Gauge invariance from on-shell massive amplitudes and tree-level unitarity

Da Liu[1,*] and Zhewei Yin[2,†]
[1]*Center for Quantum Mathematics and Physics (QMAP), University of California, Davis, California 95616, USA*
[2]*Department of Physics and Astronomy, Uppsala University, 75108 Uppsala, Sweden*

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We study the three-particle and four-particle scattering amplitudes for an arbitrary, finite number of massive scalars, spinors and vectors by employing the on-shell massive spinor formalism. We consider the most general three-particle amplitudes with energy-growing behavior at most of $\mathcal{O}(E)$. This is the special case of the requirement of tree unitarity, which states that the N-particle scattering amplitudes at tree level should grow at most as $\mathcal{O}(E^{4-N})$ in the high-energy hard-scattering limit, i.e., at fixed nonzero angles. Then the factorizable parts of the four-particle amplitudes are calculated by gluing the on-shell three-particle amplitudes together and utilizing the fact that tree-level amplitudes have only simple poles. The contact parts of the four-particle amplitudes are further determined by tree unitarity, which also puts strong constraints on the possible allowed three-particle coupling constants and the masses. The derived relations among them converge to the predictions of gauge invariance in the UV theory. This provides a purely on-shell understanding of spontaneously broken gauge theories.

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I. INTRODUCTION

In Ref. [1], Weinberg took the point of view that quantum field theory (QFT) is an inevitable outcome of the physical principles of quantum mechanics and special relativity. Starting from Wigner's definition of particles as irreducible representations of the inhomogeneous Lorentz group and by exploring the symmetries of the S-matrix, especially Lorentz invariance (covariance) as well as the cluster decomposition principle, it is possible to show that field theory is a natural framework to describe physics at sufficiently low energy. The central role in Wigner's classification is played by the little group for a given momentum, which is defined as the subgroup of the Lorentz group that leaves the momentum unchanged. The general unitary Lorentz transformation on the Hilbert space, $U(\Lambda)$, can be induced by transformations of the little group $W(\Lambda, p)$.

The S-matrix element is then given by the transition amplitude between the "in" and "out" states, which transforms as the direct product of one-particle states. The Lorentz covariance of S-matrix requires that there should exist one unitary operator, acting on both "in" and "out" states, which further leads to the commutation of the S-operator with the free Lorentz generators. Thus, one can show that in the time-dependent perturbation theory, if the interaction operator can be written as an integral of a scalar density, which commutes at spacelike or lightlike separations,

$$V(t) = \int d^3x \mathcal{V}(\mathbf{x}, t), \quad \text{and}$$
$$[\mathcal{V}(x), \mathcal{V}(x')] = 0, \quad \text{for } (x - x')^2 \le 0, \tag{1}$$

then the S-matrix is Lorentz covariant. Furthermore, the cluster-decomposition principle requires that the S-matrix factorizes for multiparticle processes which are sufficiently separated in space. It can be shown that if the Hamiltonian can be expressed as the sum of the products of annihilation and creation operators with coefficient functions only containing single three-momentum conservation delta function, the connected part of S-matrix will also only carry single momentum conservation delta function. This ensures that the S-matrix in the coordinate space satisfies the cluster-decomposition principle. Such considerations in addition to the requirement of Eq. (1) naturally call for quantum fields as building blocks of $\mathcal{V}(x)$.

However, huge progress on studying the scattering amplitudes for gauge theory and gravity has been made in recent years (see Refs. [2–8] for reviews) suggesting that pure on-shell ways to determine the S-matrix, without the notion of local quantum fields, are worth exploring. The

*daeliu@ucdavis.edu
†zhewei.yin@physics.uu.se

massless helicity amplitudes are natural functions of spinor-helicity variables [9–14]. At tree level they are rational functions of spinor products and only have simple poles. In particular, the marvellous simplicity of the maximal helicity violating (MHV) n-gluon amplitude (the amplitude with the maximal number of same helicities in the all-momenta-incoming convention) deduced by Refs. [15,16] has suggested that there may exist alternative ways to calculate the helicity amplitudes for the gauge theory other than the conventional field-theory Feynman-diagram approach.

Indeed, Witten's formulation of perturbative gauge theory in the twistor space [17] has motived the Cachazo-Svrcek-Witten construction of the tree-level amplitudes using MHV diagrams [18] and finally, the discovery of on-shell Britto-Cachazo-Feng-Witten recursion relations [19,20] has provided an efficient and elegant way to determine the tree-level amplitudes of Yang-Mills theory from their singularities. Note that Weinberg's argument of the Lorentz-covariance of the S-matrix is ensured by the recursion relations. This has motivated active research in the quest for the dual formulation of QFT, in which the symmetries and the simplicities of the amplitudes, such as the infinite-dimensional Yangian symmetry of $\mathcal{N} = 4$ planar supersymmetric Yang-Mills theory, are manifest, while the notion of locality or even space-time may not be apparent [21–25]. The formulation of planar $\mathcal{N} = 4$ SYM amplitudes as the volume of "amplituhedron" in Ref. [26] provides such an example; locality and unitary emerge from positivity geometry. See also Refs. [27–30] for deriving locality and unitary from other principles, e.g., gauge invariance or infrared behavior. Other interesting and exciting approaches include the color-kinematics duality and the double copy [31–33], the Cachazo-He-Yuan formalism based on scattering equations [34–37] etc.

From the more practical point of view, the idea of constructing amplitudes from the on-shell data has found many applications in the effective field theory (EFT), especially the EFTs that have enhanced soft limit, where on-shell constructability becomes possible [38–42]. It has nicely explained the one-loop nonrenormalization patterns of dimension-six operators in the Standard Model (SM) EFT [43] and leads to new nonrenormalization theorem of operator mixing [44]. It also leads to the noninterference between the SM 4-point amplitudes involving at least one transverse vector boson and the corresponding linear dimension-six operator contributions [45]. Calculations of anomalous dimensions of the effective operators have also been performed recently by using the on-shell amplitudes in Refs. [46–51]. Furthermore, the simplicity of on-shell helicity amplitudes can be used to enumerate the independent EFT operators, which was first demonstrated in the context of a gauge singlet scalar or vector coupled to gluons in Ref. [52] and further employed in Ref. [53–55].

Another step towards the on-shell formulation of QFT has been put forward by Ref. [56], in which the on-shell formalism for scattering amplitudes of general masses and spins has been systematically developed. The massive particles carry SU(2) little group indices in the form of completely symmetric tensor representations, and as a result, the Lorentz-covariance of the scattering amplitudes for spin-S particles has manifested itself as rank-$2S$ tensor. The on-shell three-particle amplitudes can be constructed systematically by the use of massive spinor kinematic variables λ_α^I, $\tilde{\lambda}_{\dot{\alpha}}^I$, and the four-particle scattering amplitudes can be derived by the fact that tree-level amplitudes have only simple poles and the residues are determined by unitary in the form of consistent factorization.

The presence of spurious nonlocal poles in the 3-point on-shell massless helicity amplitudes (in the complex momenta scheme) and the requirement of consistent factorization for 4-point amplitudes have put strong constraints on the possible structure of the interacting massless particles, such as the Yang-Mills structure for the multiple self-interacting massless spin-1 particles, and the universal couplings to the massless spin-2 particles [57]. In a similar fashion, one should be able to understand the structure of the spontaneously broken gauge theory as the consequence of the locality and perturbative-unitarity constraints on the massive amplitudes. It is known that the on-shell 3-point massive amplitudes can turn the spurious poles of massless amplitudes into some kind of mass singularities, and the Higgs mechanism can be understood as the infrared unification of different massless amplitudes in the ultraviolet [56]. The $1/m$ mass singularities in general lead to energy-growing behaviors for higher-point amplitudes involving the longitudinal components of the massive gauge bosons.

It was proven in the 1970s by the authors of Refs. [58–60] that any tree-unitary theory of massive vector bosons (with general interactions with scalars and spinors) is equivalent to a spontaneously broken gauge theory. Here tree unitarity means that the N-particle scattering amplitudes at tree level should scale at most as E^{4-N} in the high-energy (E) limit at fixed, nonzero angles, i.e., the hard-scattering limit. It can be argued that nontree-unitary theories will not be renormalizable or in the modern effective field theory language, it will have very low cutoff. In this paper, we aim to understand this from a completely on-shell point of view, using the aforementioned massive spinor helicity formalism. We will study the 3-point and 4-point scattering amplitudes for an arbitrary, finite spectrum of massive scalars, spinors, and vectors, deriving the consequence of tree unitarity. Note that the general 3-point on-shell massive amplitudes and four-particle contact terms for the SM and EFTs have been constructed by [61–68]. Reference [61] has derived the constraints among the relevant coupling and mass parameters from perturbative unitary on the four-point $\psi^c\psi Zh$ amplitude. Similar work for the on-shell description of

Higgs mechanism in the SM electroweak sector has been studied in Ref. [69]. See Ref. [70] for the consideration of finite number of massless and massive scalar fields with arbitrary local interactions, where linearized symmetry and unification can emerge from soft theorems and perturbative unitarity. See also Refs. [71–80] for related works.

The paper is organized as follows. In Sec. II, we first review the definition of the little group and its role in the classification of the irreducible representations of the inhomogeneous Lorentz group. Then we introduce on-shell massless and massive amplitudes and their interplay as the connection between UV and IR physics. In Sec. III, we discuss the relations between polarization functions and massive spinor variables and present the general relevant or marginal on-shell massive three-particle amplitudes involving arbitrary numbers of scalars, fermions and vector bosons. They are selected by studying their high-energy limit and imposing the tree-level unitarity at three-particle level, i.e., $\mathcal{M}_3 \lesssim \mathcal{O}(E)$. In Sec. IV, we move on to calculate the four-particle scattering amplitudes by obtaining the residues from gluing together the three-particle on-shell massive amplitudes and then imposing the tree-level unitarity constraint, $\mathcal{M}_4 \lesssim \mathcal{O}(E^0)$, in the high-energy limit. In addition to constraining possible contact parts of the four-particle amplitudes, we derive relations among the coupling constants and show that they converge to gauge invariance in the UV theory, and a spontaneously broken symmetry in the IR. Section V contains our conclusion and outlook. Several appendices collect our conventions and useful formulas.

II. THE LITTLE GROUP AND THE ON-SHELL MASSLESS AND MASSIVE AMPLITUDES

In this section we review the basic concepts of the little group and illustrate how the on-shell massless and massive amplitudes make the little group transformation manifest. The detailed discussion about the massless and massive spinor variables are presented in Appendix B and Appendix C, respectively.

A. Review of the little group

We start from the Wigner's definition of the little group [81]. In terms of Wigner's classification, the one-particle states can be defined as the irreducible representations of the inhomogeneous Lorentz group and the representations can be induced by the irreducible representations of the little group. Given a general momentum p^μ, the little group is the subgroup of the homogeneous Lorentz group SO(3,1) or its universal covering group $\mathrm{SL}(2,\mathcal{C})$, which leaves the momenta of the particles the same. The classification can be performed using the reference momentum trick.

For massless particles, the reference momentum can be chosen as $k^\mu = k(1,0,0,1)$, where the little group associated with this reference momentum is simply the isometry group of the two-dimensional Euclidean space ISO(2).[1] Actually, by using the explicit formulas in Eq. (A4), it is straightforward to show that the following combinations of the generators acting on the reference momentum will give zero four-vector,[2]

$$J^2 - K^1, \qquad -J^1 - K^2, \qquad J^3. \tag{2}$$

To avoid the continuum internal indices of the particles, only the subgroup $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ is considered. This means that the particles in the Hilbert space carry zero eigenvalues of the Hermitian operators corresponding to the first two generators. It is well known that the representation is the helicity of the particles. The general momentum can be obtained by the standard Lorentz transformation, which can be chosen as[3]

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|/k), \tag{3}$$

where the rotation $R(\hat{\mathbf{p}}) = \exp(-i\phi J^3)\exp(-i\theta J^2)$ transforms the z-axis into the direction of $\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and the boost $B(|\mathbf{p}|/k)$ is along the z-axis, with the nonzero components of B given by

$$B^0{}_0(u) = B^3{}_3(u) = \frac{u^2+1}{2u}, \qquad B^0{}_3(u) = B^3{}_0(u) = \frac{u^2-1}{2u}. \tag{4}$$

For massive particles, we can choose the reference momentum as the momentum in the rest frame $k^\mu = m(1,0,0,0)$. The little group is the rotation group SO(3) or its universal covering group SU(2). The generators are simply

$$J^1, \qquad J^2, \qquad J^3. \tag{5}$$

The standard Lorentz transformation, which boosts the standard momentum k^μ to the general momentum p^μ, can be chosen as

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|), \tag{6}$$

[1]Note that ISO(2) can be considered as the Innou-Wigner contraction of SO(3) with respect to its subgroup SO(2).

[2]Note that we have adopted the same convention for the definition of boosted generators as Peskin and Schroeder [82], which is different from that of Weinberg [1] by a minus sign. See Appendix A for details.

[3]In this subsection, the boldface letter **p** represents the three-momentum of the particle and in the following sections, we sometime use it for the massive on-shell spinors with symmetrized little group indices. As for the latter case, it is always associated with angle or square brackets, thus there should be no confusion.

TABLE I. The little group for the massless and massive particles. The reference momenta and the corresponding standard Lorentz transformations are also shown. Here, $\hat{\mathbf{p}}$ stands for the unit vector along the direction of the 3-momentum, i.e., $\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and η is the rapidity given by $\eta = \operatorname{arctanh}\frac{|\mathbf{p}|}{E}$. The explicit formulas for the Lorentz group generators in the vector representation are presented in Appendix A.

Standard momentum	Little group	Standard Lorentz transformation
$k^\mu = k(1,0,0,1)$	ISO (2) $J^2 - K^1, -J^1 - K^2, J^3$	$L(p) = R(\hat{\mathbf{p}})B(\lvert\mathbf{p}\rvert/k)$ $R(\hat{\mathbf{p}}) = e^{-i\phi J^3}e^{-i\theta J^2}$ $B^0{}_0(u) = B(u)^3{}_3 = \frac{u^2+1}{2u}$, $B(u)^0{}_3 = B(u)^3{}_0 = \frac{u^2-1}{2u}$
$k^\mu = m(1,0,0,0)$	SO(3) J^1, J^2, J^3	$L(p) = R(\hat{\mathbf{p}})B(\lvert\mathbf{p}\rvert)$ $R(\hat{\mathbf{p}}) = e^{-i\phi J^3}e^{-i\theta J^2}$, $B(\lvert\mathbf{p}\rvert) = e^{-i\eta K^3}$

where $B(|\mathbf{p}|) = \exp(-i\eta K^3)$ is the boost along the z-axis. Note that we have chosen a different standard Lorentz transformations from Ref. [1] and the reason will be clear later on. We summarize our previous discussions in Table I.

In the Hilbert space, the state vector for the general momentum p^μ with helicity σ of particle species n, $\Psi_{p,\sigma,n}$, can be obtained by the unitary transformation $U(L(p))$ on the state vectors associated with the standard momentum k^μ, $\Psi_{k,\sigma,n}$,

$$\Psi_{p,\sigma,n} = U(L(p))\Psi_{k,\sigma,n}. \tag{7}$$

It can be shown that once we normalize the states of the standard momentum,

$$(\Psi_{k',\sigma',n'}, \Psi_{k,\sigma,n}) = 2E_k\delta_{\sigma'\sigma}\delta_{nn'}\delta^3(\mathbf{k}' - \mathbf{k}), \tag{8}$$

the states at general momenta have the following normalization,

$$(\Psi_{p',\sigma',n'}, \Psi_{p,\sigma,n}) = 2E_p\delta_{\sigma'\sigma}\delta_{nn'}\delta^3(\mathbf{p}' - \mathbf{p}). \tag{9}$$

Given the definition, under the general proper orthochronous Lorentz transformation Λ, the state-vectors transform under the unitary operator $U(\Lambda)$ as

$$U(\Lambda)\Psi_{p,\sigma,n} = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma',n}. \tag{10}$$

Here $W(\Lambda, p)$ is the little group element defined as

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p). \tag{11}$$

For massive particles with spin j, $D_{\sigma'\sigma}(W(\Lambda,p)) = D^j_{\sigma'\sigma}(W(\Lambda,p))$ is an irreducible unitary representation of SU(2) with dimension $2j+1$, while for massless particles, since helicity is a Lorentz invariant quantity, $D_{\sigma'\sigma}(W(\Lambda,p))$ is diagonal with phase elements,

$$D_{\sigma'\sigma}(W(\Lambda,p)) = \exp(-i\theta(\Lambda,p)\sigma)\delta_{\sigma'\sigma}. \tag{12}$$

It is important to point out for massive particles that when Λ is the three-dimensional rotation R, the little group rotation $W(\Lambda, p)$ remains the same as R, i.e., $W(R,p) = R$, because R is independent of the momentum p. This can be directly derived by using the explicit formula of the standard Lorentz transformation in Eq. (6).

The S-matrix elements are defined as the probability transition amplitudes from the in states Ψ^+_α to the out state Ψ^-_β as follows:

$$S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha), \tag{13}$$

with the state labels collectively given by $\alpha = p_1\sigma_1n_1; p_2\sigma_2n_2; \cdots, \beta = p'_1\sigma'_1n'_1; p'_2\sigma'_2n'_2; \cdots$. The in and out states are transformed in the same way as the direct product of one-particle states. The Lorentz invariance of the S-matrix is defined as

$$S_{\beta\alpha} = (U(\Lambda)\Psi^-_\beta, U(\Lambda)\Psi^+_\alpha), \tag{14}$$

where the same unitary transformations acting on both in and out states are the essential part. This will give us the Lorentz covariant property of the S-matrix,

$$S_{\beta\alpha} = \sum_{\bar{\sigma}_1,\bar{\sigma}'_1,\cdots} D_{\bar{\sigma}_1\sigma_1}(W(\Lambda,p_1))D_{\bar{\sigma}_2\sigma_2}(W(\Lambda,p_2)) \cdots D^*_{\bar{\sigma}'_1\sigma'_1}(W(\Lambda,p'_1))D^*_{\bar{\sigma}'_2\sigma'_2}(W(\Lambda,p'_2))\cdots S_{\Lambda\bar{\beta},\Lambda\bar{\alpha}}. \tag{15}$$

Here $\Lambda\bar{\alpha}$ stands for $\Lambda p_1\bar{\sigma}_1n_1; \Lambda p_2\bar{\sigma}_2n_2; \cdots$ and the same applies to $\Lambda\bar{\beta}$. For massive particles with general spins, the Lorentz covariance tells us that the on-shell amplitudes are tensors under the little group SU(2), while for massless particles, the on-shell helicity amplitudes are subject to the U(1) little group phase transformations. Since in our convention, we take all the momenta ingoing, the final particle states $\Psi_{p,\sigma,n}$ are represented by the analytical continuation $(-p, -\sigma, n)$.

