Twisted geometry coherent states in all dimensional loop quantum gravity. II. Ehrenfest property

Gaoping Long[*]

Department of Physics, South China University of Technology, Guangzhou 510641, China and Department of Physics, Beijing Normal University, Beijing 100875, China

(Received 18 April 2022; accepted 30 August 2022; published 21 September 2022)

In the preceding paper of this series of articles we constructed the twisted geometry coherent states in all dimensional loop quantum gravity and established their peakedness properties. In this paper we establish the "Ehrenfest property" of these coherent states which are labeled by the twisted geometry parameters. By this we mean that the expectation values of the polynomials of the elementary operators as well as the operators which are not polynomial functions of the elementary operators, reproduce, to zeroth order in $\hbar$, the values of the corresponding classical functions at the twisted geometry space point where the coherent state is peaked.

DOI: 10.1103/PhysRevD.106.066021

I. INTRODUCTION

In our companion paper [1], we constructed a new family of coherent state in all dimensional loop quantum gravity (LQG) and studied its basic properties. This new family of coherent state, whose analog in $(1+3)$-dimensional $SU(2)$ LQG is proposed and studied in [2–6], is called the twisted geometry coherent state since it is labeled by the twisted geometry variables which parametrize the $SO(D+1)$ holonomy-flux phase space of all dimensional LQG. As we explained in [1], we consider the twisted geometry coherent state in all dimensional LQG instead of the heat-kernel coherent state which is frequently used in $(1+3)$-dimensional $SU(2)$ LQG [7–15], because the specific studies of the heat-kernel coherent state in all dimensional LQG are confronted with some technical problems. Nevertheless, the twisted geometry coherent state in all dimensional LQG takes a much simpler formulation than the heat-kernel one [16], which ensures that its related calculations only involve the familiar Gaussian summation and the $SO(D+1)$ coherent intertwiner which has been fully studied in [17,18]. Thus, if one can verify that the twisted geometry coherent state in all dimensional LQG possesses a well-behaved peakedness property and "Ehrenfest property" in the $SO(D+1)$ holonomy-flux phase space, then the twisted geometry coherent state can be used to study many issues involving the semiclassicality in all dimensional LQG, e.g., the effective dynamics based on coherent state. In our companion paper [1], we have shown that the twisted geometry coherent states in all dimensional LQG provide an overcompleteness basis of the kinematic Hilbert space in which the edge-simplicity constraint is solved, and the expectation values of holonomy and flux operators with respect to the twisted geometry coherent states coincide with the corresponding classical values given by the labels of the coherent states, up to some gauge degrees of freedom. Besides, the peakedness of the wave functions of the twisted geometry coherent state in holonomy, momentum, and phase space representations is studied and it is well controlled by a semiclassical parameter which is proportional to $\hbar$.

The main result of the present article is that the Ehrenfest property, to zeroth order, indeed holds for the twisted geometry coherent states in all dimensional LQG. In other words, the expectation values of polynomials of the elementary operators as well as the operators which are not polynomial functions of the elementary operators, reproduce, to zeroth order in $\hbar$, the values of the corresponding classical functions at the twisted geometry space point where the coherent state is peaked. To achieve this goal, we consider the monomials of the holonomy and flux operators. By using the completeness relation of the twisted geometry coherent state, the calculation of the monomials of the holonomy and flux operators is converted into the calculation of the matrix elements of the holonomy and flux operators in the twisted geometry coherent state basis. Then, the matrix elements of the holonomy and flux operators in the twisted geometry coherent state basis can be calculated by using the techniques developed in the calculation of the overlap function of the twisted geometry coherent state [1]. To complete this calculation, the properties of the Clebsch-Gordan coefficients related to the Perelomov type coherent state of $SO(D+1)$, as well as the derivative of the overlap functions of the Perelomov type coherent state of $SO(D+1)$, are studied as the key

[*]Corresponding author.
201731140005@mail.bnu.edu.cn

points. Besides, similar to that of the heat-kernel coherent state in $SU(2)$ LQG [11], the expectation values of nonpolynomial operators with respect to the twisted geometry coherent state in all dimensional LQG will be studied by reformulating it as the Hamburger moment problem.

This paper is organized as follows. In Sec. II, we will review the kinematic structures of all dimensional LQG and some basic properties of the twisted geometry coherent state which are necessary for the studies in this article. Beginning with the structure of classical phase spaces, including the connection phase space and the holonomy-flux phase space of all dimensional LQG, the twisted geometry parametrization and the analysis of gauge degrees of freedom with respect to the simplicity constraint will be reviewed. Then, the quantum Hilbert space with its coherent-intertwiner-spin-network basis and the elementary operators in all dimensional LQG will be pointed out. Besides, as key ingredients to construct the Ehrenfest property of the twisted geometry coherent state, the completeness relation of twisted geometry coherent states and the overlap functions of both Perelomov type coherent state and twisted geometry coherent state in all dimensional LQG will be introduced explicitly. Based on this foundation, we will construct the Ehrenfest property of the twisted geometry coherent state explicitly in Sec. III. We will first construct the Ehrenfest property for operator monomials by proving that the matrix elements of the elementary operators with respect to the twisted geometry coherent state are well evaluated by their corresponding expectation values, and then construct it for nonpolynomial operators by using the Hamburger theorem. Finally, a short conclusion will be given in Sec. IV.

II. KINEMATIC STRUCTURE OF ALL DIMENSIONAL LOOP QUANTUM GRAVITY

A. Classical phase space of all dimensional loop quantum gravity

The $(1+\mathrm{D})$-dimensional Lorentzian LQG is constructed by canonically quantizing general relativity (GR) based on the Yang-Mills phase space coordinatized by the conjugate pair (A_{aIJ}, π^{bKL}) with the nonvanishing Poisson bracket [19–21]

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} = 2\kappa\beta\delta_a^b\delta_{[I}^K\delta_{J]}^L\delta^{(D)}(x-y), \quad (1)$$

where the connection A_{aIJ} and its canonical conjugate momentum π^{bKL} are $so(D+1)$ valued fields on a D-dimensional spatial manifold Σ. κ and β represent the gravitational constant and Babero-Immirze parameter respectively. Here we use $I, J, K, \ldots$ for the internal vector index in the definition representation space of $SO(D+1)$ and $a, b, c, \ldots$ for the spatial index. The dynamics of this Hamiltonian system is governed by the Gaussian, simplicity, vector, and scalar constraints, which read

$$\mathcal{G}^{IJ} \equiv \partial_a\pi^{aIJ} + 2A_{aK}^{[I}\pi^{a|K|J]} \approx 0, \quad (2)$$

$$S^{ab[IJKL]} \equiv \pi^{a[IJ}\pi^{|b|KL]} \approx 0, \quad (3)$$

$$C_a \approx 0, \quad \text{and} \quad C \approx 0 \quad (4)$$

respectively. Based on the Poisson structure (1) of the connection phase space, one can check that these constraints obey a first class constraint algebra. Furthermore, one can also check that the Gauss constraint generates the $SO(D+1)$ gauge transformation of this Yang-Mills gauge theory, while the simplicity constraint restricts the degrees of freedom of π^{aIJ} to that of a D-frame E^{aI} which describes the spatial geometry and generates some other gauge transformations. In other words, one can solve the simplicity constraint and get the solution $\pi^{aIJ} = 2n^{[I}E^{|a|J]}$ with $n^I E_I^a = 0$, $n^I n_I = 1$ and E^{aI} being the densitized D-frame which gives the spatial metric q_{ab} by $qq^{ab} = E^{aI}E_I^b$, where q is the determinant of q_{ab} [19]. Besides, one can reconstruct the densitized extrinsic curvature $\tilde{K}_{ab} = q_{bc}\tilde{K}_a{}^c$ of the spatial manifold Σ by

$$\tilde{K}_a{}^b \approx K_{aIJ}\pi^{bIJ} \equiv \frac{1}{\beta}(A_{aIJ} - \Gamma_{aIJ})\pi^{bIJ} \quad (5)$$

on the Gaussian and simplicity constraint surface, where Γ_{aIJ} is purely constructed from π^{aIJ} and it is the spin connection of E^{aI} exactly on the simplicity constraint surface [19]. To clarify the gauge degrees of freedom corresponding to simplicity, let us decompose $K_{aIJ} := \frac{1}{\beta}(A_{aIJ} - \Gamma_{aIJ})$ as

$$K_{aIJ} = 2n^{[I}K_a^{J]} + \bar{K}_a^{IJ}, \quad (6)$$

where $\bar{K}_a^{IJ} := \bar{\eta}_K^I\bar{\eta}_L^J K_a^{KL}$ with $\bar{\eta}_I^J := \delta_I^J - n_I n^J$ and $\bar{K}_a^{IJ} n_I = 0$. Based on Eqs. (1) and (3), one can check that the component $2n^{[I}K_a^{J]}$ and π^{aIJ} Poisson commutes with the simplicity constraint while $\bar{K}_a^{IJ}$ does not. Hence, the simplicity constraint fixes both $\tilde{K}_a{}^b$ and q_{ab} and it exactly introduce extra gauge degrees of freedom represented by $\bar{K}_a^{IJ}$. The details of these discussions can be found in the Ref. [19] and it is shown that, the standard symplectic reduction procedures with respect to Gaussian and simplicity constraint in the $SO(D+1)$ connection phase space leads to the Arnowitt-Deser-Misner (ADM) [22] phase space of $(1+D)$-dimensional GR, with the coordinates $\tilde{K}_a{}^b$ and q_{ab} of the ADM phase space are Dirac observables with respect to Gaussian and simplicity constraints. It should be emphasized that $\bar{K}_a^{IJ}$ are pure gauge components with respect to the simplicity constraint, which only contributes gauge degrees of freedom in this $SO(D+1)$ Yang-Mills theory. As we will show, the counterpart of $\bar{K}_a^{IJ}$

in the discrete phase of all dimensional LQG is a critical point of the results of this paper.

