Sudakov shoulder resummation for thrust and heavy jet mass

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(Received 4 August 2022; accepted 28 September 2022; published 13 October 2022)

When the allowed range of an observable grows order by order in perturbation theory, its perturbative expansion can have discontinuities (as in the C parameter) or discontinuities in its derivatives (as in thrust or heavy jet mass) called Sudakov shoulders. We explore the origin of these logarithms using both perturbation theory and effective field theory. We show that for thrust and heavy jet mass, the logarithms arise from kinematic configurations with narrow jets and deduce the next-to-leading logarithmic series. The left-shoulder logarithms in heavy jet mass (ρ) of the form $r\alpha_s^n \ln^{2n} r$ with $r = \frac{1}{3} - \rho$ are particularly dangerous, because they invalidate fixed-order perturbation theory in regions traditionally used to extract α_s. Although the factorization formula shows there are no nonglobal logarithms, we find Landau-pole-like singularities in the resummed distribution associated with the cusp anomalous dimension and that power corrections are exceptionally important.

DOI: 10.1103/PhysRevD.106.074011

I. INTRODUCTION

It is not uncommon for an observable to have a range that grows order by order in perturbation theory. Traditional e^+e^- event shapes, such as thrust, the C parameter, and heavy jet mass [1], have this property as do some hadron-collider observables like the jet shape [2,3]. Similar behavior can also be seen in the soft-drop jet mass [4]. As observed by Catani and Webber [1], when the range grows order by order, there can be incomplete cancellations between the virtual contributions, which are confined to the lower-order range, and the real-emission contributions, which are not. The results are distributions with nonanalytic behavior at intermediate values of the observable: discontinuities, cusps or kinks at any given finite order in perturbation theory, collectively called *Sudakov shoulders*, as shown in Fig. 2. Sudakov shoulders are caused by large logarithms associated with kinematic regions not close to the absolute (nonperturbative) phase space boundary. We classify the Sudakov shoulders as either right shoulders, which have large logarithms extending into regions accessible only at higher orders in perturbation theory (i.e., to the right of the shoulder as in thrust or C parameter) or left shoulders, which have logarithms affecting regions accessible at all orders in perturbation theory (i.e., to the left of the shoulder, as in heavy jet mass). Left shoulders are particularly problematic as they can invalidate the use of fixed-order perturbation theory over a wide range of observable values.

To understand Sudakov shoulders, consider first the thrust observable [5]. Thrust is defined in the center-of-mass frame of an e^+e^- collision as

$$T \equiv \max_{\vec{n}} \frac{\sum_j |\vec{p}_j \cdot \vec{n}|}{\sum_j |\vec{p}_j|}, \tag{1}$$

where the sum is over all particles in the event and the maximum is over 3-vectors $\vec{n}$ of unit norm. It is common to use $\tau = 1 - T$ in place of T. The vector $\vec{n}$ that maximizes thrust is known as the *thrust axis*. When there are only two particles, they must be back to back, and then $\tau = 0$ exactly. If there are three massless particles, then the phase space is two dimensional and can be parametrized with $s_{ij} = (p_i + p_j)^2/Q^2$ constrained by $s_{12} + s_{23} + s_{13} = 1$ with Q the center-of-mass energy. Then

$$\tau = \min(s_{12}, s_{13}, s_{23}) \leq \frac{1}{3}. \tag{2}$$

The phase space point that saturates this bound has $s_{12} = s_{13} = s_{23} = \frac{1}{3}$ and comprises the symmetric *trijet* configuration: three particles of equal energy and angular separation, as shown in Fig. 1. Near this point the spin-summed three-body matrix element squared is not exceptional:

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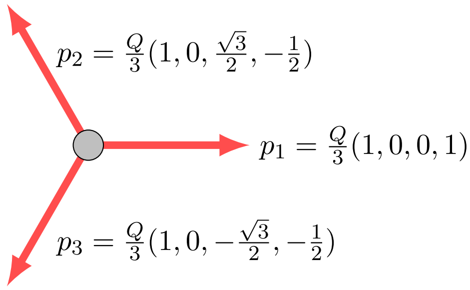


FIG. 1. The trijet configuration where $\rho = \tau = \frac{1}{3}$ has three equally spaced jets of equal energy.

$$|\mathcal{M}_1|^2 = |\mathcal{M}_0|^2 2C_F g_s^2 \frac{s_{12}^2 + s_{23}^2 + 2Q^2 s_{13}}{s_{12}s_{13}} \cong |\mathcal{M}_0|^2 64\pi C_F \alpha_s, \tag{3}$$

where $|\mathcal{M}_0|^2$ is the $\gamma^* \to q\bar{q}$ matrix element squared. Because the phase space goes to zero at $\tau = \frac{1}{3}$, the differential cross section must vanish there. The result is that

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} \cong 3 \times 48 C_F \frac{\alpha_s}{4\pi} \int_0^{\frac{1}{3}-\tau} ds_{12} = 144 C_F \frac{\alpha_s}{4\pi}\left(\frac{1}{3} - \tau\right)\theta\left(\frac{1}{3} - \tau\right) \tag{4}$$

with the factor of 3 coming from the three choices of thrust axis all of which contribute equally near $\tau = \frac{1}{3}$. Already here we can see the Sudakov shoulder: there is a discontinuity in the first derivative of the distribution from $-144 C_F \frac{\alpha_s}{4\pi}$ for $\tau < \frac{1}{3}$ to 0 for $\tau > \frac{1}{3}$.

Given the thrust axis from the maximization in Eq. (1), the event is divided into two hemispheres. We can compute the invariant masses m_1 and m_2 of all the partons in hemisphere 1 and 2 and then heavy jet mass is defined as

$$\rho = \frac{1}{Q^2}\max(m_1^2, m_2^2). \tag{5}$$

At order α_s one hemisphere must be massless and $\tau = \rho$, and thus $\frac{d\sigma}{d\rho}$ has a discontinuity in its first derivative at leading order, just like τ.

Now, consider what happens at higher order in perturbation theory. The parton in the light hemisphere will radiate gluons, making the light hemisphere massive. Since the cross section for the light jet having mass less than m after one emission scales like $\sigma \sim \alpha_s \ln^2 m^2$ there is a Sudakov enhancement to the cross section at small m^2. As the light-hemisphere jet grows, energy must be drawn away from the heavy hemisphere, making it lighter. Roughly speaking, setting $Q = 1$ for simplicity, $\rho \lesssim \frac{1}{3} - m^2$ (as we will derive). As a consequence, the cross section at $\rho = \frac{1}{3} - m^2$ will be enhanced by factors of $\ln^2 m^2 = \ln^2(\frac{1}{3} - \rho)$. Thus large Sudakov logs associated with radiation into the light hemisphere translate into Sudakov shoulder logs. This is the physical mechanism for the production of large logs in the left shoulder for heavy jet mass.

To properly and systematically resum the Sudakov shoulder logarithms, we must understand this mechanism, as well as the consequences of radiation from the heavy-hemisphere partons. At first glance, the mechanism, which transfers large logs from the light to the heavy hemisphere using energy conservation may seem difficult to reconcile with factorization. Indeed, previous work has noted the recoil sensitivity of Sudakov shoulder logarithms starting at the next-to-leading logarithmic (NLL) level [3]. Nevertheless, as we will see it is still possible to factorize the matrix elements and phase space near $\rho = \frac{1}{3}$ to isolate and extract the large logarithms, at least at the next-to-leading logarithmic level.

One may ask whether Sudakov shoulder resummation is important. For observables with only a right shoulder, such as thrust, one might argue that it is not so important, since there is not much data for $\tau > \frac{1}{3}$. However, for heavy jet mass one should generically expect that logs of the form $\alpha_s \ln^2(\frac{1}{3} - \rho)$ are as important away from the shoulder region as logs $\alpha_s \ln^2 \rho$ are away from the threshold $\rho = 0$. This leaves a rather narrow range of intermediate values of ρ where fixed-order perturbation theory might be trusted. Moreover, looking at Fig. 2 it seems that the Sudakov shoulder effects on the left shoulder of heavy jet mass curve tend to pull it down (and away from thrust), so that resumming the left Sudakov shoulder might bring the curves closer together. This difference of the left shoulder in thrust and heavy jet mass could help explain long-standing discrepancies between fits for α_s using the two event shapes [6,7].

In order to resum the Sudakov logs we first explore the regions of phase space that can contribute logarithms near the shoulder. We do this for the left shoulder of heavy jet mass in Sec. II. We find that the phase space near $\rho \lesssim \frac{1}{3}$ splits up into regions some of which generate large logarithms of $\frac{1}{3} - \rho$ and some of which do not. We find that all the logarithms come from regions with narrow jets in the light and heavy hemispheres. This is in contrast to the threshold region, for which every allowed point of phase space near $\rho \sim 0$ can contribute logarithms of ρ. It is also in contrast to nonglobal logarithms, such as for the light jet mass. There, logarithms of the light jet mass come from regions where the heavy jet side does not have to contain only narrow jets.

In Sec. III we discuss the factorization of ρ and τ near $\frac{1}{3}$. We find that near the shoulder region, the phase space and matrix elements both neatly factorize. This allows us to define a soft function, which, along with the inclusive jet function, can be used to reproduce all the logarithms at NLO and more generally the next-to-leading logarithmic series. In Sec. IV we analyze the resummed expression. We

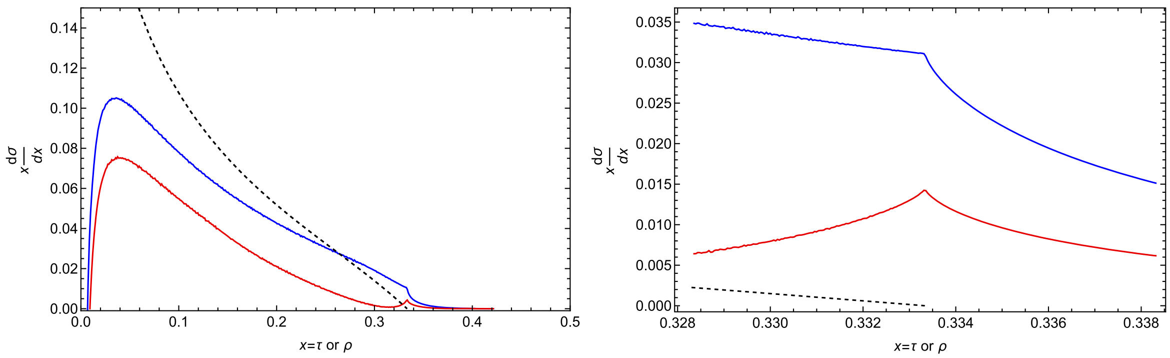


FIG. 2. Thrust (blue lines) and heavy jet mass (red lines) at next-to-leading order (NLO) compared to LO (dashed lines). The NLO curve does not have LO added in. That is, the LO is the $\frac{\alpha_s}{2\pi}$ times the "A" function and the NLO curves are $(\frac{\alpha_s}{2\pi})^2$ times the "B" functions, in the notation of Ref. [8]. Right is an enlargement of the Sudakov shoulder region near $\frac{1}{3}$. The NLO computation is performed with the program EVENT2 [9,10]. All distributions are normalized to Born cross section σ_0.

show that there are no nonglobal logarithms for the Sudakov shoulder; only regions related to the trijet configuration by soft or collinear radiation can generate the shoulder logs. We also find an unusual pole in the resummed distribution, qualitatively similar to the Landau pole in the running coupling. Unlike the QCD Landau pole, however, the singularity in the resummed heavy jet mass shoulder distribution is determined by the cusp anomalous dimension. Thus it is a kind of Sudakov Landau pole. Similar poles can be found in other observables, such as the Drell-Yan spectrum at small p_T [11–13]. We show that for the Sudakov shoulder case, the large Sudakov anomalous dimension contributing to this pole also enhances subleading power effects, making them comparable to the leading power result allowing the pole to be canceled in the full distribution. We conclude in Sec. VI.

II. NEXT-TO-LEADING-ORDER ANALYSIS

As a first step toward understanding Sudakov shoulder logarithms, we analyze the matrix elements and phase space near the shoulder region in full QCD. We concentrate here on the heavy jet mass for concreteness, but the same analysis works for thrust.

At next-to-leading order in QCD, there is the virtual contribution with three partons in the final state and a real emission contribution with four partons. The virtual contribution is proportional to the LO cross section and serves to regularize infrared and collinear divergences. Thus we focus on the real emission contributions to extract the logarithms.

To have $\rho \lesssim \frac{1}{3}$ we can have configurations which differ from the trijet configuration by soft and collinear emissions or configurations which do not. For example, one could take a nonplanar four-parton configuration with four well-separated partons and $\rho \sim 0.4$ and then adjust their momenta to lower ρ. Staring from such a configuration, one would not expect anything unusual to happen as ρ is lowered through $\frac{1}{3}$. Indeed, $\rho = \frac{1}{3}$ is only special because it is a kinematic limit for three-body phase space. Thus we expect that the only four-parton configurations which will contribute Sudakov shoulder logarithms are those close to the trijet configuration. We will find that this is in fact the case.

A. Kinematics

Let us define the momenta of the four particles in the final state as p_1^μ, p_2^μ, p_3^μ and p_4^μ. After momentum conservation, on-shell conditions and a frame choice, there are five independent degrees of freedom of these four momenta. Although we will not restrict the momenta to be soft or collinear, it is helpful to choose variables so that the soft and collinear limits are transparent. To impose the on-shell constraints, it is helpful to parametrize the momenta initially in light-cone coordinates:

$$p_1 = z_1 n^\mu + \frac{p_\perp^2}{4z_1}\bar{n}^\mu + p_\perp^\mu, \qquad p_2 = z_2 n^\mu + \frac{p_\perp^2}{4z_2}\bar{n}^\mu - p_\perp^\mu, \tag{6}$$

$$p_3 = z\omega n^\mu + \frac{q_\perp^2}{4z\omega}\bar{n}^\mu + q_\perp^\mu,$$
$$p_4 = (1-z)\omega n^\mu + \frac{q_\perp^2}{4(1-z)\omega}\bar{n}^\mu - q_\perp^\mu, \tag{7}$$

where $n^\mu = (1,0,0,1)$ and $\bar{n}^\mu = (1,0,0,-1)$ are back-to-back lightlike directions. Imposing momentum conservation and defining ϕ as the azimuthal angle between the 1-2 and 3-4 planes we can then express all the momenta in terms of

$$s_{234} = (p_2+p_3+p_4)^2, \quad s_{34} = (p_3+p_4)^2, \quad z, \quad \omega \text{ and } \phi. \tag{8}$$

We conventionally define ϕ by $p_\perp^\mu = (0, p_T \sin\phi, p_T \cos\phi, 0)$. The variables $\omega = \frac{1}{2}\bar{n}\cdot(p_3 + p_4)$ and s_{234} are hard variables, approaching $\frac{1}{3}$ at the trijet configuration. s_{34} is the invariant mass of one of the jets in the collinear limit which approaches zero in the trijet limit. The collinear momentum fraction z and the azimuthal angle are order 1 in the collinear limit, but $z \to 0$ in the limit that p_4 is soft. We also find it sometimes convenient to trade $\cos\phi$ for s_{23} using

$$s_{23} = \frac{s_{34}^2 z + 2s_{34}(1+2z\omega-2z-2\omega)\omega + 4zs_{234}\omega^2 + s_{34}s_{234}(z-1+2\omega-4z\omega)}{4\omega^2 - s_{34}} + \frac{2\sqrt{s_{34}(1-z)z(2\omega-s_{34})(1-2\omega)(2\omega-s_{234})(2\omega s_{234}-s_{34})}}{4\omega^2-s_{34}}\cos\phi. \tag{9}$$

When using s_{23} the physical constraint $-1 \le \cos\phi \le 1$ must be imposed on the region of integration. Another useful exact relation is

$$s_{12} = 1 - 2\omega + s_{34}\left(1 - \frac{1}{2\omega}\right). \tag{10}$$

We can use this relation to trade ω for ρ when $\rho = s_{12}$.

To compute thrust or heavy jet mass, we need to determine the thrust axis from the formula in Eq. (1). With four partons, the two possibilities are that three partons are in one hemisphere and one parton in the other, or two partons can be in each hemisphere. If we know that partons $p_1 \ldots p_m$ are to be clustered in the same hemisphere, then

$$\max_{\vec{n}} \sum_{j=1}^{m} |\vec{p}_j \cdot \vec{n}| = 2\max_{\vec{n}}\left(\sum_{j=1}^{m} \vec{p}_j\right)\cdot\vec{n}. \tag{11}$$

This dot product will be maximized if $\vec{n} = |\sum \vec{p}_j|^{-1}\sum\vec{p}_j$ so that the thrust axis will always align with the sum of momenta in each hemisphere. So there are seven possibilities for the thrust axis. For each axis choice

$$\sum_{j=1}^{m} |\vec{p}_j\cdot\vec{n}| = 2\frac{1}{|\sum\vec{p}_j|}\left(\sum\vec{p}_j\right)\cdot\left(\sum\vec{p}_j\right) = 2\left|\sum_{j=1}^{m}\vec{p}_j\right|. \tag{12}$$

Thus, to determine the thrust axis, we need to find which set of partons has the largest value of $2|\sum\vec{p}_j|$ or, equivalently,

$$T_j^2 \equiv 4\left|\sum\vec{p}_j\right|^2. \tag{13}$$

In terms of our variables in Eq. (8), the T_j with one parton in one hemisphere are relatively simple:

$$T_1^2 = T_{234}^2 = (1-s_{234})^2, \quad T_2^2 = T_{134}^2 = \left[\frac{2(1+s_{234}-2\omega)\omega - s_{34}}{2\omega}\right]^2, \tag{14}$$

$$T_3^2 = T_{124}^2 = \left[\frac{4z\omega^2 + s_{34}(1-z)}{2\omega}\right]^2, \quad T_4^2 = T_{123}^2 = \left[\frac{4(1-z)\omega^2 + s_{34}z}{2\omega}\right]^2. \tag{15}$$

We also have

$$T_{12}^2 = \left(\frac{4\omega^2 - s_{34}}{2\omega}\right)^2 \tag{16}$$

and

$$T_{13}^2 = \frac{1}{4\omega^2}[s_{34}^2(1-z)^2 + 4\omega^2(4s_{23} + s_{234}^2 + (1-2\omega z)^2 - 2s_{234}(1+2\omega z)) - 4s_{34}\omega(1+s_{234}-z-s_{234}z+2\omega(z^2-z-2))], \tag{17}$$

$$T_{14}^2 = \left[\frac{s_{34}z - 2\omega(1+s_{234}-2\omega(1-z))}{2\omega}\right]^2 - 4s_{23}. \tag{18}$$

All of these T_j values are exact.