B. On-shell massless and massive amplitudes

The on-shell massless helicity amplitudes are naturally functions of spinor-helicity variables, which can be introduced by exploring the equivalence between the Lorentz group and the $SL(2,\mathcal{C})$ group. The on-shell condition $p^2=0$ implies that the 2×2 matrix $p_{\alpha\dot{\alpha}}=p_\mu\sigma^\mu_{\alpha\dot{\alpha}}$ has rank 1 and can be written as the direct product of two spinor vectors (see Appendix A for the summary of the notations and conventions),

$$p_{\alpha\dot{\alpha}}=\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}. \tag{16}$$

Similar to the previous discussions about the induced general Lorentz transformation from the little group transformation, we can also specify the standard Lorentz transformation on the spinor variables [56],

$$\lambda_\alpha(p)=D(L(p))_\alpha{}^\beta\lambda_\beta(k), \tag{17}$$

and therefore, under the general Lorentz transformation Λ, the spinor variable has the following little group transformation,

$$D(\Lambda)\lambda(p)=D(W(\Lambda,p))\lambda(\Lambda p). \tag{18}$$

Here for real momenta, $D(W(\Lambda,p))$ is just the U(1) little group phase factor $e^{\frac{i}{2}\theta(\Lambda,p)}$, which corresponds to $\sigma=-\frac{1}{2}$ in Eq. (12). For general complex momenta, $D(W(\Lambda,p))$ will be a complex number $w\in\mathcal{C}$, as the complexification of the little group U(1) is $GL(1,\mathcal{C})$. In these definitions (conventions), the spinor $\lambda(\tilde{\lambda})$ has helicity weight $-(+)\frac{1}{2}$. If we consider the helicity amplitudes as functions of the spinor-helicity variables $\lambda,\tilde{\lambda}$, the Lorentz covariance of the S-matrix in Eq. (15) can be stated as follows:

$$\begin{aligned}&\mathcal{M}^{h_1,\ldots,h_n}(w_1\lambda_1,w_1^{-1}\tilde{\lambda}_1;\cdots;w_n\lambda_n,w_n^{-1}\tilde{\lambda}_n)\\&=w_1^{-2h_1}\cdots w_n^{-2h_n}\mathcal{M}^{h_1,\ldots,h_n}(\lambda_1,\tilde{\lambda}_1;\cdots;\lambda_n,\tilde{\lambda}_n),\end{aligned} \tag{19}$$

where for real momenta, $w_i=e^{\frac{i}{2}\theta(\Lambda,p_i)}$.

The beauty and power about the massless on-shell amplitudes are manifest from the fact that the 3-pt on-shell amplitudes in the scheme of complex momenta are uniquely fixed by the requirement of on-shell conditions, momentum conservation and the good behaviors under the real momentum limit. To be specific, the on-shell three-particle helicity amplitudes are given by

$$\mathcal{M}_3^{h_1,h_2,h_3}=\begin{cases}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}, & h_1+h_2+h_3<0\\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}, & h_1+h_2+h_3>0\end{cases}, \tag{20}$$

which have smooth limits when we take the momenta as real quantity, i.e., $\lambda_1\propto\lambda_2\propto\lambda_3$, or $\tilde{\lambda}_1\propto\tilde{\lambda}_2\propto\tilde{\lambda}_3$ (see Appendix B for detail). Note that under the parity transformation, the spinor-helicity variables transform as[4]

$$\lambda_\alpha\to i\tilde{\lambda}^{\dot{\alpha}},\qquad \tilde{\lambda}^{\dot{\alpha}}\to i\lambda_\alpha, \tag{21}$$

which results in the interchange between the angular and square brackets,

$$\langle 12\rangle\leftrightarrow[12]. \tag{22}$$

Since the helicities change sign under the space inversion, the two cases in Eq. (20) are related by the parity transformation.

A special case corresponds to the total helicity of ± 1,

$$|h_1+h_2+h_3|=1, \tag{23}$$

where by dimensional analysis, the coupling constants associated with the helicity amplitudes have mass dimension zero. This corresponds to the marginal interaction terms in the classification of Wilson [83,84]. Let us focus on the case of $h_1+h_2+h_3=1$, then we have

$$\mathcal{M}_3^{h_1,h_2,h_3}=[12]^{1-2h_3}[23]^{1-2h_1}[31]^{1-2h_2}. \tag{24}$$

It can be immediately seen that there are always spurious poles for amplitudes involving particles with helicities greater than or equal to one. When we try to calculate the residues of four-particle scattering amplitudes in one particular channel by gluing the three-particle amplitudes together, they will always lead to poles in other channels. This plus the requirement of the unitarity in the form of consistent factorization have put strong constraints on the allowed possible interaction types and coupling structures of three-particle on-shell amplitudes [57]. In particular, the self-interacting multiple spin-1 particles must have Yang-Mills structure, and the interactions between fermions and the vector bosons must form a representation of the Lie algebra of the vector bosons. As we will see in the following discussion, in the case of massive vector bosons and fermions, the same conclusion holds and the requirement

[4]We can check explicitly that under this transformation, $p_{\alpha\dot{\alpha}}$ changes to $p^{\dot{\alpha}\alpha}$ which is consistent with the parity transformation on the momentum $(p^0,\vec{p})\to(p^0,-\vec{p})$. The presence of the factor of i is also consistent with the reality condition for the positive energy $\tilde{\lambda}_{\dot{\alpha}}=\lambda^*_\alpha$. One can also check this explicitly by using the explicit formulas of the spinor-helicity variables as functions of (θ,ϕ) and noticing that under parity transformation, $\theta\to\pi-\theta$, $\phi\to\phi+\pi$.

of consistent factorization corresponds to imposing tree-level unitarity.

The massless spinor-helicity variables have been generalized to general masses and spins by Ref. [56]. In contrast to the spinor-helicity variables, the massive spinor variables carry little group SU(2) indices,

$$\lambda_\alpha \to \lambda_\alpha^I, \qquad \tilde{\lambda}_{\dot{\alpha}} \to \tilde{\lambda}_{\dot{\alpha}}^I, \tag{25}$$

which corresponds to the spin degrees of freedom of the particles. Similar to the massless case, one can obtain the massive spinor variables at general momentum p^μ from the standard Lorentz transformation of spinor variables at standard momentum $k^\mu = m(1,0,0,0)$,

$$\lambda_\alpha^I(p) = D(L(p))_\alpha{}^\beta \lambda_\beta^I(k), \tag{26}$$

where $\lambda_\alpha^{\frac{1}{2}}(k), \lambda_\alpha^{-\frac{1}{2}}(k)$ correspond to spin-z components of $+\frac{1}{2}, -\frac{1}{2}$, respectively. Note that in our choice of standard Lorentz transformation in Eq. (6), $\lambda_\alpha^I(p)$ represents the spin component along the momentum-axis, i.e., the helicity. Once we specify the standard transformation, the general Lorentz transformation Λ on the massive spinor variables are induced by the following little group transformation,

$$D(\Lambda)\lambda^I(p) = D(W(\Lambda,p))^I{}_J \lambda^J(\Lambda p). \tag{27}$$

Since the spin-S particle carries $2S$ completely symmetric indices of SU(2), the Lorentz covariance of the S-matrix in Eq. (15) is equivalent to the statement that the corresponding scattering amplitudes are fully symmetric rank-$2S$ tensors of the massive spinor variables $\lambda^I, \tilde{\lambda}^I$. The momentum of the particle transforms trivially under the little group, thus can be constructed as an "inner" product of $\lambda^I, \tilde{\lambda}^I$,

$$p_{\alpha\dot{\alpha}} = \epsilon_{IJ}\lambda_\alpha^I \tilde{\lambda}_{\dot{\alpha}}^J = \lambda_\alpha^I \tilde{\lambda}_{I\dot{\alpha}}, \tag{28}$$

which can also be thought of as the sum of two rank 1 matrices. As in the massless case, λ_α^I is independent of $\tilde{\lambda}_I$ for the general complex momenta and the limit of real momenta can be obtained by taking

$$\tilde{\lambda}_{I\dot{\alpha}} = \pm(\lambda_\alpha^I)^*, \tag{29}$$

where the $+(-)$ sign depends on the energy being positive or negative, respectively. Note that another advantage for the massive spinor variables is the simple relations with the massless spinor variables as the high-energy limit. To see this, we can always expand the spinors in the bases of the little group space as

$$\lambda_\alpha^I = \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I}, \qquad \tilde{\lambda}_{\dot{\alpha}}^I = \tilde{\lambda}_{\dot{\alpha}} \zeta^{+I} + \tilde{\eta}_{\dot{\alpha}} \zeta^{-I}. \tag{30}$$

In terms of the expansion above, the momentum matrix can be rewritten as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} - \eta_\alpha \tilde{\eta}_{\dot{\alpha}}. \tag{31}$$

As discussed in Appendix C, with suitable sign convention, the on-shell condition of the momentum becomes

$$\langle \lambda\eta \rangle = m, \qquad [\tilde{\lambda}\tilde{\eta}] = m. \tag{32}$$

We will choose $\lambda, \tilde{\lambda}$ as the surviving parts in the high-energy limit, which will scale like $\sqrt{E}$. On the other hand, the subleading spinor variables $\eta, \tilde{\eta}$ scale like $\frac{m}{\sqrt{E}}$. This does not mean that $\eta, \tilde{\eta}$ are totally irrelevant in the high-energy limit; actually, there are always mass singularities associated with massive vector bosons. The relation between UV-massless on-shell amplitude and IR-massive amplitudes can be described as "unbold" to "bold", with the subtlety for the massive spin-one or higher-spin particles. For example, for fermion-fermion-scalar amplitudes, we have

$$\langle 12 \rangle \leftrightarrow \langle \mathbf{12} \rangle, \qquad [12] \leftrightarrow [\mathbf{12}], \tag{33}$$

while for vector-vector-scalar amplitudes, we have

$$\left(\frac{[12][23]}{[31]}, \frac{\langle 12\rangle\langle 23\rangle}{\langle 31\rangle}\right) \leftrightarrow \sqrt{2}\frac{[\mathbf{12}]\langle \mathbf{12}\rangle}{m_2}. \tag{34}$$

Here the bold notation means that the little group indices are completely symmetrized with appropriate Clebsch-Gordan coefficients [56,61]. To be more explicit, we have

$$[\mathbf{12}]\langle \mathbf{12}\rangle \equiv \frac{1}{\sqrt{2}}([1^{I_1}\mathbf{2}]\langle 1^{I_2}\mathbf{2}\rangle + [1^{I_2}\mathbf{2}\langle 1^{I_1}\mathbf{2}\rangle), \quad I_1 \neq I_2 \tag{35}$$

In the latter case, we can choose the suitable $\eta, \tilde{\eta}$ as the reference spinor in order to IR-deform the massless 3-point amplitudes (see Appendix D for detail).

III. GENERAL RELEVANT AND MARGINAL THREE-PARTICLE AMPLITUDES

In this section we will present the general on-shell massive three-particle amplitudes relevant in our calculation. We only consider the particles with spin less than or equal to one and leave other cases for future possible work. We adopt a bottom-up approach and allow the coupling constants to be arbitrary and eventually we will see that the group structure and gauge invariance will emerge from the requirement of tree-level unitarity, i.e., $\mathcal{M}_n \lesssim \mathcal{O}(E^{4-n})$. In the same spirit of Ref. [59], we consider arbitrary finite number of scalars, fermions and vectors, by which we mean the Hilbert space consists of one-particle states labeled by their momenta, helicities and species $\Psi_{p,\sigma,n}$. Note that Ref. [61] has studied the on-shell 3-point massive amplitude bases with the particle spectrum of the electroweak sector in SM in addition to one generation of fermions. We start from the discussion of the polarization functions and their relations with massless and massive spinor variables.

A. Polarization functions

In Ref. [1], the polarization functions are obtained by requiring that the quantum fields constructed out of them and the annihilation (creation) operators transform linearly under the Lorentz group and especially independent of the space-time coordinates. Specifically, the polarization functions satisfy the following conditions,

$$\begin{aligned}\sum_{\bar\sigma} u_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n)D^{(j_n)}_{\bar\sigma\sigma}(W(\Lambda,p)) &= \sum_{\ell} D_{\bar\ell\ell}(\Lambda)u_\ell(\mathbf{p},\sigma,n),\\ \sum_{\bar\sigma} v_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n)D^{(j_n)*}_{\bar\sigma\sigma}(W(\Lambda,p)) &= \sum_{\ell} D_{\bar\ell\ell}(\Lambda)v_\ell(\mathbf{p},\sigma,n),\end{aligned} \tag{36}$$

where u_ℓ and v_ℓ are the polarization functions associated with annihilation and creation operator respectively and $D_{\bar\ell\ell}(\Lambda)$ belongs to any irreducible representation of the Lorentz group.

The consequences of the above formulas can be explored by the special cases. For $\mathbf{p}=0$ and $\Lambda = L(q)$ such that $W(\Lambda,p)=1$, we have the following useful identities,

$$\begin{aligned}u_{\bar\ell}(\mathbf{q},\sigma,n) &= \sum_\ell D_{\bar\ell\ell}(L(q))u_\ell(0,\sigma,n),\\ v_{\bar\ell}(\mathbf{q},\sigma,n) &= \sum_\ell D_{\bar\ell\ell}(L(q))v_\ell(0,\sigma,n),\end{aligned} \tag{37}$$

which just tell us that the wave functions at general momentum can be obtained by the Lorentz transformations of the wave functions at zero momentum. To obtain the wave functions at zero momentum, we can take again $\mathbf{p}=0$ but $\Lambda = R$, which is a three-dimensional rotation. This time, we have

$$\begin{aligned}\sum_{\bar\sigma} u_{\bar\ell}(0,\bar\sigma,n)D^{(j_n)}_{\bar\sigma\sigma}(R) &= \sum_\ell D_{\bar\ell\ell}(R)u_\ell(0,\sigma,n),\\ \sum_{\bar\sigma} v_{\bar\ell}(0,\bar\sigma,n)D^{(j_n)*}_{\bar\sigma\sigma}(R) &= \sum_\ell D_{\bar\ell\ell}(R)v_\ell(0,\sigma,n).\end{aligned} \tag{38}$$

This establishes the relations between the representations of the little group (spins or helicities) and the representations of the polarization functions under the Lorentz group (or more precisely, the rotation subgroup). The solutions of the above equation for spin-1 and spin-1/2 can be found by exploring the explicit formulas of the representation matrices and the results are shown in Table II.

To establish the relation between the polarization functions and the massive spinor variables, we first recall the fact that the spin-j representations of the rotation group can be treated as symmetrized direct products of $2j$ spin-$1/2$ representations. The normalized tensor state of $|j,\sigma\rangle$ corresponds to the following tensor components with $2j$ indices [85],

$$\binom{2j}{j+\sigma}^{-1/2} v^{s_1\cdots s_{2j}}_{j,\sigma}, \tag{39}$$

where the completely symmetric tensor $v^{s_1\cdots s_{2j}}_{j,\sigma}$ is equal to one if there are $j+\sigma$ values of spin $\frac{1}{2}$ and $j-\sigma$ values of spin $-\frac{1}{2}$ and zero otherwise. The normalization prefactor comes from the fact that there are $\binom{2j}{j+\sigma}$ of possibilities. It also applies that the general irreducible representation of proper orthochronous Lorentz group can be thought as direct sum of spins of two particles (j_1,j_2), which are representations of complexified direct sum of two SU(2) Lie algebra $\mathfrak{su}(2)\oplus_C \mathfrak{su}(2)$,

$$\mathbf{J}_1 = \frac{1}{2}(\mathbf{J}+i\mathbf{K}),\qquad \mathbf{J}_2 = \frac{1}{2}(\mathbf{J}-i\mathbf{K}). \tag{40}$$

In the formalism of completely symmetric tensor representations of SU(2), we can think of the indices $\ell,\bar\ell$ in Eq. (38) as collections of $2j_1$ two-value indices $\alpha_1\cdots\alpha_{2j_1}$ and $2j_2$ two-value indices $\dot\alpha_1\cdots\dot\alpha_{2j_2}$. The spin label σ for particle of spin-j_n can be treated as $2j_n$ two-value indices $I_1,\cdots I_{2j_n}$. Actually, we only need the special case of $j_n = j_1 + j_2$. In this special case, the solutions to Eq. (38) can be obtained by the completely symmetric product of the building blocks $u_\alpha(0,I), u_{\dot\alpha}(I,0)$,

$$u_\alpha(0,I) = \sqrt{m}\delta_{I\alpha},\qquad u_{\dot\alpha}(I,0) = \sqrt{m}\delta_{I\dot\alpha}. \tag{41}$$

These are the massive spinor variables $\lambda,\tilde\lambda$ with the following index convention,

$$\lambda^I_\alpha(0) = u_\alpha(0,I),\qquad \tilde\lambda^{I\dot\alpha}(0) = u_{\dot\alpha}(0,I). \tag{42}$$

The normalization is chosen such that

$$\lambda^I_\alpha(0)\tilde\lambda_{I\dot\alpha}(0) = m\sigma^0_{\alpha\dot\alpha} = (p_\mu\sigma^\mu)_{\mathbf{p}=0}, \tag{43}$$

where $I,\dot\alpha$ are lowered by the antisymmetric tensor $\varepsilon_{IJ},\varepsilon_{\dot\alpha\dot\beta}$. The massive spinor variables at general momentum are given by the standard Lorentz transformation of the zero-momentum spinors

$$\begin{aligned}\lambda^I_\alpha(\mathbf{p}) &= (e^{-i\phi\frac{\sigma^3}{2}}e^{-i\theta\frac{\sigma^2}{2}}e^{-\eta\frac{\sigma^3}{2}})_{\alpha\alpha'}\lambda^I_{\alpha'}(0),\\ \tilde\lambda^{I\dot\alpha}(\mathbf{p}) &= (e^{-i\phi\frac{\sigma^3}{2}}e^{-i\theta\frac{\sigma^2}{2}}e^{\eta\frac{\sigma^3}{2}})^{\dot\alpha\dot\alpha'}\tilde\lambda^{I\dot\alpha'}(0),\end{aligned} \tag{44}$$

where η is the rapidity defined as $\cosh\eta = E/m$ and as a consequence

TABLE II. Polarization functions and spinor variables at standard and general momentum. See the main text for detailed discussion. For the general momentum for the massless particles, λ_α coincides with the high energy limit of $\lambda_\alpha^{-\frac{1}{2}}$.