Apart from the different gauge group which however is compact and the additional simplicity constraint, the $SO(D+1)$ connection formulation of (1 + D)-dimensional GR is precisely the same as $SU(2)$ connection formulation of (1 + 3)-dimensional GR, and the quantization of the $SO(D+1)$ connection formulation is therefore in complete analogy with (1 + 3)-dimensional $SU(2)$ LQG [23–27]. By following any standard text on LQG such as [24,25], the loop quantization of the $SO(D+1)$ connection formulation of (1 + D)-dimensional GR leads to a kinematical Hilbert space $\mathcal{H}$ [21], which can be regarded as a union of the Hilbert spaces $\mathcal{H}_\gamma = L^2((SO(D+1))^{|E(\gamma)|}, d\mu_{\text{Haar}}^{|E(\gamma)|})$ on all possible finite graphs γ embedded in Σ, where $E(\gamma)$ denotes the set composed by the independent edges of γ and $d\mu_{\text{Haar}}^{|E(\gamma)|}$ denotes the product of the Haar measure on $SO(D+1)$. In this sense, on each given γ there is a discrete phase space $(T^*SO(D+1))^{|E(\gamma)|}$, which is coordinatized by the elementary discrete variables—holonomies and fluxes. The holonomy of A_{aIJ} along an edge $e \in \gamma$ is defined by

$$h_e[A] \coloneqq \mathcal{P}\exp\left(\int_e A\right) = 1 + \sum_{n=1}^{\infty}\int_0^1 dt_n \int_0^{t_n} dt_{n-1}\ldots\int_0^{t_2} dt_1 A(t_1)\ldots A(t_n), \tag{7}$$

where $A(t) \coloneqq \frac{1}{2}\dot{e}^a A_{aIJ}\tau^{IJ}$, $\dot{e}^a$ is the tangent vector field of e, τ^{IJ} is a basis of $so(D+1)$ given by $(\tau^{IJ})^{\text{def.}}_{KL} = 2\delta^{[I}_K\delta^{J]}_L$ in definition representation space of $SO(D+1)$, and $\mathcal{P}$ denoting the path-ordered product. The flux F^{IJ}_e of π^{aIJ} through the (D-1)-dimensional face dual to edge e is defined by

$$F^{IJ}_e \coloneqq -\frac{1}{4}\operatorname{tr}\left(\tau^{IJ}\int_{e^\star}\epsilon_{aa_1\ldots a_{D-1}}h(\rho^s_e(\sigma))\pi^{aKL}(\sigma)\tau_{KL}h(\rho^s_e(\sigma)^{-1})\right), \tag{8}$$

where $e^\star$ is the (D-1)-face traversed by e in the dual lattice of γ, $\rho^s(\sigma): [0,1] \to \Sigma$ is a path connecting the source point $s_e \in e$ to $\sigma \in S_e$ such that $\rho^s_e(\sigma): [0,\frac{1}{2}] \to e$ and $\rho^s_e(\sigma): [\frac{1}{2},1] \to S_e$. Similarly, we can define the dimensionless flux X^{IJ}_e as

$$X^{IJ}_e = -\frac{1}{4\beta a^{D-1}}\operatorname{tr}\left(\tau^{IJ}\int_{e^\star}\epsilon_{aa_1\ldots a_{D-1}}h(\rho^s_e(\sigma))\pi^{aKL}(\sigma)\tau_{KL}h(\rho^s_e(\sigma)^{-1})\right), \tag{9}$$

where a is an arbitrary but fixed constant with the dimension of length. Since $SO(D+1)\times so(D+1) \cong T^*SO(D+1)$, this new discrete phase space $\times_{e\in\gamma}(SO(D+1)\times so(D+1))_e$, called the phase space of $SO(D+1)$ loop quantum gravity on the fixed graph γ, is a direct product of $SO(D+1)$ cotangent bundles. Finally, the complete phase space of the theory is given by taking the union over the phase spaces of all possible graphs. In the discrete phase space associated to γ, the constraints are expressed by the smeared variables. The discretized Gauss constraint is given by

$$G_v \coloneqq \sum_{b(e)=v} X_e - \sum_{t(e')=v} h^{-1}_{e'}X_{e'}h_{e'} \approx 0. \tag{10}$$

The discretized simplicity constraints are separated as two sets. The first one is the edge-simplicity constraint $S^{IJKL}_e \approx 0$ which takes the form [21,28]

$$S^{IJKL}_e \equiv X^{[IJ}_e X^{KL]}_e \approx 0, \quad \forall\, e \in \gamma \tag{11}$$

and the second one is the vertex-simplicity constraint $S^{IJKL}_{v,e,e'} \approx 0$ which is given by [21,28]

$$S^{IJKL}_{v,e,e'} \equiv X^{[IJ}_e X^{KL]}_{e'} \approx 0, \quad \forall\, e, e' \in \gamma, \qquad s(e) = s(e') = v. \tag{12}$$

The symplectic structure of the discrete phase space can be expressed by the Poisson algebra between the elementary variables (h_e, X^{IJ}_e), which reads

$$\{h_e, h_{e'}\} = 0, \qquad \{h_e, X^{IJ}_{e'}\} = \delta_{e,e'}\frac{\kappa}{a^{D-1}}\frac{d}{dt}(e^{t\tau^{IJ}}h_e)\Big|_{t=0},$$
$$\{X^{IJ}_e, X^{KL}_{e'}\} = \delta_{e,e'}\frac{\kappa}{a^{D-1}}(\delta^{IK}X^{JL}_e + \delta^{JL}X^{IK}_e - \delta^{IL}X^{JK}_e - \delta^{JK}X^{IL}_e). \tag{13}$$

Then, by using this Poisson algebra, it is easy to verify that $G_v \approx 0$ and $S_e \approx 0$ form a first class constraint system as

$$\{S_e, S_e\} \propto S_e, \qquad \{S_e, S_v\} \propto S_e, \qquad \{G_v, G_v\} \propto G_v,$$
$$\{G_v, S_e\} \propto S_e, \qquad \{G_v, S_v\} \propto S_v, \qquad b(e) = v, \tag{14}$$

where the algebras among $G_v \approx 0$ are isomorphic to the $so(D+1)$ algebra, and the ones involving $S_e \approx 0$ weakly vanish. Especially, the algebras among the vertex-simplicity constraint are the problematic ones, with the open anomalous brackets [29]

$$\{S_{v,e,e'}, S_{v,e,e''}\} \propto \text{anomaly terms} \tag{15}$$

where the "anomaly terms" are not proportional to any of the existing constraints in the phase space.

In fact, a similar simplicity constraint is widely studied in the 4-dimensional spin-foam theory [18,30–34]. In all dimensional LQG, the treatment of this anomalous vertex simplicity constraint in both quantum theory and classical discrete theory is a critical problem. It has been shown that the strong imposition of vertex simplicity constraint eliminates the physical degrees of freedom erroneously [35,36]. Thus, one needs to construct a new treatment of the anomalous simplicity constraint and explain its geometric meaning to ensure the correctness. The generalized twisted geometric parametrization of the discrete phase space of all dimensional LQG is such a scheme, which leads us to solve the anomalous simplicity constraint and ensure the physical degrees of freedom and gauge degrees of freedom are separated correctly. As shown in Ref. [37], with the Gaussian constraint, simplicity constraint, and the $(D-1)$-faces' shape matching condition being imposed appropriately, the generalized twisted geometric parameters reproduce the Regge geometries on the D-dimensional spatial manifold Σ correctly. Let us review the generalized twisted geometric parametrization of the discrete phase space in all dimensional LQG briefly as follows. Recall that the discrete phase space associated to a given graph γ is denoted by $\times_{e\in\gamma}T^*SO(D+1)_e$. In this phase space, one can first solve the edge-simplicity constraint equation and it leads to the constraint surface defined by

$$\times_{e\in\gamma}T^*_{\rm s}SO(D+1)_e := \{(\ldots,(h_e,X_e),\ldots)\in\times_{e\in\gamma}T^*SO(D+1)_e | X_e^{[IJ}X_e^{KL]}=0\}. \tag{16}$$