Now we would like to consider the region $\rho < \frac{1}{3}$. The heavy hemisphere can have either two partons or three partons. We can therefore choose it to be $\rho = s_{234}$ with T_1 maximal or s_{12} with T_{12} maximal. The other cases are given by permutation of the indices. Figure 3 shows examples of the phase space regions labeled by which T_j is greatest. All regions in these plots contribute to some value of ρ. However, to avoid overcounting we only need to consider the green region on the left plot and the blue region in the right plot.

B. Matrix elements

Let us define

$$r \equiv \frac{1}{3} - \rho. \tag{19}$$

As we have discussed, we expect contributions to the NLO heavy jet mass cross section with factors of $\ln r$ or $\ln^2 r$ to

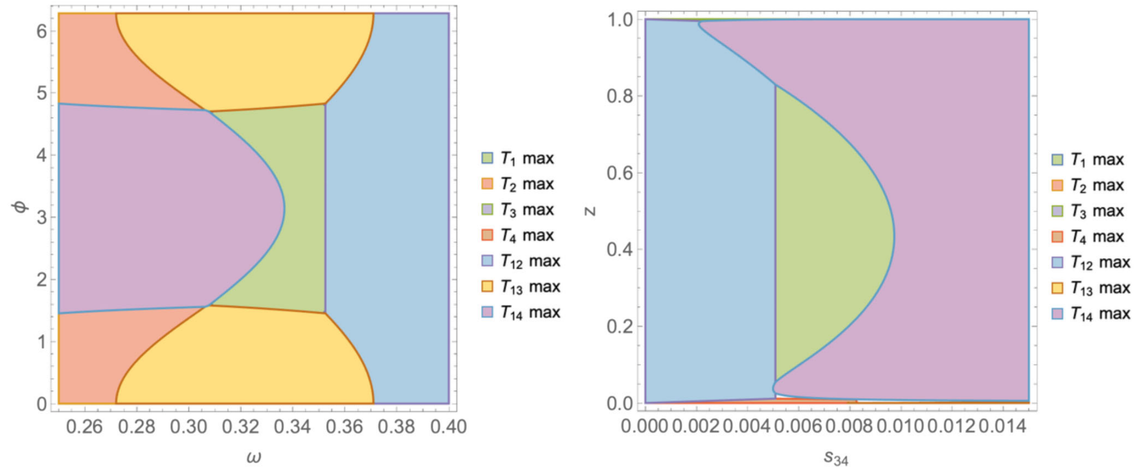


FIG. 3. Example slices of 5D phase space for four massless partons. The colors indicate which collection of momenta determines the thrust axis. The left plot has $z = 0.06$, $s_{34} = 0.02$ and $s_{234} = \frac{1}{3} - 0.01$. The green region in this plot would contribute to the heavy jet mass distribution at $r = \frac{1}{3} - \rho = 0.01$. The right plot has $\phi = \pi$, $s_{234} = \frac{1}{3}$ and $s_{12} = \frac{1}{3} - 0.01$. The blue region in this plot contributes also at $r = 0.01$.

come from soft or collinear regions of phase space close to the trijet configuration. We can therefore power expand the matrix elements and phase space constraints in soft and collinear limits. This dramatically simplifies the calculation. There are two ways to confirm that only soft and collinear limits are relevant. First, we can extend the integration limits to the full phase space and verify that no additional logarithms can be generated. Second, we can compare the logarithms we extract with a numerical computation of the heavy jet mass distribution at NLO.

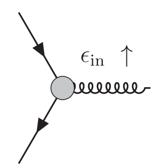
For power counting we take $r \sim \lambda \ll 1$. In the collinear limit where $p_4 \| p_3$, the phase space variables scale as

$$s_{34} \sim \lambda, \qquad x \equiv \omega - \frac{1}{3} \sim \lambda, \qquad y \equiv s_{234} - \frac{1}{3} \sim \lambda,$$
$$z \sim \lambda^0, \qquad \phi \sim s_{23} \sim \lambda^0. \tag{20}$$

In the soft limit, where p_3 is soft, the scaling is the same except that $z \sim \lambda$ instead of $z \sim \lambda^0$.

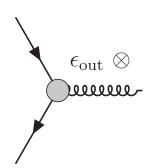
First we compute the matrix elements squared at leading power. We do this by summing all the relevant Feynman diagrams, squaring the amplitudes and summing over spins, after which we take the leading power expansion. We cross-check the results against the expectation for soft and collinear limits from factorization.

The $\gamma^\star \to q\bar{q}g$ matrix element depends on whether the gluon is polarized in the plane of scattering or out of the plane. We find

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{in}}_{\gamma^* \to q\bar{q}g}\right|^2 =$$


$$= |\mathcal{M}_0|^2 2 g_s^2 C_F \tag{21}$$

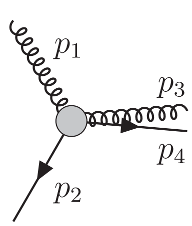
when the gluon polarization $\epsilon_{\text{in}} = (0,0,1,0)$ in the conventions of Fig. 1, where $p_g = \frac{Q}{3}(1,0,0,1)$, and

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{out}}_{\gamma^* \to q\bar{q}g}\right|^2 =$$


$$= |\mathcal{M}_0|^2 14 g_s^2 C_F \tag{22}$$

when the gluon polarization is $\epsilon_{\text{out}} = (0,1,0,0)$. The sum of these agrees with Eq. (3).

The matrix elements depend on which partons are gluons and which are quarks. If p_2 is a quark and p_4 is a gluon, then to leading power in collinear scaling

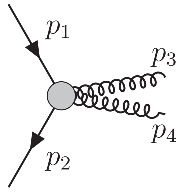
$$\left|\mathcal{M}^{\text{collinear}}_{\gamma^* \to qg\bar{q}g}\right|^2 =$$


$$\cong |\mathcal{M}_0|^2\, 32 g_s^4 C_F^2 \frac{1}{s_{34}} \frac{1+z^2}{1-z}\,. \tag{23}$$

Here the blob represents all the diagrams that can contribute. We derive this by squaring the full matrix element for $\gamma^\star \to qg\bar{q}g$ using Qgraf [14] and FORM [15] or FeynCalc [16–18], summing over spins and then power expanding in the small s_{34} limit. The splitting function naturally appears.

When p_3 and p_4 are gluons, then we find

$$\left|\mathcal{M}^{\text{collinear}}_{\gamma^*\to q\bar{q}gg}\right|^2 =$$

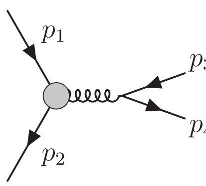


$$\cong \left|\mathcal{M}_0\right|^2 C_F C_A g_s^4 \frac{64}{s_{34}}\left[\frac{(1-z+z^2)^2}{z(1-z)} + \frac{3}{4}z(1-z)\cos 2\phi\right]. \tag{24}$$

Note the azimuthal angle dependence is due to the polarization of the gluons. Indeed, the leading-order $\gamma^\star \to q\bar{q}g$ matrix element is polarized, and we must therefore use polarized splitting functions (see [19] for example). We have checked that summing the polarized leading-order matrix elements in Eqs. (21) and (22) with the polarized splitting functions (see [19] for example) reproduces Eq. (24).

And finally when p_3 and p_4 are quarks (or antiquarks), the leading power result is the same whether they are identical or not:

$$\left|\mathcal{M}^{\text{collinear}}_{\gamma^*\to q\bar{q}q'\bar{q}'}\right|^2 \cong \left|\mathcal{M}^{\text{collinear}}_{\gamma^*\to q\bar{q}q\bar{q}}\right|^2 = \quad . \tag{25}$$



$$\cong |\mathcal{M}_0|^2 C_F T_f n_f g_s^4 \frac{16}{s_{34}}(2 - z - z^2 - 6z(1-z)\cos^2\phi) \tag{26}$$

This expression also depends on the azimuthal angle and, like the gluon case, is consistent with using the polarized three-parton matrix elements and polarization-dependent splitting functions.

For the soft limits, we can power expand the full matrix elements in the soft limit. When z is soft, we cannot drop s_{34} with respect to z, or vice versa. When p_3 and p_4 are both gluons, the result can be written as

$$|\mathcal{M}^{\text{soft}}_{\gamma^*\to q\bar{q}gg}|^2 \cong |\mathcal{M}_0|^2 C_F g_s^4 \frac{64}{3}\left[\left(C_F - \frac{1}{2}C_A\right)\frac{1}{s_{14}s_{24}} + \frac{C_A}{2}\frac{1}{s_{14}s_{34}} + \frac{C_A}{2}\frac{1}{s_{24}s_{34}}\right], \tag{27}$$

where

$$s_{14}s_{24} = 9s_{34}^2 + 16z^2 - 24s_{34}z\cos(2\phi), \tag{28}$$

$$s_{14}s_{34} = 3s_{34}^2 + 4s_{34}z - 4s_{34}\sqrt{3s_{34}z}\cos\phi, \tag{29}$$

$$s_{24}s_{34} = 3s_{34}^2 + 4s_{34}z + 4s_{34}\sqrt{3s_{34}z}\cos\phi. \tag{30}$$

This is consistent with the eikonal approximation.

To avoid double counting we also need the soft collinear matrix elements which come from taking the soft limit (small z) of the collinear matrix elements or, equivalently, the collinear limit ($s_{34} \ll z$) of the soft matrix elements. These are therefore the same as the soft matrix elements but keeping only the final term in Eqs. (28)–(30).

C. Phase space

For the phase space limits, we will first examine the soft-collinear limit where $z \sim \lambda$. To leading power in the soft-collinear limit

$$T_1 \cong \frac{1}{9} - \frac{y}{3}, \qquad T_2 \cong \frac{1}{9} - \frac{s_{34}}{2} - \frac{2x}{3} + \frac{y}{3},$$
$$T_3 \cong 0, \qquad T_4 \cong \frac{1}{9} + \frac{2x}{3} - \frac{2z}{9}, \tag{31}$$

$$T_{12} \cong \frac{1}{9} - \frac{s_{34}}{2} + \frac{2x}{3}, \qquad T_{13} \cong \frac{1}{9} + s_{23} - \frac{y}{3} - \frac{4z}{9},$$
$$T_{23} \cong \frac{1}{9} - s_{23} - \frac{2x}{3} + \frac{y}{3} + \frac{2z}{9}. \tag{32}$$

For the case where T_1 is maximal, $\rho = s_{234}$ and $y = r$. We can then impose the constraints $T_1 > T_2$, $T_1 > T_3$, and so on. Since we are using the variable s_{23} instead of $\cos\phi$ we also have to impose $-1 \le \cos\phi \le 1$. Reducing these constraints leads to five integration regions:

$$\begin{aligned}\int d\Pi_1 = {} & 2\int_0^{\frac{1}{2}s_{34}^+} ds_{34}\int_{z^+}^{1-z^+} dz \int_{x_A}^{x_B} dx \int_{s_{23}^{\phi-}}^{s_{23}^{\phi+}} ds_{23} J \\ & + 2\int_0^{s_{34}^+} ds_{34}\int_{\frac{9}{4}s_{34}}^{z^+} dz \int_{x_C}^{x_B} dx \int_{s_{23}^{\phi-}}^{\frac{4}{9}z} ds_{23} J \\ & + 2\int_0^{s_{34}^+} ds_{34}\int_{\frac{9}{4}s_{34}}^{z^+} dz \int_{x_A}^{x_C} dx \int_{s_{23}^{A}}^{\frac{4}{9}z} ds_{23} J \\ & + 2\int_0^{s_{34}^+} ds_{34}\int_{z^-}^{\frac{9}{4}s_{34}} dz \int_{x_C}^{x_D} dx \int_{s_{23}^{\phi-}}^{\frac{4}{9}z} ds_{23} J \\ & + \int_0 ds_{34}\int_{z^-}^{\frac{9}{4}s_{34}} dz \int_{x_E}^{x_C} dx \int_{s_{23}^{A}}^{\frac{4}{9}z} ds_{23} J,\end{aligned} \tag{33}$$

where

$$s_{34}^{+} = \frac{4}{9}(7 - 4\sqrt{3}), \qquad z^{\pm} = \frac{9}{4}(7 \pm 4\sqrt{3}) s_{34}, \tag{34}$$

$$s_{23}^{\phi\pm} = \left(\sqrt{\frac{s_{34}}{4}} \pm \sqrt{\frac{z}{3}}\right)^2, \quad s_{23}^{A} = -\frac{2}{3}x + \frac{2}{3}y + \frac{2}{9}z, \tag{35}$$

$$x_A = -\frac{3}{4}s_{34} + y, \qquad x_B = \frac{3}{4}s_{34} - \frac{y}{2},$$
$$x_C = -\frac{3}{8}s_{34} + y + \frac{\sqrt{3 s_{34} z}}{2} - \frac{z}{6}, \tag{36}$$

$$x_D = -\frac{y}{2} + \frac{z}{3}, \qquad x_E = y - \frac{z}{3}. \tag{37}$$

The Jacobian

$$J = \frac{1}{\sqrt{(s_{23}^{\phi+} - s_{23})(s_{23} - s_{23}^{\phi-})}} \tag{38}$$

scales like $J \sim \lambda^0$. For the leading double log we need to compute

$$\int d\Pi_1 |\mathcal{M}|^2 \sim \int d\Pi_1 \frac{1}{s_{34} z}. \tag{39}$$

Analyzing the integrals we find that none of them generate $\ln r$ terms; the limit $r \to 0$ in each of the integrals is smooth. Thus the region with T_1 max does not contribute to the Sudakov shoulder at NLO. The logs must therefore come from regions with two partons in each hemisphere.

Next, we consider configurations where T_{12} is maximal. As before, we expand first assuming collinear scaling. In this case, we no longer have $r = \frac{1}{3} - \rho = y$ but instead

$$\rho = s_{12} = 1 + s_{34} - \frac{s_{34}}{2\omega} - 2\omega \approx \frac{1}{3} - \frac{1}{2}s_{34} - 2x \tag{40}$$

so that $r = \frac{1}{2}s_{34} + 2x$. To hold r fixed we then can use r, s_{34}, z, y, and s_{23} as independent variables (instead of r, s_{34}, z, x, and s_{23} in the T_1 max case). Now we find 40 relevant integration regions. In most of these r can be set to zero without consequence. Only four can possibly generate logs of r:

$$\begin{aligned}\int d\Pi_{12} = &\int_0^r ds_{34} \int_{z^+}^{1-z^+} dz \int_{y_A}^{y_B} dy \int_0^{2\pi} d\phi \\ &+ \int_0^{\frac{r}{3}} ds_{34} \int_{\frac{9s_{34}}{4}}^{z^+} dz \int_{y_D}^{y_C} dy \int_0^{2\pi} d\phi \\ &+ \int_0^{\frac{r}{3}} ds_{34} \int_{\frac{9s_{34}}{4}}^{z^+} dz \int_{y_C}^{y_B} dy \int_{s_{23}^B}^{s_{\phi}^{+}} ds_{23} J \\ &+ \int_0^{\frac{r}{3}} ds_{34} \int_{\frac{9s_{34}}{4}}^{z^+} dz \int_{y_A}^{y_D} dy \int_{s_{23}^{\phi-}}^{s_{23}^C} ds_{23} J,\end{aligned} \tag{41}$$

where

$$y_A = -r + 2s_{34}, \quad y_B = 2r - s_{34},$$
$$y_C = 2r - \frac{7s_{34}}{4} - \sqrt{3s_{34}z} + \frac{z}{3},$$
$$y_D = -r + \frac{11 s_{34}}{4} + \sqrt{3 s_{34} z} - \frac{z}{3},$$
$$s_{23}^B = -\frac{2r}{3} + \frac{5s_{34}}{6} + \frac{y}{3} + \frac{2z}{9}, \quad s_{23}^C = \frac{r}{3} - \frac{2s_{34}}{3} + \frac{y}{3} + \frac{4z}{9}. \tag{42}$$

For the C_F^2 color structure, using the power-expanded matrix elements in the collinear limit, Eq. (23), only the first two integrals in Eq. (41) contribute. We find

$$\begin{aligned}\mathcal{S}_{\mathrm{c}}^{(C_F)} &= 4\int d\Pi_{12} |\mathcal{M}_{\gamma^* \to qg g\bar{q}}^{\mathrm{collinear}}|^2 \\ &= \frac{6\alpha_s^2}{\pi^2} C_F^2 r \left[-2\ln^2 r + \left(1 - 8\ln\frac{3}{2}\right)\ln r + \cdots\right].\end{aligned} \tag{43}$$

Similarly, integrating against the soft matrix element and the soft-collinear overlap region, we find

$$\begin{aligned}\mathcal{S}_{\mathrm{s}}^{(C_F)} &= \int d\Pi_{12} (4|\mathcal{M}_{\gamma^* \to qgg\bar{q}}^{\mathrm{soft}}|^2 + 2|\mathcal{M}_{\gamma^* \to q\bar{q}gg}^{\mathrm{soft}}|^2) \\ &= \frac{12\alpha_s^2}{\pi^2} C_F^2 r \left[-\ln^2 r + 2\left(1 + \ln\frac{4}{3}\right)\ln r + \cdots\right],\end{aligned} \tag{44}$$

$$\begin{aligned}\mathcal{S}_{\mathrm{sc}}^{(C_F)} &= 4\int d\Pi_{12} |\mathcal{M}_{\gamma^* \to qgg\bar{q}}^{\mathrm{soft-coll}}|^2 \\ &= \frac{12\alpha_s^2}{\pi^2} C_F^2 r \left[-\ln^2 r + 2\left(1 - 2\ln\frac{3}{2}\right)\ln r + \cdots\right].\end{aligned} \tag{45}$$

The constants in the integrals come from the permutations of final state particles and we have accounted the symmetry factor for identical gluons. The total is

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma^{(C_F)}}{dr} &= \mathcal{S}_{\mathrm{c}}^{(C_F)} + \mathcal{S}_{\mathrm{s}}^{(C_F)} - \mathcal{S}_{\mathrm{sc}}^{(C_F)} \\ &= \left(\frac{\alpha_s}{4\pi}\right)^2 C_F^2 r [-192 \ln^2 r \\ &\quad + (96 + 768\ln 2 - 384\ln 3)\ln r + \cdots].\end{aligned} \tag{46}$$

This is compared to the exact (numerical) NLO calculation in the shoulder region in Fig. 6.