Standard momentum	Polarization functions	Spinor variables
$k^\mu = k(1,0,0,1)$	$\epsilon^{+\mu} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$, $\epsilon^{-\mu} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$	$\lambda_\alpha = \sqrt{2k}\begin{pmatrix}0\\1\end{pmatrix}$, $\tilde{\lambda}^{\dot\alpha} = \sqrt{2k}\begin{pmatrix}1\\0\end{pmatrix}$, $\epsilon^-_{\alpha\dot\alpha} = \sqrt{2}\frac{\lambda_\alpha\tilde{\mu}_{\dot\alpha}}{[\tilde{\lambda}\tilde{\mu}]}$, $\epsilon^+_{\alpha\dot\alpha} = \sqrt{2}\frac{\mu_\alpha\tilde{\lambda}_{\dot\alpha}}{\langle\mu\lambda\rangle}$
$k^\mu = m(1,0,0,0)$	$\epsilon^{0\mu} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $\epsilon^{+\mu} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$, $\epsilon^{-\mu} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$ $u^{+\frac{1}{2}} = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, $u^{-\frac{1}{2}} = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$, $v^{+\frac{1}{2}} = \sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$, $v^{-\frac{1}{2}} = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$	$\lambda^I_\alpha = \sqrt{m}\delta^I_\alpha$, $\tilde{\lambda}^{I\dot\alpha} = \sqrt{m}\delta^{I\dot\alpha}$ $\epsilon^{IJ}_{\alpha\dot\alpha} = \frac{\sqrt{2}}{m}\lambda_\alpha^{\{I_1}\tilde{\lambda}_{\dot\alpha}^{I_2\}}$ $u^I = \begin{pmatrix}\lambda^I_\alpha\\ \tilde{\lambda}^{I\dot\alpha}\end{pmatrix}$ $v^I = \begin{pmatrix}\lambda^I_\alpha\\ -\tilde{\lambda}^{I\dot\alpha}\end{pmatrix}$ $\bar{u}_I = (-\lambda^\alpha_I, \tilde{\lambda}_{I\dot\alpha})$, $\bar{v}_I = (\lambda^\alpha_I, \tilde{\lambda}_{I\dot\alpha})$
General momentum $p^\mu = (E, \mathbf{p})$	$\epsilon^\mu(p) = L^\mu_\nu(p)\epsilon^\nu(k)$, $u^a(p) = D(L(p))^a_b u^b(k)$, $v^a(p) = D(L(p))^a_b v^b(k)$	$\lambda^I_\alpha(p) = D(L(p))^\beta_\alpha\lambda^I_\beta(k)$ $\tilde{\lambda}_{I\dot\alpha}(p) = D^*(L(p))^{\dot\beta}_{\dot\alpha}\tilde{\lambda}_{I\dot\beta}(k)$ $\lambda_\alpha^{\frac{1}{2}} = \sqrt{E-p}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$ $\lambda_\alpha^{-\frac{1}{2}} = \sqrt{E+p}\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$ $\tilde{\lambda}_{I\dot\alpha} = (\lambda^I_\alpha)^*$

$$\lambda^I_\alpha(\mathbf{p})\tilde{\lambda}_{I\dot\alpha}(\mathbf{p}) = p_\mu\sigma^\mu. \tag{45}$$

We then make some comments on the massless particles. For the scalar and spinor representations, satisfying Eq. (36) in the massless version is straightforward and there is no subtlety in taking the massless limit. For spin larger than or equal to one, it is not possible to satisfy the massless version of Eq. (36),

$$u_{\bar\ell}(\mathbf{p}_\Lambda, \bar\sigma)e^{-i\sigma\theta(p,\Lambda)} = \sum_\ell D_{\bar\ell\ell}(\Lambda)u_\ell(\mathbf{p},\sigma), \tag{46}$$

where $\theta(p,\Lambda)$ is the rotation angle of the little group transformation,

$$\begin{aligned} W(\Lambda, p) &= L^{-1}(\Lambda p)\Lambda L(p) \\ &= S(\alpha(p,\Lambda), \beta(p,\Lambda))R(\theta(p,\Lambda)). \end{aligned} \tag{47}$$

Here $S(\alpha,\beta)$ is the invariant Abelian subgroup of the little group ISO(2) and $R(\theta)$ is the rotation around the z-axis. To be more specific, we have

$$\begin{aligned} S(\alpha(p,\Lambda), \beta(p,\Lambda)) &= e^{-i\alpha(p,\Lambda)(J^2-K^1)}e^{-i\beta(p,\Lambda)(-J^1-K^2)}, \\ R(\theta(p,\Lambda)) &= e^{-i\theta(p,\Lambda)J^3}. \end{aligned} \tag{48}$$

To see this, let us take spin-1 as an example. We can set the momentum to the standard momentum $k^\mu = k(1,0,0,1)$ and take $\Lambda^\mu{}_\nu$ as $S^\mu{}_\nu$ or $R(\theta)$, and the conditions become respectively

$$\epsilon^\mu(\mathbf{k},\sigma)e^{-i\sigma\theta} = R^\mu{}_\nu\epsilon^\nu(\mathbf{k},\sigma), \quad \epsilon^\mu(\mathbf{k},\sigma) = S^\mu{}_\nu\epsilon^\nu(\mathbf{k},\sigma). \tag{49}$$

The solutions to the first equation read

$$\epsilon^\mu(\mathbf{k},\pm 1) = \frac{1}{\sqrt{2}}(0,1,\pm i,0), \tag{50}$$

TABLE III. The vertices in the Lagrangian and the corresponding on-shell massive amplitudes. Here c.p.t. means cyclic permutation terms. All the momenta are taking to be ingoing. The on-shell amplitudes are entirely fixed by tree unitarity; the vertices are listed only to match the conventional normalization of the coupling constants. For on-shell massive amplitudes involving fermions, our convention is $(1_{W/\phi}, 2_{\psi}, 3_{\bar{\psi}})$, and we suppress the fermion internal quantum number i_2 and i_3. The last column shows the corresponding helicity amplitudes in the high-energy limit.

Vertices	On-shell amplitude	High-energy limit
$-C_{abc}\partial_\nu W^a_\mu W^{b\mu} W^{c\nu}$	$\sqrt{2} i C_{a_1 a_2 a_3}\left(\frac{\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{31}]}{m_1 m_3} + \text{c.p.t.}\right)$	$(1^{+1}2^{-1}3^{+1}): \sqrt{2} i C_{a_1 a_2 a_3}\frac{[13]^3}{[12][23]}$ $(1^{+1}2^{0}3^{0}): i\sqrt{2} C_{a_1 a_2 a_3}\frac{(m_1^2-m_2^2-m_3^2)}{2m_2 m_3}\frac{[12][13]}{[23]}$
$-\bar{\psi}_R \not{W}_a R^a \psi_R - \bar{\psi}_L \not{W}_a L^a \psi_L$	$\frac{\sqrt{2}}{m_1}(R^{a_1}[\mathbf{12}]\langle\mathbf{13}\rangle + L^{a_1}\langle\mathbf{12}\rangle[\mathbf{13}])$	$(1^{+1}2^{+\frac{1}{2}}3^{-\frac{1}{2}}): R^{a_1}\frac{[12]^2}{[23]}$ $(1^{-1}2^{-\frac{1}{2}}3^{+\frac{1}{2}}): L^{a_1}\frac{\langle 12\rangle^2}{\langle 23\rangle}$ $(1^{0}2^{+\frac{1}{2}}3^{+\frac{1}{2}}): -\frac{m_2 L^{a_1} - m_3 R^{a_1}}{m_1}[23]$
$F_{abi} W_{a\mu} W^{b\mu}\phi_i$	$2F_{a_1 a_2 i_3}\frac{[\mathbf{12}]\langle\mathbf{21}\rangle}{m_1 m_2}$	$(1^{+1}2^{0}3^{0}): -\sqrt{2}\frac{F_{a_1 a_2 i_3}}{m_2}\frac{[12][13]}{[23]}$
$-G_{aij} W_{a\mu}\partial^\mu \phi_i \phi_j$	$\frac{i}{\sqrt{2}m_1} G_{a_1 i_2 i_3}\langle\mathbf{1}\vert p_2 - p_3\vert\mathbf{1}]$	$(1^{+1}2^{0}3^{0}): -i\sqrt{2} G_{a_1 i_2 i_3}\frac{[12][13]}{[23]}$
$-\frac{1}{6}P_{ijk}\phi_i\phi_j\phi_k$	$-P_{i_1 i_2 i_3}$	$(1^{0}2^{0}3^{0}): -P_{i_1 i_2 i_3}$
$-(\bar{\psi}_L H_i \psi_R + \bar{\psi}_R H_i^\dagger \psi_L)\phi_i$	$H_{i_1}[\mathbf{23}] + H^\dagger_{i_1}\langle\mathbf{23}\rangle$	$(1^{0}2^{+\frac{1}{2}}3^{+\frac{1}{2}}): H_{i_1}[23]$ $(1^{0}2^{-\frac{1}{2}}3^{-\frac{1}{2}}): H^\dagger_{i_1}\langle 23\rangle$

but then the second equation can't be satisfied for general parameters α, β. Instead, applying little group transformation $S(\alpha, \beta)$ will give us the polarization functions as follows:

$$\epsilon^\mu(\mathbf{k}, \pm) \to \epsilon^\mu(\mathbf{k}, \pm) - \frac{\alpha \pm i\beta}{\sqrt{2}k} k^\mu. \tag{51}$$

This is the origin of the necessity of gauge invariance. Nevertheless, the polarization vectors at the general momentum can still be obtained by the standard Lorentz transformation,

$$\epsilon^\mu(\mathbf{p}, \pm) = R(\hat{\mathbf{p}}) B(|\mathbf{p}|/k)\epsilon^\mu(\mathbf{k}, \pm) = R(\hat{\mathbf{p}})\epsilon^\mu(\mathbf{k}, \pm), \tag{52}$$

where we have used the fact the boost along the z-axis doesn't affect the x, y components. The resulting polarization vectors are the same as Eq. (B21). Under general Lorentz transformation, they transform as a vector plus an additional term proportional to the momentum,

$$\epsilon^\mu(\mathbf{p}, \pm) \to e^{-i\theta(p,\Lambda)}\epsilon^\mu(\mathbf{p}, \pm) - \frac{\alpha(p,\Lambda) \pm i\beta(p,\Lambda)}{\sqrt{2}k} p^\mu. \tag{53}$$

B. 3-point on-shell massive amplitudes

We start by listing in Table III all the interactions and the corresponding 3-point tree-unitary on-shell amplitudes for an arbitrary, finite number of massive scalars ϕ_i, fermions ψ_{i} and vectors W^a_μ, and present the derivations in the following.[5] We are adopting a purely on-shell approach in this paper, thus it is sufficient to impose tree unitarity on the complete basis of 3-point massive amplitudes given by Ref. [62]. On the other hand, we would like to make connections to the results computed using Feynman rules, thus we also calculate the same amplitudes using the Lagrangian in Ref. [59] and the polarization functions derived in Sec. III A, just to match the normalization of the coupling constants.

Let's start from the WWW on-shell amplitudes and assume that all the vector bosons are massive. A complete independent basis including seven different terms has been derived in Ref. [61] and the requirement of tree-level unitarity for the 3-point on-shell amplitude $\mathcal{M}_3 \lesssim \mathcal{O}(E)$ has singled out the following unique structure,

$$\frac{\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{31}]}{m_3 m_1} + \frac{\langle\mathbf{23}\rangle\langle\mathbf{31}\rangle[\mathbf{12}]}{m_1 m_2} + \frac{\langle\mathbf{31}\rangle\langle\mathbf{12}\rangle[\mathbf{23}]}{m_2 m_3}, \tag{54}$$

or simply

$$\frac{\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{31}]}{m_3 m_1} + \text{c.p.t.}, \tag{55}$$

where c.p.t. means cyclic permutation terms. The above is clearly totally antisymmetric in exchanging the external

[5] Notice that we are using the sans serif $\{\mathsf{i}, \mathsf{j}, \cdots\}$ to denote the fermionic internal quantum numbers, to differentiate from the scalar state labels $\{i, j, \cdots\}$.

particle labels. Therefore, the uniqueness of the on-shell amplitude and its permutation symmetry tells us that after adding the vector indices, the coupling constant $C_{a_1a_2a_3}$ will be completely antisymmetric.

This can also be seen by plugging the polarization vectors obtained in the previous section into the interaction in the Lagrangian,

$$-C_{abc}\partial_\nu W^a_\mu W^{b\mu} W^{c\nu}, \tag{56}$$

and the resulting amplitude reads[6]

$$\mathcal{M}_3(1^{a_1}, 2^{a_2}, 3^{a_3}) = \frac{i}{\sqrt{2}}\left(C_{a_1a_2a_3}\frac{\langle\mathbf{3}|p_1|\mathbf{3}][\mathbf{12}]\langle\mathbf{21}\rangle}{m_1m_2m_3} + \text{p.t.}\right). \tag{58}$$

Here p.t. means permutation terms. We first realize that the amplitude vanishes for the symmetric part of indices (a_1, a_2), i.e.,

$$C_{\{a_1a_2\}a_3} = 0. \tag{59}$$

Secondly, by using Schouten identity

$$|\mathbf{3}][\mathbf{12}] + |\mathbf{1}][\mathbf{23}] + |\mathbf{2}][\mathbf{31}] = 0, \tag{60}$$

and Dirac equations

$$p|\boldsymbol{p}] = m|\boldsymbol{p}\rangle, \qquad \langle\boldsymbol{p}|p = -m[\boldsymbol{p}|, \tag{61}$$

we can bring Eq. (58) in the form of Eq. (54), and the requirement of proportionality to Eq. (54) leads to

$$C_{[ab]c} = C_{[bc]a} = C_{[ca]b}. \tag{62}$$

Combining this with Eq. (59) again tells us that $C_{a_1a_2a_3}$ is fully antisymmetric. This leads to the following normalization of the on-shell WWW massive amplitude,

$$\mathcal{M}_3(1^{a_1}, 2^{a_2}, 3^{a_3}) = \sqrt{2}iC_{a_1a_2a_3}\left(\frac{\langle\mathbf{12}\rangle\langle\mathbf{23}\rangle[\mathbf{31}]}{m_1m_3} + \text{c.p.t.}\right). \tag{63}$$

The similar consideration of the other marginal operator that one may write down in the Lagrangian,

$$-A_{abc}\epsilon_{\mu\nu\rho\sigma}\partial^\mu W^\nu_a W^\rho_b W^\sigma_c, \qquad A_{abc} = -A_{acb}, \tag{64}$$

enforces the following relations,

$$A_{a_1a_2a_3} = A_{a_2a_3a_1} = A_{a_3a_1a_2}, \tag{65}$$

and as a result, the on-shell amplitude vanishes. Actually, one can verify that in this case, the Lagrangian is a total derivative. We arrive at Eq. (63) as our only three-vector-boson on shell massive amplitude. We will further impose that the coupling constant C_{abc} to be real, as required by the optical theorem, which demands the imaginary part of the forward scattering amplitude to be proportional to the cross section to every possible final state,

$$\text{Im}\mathcal{M}_{\alpha,\alpha} \sim \sum_\beta \sigma(\alpha \to \beta). \tag{66}$$

Since the cross section usually starts at the 3-point coupling to the fourth order for $2 \to 2$ scattering, this relation immediately tells that the imaginary part of the of four-particle amplitudes should start at loop level. By studying all possible scattering processes, it is possible to show that all coupling constants in the three-particle amplitudes should be the case to make the Lagrangian real. In the following discussion, we impose these constraints on all the couplings.

We can take the high-energy limit by specifying the spin components along the three-momentum direction and the resulting polarization amplitudes are functions of $(\lambda, \tilde{\lambda}, \eta, \tilde{\eta})$ in Eq. (30). Let's take the helicity configuration $(1^+, 2^-, 3^+)$ as an example,

$$\begin{aligned}\mathcal{M}_3(1^+, 2^-, 3^+) &= \mathcal{M}_3(1^{\frac{1}{2},\frac{1}{2}}, 2^{-\frac{1}{2},-\frac{1}{2}}, 3^{\frac{1}{2},\frac{1}{2}}) \\ &= \sqrt{2}iC_{a_1a_2a_3}\left(\frac{\langle\eta_1 2\rangle\langle 2\eta_3\rangle[31]}{m_1m_3}\right) + \mathcal{O}(m).\end{aligned} \tag{67}$$

By using the fact that for the total-plus 3-point on-shell massless amplitudes, the three angular spinors are proportional to each other, we have

$$\frac{[23]}{[31]} = \frac{\langle 1\eta_1\rangle}{\langle 2\eta_1\rangle}, \qquad \frac{[31]}{[12]} = \frac{\langle 2\eta_3\rangle}{\langle 3\eta_3\rangle}, \tag{68}$$

and it is easy to see that the helicity amplitude becomes

$$\mathcal{M}_3(1^+, 2^-, 3^+) = \sqrt{2}iC_{a_1a_2a_3}\frac{[13]^3}{[12][23]}. \tag{69}$$

Alternatively, we can start from the above on-shell massless amplitude and invert the procedure to IR deform it to the massive case (see Appendix D for detail and see also Ref. [63]). In addition to the above three massless vector amplitude, the SU(2) covariant massive amplitude in Eq. (63) naturally consists of massless vector-scalar-scalar amplitude. After an involved but straightforward calculation,

[6]Our convention for the amplitudes is the same as Peskin and Schroeder [82],

$$S_{\beta,\alpha} = \delta_{\beta,\alpha} + i(2\pi)^4\delta^{(4)}(p_\alpha - p_\beta)\mathcal{M}_{\beta,\alpha}. \tag{57}$$

but with all the momenta ingoing.

we can show that

$$\mathcal{M}_3(1^+,2^0,3^0)=i\sqrt{2}C_{a_1a_2a_3}\frac{(m_1^2-m_2^2-m_3^2)}{2m_2m_3}\frac{[12][13]}{[23]}. \quad (70)$$

Next, we consider the $W\psi\bar{\psi}$ amplitude. The complete basis for such an amplitude is given by the following four terms [62],

$$\langle\mathbf{12}\rangle\langle\mathbf{13}\rangle, \qquad [\mathbf{12}][\mathbf{13}], \qquad \langle\mathbf{12}\rangle[\mathbf{13}], \qquad \langle\mathbf{13}\rangle[\mathbf{12}], \quad (71)$$

and it is known that the former two scale as $\mathcal{O}(E^2)$ in the high-energy limit [61], thus should be dropped when imposing tree unitarity; the latter two, on the other hand, scale as $\mathcal{O}(E)$ and should remain. Now we can match the basis to the following interaction terms in the Lagrangian

$$-\bar{\psi}_R \not{W}_a R^a \psi_R - \bar{\psi}_L \not{W}_a L^a \psi_L, \quad (72)$$

where L^a, R^a are the Hermitian matrices in the space of fermion internal quantum numbers with labels $\{\mathsf{i},\mathsf{j},\cdots\}$, which we usually suppress, i.e., $\bar{\psi}L^a\psi \equiv \bar{\psi}_\mathsf{i} L^a_{\mathsf{ij}}\psi_\mathsf{j}$, etc. After the substitution of the following polarization functions,

$$u_R(p) = (0, |\boldsymbol{p}])^T, \qquad u_L(p) \to (|\boldsymbol{p}\rangle, 0)^T,$$
$$\bar{v}_L(p) \to (0, [\boldsymbol{p}|), \qquad \bar{v}_R(p) \to (\langle\boldsymbol{p}|, 0), \quad (73)$$

as shown in Table II, we obtain the on-shell massive amplitude as

$$\mathcal{M}_3(1^{a_1}, 2_\psi, 3_{\bar{\psi}}) \equiv \mathcal{M}_3(1^{a_1}, 2, \bar{3})$$
$$= \frac{\sqrt{2}}{m_1}(R^{a_1}[\mathbf{12}]\langle\mathbf{13}\rangle + L^{a_1}\langle\mathbf{12}\rangle[\mathbf{13}]). \quad (74)$$

Note that the two terms are related by parity transformation[7]

$$\lambda_\alpha^I \to i\tilde{\lambda}^{-I\dot{\alpha}}, \qquad \tilde{\lambda}^{I\dot{\alpha}} \to i\lambda_\alpha^{-I}. \quad (75)$$

Alternatively, one can obtain the same amplitudes by starting from the UV massless amplitudes,

$$\mathcal{M}_{3,R}(1^{+1}, 2^{+\frac{1}{2}}, 3^{-\frac{1}{2}}) = \frac{[12]^2}{[23]},$$
$$\mathcal{M}_{3,L}(1^{-1}, 2^{-\frac{1}{2}}, 3^{+\frac{1}{2}}) = \frac{\langle 12\rangle^2}{\langle 23\rangle}, \quad (76)$$

and following the procedure outlined in Appendix D to IR deform them to the massive on-shell amplitudes. It also can be shown that the IR-unified on-shell massive amplitude in Eq. (74) contains the following UV fermion-fermion-scalar massless amplitude,

$$\mathcal{M}_3(1^0, 2^{+\frac{1}{2}}, 3^{+\frac{1}{2}}) = -\frac{m_2 L^{a_1} - m_3 R^{a_1}}{m_1}[23]. \quad (77)$$

Note that the coupling factor is proportional to m_ψ/m_W, which is indeed in the form that one would expect from the Higgs mechanism. This is consistent with the understanding that Higgs mechanism can be thought as IR unification of different massless UV amplitudes [56].

Now we turn to the interaction terms involved scalars. The $WW\phi$ amplitude has the following 3-term basis,

$$\langle\mathbf{12}\rangle^2, \qquad [\mathbf{12}]^2, \qquad \langle\mathbf{12}\rangle[\mathbf{21}], \quad (78)$$

where only the last term satisfies tree unitarity, and it is symmetric in exchanging the two vector labels, thus the corresponding (real) coupling constant $F_{a_1a_2i_3}$ needs to be symmetric in $\{a_1, a_2\}$. The $W\phi\phi$ amplitude has a 1-term basis

$$\langle\mathbf{1}|p_2 - p_3|\mathbf{1}], \quad (79)$$

which already satisfies tree unitarity; it is antisymmetric in the two external scalar labels, thus the associated (real) coupling constant $G_{a_1i_2i_3}$ needs to be antisymmetric in $\{i_2, i_3\}$. The $\phi\phi\phi$ amplitude has to be a (real) constant $P_{i_1i_2i_3}$, which satisfies tree unitarity and needs to be totally symmetric.