To simplify the statements, one can first consider the edge-simplicity constraint surface $T^*_{\rm s}SO(D+1)$ for one copy of the edge only. Then, the generalized twisted geometry variables $(V,\tilde{V},\xi^o,\eta,\bar{\xi}^\mu)$ can be introduced to reparametrize the edge-simplicity constraint surface $T^*_{\rm s}SO(D+1)$. These generalized twisted geometry variables $(V,\tilde{V},\xi^o,\eta,\bar{\xi}^\mu)$ and their space

$$P := Q_{D-1}\times Q_{D-1}\times T^*S^1\times SO(D-1) \tag{17}$$

are constructed as follows. The bivector V or $\tilde{V}$ with fixed norm constitutes the space $Q_{D-1} := SO(D+1)/(SO(2)\times SO(D-1))$, where $SO(2)\times SO(D-1)$ is the maximum subgroup of $SO(D+1)$ which preserves the bivector $\tau_o := 2\delta_1^{[I}\delta_2^{J]}$. The real number η combining with $\xi^o\in[-\pi,\pi)$ constitutes the space T^*S^1. Besides, $e^{\bar{\xi}^\mu\bar{\tau}_\mu} =: \bar{u}$ is an element of $SO(D-1)$ which preserves the bivector τ_o, with $\bar{\tau}_\mu$ being a basis of $so(D-1)$ and $\mu\in\{1,\ldots,\frac{(D-1)(D-2)}{2}\}$. In order to capture the intrinsic curvature by these parameters, it is necessary to specify one pair of the $SO(D+1)$ valued Hopf sections $u(V)$ and $\tilde{u}(\tilde{V})$, which satisfy $V=u(V)\tau_o u(V)^{-1}$ and $\tilde{V}=-\tilde{u}(\tilde{V})\tau_o\tilde{u}(\tilde{V})^{-1}$. Then, the generalized twisted geometry parametrization for one copy of the edge can be established by the map

$$P\ni(V,\tilde{V},\xi^o,\eta,\bar{\xi}^\mu)\mapsto(h,X)\in T^*_{\rm s}SO(D+1):\ X=\frac{1}{2}\eta V=\frac{1}{2}\eta u(V)\tau_o u(V)^{-1},\quad h=u(V)e^{\bar{\xi}^\mu\bar{\tau}_\mu}e^{\xi^o\tau_o}\tilde{u}(\tilde{V})^{-1}. \tag{18}$$

It is easy to check that, the two points $(V,\tilde{V},\xi^o,\eta,\bar{\xi}^\mu)$ and $(-V,-\tilde{V},-\xi^o,-\eta,\dot{\bar{\xi}}^\mu)$ in P are mapped to the same point $(h,X)\in T^*_sSO(D+1)$ by the map (18), where $e^{\dot{\bar{\xi}}^\mu\bar{\tau}_\mu}=e^{-2\pi\tau_{13}}e^{\bar{\xi}^\mu\bar{\tau}_\mu}e^{2\pi\tau_{13}}$ and $\tau_{13}=\delta_1^{[I}\delta_3^{J]}$. Thus, the map (18) gives a two-to-one double covering of the image. A more detailed study shows that [37] a bijection map can be constructed in the region $|X|\neq 0$ by selecting either branch among the two-to-one double covering (18). Moreover, the new parameters also simplify the Poisson structures of the discrete phase space. For instance, the nonvanishing Poisson bracket between ξ^o and η can be given by

$$\{\xi^o,\eta\}=\frac{2\kappa}{a^{D-1}}, \tag{19}$$

with ξ^o and η representing a portion of the degrees of freedom of extrinsic and intrinsic geometry respectively. Now we can get back to the discrete phase space of all dimensional LQG on the whole graph γ, which is just the Cartesian product of the discrete phase space on each single edge of γ. Then, the twisted geometry parametrization of the discrete phase space on one copy of the edge can be generalized to that of the whole graph γ directly. Furthermore, the twisted geometry parameters $(V,\tilde{V},\xi^o,\eta)$ take the interpretation of the discrete geometry describing the dual lattice of γ, which can be explained explicitly as follows. We first note that $\frac{1}{2}\eta_e V_e$ and $\frac{1}{2}\eta_e\tilde{V}_e$ represent the area-weighted outward normal bivectors of the $(D-1)$-face dual to e in the perspective of source and target points of e respectively, with $\frac{1}{2}\eta_e$ being the dimensionless area of the $(D-1)$-face dual to e. Then, the holonomy $h_e = u_e(V_e)e^{\bar{\xi}^\mu_e\bar{\tau}_\mu}e^{\xi^o_e\tau_o}\tilde{u}_e^{-1}(\tilde{V}_e)$ takes the interpretation that it rotates the inward normal $-\frac{1}{2}\eta_e\tilde{V}_e$ of the (D-1)-face dual to e in the perspective of the target point of e, into the outward normal $\frac{1}{2}\eta_e V_e$ of the (D-1)-face dual to e in the perspective of the source point of e, wherein $u_e(V_e)$ and $\tilde{u}_e(\tilde{V}_e)$ capture the contribution of intrinsic curvature, and $e^{\xi^o_e\tau_o}$ captures the contribution of extrinsic curvature to this rotation. Moreover, $e^{\bar{\xi}^\mu_e\bar{\tau}_\mu}$ represents some redundant degrees of freedom for reconstructing the discrete geometry. Finally, with the Gaussian and vertex simplicity constraint being imposed at the vertices of γ, one can get the closed twisted geometry which describes the D-polytopes and their gluing method in the dual lattice of γ [6,37,38].

The discrete geometric interpretation of the twisted geometry parametrization points out a proper treatment of the anomalous vertex simplicity constraint in the discrete phase space of all dimensional LQG. It has been shown that, by considering some kinds of the continuum limit, one can establish the relation between the constraint surface defined by both of the edge simplicity and anomalous vertex simplicity constraints (CSEVSC) in the discrete phase space and the constraint surface defined by the nonanomalous simplicity constraint (CSNASC) in the connection phase space. Especially, on these two constraint surfaces, the gauge transformation induced by the edge simplicity constraint corresponds to that induced by the nonanomalous simplicity constraint exactly in the continuum limit, which can be illustrated as

$$\bar{\xi}^\mu_e|_{\text{CSEVSC}} \xrightarrow{\text{continuum limit}} \bar{K}_{aIJ}|_{\text{CSNASC}} \tag{20}$$

where $\bar{\xi}^\mu_e$ and $\bar{K}_{aIJ}$ capture the pure gauge degrees of freedom with respect to the simplicity constraint in holonomy h_e and connection A_{aIJ} respectively. In summary, the implementation of the Gaussian and anomalous simplicity constraints in discrete phase space contains two steps: (i) execute the symplectic reduction with respect to edge simplicity constraint and Gaussian constraint; (ii) solve the vertex simplicity constraint equation to get the constraint surface. As we mentioned before, the resulting space is parametrized by the so-called constrained twisted geometry space, which covers the degrees of freedom of internal and external Regge geometry on the D-dimensional spatial manifold Σ, with the twisted geometry parameters being endowed with certain geometric interpretations in Regge geometry [37].

B. Spin network basis of the kinematic Hilbert space in all dimensional loop quantum gravity

The Hilbert space $\mathcal{H}$ of all dimensional LQG is given by the completion of the space of cylindrical functions on the quantum configuration space, which can be decomposed into the sectors—the Hilbert spaces constructed on graphs. For a given graph γ with $|E(\gamma)|$ edges, the related Hilbert space is given by $\mathcal{H}_\gamma = L^2((SO(D+1))^{|E(\gamma)|}, d\mu_{\text{Haar}}^{|E(\gamma)|})$. This Hilbert space associates to the classical phase space $\times_{e\in\gamma}T^*SO(D+1)_e$ aforementioned. A basis of this space is given by the spin-network functions which are labeled by (i) an $SO(D+1)$ representation Λ assigned to each edge, and (ii) an intertwiner i_v assigned to each vertex v. Each basis state $\Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}_e)$, as a wave function on $\times_{e\in\gamma}SO(D+1)_e$, is then given by

$$\Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A)) \equiv \bigotimes_{v\in\gamma} i_v \rhd \bigotimes_{e\in\gamma}\pi_{\Lambda_e}(h_e(A)), \tag{21}$$

where $\vec{h}(A) \coloneqq (\ldots, h_e(A), \ldots)$, $\vec{\Lambda} \coloneqq (\ldots, \Lambda_e, \ldots)$, $e \in \gamma$, $\vec{i} \coloneqq (\ldots, i_v, \ldots)$, $v \in \gamma$, $\pi_{\Lambda_e}(h_e)$ denotes the matrix of holonomy h_e associated to edge e in the representation labeled by Λ_e, and $\rhd$ denotes the contraction of the representation matrixes of holonomies with the intertwiners. Hence, the wave function $\Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A))$ is simply the product of the functions given by specified components of the holonomy matrices, selected by the intertwiners at the vertices. The action of the elementary operators—holonomy operator and flux operator—on the spin-network functions can be given as

$$\begin{aligned} \hat{h}_e(A) \circ \Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A)) &= h_e(A)\Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A)), \\ \hat{F}^{IJ}_e \circ \Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A)) &= -\mathbf{i}\hbar\kappa\beta R^{IJ}_e\Psi_{\gamma,\vec{\Lambda},\vec{i}}(\vec{h}(A)), \end{aligned} \tag{22}$$

with $R^{IJ}_e \coloneqq \text{tr}((\tau^{IJ}h_e)^{\text{T}}\frac{\partial}{\partial h_e})$ being the right invariant vector fields on $SO(D+1)$ associated to the edge e, and T denoting the transposition of the matrix. Then, the other operators in all dimensional LQG, such as spatial geometric operators and Hamiltonian operator, can be constructed based on these elementary operators [39–41].