For the $C_F C_A$ color structure, there can be single logarithms coming from both the $z \sim 0$ and $z \sim 1$ regions. Moreover, the splitting functions in this case depend on the polarization of the gluon that splits. However, because the only integration regions that contribute logarithms are uniform in ϕ [the first two in Eq. (41)], one can simply azimuthally average the splitting functions, reducing them to the unpolarized case. The final resums we find are

$$\mathcal{S}_c^{(C_A)} = \frac{\alpha_s^2}{\pi^2} C_F C_A r \left[-6\ln^2 r + \left(1 - 24\ln\frac{3}{2}\right)\ln r + \cdots \right],$$
$$\mathcal{S}_s^{(C_A)} = 6\frac{\alpha_s^2}{\pi^2} C_F C_A r \left[-\ln^2 r + 2\left(1 + \ln\frac{4}{3}\right)\ln r + \cdots \right],$$
$$\mathcal{S}_{sc}^{(C_A)} = 6\frac{\alpha_s^2}{\pi^2} C_F C_A r \left[-\ln^2 r + 2\left(1 - 2\ln\frac{3}{2}\right)\ln r + \cdots \right], \quad (47)$$

which gives

$$\frac{1}{\sigma_0}\frac{d\sigma^{(C_A)}}{dr} = \left(\frac{\alpha_s}{4\pi}\right)^2 C_F C_A r[-96\ln^2 r + (16 + 384\ln 2 - 192\ln 3)\ln r + \cdots]. \quad (48)$$

Again, this is compared to NLO in the shoulder region in Fig. 6.

The $n_f T_F C_F$ color structure only contains a single logarithm since there is no soft region. Integrating the collinear matrix element Eq. (26) over power-expanded phase space gives

$$\frac{1}{\sigma_0}\frac{d\sigma^{(n_f)}}{dr} = 64\left(\frac{\alpha_s}{4\pi}\right)^2 C_F T_f n_f r \ln r + \cdots. \quad (49)$$

No overlap subtraction is needed. This is also shown in Fig. 6.

One can perform a similar leading-power computation for the right shoulder for thrust and heavy jet mass. For these cases, we find it is only the phase space regions with one parton in one hemisphere and three partons in the other hemisphere that contribute. Since the equivalent calculation is significantly easier using soft-collinear effective theory, we skip the details of the right-shoulder cases using the full theory and turn instead to the effective theory approach.

III. FACTORIZATION AND RESUMMATION

In Sec. II, we computed the Sudakov shoulder logs for heavy jet mass and thrust at NLO using full QCD expanded to leading power. We now want to generalize the analysis to all orders leading to a factorization formula. To do so, we first review the approach of [1] and discuss recoil sensitivity. We then demonstrate a different approach inspired by the NLO calculation that leads to a systematically improvable factorization formula.

A. Recoil sensitivity

One approach to resummation of Sudakov shoulders [1] is that emissions from one of the hard partons will cause an additive shift in heavy jet mass (or thrust) from $\rho \to \rho + m^2$. Then one could write the resummed distribution as a convolution. Heuristically,

$$\sigma_{\text{resummed}}(\rho) \sim \int dm^2 \sigma_{\text{LO}}(\rho - m^2) J(m^2) \quad (50)$$

with $J(m^2)$ representing some sort of jet function and $\sigma_{\text{LO}}(\rho)$ the leading-order cross section.

Unfortunately, when one tries to make this formula more precise it produces ambiguities beyond the leading logarithmic order. To see this, consider how ρ changes due to emissions in the light hemisphere making the light hemisphere have a mass m^2. With three massless partons taking p_1 and p_2 in the heavy hemisphere and p_3 in the light hemisphere for concreteness, the heavy jet mass is

$$\rho = (p_1^\mu + p_2^\mu)^2 = (p_{\text{tot}}^\mu - p_3^\mu)^2 = 1 - 2E_3 \quad (51)$$

with E_3 the energy of the light-hemisphere parton. Now say the p_3 parton becomes massive (i.e., turns into a jet) with $p_3^2 = m^2$. Then we have the exact relation

$$\rho = (p_1^\mu + p_2^\mu)^2 = (p_{\text{tot}}^\mu - p_3^\mu)^2 = 1 + m^2 - 2E_3. \quad (52)$$

So it seems $\rho \to \rho + m^2$, as in Eq. (50). However, this was a little too quick. Suppose instead of expressing ρ in terms of E_3 we expressed it in terms of $|\vec{p}_3|$. Then, when p_3 is massless,

$$\rho = 1 - 2|\vec{p}_3|. \quad (53)$$

However, after the emissions,

$$\rho = 1 + m^2 - 2E_3 = 1 + m^2 - 2\sqrt{\vec{p}_3^{\,2} + m^2}$$
$$\cong 1 + m^2 - 2|\vec{p}_3| - \frac{m^2}{|\vec{p}_3|} + \cdots. \quad (54)$$

Now, near threshold $|\vec{p}_3| \sim \frac{1}{3}$, so $\frac{m^2}{|\vec{p}_3|} \cong 3m^2$ and we find $\rho \to \rho - 2m^2$ instead of $\rho \to \rho + m^2$. Thus the way ρ shifts depends on whether we hold the energy or the momentum of the jet fixed after the emission. This recoil sensitivity seems to violate factorization. Moreover, if $\rho \to \rho - 2m^2$, one cannot write down a convolution for the distribution as in (50), since the shift implies that emissions only decrease the value of the heavy jet mass. Thus it becomes clear that while one might use the emission picture for the double-logarithmic analysis of [1], it is inadequate for NLL resummation.

B. Factorization

To proceed, recall from Sec. II which configurations contributed to the NLO logs. With four partons, we can have either two in each hemisphere or one in one light hemisphere and three in the heavy hemisphere. For the left shoulder of heavy jet mass at NLO we found that only the case with two partons in each hemisphere contributed. Moreover, the two partons in the heavy hemisphere were hard, with invariant mass $\rho \sim \frac{1}{3}$, while the two partons in the light hemisphere formed a jet of small invariant mass, $s_{34} \sim \frac{1}{3} - \rho \ll 1$. In contrast, for the right shoulder of heavy jet mass or thrust, only the region with one parton in the light hemisphere contributed. Moreover, the configuration in the heavy hemisphere had two hard partons and one parton which was soft or collinear to one of the hard partons.

In the $r = \frac{1}{3} - \rho \ll 1$ region, we found integrals like

$$I \sim |\mathcal{M}_0|^2 \frac{\alpha_s}{4\pi} C_F^2 \int_0^r \frac{ds_{34}}{s_{34}} \int_{\frac{9}{4}s_{34}}^1 \frac{dz}{z} \int_{\frac{1}{3}-r}^{\frac{1}{3}+2r} ds_{234} \quad (55)$$

$$\cong |\mathcal{M}_0|^2 \frac{\alpha_s}{4\pi} C_F^2 r \ln^2 r. \quad (56)$$

The integrals over s_{34} (the invariant mass of the 34 jet) and z (the collinear splitting fraction in the 34 jet) are similar to what we would have in an inclusive jet function. The s_{234} variable is a hard phase space variable, equal to s_{23} at leading power. The last integral gives the factor of r which is the same factor in the leading-order cross section, as in Eq. (4). Thus at higher orders it is natural to expect the generalization of this integral to one with a single integral over hard kinematic phase space and an integral over the kinematics of the light jet. Thus, instead of convolution of the hard cross section with the emission cross section, as in (50), we should expect the phase space to factorize into a part which depends on the hard kinematics and a part which depends on the emissions.

The first observation allowing us to factorize the cross section in the region $r = \frac{1}{3} - \rho \ll 1$ is that only configurations which differ from the trijet configuration by soft or collinear emissions can generate logarithms of r. The reason for this is that $r = 0$ is only special from the point of view of three-body massless kinematics. One can have four-parton configurations with ρ close to $\frac{1}{3}$ that are not close to the trijet configuration. However, such configurations contribute to the cross section both for $r < 0$ and $r > 0$ and will be smooth across $r = 0$. Hence they cannot produce large logarithms (in Sec. IV B we use this same argument to show there are no nonglobal logs in the Sudakov shoulders).

So let use consider a generic configuration with three jets pointing in the n_1, n_2 and n_3 directions. Such a configuration can have particles collinear to the three directions as well as soft partons scattered throughout phase space. At leading power, we can treat the collinear radiation as generating masses m_1, m_2 and m_3 for the three jets. Thus we can approximate the state as having three hard, massive particles with momenta p_1, p_2 and p_3 and soft radiation.

To compute heavy jet mass and thrust, we need to know which direction the thrust axis points for a given amount of collinear and soft radiation. To determine this, we first observe that, as in Eq. (11), the thrust axis is determined by the set of momenta in a given hemisphere that maximize

$$T_{\{p_i\}} = 2\left|\sum_{p_i} \vec{p}\right|. \quad (57)$$

Then τ and ρ can be computed from the set $\{p_i\}$.

Let us begin with the case where there is only collinear momenta, so we only have the three massive momenta to consider. In this case, phase space is described by s_{12}, s_{13} and s_{23} subject to $s_{12} + s_{13} + s_{23} = 1 + m_1^2 + m_2^2 + m_3^2$, where $s_{ij} = (p_i + p_j)^2$. Then

$$T_1^2 = 4\vec{p}_1^{\,2} = (1 - s_{23})^2 - 2m_1^2(1 + s_{23}) + m_1^4 \quad (58)$$

and similarly for T_2^2 and T_3^2 by permutation. Let us take the case where T_1 sets the thrust axis, so that $r = \frac{1}{3} - s_{23}$. Then at leading power (assuming $m_i^2 \sim r \sim s_{12} - \frac{1}{3}$)

$$T_1 \cong \frac{2}{3} + r - 2m_1^2, \qquad T_2 \cong \frac{1}{3} - r + s_{12} - m_1^2 - 3m_2^2 - m_3^2,$$
$$T_3 \cong 1 - s_{12} - 2m_3^2. \quad (59)$$

So the conditions $T_1 > T_2$ and $T_1 > T_3$ imply

$$\frac{1}{3} - r + 2m_1^2 - 2m_3^2 < s_{12} < \frac{1}{3} + 2r - m_1^2 + 3m_2^2 + m_3^2. \quad (60)$$

These limits on s_{12} pinch off when $r = m_2^2 + m_3^2 - m_1^2$. At $m = 0$ the linear scaling with r of the s_{12} integration region is what generates the linear falloff of the thrust or heavy jet mass cross section as in Eq. (4). For the integration region to be nonzero we therefore have

$$m_1^2 < r + m_2^2 + m_3^2. \quad (61)$$

In other words, at fixed m_2, m_3 and r, there is an upper limit on the light-hemisphere jet mass. The probability of finding a light jet of mass at most m_1 at leading power is proportional to $\ln^2 m_1$, so for $m_2 = m_3 = 0$ the integral over m_1 up to r will give the $\ln^2 r$ left Sudakov shoulder logarithms. Combined with the factor of r from the s_{12}

integration gives an overall $r \ln^2 r$ behavior. If m_2 and m_3 are parametrically larger than r, then we can drop r in Eq. (61). In that case, no logs are generated. Thus the shoulder logs are determined by the region of small m_1, m_2 and m_3, consistent with a global observable.

The right shoulder for heavy jet mass is constrained by Eq. (61), but with $r < 0$. For the right shoulder we define $s = \rho - \frac{1}{3}$. Then

$$m_1^2 + s < m_2^2 + m_3^2 \tag{62}$$

replaces Eq. (61).

For the thrust case, we define $t = \tau - \frac{1}{3}$. When T_1 determines the thrust axis, then at leading power

$$t = \frac{2}{3} - T_1 \cong 2m_1^2 - r \tag{63}$$

and Eq. (61) becomes

$$t < m_1^2 + m_2^2 + m_3^2. \tag{64}$$

Thus the right shoulder for thrust is defined by integrals over any of the masses with a lower limit of t. Since the inclusive integral, without this constraint, has no t dependence, one can equivalently get the right Sudakov shoulder logarithms by integrating over the masses constrained by $m_1^2 + m_2^2 + m_3^2 < t$.

For the soft radiation, we first need to determine when it affects the thrust axis. Let us start with the configuration with three massive partons and suppose some soft radiation k enters hemisphere 1. We want to know whether the thrust axis should shift so that hemisphere 1 excludes k or if it should stay fixed, to include k. To find out, we need to compare T_{1k}, the thrust value with p_1 and k included in the hemisphere, to T_1 where k is not the 1 hemisphere, but is still included overall. A quick calculation shows that

$$T_{1k}^2 \cong T_1^2 + \frac{8}{3}(p_2 \cdot k + p_3 \cdot k - 2p_1 \cdot k). \tag{65}$$

Defining $p_{\bar{j}}$ as p_j with its 3-momentum reversed, so

$$p_{\bar{1}} = \frac{Q}{3}(1, 0, 0, -1) = \frac{2}{3}p_2 + \frac{2}{3}p_3 - \frac{1}{3}p_1, \tag{66}$$

we can write

$$T_{1k}^2 \cong T_1^2 + 4(p_{\bar{1}} \cdot k - p_1 \cdot k). \tag{67}$$

When k is in the 1 hemisphere, it must be closer to p_1 than $p_{\bar{1}}$. In that case $p_{\bar{1}} \cdot k > p_1 \cdot k$. We conclude that thrust is maximized when all the soft radiation in the hemisphere centered on p_1 is included. In other words, if radiation is slightly on the opposite side of the hemisphere boundary, the thrust axis should not shift to cluster k with p_1.

Now suppose there is a lot of soft radiation with momenta with $\{k_i^\mu\}$. Since the thrust value goes up when radiation is included in a given hemisphere, to find the thrust axis we only have to consider three sets of momenta: for each j the set includes a hard jet's momentum p_j and all the soft radiation k_j^{hemi} in the jet's hemisphere. That is, the maximal value of thrust for a hemisphere containing p_j will be given by

$$T_j^{\max} = T_j + 3(p_{\bar{j}} \cdot k_j^{\text{hemi}} - p_j \cdot k_j^{\text{hemi}}). \tag{68}$$

Since the jet hemispheres overlap, there will be some soft radiation included in both k_1^{hemi} and k_2^{hemi}, for example. To avoid overcounting, let us decompose the soft momenta into six regions, as shown in Fig. 4. So

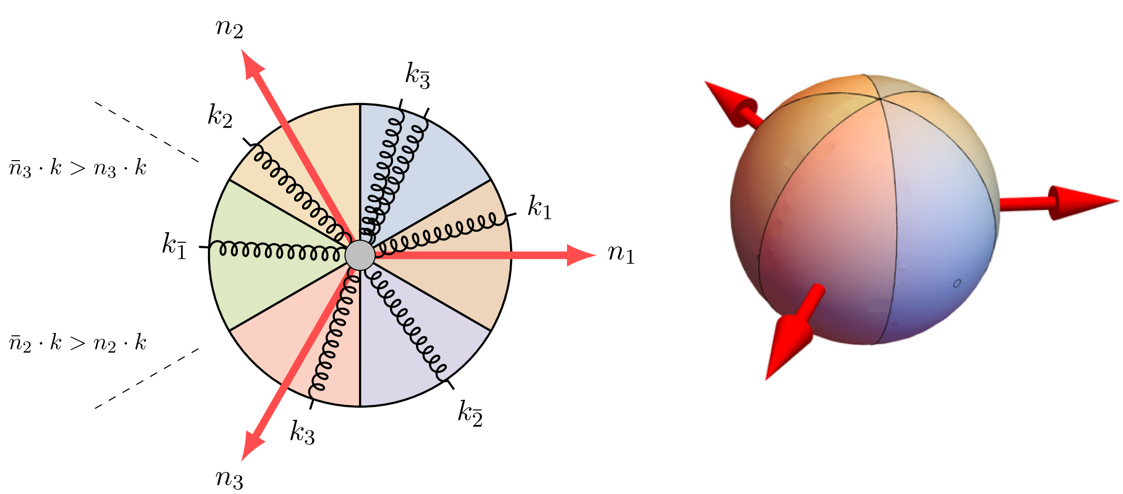


FIG. 4. Soft radiation from the trijet configuration can be categorized as entering one of six sextant wedges shaped like carpels of an orange. The boundary of each sextant is determined by two planes orthogonal to the jet directions n_1, n_2 and n_3. For example, radiation in the sextant labeled $k_{\bar{1}}$ (backward to the 1-jet) is characterized by $\bar{n}_2 \cdot k > n_2 \cdot k$ and $\bar{n}_3 \cdot k > n_3 \cdot k$.

$$k_1^{\text{hemi}} = k_1 + k_{\bar{2}} + k_{\bar{3}} \tag{69}$$

and so on. Here k_j is the soft radiation in the sextant centered on p_j and $k_{\bar{j}}$ is the soft radiation in the sextant opposite to p_j.