On the other hand, the $\phi\psi\bar{\psi}$ amplitude has the following basis

$$\langle\mathbf{23}\rangle, \qquad [\mathbf{23}], \quad (80)$$

where both terms satisfy tree unitarity. We can write down the following amplitude,

$$\mathcal{M}_3(1^{i_1}, 2, \bar{3}) = H_{i_1}[\mathbf{23}] + H^\dagger_{i_1}\langle\mathbf{23}\rangle, \quad (81)$$

where we have suppressed the indices $\{\mathsf{i}_2, \mathsf{i}_3\}$ for the fermionic internal quantum numbers of $\{\psi_{\mathsf{i}_2}, \bar{\psi}_{\mathsf{i}_3}\}$, i.e., $H_{i_1} \equiv (H_{i_1})_{\mathsf{i}_2\mathsf{i}_3}$ etc. The coupling constants in front of $\langle\mathbf{23}\rangle$ and $[\mathbf{23}]$ are related by Hermitian conjugation because of the aforementioned optical theorem of Eq. (66). The relevant three-particle operators in the Lagrangian is as follows:

[7]Similar to the massless spinor-helicity variables, one can check this explicitly by using the formulas in Eq. (C16) and perform the parity transformation: $\theta \to \pi - \theta, \phi \to \phi + \pi$. We can also show that under this transformation, $p_{\alpha\dot{\alpha}} = \lambda_\alpha^I\tilde{\lambda}_{I\dot{\alpha}}$ changes to $p^{\dot{\alpha}\alpha} = \lambda^{I\alpha}\tilde{\lambda}_I^{\dot{\alpha}}$ which is consistent with the parity transformation on the momentum $(p^0, \vec{p}) \to (p^0, -\vec{p})$. The change of sign of the little group index I is also consistent with interpretation that it corresponds to the spin eigenstates along the momentum direction under our choice of standard Lorentz transformation.

$$F_{abi}W_{a\mu}W^{b\mu}\phi_i - G_{aij}W_{a\mu}\partial^\mu\phi_i\phi_j - \frac{1}{3!}P_{ijk}\phi_i\phi_j\phi_k - (\bar{\psi}_L H_i \psi_R + \bar{\psi}_R H_i^\dagger \psi_L)\phi_i. \quad (82)$$

It is straightforward to derive the on-shell massive amplitudes from the above, which fixed the normalization of the coupling constants as given by Table III.

IV. FOUR-PARTICLE AMPLITUDES AND THE TREE-LEVEL UNITARITY

Now we construct 4-point amplitudes from unitarity and locality. Locality tells us that when one internal momentum is going on-shell, the amplitudes have simple poles in terms of Mandelstam variables, and unitarity requires that the residue is the product of lower-point amplitudes. To be more explicit, one can write the 4-point amplitudes as

$$\mathcal{M}_4 = \mathcal{M}_{4,\mathrm{f}} + \mathcal{M}_{4,\mathrm{c}}, \quad (83)$$

where $\mathcal{M}_{4,\mathrm{f}}$ contains the nonlocal parts of different factorization channels, while $\mathcal{M}_{4,c}$ are the possible additional contact terms. The latter is a linear combination of all local terms given the particle contents, expressed in the stripped-contact-term (SCT) basis $\{\mathcal{M}_{4,\mathrm{c}}^{(i)}\}$ given by Ref. [62],

$$\mathcal{M}_{4,\mathrm{c}} = \sum_i c_i \mathcal{M}_{4,\mathrm{c}}^{(i)}, \quad (84)$$

where c_i are polynomials of Mandelstam variables. Apparently, the slowest-energy growing behavior for these terms are achieved when c_i are constants. On the other hand, the factorizable part $\mathcal{M}_{4,\mathrm{f}}$ is fixed by unitarity,

$$\mathcal{M}_{4,\mathrm{f}} = -\sum_{\mathcal{I}}\sum_{i=2}^{4}\frac{1}{s_{1i}^2 - m_{\mathcal{I}}^2}\mathcal{M}_{3,i_L}^{\{I_1,I_2,\ldots,I_{2s_{\mathcal{I}}}\}}\epsilon_{I_1J_1}\cdots\epsilon_{I_{2s_{\mathcal{I}}}J_{2s_{\mathcal{I}}}}\mathcal{M}_{3,i_R}^{\{J_1,J_2,\ldots,J_{2s_{\mathcal{I}}}\}}, \quad (85)$$

where $s_{ij} \equiv (p_i + p_j)^2$, and we sum over all possible states $\mathcal{I}$ of mass $m_{\mathcal{I}}$ and spin $s_{\mathcal{I}}$ as well as all possible factorization channels. Here again, we take all the momenta as ingoing, which means that in the real momentum limit, some of the momenta have negative energy. We have the following analytical continuation,

$$\lambda^I(-p) = -\lambda^I(p), \qquad \tilde{\lambda}_I(-p) = \tilde{\lambda}_I(p). \quad (86)$$

In the above convention, the 3-point amplitudes $\mathcal{M}_{3,i_L}$ has momenta p_1, p_i and $-p_1 - p_i$, while $\mathcal{M}_{3,i_R}$ has momenta $\{p_j\}$ with $j \in \{1,2,3,4\}\backslash\{1,i\}$ and $p_1 + p_i$. Notice that in Sec. III, on-shell massive amplitudes are considered equivalent if they are related by equations of, but different forms of 3-point on-shell amplitudes certainly lead to different formulas for the local terms with different coefficients c_i.

In order to obtain the coefficient c_i and the coupling relations, we will take the high-energy limit of the amplitude $\mathcal{M}_4$ at fixed, nonzero angles and impose the tree-level unitary criterion, which requires that the energy growing behavior of the four-particle amplitude should be at most a constant. As discussed in detail in Appendix C, we can expand the massive spinors for the external states in the little group space as

$$\lambda_\alpha^I = \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I}, \qquad \tilde{\lambda}_{\dot{\alpha}}^I = \tilde{\lambda}_{\dot{\alpha}}\zeta^{+I} + \tilde{\eta}_{\dot{\alpha}}\zeta^{-I}, \quad (87)$$

and the helicity amplitudes for particle with spin S in a particular frame can be obtained by extracting the coefficients of $((\zeta^+)^{S+h}(\zeta^-)^{S-h})^{I_1\cdots I_{2s}}$. The resulting helicity amplitudes are functions of $(\lambda_i, \tilde{\lambda}_i, \eta_i, \tilde{\eta}_i)$ with explicit formulas as follows:

$$\lambda_{i,\alpha} = \sqrt{E_i + p_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \qquad \tilde{\lambda}_{i,\dot{\alpha}} = \sqrt{E_i + p_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix},$$
$$\eta_{i,\alpha} = \sqrt{E_i - p_i}\begin{pmatrix} c_i^* \\ s_i \end{pmatrix}, \qquad \tilde{\eta}_{i,\dot{\alpha}} = -\sqrt{E_i - p_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix}, \quad (88)$$

where E_i and p_i are the energy and the magnitude of 3-momentum for each external particle i, and c_i, s_i are defined as (see the Appendix B for the discussion of the phase convention)

$$c_i = \cos\frac{\theta_i}{2}e^{\frac{i}{2}\phi_i}, \qquad s_i = \sin\frac{\theta_i}{2}e^{\frac{i}{2}\phi_i}. \quad (89)$$

We have assumed that the energy of the particle is positive and for negative energy, the spinors are obtained by the analytic continuation in Eq. (86) and in all cases, $E_i = +\sqrt{m_i^2 + p_i^2}$. To simplify the derivation, we will work in the center-of-mass frame, which is obtained by setting the angles as follows:

$$\theta_1 = 0, \qquad \theta_2 = \pi, \qquad \theta_3 = \theta, \qquad \theta_4 = \pi - \theta,$$
$$\phi_1 = \phi_2 = 0, \qquad \phi_3 = \phi, \qquad \phi_4 = \phi + \pi, \quad (90)$$

The magnitude of the 3-momenta p_i can be obtained using momentum conservation and on-shell condition as

$$p_1 = p_2 = E\sqrt{\left(1 - \frac{m_1^2 + m_2^2}{4E^2}\right)^2 - \frac{m_1^2 m_2^2}{E^4}},$$
$$p_3 = p_4 = E\sqrt{\left(1 - \frac{m_3^2 + m_4^2}{4E^2}\right)^2 - \frac{m_3^2 m_4^2}{E^4}}, \quad (91)$$

where $2E = \sqrt{s_{12}}$ is the total center-of-mass energy. All the helicity amplitudes will be functions of energy E and scattering angles (θ, ϕ). We will take the fixed-angle high-energy limit and extract the tree-level unitary conditions by setting to zero the coefficients of linearly independent functions of (θ, ϕ) with energy growing behavior faster than $\mathcal{O}(E^0)$. Table 1 in Ref. [62] shows the SCT basis for the contact terms, organized by helicity components in which these contact terms have the fastest energy growing behavior. We will see that only the contact terms corresponding to dimension-4 operators in an EFT Lagrangian will survive the tree unitarity constraints.

A. Constraining the 4-point amplitudes

Let us enumerate all the possible 4-point amplitudes.

1. *WWWW*

We start with the *WWWW* amplitude. As discussed above, the factorizable part of the amplitude $\mathcal{M}_{4,\mathrm{f}}$ can be obtained by gluing the 3-point on-shell massive amplitudes together and adding back the simple pole structures $1/(s_{ij} - m_{\mathcal{I}}^2)$. By naive energy scaling counting, the energy growing behavior of $\mathcal{M}_{4,\mathrm{f}}$ is at most $\mathcal{O}(E^4)$, while in contrast the contact terms of helicity components $(+000)$, $(++\,+0)$, and $(++\,-0)$ are at least $\mathcal{O}(E^5)$, thus the coefficients c_i of these contact terms must vanish. One example of such contact terms is $[\mathbf{12}][\mathbf{34}]\langle\mathbf{241}]\langle\mathbf{34}\rangle$, which is in the $(+000)$ category. The $\mathcal{O}(E^4)$ energy growing behavior of $\mathcal{M}_{4,\mathrm{f}}$ arises only from (0000) helicity category. This eliminates the possibility of adding contact terms for helicity configurations $(+\,+00)$, $(+\,-00)$, $(+\,+\,++)$, and $(+\,+\,+-)$, as they are at least $\mathcal{O}(E^4)$. On the other hand, we need (0000) contact terms in $\mathcal{M}_{4,\mathrm{c}}$, which can be parametrized as

$$[\mathbf{12}][\mathbf{34}](c_{W^4,1}\langle\mathbf{12}\rangle\langle\mathbf{34}\rangle + c_{W^4,2}\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle) + [\mathbf{13}][\mathbf{24}](c_{W^4,3}\langle\mathbf{12}\rangle\langle\mathbf{34}\rangle + c_{W^4,4}\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle). \quad (92)$$

The requirement that $\mathcal{M}_4 = \mathcal{M}_{4,\mathrm{f}} + \mathcal{M}_{4,\mathrm{c}}$ satisfies tree unitarity then uniquely fix the coefficients $c_{W^4,i}$. This means that we have completely determined the form of $\mathcal{M}_{4,\mathrm{c}}$. [The only helicity configuration for the contact terms that we have not discussed is $(+\,+\,+-)$, where contact terms in the category simply cannot exist.] The amplitude is determined to be

$$\begin{aligned}\mathcal{M}_4(1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}) &= \mathcal{M}_{4,s}(1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}) + \mathcal{M}_{4,s}(2^{a_2}, 3^{a_3}, 1^{a_1}, 4^{a_4}) \\ &\quad + \mathcal{M}_{4,s}(3^{a_3}, 1^{a_1}, 2^{a_2}, 4^{a_4}),\end{aligned} \quad (93)$$

where the s-channel component is

$$\begin{aligned}\mathcal{M}_{4,s}(1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}) &= -\sum_b \frac{C_{a_1a_2b}C_{a_3a_4b}N_{W^4,b}}{m_{a_1}m_{a_2}m_{a_3}m_{a_4}(s_{12}-m_b^2)} \\ &\quad - \sum_i \frac{4F_{a_1a_2i}F_{a_3a_4i}\langle\mathbf{12}\rangle\langle\mathbf{34}\rangle[\mathbf{12}][\mathbf{34}]}{m_{a_1}m_{a_2}m_{a_3}m_{a_4}(s_{12}-m_i^2)},\end{aligned} \quad (94)$$

and

$$\begin{aligned}N_{W^4,b} &= (\langle\mathbf{343}\rangle]\langle\mathbf{4}(\mathbf{2}-\mathbf{1})\mathbf{4}] + \langle\mathbf{434}]\langle\mathbf{3}(\mathbf{1}-\mathbf{2})\mathbf{3}])\langle\mathbf{12}\rangle[\mathbf{12}] \\ &\quad + (\langle\mathbf{212}]\langle\mathbf{1}(\mathbf{3}-\mathbf{4})\mathbf{1}] + \langle\mathbf{121}]\langle\mathbf{2}(\mathbf{4}-\mathbf{3})\mathbf{2}])\langle\mathbf{34}\rangle[\mathbf{34}] \\ &\quad + 2(\langle\mathbf{212}](\langle\mathbf{343}]\langle\mathbf{14}\rangle[\mathbf{14}] - \langle\mathbf{434}]\langle\mathbf{13}\rangle[\mathbf{13}]) + \langle\mathbf{121}](\langle\mathbf{434}]\langle\mathbf{23}\rangle[\mathbf{23}] - \langle\mathbf{343}]\langle\mathbf{24}\rangle[\mathbf{24}])) \\ &\quad + \langle\mathbf{12}\rangle\langle\mathbf{34}\rangle[\mathbf{12}][\mathbf{34}]\left(\frac{(m_{a_1}^2 - m_{a_2}^2)(m_{a_3}^2 - m_{a_4}^2)}{m_b^2} + s_{12} + 2s_{13} - m_{a_1}^2 - m_{a_2}^2 - m_{a_3}^2 - m_{a_4}^2\right) \\ &\quad + (\langle\mathbf{13}\rangle\langle\mathbf{24}\rangle[\mathbf{13}][\mathbf{24}] - \langle\mathbf{14}\rangle\langle\mathbf{23}\rangle[\mathbf{14}][\mathbf{23}])(s_{12} - m_b^2).\end{aligned} \quad (95)$$

The t- and u-channels are given by $\mathcal{M}_{4,s}(2^{a_2}, 3^{a_3}, 1^{a_1}, 4^{a_4})$ and $\mathcal{M}_{4,s}(3^{a_3}, 1^{a_1}, 2^{a_2}, 4^{a_4})$, respectively. The terms on the right-hand side of Eq. (93), as well as similar expressions below, are organized according to the internal states of the factorization channels, as indicated by the particle label that is summed, or the masses in the propagators. For example, the first term on the right-hand side of Eq. (93) has a sum of vector index b, and together with the mass m_b^2 indicate that this is the contribution of a vector boson exchange. Notice that we have absorbed the contact terms into the different factorization channels in a symmetric way.

Next, the $\mathcal{O}(E^3)$ terms need to vanish as well. Here it is convenient to calculate in the center of mass frame. We find that $\mathcal{O}(E^3)$ terms only exist for $(+000)$.This leads to the following constraint on the coefficient C_{abc} of the *WWW* amplitudes,

$$C_{a_1a_2b}C_{a_3a_4b} + C_{a_1a_3b}C_{a_4a_2b} + C_{a_1a_4b}C_{a_2a_3b} = 0, \quad (96)$$

which has the form of the Jacobi identity. Therefore, we can identify the totally antisymmetric C_{abc} as the structure constants for some compact Lie group G.

We then proceed to consider the $\mathcal{O}(E^2)$ terms, which only exist for (0000). This leads to the following constraint for C_{abc} as well as the coefficients of $WW\phi$ amplitudes F_{abi},

$$\begin{aligned} &-4(F_{a_1a_3i}F_{a_2a_4i}-F_{a_1a_4i}F_{a_2a_3i})\\ &=\sum_b\{(C_{a_1a_3b}C_{a_2a_4b}-C_{a_1a_4b}C_{a_2a_3b})\\ &\quad\times(3m_b^2-m_{a_1}^2-m_{a_2}^2-m_{a_3}^2-m_{a_4}^2)\\ &\quad+\frac{1}{m_b^2}[C_{a_1a_3b}C_{a_2a_4b}(m_{a_1}^2-m_{a_3}^2)(m_{a_2}^2-m_{a_4}^2)\\ &\quad-C_{a_1a_4b}C_{a_2a_3b}(m_{a_1}^2-m_{a_4}^2)(m_{a_2}^2-m_{a_3}^2)]\}. \end{aligned} \tag{97}$$

This relation agrees with Eq. (7) in Ref. [58]. One can check explicitly that as long as the constraints in Eqs. (96) and (97) are satisfied, the amplitude in Eq. (93) behaves as $\mathcal{O}(E^0)$, we have extracted all information given by tree unitarity.

2. $WWW\phi$

We next consider the $WWW\phi$ amplitude. In this case, the factorizable part $\mathcal{M}_{4,\mathrm{f}}$ at high-energy limit grows at most at $\mathcal{O}(E^2)$, but the energy-growing behavior of the contact terms $\mathcal{M}_{4,\mathrm{c}}$, if nonvanishing, are at least of $\mathcal{O}(E^3)$. Therefore, we don't need the contact terms in this case, and the amplitude is fully determined by the factorizable part $\mathcal{M}_{4,\mathrm{f}}$,

$$\begin{aligned} &\mathcal{M}_4(1^{a_1},2^{a_2},3^{a_3},4^i)\\ &=\mathcal{M}_{4,s}(1^{a_1},2^{a_2},3^{a_3},4^i)+\mathcal{M}_{4,s}(2^{a_2},3^{a_3},1^{a_1},4^i)\\ &\quad+\mathcal{M}_{4,s}(3^{a_3},1^{a_1},2^{a_2},4^i), \end{aligned} \tag{98}$$

where the s-channel component is given by

$$\begin{aligned} \mathcal{M}_{4,s}(1^{a_1},2^{a_2},3^{a_3},4^i)=-i\sqrt{2}\Bigg(&\sum_b\frac{C_{a_1a_2b}F_{ba_3i}N_{W^3\phi,b}}{m_{a_1}m_{a_2}m_{a_3}(s_{12}-m_b^2)}\\ &+\sum_j\frac{2F_{a_1a_2j}G_{a_3ji}\langle\mathbf{343}]\langle\mathbf{12}\rangle[\mathbf{12}]}{m_{a_1}m_{a_2}m_{a_3}(s_{12}-m_j^2)}\Bigg), \end{aligned} \tag{99}$$

with

$$\begin{aligned} N_{W^3\phi,b}=&\langle\mathbf{3}(\mathbf{1}-\mathbf{2})\mathbf{3}]\langle\mathbf{12}\rangle[\mathbf{12}]+2(\langle\mathbf{121}]\langle\mathbf{23}\rangle[\mathbf{23}]\\ &-\langle\mathbf{212}]\langle\mathbf{13}\rangle[\mathbf{13}])+\frac{m_{a_1}^2-m_{a_2}^2}{m_b^2}\langle\mathbf{343}]\langle\mathbf{12}\rangle[\mathbf{12}]. \end{aligned} \tag{100}$$

Now, tree unitarity requires that the $\mathcal{O}(E^2)$ terms in Eq. (98) vanish, which turn out to only exist for the (0000) component. This leads to the following constraints on C_{abc}, F_{abi} as well as the coefficient for the $W\phi\phi$ amplitude G_{aij},

$$\begin{aligned} &\sum_b\frac{1}{2m_b^2}[C_{a_2a_3b}(m_b^2+m_{a_2}^2-m_{a_3}^2)\\ &\quad\times F_{ba_1i}-C_{a_2a_1b}(m_b^2+m_{a_2}^2-m_{a_1}^2)F_{a_Ia_3i}]\\ &=F_{a_2a_3j}G_{a_1ij}-F_{a_1a_2j}G_{a_3ij}-C_{a_1a_3b}F_{ba_2i}. \end{aligned} \tag{101}$$

One can check that upon this constraint, the amplitude in Eq. (98) is $\mathcal{O}(E^0)$.