In order to obtain the kinematic physical Hilbert space, one needs to solve the kinematic constraints, including Gaussian constraint, edge-simplicity constraint, and vertex simplicity constraint in $\mathcal{H}$. Following the results given in Sec. II A, the Gaussian constraint and edge-simplicity constraint are imposed strongly. The resulting space is spanned by the edge-simple and gauge invariant spin-network states, whose edges are labeled by the simple representations of $SO(D+1)$ and vertices are labeled by the gauge invariant intertwiners. Besides, the anomalous vertex simplicity constraints are imposed weakly and the corresponding weak solutions are given by the spin-network states whose vertices are labeled by the simple coherent intertwiners [42]. A typical spin-network state whose vertices are labeled by the gauge invariant simple coherent intertwiners can be given as

$$\Psi_{\gamma,\vec{N},\vec{\mathcal{I}}_{\text{s.c.}}}(\vec{h}(A)) = \text{tr}(\otimes_{e\in\gamma}\pi_{N_e}(h_e(A))\otimes_{v\in\gamma}\mathcal{I}^{\text{s.c.}}_v) \tag{23}$$

where $\pi_{N_e}(h_e(A))$ denotes the representation matrix of $h_e(A)$ with N_e a non-negative integer labeling a simple representation of $SO(D+1)$, and $\vec{\mathcal{I}}_{\text{s.c.}}$ is defined by $\vec{\mathcal{I}}_{\text{s.c.}} \coloneqq (\ldots, \mathcal{I}_v^{\text{s.c.}}, \ldots)$ with $\mathcal{I}_v^{\text{s.c.}}$ being the so-called gauge invariant simple coherent intertwiner labeling the vertex $v \in \gamma$.

C. Perelomov type coherent state of $SO(D+1)$ and coherent intertwiner

In order to give the details of the construction of simple coherent intertwiners, we must first introduce some concepts of the simple representation of $SO(D+1)$ and the homogeneous harmonic functions on the D-sphere. The homogeneous harmonic functions with degree N on the D-sphere (S^D) compose a space $\mathfrak{H}^N_{D+1}$ with dimensionality

$$\dim(\mathfrak{H}^N_{D+1}) = \dim(\pi_N) = \frac{(D+N-2)!(2N+D-1)}{(D-1)!N!}, \qquad (24)$$

and $\mathfrak{H}^N_{D+1}$ is a realization of the simple representation space of $SO(D+1)$ labeled by N. Introduce a subgroup series $SO(D+1) \supset SO(D) \supset SO(D-1) \supset \ldots \supset SO(2)_{\delta_1^{[I}\delta_2^{J]}}$ with $SO(2)_{\delta_1^{[I}\delta_2^{J]}}$ being the one-parameter subgroup of SO$(D+1)$ generated by $\tau_o \coloneqq 2\delta_1^{[I}\delta_2^{J]}$. Then, an orthogonal and normalized basis of the space $\mathfrak{H}^N_{D+1}$ can be given as

$$\{\Xi^{N,\mathbf{M}}_{D+1}(\boldsymbol{x}) | \mathbf{M} \coloneqq M_1, M_2, \ldots, M_{D-1}, \quad N \geq M_1 \geq M_2 \geq \ldots \geq |M_{D-1}|, \boldsymbol{x} \in S^D\} \qquad (25)$$

or equivalently, in Dirac bracket notation can be denoted by $|N, \mathbf{M}\rangle$, where $\mathbf{M} \coloneqq M_1, M_2, \ldots, M_{D-1}$ with $N \geq M_1 \geq M_2 \geq \ldots \geq |M_{D-1}|$, and $N, M_1, \ldots M_{D-2} \in \mathbb{N}$, $M_{D-1} \in \mathbb{Z}$. The labels N, $\mathbf{M}$ of the function $\Xi^{N,\mathbf{M}}_{D+1}(\boldsymbol{x})$ is interpreted such that $\Xi^{N,\mathbf{M}}_{D+1}(\boldsymbol{x})$ belongs to the series of subspaces $\mathfrak{H}_2^{M_{D-1}} \subset \mathfrak{H}_3^{M_{D-2}} \subset \ldots \subset \mathfrak{H}_D^{M_1} \subset \mathfrak{H}^N_{D+1}$ which are the irreducible simple representation spaces labeled by $M_{D-1}, \ldots, M_2, M_1, N$ of the series of subgroups $SO(2)_{\delta_1^{[I}\delta_2^{J]}} \subset SO(3) \subset \ldots \subset SO(D) \subset SO(D+1)$ respectively [43]. With this convention, the orthogonal and normalized property of this basis can be expressed as

$$\langle N, \mathbf{M} | N, \mathbf{M}' \rangle \coloneqq \int_{S^D} d\boldsymbol{x} \overline{\Xi^{N,\mathbf{M}}_{D+1}(\boldsymbol{x})} \Xi^{N,\mathbf{M}'}_{D+1}(\boldsymbol{x}) = \delta_{\mathbf{M},\mathbf{M}'} \qquad (26)$$

with $\delta_{\mathbf{M},\mathbf{M}'} = 1$ if $\mathbf{M} = \mathbf{M}'$ and zero otherwise, where $d\boldsymbol{x}$ is the normalized invariant measure on S^D. An element $g \in \text{SO}(D+1)$ acts on a spherical harmonic function $f(\boldsymbol{x})$ on D-sphere as

$$g \circ f(\boldsymbol{x}) = f(g^{-1} \circ \boldsymbol{x}), \qquad (27)$$

where $g \circ \boldsymbol{x}$ denotes the action of $g \in SO(D+1)$ on the point $\boldsymbol{x} \in S^D$ by its definition. Correspondingly, the basis $\{\tau_{IJ}\}$ of $so(D+1)$, defined by $(\tau_{IJ})^{\text{def.}} \coloneqq 2\delta_I^{[K}\delta_J^{L]}$ in the definition representation space of $SO(D+1)$, are operators in $\mathfrak{H}^N_{D+1}$ and they act on the spherical harmonic function as

$$\tau_{IJ} \circ f(\boldsymbol{x}) \coloneqq \frac{d}{dt} f(e^{-t\tau_{IJ}} \circ \boldsymbol{x})|_{t=0}. \qquad (28)$$

This action also gives a representation of the Lie algebra

$$[\tau_{IJ}, \tau_{KL}] = \delta_{IL}\tau_{JK} + \delta_{JK}\tau_{IL} - \delta_{IK}\tau_{JL} - \delta_{JL}\tau_{IK}. \qquad (29)$$

The general scheme of the construction of Perelomov type coherent state for compact Lie algebra is introduced in Ref. [44]. For the case of $SO(D+1)$ involved in this article, let us consider the state $|N, \mathbf{N}\rangle \in \mathfrak{H}^N_{D+1}$ which corresponds to the highest weight vector with $\mathbf{N} = \mathbf{M}|_{M_1=\ldots=M_{D-1}=N}$. Then, the Perelomov type coherent states in $\mathfrak{H}^N_{D+1}$ can be defined by [17]

$$|N, V\rangle \coloneqq u(V)|N, \mathbf{N}\rangle. \qquad (30)$$

Equivalently, the Perelomov type coherent state $|N, V\rangle$ can also be defined by

$$|N, V\rangle \coloneqq u(-V)|N, \bar{\mathbf{N}}\rangle, \qquad (31)$$

wherein the state $|N, \bar{\mathbf{N}}\rangle \in \mathfrak{H}^N_{D+1}$ corresponds to the lowest weight vector with $\mathbf{N} = \mathbf{M}|_{M_1=\ldots=M_{D-2}=N, M_{D-1}=-N}$. It has been proved that the Perelomov type coherent state $|N, V\rangle$ of $SO(D+1)$ processes well-peakedness properties for the operators τ_{IJ} [17], i.e. minimizes the uncertainty of the expectation value $\langle N, V|\tau_{IJ}|N, V\rangle = \mathbf{i}NV_{IJ}$ and the Heisenberg uncertainty relation of the operators τ_{IJ}: the inequality

$$(\Delta\langle\tau_{IJ}\rangle)^2(\Delta\langle\tau_{KL}\rangle)^2 \geq \frac{1}{4}|\langle[\tau_{IJ}, \tau_{KL}]\rangle|^2 \qquad (32)$$

is saturated for the Perelomov type coherent state $|N, V\rangle$, where we used the shorthand $\langle\hat{o}\rangle \equiv \langle N, V|\hat{o}|N, V\rangle$ and $\Delta\langle\hat{o}\rangle \equiv \sqrt{\langle\hat{o}^2\rangle - \langle\hat{o}\rangle^2}$. The family of Perelomov type coherent states $\{|N, V\rangle\}$ also composes an over-complete basis of $\mathfrak{H}^N_{D+1}$, which reads

$$\dim(\mathfrak{H}^N_{D+1}) \int_{Q_{D-1}} dV |N, V\rangle\langle N, V| = \mathbb{I}_{\mathfrak{H}^N_{D+1}}, \qquad (33)$$

where $\int_{Q_{D-1}} dV = 1$ with dV being the invariant measure on Q_{D-1} induced by the Haar measure on $SO(D+1)$. One can also check the nonorthogonal property of this type of coherent state, that is, the coherent states $|N, V\rangle$ and