Assuming p_1 is the thrust axis, then heavy jet mass

$$\rho = (p_2 + p_3 + k_{\bar{1}} + k_2 + k_3)^2. \tag{70}$$

For $\rho < \frac{1}{3}$ we want to express constraints in terms of $r = \frac{1}{3} - \rho$. For the hard kinematic variable, we can use anything equal to s_{12} at leading power. A convenient choice is

$$\begin{aligned}\xi \equiv s_{12} - \frac{1}{3} + r - 2m_1^2 + 2m_3^2 \\ + 2(p_2 - p_1)\cdot(k_1 + k_{\bar{3}}) + 2k_2\cdot(p_1 + p_2) \\ - 4k_{\bar{2}}\cdot(p_1 - p_3) + 4k_3\cdot p_3 + 4k_{\bar{1}}\cdot p_3.\end{aligned} \tag{71}$$

The variable ξ is defined so that $T_{1k}^{\max} = T_{2k}^{\max}$ at $\xi = 0$. Then phase space where $T_{1k}^{\max} = T_{2k}^{\max}$ and $T_{1k}^{\max} = T_{3k}^{\max}$ is $0 \le \xi \le \frac{1}{3}W$, where

$$\begin{aligned}W(r, m_j, k_i) = r - m_1^2 + m_2^2 + m_3^2 - 2p_1k_1 + 2p_2k_2 \\ + 2p_3k_3 + 2v_{\bar{1}}k_{\bar{1}} - 2v_{\bar{2}}k_{\bar{2}} - 2v_{\bar{3}}k_{\bar{3}}\end{aligned} \tag{72}$$

with

$$v_{\bar{1}} = -\frac{1}{3}p_1 + \frac{2}{3}p_2 + \frac{2}{3}p_3 = \frac{Q}{3}(1, 0, 0, -1), \tag{73}$$

$$v_{\bar{2}} = \frac{4}{3}p_1 + \frac{1}{3}p_2 - \frac{2}{3}p_3 = \frac{Q}{3}\left(1, 0, \frac{\sqrt{3}}{2}, \frac{3}{2}\right), \tag{74}$$

$$v_{\bar{3}} = \frac{4}{3}p_1 - \frac{2}{3}p_2 + \frac{1}{3}p_3 = \frac{Q}{3}\left(1, 0, -\frac{\sqrt{3}}{2}, \frac{3}{2}\right). \tag{75}$$

We have fixed the signs of the $v_{\bar{j}}$ so that they all have positive energy. Since $v_{\bar{1}} = p_{\bar{1}}$, and $k_{\bar{1}}$ is close to $p_{\bar{1}}$, we will have $v_{\bar{1}}\cdot k_{\bar{1}} \ge 0$ for all $k_{\bar{1}}$. For the other directions, $\vec{v}_{\bar{2}}\cdot\vec{p}_2 = 0$ and $\vec{v}_{\bar{3}}\cdot\vec{p}_3 = 0$, and they will also have $v_{\bar{2}}\cdot k_{\bar{2}} \ge 0$ and $v_{\bar{3}}\cdot k_{\bar{3}} \ge 0$.

For the integration range over ξ to be nonzero we therefore need

$$\begin{aligned}m_1^2 + 2p_1k_1 + 2v_{\bar{2}}k_{\bar{2}} + 2v_{\bar{3}}k_{\bar{3}} \\ < r + m_2^2 + 2p_2k_2 + m_3^2 + 2p_3k_3 + 2v_{\bar{1}}k_{\bar{1}},\end{aligned} \tag{76}$$

which is the same as $W(r, m_j, k_i) > 0$, with W in Eq. (72). Every term in this expression is a positive quantity. This inequality applies to both the left and right shoulders for heavy jet mass (for the right shoulder we prefer to use $s = -r = \rho - \frac{1}{3} > 0$).

For thrust, defining $t = \tau - \frac{1}{3} = \frac{2}{3} - T_{1k}$ the bound is $0 \le x \le T$, where

$$\begin{aligned}T(t, m_j, k_i) = m_1^2 + m_2^2 + m_3^2 + 2p_1k_1 + 2p_2k_2 + 2p_3k_3 \\ + 2v_{\bar{1}}k_{\bar{1}} + 2v'_{\bar{2}}k_{\bar{2}} + 2v'_{\bar{3}}k_{\bar{3}} - t,\end{aligned} \tag{77}$$

where

$$v'_{\bar{2}} = 2p_1 - v_{\bar{2}} = \frac{Q}{3}\left(1, 0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{Q}{3}\bar{n}_2, \tag{78}$$

$$v'_{\bar{3}} = 2p_1 - v_{\bar{3}} = \frac{Q}{3}\left(1, 0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{Q}{3}\bar{n}_3 \tag{79}$$

so that

$$\begin{aligned}t < m_1^2 + m_2^2 + m_3^2 + 2p_1k_1 + 2p_2k_2 + 2p_3k_3 + 2v_{\bar{1}}k_{\bar{1}} \\ + 2v'_{\bar{2}}k_{\bar{2}} + 2v'_{\bar{3}}k_{\bar{3}}.\end{aligned} \tag{80}$$

For thrust, as for heavy jet mass, every term in this inequality is positive.

As observed in Sec. II, we can set $m_j = k_i = 0$ to zero in the hard matrix elements at leading power. Then the integral over hard phase space simply gives the maximum value of ξ from Eq. (72) or (77). That is, each channel of the LO integral in Eq. (4) gets modified as

$$48C_F\frac{\alpha_s}{4\pi}\int_0^{\frac{1}{3}-\tau} ds_{12} \to 48C_F\frac{\alpha_s}{4\pi}\int_0^{R/3} d\xi = 48C_F\frac{\alpha_s}{4\pi}R\theta(R) \tag{81}$$

with $\theta(x)$ the Heaviside step function.

The rate for producing collinear radiation is given by splitting functions, and the cross section for producing collinear radiation of mass m is given by the inclusive jet function $J(m^2)$. The rate for soft radiation is given by a soft function, defined as an integral over emissions from Wilson lines using a measurement function (see Sec. III C). The key equation, Eq. (76), lets us then write the factorized expression for the heavy jet mass Sudakov shoulder as

$$\begin{aligned}\frac{1}{\sigma_1}\frac{d\sigma}{dr} = H(Q)\int d^3m^2 d^6q J(m_1^2)J(m_2^2)J(m_3^2)S_6(q_i) \\ \times W(m_j, q_i, r)\theta[W(m_j, q_i, r)],\end{aligned} \tag{82}$$

where

$$\sigma_1 = 48C_F\frac{\alpha_s}{4\pi}\sigma_0. \tag{83}$$

The arguments of the six-parameter soft function $S_6(q_i)$ are the projections $q_i = n_i\cdot k_i$ and $q_{\bar{i}} = v_{\bar{i}}\cdot k_{\bar{i}}$. In terms of the q_i, Eq. (72) becomes

$$W(m_j, q_i, r) = r - m_1^2 + m_2^2 + m_3^2 + \frac{2Q}{3}(q_2 + q_3 + q_{\bar{1}} - q_1 - q_{\bar{3}} - q_{\bar{2}}). \quad (84)$$

We can simplify the factorized expression by defining a two-parameter trijet hemisphere soft function

$$S(q_\ell, q_h) = \int d^6 q_i S_6(q_i) \delta(q_\ell - q_1 - q_{\bar{2}} - q_{\bar{3}}) \times \delta(q_h - q_{\bar{1}} - q_2 - q_3), \quad (85)$$

where q_ℓ and q_h represent the soft radiation in the light and heavy hemispheres, respectively. This soft function contributes to the doubly differential distribution of the hemisphere masses as

$$\frac{d^2\sigma}{dm_\ell^2 dm_h^2} = H(Q,\mu) \int dm_1^2 dm_2^2 dm_3^2 dq_\ell dq_h J(m_1^2,\mu) \times J(m_2^2,\mu) J(m_3^2,\mu) S(q_\ell, q_h, \mu) \times \delta\left(m_\ell^2 - m_1^2 - \frac{2}{3} q_\ell Q\right) \times \delta\left(m_h^2 - m_2^2 - m_3^2 - \frac{2}{3} q_h Q\right). \quad (86)$$

And then

$$\frac{d\sigma}{dr} = \int dm_h^2 dm_\ell^2 \frac{d^2\sigma}{dm_\ell^2 dm_h^2}(r + m_h^2 - m_\ell^2)\Theta(r + m_h^2 - m_\ell^2). \quad (87)$$

One also must sum over channels, corresponding to which jet is the quark jet, which is antiquark and which is gluon.

The factorization formula for thrust is similar:

$$\frac{d\sigma}{dt} = \int dm_h^2 dm_\ell^2 \frac{d^2\sigma}{dm_\ell^2 dm_h^2}(m_h^2 + m_\ell^2 - t)\Theta(m_h^2 + m_\ell^2 - t). \quad (88)$$

The soft function for thrust is the same as for heavy jet mass after changing $v \to v'$. As we will show in the Appendix, changing $v \to v'$ has no effect on the parts of the soft function relevant to NLL resummation, so we will treat the heavy jet mass and thrust trijet hemisphere soft functions as being the same.

C. Soft function

According to the analysis in the previous section, the factorization formula requires a soft function giving the rate for producing gluons k_i entering one of six sextants, as in Fig. 4. In each sextant we need the projection $p_i \cdot k_i$ for $i = 1, 2, 3$ (sextants containing a jet) or $v_i \ldots k_i$ for $i = \bar{1}, \bar{2}, \bar{3}$ (sextants between jets). For NLL resummation, we only need the anomalous dimension of the soft function at one loop. This can be determined by RG invariance. However, as a cross-check on the factorization formula, it is important to compute the soft function explicitly.

It is convenient to introduce the scaleless vectors for the six directions that appear in the measurement function:

$$p_i = \frac{Q}{3} n_i, \qquad v_{\bar{i}} = \frac{Q}{6} N_i, \quad i = 1, 2, 3, \quad (89)$$

where the n_i can be read off from Fig. 1 and the N_i from Eqs. (73)–(75) (or see the Appendix). The vectors p_i, n_i, $v_{\bar{1}}$ and $N_{\bar{1}}$ are lightlike while $v_{\bar{2}}$, $v_{\bar{3}}$, $N_{\bar{2}}$ and $N_{\bar{3}}$ are spacelike. For heavy jet mass, the measurement function $M(k, q_i)$ is

$$M(k,q_i) = \theta(n_{\bar{2}} \cdot k - n_2 \cdot k)\theta(n_{\bar{3}} \cdot k - n_3 \cdot k)\delta\left(q_1 - \frac{2}{3} n_1 \cdot k\right) \quad (90)$$

$$+ \theta(n_{\bar{3}} \cdot k - n_3 \cdot k)\theta(n_{\bar{1}} \cdot k - n_1 \cdot k)\delta\left(q_2 - \frac{2}{3} n_2 \cdot k\right) \quad (91)$$

$$+ \theta(n_{\bar{1}} \cdot k - n_1 \cdot k)\theta(n_{\bar{2}} \cdot k - n_2 \cdot k)\delta\left(q_3 - \frac{2}{3} n_3 \cdot k\right) \quad (92)$$

$$+ \theta(n_2 \cdot k - n_{\bar{2}} \cdot k)\theta(n_3 \cdot k - n_{\bar{3}} \cdot k)\delta\left(q_{\bar{1}} - \frac{2}{3} N_1 \cdot k\right) \quad (93)$$

$$+ \theta(n_3 \cdot k - n_{\bar{3}} \cdot k)\theta(n_1 \cdot k - n_{\bar{1}} \cdot k)\delta\left(q_{\bar{2}} - \frac{2}{3} N_2 \cdot k\right) \quad (94)$$

$$+ \theta(n_1 \cdot k - n_{\bar{1}} \cdot k)\theta(n_2 \cdot k - n_{\bar{2}} \cdot k)\delta\left(q_{\bar{3}} - \frac{2}{3} N_3 \cdot k\right). \quad (95)$$

The matrix element for eikonal emission of one gluon off of three Wilson lines is the same as for direct photon production [20,21] or hard W/Z production [22,23]. There are three Wilson lines in the trijet configuration, pointing in the n_1, n_2 and n_3 directions (see Fig. 1). When the jet in the 1 direction is a gluon, the one-loop soft function is

$$S_{6g}(q_i) = 2 g_s^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2)\theta(k_0) M(k, q_i) \times \left[\left(C_F - \frac{1}{2} C_A\right) \frac{n_2 \cdot n_3}{(n_2 \cdot k)(n_3 \cdot k)} + \frac{1}{2} C_A \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} + \frac{1}{2} C_A \frac{n_1 \cdot n_3}{(n_1 \cdot k)(n_3 \cdot k)}\right]. \quad (96)$$

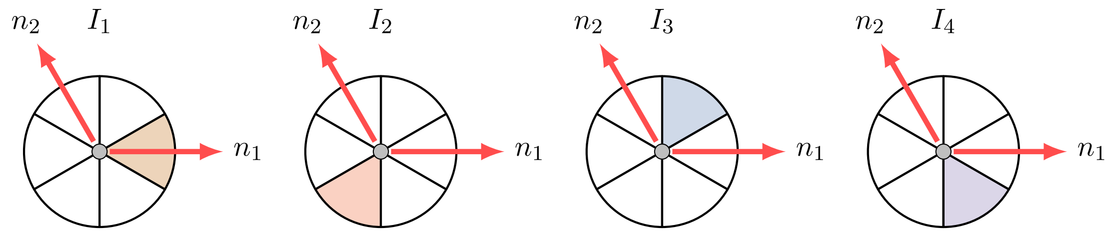


FIG. 5. There are four independent integrals needed for the soft function. The shaded region indicates the measurement region which can be in various positions relative to the Wilson lines.

The soft function with a quark Wilson line in the 1 direction has the color structures interchanged:

$$S_{6q}(q_i) = 2g_s^2\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^{d-1}}\delta(k^2)\theta(k_0)M(k,q_i) \times \left[\left(C_F - \frac{1}{2}C_A\right)\frac{n_1\cdot n_2}{(n_1\cdot k)(n_2\cdot k)} + \frac{1}{2}C_A\frac{n_1\cdot n_3}{(n_1\cdot k)(n_3\cdot k)} + \frac{1}{2}C_A\frac{n_2\cdot n_3}{(n_2\cdot k)(n_3\cdot k)}\right]. \tag{97}$$

Despite the preponderance of directions, the integrals required are all of the same general form. By rotational invariance, we can always take the Wilson lines to be the n_1 and n_2 directions. Then all the required integrals are special cases of the general form

$$I_{n_a,n_b,N}(q) = \int d^dk\frac{n_1\cdot n_2}{(n_1\cdot k)(n_2\cdot k)}\delta(k^2)\theta(k^0) \times \delta\left(q - \frac{2}{3}N\cdot k\right)\theta(n_a\cdot k - \bar{n}_a\cdot k) \times \theta(n_b\cdot k - \bar{n}_b\cdot k). \tag{98}$$

This integral is Lorentz invariant, so it can only depend on dot products of the 4-vectors involved and is also invariant under separate rescaling of all the n_i. Related integrals, with a single θ function, appear in the iterative solution of the Banfi-Marchesini-Smye equation [24] for nonglobal logarithms of the light-jet mass distribution. There, a larger $\mathrm{SL}(2,R)$ symmetry constrains the functional form even more [25]. Here, the $\mathrm{SL}(2,R)$ is broken by the second θ function, so the integration region is a cats-eye-shaped wedge inside the Poincaré disk. However, the conformal coordinates proposed in [25] can still provide a useful change of variables which we used to understand and simplify the integrals.