3. $WW\phi\phi$

We then turn to the $WW\phi\phi$ amplitude. In the high-energy limit, the factorizable part $\mathcal{M}_{4,\mathrm{f}}$ is of $\mathcal{O}(E^2)$, while contact terms for helicity components (+000) and (+ − 00) start at $\mathcal{O}(E^3)$ and $\mathcal{O}(E^4)$, respectively, thus they are eliminated by tree unitarity. The contact terms for (+ + 00) start at $\mathcal{O}(E^2)$, but terms in $\mathcal{M}_{4,\mathrm{f}}$ that are in this helicity category are only $\mathcal{O}(E^0)$, thus contact terms for (+ + 00) cannot exist either. On the other hand, for the (0000) component $\mathcal{M}_{4,\mathrm{f}}$ is $\mathcal{O}(E^2)$, while the contact terms of $\mathcal{O}(E^2)$ can be parametrized as

$$c_{W^2\phi^2}\langle\mathbf{12}\rangle[\mathbf{12}]. \tag{102}$$

The requirement that the $\mathcal{O}(E^2)$ contributions of the full amplitude vanish completely determines the coefficient $c_{W^2\phi^2}$, so that the total amplitude is calculated to be

$$\mathcal{M}_4(1^{a_1},2^{a_2},3^{i_3},4^{i_4}) = \sum_b \left(\frac{C_{a_1a_2b}G_{bi_3i_4}N_{W^2\phi^2,b}}{m_{a_1}m_{a_2}(s_{12}-m_b^2)} + \frac{2F_{ba_1i_3}F_{ba_2i_4}(\langle\mathbf{131}]\langle\mathbf{242}]-2m_b^2\langle\mathbf{12}\rangle[\mathbf{12}])}{m_{a_1}m_{a_2}m_b^2(s_{13}-m_b^2)} + \frac{2F_{ba_1i_4}F_{ba_2i_3}(\langle\mathbf{141}]\langle\mathbf{232}]-(s_{23}+m_b^2)\langle\mathbf{12}\rangle[\mathbf{12}])}{m_{a_1}m_{a_2}m_b^2(s_{23}-m_b^2)} \right) + \sum_j \left(-\frac{2F_{a_1a_2j}P_{ji_3i_4}\langle\mathbf{12}\rangle[\mathbf{12}]}{m_{a_1}m_{a_2}(s_{12}-m_j^2)} + \frac{2G_{a_1ji_3}G_{a_2ji_4}\langle\mathbf{131}]\langle\mathbf{242}]}{m_{a_1}m_{a_2}(s_{13}-m_j^2)} + \frac{2G_{a_1ji_4}G_{a_2ji_3}(\langle\mathbf{141}]\langle\mathbf{232}]-(s_{23}-m_j^2)\langle\mathbf{12}\rangle[\mathbf{12}])}{m_{a_1}m_{a_2}(s_{23}-m_j^2)} \right), \tag{103}$$

with

$$N_{W^2\phi^2,b} = \left(\frac{(m_{a_1}^2-m_{a_2}^2)(m_{i_3}^2-m_{i_4}^2)}{m_b^2} - m_b^2 + m_{a_1}^2 + m_{a_2}^2 + m_{i_3}^2 + m_{i_4}^2 - 2s_{23} \right)\langle\mathbf{12}\rangle[\mathbf{12}] + 2(\langle\mathbf{131}]\langle\mathbf{212}] - \langle\mathbf{121}]\langle\mathbf{232}]). \tag{104}$$

However, fixing $c_{W^2\phi^2}$ is necessary but not sufficient to make the $\mathcal{O}(E^2)$ terms vanish in the above; we need an additional constraint on the coupling constants C_{abc}, F_{abi}, and G_{aij},

$$-\sum_b \frac{1}{m_b^2}(F_{a_1bi_3}F_{a_2bi_4} - F_{ba_1i_4}F_{ba_2i_3}) = G_{a_1i_3j}G_{a_2i_4j} - G_{a_1i_4j}G_{a_2i_3j} + C_{a_1a_2b}G_{bi_3i_4}, \tag{105}$$

which agrees with Eq. (8) in Ref. [58]. One can check that the constraint above will make the amplitude in Eq. (103) satisfy tree unitarity. Notice that no constraint has been put on the coefficient P_{ijk} for the ϕ^3 amplitude. Actually, in order to obtain nontrivial constraint on the pure-scalar interactions, one need go to higher-point amplitudes [86,87].

4. $W\psi W\bar{\psi}$

We now turn to the 4-point amplitudes involving fermions. First, we consider the case of $W\psi W\bar{\psi}$. In the high-energy limit at fixed, nonzero angle, the factorizable part $\mathcal{M}_{4,\mathrm{f}}$ is growing at most at $\mathcal{O}(E^2)$ while a nonvanishing $\mathcal{M}_{4,\mathrm{c}}$ would be at least $\mathcal{O}(E^3)$. Therefore, the possible contact terms are forbidden by tree unitarity, and the amplitude is fully determined by the factorizable part,

$$\mathcal{M}_4(1^{a_1},2^{\mathrm{i}_2},3^{a_3},\bar{4}^{\mathrm{i}_4}) = \sum_b \frac{iC_{a_1a_3b}N_{W^2\psi^2,b}}{m_{a_1}m_{a_3}(s_{13}-m_b^2)} + \sum_i \frac{2F_{a_1a_3i}\langle\mathbf{13}\rangle[\mathbf{13}]((H_i)_{\mathrm{i}_4\mathrm{i}_2}[\mathbf{24}]+(H_i^\dagger)_{\mathrm{i}_4\mathrm{i}_2}\langle\mathbf{24}\rangle)}{m_{a_1}m_{a_3}(s_{13}-m_j^2)} - \frac{2}{m_{a_1}m_{a_3}}\sum_{\mathrm{j}}\left(\frac{1}{(s_{23}-m_{\mathrm{j}}^2)}[L^{a_1}_{\mathrm{i}_4\mathrm{j}}L^{a_3}_{\mathrm{j}\mathrm{i}_2}\langle\mathbf{23}\rangle[\mathbf{14}]([\mathbf{13}]m_{a_1}-\langle\mathbf{143}]) + R^{a_1}_{\mathrm{i}_4\mathrm{j}}R^{a_3}_{\mathrm{j}\mathrm{i}_2}\langle\mathbf{14}\rangle[\mathbf{23}]([\mathbf{13}]m_{a_3}+\langle\mathbf{321}]) + m_{\mathrm{j}}(L^{a_1}_{\mathrm{i}_4\mathrm{j}}R^{a_3}_{\mathrm{j}\mathrm{i}_2}\langle\mathbf{13}\rangle[\mathbf{14}][\mathbf{23}] + R^{a_1}_{\mathrm{i}_4\mathrm{j}}L^{a_3}_{\mathrm{j}\mathrm{i}_2}\langle\mathbf{14}\rangle\langle\mathbf{23}\rangle[\mathbf{13}])] + 1\leftrightarrow 3\right), \tag{106}$$

with the numerator factor as follows:

$$N_{W^2\psi^2,b} = L^b_{\mathrm{i}_4\mathrm{i}_2}\left(\langle\mathbf{2}(\mathbf{1}-\mathbf{3})\mathbf{4}]\langle\mathbf{13}\rangle[\mathbf{13}] + 2(\langle\mathbf{313}]\langle\mathbf{12}\rangle[\mathbf{14}] + \langle\mathbf{131}]\langle\mathbf{23}\rangle[\mathbf{34}]) + \frac{m_{a_1}^2-m_{a_3}^2}{m_b^2}\langle\mathbf{13}\rangle[\mathbf{13}](m_{\mathrm{i}_4}\langle\mathbf{24}\rangle - m_{\mathrm{i}_2}[\mathbf{24}])\right) + R^b_{\mathrm{i}_4\mathrm{i}_2}(\mathbf{4}(\mathbf{1}-\mathbf{3})\mathbf{2}]\langle\mathbf{13}\rangle[\mathbf{13}] - 2(\langle\mathbf{313}]\langle\mathbf{14}\rangle[\mathbf{12}] + \langle\mathbf{131}]\langle\mathbf{34}\rangle[\mathbf{23}]) + \frac{m_{a_1}^2-m_{a_3}^2}{m_b^2}\langle\mathbf{13}\rangle[\mathbf{13}](m_{\mathrm{i}_4}[\mathbf{24}] - m_{\mathrm{i}_2}\langle\mathbf{24}\rangle)). \tag{107}$$

The $\mathcal{O}(E^2)$ terms only exist for $(0-0+)$ and $(0+0-)$, and for them to vanish we arrive at the following constraints,

$$iC_{a_1a_3b}L^b = [L^{a_1}, L^{a_3}], \qquad iC_{a_1a_3b}R^b = [R^{a_1}, R^{a_3}]. \tag{108}$$

As C_{abc} has been identified in Eq. (96) as structure constants in some Lie group G, the above commutation relations indicates that L^a and R^a are generators in some representations of G.

Upon identifying the commutation relations, the amplitude in Eq. (106) behaves as $\mathcal{O}(E^0)$ for all helicity components except for $(0+0+)$, which are $\mathcal{O}(E)$. Tree unitarity then imposes another constraint

$$\begin{aligned} &2F_{a_1a_3i}(H_i)_{\mathrm{i}_4\mathrm{i}_2} - m_{\mathrm{i}_2}\{L^{a_1},L^{a_3}\}_{\mathrm{i}_4\mathrm{i}_2} - m_{\mathrm{i}_4}\{R^{a_1},R^{a_3}\}_{\mathrm{i}_4\mathrm{i}_2} + \sum_{\mathrm{j}} 2m_{\mathrm{j}}(L^{a_1}_{\mathrm{i}_4\mathrm{j}}R^{a_3}_{\mathrm{j}\mathrm{i}_2} + L^{a_3}_{\mathrm{i}_4\mathrm{j}}R^{a_1}_{\mathrm{j}\mathrm{i}_2}) \\ &= \sum_b iC^{a_1a_3b}\frac{(m_{a_1}^2 - m_{a_3}^2)}{m_b^2}(m_{\mathrm{i}_2}L^b_{\mathrm{i}_4\mathrm{i}_2} - m_{\mathrm{i}_4}R^b_{\mathrm{i}_4\mathrm{i}_2}), \end{aligned} \tag{109}$$

which ensures that the full amplitude is $\mathcal{O}(E^0)$.

5. $W\psi\phi\bar{\psi}$

Next, we study the $W\psi\phi\bar{\psi}$ amplitude. Similar to the above case, the full amplitude is fully determined by the factorizable part $\mathcal{M}_{4,\mathrm{f}}$, which in the high-energy limit is growing at most at $\mathcal{O}(E)$. On the other hand, $\mathcal{M}_{4,\mathrm{c}}$ is at least $\mathcal{O}(E^2)$, thus set to 0 by tree unitarity. The amplitude is listed as follows:

$$\begin{aligned} \mathcal{M}_4(1^a,2^{\mathrm{i}_2},3^i,\bar{4}^{\mathrm{i}_4}) = \sum_{\mathrm{j}}\frac{\sqrt{2}}{m_a}\Bigg(&\frac{1}{s_{12}-m_{\mathrm{j}}^2}\Big[\langle\mathbf{12}\rangle\Big(m_{\mathrm{j}}[\mathbf{14}](H_i)_{\mathrm{i}_4\mathrm{j}} + (m_{\mathrm{i}_4}[\mathbf{14}] + \langle\mathbf{431}])(H_i^\dagger)_{\mathrm{i}_4\mathrm{j}}\Big)L^a_{\mathrm{j}\mathrm{i}_2} \\ &+ [\mathbf{12}]\Big(m_{\mathrm{j}}\langle\mathbf{14}\rangle(H_i^\dagger)_{\mathrm{i}_4\mathrm{j}} + (m_{a_1}[\mathbf{14}] - \langle\mathbf{124}])(H_i)_{\mathrm{i}_4\mathrm{j}}\Big)R^a_{\mathrm{j}\mathrm{i}_2}\Big] \\ &+ \frac{1}{s_{23}-m_{\mathrm{j}}^2}\Big[L^a_{\mathrm{i}_4\mathrm{j}}[\mathbf{14}]\Big(m_{\mathrm{j}}\langle\mathbf{12}\rangle(H_i^\dagger)_{\mathrm{j}\mathrm{i}_2} + (m_{a_1}[\mathbf{12}] - \langle\mathbf{142}])(H_i)_{\mathrm{i}_4\mathrm{j}}\Big) \\ &+ R^a_{\mathrm{i}_4\mathrm{j}}\langle\mathbf{14}\rangle\Big(m_{\mathrm{j}}[\mathbf{12}](H_i)_{\mathrm{j}\mathrm{i}_2} + (m_{\mathrm{i}_4}[\mathbf{12}] + \langle\mathbf{231}])(H_i^\dagger)_{\mathrm{j}\mathrm{i}_2}\Big)\Big]\Bigg) \\ &+ \sum_b\frac{\sqrt{2}F_{bai}}{m_a m_b^2(s_{13}-m_b^2)}[L^b_{\mathrm{i}_4\mathrm{i}_2}(2m_b^2\langle\mathbf{12}\rangle[\mathbf{14}] + \langle\mathbf{131}](m_{\mathrm{i}_2}[\mathbf{24}] - m_{\mathrm{i}_4}\langle\mathbf{24}\rangle)) \\ &+ R^b_{\mathrm{i}_4\mathrm{i}_2}(2m_b^2\langle\mathbf{14}\rangle[\mathbf{12}] + \langle\mathbf{131}](m_{\mathrm{i}_2}\langle\mathbf{24}\rangle - m_{\mathrm{i}_4}[\mathbf{24}]))] \\ &+ \sum_j\frac{\sqrt{2}iG_{aji}\langle\mathbf{131}]((H_j)_{\mathrm{i}_4\mathrm{i}_2}[\mathbf{24}] + (H_j^\dagger)_{\mathrm{i}_4\mathrm{i}_2}\langle\mathbf{24}\rangle)}{m_a(s_{13}-m_j^2)}. \end{aligned} \tag{110}$$

The $\mathcal{O}(E)$ contributions come from the $(0+0+)$ helicity components, and for them to vanish we need to impose the following relation,

$$\sum_b\frac{1}{m_b^2}F_{abi}(m_{\mathrm{i}_4}R^b_{\mathrm{i}_4\mathrm{i}_2} - m_{\mathrm{i}_2}L^b_{\mathrm{i}_4\mathrm{i}_2}) = iG_{aij}(H_j)_{\mathrm{i}_4\mathrm{i}_2} - (L^aH_i)_{\mathrm{i}_4\mathrm{i}_2} + (H_iR^a)_{\mathrm{i}_4\mathrm{i}_2}, \tag{111}$$

which ensures that the full amplitude is tree unitary.

6. Amplitudes for other processes

For all of the other four-particle processes, $\mathcal{M}_{4,\mathrm{f}}$ is already $\mathcal{O}(E^0)$, thus the only possible contact term is the constant term in the $\phi\phi\phi\phi$ amplitude. There are no nontrivial relations obtained in these processes, but for completeness, we list them below. First, the amplitude for $W\phi\phi\phi$ is given by

$$\mathcal{M}_4(1^a,2^{i_2},3^{i_3},4^{i_4}) = \mathcal{M}_{4,s}(1^a,2^{i_2},3^{i_3},4^{i_4}) + \mathcal{M}_{4,s}(1^a,4^{i_4},2^{i_2},3^{i_3}) + \mathcal{M}_{4,s}(1^a,3^{i_3},4^{i_4},2^{i_2}), \tag{112}$$

with s-channel contribution as

$$\mathcal{M}_{4,s}(1^a,2^{i_2},3^{i_3},4^{i_4}) = i\sqrt{2}\Bigg(\sum_b\frac{F_{bai_2}G_{bi_3i_4}(2\langle\mathbf{131}]m_b^2 + \langle\mathbf{121}](m_b^2 + m_{i_3}^2 - m_{i_4}^2))}{m_b^2 m_a(s_{12}-m_b^2)} - \sum_j\frac{G_{aji_2}P_{ji_3i_4}\langle\mathbf{121}]}{m_a(s_{12}-m_j^2)}\Bigg). \tag{113}$$

The $\phi\phi\phi\phi$ amplitude reads

$$\mathcal{M}_4(1^{i_1},2^{i_2},3^{i_3},4^{i_4}) = \mathcal{M}_{4,s}(1^{i_1},2^{i_2},3^{i_3},4^{i_4}) + \mathcal{M}_{4,s}(2^{i_2},3^{i_3},1^{i_1},4^{i_4}) + \mathcal{M}_{4,s}(3^{i_3},1^{i_1},2^{i_2},4^{i_4}) - K_{i_1 i_2 i_3 i_4}, \quad (114)$$

where the s-channel contribution is given by

$$M_{4,s}(1^{i_1},2^{i_2},3^{i_3},4^{i_4}) = -\sum_a \frac{G_{ai_1i_2}G_{ai_3i_4}[m_a^2(s_{13}+m_a^2-m_{i_1}^2-m_{i_2}^2-m_{i_3}^2-m_{i_4}^2)+(m_{i_1}^2-m_{i_2}^2)(m_{i_3}^2-m_{i_4}^2)]}{m_b^2(s_{12}-m_a^2)} - \sum_j \frac{P_{i_1i_2j}P_{i_3i_4j}}{(s_{12}-m_j^2)}, \quad (115)$$

and $K_{i_1 i_2 i_3 i_4}$ is the constant contact term, which needs to be totally symmetric because of Bose symmetry.