$|N, V'\rangle$ with $V \neq V'$ are not mutually orthogonal unless $[V^{IJ}\tau_{IJ}, V'^{KL}\tau_{KL}] = 0$. This means that we have

$$0 \leq |\langle N, V|N, V'\rangle| \leq 1 \tag{34}$$

with $\langle N, V|N, V'\rangle = 1$ if $V = V'$, $\langle N, V|N, V'\rangle = 0$ if $[V^{IJ}\tau_{IJ}, V'^{KL}\tau_{KL}] = 0$ and $V \neq V'$. Moreover, it is worth introducing is the matrix element $\langle N, V|\tau_{IJ}|N, V'\rangle$ of the operator τ_{IJ} in the Perelomov type coherent state basis. Let us take $V'^{KL} = 2\delta_1^{[K}\delta_2^{L]}$ without loss of generality, then we have

$$\langle N, V|\tau_{12}|N, V'\rangle = \mathbf{i}N\langle N, V|N, V'\rangle, \tag{35}$$

$$\langle N, V|\tau_{IJ}|N, V'\rangle = 0, \quad \text{for } I, J \neq 1, 2, \tag{36}$$

$$\langle N, V|\tau_{IJ}|N, V'\rangle = N\langle 1, V|\tau_{IJ}|1, V'\rangle\langle N-1, V|N-1, V'\rangle, \quad \text{for } I \in \{1, 2\} \text{ and } J \neq 1, 2, \tag{37}$$

where $\langle 1, V|\tau_{IJ}|1, V'\rangle = 0$ if $V = V'$ with $I \in \{1, 2\}$ and $J \neq 1,\ 2$. We are interested in the derivatives of $\langle 1, V|\tau_{IJ}|1, V'\rangle$ with $I \in \{1, 2\}$ and $J \neq 1,\ 2$, which can be evaluated by

$$0 \leq |\langle 1, V|\tau_{KL}\tau_{IJ}|1, V'\rangle| \leq 1, \quad \text{for } K, L \in \{1, \ldots, D+1\}, \quad I \in \{1, 2\} \text{ and } J \neq 1, 2. \tag{38}$$

Thus, we can conclude that $\langle 1, V|\tau_{IJ}|1, V'\rangle$ with $I \in \{1, 2\}$ and $J \neq 1, 2$ as functions of V on Q_{D-1} vanish at $V = V'$ and the growth of their modules are restricted by their derivatives evaluated by Eq. (38) as V being transformed by $e^{t\tau_{KL}} \in SO(D+1)$.

Now let us introduce the details of the coherent intertwiner constructed by the Perelomov type coherent state of $SO(D+1)$ at a vertex $v \in \gamma$. Without loss of generality, we reorient the edges linked to v to be outgoing at v in γ. With this setting, the gauge fixed coherent intertwiners, as elements of the tensor product space $\mathcal{H}_v^{\vec{N}_e} \coloneqq \otimes_{b(e)=v}\bar{\mathfrak{H}}^{N_e, D+1}$, are defined as $\check{\mathcal{I}}_v^{\mathrm{c.}}(\vec{N}, \vec{V}) \coloneqq \otimes_{e:b(e)=v}\langle N_e, V_e|$, where $\bar{\mathfrak{H}}^{N_e, D+1}$ is the dual space of homogeneous harmonic functions with degree N_e on the D-sphere and $|N_e, V_e\rangle \coloneqq u(V_e)|N_e, \mathbf{N}_e\rangle$ with $u(V_e)$ being a specific $SO(D+1)$ valued function of V_e satisfying $V_e = u(V_e)\tau_o u(V_e)^{-1}$. Then, the gauge invariant coherent intertwiners $\mathcal{I}_v^{\mathrm{c.}}$ can be defined as the group averaging of $\check{\mathcal{I}}_v^{\mathrm{c.}}$, which means $\mathcal{I}_v^{\mathrm{c.}}(\vec{N}, \vec{V}) \coloneqq \int_{SO(D+1)} dg \otimes_{e:b(e)=v} \langle N_e, V_e|g$. Specifically, the so-called simple coherent intertwiners $\check{\mathcal{I}}_v^{\mathrm{s.c.}}$ (or $\mathcal{I}_v^{\mathrm{s.c.}}$ in gauge invariant case) are defined by requiring $V_e^{[IJ}V_{e'}^{KL]} = 0$ with $b(e) = b(e') = v$ in their definitions. It has been proved that the expectation value of the vertex simplicity constraint operator vanishes with respect to the simple coherent intertwiners, hence they weakly solve the vertex simplicity constraint [42]. Besides, it has been shown that the gauge invariant simple coherent intertwiners can be regarded as quantum D-polytopes in all dimensional LQG [40], hence it is reasonable to weakly solve the anomalous vertex simplicity constraint.

D. Generalized twisted geometry coherent states in all dimensional loop quantum gravity

1. Construction of the coherent states

As mentioned before, the generalized twisted geometry coherent state is considered in all dimensional LQG, instead of the heat-kernel coherent state which is frequently used in $(1+3)$-dimensional $SU(2)$ LQG, since the specific studies of heat-kernel coherent state in all dimensional LQG are confronted with some technical problems. Indeed, the construction of generalized twisted geometry coherent states in all dimensional loop quantum gravity is inspired by the analysis of the heat-kernel coherent state of $SO(D+1)$. The heat-kernel coherent state of $SO(D+1)$ is obtained by the analytic continuation of the solution of the heat equation on $SO(D+1)$, which reads

$$K_t(h, H) = \sum_{\Lambda} \dim(\pi_\Lambda) e^{t\Delta} \chi^{\pi_\Lambda}(hH^{-1}), \tag{39}$$

where t is the time in the heat equation, $h \in SO(D+1)$, $H \in SO(D+1)_{\mathbb{C}} \cong T^*SO(D+1)$ with $SO(D+1)_{\mathbb{C}}$ being the complexification $SO(D+1)$, and $\chi^{\pi_\Lambda}(hH^{-1})$ is the trace of hH^{-1} in the representation π_Λ. The heat-kernel coherent states in the Hilbert space $\mathcal{H}_\gamma$ of all dimensional LQG are given as the product of heat-kernel coherent states associated to each edge $e \in \gamma$, with the heat-kernel time $t = \frac{\kappa\hbar}{a^{D-1}}$. Since the appearance of the simplicity constraint in all dimensional LQG, to simplify the analysis of the properties of the heat-kernel coherent state, one can restrict the representations of holonomies to be the simple ones to vanish the edge simplicity constraint, and the labeling H of heat-kernel coherent states can be restricted to be H^o which takes values in the edge-simple constraint surface $SO(D+1)_{\mathbb{C}}^{\mathrm{s}} \cong T_{\mathrm{s}}^*SO(D+1)$. This procedures give the simple heat-kernel coherent states of $SO(D+1)$ as

$$K_t(h, H^o) = \sum_N \dim(\pi_N) e^{-N(N+D-1)t} \chi^{\pi_N}(hH^{o-1}), \tag{40}$$

where π_N denotes the simple representation of $SO(D+1)$ labeled by the non-negative integer N.

The analysis of this simple heat-kernel coherent state follows a decomposition of the element $H^o \in T_{\mathrm{s}}^*SO(D+1)$. Following the polar decomposition of $SO(D+1)_{\mathbb{C}}$, an element $H^o \in SO(D+1)_{\mathbb{C}}^{\mathrm{s}}$ can be rewritten as

$$H^o = g\exp(\mathbf{i}\eta\tau_o)\tilde{g}^{-1}, \tag{41}$$

where η is a positive real number, g and $\tilde{g}$ are two independent $SO(D+1)$ group elements. Further, let us choose and fix two Hopf sections $u(V): V \mapsto u(V) \in SO(D+1)$ and $\tilde{u}(\tilde{V}): \tilde{V} \mapsto \tilde{u}(\tilde{V}) \in SO(D+1)$. Then, an arbitrary element $g \in SO(D+1)$ or $\tilde{g} \in SO(D+1)$ can be uniquely decomposed as

$$g = u(V)e^{\phi\tau_o}\bar{g} \quad \text{or} \quad \tilde{g} = \tilde{u}(\tilde{V})e^{\tilde{\phi}\tau_o}\bar{\tilde{g}} \tag{42}$$

with an angle ϕ or $\tilde{\phi}$, an element $\bar{g}$ or $\bar{\tilde{g}}$ of $SO(D-1)$ preserving τ_o and an unit bivector $V \in Q_{D-1}$ or $\tilde{V} \in Q_{D-1}$ satisfying $V = u(V)\tau_o u^{-1}(V)$ or $\tilde{V} = -\tilde{u}(\tilde{V})\tau_o\tilde{u}^{-1}(\tilde{V})$. Based on these expressions, H^o is finally decomposed as

$$\begin{aligned} H^o &= u(V)e^{\phi\tau_o}\bar{g}\exp(\mathbf{i}\eta\tau_o)e^{-\tilde{\phi}\tau_o}\bar{\tilde{g}}^{-1}\tilde{u}^{-1}(\tilde{V}) \\ &= u(V)\bar{g}\bar{\tilde{g}}^{-1}\exp(z\tau_o)\tilde{u}^{-1}(\tilde{V}), \end{aligned} \tag{43}$$