In the regions without a Wilson line, the anomalous dimension of the soft function is insensitive to the projection vectors N_i; it only depends on the location of the measurement region relative to the Wilson lines. Thus for NLL resummation there are only four independent integrals, as illustrated in Fig. 5. A detailed calculation of the soft integrals can be found in the Appendix. Here we just summarize the results. We find for the four integrals

$$I_1(q) = I_{n_2,n_3,n_1}(q) = \frac{1}{q^{1+2\epsilon}}\left(\frac{1}{\epsilon} - \frac{7}{2}\ln 2 + \ln 3 - \frac{3\kappa}{2\pi}\right), \tag{99}$$

$$I_2(q) = I_{n_1,n_2,n_3}(q) = \frac{1}{q^{1+2\epsilon}}\left(-\ln 2 + \frac{3\kappa}{\pi}\right), \tag{100}$$

$$I_3(q) = I_{\bar{n}_1,\bar{n}_2,N_i}(q) = \frac{1}{q^{1+2\epsilon}}\left(\ln 2 + \frac{3\kappa}{\pi}\right), \tag{101}$$

$$I_4(q) = I_{\bar{n}_1,\bar{n}_3,N_i}(q) = \frac{1}{q^{1+2\epsilon}}\left(\frac{3}{2}\ln 2 - \frac{3\kappa}{2\pi}\right), \tag{102}$$

where

$$\kappa = \mathrm{ImLi}_2 e^{\frac{\pi i}{3}} \approx 1.0149 \tag{103}$$

is Gieseking's constant. Gieseking's constant is a transecendentality-2 number[1] in the family with Catalan's constant $C = \mathrm{ImLi}_2 e^{\frac{\pi i}{2}}$ and $\pi^2 = 6\mathrm{Li}_2(1)$.

Then, when we add in the color structures, the soft function is

$$I_{6g}(q_i) \propto \delta(q_1)\delta(q_2)\delta(q_2)\delta(q_{\bar{1}})\delta(q_{\bar{2}})\delta(q_{\bar{3}}) + \delta(q_2)\delta(q_2)\delta(q_{\bar{1}})\delta(q_{\bar{2}})\delta(q_{\bar{3}})\left[\left(C_F - \frac{1}{2}C_A\right)I_2(q_1) + C_AI_1(q_1)\right] + \delta(q_1)\delta(q_2)\delta(q_3)\delta(q_{\bar{2}})\delta(q_{\bar{3}})\left[\left(C_F - \frac{1}{2}C_A\right)I_3(q_{\bar{1}}) + C_AI_4(q_{\bar{1}})\right] + \cdots \tag{104}$$

and so on for the other four q_i sectors and for $I_{6q}(q_i)$. For the trijet hemisphere soft function in Eq. (85), we can set all the q_i in each hemisphere equal. For the channel with a gluon jet in the light hemisphere we find

[1]It has not been proven whether Gieseking's constant or Catalan's constant are transcendental, or even irrational. In this context, transecendentality-2 refers to the representation of κ as a twofold iterated polylogarithmic integral.

$$
\begin{aligned}
S_g^{\mathrm{hemi}}(q_\ell, q_h, \mu) &\propto \delta(q_\ell)\delta(q_h) \\
&+ \delta(q_\ell)\left[\left(C_F - \frac{1}{2}C_A\right)(I_2(q_h) + 2I_3(q_h)) + C_A(2I_1(q_h) + I_4(q_h))\right] \\
&+ \delta(q_h)\left[\left(C_F - \frac{1}{2}C_A\right)(I_2(q_\ell) + 2I_3(q_\ell)) + C_A(2I_1(q_\ell) + I_4(q_\ell))\right] \\
&= \delta(q_\ell)\delta(q_h) + \frac{\alpha_s(\mu)}{4\pi}\delta(q_\ell)\left[\frac{-4C_F\Gamma_0 \ln\frac{q_h}{\mu} + 2\gamma_{sqq}}{q_h}\right]_\star + \frac{\alpha_s(\mu)}{4\pi}\delta(q_h)\left[\frac{-2C_A\Gamma_0 \ln\frac{q_\ell}{\mu} + 2\gamma_{sg}}{q_\ell}\right]_\star
\end{aligned}
\tag{105}
$$

with $\Gamma_0 = 4$ and

$$
\gamma_{sqq} = -4C_F \ln 6, \qquad \gamma_{sg} = -2C_A \ln 3 + 4C_F \ln 2. \tag{106}
$$

Notation for the $\star$ distributions can be found in [20,22,26–28].

In the channel where the light hemisphere has quark jet, the trijet hemisphere soft function has terms of the form

$$
\begin{aligned}
&S_q^{\mathrm{hemi}}(q_\ell, q_h, \mu) \\
&\propto \delta(q_\ell)\delta(q_h) + \frac{\alpha_s(\mu)}{4\pi}\delta(q_\ell)\left[\frac{-2(C_F + C_A)\Gamma_0 \ln\frac{q_h}{\mu} + 2\gamma_{sqg}}{q_h}\right]_\star \\
&+ \frac{\alpha_s(\mu)}{4\pi}\delta(q_h)\left[\frac{-2C_F\Gamma_0 \ln\frac{q_\ell}{\mu} + 2\gamma_{sq}}{q_\ell}\right]_\star,
\end{aligned}
\tag{107}
$$

where

$$
\begin{aligned}
\gamma_{sqg} &= -2(C_A + C_F)\ln 6, \\
\gamma_{sq} &= -2C_F \ln\frac{3}{2} + 2C_A \ln 2.
\end{aligned}
\tag{108}
$$

D. Resummation

To resum the large Sudakov shoulder logarithms, we convolve the resummed hard, jet and soft function. The resummation of these individual functions is the same as for thrust in the threshold limit [27–29] and other processes [7,20,26,30–33].

The resummed quark and gluon jet functions have the form [20]

$$
\begin{aligned}
J_i(m^2, \mu) &= \exp[-4C_i S(\mu_j, \mu) + 2A_{\gamma_j}(\mu_j, \mu)]\tilde{j}_i(\partial_{\eta_j}) \\
&\times \frac{1}{m^2}\left(\frac{m^2}{\mu_j^2}\right)^{\eta_j} \frac{e^{-\gamma_E \eta_j}}{\Gamma(\eta_j)},
\end{aligned}
\tag{109}
$$

where the Laplace transform of the one-loop jet functions is

$$
\tilde{j}_i(L) = 1 + \left(\frac{\alpha_s(\mu_j)}{4\pi}\right)\left[C_i\Gamma_0 \frac{L^2}{2} + \gamma_i L\right] \tag{110}
$$

and the Casimirs and one-loop anomalous dimensions are

$$
C_q = C_F, \quad C_g = C_A, \quad \gamma_{jq} = -3C_F, \quad \gamma_{jg} = -\beta_0. \tag{111}
$$

The Sudakov RG kernel is

$$
\begin{aligned}
S(\nu, \mu) &= -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{\mathrm{cusp}}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \\
&= -\frac{\alpha_s}{8\pi}\Gamma_0 \ln^2\frac{\nu}{\mu} + \cdots
\end{aligned}
\tag{112}
$$

with

$$
\gamma_{\mathrm{cusp}}(\alpha_s) = \left(\frac{\alpha_s}{4\pi}\right)\Gamma_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma_1 + \cdots, \tag{113}
$$

$$
\beta(\alpha_s) = -2\alpha_s\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_1 + \cdots\right], \tag{114}
$$

where

$$
\Gamma_0 = 4, \qquad \Gamma_1 = 4\left[C_A\left(\frac{67}{9} - \frac{\pi^2}{3}\right) - \frac{20}{9}T_F n_f\right], \tag{115}
$$

$$
\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_f, \\
\beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_f - 4C_F T_F n_f.
\end{aligned}
\tag{116}
$$

To NLL order

$$
A_{\gamma_j}(\nu, \mu) = -\gamma_j \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\alpha}{4\pi\beta(\alpha)} = \frac{\gamma_j}{2\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\nu)}. \tag{117}
$$

Finally,

$$
\eta_{jq} = 2C_F A_\Gamma(\mu_j, \mu), \qquad \eta_{jg} = 2C_A A_\Gamma(\mu_j, \mu), \tag{118}
$$

where

$$
A_\Gamma(\nu, \mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{\mathrm{cusp}}(\alpha)}{\beta(\alpha)} = \frac{\alpha_s}{4\pi}\Gamma_0 \ln\frac{\nu}{\mu} + \cdots. \tag{119}
$$

The hard function can be extracted from [34] or using the general forms for hard functions in [35] or from the hard function for n-jettiness [32]. It is

$$H(Q,\mu) = \exp[(4C_F + 2C_A)S(\mu_h,\mu) - 2A_{\gamma h}(\mu_h,\mu)] \times \left(\frac{Q^2}{\mu_h^2}\right)^{-(2C_F+C_A)A_\Gamma(\mu_h,\mu)} H(Q,\mu_h), \quad (120)$$

where

$$H(Q,\mu_h) = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[-(2C_F+C_A)\frac{\Gamma_0}{4}\ln^2\frac{Q^2}{\mu_h^2} - \gamma_h \ln\frac{Q^2}{\mu_h^2}\right], \quad (121)$$

with

$$\gamma_h = -2(2C_F + C_A)\ln 3 - 6C_F - \beta_0. \quad (122)$$

The trijet hemisphere soft functions can be resummed in exactly the same manner as the hemisphere soft function [7,27–29,36]. At NLL level they factorize into the product of soft functions for each hemisphere:

$$S_g^{\text{hemi}}(k_\ell, k_h, \mu) = S_g(k_\ell,\mu) S_{qq}(k_h,\mu), \quad (123)$$

$$S_q^{\text{hemi}}(k_\ell, k_h, \mu) = S_q(k_\ell,\mu) S_{qg}(k_h,\mu). \quad (124)$$

The single-variable soft functions all have the same form:

$$S_i(k,\mu) = \exp[2C_i S(\mu_s,\mu) + 2A_{\gamma_i}(\mu_s,\mu)] \times \tilde{s}_i(\partial_{\eta_i}) \frac{1}{k}\left(\frac{k}{\mu_s}\right)^{\eta_i} \frac{e^{-\gamma_E \eta_i}}{\Gamma(\eta_i)}, \quad (125)$$

where

$$\tilde{s}_i(L) = 1 + \left(\frac{\alpha_s(\mu_s)}{4\pi}\right)\left[-2C_i\Gamma_0 \frac{L^2}{2} + 2\gamma_i L\right] \quad (126)$$

and

$$\eta_i = -2C_i A_\Gamma(\mu_s,\mu). \quad (127)$$

The only difference is the anomalous dimensions. The coefficient of the Sudakov logs are determined by Casimir scaling as the sum of the color factors for each parton in the hemisphere:

$$C_g = C_A, \quad C_{qq} = 2C_F, \quad C_q = C_F, \quad \text{and} \quad C_{qg} = C_F + C_A. \quad (128)$$

The anomalous dimensions γ_{sg}, γ_{sq}, γ_{sqq} and γ_{sqg} are in Eqs. (106) and (108).

Now we just have to put everything together and perform the integrals in Eqs. (86)–(88). Since the various functions after resummation are simply powers, e.g., $J(m^2) \sim (m^2)^{\eta_j - 1}$, the integrals are all products or convolutions of powers, which can be done directly or through Laplace transforms.

For thrust, with $t = \tau - \frac{1}{3} > 0$, the core measurement function integral following from Eq. (80) is

$$\int_0^\infty dx \int_0^\infty dy x^{a-1} y^{b-1} (x+y-t)\theta(x+y-t) = t^{1+a+b} \frac{\Gamma(a)\Gamma(b)}{\Gamma(2+a+b)}. \quad (129)$$

For the left shoulder of heavy jet mass, the integral is similar, but the sign flip in Eq. (76) as compared to Eq. (80) gives an important change:

$$\int_0^\infty dx \int_0^\infty dy x^{a-1} y^{b-1} (r+y-x)\theta(r+y-x) = r^{1+a+b} \frac{\Gamma(a)\Gamma(b)}{\Gamma(2+a+b)} \frac{\sin(\pi a)}{\sin(\pi(a+b))}. \quad (130)$$

For the right shoulder of heavy jet mass we define $s = -r = \rho - \frac{1}{3} > 0$. Then the core integral is

$$\int_0^\infty dx \int_0^\infty dy x^{a-1} y^{b-1} (y-x-s)\theta(y-x-s) = s^{1+a+b} \frac{\Gamma(a)\Gamma(b)}{\Gamma(2+a+b)} \frac{\sin(\pi b)}{\sin(\pi(a+b))}. \quad (131)$$

These integrals are all UV and IR divergent, and so analytic continuation has been used to complete them. We discuss the integrals in more detail in Sec. IV B.

Putting everything together and applying algebraic simplifications as in [20,28], we find that all three observables can be written in terms of the same RG evolution kernel. For the gluon channels

$$\frac{1}{\sigma_1}\frac{d\sigma_g}{dt} = \Pi_g(\partial_{\eta_\ell}, \partial_{\eta_h}) t \left(\frac{tQ}{\mu_s}\right)^{\eta_\ell} \left(\frac{tQ}{\mu_s}\right)^{\eta_h} \frac{e^{-\gamma_E(\eta_\ell+\eta_h)}}{\Gamma(2+\eta_\ell+\eta_h)}, \quad (132)$$

$$\frac{1}{\sigma_1}\frac{d\sigma_g}{dr} = \Pi_g(\partial_{\eta_\ell}, \partial_{\eta_h}) r \left(\frac{rQ}{\mu_s}\right)^{\eta_\ell} \left(\frac{rQ}{\mu_s}\right)^{\eta_h} \frac{e^{-\gamma_E(\eta_\ell+\eta_h)}}{\Gamma(2+\eta_\ell+\eta_h)} \times \frac{\sin(\pi\eta_\ell)}{\sin(\pi(\eta_\ell+\eta_h))}, \quad (133)$$

$$\frac{1}{\sigma_1}\frac{d\sigma_g}{ds} = \Pi_g(\partial_{\eta_\ell}, \partial_{\eta_h}) s \left(\frac{sQ}{\mu_s}\right)^{\eta_\ell} \left(\frac{sQ}{\mu_s}\right)^{\eta_h} \frac{e^{-\gamma_E(\eta_\ell+\eta_h)}}{\Gamma(2+\eta_\ell+\eta_h)} \times \frac{\sin(\pi\eta_h)}{\sin(\pi(\eta_\ell+\eta_h))}, \quad (134)$$

where

$$\Pi_g(\partial_{\eta_\ell},\partial_{\eta_h}) = \exp[4C_F S(\mu_h,\mu_j) + 4C_F S(\mu_s,\mu_j) + 2C_A S(\mu_h,\mu_j) + 2C_A S(\mu_s,\mu_j)] \times \exp[2A_{\gamma sg}(\mu_s,\mu_h) + 2A_{\gamma sqq}(\mu_s,\mu_h) + 2A_{\gamma jg}(\mu_j,\mu_h) + 4A_{\gamma jq}(\mu_j,\mu_h)] \times H(Q,\mu_h)\tilde{j}_q\left(\partial_{\eta_h} + \ln\frac{Q\mu_s}{\mu_j^2}\right)\tilde{j}_{\bar{q}}\left(\partial_{\eta_h} + \ln\frac{Q\mu_{sh}}{\mu_j^2}\right)\tilde{j}_g\left(\partial_{\eta_\ell} + \ln\frac{Q\mu_{s\ell}}{\mu_j^2}\right)\tilde{s}_{qq}(\partial_{\eta_h})\tilde{s}_g(\partial_{\eta_\ell}) \tag{135}$$

and

$$\eta_\ell = \eta_{jg} + \eta_{sg} = 2C_A A_\Gamma(\mu_j,\mu_s), \tag{136}$$

$$\eta_h = 2\eta_{j_q} + \eta_{sqq} = 4C_F A_\Gamma(\mu_j,\mu_s). \tag{137}$$

We have chosen the same jet scales for the light and heavy hemispheres although one could also choose them to be different. Similarly, we have taken the same soft scales for the left and right hemispheres.

One can read off from Eq. (133) that the large logs will be resummed for the left shoulder of heavy jet mass with the canonical scale choices

$$\mu_h = Q, \qquad \mu_j = \sqrt{r}Q, \qquad \mu_s = rQ. \tag{138}$$

For thrust or the right shoulder of heavy jet mass, the canonical scale choices are the same with r replaced by t or s, respectively. We have verified that the expansion of the resummed distribution is independent of the matching scales μ_h, μ_j and μ_s at order α_s.

The quark channels have the same form as Eqs. (132)–(134) but with

$$\Pi_q(\partial_{\eta_\ell},\partial_{\eta_h}) = \exp[(2C_F + C_A)[S(\mu_h,\mu_j) + S(\mu_s,\mu_j)] + 2C_F[S(\mu_h,\mu_j) + S(\mu_s,\mu_j)]] \times \exp[2A_{\gamma sq}(\mu_s,\mu_h) + 2A_{\gamma sqg}(\mu_s,\mu_h) + 2A_{\gamma jq}(\mu_j,\mu_h) + 2A_{\gamma jq}(\mu_j,\mu_h) + 2A_{\gamma jq}(\mu_j,\mu_h)] \times H(Q,\mu_h)\tilde{j}_q\left(\partial_{\eta_h} + \ln\frac{Q\mu_s}{\mu_j^2}\right)\tilde{j}_g\left(\partial_{\eta_h} + \ln\frac{Q\mu_s}{\mu_j^2}\right)\tilde{j}_q\left(\partial_{\eta_\ell} + \ln\frac{Q\mu_s}{\mu_j^2}\right)\tilde{s}_{qg}(\partial_{\eta_h})\tilde{s}_q(\partial_{\eta_\ell}) \tag{139}$$

and

$$\eta_\ell = \eta_{jq} + \eta_{sq} = 2C_F A_\Gamma(\mu_j,\mu_s), \tag{140}$$

$$\eta_h = \eta_{jq} + \eta_{jg} + \eta_{sqg} = (2C_F + 2C_A)A_\Gamma(\mu_j,\mu_s). \tag{141}$$

The final resummed distribution for thrust is

$$\frac{d\sigma}{dt} = \frac{d\sigma_g}{dt} + 2\frac{d\sigma_q}{dt} \tag{142}$$

and similarly for heavy jet mass.