Finally, we have the $\phi\psi\phi\bar{\psi}$ amplitude as

$$\begin{aligned} M_4(1^{i_1},2^{\mathrm{i}_2},3^{i_3},\bar{4}^{\mathrm{i}_4}) = &\sum_{\mathrm{j}}\Big[\frac{1}{s_{12}-m_{\mathrm{j}}^2}((H_{i_3})_{\mathrm{i}_4\mathrm{j}}(H_{i_1}^\dagger)_{\mathrm{j}\mathrm{i}_2}(m_{\mathrm{i}_2}[\mathbf{24}]-\langle\mathbf{214}])+(H_{i_3}^\dagger)_{\mathrm{i}_4\mathrm{j}}(H_{i_1})_{\mathrm{j}\mathrm{i}_2}(m_{\mathrm{i}_4}[\mathbf{24}]+\langle\mathbf{432}]) \\ &\times(H_{i_3})_{\mathrm{i}_4\mathrm{j}}(H_{i_1})_{\mathrm{j}\mathrm{i}_2}m_{\mathrm{i}_2}[\mathbf{24}]+(H_{i_3}^\dagger)_{\mathrm{i}_4\mathrm{j}}(H_{i_1}^\dagger)_{\mathrm{j}\mathrm{i}_2}(m_{\mathrm{i}_2}\langle\mathbf{24}\rangle)+1\leftrightarrow 3\Big] \\ &-\sum_a \frac{iG_{ai_1i_3}}{m_a^2(s_{13}-m_a^2)}[L^a_{\mathrm{i}_4\mathrm{i}_2}(2m_a^2\langle\mathbf{214}]-(m_a^2+m_{i_1}^2-m_{i_3}^2)(m_{\mathrm{i}_2}[\mathbf{24}]-m_{\mathrm{i}_4}\langle\mathbf{24}\rangle)) \\ &+R^a_{\mathrm{i}_4\mathrm{i}_2}(2m_a^2\langle\mathbf{412}]-(m_a^2+m_{i_1}^2-m_{i_3}^2)(m_{\mathrm{i}_2}\langle\mathbf{24}\rangle-m_{\mathrm{i}_4}[\mathbf{24}]))]+\sum_j\frac{P_{ji_1i_3}((H_j)_{\mathrm{i}_4\mathrm{i}_2}[\mathbf{24}]+(H_j^\dagger)_{\mathrm{i}_4\mathrm{i}_2}\langle\mathbf{24}\rangle)}{(s_{13}-m_j^2)}, \end{aligned} \quad (116)$$

and the $\psi\bar{\psi}\psi\bar{\psi}$ amplitude reads

$$\begin{aligned} M_4(1^{\mathrm{i}_1},\bar{2}^{\mathrm{i}_2},3^{\mathrm{i}_3},\bar{4}^{\mathrm{i}_4}) = &\Big(\sum_a\frac{1}{m_a^2(s_{12}-m_a^2)}[(L^a_{\mathrm{i}_4\mathrm{i}_3}(m_{\mathrm{i}_3}[\mathbf{34}]-m_{\mathrm{i}_4}\langle\mathbf{34}\rangle)+R^a_{\mathrm{i}_4\mathrm{i}_3}(m_{\mathrm{i}_3}\langle\mathbf{34}\rangle-m_{\mathrm{i}_4}[\mathbf{34}])) \\ &\times(L^a_{\mathrm{i}_2\mathrm{i}_1}(m_{\mathrm{i}_1}[\mathbf{12}]-m_{\mathrm{i}_2}\langle\mathbf{12}\rangle)+R^a_{\mathrm{i}_2\mathrm{i}_1}(m_{\mathrm{i}_1}\langle\mathbf{12}\rangle-m_{\mathrm{i}_2}[\mathbf{12}])) \\ &-2m_a^2(L^a_{\mathrm{i}_4\mathrm{i}_3}R^a_{\mathrm{i}_2\mathrm{i}_1}\langle\mathbf{23}\rangle[\mathbf{14}]+L^a_{\mathrm{i}_2\mathrm{i}_1}R^a_{\mathrm{i}_4\mathrm{i}_3}\langle\mathbf{14}\rangle[\mathbf{23}]+L^a_{\mathrm{i}_2\mathrm{i}_1}L^a_{\mathrm{i}_4\mathrm{i}_3}\langle\mathbf{13}\rangle[\mathbf{24}]+R^a_{\mathrm{i}_2\mathrm{i}_1}R^a_{\mathrm{i}_4\mathrm{i}_3}\langle\mathbf{24}\rangle[\mathbf{13}])] \\ &-\sum_i\frac{((H_i)_{\mathrm{i}_2\mathrm{i}_1}[\mathbf{12}]+(H_i^\dagger)_{\mathrm{i}_2\mathrm{i}_1}\langle\mathbf{12}\rangle)((H_i)_{\mathrm{i}_4\mathrm{i}_3}[\mathbf{34}]+(H_i^\dagger)_{\mathrm{i}_4\mathrm{i}_3}\langle\mathbf{34}\rangle)}{s_{12}-m_i^2}\Big)+1\leftrightarrow 3. \end{aligned} \quad (117)$$

B. Interpretation of the constraints

We see that tree unitarity completely fixes the 4-point amplitudes in terms of 3-point amplitudes, with the exception of the additional parameter K_{ijkl} as the constant scalar contact term. Moreover, tree unitarity puts additional constraints to all parameters of 3-point amplitudes except for the $\phi\phi\phi$ interaction P_{ijk}. We obtain the relations in Eqs. (96), (97), (101), (105), (108), (109), and (111). It is easy to see that Eqs. (96) and (108) indicate that the totally antisymmetric coupling constants C_{abc} are structure constants of some Lie group G, and the $W\psi\bar{\psi}$ coupling matrix L^a and R^a are generators of some representations of the same Lie group. The other relations put constraints on $WW\phi$ couplings F_{abi}, $W\phi\phi$ couplings G_{aij} and the $\phi\psi\bar{\psi}$ couplings H_i. To see the meaning of these relations clearly, we define the following coupling matrices,

$$T^a_{ij} = iG_{aij}, \qquad T^a_{ib} = -T^a_{bi} = \frac{i}{m_b}F_{abi},$$

$$T^a_{bc} = iC_{abc}\frac{m_a^2-m_b^2-m_c^2}{2m_bm_c}. \quad (118)$$

Then Eqs. (97), (101), (105) become

$$iC_{abe}T^e_{cd} = T^a_{c\tilde{k}}T^b_{\tilde{k}d} - T^b_{c\tilde{k}}T^a_{\tilde{k}d}, \quad (119)$$

$$iC_{abd}T^d_{ic} = T^a_{i\tilde{k}}T^b_{\tilde{k}c} - T^b_{i\tilde{k}}T^a_{\tilde{k}c}, \quad (120)$$

$$iC_{abc}T^c_{ij} = T^a_{i\tilde{k}}T^b_{\tilde{k}j} - T^b_{i\tilde{k}}T^a_{\tilde{k}j}, \quad (121)$$

where the index $\tilde{k}$ runs over both the vector indices $\{a\}$ and the scalar indices $\{i\}$. We will see that this corresponds to all of the scalar states in the UV, including the longitudinal components of the massive vector bosons. This motivates us to group T^a_{ij}, T^a_{ib}, and T^a_{bc} together as an anti-symmetric matrix $T^a_{\tilde{i}\tilde{j}}$, and the above is just

$$iC_{abc}T^c_{\tilde{i}\tilde{j}} = [T^a, T^b]_{\tilde{i}\tilde{j}}. \tag{122}$$

In other words, the interactions between the vector and scalar states together form a generator of the representation of the Lie group G. If a generator a belongs to the Abelian invariant subgroup of G, the structure constant C_{abc} vanishes for all b, c and we can have additional Stü ckelberg mass terms for the corresponding Abelian vector bosons.

Similarly, for the fermion Yukawa interactions, one can generalize H_i to $H_{\tilde{i}}$, by the following definition when $\tilde{i}$ is a vector index a,

$$(H_a)_{\mathrm{ij}} \equiv \frac{i}{m_a}(m_{\mathrm{j}}L^a - m_{\mathrm{i}}R^a)_{\mathrm{ij}}. \tag{123}$$

After the extension, the relations in Eqs. (109) and (111) then become

$$L^aH_b - H_bR^a - H_iT^a_{ib} - H_cT^a_{cb} = 0, \tag{124}$$

$$L^aH_i - H_iR^a - H_jT^a_{ji} - H_bT^a_{bi} = 0, \tag{125}$$

which can be combined into the following identity,

$$L^aH_{\tilde{i}} - H_{\tilde{i}}R^a - H_{\tilde{j}}T^a_{\tilde{j}\tilde{i}} = 0. \tag{126}$$

This tells us that the coupling matrices $(H_{\tilde{i}})_{\mathrm{ij}}$ are rank-3 invariant tensors of Lie group G, where the indices $\tilde{i}$, i and j transform in the representation associated with T^a, L^a, and R^a, respectively. It means that the three-particle on-shell amplitudes involving the fermions, physical scalar states and the longitudinal components of the massive gauge boson respect the symmetry generated by Lie group G.

One can understand the definitions in Eqs. (118) and (123) at the Lagrangian level, which we discuss in Appendix E; see also Refs. [88,89] where similar relations are derived using current conservation. However, there is a much more straightforward, on-shell way to arrive at these definitions. In Table III we presented the 3-point UV massless amplitudes contained in the high-energy limit of the IR massive amplitudes. In particular, the same massless vector-scalar-scalar helicity amplitude in the UV can be generated by three different massive amplitudes in the IR,

$$\mathcal{M}_3(1^+,2^0,3^0) = \begin{cases} i\sqrt{2}C_{a_1a_2a_3}\frac{(m_1^2-m_2^2-m_3^2)}{2m_2m_3}\frac{[12][13]}{[23]} & \text{from } \mathcal{M}_3(1^{a_1},2^{a_2},3^{a_3}) \\ -\sqrt{2}\frac{F_{a_1a_2i_3}}{m_2}\frac{[12][13]}{[23]} & \text{from } \mathcal{M}_3(1^{a_1},2^{a_2},3^{i_3}) \\ -i\sqrt{2}G_{a_1i_2i_3}\frac{[12][13]}{[23]} & \text{from } \mathcal{M}_3(1^{a_1},2^{i_2},3^{i_3}) \end{cases}, \tag{127}$$

i.e., the longitudinal modes of the vector states in the IR amplitudes can be identified with scalar external states in the UV. Now, in the spirit of the Goldstone boson equivalence theorem [59,90,91], we want to unify all the scalar states in the UV, including the physical scalar and the longitudinal components of vector states in the IR, under a universal coupling and a single group representation of the gauge symmetry, then the redefinition in Eq. (118) is completely natural. Note that the definition of Eq. (118) has also taken into account the factor of $-i$ between the amplitude of longitudinal component of massive vector boson in the high-energy limit and corresponding Goldstone boson amplitude. Conversely, our ability to use the redefinition in Eq. (118) to arrive at the unified commutation relation of Eq. (122) suggests a spontaneously broken symmetry, as the (longitudinal components of) vector states and scalar states are clearly distinct in the IR, which are only unified in the UV. Similarly, for 3-pt amplitudes involving fermions, the massless scalar-fermion-fermion amplitude in the UV can be obtained by two distinct massive amplitudes in the IR,

$$\mathcal{M}(1^0,2^{+\frac{1}{2}},3^{+\frac{1}{2}}) = \begin{cases} -\frac{m_2L^{a_1}-m_3R^{a_1}}{m_1}[23] & \text{from } \mathcal{M}_3(1^{a_1},2_\psi,3_{\bar{\psi}}) \\ H_{i_1}[23] & \text{from } \mathcal{M}_3(1^{i_1},2_\psi,3_{\bar{\psi}}) \end{cases}. \tag{128}$$

Again, imposing the Goldstone boson equivalence in the UV makes the definition in Eq. (123) completely natural, which again manifests the existence of a spontaneously broken symmetry that unifies the vector and scalar states.

One can also compare our general setting with the special case of the electroweak theory in the SM. For example, Ref. [69] studied the 4-point bosonic amplitudes in the electroweak theory. As we are considering all external states to be massive, to compare with their results we need to decouple the photons in Ref. [69], i.e., setting the coupling to photons $e = 0$, and as a result the W and Z boson have the same mass: $m_Z = m_W$. Then relevant coupling constants are identified as

$$-\frac{1}{\sqrt{2}}C_{abc} \to e_W \quad \text{for } W^+W^-Z,$$

$$-\frac{2F_{abi}}{m_am_b} \to \begin{cases} \frac{e_{ZZH}}{m_Z} & \text{for } ZZh \\ \frac{e_{WWH}}{m_W} & \text{for } W^+W^-h \end{cases}, \qquad G_{aij} \to 0, \tag{129}$$

and the tree-unitarity relations in Eq. (97) and Eq. (101) becomes

$$e_{WWH}^2 = 2e_W^2, \qquad e_{WWH}e_{ZZH} = 2e_W^2, \qquad e_{WWH} = e_{ZZH}. \tag{130}$$

Our results of the bosonic 4-point amplitudes as well as the above relations agree with Ref. [69]. We see that the above relations are the result of a spontaneously broken symmetry, where the vector and scalar states are unified under the same group representation.

Another example is Ref. [61], which, in addition to the electroweak sector in SM, also considered a single generation of fermions, and studied the 4-point $\psi^c\psi Zh$ amplitude, which is a special case of our consideration with the following values for the coupling constants,

$$\frac{2}{m_a}F_{abi} \to -c_{ZZh}^{00}, \quad \frac{\sqrt{2}}{m_a}R^a \to \frac{1}{m_Z}c_{\psi^c\psi Z}^{LR0}, \quad \frac{\sqrt{2}}{m_a}L^a \to \frac{1}{m_Z}c_{\psi^c\psi Z}^{RL0},$$
$$G_{aij} \to 0, \quad H_i \to -c_{\psi^c\psi h}^{RR}, \quad H_i^\dagger \to -c_{\psi^c\psi h}^{LL}. \tag{131}$$

Then Eq. (111) becomes

$$(c_{\psi^c\psi Z}^{LR0} - c_{\psi^c\psi Z}^{RL0})\left(\frac{m_\psi}{2m_Z}c_{ZZh}^{00} - c_{\psi^c\psi h}^{RR}\right) = 0. \tag{132}$$

Again, our amplitude in Eq. (110) agrees with Ref. [61] upon the proper identification of the coupling constants, and we agree on the above relation as well. We see clearly that the above relation comes from the constraint that the Yukawa coupling needs to be an invariant tensor, again a consequence of a spontaneously broken symmetry.

V. CONCLUSION AND OUTLOOK

In this paper, we have considered the most general 3-point on-shell massive amplitudes with energy scaling at most $\mathcal{O}(E)$, involving an arbitrary, finite number of scalar, spinor, and vector particle states defined as irreducible representations of the little group. Starting from 3-point on-shell amplitudes, we have calculated the full 4-point amplitudes from unitary and locality, which lead to the formulas to construct the 4-point on-shell amplitudes in Eq. (83) and Eq. (85). The contact terms are further determined by the requirement of tree unitarity, which states that the energy growing behaviors of n-point amplitudes in the fixed-angle high-energy limit should not exceed $\mathcal{O}(E^{4-n})$. For 4-point amplitudes, the leading energy growing behavior should be at most a constant. Moreover, the requirement of tree unitarity further imposes relations on the 3-point couplings constants and the masses of the particles. In Table IV, we summarize the processes and relations obtained in this approach and they coincide with Ref. [59]. We can see that the fastest energy growing behaviors happen in the longitudinal modes of the massive vectors, which is consistent with the fact that the Stü eckelberg scalars are always associated with derivatives. As discussed in Sec. IV B and Appendix E, the relations can be understood from the point of view of the Lie algebra. This includes the Jacobi identity for the triple-vector couplings, commutation relations for vector-fermions couplings, and the predictions of the Higgs mechanism for the scalar-vector and scalar fermion couplings. They all converge to the gauge invariance from the UV interactions with possible modifications by the vector mass terms of the invariant Abelian subgroups.

TABLE IV. Summary of couplings, processes and the corresponding relations considered in the paper. The superscripts in the particle-type labels in the processes indicate the helicities of the corresponding particles in the high-energy limit, and we also indicate the energy growing behaviors for each case. Relations among the coupling constants and the masses are schematically displayed in the last column.

Particles	Couplings	Processes	Relations
WWW	C_{abc}	$W^{(\pm)}W^{(0)}W^{(0)}W^{(0)}\mathcal{O}(E^3)$ $W^{(0)}W^{(0)}W^{(0)}W^{(0)}\mathcal{O}(E^2)$ $W^{(0)}W^{(0)}W^{(0)}\phi\mathcal{O}(E^2)$ $W^{(0)}W^{(0)}\phi\phi\mathcal{O}(E^2)$	Jacobi identity, Eq. (96) $C_{abc} \sim \frac{m_b m_c}{m_a^2-m_b^2-m_c^2}T_{bc}^a$, Lie algebra for T^a, Eqs. (97), (101), and (105)
$WW\phi$	F_{abi}	$W^{(0)}W^{(0)}W^{(0)}W^{(0)}\mathcal{O}(E^2)$ $W^{(0)}W^{(0)}W^{(0)}\phi\mathcal{O}(E^2)$ $W^{(0)}W^{(0)}\phi\phi\mathcal{O}(E^2)$	$F_{abi} \sim m_a T_{ia}^b$, Lie algebra for T^a, Eqs. (97), (101), and (105)
$W\phi\phi$	G_{aij}	$W^{(0)}W^{(0)}W^{(0)}\phi\mathcal{O}(E^2)$ $W^{(0)}W^{(0)}\phi\phi\mathcal{O}(E^2)$	$G_{aij} \sim T_{ij}^a$, Lie algebra for T^a Eqs. (101) and (105)
$W\psi\bar{\psi}$	$L_{\mathrm{ij}}^a, R_{\mathrm{ij}}^a$	$W^{(0)}\psi^{(\pm)}W^{(0)}\bar{\psi}^{(\mp)}\mathcal{O}(E^2)$	Lie algebra for L^a, R^a, Eq. (108)
$\phi\psi\bar{\psi}$	H_{ij}^i	$W^{(0)}\psi^{(\pm)}W^{(0)}\bar{\psi}^{(\pm)}\mathcal{O}(E)$ $W^{(0)}\psi^{(\pm)}\phi\bar{\psi}^{(\pm)}\mathcal{O}(E)$	H_{ij}^i part of an invariant tensor, Eqs. (109) and (111)

From the study, we have shown that the on-shell massive and massless amplitudes are manifestly little group covariant and they further unleash the power of quantum mechanics and special relativity. With analytical continuation into complex momenta, we are able to discuss 3-point on-shell massive spin amplitudes and massless helicity amplitudes. The requirement of little group covariance puts strong constraints on the possible structures and especially for the massless case, it uniquely determines the helicity amplitudes. In this paper, we have seen that the tree unitarity at the 3-point on-shell massive amplitudes level already constrains the coupling constants, like that 3-vector couplings C_{abc} should be fully antisymmetric. As also illustrated in Appendix D, 3-point on-shell massive amplitudes can be obtained from the IR deformation of the corresponding massless helicity amplitudes, and one interesting observation is that the spurious poles in the massless vector amplitudes turn into vector mass singularities in the on-shell massive amplitudes. It further induces the energy growing behaviors in the longitudinal modes of the massive vectors. In other words, this translates the requirement of consistent factorization for the 4-point massless amplitudes into the requirement of no faster energy growing behaviors than it should be in the tree unitary theory.

Our study can be generalized in several ways. Firstly, one can go beyond the 4-point scattering amplitudes and determine the form of the scalar potential from tree unitarity. As we have seen, tree unitarity at 4-point does not impose any constraints on the $\phi\phi\phi$ coupling or the $\phi\phi\phi\phi$ contact term apart from being totally symmetric, and the relations that they satisfy can only be derived at the 5-point level. A computation of all 5-point processes when all external states are bosonic, and at least one of them is a scalar, should fully determine the relations satisfied by the scalar self-interactions. Secondly, it would be nice to explore how Higgsless theories can be embedded in the on-shell formalism and in that cases, no scalar degrees of freedom are involved and one needs Kaluza-Klein towers of massive vectors and fermions [92,93]. Finally, one can try to include the massive spin-3/2 and spin-2 particles to see what nontrivial tree-unitary theory can be obtained.

Last but not least, efforts have been made to extend the color-kinematics duality and the modern double copy program to include massive gauge bosons [94–99]. The variety of coupling relations that we present here greatly extends the meaning of “color” relations in the usual sense of color-kinematics duality, which may help us understand the possible double copy structures of spontaneously broken gauge theories.