where $z = (\phi - \tilde{\phi}) + \mathbf{i}\eta =: \xi^o + \mathbf{i}\eta$, $\bar{g}, \bar{\tilde{g}} \in SO(D-1)$, $u(V), \tilde{u}(\tilde{V}) \in Q_{D-1}$. It is easy to see that this decomposition recovers the twisted geometry parametrization of $T^*_{\rm s}SO(D+1)$ by $(\eta, V, \tilde{V}, \xi^o, \bar{\xi}^\mu)$ introduced in Sec. II, with $\bar{g}\bar{\tilde{g}}^{-1} = e^{\bar{\xi}^\mu\bar{\tau}_\mu}$. Making use of this decomposition, one can consider the large η_e limit of the $SO(D+1)$ heat-kernel coherent state $K_t(h_e, H^o_e)$ constructed for a given edge $e \in \gamma$. Let us focus on the curious cases with $\eta_e \gg 1$. In this case, by choosing a proper basis, the matrix of $\exp(-z_e\tau_o)$ appearing in the decomposition of H^{o-1}_e can be simplified as

$$\begin{aligned} &\langle N_e, \mathbf{M}|\exp(-z_e\tau_o)|N_e, \mathbf{M}'\rangle \\ &\quad = \delta^{\mathbf{M}}_{\mathbf{M}'}e^{-\mathbf{i}z_eM_{D-1}} \\ &\quad = \delta^{\mathbf{M}}_{\mathbf{M}'}\exp(\eta_eN_e)(\delta_{\mathbf{M},\mathbf{N}_e}e^{-\mathbf{i}\xi^o_eN_e} + \mathcal{O}(e^{-\eta_e})), \end{aligned} \tag{44}$$

where $\mathbf{N}_e = \mathbf{M}|_{M_1=\ldots=M_{D-1}=N_e}$. Hence, we get the approximation

$$\begin{aligned} &\sum_{\mathbf{M},\mathbf{M}'}|N_e, \mathbf{M}\rangle\langle N_e, \mathbf{M}|\exp(-z_e\tau_o)|N_e, \mathbf{M}'\rangle\langle N_e, \mathbf{M}'| \\ &\quad \approx e^{\eta_eN_e}e^{-\mathbf{i}\xi^o_eN_e}|N_e, \mathbf{N}_e\rangle\langle N_e, \mathbf{N}_e|. \end{aligned} \tag{45}$$

Now let us insert Eq. (45) into H^{o-1}_e. Notice that $\bar{g}_e\bar{\tilde{g}}^{-1}_e$ fixes $|N_e, \mathbf{N}_e\rangle$ as $\bar{g}_e\bar{\tilde{g}}^{-1}_e|N_e, \mathbf{N}_e\rangle = |N_e, \mathbf{N}_e\rangle$. Then we have

$$\begin{aligned} K_t(h_e, H^o_e) &\overset{\text{large}\,\eta_e}{=} \tilde{\Psi}_{\mathbb{H}^o_e}(h_e) \\ &= \sum_{N_e}\dim(\pi_{N_e})e^{-tN_e(N_e+D-1)} \\ &\quad \times e^{(\eta_e-\mathbf{i}\xi^o_e)N_e}\langle N_e, \mathbf{N}|u^{-1}_eh_e\tilde{u}_e|N_e, \mathbf{N}\rangle, \end{aligned} \tag{46}$$

where we define $\mathbb{H}^o_e := (\eta_e, \xi^o_e, V_e, \tilde{V}_e)$. In fact, the state $\tilde{\Psi}_{\mathbb{H}^o_e}(h_e)$ is just the superposition type coherent state on an edge e in all dimensional LQG [16]. It has been shown that the superposition type coherent state $\tilde{\Psi}_{\gamma,\vec{\mathbb{H}}^o}$ on the graph γ in all dimensional LQG provides a resolution of identity of the space $\mathcal{H}^{\rm s}_\gamma$ if the range of labeling η_e is extended to be $\mathbb{R}$, with $\mathcal{H}^{\rm s}_\gamma$ being the space spanned by the spin-network functions constructed on γ and labeled by simple representations on their edges. Additionally, the peakedness property of this coherent state is studied based on the simplest one loop graph [16]. However, one finds that only a fraction of the superposition type coherent states in the overcomplete basis of $\mathcal{H}^{\rm s}_\gamma$ have well-behaved peakedness property, which leads that this type of coherent states is not applied in many specific calculations. Hence, an improvement on the construction of the coherent state based on twisted geometry parametrization is desired.

In fact, the superposition type coherent state is given by selecting the terms corresponding to the highest weight vector of representation of $SO(D+1)$ in the simple heat-kernel coherent state. These terms give the superpositions over quantum numbers and holonomy matrix element selected by the Perelomov type coherent state of $SO(D+1)$. Inspired by the twisted geometric coherent state in the $(1+3)$-dimensional $SU(2)$ LQG [5], we propose the generalized twisted geometry coherent states in all dimensional LQG, which contains the terms corresponding to both the highest and lowest weight vector of representation of $SO(D+1)$ in the simple heat-kernel coherent state. This generalized twisted geometry coherent state in all dimensional LQG was introduced in our companion paper [1] first, which is given by

$$\breve{\Psi}_{\gamma,\vec{\mathbb{H}}^o_e}(\vec{h}_e) := \prod_e\breve{\Psi}_{\mathbb{H}^o_e}(h_e) \tag{47}$$

with

$$\begin{aligned} \breve{\Psi}_{\mathbb{H}^o_e}(h_e) := &\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}e^{-tN_e(N_e+D-1)} \\ &\times(e^{(\eta_e-\mathbf{i}\xi^o_e)(N_e+\frac{D-1}{2})}\langle N_e, \mathbf{N}|u^{-1}_eh_e\tilde{u}_e|N_e, \mathbf{N}\rangle \\ &+ e^{(-\eta_e+\mathbf{i}\xi^o_e)(N_e+\frac{D-1}{2})}\langle N_e, \bar{\mathbf{N}}|u^{-1}_eh_e\tilde{u}_e|N_e, \bar{\mathbf{N}}\rangle). \end{aligned} \tag{48}$$

This coherent state associated to edge e can also be reformulated as

$$\breve{\Psi}_{\mathbb{H}_e^o}(h_e) := \sum_{N_e} (\dim(\pi_{N_e}))^{3/2} e^{\frac{(\eta_e)^2 + t^2(D-1)^2}{4t}} \left(\exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2\right) e^{-\mathbf{i}\xi_e^o d_{N_e}} \langle N_e, \mathbf{N} | u_e^{-1} h_e \tilde{u}_e | N_e, \mathbf{N} \rangle + \exp\left(-t\left(\frac{\eta_e}{2t} + d_{N_e}\right)^2\right) e^{\mathbf{i}\xi_e^o d_{N_e}} \langle N_e, \bar{\mathbf{N}} | u_e^{-1} h_e \tilde{u}_e | N_e, \bar{\mathbf{N}} \rangle \right) \tag{49}$$

where $d_{N_e} \equiv (N_e + \frac{D-1}{2})$, $\mathbb{H}_e^o := (V_e, \tilde{V}_e, \xi_e^o, \eta_e)$ are the twisted geometry parameters, η_e represents the module of the dimensionless flux X_e, and $t \equiv \frac{\kappa\hbar}{a^{D-1}}$. It is easy to see that the first term in the right-hand side of Eq. (48) is identical with the right-hand side of Eq. (46) up to some prefactors, and the second term in the right-hand side of Eq. (48) vanishes in the large η_e limit; see more details in [1].

2. Resolution of the identity

The system of the twisted geometry coherent state spans an overcomplete basis of the solution space of the edge simplicity constraint. Denoted by $\mathcal{H}_\gamma^s$ the space is composed by the spin-network functions constructed on γ and labeled by simple representations on their edges. Then, the system of the generalized twisted geometry coherent state provides a resolution of the identity in $\mathcal{H}_\gamma^s$, which reads

$$\mathbb{1}_{\mathcal{H}_\gamma^s} = \int_{\times_{e\in\gamma}\check{P}_e} d\vec{\mathbb{H}}_e^o |\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}\rangle\langle\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}|, \tag{50}$$

wherein $\check{P}_e := (\mathbb{R}_+ \times S^1 \times Q_{D-1} \times Q_{D-1})_e$ is the space of the twisted geometry parameters $\mathbb{H}_e^o$, and the measure $d\vec{\mathbb{H}}^o$ is defined by

$$d\vec{\mathbb{H}}_e^o := \prod_{e\in\gamma} \frac{d\eta_e}{\sqrt{2\pi t}} e^{-\frac{\eta_e^2 + t^2(D-1)^2}{2t}} \prod_{e\in\gamma} du_e d\tilde{u}_e \prod_{e\in\gamma} \frac{d\xi_e^o}{2\pi}, \tag{51}$$

where $d\eta_e$ is the Lebesgue measure on $\mathbb{R}$, $d\xi_e^o$ is the measure on S^1, and du_e or $d\tilde{u}_e$ is the measure on Q_{D-1}. We are also interested in the Hilbert space $\mathcal{H}_e^s$ spanned by the spin-network functions constructed on a single edge e and labeled by simple representations. The twisted geometry coherent state associated to edge e also provides an overcomplete basis of $\mathcal{H}_e^s$, which reads

$$\mathbb{1}_{\mathcal{H}_e^s} = \int_{\check{P}_e} d\mathbb{H}_e^o |\breve{\Psi}_{\mathbb{H}_e^o}\rangle\langle\breve{\Psi}_{\mathbb{H}_e^o}|, \tag{52}$$

wherein the measure $d\mathbb{H}_e^o$ is defined by

$$d\mathbb{H}_e^o := \frac{d\eta_e}{\sqrt{2\pi t}} e^{-\frac{\eta_e^2 + t^2(D-1)^2}{2t}} du_e d\tilde{u}_e \frac{d\xi_e^o}{2\pi}. \tag{53}$$

Though the terms corresponding to lowest weights in the twisted geometry coherent states (47) are exponentially suppressed, they still play a key role in the resolution of the identity in $\mathcal{H}_\gamma^s$. Nevertheless, the terms corresponding to lowest weights in the twisted geometry coherent states (47) will be neglected in the following analysis of this paper, since they always contribute exponentially suppressed small terms to the results in our discussion.