IV. ANALYSIS

In Sec. III we derived a factorization formula for the left and right Sudakov shoulders for heavy jet mass as well as the right Sudakov shoulder for thrust (thrust has no left shoulder). We will now perform some cross-checks on those results. We first perform the fixed-order expansion and compare to a numerical computation of the exact NLO expression to verify the singular behavior. Then we demonstrate that there are no nonglobal logarithms and discuss power corrections.

A. Fixed-order expansions

First of all, we observe that the full resummed distributions are renormalization-group invariant. This invariance has let us write the evolution kernels in Eqs. (135) and (139) in a form that depends only on the hard, jet and soft matching scales μ_i, and not on μ. The cancellation of the μ dependence is nontrivial and requires the Casimirs associated with the Sudakov double logs to cancel and the anomalous dimensions to satisfy

$$\gamma_h = \gamma_{jg} + 2\gamma_{jq} + \gamma_{sqq} + \gamma_{sg} = \gamma_{jg} + 2\gamma_{jq} + \gamma_{sqg} + \gamma_{sq}. \tag{143}$$

These relations can be checked explicitly using Eqs. (122), (111), (106), and (108).

Expanding the resummed distributions to order α_s we find

$$\frac{1}{\sigma_1}\frac{d\sigma}{dt} = 3t + \frac{\alpha_s}{4\pi}\Big\{B_1 t - \frac{3}{2}(2C_F + C_A)\Gamma_0 t\ln^2 t + [3\gamma_{jg} + 6\gamma_{jq} + 2\gamma_{sg} + 4\gamma_{sq} + 2\gamma_{sqq} + 4\gamma_{sqg} + 3(C_A + 2C_F)\Gamma_0]t\ln t\Big\} \tag{144}$$

for some B_1. The linear terms $3t$ and $B_1 t$ are not predicted with NLL resummation. So to be consistent we should remove all the terms linear in t. This can be done to all orders by subtracting from the full resummed distribution $\sigma(t)$ the boundary condition $t\sigma(1)$. That is, we consider

$$\frac{1}{\sigma_1}\frac{d\sigma^{\rm sub}}{dt} \equiv \frac{1}{\sigma_1}\frac{d\sigma}{dt} - t\left[\frac{1}{\sigma_1}\frac{d\sigma}{dt}\right]_{t=1}, \tag{145}$$

which has only terms of the form $t\ln^n t$ to all orders in α_s. We use an analogous definition with t replaced by r or s for the subtracted form of the heavy jet mass distribution. Plugging in the anomalous dimensions

$$\begin{aligned}\frac{1}{\sigma_1}\frac{d\sigma^{\rm sub}}{dt} = \frac{\alpha_s}{4\pi}\{&-6(2C_F+C_A)t\ln^2 t \\ &+[6C_F(1-4\ln 3)+C_A(1-12\ln 3) \\ &+4n_f T_F]t\ln t\} + \mathcal{O}(\alpha_s^2).\end{aligned} \tag{146}$$

This is shown in comparison to the NLO calculation in Fig. 6.

For the left shoulder of heavy jet mass, the expansion gives

$$\begin{aligned}\frac{1}{\sigma_1}\frac{d\sigma^{\rm sub}}{dr} &= \frac{\alpha_s}{4\pi}\left\{-\frac{1}{2}(2C_F+C_A)\Gamma_0 r\ln^2 r + [(C_A+2C_F)\Gamma_0 \right. \\ &\quad \left. +\gamma_{jg}+2\gamma_{jq}+2\gamma_{sg}+4\gamma_{sq}]r\ln r\right\} \\ &= \frac{\alpha_s}{4\pi}\left\{-2(2C_F+C_A)r\ln^2 r + \left[2C_F\left(1+4\ln\frac{4}{3}\right)\right.\right. \\ &\quad \left.\left. +C_A\left(\frac{1}{3}+4\ln\frac{4}{3}\right)+\frac{4}{3}n_f T_F\right]r\ln r\right\}.\end{aligned} \tag{147}$$

This agrees with our fixed-order computation in Sec. II and with the leading shoulder logarithms at NLO as can be seen in Fig. 6.

Breaking down the expression in Eq. (147) the anomalous dimensions which appear are $\gamma_{jg}+2\gamma_{sg}$ from the gluon channel and $\gamma_{jq}+2\gamma_{sq}$ from the quark and antiquark channels. So in each channel only anomalous dimensions associated with light-hemisphere side are contributing logarithms as order α_s. This is a somewhat remarkable feature of the factorization formula: although both sides contribute one-loop anomalous dimensions, as is required for renormalization-group invariance, Eq. (143) only one side contributes logarithms. Mechanically, what happens is

$$(\gamma_h\partial_{\eta_h}+\gamma_\ell\partial_{\eta_\ell})[r^{\eta_\ell+\eta_h}-1]\frac{\sin(\pi\eta_\ell)}{\sin(\pi(\eta_\ell+\eta_h))} = \gamma_\ell r\ln r. \tag{148}$$

So the $\frac{\sin(\pi\eta_\ell)}{\sin(\pi(\eta_\ell+\eta_h))}$ factor replaces the full anomalous dimension $\gamma_\ell+\gamma_h$ with just γ_ℓ.

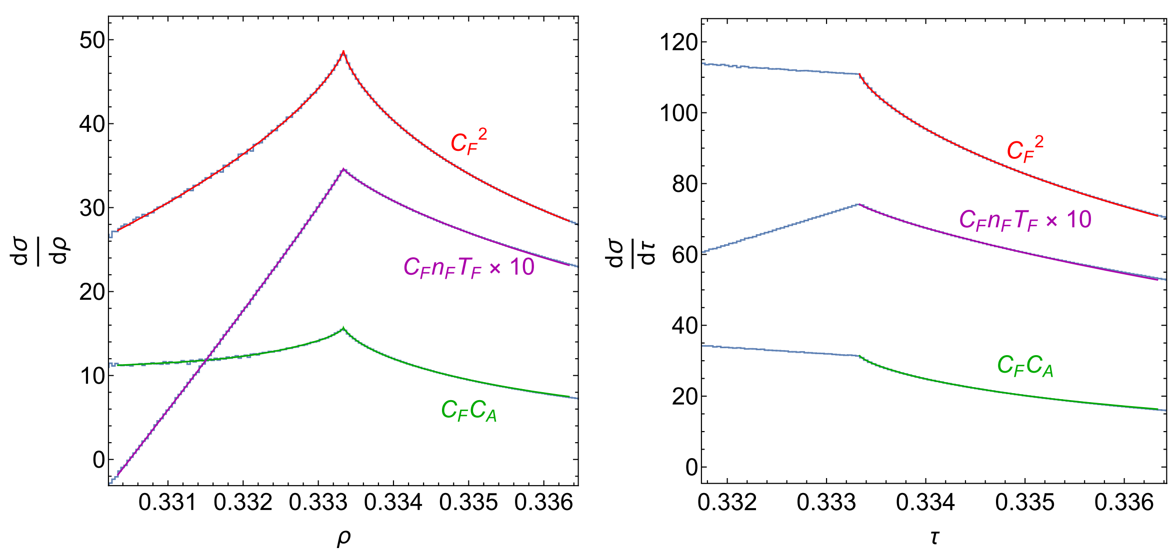


FIG. 6. Comparison of the resummed distribution expanded to NLO (colored curves) to the exact NLO distribution in the Sudakov shoulder region (blue histograms) for heavy jet mass (left) and thrust (right). We include in the prediction an offset and a linear term which are fit separately on either side of the peak. The $C_F n_f T_F$ color structure has been scaled up by a factor of 10 for clarity. The NLO histograms is this figure were computed using EVENT2 [9,10] with a cutoff of 10^{-12} and 12 trillion events and normalized to Born cross section σ_0.

For the right shoulder of heavy jet mass

$$\frac{1}{\sigma_1}\frac{d\sigma^{\rm sub}}{ds} = \frac{\alpha_s}{4\pi}\{-(2C_F + C_A)\Gamma_0 s\ln^2 s + [2(C_A + 2C_F)\Gamma_0 + 2\gamma_{jg} + 4\gamma_{jq} + 2\gamma_{sqq} + 4\gamma_{sqg}]s\ln s\}$$
$$= \frac{\alpha_s}{4\pi}\left\{-4(2C_F + C_A)s\ln^2 s + \left[4C_F(1 - 4\ln 6) + \frac{2C_A}{3}(1 - 12\ln 6) + \frac{8}{3}n_f T_F\right]s\ln s\right\}. \quad (149)$$

In this case, only the anomalous dimensions in the *heavy* hemisphere contribute at NLO. This distribution is also shown in Fig. 6 and compared to the exact NLO calculation.

B. Nonglobal logarithms

When observables are sensitive to emissions only in a restricted region of phase space, there can be an incomplete cancellation of virtual and real emissions leading to nonglobal logarithms [37]. The classic example is the light-hemisphere mass in e^+e^- collisions. For the light-hemisphere mass, emissions into the heavy hemisphere do not affect the value of the light-hemisphere mass, making it nonglobal. Generally, to be able to resum logarithms using a factorization formula, one would like the condition that the observable be small to force soft and collinear kinematics. This does not happen with the light-hemisphere mass, for example, since demanding it be small does not prevent additional hard emissions into the heavy hemisphere. For light-jet mass, the leading nonglobal logarithm contributes to the cross section at order $d\sigma/d\rho_\ell \sim \alpha_s^2 \ln^2\rho_\ell$ so it is the same order as terms in NLL resummation. The leading logarithmic series of nonglobal logs for the light jet mass and related observables is understood and can be resummed [24,25]. Progress has also been made on systematic higher-order resummation of nonglobal logarithms [38–42]. Thus, if there were nonglobal logs in the Sudakov shoulders, it would not pose an insurmountable obstacle. Nevertheless, we will show that for the Sudakov shoulders of thrust and heavy jet mass, nonglobal logs are absent.

For the right shoulder of thrust, the constraint in Eq. (80) is of the form $t < x + y$, where x and y represent contributions to the mass of the heavy or light hemispheres, respectively, from soft and collinear radiation near the trijet region (i.e., $x = m_2^2 + m_3^2 + 2p_2k_2 + 2p_3k_3 + 2v_{\bar{1}}k_{\bar{1}}$ and $y = m_1^2 + 2p_1k_1 + v'_{\bar{2}}k_{\bar{2}} + v'_{\bar{3}}k_{\bar{3}}$ when the light jet is in the 1 direction). Since the constraint imposes a *lower* bound on t, demanding $t \ll 1$ does not force x and y to be small, suggesting that the Sudakov shoulder for thrust might be nonglobal. However, we can rewrite the core convolution integral in Eq. (129) as

$$\int_0^\infty dx \int_0^\infty dy x^{a-1}y^{b-1}(x + y - t)\Theta(x + y - t) \quad (150)$$

$$= \int_0^\infty dz\left[\int_0^\infty dx\int_0^\infty dy x^{a-1}y^{b-1}\delta(x + y - z)\right] \times (z - t)\Theta(z - t) \quad (151)$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left[\underbrace{\int_0^\infty dz z^{a+b-1}(z - t)}_{\text{scaleless}} - \underbrace{\int_0^t dz z^{a+b-1}(z - t)}_{\text{global}}\right] \quad (152)$$

$$= t^{1+a+b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(2 + a + b)}. \quad (153)$$

The first integral in brackets in Eq. (152) is divergent but either independent of t or linear in t, so it is smooth across $t = 0$ and does not generate Sudakov shoulder logarithms. The remaining integral has $z = x + y < t$, so taking $t \ll 1$ does force $x, y \ll 1$. We conclude that the right shoulder of thrust should be free of nonglobal logarithms. The actual divergence is an artifact of expanding the phase space limits to leading power. In the full theory, the divergences would cut off by the hard scale Q but still would not generate logarithms of t.

It is also worth noting that the scaleless integral in Eq. (152) does generate a divergent term proportional to t. This would be the same order as terms in the NNLL resummation of the Sudakov shoulder. The presence of such a term does not imply that the factorization formula is valid only to NLL. Indeed, this divergent contribution is smooth across $t = 0$, suggesting that it contributes similarly to the left and right sides of $\tau = \frac{1}{3}$ and therefore does not give a discontinuity or a kink at $\tau = \frac{1}{3}$. In any case, since we are only working to NLL in this paper, we can safely ignore it.

For heavy jet mass, the analogous constraint is in Eq. (76) which corresponds to $x < r + y$ for the left shoulder or $x + s < y$ for the right shoulder, as in Eqs. (130) and (131). We can rewrite Eq. (130) as

$$f(r) = \int_0^\infty dx\int_0^\infty dy x^{a-1}y^{b-1}(r + y - x)\theta(r + y - x)$$
$$= \frac{1}{a(a+1)}\int_0^\infty dy(r + y)^{a+1}y^{b-1}. \quad (154)$$

This integral is both UV and IR divergent (for $a, b > 0$) and gets contributions from all scales, suggesting, again, that it may generate nonglobal logarithms. To separate out the UV and IR divergences, we can take two derivatives with respect to r, leaving an integral which is UV finite for a, $b > 0$. We also introduce a new scale R to separate small r from large r. Then we have

$$f''(r) = \underbrace{\int_0^R dy (r+y)^{a-1} y^{b-1}}_{\text{global}} + \underbrace{\int_R^\infty dy (r+y)^{a-1} y^{b-1}}_{\text{regular in } r}. \tag{155}$$

Since we are interested in the region with $r \ll 1$, we can take $0 < r < R \ll 1$. Then the first integral in Eq. (155) is global, since it gets contributions only from the region where $y \ll 1$ and $x \ll 1$ [we had integrated x from 0 to $r + y \ll 1$ in Eq. (154)]. The second integral in Eq. (155) is regular as $r \to 0$. Thus it does not contribute to any discontinuities or kinks near the shoulder, at $r = 0$. As with thrust, it may contribute terms linear in r but will not give any Sudakov shoulder logs. So only the soft and collinear regions should contribute to the Sudakov shoulder logs for heavy jet mass, as with thrust, and there are no nonglobal logarithms.

To complete the computation, as far as the Sudakov shoulder logs are concerned, we have

$$f''(r) \cong \int_0^R dy (r+y)^{a-1} y^{b-1} \tag{156}$$

$$= \int_0^\infty dy (r+y)^{a-1} y^{b-1} - \underbrace{\int_R^\infty dy (r+y)^{a-1} y^{b-1}}_{\text{regular in } r} \tag{157}$$

$$\cong \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\sin(\pi a)}{\sin(\pi(a+b))} r^{a+b-1} - R^{1-a-b} \frac{1}{a+b-1}, \tag{158}$$

where we have taken $R \gg r$ to simplify the second integral. Integrating twice with respect to r then gives

$$f(r) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(2+a+b)} \frac{\sin(\pi a)}{\sin(\pi(a+b))} r^{1+a+b} - \frac{r^2}{2} R^{a+b-1} \frac{1}{1-a-b} + c_1 r + c_0. \tag{159}$$

At small a and b (these are proportional to α_s), the second term on the right-hand side is suppressed by a factor of $\frac{r}{R}$ compared to the first term, so it only gives power corrections and no Sudakov shoulder logs, as anticipated. We should fix the integration constants c_1 and c_2 so that the expansion of $f(r)$ at small a and b only has terms of the form $r \ln^n r$ with $n > 0$. The constant term we can simply discard, $c_0 = 0$. To fix c_1 we should set $c_1 = -f(1)$. This corresponds to integrating $f'(r)$ from 1 to r. These integration constants were used in Eq. (145).

In summary, the heavy jet mass distribution at a value of $\rho \approx \frac{1}{3}$ does get contributions from phase space regions with jets whose masses are not small. In this sense it is similar to light jet mass near $\rho_\ell = 0$ which gets contributions from phase space regions where ρ is not small. However, the contributions corresponding to heavy jets for the Sudakov shoulder do not generate large logarithms. This is because the phase space regions with heavy jets can contribute to both $\rho \lesssim \frac{1}{3}$ and $\rho \gtrsim \frac{1}{3}$ and are smooth across $\rho = \frac{1}{3}$. All the contributions to the distribution that are not smooth across $\rho = \frac{1}{3}$ come from the regions with one nearly massless jet in the light hemisphere and two nearly massless jets in the heavy hemisphere. There is no analog of this continuity argument for light jet mass, which cannot have $\rho_\ell < 0$. Thus, the Sudakov shoulders of heavy jet mass (and thrust) are free of nonglobal logarithms.

C. Power corrections

In resummed distributions, there are typically different types of power corrections. For threshold resummation, near $\rho = 0$ for example, there can be power corrections of order $\frac{\Lambda_{\rm QCD}}{Q}$ associated with the strong dynamics of QCD. There can also be hard power corrections, suppressed by additional powers of $\rho = \frac{m_H^2}{Q^2}$, where m_H is the mass of the heavy jet. The $\Lambda_{\rm QCD}$ power corrections are often modeled with parameters fit to data. This allows for predictivity closer to threshold than with just the resummed distribution alone, although one cannot get too close to threshold since more and more nonperturbative parameters then become relevant. The hard power corrections are typically accounted for in matching to an exact fixed-order expression at large ρ.

