_{ab}K^{ab} + K^2). \tag{A3}$$

Using Eqs. (A1) and (A2), we can compute $r^a D_a(k + \kappa)$, and then, eliminating ${}^{(3)}R$ via Eq. (A3), we can derive Eq. (8).

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