3. The overlap function of the coherent states

Notice that the twisted geometry coherent state $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$ on γ is the product of the twisted geometry coherent state $\breve{\Psi}_{\mathbb{H}_e^o}(h_e)$ on each edge $e \in \gamma$. Thus the overlap function for $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$ can be given by

$$i^t((\gamma, \vec{\mathbb{H}}_e^o), (\gamma, \vec{\mathbb{H}}_e'^o)) := \frac{|\langle\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}|\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e'^o}\rangle|^2}{||\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}||^2 \cdot ||\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e'^o}||^2} = \prod_{e\in\gamma} i^t(\mathbb{H}_e^o, \mathbb{H}_e'^o) \tag{54}$$

with

$$i^t(\mathbb{H}_e^o, \mathbb{H}_e'^o) := \frac{|\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\vec{\mathbb{H}}_e'^o}\rangle|^2}{||\breve{\Psi}_{\mathbb{H}_e^o}||^2 ||\breve{\Psi}_{\vec{\mathbb{H}}_e'^o}||^2} \tag{55}$$

being the overlap function for the coherent state $\breve{\Psi}_{\mathbb{H}_e^o}(h_e)$ on an edge e, where

$$||\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}||^2 := |\langle\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}|\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}\rangle|^2 \tag{56}$$

and

$$||\breve{\Psi}_{\mathbb{H}_e^o}||^2 := |\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle|^2 \tag{57}$$

are the module squares of $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$ and $\breve{\Psi}_{\mathbb{H}_e^o}(h_e)$, respectively. In the following calculations and analysis, we will only consider $i^t(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ without loss of generality to simplify our expressions.

We first find that

$$||\breve{\Psi}_{\mathbb{H}_e^o}||^2 \overset{\text{large}\,\eta_e}{=} \sqrt{\frac{\pi}{2t}} e^{\frac{(\eta_e)^2 + t^2(D-1)^2}{2t}} \left(\check{\mathrm{P}}\left(\frac{\eta_e}{2t}\right)\right)^2 \times \left(1 + \mathcal{O}\left(e^{-\frac{1}{t}}\right) + \mathcal{O}\left(\frac{t}{\eta_e}\right)\right) \tag{58}$$

with $\check{\mathrm{P}}(N) = \dim(\pi_N)$ is a polynomial of N. Notice that $\langle N_e, V_e' | N_e, V_e\rangle = 0$ or $\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}_e'\rangle = 0$ leads to

$\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle = 0$. Hence, we only consider the case of $\langle N_e, V'_e | N_e, V_e \rangle \neq 0$ and $\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle \neq 0$ in the following part of this paper. Then, we have

$$\begin{aligned}
&\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle e^{-\frac{(\eta_e)^2 + (\eta'_e)^2 + 2t^2(D-1)^2}{4t}} \\
&\quad = e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o - \xi'^o_e)} \sum_{N_e} (\dim(\pi_{N_e}))^2 \\
&\qquad \times \exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\qquad \cdot e^{\mathbf{i}N_e(\xi_e^o - \xi'^o_e + \varphi(u_e, u'_e) + \varphi(\tilde{u}_e, \tilde{u}'_e))} \exp(-N_e \tilde{\Theta}_e) + \frac{1}{\sqrt{t}} \mathcal{O}\left(e^{-\frac{\eta'^2_e}{8t}}\right)
\end{aligned} \tag{59}$$

for large η'_e, where $\tilde{\Theta}_e \coloneqq \Theta(u_e, u'_e) + \Theta(\tilde{u}_e, \tilde{u}'_e)$ and we used the invention

$$\langle N_e, V'_e | N_e, V_e \rangle = \exp\left(-N_e \Theta(u_e, u'_e)\right) e^{\mathbf{i}N_e \varphi(u_e, u'_e)}, \tag{60}$$

$$\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle = \exp\left(-N_e \Theta(\tilde{u}_e, \tilde{u}'_e)\right) e^{\mathbf{i}N_e \varphi(\tilde{u}_e, \tilde{u}'_e)}, \tag{61}$$

where $\Theta(u_e, u'_e) \coloneqq -\frac{\ln|\langle N_e, V'_e | N_e, V_e \rangle|}{N_e} \geq 0$, $\Theta(\tilde{u}_e, \tilde{u}'_e) \coloneqq -\frac{\ln|\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle|}{N_e} \geq 0$ with $\Theta(u_e, u'_e) = 0$, $\Theta(\tilde{u}_e, \tilde{u}'_e) = 0$ for $V_e = V'_e$, $\tilde{V}_e = \tilde{V}'_e$ respectively, and $e^{\mathbf{i}N_e \varphi(u_e, u'_e)} \coloneqq \frac{\langle N_e, V'_e | N_e, V_e \rangle}{|\langle N_e, V'_e | N_e, V_e \rangle|}$, $e^{\mathbf{i}N_e \varphi(\tilde{u}_e, \tilde{u}'_e)} \coloneqq \frac{\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle}{|\langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle|}$ with $\varphi(u_e, u'_e) = 0$, $\varphi(\tilde{u}_e, \tilde{u}'_e) = 0$ for $V_e = V'_e$, $\tilde{V}_e = \tilde{V}'_e$ respectively. In order to study Eq. (59), the cases of $\tilde{\Theta}_e \ll \eta_e + \eta'_e$ and $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$ are considered respectively.

(i) For the case of $\tilde{\Theta}_e \ll \eta_e + \eta'_e$, the overlap function is expressed as

$$\begin{aligned}
i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e) &\coloneqq \frac{|\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle|^2}{||\breve{\Psi}_{\mathbb{H}_e^o}||^2 ||\breve{\Psi}_{\mathbb{H}'^o_e}||^2} \\
&= \frac{(f_{\text{Poly}}(\frac{\eta'_e}{t}, \frac{\eta_e}{t}, \frac{\tilde{\Theta}_e}{t}))^2}{(\breve{\text{P}}(\frac{\eta_e}{2t}))^2 (\breve{\text{P}}(\frac{\eta'_e}{2t}))^2} e^{-2t(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2 + 4t(\frac{\eta'_e}{4t} - \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t})^2} e^{-2(\frac{\eta_e}{2t} - \frac{D-1}{2})\tilde{\Theta}_e} \exp\left(-\frac{(\xi_e^o - \xi'^o_e + \tilde{\varphi}_e)^2}{4t}\right) \\
&\quad \cdot \left(1 + \mathcal{O}\left(\frac{t}{\eta'_e}\right) + \mathcal{O}(e^{-\frac{1}{t}})\right)
\end{aligned} \tag{62}$$

for large η'_e and $\tilde{\Theta}_e \ll \eta_e + \eta'_e$, where $f_{\text{Poly}}(\frac{\eta'_e}{t}, \frac{\eta_e}{t}, \frac{\tilde{\Theta}_e}{t})$ is a polynomial of the three variables $\frac{\eta'_e}{t}, \frac{\eta_e}{t}, \frac{\tilde{\Theta}_e}{t}$ which satisfies

$$f_{\text{Poly}}\left(\frac{\eta'_e}{t}, \frac{\eta_e}{t}, \frac{\tilde{\Theta}_e}{t}\right) = \left(\breve{\text{P}}\left(\frac{\eta'_e}{4t} + \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t}\right)\right)^2 \left(1 + \mathcal{O}\left(\frac{t}{\eta'_e}\right)\right) \tag{63}$$

for large η'_e and $\tilde{\Theta}_e \ll \eta_e + \eta'_e$. Then one can conclude the peakedness property of the overlap function in the case of $\tilde{\Theta}_e \ll \eta_e + \eta'_e$. For the overlap function $i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e)$ given by Eq. (62), one first finds that it is sharply peaked at $\tilde{\Theta}_e = 0$ by the factor $e^{-2(\frac{\eta_e}{2t} - \frac{D-1}{2})\tilde{\Theta}_e}$. Notice that $\tilde{\varphi}_e = 0$ if $\tilde{\Theta}_e = 0$ by their definition, one can further conclude that the overlap function $i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e)$ is sharply peaked at $\xi_e^o = \xi'^o_e$ and $\eta_e = \eta'_e$ by the factors $\exp(-\frac{(\xi_e^o - \xi'^o_e + \tilde{\varphi}_e)^2}{4t})$ and $e^{-2t(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2 + 4t(\frac{\eta'_e}{4t} - \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t})^2}$ respectively.