For Sudakov shoulder resummation, it is not clear whether $\frac{\Lambda_{\rm QCD}}{Q}$ power corrections are important near the trijet threshold. On the one hand, the resummed distribution involves evaluating α_s at scales such as $\mu_s = Qr$ which can reach $\Lambda_{\rm QCD}$ for small enough r. On the other hand, the shoulder is intrinsically perturbative, associated with fixed-order phase space boundaries, so one might expect that it might be invisible to nonperturbative physics.

The hard power corrections for the Sudakov shoulder are more interesting. In Sec. IV B we argued that at leading power all the nonanalytic behavior near the shoulder is determined by soft and collinear physics. That is, there are no nonglobal logarithms. One can see this from Eq. (159). The quantities a and b are to be replaced by η_ℓ and η_h in the resummed distribution, which are parametrically of the form $\eta \sim \alpha_s \Gamma_0 \ln r$. Thus at small α_s, all the terms of the form $r \ln^n r$ will come from the expansion of the first term on the right-hand side in Eq. (159). On the other hand, if r is sufficiently small, then $a + b$ can be of order 1. As $a + b$ nears 1 a pole from the $\sin^{-1}(\pi(a+b))$ factor in Eq. (159) is approached. However, when $a + b \approx 1$, the

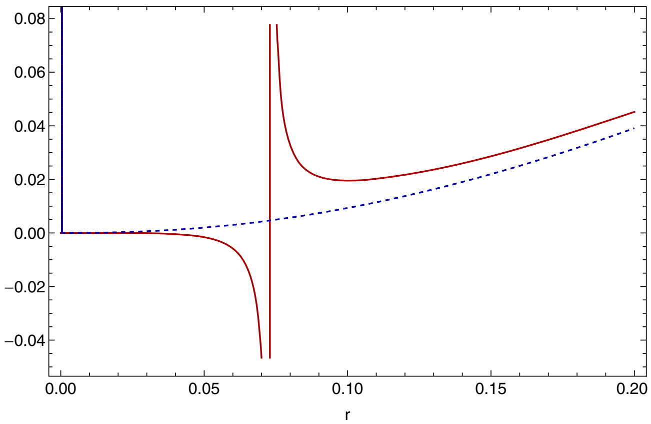


FIG. 7. The solid red curve shows $r^{1+a+b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(2+a+b)}\frac{\sin(\pi a)}{\sin(\pi(a+b))}$ with $a=b=-6\frac{\alpha_s}{\pi}\ln r$ and $\alpha_s=0.1$. The pole is at $a+b=1$. The dashed curve shows this same function once the power-suppressed term $-\frac{r^2}{2}R^{a+b-1}\frac{1}{1-a-b}$ is added with $R=1$. The power suppression is apparent at small r, where the difference between the two curves is negligible. The pole at $a+b=2$ is not canceled and appears as the spike at $r\approx 0.0003$.

power-suppressed term is no longer power suppressed. Indeed, it has precisely the behavior needed to remove the singular behavior from the leading power term. We show this in Fig. 7.

To see what is happening analytically, noting that the leading power expression scales like r^{1+a+b} we can write the subleading power expression as

$$r^2R^{a+b-1}=r^{1+a+b}\left(\frac{r}{R}\right)^{1-a-b}. \quad (160)$$

For $a+b\ll 1$ there is a $\frac{r}{R}$ linear power suppression. However for $a+b\sim 1$ there is no power suppression at all; for $a+b=1$ this expressions reduces to r^2 as does r^{1+a+b}. In effect, the scaling dimensions of the leading power and subleading power pick up such large anomalous contributions that their relative scaling changes.

Taking $R\to\infty$ gives the leading contribution. The first subleading power contribution in this limit cancels the pole at $a+b=1$. To cancel subsequent poles, one can use the exact integrated form of Eq. (155). This effectively replaces $f(r)$ in Eq. (159) by

$$f(r)=r^{1+a+b}e^{-ib\pi}\frac{1}{a(a+1)}B_{\frac{R}{r}}(b,2+a)+c_1r+c_0, \quad (161)$$

where $B_z(x,y)$ is the incomplete Euler β function and c_0 and c_1 are again integration constants to be fixed with physical boundary conditions.

V. DISCUSSION

Next, we want to evaluate the resummed distribution numerically and compare to fixed order, to see the effect of the higher-order logarithms. There are a number of issues which complicate the analysis, compared with typical threshold resummation of large logarithms.

First, the relevant domain of the observable is rather small for Sudakov shoulders. For example, for thrust in the threshold limit, although the logarithms are largest at small τ, power corrections and subleading logs are also large there. Typical fits restrict $\tau\gtrsim 0.1$ where perturbative control is best. For example, with $Q=92$ GeV, Ref. [28] used $0.1<\tau<0.24$ for their α_s fits to thrust while Ref. [43] took $\tau\gtrsim\frac{6\ \mathrm{GeV}}{Q}=0.066$. For the right shoulder of thrust which begins at the three-parton maximum $\tau=\frac{1}{3}$ if one excludes the region up to $\frac{1}{3}+0.1=0.43$ there is no cross section or phase space left. Moreover, the four-particle phase space forces $\tau\lesssim 0.42$, so there is another Sudakov shoulder at this thrust value whose logs must be resummed separately. So it is not clear if there is a region on the right shoulder where the resummed formula might even be valid. For the left shoulder, in contrast, one can exclude the region with $r=\frac{1}{3}-\rho<0.1$ which still leaves a region of $0.1\lesssim\rho\lesssim 0.23$ in which Sudakov shoulder logarithms might be important and renormalization-group improved perturbation theory could be valid.

Second, in the threshold region, the logarithms of thrust are of the form $\frac{d\sigma}{d\tau}\sim\alpha_s^n\frac{\ln^m\tau}{\tau}$. In contrast, the logarithms near the shoulder region are of the form $\frac{d\sigma}{dt}\sim\alpha_s^n t\ln^m t$. So they are suppressed effectively by t^2 compared to the threshold region. The thrust and heavy mass distributions are indeed finite at the trijet threshold to all orders while they are divergent at the dijet threshold. Despite this additional suppression, the logarithms are noticeable, as can be been in Fig. 2.

Third, in the important left-shoulder region for heavy jet mass, the resummed distribution has usually singular behavior. Let us recall the form of the resummed heavy jet mass distribution in the region $r=\frac{1}{3}-\rho\ll 1$ when the light hemisphere has a gluon jet from Eq. (133):

$$\frac{1}{\sigma_1}\frac{d\sigma_g}{dr}=\Pi_g(\partial_{\eta_\ell},\partial_{\eta_h})r\left(\frac{rQ}{\mu_{s\ell}}\right)^{\eta_\ell}\left(\frac{rQ}{\mu_{sh}}\right)^{\eta_h}\frac{e^{-\gamma_E(\eta_\ell+\eta_h)}}{\Gamma(2+\eta_\ell+\eta_h)}\times\frac{\sin(\pi\eta_\ell)}{\sin(\pi(\eta_\ell+\eta_h))} \quad (162)$$

with η_h and η_ℓ in Eqs. (141) and (140). This expression has singularities whenever $\eta_\ell+\eta_h\in\mathbb{Z}$.

Choosing canonical scales as in Eq. (138) at leading logarithmic level gives

$$\eta_\ell+\eta_h=-\frac{\alpha_s}{2\pi}(C_A+2C_F)\Gamma_0\ln\frac{\mu_j}{\mu_s}=\frac{\alpha_s}{4\pi}(C_A+2C_F)\Gamma_0\ln r. \quad (163)$$

The singularity $\eta_\ell+\eta_h=0$ occurs when $\mu_j=\mu_s$, which happens at $r=1$. At $r=1$ there are no logarithms, so this singularity is entirely removed by the subtraction in Eq. (145). That is,

$$\frac{d\sigma^{\rm sub}}{dr} = \frac{d\sigma}{dr} - r\left[\frac{d\sigma}{dr}\right]_{r=1} \tag{164}$$

is regular at $r = 1$. Note, however, if the soft and jet scales meet at some lower scale, this singularity may be reintroduced.

The singularity at $\eta_\ell + \eta_h = 1$ is more troublesome. Similar singularities have been seen in other processes, such as Drell-Yan or Higgs production at small p_T [11–13,44,45] or the jet shape [2,46]. Writing $L = \ln\frac{1}{r}$, resummation at order NLL is meant to get right all terms of order $\alpha_s^n L^j$ with $j \geq 2n - 1$ in $R(r)$ or equivalently all terms of order $\alpha_s^n L^j$ with $j \geq n$ in the exponent, i.e., in $\ln R(r)$. In the notation of [47], we can write

$$\ln\frac{d\sigma^{\rm sub}}{dr} = Lg_1(\alpha_s L) + g_2(\alpha_s L) + \cdots$$
$$= \sim \cdots - \ln\ \sin(\pi(\eta_\ell + \eta_h)) + \cdots \tag{165}$$

with $g_1(\alpha_s L)$ and $g_2(\alpha_s L)$ completely fixed by the expansion and reorganization of our resummed expression. Normally, when $\alpha_s L \sim 1$ then we must go to higher order in RG-improved perturbation theory; at NNLL level, we would have additionally $Lg_3(\alpha_s L)$ which would extend the validity of the theoretical prediction. Here, instead we find a singularity in the exponent: $\ln R^{\rm sub}$ is infinite at $\alpha_s L \sim 1$ due to the $\eta_\ell + \eta_h = 1$ singularity. Therefore, going beyond NLL would *not* allow us to make perturbative predictions beyond where the singularity occurs. Instead, the singularity is canceled by including subleading power effects, as discussed in Sec. IV C and shown in Fig. 7.

The singularity at $\alpha \ln r \sim 1$ is reminiscent of the Landau pole in QCD. There, already at one loop one can see a pole in the running coupling at $\mu = \Lambda_{\rm QCD}$. With two-loop or higher-order running, the precise location of $\Lambda_{\rm QCD}$ moves around but cannot be surpassed. Thus what we see here is a kind of Sudakov Landau pole. Using the LL form in Eq. (163) it occurs at

$$r = \exp\left[-\frac{4\pi}{(C_A + 2C_F)\alpha_s\Gamma_0}\right] = \exp\left[-\frac{3\pi}{17\alpha_s}\right]. \tag{166}$$

For $\alpha_s = 0.119$ this gives $r \approx 0.01$. Using the NLL expressions for η_ℓ and η_h with canonical scale choices [Eq. (138)] the pole ascends to $r \approx 0.06$. Thus we cannot expect the leading-power NLL resummed distribution to be predictive between $0.27 < \rho < 0.39$. This essentially excludes the entire region on the right shoulder but leaves the region with $\rho \lesssim 0.27$ as potentially viable for a precise prediction. To stay well away from the singular region, however, one must take ρ smaller, $\rho \lesssim 0.2$ where the logarithms are no longer particularly large.

We emphasize that the excluded range is larger than that associated with strong coupling. With two-loop running and $\alpha_s(m_Z) = 0.119$, we find $\alpha_s(m_s) = 1$ at $r = 0.005$. Thus the singularity comes in at a factor of 10 larger values of r than where the soft scale probes strong dynamics. This is because the singularity is associated with the cusp anomalous dimension, not the QCD β function: the two Landau poles are unrelated.

Because of the Sudakov Landau pole in the resummed distribution it is difficult to make quantitative predictions, particularly at the NLL level, without a better understanding of the power corrections. There are a number of approaches that could be applied to ameliorate the problem. In [11], a similar pole in the Drell-Yan spectrum at small p_T [at $q^\star = m_Z \exp(-\frac{3\pi}{8\alpha_s})$ [48]] was shown to be associated with a power-suppressed region of small impact parameter but could be softened with higher-order resummation. In [13] it is argued that one could also do resummation in momentum space directly with a modified expansion of the Sudakov radiator. Related ideas can be found in [46,49]. It will be important to understand which of these approaches might apply for Sudakov shoulder resummation, but we do not attempt a complete analysis here.

At the LL level, however, because the Sudakov Landau pole is very close to the shoulder, we can at least begin to get a quantitative feel of how important resummation is. Consider the LL distribution using canonical scales in Eq. (138). When the jet in the light hemisphere is a gluon, it has the form as in Eq. (162) with Π_g from Eq. (135) becoming

$$\Pi_g = e^{-\frac{\alpha_s}{8\pi}\Gamma_0(C_A+2C_F)\log^2 r}\left[1 - \frac{\alpha_s}{8\pi}\Gamma_0(C_A\partial^2_{\eta_\ell} + 2C_F\partial^2_{\eta_h})\right]. \tag{167}$$

Note that we include every term with Γ_0 in it for leading-log resummation, not just the exponential prefactor. Including only the prefactor would give the double-logarithmic approximation, as used in previous work on resummation of the C parameter Sudakov shoulder [1]. We subtract off from the resummed distribution r times its $r \to 1$ limit as done in Eq. (145). Note that this subtraction must be done before setting canonical scales. We then match to the fixed-order LO + NLO calculation by subtracting from the resummed distribution its expansion to order α_s. In this case, the matching subtraction is

$$\frac{1}{\sigma_1}\frac{d\sigma^{\rm match}}{dr} = \frac{\alpha_s}{4\pi}(C_A + 2C_F)\Gamma_0\left(r\ln r - \frac{1}{2}r\ln^2 r\right). \tag{168}$$

Finally we include the subtraction of the first subleading power contribution. For the gluon channel this amounts to subtracting

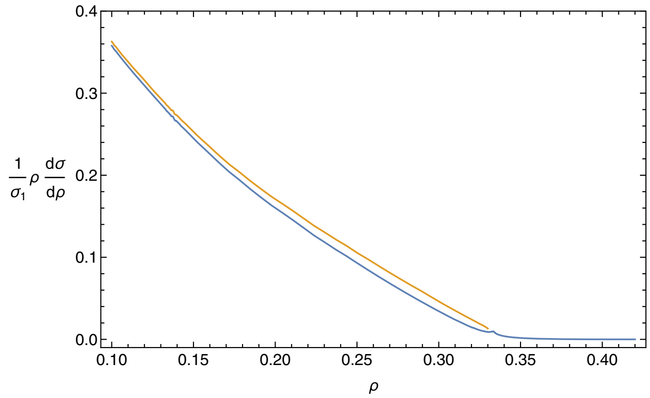


FIG. 8. Resummation of the left Sudakov shoulder for heavy jet mass at leading-logarithmic level (upper curve) compared to NLO (lower curve). The strong coupling constant is fixed to $\alpha_s = 0.119$.

$$\frac{1}{\sigma_1}\frac{d\sigma_g^R}{dr} = \frac{r^2}{2R}\Pi_g(\partial_{\eta_\ell},\partial_{\eta_h})\left(\frac{RQ}{\mu_{s\ell}}\right)^{\eta_\ell}\left(\frac{RQ}{\mu_{sh}}\right)^{\eta_h} \times \frac{e^{-\gamma_E(\eta_\ell+\eta_h)}}{\Gamma(\eta_\ell)\Gamma(\eta_h)}\frac{1}{(1-\eta_\ell-\eta_h)} \tag{169}$$

from the resummed distribution. The resulting LL resummed and matched result at $Q = m_Z$ and $\alpha_s(Q) = 0.119$ with $R = \frac{1}{3}$ for the left shoulder is shown in Fig. 8. To be clear, this is not the complete power correction but amounts to integrating the leading power soft and collinear matrix elements outside of their formal region of validity up to the kinematic limit of $r = \frac{1}{3}$.

VI. CONCLUSION

Thrust τ and heavy jet mass ρ are two of the most important observables at e^+e^- colliders. They have been used for decades for tests of precision QCD and measurements of α_s. At leading order in perturbation theory, both ρ and τ are phase-space limited to be less than $\frac{1}{3}$ and have a nonvanishing slope as $\frac{1}{3}$ is approached. At next-to-leading order, thrust behaves like $\alpha_s^2(\tau-\frac{1}{3})\ln^2(\tau-\frac{1}{3})$ for $\tau > \frac{1}{3}$ so that the slope diverges as $\frac{1}{3}$ is approached from the right. This behavior is called a right Sudakov shoulder. Heavy jet mass has a slope which diverges as ρ nears $\frac{1}{3}$ both from the left and the right: it has two Sudakov shoulders. The left shoulder of heavy jet mass is particularly important as the large logarithms can extend well into the region where α_s fits are typically done ($0.1 \lesssim \rho \lesssim 0.24$). Thus understanding and resumming its Sudakov shoulders could be very important for improving agreement of theoretical predictions with data and subsequent extractions of α_s. We also point out that it has been noted recently in the literature that in the context of other event shape observables such as fractional moments of energy-energy correlation [45] or projected energy correlators [50], one must resort to a joint resummation of the Sudakov shoulders and end point peaks.

We derived a factorization formula for both thrust and heavy jet mass in the Sudakov shoulder region. The basic mechanism for generating Sudakov shoulder logs is when a soft or collinear emission goes into one hemisphere a global constraint such as $m^2 < \rho - \frac{1}{3}$ transfers large logs from the emissions to the shoulder. Although the constraint seems nonlocal, involving both hemispheres, and therefore might violate factorization, we show that it does not. Moreover regions of large jet mass do not contribute Sudakov shoulder logs, showing that there is no nonglobal log contribution in the shoulder region. We checked our factorization formula by expanding to NLO and comparing to the exact numerical NLO calculation very close to the shoulder region. As can be seen in Fig. 6 the agreement is excellent.

The calculation involves some unusual ingredients. Since the emissions come off a trijet configuration with two quarks and one gluon, there is no azimuthal symmetry (unlike the threshold case), and the polarization of the gluon affects the spectrum. At leading order, only a uniform azimuthal angle integral was needed for the resummed expression, but in general polarized splitting function may be necessary. We also saw the appearance of Gieseking's constant, a transcendentally two number. Although it also drops out of the NLL expression, at higher orders it or related constants may be involved.