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APPENDIX A: NOTATIONS AND CONVENTIONS

In this appendix, we collect the notations and conventions used throughout the paper. We will use the mostly minus metric,

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{A1}$$

Our momenta are parametrized as

$$p^\mu = (E, \vec{p}) = (E, p_x, p_y, p_z). \tag{A2}$$

Note that we will take all the momenta ingoing, which means that E can be either positive or negative. The matrix generators of the Lorentz group in the vector representation are given as follows [82]:

$$(\mathcal{J}^{\mu\nu})^\rho_\sigma = i(\eta^{\rho\mu}\delta^\nu_\sigma - \delta^\mu_\sigma\eta^{\rho\nu}). \tag{A3}$$

To be more explicit, the expressions for the rotation generators $J^1 = \mathcal{J}^{23}, J^2 = \mathcal{J}^{31}, J^3 = \mathcal{J}^{12}$ and the boost generators $K^i = \mathcal{J}^{0i}$ read

$$J^1 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad J^2 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$J^3 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K^1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$K^2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K^3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A4}$$

The finite Lorentz transformation can be obtained by the exponential mapping

$$\Lambda^\rho{}_\sigma = (e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}})^\rho{}_\sigma \tag{A5}$$

with the rotation angles as $\theta^1 = \omega_{23}, \cdots$ and the boost parameters (rapidities) as $\eta^i = \omega_{0i}$. For the spinor representation in the Weyl basis of Dirac matrices, the generator matrices are:

$$J^k = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad K^k = -\frac{i}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}. \tag{A6}$$

We know that $\mathrm{SL}(2,\mathcal{C})$ is the double cover of the proper, orthochronous Lorentz group SO(3,1), similar to the fact that the SU(2) is the double cover of the rotation group SO(3) [100]. This can be seen by defining the 2×2 matrix for each four-momentum,

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}, \qquad p^{\dot{\alpha}\alpha} = p_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha}, \tag{A7}$$

where the sigma matrices are defined as

$$\sigma^\mu = (\mathbf{1}_{2\times 2}, \vec{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbf{1}_{2\times 2}, -\vec{\sigma}), \tag{A8}$$

with σ^i as Pauli matrices. The four-momentum vector can be obtained by exploring the following identity [101],

$$\mathrm{Tr}[\sigma^\mu \bar{\sigma}^\nu] = \mathrm{Tr}[\bar{\sigma}^\mu \sigma^\nu] = 2\eta^{\mu\nu}, \tag{A9}$$

which yields

$$p^\mu = \frac{1}{2} p_{\alpha\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} = \frac{1}{2} p^{\dot{\alpha}\alpha} \sigma^\mu_{\alpha\dot{\alpha}}. \tag{A10}$$

From the same identity, it can also be shown that the determinant of the momentum matrix gives the scalar product of the momentum

$$\det p_{\alpha\dot{\alpha}} = \frac{1}{2} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} p_{\alpha\dot{\alpha}} p_{\beta\dot{\beta}} = p_\mu p^\mu, \tag{A11}$$

which can be generalized to any two momentum vectors,

$$p_{1\alpha\dot{\alpha}} p_2^{\dot{\alpha}\alpha} = 2 p_1 \cdot p_2. \tag{A12}$$

For any $\mathcal{L} \in \mathrm{SL}(2,\mathcal{C})$, the momentum matrix $p_{\alpha\dot{\alpha}}$ transforms as

$$p \to \mathcal{L}^\dagger p \mathcal{L}, \tag{A13}$$

which leaves the determinant invariant. This establishes the connection between Lorentz group and $\mathrm{SL}(2,\mathcal{C})$. We can also see that $\mathcal{L}$ and $-\mathcal{L}$ gives the same Lorentz transformation. The $\mathrm{SL}(2,\mathcal{C})$ indices $\alpha, \dot{\alpha}$ can be raised or lowered by the antisymmetric tensor $\epsilon^{\alpha\beta}$ and its inverse $\epsilon_{\alpha\beta}$,

$$\epsilon^{12} = \epsilon_{21} = 1, \qquad \epsilon_{12} = \epsilon^{21} = -1, \qquad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^\gamma_\alpha, \tag{A14}$$

and the same definition applies to $\epsilon^{\dot{\alpha}\dot{\beta}}, \epsilon_{\dot{\alpha}\dot{\beta}}$.

APPENDIX B: MASSLESS SPINOR-HELICITY VARIABLES

For massless particles, $p^2 = 0$, and the matrix $p_{\alpha\dot{\alpha}}$ has rank one, which can be always factorized as direct product of two spinors,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{B1}$$

For real momenta in the Minkowski space, $p_{\alpha\dot{\alpha}}$ is Hermitian, and we have

$$\tilde{\lambda}_{\dot{\alpha}} = \pm(\lambda_\alpha)^*, \tag{B2}$$

with the sign determined by whether the energy is taken to be positive (+) or negative (−). It is clear that the helicity variables $\lambda, \tilde{\lambda}$ satisfy the massless Weyl equations,

$$p_{\alpha\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} = 0, \qquad p^{\dot{\alpha}\alpha} \lambda_\alpha = 0. \tag{B3}$$

From the definition, it is also clear that given a particular momentum p, λ, and $\tilde{\lambda}$ are not uniquely determined but up to a scaling,

$$\lambda \to w\lambda, \qquad \tilde{\lambda} \to w^{-1}\tilde{\lambda}, \tag{B4}$$

with $w \in \mathcal{C}$ being a nonzero complex number. In fact, there is no continuous way to define λ as a function of $\vec{p}$ [17], as will be seen later on from the concrete formulas. The angular and square spinor products are defined as follows:

$$\begin{aligned} \langle 12\rangle &\equiv \langle \lambda_1 \lambda_2 \rangle = \lambda_1^\alpha \lambda_{2\alpha} = \epsilon_{\alpha\beta} \lambda_1^\alpha \lambda_2^\beta, \\ [12] &\equiv [\tilde{\lambda}_1 \tilde{\lambda}_2] = \tilde{\lambda}_{1\dot{\alpha}} \tilde{\lambda}_2^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{1\dot{\beta}} \tilde{\lambda}^{2\dot{\alpha}}. \end{aligned} \tag{B5}$$

For particle i, we also define the “half-brackets” [61],

$$|i\rangle = \lambda_{i\alpha}, \qquad \langle i| = \lambda_i^\alpha, \qquad |i] = \tilde{\lambda}_i^{\dot{\alpha}}, \qquad [i| = \tilde{\lambda}_{i\dot{\alpha}}, \tag{B6}$$

and the spinor products can also be understood as follows:

$$\langle 12\rangle = \langle 1|^\alpha |2\rangle_\alpha, \qquad [12] = [1|_{\dot{\alpha}} |2]^{\dot{\alpha}}. \tag{B7}$$

Note that in our convention, for the real momenta with same sign of energy, we have the following relation,

$$\langle 12\rangle = -[12]^*, \tag{B8}$$

as can be directly verified by using the definition Eq. (B5) and Eq. (B2). By using the fact that any fully anti-symmetric rank-2 tensor is proportional to the Levi-Civita tensor ϵ, we can obtain the identities

$$\lambda_{1[\alpha} \lambda_{2\beta]} = \langle 12\rangle \epsilon_{\alpha\beta}, \qquad \tilde{\lambda}_{1[\dot{\alpha}} \tilde{\lambda}_{2\dot{\beta}]} = -[12] \epsilon_{\dot{\alpha}\dot{\beta}}. \tag{B9}$$

We have some useful identities,

$$\langle ij\rangle [ji] = \langle i|p_j|i] = 2 p_i \cdot p_j, \tag{B10}$$

and the Schouten-identity,

$$\langle 12\rangle|3\rangle + \langle 23\rangle|1\rangle + \langle 31\rangle|2\rangle = 0$$
$$[12]|3] + [23]|1] + [31]|2] = 0, \tag{B11}$$

which can be proved by using the fact that the spinor space is two-dimensional and any two spinors can provide a basis as long as their angular/square inner product is not vanishing.

For the parametrization (E,θ,ϕ) in the real momenta (let us assume the energy is positive for the moment, $E>0$),

$$p_x = E\sin\theta\cos\phi, \qquad p_y = E\sin\theta\sin\phi, \qquad p_z = E\cos\theta. \tag{B12}$$

Then

$$p_{\alpha\dot\alpha} = \begin{pmatrix} E(1-\cos\theta) & -E\sin\theta e^{-i\phi} \\ -E\sin\theta e^{i\phi} & E(1+\cos\theta), \end{pmatrix} = 2E\begin{pmatrix} ss^* & -c^*s^* \\ -cs & cc^* \end{pmatrix}, \tag{B13}$$

where we have defined

$$c \equiv \cos\frac{\theta}{2} e^{\frac{i}{2}\phi}, \qquad s \equiv \sin\frac{\theta}{2} e^{\frac{i}{2}\phi}. \tag{B14}$$

We can choose the spinor-helicity variables as[8]

$$\lambda_\alpha = \sqrt{2E}\begin{pmatrix} -s^* \\ c \end{pmatrix}, \qquad \tilde\lambda_{\dot\alpha} = \sqrt{2E}\begin{pmatrix} -s \\ c^* \end{pmatrix}. \tag{B15}$$

It is straightforward to verify that the spinors $\lambda_\alpha, \tilde\lambda^{\dot\alpha}$ are the eigenvectors of the helicity operator with eigenvalues of $-\frac{1}{2}\left(+\frac{1}{2}\right)$,[9]

$$h_{\mathcal{O}} = \frac{1}{2}\vec\sigma\cdot\hat{\vec p} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}, \tag{B16}$$

which confirms that $\lambda(\tilde\lambda)$ carries helicity weight $-\frac{1}{2}(+\frac{1}{2})$. The presence of functions $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ indicates that the spinor-helicity variables are not continuous function of the momenta.

For massless spin-1 particles, the polarization vectors can be written as

$$\epsilon^-_{\alpha\dot\alpha} = \epsilon^-_\mu\sigma^\mu_{\alpha\dot\alpha} = \sqrt{2}\frac{\lambda_\alpha\tilde\mu_{\dot\alpha}}{[\tilde\lambda\tilde\mu]}, \quad \epsilon^+_{\alpha\dot\alpha} = \epsilon^+_\mu\sigma^\mu_{\alpha\dot\alpha} = \sqrt{2}\frac{\mu_\alpha\tilde\lambda_{\dot\alpha}}{\langle\mu\lambda\rangle}, \tag{B17}$$

where $\mu, \tilde\mu$ are any reference spinors linearly independent of $\lambda, \tilde\lambda$, which is reflecting the gauge redundancy. Indeed, any transformation of $\tilde\mu$ can be written as [7]

$$\tilde\mu \to \tilde\mu + z\tilde\mu + z'\tilde\lambda, \tag{B18}$$

with z and z' being complex numbers. Since the polarization vector is invariant under the scaling of $\tilde\mu$, it becomes

$$\epsilon^-_{\alpha\dot\alpha} \to \epsilon^-_{\alpha\dot\alpha} + \frac{\sqrt{2}z'}{1+z}\frac{p_{\alpha\dot\alpha}}{[\tilde\lambda\tilde\mu]}, \tag{B19}$$

which is just the residual gauge transformation preserving the condition $p_\mu\epsilon^\mu = 0$. For example, we can choose

$$\mu_\alpha = \begin{pmatrix} c^* \\ s \end{pmatrix}, \qquad \tilde\mu_{\dot\alpha} = \begin{pmatrix} c \\ s^* \end{pmatrix}, \tag{B20}$$

which correspond to the explicit formulas for polarization vectors,

$$\epsilon^{+\mu} = \frac{1}{\sqrt{2}}(0, \cos\theta\cos\phi - i\sin\phi, \cos\theta\sin\phi + i\cos\phi, -\sin\theta),$$
$$\epsilon^{-\mu} = \frac{1}{\sqrt{2}}(0, \cos\theta\cos\phi + i\sin\phi, \cos\theta\sin\phi - i\cos\phi, -\sin\theta). \tag{B21}$$

The n-point helicity amplitudes are not continuous functions of the momenta, but rather the functions of the spinor-helicity variables,

$$\mathcal{M}_n(\lambda_a, \tilde\lambda_a) \tag{B22}$$

where $a = 1, \ldots, n$ denotes the particle indices and satisfy the covariant constraint,

$$\mathcal{M}_n(\omega_a\lambda_a, \omega_a^{-1}\tilde\lambda_a) = (\omega_a)^{-2h_a}\mathcal{M}_n(\lambda_a, \tilde\lambda_a). \tag{B23}$$

Let us consider the geometry of n-particle momentum conservation,

$$\sum_{a=1}^n p_a^\mu = 0 \Leftrightarrow \sum_a \lambda_{a\alpha}\tilde\lambda_{a\dot\alpha} = 0. \tag{B24}$$

We can think of this condition as imposing a constraint on the spinor vector space $\{\lambda_{a\alpha}\}$ or $\{\tilde\lambda_{a\dot\alpha}\}$. The condition can be fully explored by projecting into two linearly independent spinors. For $n=3$, this is easy to solve, as we can choose either λ_a or $\tilde\lambda_a$ as generic. For example, in the first case, we can project into the $|1\rangle$ subspace and the nonvanishing of $\langle 12\rangle, \langle 13\rangle$ implies the proportionality of $|2]$ and $|3]$. Similarly, in the other case, nonvanishing of $[12], [13]$ implies the proportionality of $|2\rangle$ and $|3\rangle$. Finally we have two solutions,

[8]Here we have adopted a phase convention commonly used in quantum mechanics [102].
[9]Note that $\dot\alpha$ is the upper index.

$$\text{Generic } \lambda \Rightarrow \tilde{\lambda}_1 = \langle 23\rangle\tilde{\xi}, \qquad \tilde{\lambda}_2 = \langle 31\rangle\tilde{\xi}, \qquad \tilde{\lambda}_3 = \langle 12\rangle\tilde{\xi},$$
$$\text{Generic } \tilde{\lambda} \Rightarrow \lambda_1 = [23]\xi, \qquad \lambda_2 = [31]\xi, \qquad \lambda_3 = [12]\xi, \tag{B25}$$

with $\xi, \tilde{\xi}$ being some reference spinors. For the three-particle amplitudes, we have

$$\mathcal{M}_3 = \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}, & h_1+h_2+h_3<0 \\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}, & h_1+h_2+h_3>0 \end{cases}, \tag{B26}$$

where we also demand that the amplitudes have a smooth limit in Minkowski signature where the brackets also go to zero.

APPENDIX C: MASSIVE SPINOR VARIABLES

The massive spinor variables are a bit more complicated than the massive case, where the little group is SU(2) instead of ISO(2). Consequently, the spinor variables carry the little group index I, which transform as the fundamental representation of SU(2),

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^I\tilde{\lambda}_{I\dot{\alpha}} = |p^I\rangle[p_I|, \tag{C1}$$

which can be thought of as the sum of two rank-1 matrices $\lambda^1_\alpha\tilde{\lambda}_{1\dot{\alpha}}$, $\lambda^2_\alpha\tilde{\lambda}_{2\dot{\alpha}}$. For general complex momenta, the spinor variables transform under the fundamental representation of $W \in \mathrm{SL}(2,\mathcal{C})$,

$$\lambda^I \to \lambda^J (W^{-1})_J{}^I, \qquad \tilde{\lambda}_I \to W_I{}^J\tilde{\lambda}_J. \tag{C2}$$

We adopt the following analytic continuation,

$$\lambda^I(-p) = -\lambda^I(p), \qquad \tilde{\lambda}_I(-p) = \tilde{\lambda}_I(p). \tag{C3}$$

The case of real momenta for positive energy can be obtained by imposing

$$(\lambda^I_\alpha)^* = \tilde{\lambda}_{I\dot{\alpha}}, \tag{C4}$$

which implies

$$(\tilde{\lambda}^{I\dot{\alpha}})^* = -\lambda^\alpha_I. \tag{C5}$$

Note that the little group index I is naturally raised or lowered, which is consistent with the fact that the fundamental representation of SU(2) is self-conjugate. We can regard the massive spinor variables as two matrices (I is the column index in λ and row index in $\tilde{\lambda}$).

Unlike the massless case, the on-shell condition $p^2 = m^2$ is not manifest in this decomposition, but rather a constraint on the spinor variables,

$$\det p = \det\lambda \times \det\tilde{\lambda} = m^2. \tag{C6}$$

Without loss of any generality, we can always choose

$$\det\lambda = m, \qquad \det\tilde{\lambda} = m, \tag{C7}$$

where $m = \sqrt{E^2 - \vec{p}^2}$. With this convention, we have the following identities,

$$\lambda_\alpha{}^I\lambda_{\beta I} = m\varepsilon_{\alpha\beta}, \qquad \tilde{\lambda}^I{}_{\dot{\alpha}}\tilde{\lambda}_{I\dot{\beta}} = m\varepsilon_{\dot{\alpha}\dot{\beta}},$$
$$\lambda^{\alpha I}\lambda_\alpha{}^J = -m\varepsilon^{IJ}, \qquad \tilde{\lambda}_{I\dot{\alpha}}\tilde{\lambda}_J{}^{\dot{\alpha}} = -m\varepsilon_{IJ}. \tag{C8}$$

By using the above formulas, it is straightforward to obtain the spinor version of the Dirac equations,

$$p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}I} = m\lambda^I_\alpha, \qquad p_{\alpha\dot{\alpha}}\lambda^{\alpha I} = -m\tilde{\lambda}^I_{\dot{\alpha}}. \tag{C9}$$

We can always expand the spinor variables in the bases of the little group space as

$$\lambda^I_\alpha = \lambda_\alpha\zeta^{-I} + \eta_\alpha\zeta^{+I}, \qquad \tilde{\lambda}_{I\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\zeta^+_I + \tilde{\eta}_{\dot{\alpha}}\zeta^-_I, \tag{C10}$$

where the eigenvectors of the z-component spin operator with eigenvalues of $\pm\frac{1}{2}$ are given by

$$\zeta^{+I} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \zeta^{-I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{C11}$$

which satisfies

$$\zeta^{-I}\zeta^+_I = 1. \tag{C12}$$

By using the above identity, the momentum matrix can be written in terms of the expansion spinors $\lambda(\tilde{\lambda})$, $\eta(\tilde{\eta})$,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} - \eta_\alpha\tilde{\eta}_{\dot{\alpha}}, \tag{C13}$$

and the on-shell condition becomes

$$\langle\lambda\eta\rangle = m, \qquad [\tilde{\lambda}\tilde{\eta}] = m. \tag{C14}$$

Similar to the massless case, the real momentum matrix for positive energy can be parametrized as

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} E - p\cos\theta & -p\sin\theta e^{-i\phi} \\ -p\sin\theta e^{i\phi} & E + p\cos\theta \end{pmatrix}$$
$$= (E+p)\begin{pmatrix} ss^* & -c^*s^* \\ -cs & cc^* \end{pmatrix} + (E-p)\begin{pmatrix} cc^* & c^*s^* \\ cs & ss^* \end{pmatrix}, \tag{C15}$$

with $p = |\vec{p}|$, and the on-shell condition is $E^2 - p^2 = m^2$. It is not difficult to see that the following choices of the spinor variables can do the job,

$$\lambda_\alpha = \sqrt{E+p}\begin{pmatrix} -s^* \\ c \end{pmatrix}, \quad \tilde{\lambda}_{\dot{\alpha}} = \lambda^*_{\alpha} = \sqrt{E+p}\begin{pmatrix} -s \\ c^* \end{pmatrix},$$
$$\eta_\alpha = \sqrt{E-p}\begin{pmatrix} c^* \\ s \end{pmatrix}, \quad \tilde{\eta}_{\dot{\alpha}} = -\eta^*_\alpha = -\sqrt{E-p}\begin{pmatrix} c \\ s^* \end{pmatrix}, \tag{C16}$$

which satisfy the on-shell condition in Eq. (C14) as can be verified directly. In the high-energy limit, we have

$$\lambda \sim \mathcal{O}(\sqrt{E}), \qquad \eta \sim \mathcal{O}\left(\frac{m}{\sqrt{E}}\right), \tag{C17}$$

and λ, $\tilde{\lambda}$ coincide with Eq. (B15) of the massless case. The polarization vectors for the spin-1 particles transform as symmetric rank-2 tensors under the SU(2) little group and they can be constructed by the tensor-product of λ^I and $\tilde{\lambda}^I$,

$$\boldsymbol{\epsilon}_{\alpha\dot{\alpha}} \equiv \epsilon^{I_1 I_2}_{\alpha\dot{\alpha}} = \frac{\sqrt{2}}{m}\lambda^{\{I_1}_\alpha \tilde{\lambda}^{I_2\}}_{\dot{\alpha}} = \begin{cases} \frac{\sqrt{2}}{m}\lambda^{I_1}_\alpha \tilde{\lambda}^{I_2}_{\dot{\alpha}}, & I_1 = I_2 \\ \frac{1}{m}(\lambda^{I_1}_\alpha \tilde{\lambda}^{I_2}_{\dot{\alpha}} + \lambda^{I_2}_\alpha \tilde{\lambda}^{I_1}_{\dot{\alpha}}), & I_1 \neq I_2 \end{cases}. \tag{C18}$$

where we have adopted the same convention as [61]. The longitudinal and transverse polarization components can be extracted as the coefficients of $\zeta^{+I_1}\zeta^{-I_2}$, $\zeta^{+I_1}\zeta^{+I_2}$, $\zeta^{-I_1}\zeta^{-I_2}$ and they are found to be

$$\epsilon^0_{\alpha\dot{\alpha}} = \epsilon^{-\frac{1}{2},\frac{1}{2}}_{\alpha\dot{\alpha}} = \frac{\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} + \eta_\alpha\tilde{\eta}_{\dot{\alpha}}}{m}, \qquad \epsilon^-_{\alpha\dot{\alpha}} = \epsilon^{-\frac{1}{2},-\frac{1}{2}}_{\alpha\dot{\alpha}} = \sqrt{2}\frac{\lambda_\alpha\tilde{\eta}_{\dot{\alpha}}}{m},$$
$$\epsilon^+_{\alpha\dot{\alpha}} == \epsilon^{\frac{1}{2},\frac{1}{2}}_{\alpha\dot{\alpha}} = -\sqrt{2}\frac{\eta_\alpha\tilde{\lambda}_{\dot{\alpha}}}{m}. \tag{C19}$$

Plugging in the formulas in Eq. (C16), we find the explicit formulas for polarization vectors as follows:

$$\epsilon^{0\mu} = \frac{1}{m}(p, E_p\sin\theta\cos\phi, E_p\sin\theta\sin\phi, E_p\cos\theta),$$
$$\epsilon^{\pm\mu} = \frac{1}{\sqrt{2}}(0, \cos\theta\cos\phi \mp i\sin\phi, \cos\theta\sin\phi \pm i\cos\phi, -\sin\theta). \tag{C20}$$

The amplitudes for massive particles are functions of $\lambda^I, \tilde{\lambda}_I$, which are fully symmetric rank $2S$ tensors for spin S particles.

APPENDIX D: MASSIVE AMPLITUDES AS IR-DEFORMATION OF THE MASSLESS AMPLITUDES

Under our parametrization of massive spinor variables, in the high-energy limit, they approach the massless spinor-helicity variables as follows:

$$\lambda_\alpha{}^I \to \lambda_\alpha\zeta^{-I} + \mathcal{O}\left(\frac{m}{\sqrt{E}}\right), \qquad \tilde{\lambda}_{I\dot{\alpha}} \to \tilde{\lambda}_{\dot{\alpha}}\zeta^+_I + \mathcal{O}\left(\frac{m}{\sqrt{E}}\right). \tag{D1}$$

This may provide a way to think that the massive amplitudes as appropriate IR-deformation of the UV massless amplitudes, especially for particles with spins. For scalar particles, the transformations under the little group are trivial and they don't provide too much insight. We also confine ourselves to the on-shell three-particle amplitudes and leave the higher-point amplitudes for the future. In addition, we are satisfied with considering about the relevant and marginal interactions. This corresponds to the total helicity of the 3-point massless amplitudes smaller than or equal to one,

$$|h| = |h_1 + h_2 + h_3| \leq 1, \tag{D2}$$

as the mass dimensions of the associated couplings are given by

$$[g_h] = 1 - |h|. \tag{D3}$$

The first nontrivial example involves the fermion-fermion-scalar amplitudes and as shown in Eq. (B26), there are two kinds of marginal on-shell amplitudes,

$$\mathcal{M}(1^{-\frac{1}{2}}, 2^{-\frac{1}{2}}, 3^0) = \langle 12\rangle, \qquad \mathcal{M}(1^{+\frac{1}{2}}, 2^{+\frac{1}{2}}, 3^0) = [12]. \tag{D4}$$

The IR deformation is straightforward,

$$\langle 12\rangle \to \langle 12\rangle\zeta_1^{-I}\zeta_2^{-J} \to \langle 1^I 2^J\rangle \equiv \langle \mathbf{12}\rangle, \tag{D5}$$

where symmetrization is implicitly assumed. In Ref. [56], this has been denoted as "bolding". Similarly, for plus-helicity amplitude, we have

$$[12] \to [\mathbf{12}]. \tag{D6}$$

The case of massive spin-1 particles is more interesting, as it is famously known that an extra degree of freedom is needed to go from massless to massive. We will pursue it by first noticing the following properties of the massless spinor variables for total-plus 3-point on-shell amplitudes,

$$|1\rangle = \frac{[23]}{[12]}|3\rangle = \frac{[23]}{[31]}|2\rangle, \tag{D7}$$

which can be expressed as

$$\frac{[23]}{[31]}=\frac{\langle 1\eta\rangle}{\langle 2\eta\rangle},\qquad \frac{[12]}{[23]}=\frac{\langle 3\eta\rangle}{\langle 1\eta\rangle},\qquad \frac{[31]}{[12]}=\frac{\langle 2\eta\rangle}{\langle 3\eta\rangle},\qquad \text{(D8)}$$

where η is any reference spinor linearly independent with angle-bracket on-shell spinors $|1\rangle$, $|2\rangle$, $|3\rangle$. The first set of on-shell massless amplitudes we are interested in are

$$\mathcal{M}(1^0,2^{+1},3^0)=\frac{[12][23]}{[31]},\quad \mathcal{M}(1^0,2^{-1},3^0)=\frac{\langle 12\rangle\langle 23\rangle}{\langle 31\rangle},\qquad \text{(D9)}$$

and for simplicity, we have set the coupling constant to one. We can deform them to the massive amplitudes involving two vectors and one scalar by employing the relations in Eq. (D8). Naturally, we will choose η as the expansion spinor variables of different particles with on-shell constraint as in Eq. (C14). To be more specific, for the total-plus helicity amplitude, the procedure reads

$$\frac{[12][23]}{[31]}\to\frac{[12]\langle 1\eta_2\rangle}{m_2}\to\sqrt{2}\frac{\langle \mathbf{12}\rangle[\mathbf{12}]}{m_2},\qquad \text{(D10)}$$

and similarly for the CPT conjugate amplitude, we have

$$\frac{\langle 12\rangle\langle 23\rangle}{\langle 31\rangle}\to\frac{\langle 12\rangle[1\eta_2]}{m_2}\to\sqrt{2}\frac{\langle \mathbf{12}\rangle[\mathbf{12}]}{m_2}.\qquad \text{(D11)}$$

Remarkably, the two helicity amplitudes are unified into one massive object, and the spurious poles in the massless amplitudes have turned into mass singularities for the massive amplitudes.

The final example we are presenting here is three vector on-shell amplitude, and up to permutation and CPT conjugation, the relevant one is

$$\mathcal{M}(1^{+1},2^{+1},3^{-1})=\frac{[12]^3}{[23][31]},\qquad \text{(D12)}$$

and the deformation gives us two mass singularities,

$$\frac{[12]^3}{[23][31]}\to\frac{[12]\langle 3\eta_1\rangle\langle 3\eta_2\rangle}{m_1m_2}\to\sqrt{2}\frac{[\mathbf{12}]\langle \mathbf{31}\rangle\langle \mathbf{32}\rangle}{m_1m_2}.\qquad \text{(D13)}$$

The systematic way to IR-deform the on-shell massless amplitudes to massive ones has been explored recently in Ref. [63].

APPENDIX E: CONSTRAINTS OF THE COUPLING CONSTANTS FROM THE LAGRANGIAN

In Ref. [59], the Lagrangian of the most general tree unitary theory with a finite spectrum of spin-0, 1/2, and 1 states is given as

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}(F^a_{\mu\nu})^2+\bar q_R i(\partial\!\!\!/+i A\!\!\!/_a\bar R^a)q_R+\bar q_L i(\partial\!\!\!/+iA\!\!\!/_a\bar L^a)q_L\\&+\frac{1}{2}[(\partial_\mu+iA_{a\mu}\bar T^a)\bar\pi]^2-V(\bar\pi)-\bar q_L Y(\bar\pi)q_R-\bar q_R Y^\dagger(\bar\pi)q_L\\&+\sum_{a=1}^{N_0}\frac{1}{2}(M_0^a)^2\left(A_{a\mu}+\frac{1}{M_0^a}\partial_\mu\theta_a\right)^2,\end{aligned}\qquad \text{(E1)}$$

where $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-f^{abc}A^b_\mu A^c_\nu$ is the field strength tensor for the gauge field A^a_μ, and $\bar T$, $\bar L$, and $\bar R$ are the generators associated with the representation for the scalars, left-handed and right-handed fermions, respectively. It is written in the following basis, which we will call the "gauge basis", of scalars and vector states:

(i) The generators of the broken group G, and consequently the basis of the vector bosons, are organized according to invariant subgroups of G. In particular, the structure constants f^{abc} are in the "Cartesian" basis such that $f^{ade}f^{bde}=0$ for $a\neq b$. This tells us that if the index a belongs to the invariant Abelian subgroup, the structure constants f^{abc} vanish for all the indices b, c.

(ii) The generators $\bar T^a$ associated with the scalars are block diagonalized so that each diagonal block corresponds to an irreducible representation of G.

The N_V vector fields are labeled by a, and the indices $1\leq a\leq N_0$ are for the invariant Abelian subgroups that have an explicit mass term, the explicit mass matrix being diagonalized to be $(M_0^2)_{ab}=\delta_{ab}(M_0^a)^2$. θ_a with a running from 1 to N_0 are the redundant scalars in the Stü ckelberg formalism for the massive invariant Abelian vectors. All the other physical or Stückelberg scalars are grouped by $\bar\pi_p=\pi_p+\eta_p$ with $p=N_0+1,\ldots,\bar N_S$, where the constants η_p are the vacuum expectation values (vev's). $V(\bar\pi)$ and $Y(\bar\pi)$ are quartic and linear functions of $\bar\pi$, respectively.

The full set of constraints on the coupling constants in Eq. (E1) include not only the Lie algebra for the structure constants f^{abc} and the generators $\bar T$, $\bar L$, and $\bar R$,

$$\begin{aligned}&f^{abe}f^{cde}+f^{ace}f^{dbe}+f^{ade}f^{bce}=0,\qquad [\bar T^a,\bar T^b]=if^{abc}\bar T^c,\\&[\bar L^a,\bar L^b]=if^{abc}\bar L^c,\qquad [\bar R^a,\bar R^b]=if^{abc}\bar R^c,\end{aligned}\qquad \text{(E2)}$$

but also the following conditions on the scalar interactions

$$V_{,p}(\eta)=0,\qquad \text{(E3)}$$

$$\bar T^a\bar\lambda^b-\bar T^b\bar\lambda^a=if^{abc}\bar\lambda^c,\qquad \text{(E4)}$$

$$V_{,p}(\bar\pi)(\bar T^a\bar\pi)_p=0,\qquad \text{(E5)}$$

$$\bar L^aY(\bar\pi)-Y(\bar\pi)\bar R^a-Y_{,p}(\bar\pi)(\bar T^a\bar\pi)_p=0,\qquad \text{(E6)}$$

where we have defined N_V column vectors in the $\bar{N}_S$-scalars space

$$\bar{\lambda}^a_p = \begin{cases} \delta^{ap} M^a_0, & 1 \le p \le N_0 \\ i\bar{T}^a_{pq}\eta_q, & N_0 < p \le \bar{N}_S, \end{cases}. \tag{E7}$$

and the generators $\bar{T}$ in the N_0-scalars subspace are zero, i.e., $\bar{T}_{pq} = 0$ for $p, q = 1, \ldots, N_0$. Eq. (E3) states that the scalar potential has a local minimum at $\bar{\pi}_p = \eta_p$, while Eqs. (E4), (E5), and (E6) guarantee that the various coupling constants involved are invariant tensors of G. At the current stage, one can already diagonalize the fermion mass terms so that we have

$$Y(\eta) = Y^\dagger(\eta) = \delta^{\mathrm{ij}} m_{\mathrm{i}}. \tag{E8}$$

The field variables in Eq. (E1) in general do not correspond to the mass eigenstates of the bosons that we use in the on-shell calculations. The Lagrangian corresponding to our parameterization of the coupling constants in the "mass basis" is given by

$$\begin{aligned} \mathcal{L} \supset & -\frac{1}{4}(\partial_\mu W_{a\nu} - \partial_\nu W_{a\mu})^2 - C_{abc}\partial_\nu W_{a\mu} W^\mu_b W^\nu_c + \sum_{a=1}^{N_V} \frac{1}{2} m_a^2 W_{a\mu} W^\mu_a \\ & + \bar{\psi}_R i(\partial\!\!\!/ + i W\!\!\!\!/_a R^a)\psi_R + \bar{\psi}_L i(\partial\!\!\!/ + i W\!\!\!\!/_a L^a)\psi_L - \sum_{\mathrm{i}} m_{\mathrm{i}}(\bar{\psi}_{\mathrm{i}L}\psi_{\mathrm{i}R} + \bar{\psi}_{\mathrm{i}R}\psi_{\mathrm{i}L}) \\ & + \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}\sum_{i=1}^{N_S} m_i^2\phi_i^2 + F_{abi}W_{a\mu}W^{b\mu}\phi_i - G_{aij}W_{a\mu}\partial^\mu\phi_i\phi_j \\ & - \frac{1}{6}P_{ijk}\phi_i\phi_j\phi_k - \frac{1}{24}K_{ijkl}\phi_i\phi_j\phi_k\phi_l - (\bar{\psi}_L H_i \psi_R + \bar{\psi}_R H_i^\dagger \psi_L)\phi_i, \end{aligned} \tag{E9}$$

where we have N_S physical massive scalar states. We have neglected all 4-point interactions except for ϕ^4, as the others will be determined by the 3-point interactions by unitary and locality. The parameters in the two Lagrangians can be related by performing appropriate transformations for the scalar and vector fields. The vector boson mass matrix in Eq. (E1) is given by

$$(M^2)_{ab} = \sum_{p=N_0+1}^{\bar{N}_S} (i\bar{T}^a\eta)_p (i\bar{T}^b\eta)_p + \sum_{a=1}^{N_0} (M^a_0)^2\delta_{ab}. \tag{E10}$$

The above symmetric matrix can be diagonalized by some orthogonal transformation O_{ab} on the vector fields,

$$(OM^2O^{-1})_{ab} = M_a^2\delta_{ab}. \tag{E11}$$

After the transformation, we can work out the linear mixing terms between the vector and scalar states and this will give us the Goldstone boson fields as follows:

$$\sigma_a = \sum_{b=1}^{N_0} \frac{O_{ab}M^b_0}{m_a}\theta_b + \sum_{p=N_0+1}^{N_S} \frac{iO_{ab}\bar{D}^b_{pq}\eta_q}{m_a}\pi_p = Q_{ap}\Pi_p, \tag{E12}$$

where we have grouped the scalar fields θ_a, π_p into one array Π_p with

$$\Pi_p = \theta_p, \qquad p = 1, \cdots N_0; \qquad \Pi_p = \pi_p, \quad p = N_0 + 1, \cdots \bar{N}_S, \tag{E13}$$

and the rotation matrix is given by

$$Q_{ap} = \begin{cases} \frac{O_{ap}M^p_0}{m_a}, & 1 \le p \le N_0 \\ \frac{iO_{ab}\bar{D}^b_{pq}\eta_q}{m_a}, & N_0 < p \le \bar{N}_S \end{cases}. \tag{E14}$$

One can see that $Q_{ap} = (O_{ab}/m_a)\bar{\lambda}^b_p$, and $Q_{ap}Q_{bp} = \delta_{ab}$. We can treat Q_{ap} as N_V orthonormal vectors in the scalar space, expressed in the gauge basis. Then one can find another $N_S = \bar{N}_S - N_V$ orthonormal vectors Q_{ip}, such that $Q_{\bar{i}q}$, which includes both Q_{ap} and Q_{ip}, forms a new, complete orthonormal basis in the scalar space. The physical scalar bosons are then given by

$$\phi_i = Q_{ip}\Pi_p, \tag{E15}$$

and together with the Goldstone bosons σ_a they form a new scalar basis $\Phi_{\bar{i}}$, which is related to the gauge basis by the rotation

$$\Phi_{\bar{i}} = Q_{\bar{i}q}\Pi_q, \tag{E16}$$

where the square matrix $Q_{\bar{i}q}$ is orthogonal. (One of course has the freedom to choose Q_{ip} so that the mass matrix of the physical scalars are diagonalized.)

We have thus figured out the rotation matrices O_{ab} and $Q_{\tilde{i}p}$ needed to transform the vector and scalar states to their mass basis. To arrive at Eq. (E9) where all Goldstone scalars are eliminated, we need to use the form invariance of Eq. (E1) under gauge transformations. The actual transformations used to relate field variables in Eqs. (E1) and (E9) are [59]

$$\theta_p \equiv \phi_i Q_{ip} + \sum_{a=1}^{N_V} m_a Q_{ap} \sigma_a, \quad \bar{\pi}_p = [e^{i\sigma\cdot\bar{T}}]_{pq}(\eta_q + Q_{iq}\phi_i),$$
$$A_{a\mu} = [e^{\sigma\cdot f}]_{ab} O_{cb} W_{c\mu} - \left[\frac{e^{\sigma\cdot f}}{\sigma\cdot f}\right]_{ab} \partial_\mu \sigma_b,$$
$$q_R = e^{i\sigma\cdot\bar{R}}\psi_R, \quad q_L = e^{i\sigma\cdot\bar{L}}\psi_L, \tag{E17}$$

where we have defined

$$(\sigma\cdot f)_{ab} \equiv \sigma^c O_{cd} f^{abd}, \tag{E18}$$

and for any generator $\bar{T}_{\mathrm{r}}$ of representation R_{r} in the gauge basis,

$$\sigma\cdot\bar{T}_{\mathrm{r}} \equiv \sigma^a O_{ab} \bar{T}^b_{\mathrm{r}}. \tag{E19}$$

Notice that the fermion masses remain diagonalized as a consequence of Eq. (E6). Upon the above transformations, the mass basis couplings in Eq. (E9) can be expressed in terms of the gauge basis couplings in Eq. (E1) as

$$C_{abc} = f^{a'b'c'} O_{aa'} O_{bb'} O_{cc'}, \quad R^a = O_{ab}\bar{R}^b, \quad L^a = O_{ab}\bar{L}^b,$$
$$G_{aij} = -iT^a_{ij}, \quad F_{abi} = -\frac{i}{2}(m_a T^b_{ia} + m_b T^a_{ib}), \quad H_i = Q_{ip} Y_{,p}, \tag{E20}$$

where the scalar generator T^a_{ij} in the mass basis is given by

$$T^a_{\tilde{i}\tilde{j}} = O_{ab} Q_{\tilde{i}p} Q_{\tilde{j}q} \bar{T}^b_{pq}. \tag{E21}$$

Now we can figure out the constraints corresponding to Eqs. (E2), (E4), and (E6) in the mass basis. The coupling matrices T, L, R, will still satisfy the commutation relations with C_{abc} as the structure constants,

$$C_{abe}C_{cde} + C_{ace}C_{dbe} + C_{ade}C_{bce} = 0, \quad [T^a, T^b] = iC_{abc}T^c,$$
$$[L^a, L^b] = iC_{abc}L^c, \quad [R^a, R^b] = iC_{abc}R^c. \tag{E22}$$

In addition, by using

$$O_{ab} Q_{\tilde{i}p} \bar{\lambda}^b_p = m_a \delta^{a\tilde{i}}, \tag{E23}$$

we can rewrite Eq. (E4) in the mass basis as follows:

$$T^a_{ib} m_b - T^b_{ia} m_a = 0, \quad T^a_{cb} m_b - T^b_{ca} m_a = iC_{abc} m_c, \tag{E24}$$

which leads to

$$F_{abi} = -im_a T^b_{ia}, \qquad T^a_{bc} = iC_{abc}\frac{m_a^2 - m_b^2 - m_c^2}{2m_b m_c}. \tag{E25}$$

To summarize, the generator $T^a_{\tilde{i}\tilde{j}}$ can be completely expressed in terms of C_{abc}, G_{aij}, F_{abi} and the gauge boson masses, as in Eq. (118). Similarly, Eq. (E6) leads to Eqs. (123) and (126).

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