The resummed distribution for heavy jet mass has a term of the form $\sin^{-1}(\pi\eta)$ with $\eta \sim \alpha_s\Gamma_0 \ln r$, where $r = \frac{1}{3} - \rho$. The expansion near $\alpha_s = 0$ (or $\ln r = 0$) produces the leading and next-to-leading logarithmic series: terms like $\alpha^n \ln^{2n} r$. However, there is also a pole at $\eta = 1$. This pole in the resummed distribution is not due to the running coupling—it is present even with $\beta(\alpha_s) = 0$—but due to the cusp anomalous dimension. Thus it is a kind of Sudakov Landau pole. Similar behavior has been seen before, in the Drell-Yan process at small p_T, for example [11,13,44]. In both cases there is a connection between the pole and subleading power effects (subleading in r for the shoulder, or in impact parameter b for Drell-Yan). We show that subleading power terms can in fact cancel the $\eta = 1$ pole but do not affect the NLL series. This implies that a better understanding of power corrections will be necessary to establish proper theoretical uncertainty on the resummed distribution. There are many approaches that may help improve the convergences of the resummed distribution [13,30,49].

Although our results are only valid to NLL level, the factorization formula applies to all orders. In fact, since the anomalous dimensions of the jet and hard functions are known to two loops, and therefore the soft function anomalous dimension as well by renormalization-group invariance, NNLL resummation should be possible.

At NNLL level, terms linear in ρ or τ are determined. The slope can be discontinuous from the left to right side of the shoulder, as it is already at LO. This discontinuity should be computable. However, because there is also a linear term in the distribution not associated with the shoulder, confirming the predictions at NNLL will be challenging. Nevertheless, pushing the limits of Sudakov shoulder resummation, not just for e^+e^- event shapes but for collider observables more broadly, provides opportunities to improve our understanding of precision QCD.

ACKNOWLEDGMENTS

The authors thank Vicent Mateu for valuable conversations that helped inspire this work, as well as Thomas Becher, Stafano Catani, Pier Monni and Bryan Webber for valuable feedback. This study was supported by the U.S. Department of Energy under Contract No. DE-SC0013607.

APPENDIX: CALCULATION OF THE ONE-LOOP TRIJET SOFT FUNCTION

Before using rotational invariance all the integrals needed for the one-loop trijet soft functions are of the form

$$I_{n_a,n_b,n_c,n_d,N}(q) = \int d^dk \frac{n_a\cdot n_b}{(n_a\cdot k)(n_b\cdot k)}\delta(k^2)\theta(k^0) \times \delta\left(q-\frac{2}{3}N\cdot k\right)\theta(n_c\cdot k-\bar{n}_c\cdot k) \times \theta(n_d\cdot k-\bar{n}_d\cdot k), \tag{A1}$$

where the N is selected from the direction vectors

$$n_1 = (1,0,0,1), \qquad n_2 = \left(1,0,\frac{\sqrt{3}}{2},-\frac{1}{2}\right), \qquad n_3 = \left(1,0,-\frac{\sqrt{3}}{2},-\frac{1}{2}\right), \tag{A2}$$

$$N_1 = (2,0,0,-2), \qquad N_2 = (2,0,\sqrt{3},3), \qquad N_3 = (2,0,-\sqrt{3},3) \tag{A3}$$

and $n_a\cdots n_d$ only from the n_j in Eq. (A2). The θ functions restrict the phase space to one of the sextants in Fig. 4.

The Wilson lines in the trijet configuration have an S_3 symmetry which includes a Z_3 rotational invariance and a reflection symmetry. We can use the rotational invariance to rotate the Wilson lines so that they always point in the n_1 and n_2 directions. Thus we only need to consider integrals as in Eq. (98):

$$I_{n_a,n_b,N}(q) = \int d^dk \frac{n_1\cdot n_2}{(n_1\cdot k)(n_2\cdot k)}\delta(k^2)\theta(k^0) \times \delta\left(q-\frac{2}{3}N\cdot k\right)\theta(n_a\cdot k-\bar{n}_a\cdot k) \times \theta(n_b\cdot k-\bar{n}_b\cdot k). \tag{A4}$$

When $N=n_i$ is one of the Wilson line directions, then only two sextants are relevant, $I_1(q)$ and $I_2(q)$ from Fig. 5:

$$I_1(q) = I_{n_2,n_3,n_1}(q), \qquad I_2(q) = I_{n_1,n_2,n_3}(q). \tag{A5}$$

When $N=N_1=2\bar{n}_1$, there are two configurations relevant, with both Wilson lines adjacent to the $n_{\bar{1}}$ measurement region or just one of them adjacent two it. The two integrals are, as in Fig. 5,

$$I_3(q) = I_{\bar{n}_1,\bar{n}_2,2\bar{n}_3}(q), \qquad I_4(q) = I_{\bar{n}_1,\bar{n}_3,2\bar{n}_2}(q), \tag{A6}$$

where $2\bar{n}_3$ comes from rotating N_1 as the Wilson lines are rotated to the n_1 and n_2 directions.

The remaining integrals involve N_2 and N_3. These vectors are not lightlike, but they are related by a Z_2 symmetry. (Recall that the origin of the asymmetry between N_1 and N_2/N_3 is that n_1 points to the light hemisphere which affects the soft projections in the factorization formula.) Since N_2 and N_3 are related by a reflection in the y direction, which is a symmetry of the Wilson lines, if we know the integral for all Wilson line configurations for N_2 we know it for N_3 as well. So there are three possibilities, corresponding to the location of the three measurement regions with respect to the Wilson line. Rotating the N_2 Wilson line by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ gives

$$N_2' = (2,0,2\sqrt{3},0), \qquad N_2'' = (2,0,-\sqrt{3},-3). \tag{A7}$$

Thus the last three integrals we need are

$$I_5(q) = I_{\bar{n}_1,\bar{n}_2,N_3}(q), \qquad I_6(q) = I_{\bar{n}_1,\bar{n}_3,N_3'}(q), \qquad I_7(q) = I_{\bar{n}_2,\bar{n}_3,N_3''}(q). \tag{A8}$$

To perform the integrals, we parametrize the phase space with light-cone components in some direction n_1:

$$k^\mu = k_+\frac{n_1^\mu}{2} + k_-\frac{\bar{n}_1^\mu}{2} + k_\perp^\mu \tag{A9}$$

so that

$$d^dk = \frac{1}{2}d\Omega_{d-2}k_\perp^{d-3}dk_\perp dk_+ dk_- = \frac{1}{2}d\Omega_{d-3}\sin^{d-4}\theta d\theta k_\perp^{d-3}dk_\perp dk_+ dk_- \tag{A10}$$

and

$$\delta(k^2) = \delta(\vec{k}_\perp^2 - k_+ k_-). \quad (A11)$$

The integral over $k_\perp$ can be calculated using the δ function and then k_+ and k_- rescaled by q to obtain the q dependence q^{d-5} as expected by dimensional analysis. We also introduce $t = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ to rationalize $\sin\theta = \sqrt{1-\cos^2\theta}$.

With these preliminaries, the integral $I_1(q) = I_{n_2,n_3,n_1}(q)$ takes the form in $d = 4 - 2\epsilon$ dimensions

$$I_1 = \frac{3^{1-2\epsilon}\Omega_{1-2\epsilon}}{q^{1+2\epsilon}} \int_0^\infty \frac{dt}{t^{2\epsilon}(1+t^2)^{1-2\epsilon}} \times \int_0^\infty \frac{dk_+}{k_+^\epsilon} \frac{1}{1 + 3k_+ - \frac{2\sqrt{3k_+}(1-t^2)}{1+t^2}} \quad (A12)$$

$$\times \theta(1+k_+)\theta\left(-1 + k_+ - \frac{2\sqrt{3k_+}(1-t^2)}{1+t^2}\right) \times \theta\left(-1 + k_+ + \frac{2\sqrt{3k_+}(1-t^2)}{1+t^2}\right). \quad (A13)$$

The θ functions impose that

$$0 < t < \infty, \quad k_+ > \frac{7 - 10t^2 + 7t^4}{(1+t^2)^2} + \frac{4\sqrt{3(1-t^2)^2(1-t^2+t^4)}}{(1+t^2)^2} \equiv k_c. \quad (A14)$$

To handle the UV divergence as $k_+ \to \infty$, we can add and subtract the integral I_2^{div} over the integrand expanded at large k^+. This subtraction term requires the integral

$$\int_0^\infty \frac{dt}{t^{2\epsilon}(1+t^2)^{1-2\epsilon}} \int_{k_c}^\infty \frac{dk_+}{k_+^\epsilon} \frac{1}{3k_+} = \frac{\pi}{6}\frac{1}{\epsilon} + \frac{1}{9}\left(-5\kappa + \pi \ln\frac{64}{3}\right) + \mathcal{O}(\epsilon), \quad (A15)$$

where $\kappa = \mathrm{Im}\,\mathrm{Li}_2 e^{\frac{\pi i}{3}}$ is Gieseking's constant. Then $I_1^{\mathrm{fin}} = I_1 - I_1^{\mathrm{div}}$ is finite and can be expanded in ϵ and integrated order by order. Eventually, we arrive at

$$I_1(q) = \pi^{-\epsilon} e^{-\gamma_E \epsilon}\left(\frac{2}{3}\right)^{2\epsilon} \times \frac{1}{q^{1+2\epsilon}}\left(\frac{1}{\epsilon} + \ln 3 - \frac{7}{2}\ln 2 - \frac{3}{2\pi}\kappa + \mathcal{O}(\epsilon)\right). \quad (A16)$$

The calculations for other soft integrals are similar, and the results up to order $\mathcal{O}(\epsilon)$ are summarized as follows:

$$I_1(q) = \mathcal{N}\left[\frac{1}{\epsilon} + \ln 3 - \frac{7}{2}\ln 2 - \frac{3}{2\pi}\kappa + \epsilon\left(\frac{18}{5\pi}c_1 + \frac{8}{\pi}c_2 - \frac{103}{180}\pi^2 + \frac{3}{\pi}\kappa\ln 2 + \frac{10}{3}\ln^2 2 - \frac{7}{2}\ln 2\,\ln 3 + \frac{17}{40}\ln^2 3 + \frac{5}{12}\mathrm{Li}_2\left(\frac{1}{4}\right)\right) + \mathcal{O}(\epsilon^2)\right], \quad (A17)$$

$$I_2(q) = \mathcal{N}\left[\frac{3}{\pi}\kappa - \ln 2 + \epsilon\left(2\ln^2 2 + \frac{3}{2\pi}c_3 - \frac{6}{\pi}\kappa\ln 2\right) + \mathcal{O}(\epsilon^2)\right], \quad (A18)$$

$$I_3(q) = \mathcal{N}\left[\frac{3}{\pi}\kappa + \ln 2 + \epsilon\left(-2\ln^2 2 - \frac{6}{\pi}\kappa\ln 2 + \frac{3}{2\pi}c_5\right) + \mathcal{O}(\epsilon^2)\right], \quad (A19)$$

$$I_4(q) = \mathcal{N}\left[-\frac{3}{2\pi}\kappa + \frac{3}{2}\ln 2 + \epsilon\left(-3\ln^2 2 + \frac{3}{2\pi}c_4 + \frac{3}{\pi}\kappa\ln 2\right) + \mathcal{O}(\epsilon^2)\right], \quad (A20)$$

$$I_5(q) = \mathcal{N}\left[\frac{3}{\pi}\kappa + \ln 2 + \epsilon\left(-2\ln^2 2 + \frac{3}{2\pi}c_7 - \frac{6}{\pi}\kappa\ln 2\right) + \mathcal{O}(\epsilon^2)\right], \quad (A21)$$

$$I_6(q) = \mathcal{N}\left[-\frac{3}{2\pi}\kappa + \frac{3}{2}\ln 2 + \epsilon\left(-3\ln^2 2 + \frac{3}{2\pi}c_6 + \frac{3}{\pi}\kappa\ln 2\right) + \mathcal{O}(\epsilon^2)\right], \quad (A22)$$

$$I_7(q) = \mathcal{N}\left[-\frac{3}{2\pi}\kappa + \frac{3}{2}\ln 2 + \epsilon\left(-3\ln^2 2 + \frac{3}{2\pi}c_8 + \frac{3}{\pi}\kappa\ln 2\right) + \mathcal{O}(\epsilon^2)\right]. \quad (A23)$$

Here the normalization factor is

$$\mathcal{N} = \pi^{-\epsilon} e^{-\gamma_E \epsilon}\left(\frac{2}{3}\right)^{2\epsilon} \frac{1}{q^{1+2\epsilon}} \quad (A24)$$

with

$$c_1 = \mathrm{Im}\left[\mathrm{Li}_3\left(\frac{i}{\sqrt{3}}\right)\right], \qquad c_2 = \mathrm{Im}[\mathrm{Li}_3(1 + i\sqrt{3})], \quad (A25)$$

and

$$c_3 = -0.949789416853385, \quad c_4 = -0.305492372030520,$$
$$c_5 = -1.34784474998184, \quad c_6 = 4.39528715012114,$$
$$c_7 = 12.8006547420728, \quad c_8 = 3.37308464401608. \tag{A26}$$

As a cross-check, we can add the three sextants in one hemisphere with the same axis projection and compare to the hemisphere soft function [7,20,29,51,52]:

$$I_{\bar{n}_1,\bar{n}_1,n_1}(q) = I_1(q) + I_{\bar{n}_1,\bar{n}_3,n_1}(q) + I_{\bar{n}_1,\bar{n}_2,n_1}(q). \tag{A27}$$

The extra two soft integrals are

$$\begin{aligned} I_{\bar{n}_1,\bar{n}_3,n_1}(q) = \mathcal{N}\bigg[&\frac{3}{2}\ln 2 - \frac{3}{2\pi}\kappa \\ &+ \epsilon\left(\frac{71}{540}\pi^2 + \frac{18}{5\pi}c_1 - \frac{4}{\pi}c_2 + \frac{3}{\pi}\kappa\ln 2 - \frac{8}{3}\ln^2 2 \right.\\ &\left.+ \frac{3}{2}\ln 2\ln 3 - \frac{3}{40}\ln^2 3 - \frac{13}{12}\mathrm{Li}_2\left(\frac{1}{4}\right)\right) + \mathcal{O}(\epsilon^2)\bigg], \\ I_{\bar{n}_1,\bar{n}_2,n_1}(q) = \mathcal{N}\bigg[&\ln 2 + \frac{3}{\pi}\kappa \\ &+ \epsilon\left(\frac{119}{270}\pi^2 - \frac{36}{5\pi}c_1 - \frac{4}{\pi}c_2 - \frac{6}{\pi}\kappa\ln 2 - \frac{2}{3}\ln^2 2 \right.\\ &\left.+ \ln 2\ln 3 + \frac{3}{20}\ln^2 3 + \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{4}\right)\right) + \mathcal{O}(\epsilon^2)\bigg], \end{aligned} \tag{A28}$$

which leads to the same hemisphere soft function as in Ref. [20].

Another interesting fact is that the divergent part of our trijet soft function does not depend on the projection vector N. The soft function integral is

$$I_{n_a,n_b,n_c,n_d,N}(q) \sim \int d^d k \frac{n_a\cdot n_b}{(n_a\cdot k)(n_b\cdot k)}\delta(k^2)\delta\left(q - \frac{2}{3}N\cdot k\right) \times [\cdots]. \tag{A29}$$

If we rotate the projection vector N to another direction N', then the δ function transforms as

$$\begin{aligned} \delta\left(q - \frac{2}{3}N'\cdot k\right) &= \delta\left(q - \frac{2}{3}(N\cdot k)\frac{N'\cdot k}{N\cdot k}\right) \\ &= \frac{N\cdot k}{N'\cdot k}\delta\left(\frac{N\cdot k}{N'\cdot k}q - \frac{2}{3}N\cdot k\right). \end{aligned} \tag{A30}$$

Then after rescaling

$$k \to \frac{N\cdot q}{N'\cdot q}k, \tag{A31}$$

the integral becomes

$$I_{n_a,n_b,n_c,n_d,N}(q) \sim \left(\frac{N\cdot q}{N'\cdot q}\right)^{2\epsilon} q^{d-5}\int d^d k \frac{n_a\cdot n_b}{(n_a\cdot k)(n_b\cdot k)}\delta(k^2) \times \delta\left(q - \frac{2}{3}N\cdot k\right) \times [\cdots]. \tag{A32}$$

So the effect of using different N's only shows up at order ϵ in the expansion. This only affects the anomalous dimension for the integrals which have soft-collinear divergences. This is only $I_1(q)$, since that is the only integral where a Wilson line is in within the integration region. However, for $I_1(q)$ the projection is on n_1 (the Wilson line direction) in both thrust and heavy jet mass. For the others, using the definitions in Eqs. (A5), (A6), and (A8), we see that at NLL level after rescaling the projection vector N

$$I_5(q) = I_3(q), \quad I_6(q) = I_4(q), \text{ and } I_7(q) = I_6(q), \tag{A33}$$

where we need a reflection with respect to the z axis to see the third equation. This agrees with our explicit calculations in Eqs. (A17)–(A23) to order ϵ^0. Note however, that the ϵ^1 terms differ, as expected. In summary, at the NLL level, the thrust and heavy jet mass trijet soft function can be taken to be the same. For NNLL resummation and beyond, they will generically be different.

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