(ii) For the case of $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$, the overlap function is expressed as

$$\begin{aligned}
i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e) &\coloneqq \frac{|\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle|^2}{||\breve{\Psi}_{\mathbb{H}_e^o}||^2 ||\breve{\Psi}_{\mathbb{H}'^o_e}||^2} \\
&\lesssim \frac{\left(\sqrt{\frac{2t}{\pi}}(e^{-t((\frac{\eta_e}{2t})^2 + (\frac{\eta'_e}{2t})^2)} + f(\eta_e, \eta'_e) e^{-t(\frac{\eta_e}{4t})^2} e^{-\tilde{\Theta}_e}) + (\breve{\text{P}}(\frac{\eta_e}{4t} + \frac{\eta'_e}{4t}))^2 e^{-\frac{t}{2}(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2} e^{-[\frac{\eta_e}{4t}]\tilde{\Theta}_e}\right)^2}{(\breve{\text{P}}(\frac{\eta_e}{2t}))^2 (\breve{\text{P}}(\frac{\eta'_e}{2t}))^2}
\end{aligned} \tag{64}$$

for large η'_e, where $f(\eta_e, \eta'_e) = [\eta_e/4t] \exp(-t(\frac{\eta'_e}{2t} - \frac{\eta_e}{4t} - \frac{D+1}{2})^2)(\breve{\text{P}}(\frac{\eta_e}{4t}))^2$. Note that we considered $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$ here; it is obviously that the overlap function $i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e)$ is suppressed exponentially by the factors $e^{-t((\frac{\eta_e}{2t})^2 + (\frac{\eta'_e}{2t})^2)}$, $e^{-t(\frac{\eta_e}{4t})^2}$ and $e^{-[\frac{\eta_e}{4t}]\tilde{\Theta}_e}$ in Eq. (64).

Finally, let us combine the analysis of the overlap function $i^t(\mathbb{H}_e^o, \mathbb{H}'^o_e)$ given by Eqs. (62) and (64); one can conclude the peakedness property that the overlap function $i^t(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ is sharply peaked at $\xi_e^o = \xi_e'^o$, $\eta_e = \eta_e'$, and $V_e = V_e'$, $\tilde{V}_e = \tilde{V}_e'$ for large η_e'.

III. EHRENFEST PROPERTY OF TWISTED GEOMETRY COHERENT STATE

To establish the "Ehrenfest property" of the twisted geometry coherent states, one needs to consider the expectation value of all elementary quantum operators in all dimensional LQG. In fact, for a given graph γ and corresponding Hilbert space $\mathcal{H}_\gamma^s$, every polynomial of the elementary operators $\{\hat{h}_e, \hat{X}_e^{IJ}\}_{e\in\gamma}$ can be reduced to sums of monomials of the form

$$\hat{O}_\gamma = \prod_{e\in\gamma} \hat{O}_e, \tag{65}$$

where the operator $\hat{O}_e$ defined on $\mathcal{H}_\gamma^s$ is a certain polynomial of the elementary operators $\hat{h}_e$, $\hat{X}_e^{IJ}$ on the edge e. The expectation value of $\hat{O}_\gamma$ with respect to the twisted geometry coherent states (47) is given by

$$\frac{\langle \breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o} | \hat{O}_\gamma | \breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o} \rangle}{\| \breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o} \|^2} = \prod_{e\in\gamma} \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{O}_e | \breve{\Psi}_{\mathbb{H}_e^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_e^o} \|^2}. \tag{66}$$

As discussed in [11], it is shown that in order to establish the Ehrenfest property it will be completely sufficient to consider this problem for one copy of the edge only. In the following part of this paper, we will concentrate on the issues on a single edge e.

A. Expectation values of operator monomials

To establish the Ehrenfest property of the twisted geometry coherent state for operator monomials, one needs to calculate the expectation values of operator monomials with respect to the twisted geometry coherent state. In this subsection we will reduce the computation of expectation values of operator monomials to the computation of matrix elements of elementary operators between the twisted geometry coherent states. Recall the completeness relation (52), which reads

$$\mathbb{1}_{\mathcal{H}_e^s} = \int_{\breve{P}_e} d\mathbb{H}_e^o | \breve{\Psi}_{\mathbb{H}_e^o} \rangle \langle \breve{\Psi}_{\mathbb{H}_e^o} |. \tag{67}$$

Let us consider an operator monomial $\hat{O}_e = \hat{O}_{e,1} \dots \hat{O}_{e,n}$ where each of the $\hat{O}_{e,k}$, $k = 1, \dots, n < \infty$ represents one of the elementary operators $\hat{h}_e$, $\hat{X}_e^{IJ}$. Then, by using (52), we can write the expectation value of $\hat{O}_e$ as

$$\begin{aligned} &\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{O}_e | \breve{\Psi}_{\mathbb{H}_e^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_e^o} \|^2} \\ &= \frac{1}{\| \breve{\Psi}_{\mathbb{H}_e^o} \|^2} \int_{\breve{P}_e} d\mathbb{H}_{e,1}^o \dots \int_{\breve{P}_e} d\mathbb{H}_{e,n-1}^o \prod_{k=1}^n \langle \breve{\Psi}_{\mathbb{H}_{e,k-1}^o} | \hat{O}_{e,k} | \breve{\Psi}_{\mathbb{H}_{e,k}^o} \rangle \\ &= \int_{\breve{P}_e} d\mathbb{H}_{e,1}^o \dots \int_{\breve{P}_e} d\mathbb{H}_{e,n-1}^o \left(\prod_{k=1}^{n-1} \| \breve{\Psi}_{\mathbb{H}_{e,k}^o} \|^2 \right) \\ &\quad \times \left(\prod_{k=1}^n \frac{\langle \breve{\Psi}_{\mathbb{H}_{e,k-1}^o} | \hat{O}_{e,k} | \breve{\Psi}_{\mathbb{H}_{e,k}^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_{e,k-1}^o} \| \| \breve{\Psi}_{\mathbb{H}_{e,k}^o} \|} \right), \end{aligned} \tag{68}$$

where we have set $\mathbb{H}_{e,0}^o = \mathbb{H}_{e,n}^o = \mathbb{H}_e^o$. Notice that the quantity

$$j^t(\mathbb{H}_e^o, \mathbb{H}'^o_e) \coloneqq \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle}{\| \breve{\Psi}_{\mathbb{H}_e^o} \| \| \breve{\Psi}_{\mathbb{H}'^o_e} \|} \tag{69}$$

is exponentially small in the sense of a Gaussian needle of width $\sqrt{t}$ unless $\mathbb{H}_e^o = \mathbb{H}_e'^o$ (where it equals unity). Thus, it is conceivable that

$$\frac{\langle \breve{\Psi}_{\mathbb{H}_{e,k-1}^o} | \hat{O}_{e,k} | \breve{\Psi}_{\mathbb{H}_{e,k}^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_{e,k-1}^o} \| \| \breve{\Psi}_{\mathbb{H}_{e,k}^o} \|} \approx \frac{\langle \breve{\Psi}_{\mathbb{H}_{e,k}^o} | \hat{O}_{e,k} | \breve{\Psi}_{\mathbb{H}_{e,k}^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_{e,k}^o} \| \| \breve{\Psi}_{\mathbb{H}_{e,k}^o} \|} j^t(\mathbb{H}_{e,k-1}^o, \mathbb{H}_{e,k}^o). \tag{70}$$

By substituting Eq. (70) into Eq. (68), we would have indeed shown that

$$\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{O}_e | \breve{\Psi}_{\mathbb{H}_e^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_e^o} \|^2} \approx \prod_{k=1}^n \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{O}_{e,k} | \breve{\Psi}_{\mathbb{H}_e^o} \rangle}{\| \breve{\Psi}_{\mathbb{H}_e^o} \|^2}. \tag{71}$$

Thus, in order to prove the desired result (71) it is sufficient to prove (70) together with the precise meaning of "≈". In the following parts of this section, we will calculate and discuss the matrix elements of the elementary holonomy and flux operators in the twisted geometry coherent state basis, to gives a reliable proof of (70).

B. Matrix elements of the elementary operators

Since the expectation values of holonomy and flux operators with respect to twisted geometry coherent states are well evaluated by their corresponding classical values up to $\mathcal{O}(t)$ [1], we can prove (70) by showing that

$$\begin{aligned} &\left| \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{O}_e | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle}{\| \breve{\Psi}_{\mathbb{H}'^o_e} \| \| \breve{\Psi}_{\mathbb{H}_e^o} \|} - O_e(\mathbb{H}'^o_e) \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle}{\| \breve{\Psi}_{\mathbb{H}'^o_e} \| \| \breve{\Psi}_{\mathbb{H}_e^o} \|} \right| \\ &\quad \lesssim t |f_{O_e}(\mathbb{H}_e^o, \mathbb{H}'^o_e)| \cdot \left| \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}'^o_e} \rangle}{\| \breve{\Psi}_{\mathbb{H}'^o_e} \| \| \breve{\Psi}_{\mathbb{H}_e^o} \|} \right|, \end{aligned} \tag{72}$$

with $\hat{O}_e$ representing holonomy operator or flux operator here, $O_e(\mathbb{H}_e'^o)$ being the corresponding classical values of

$\hat{O}_e$ given by $\mathbb{H}_e'^o$, and $f_{O_e}(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ being a function whose growth is always suppressed by $\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}\right|$ exponentially as $|\eta_e-\eta_e'|$, $\tilde{\Theta}_e$ and $|\xi_e^o-\xi_e'^o|$ going large.

We first note that Eq. (72) gives the expectation values of $\hat{O}_e$ with respect to the twisted geometry coherent state if $\mathbb{H}_e^o=\mathbb{H}_e'^o$. Thus, $|f_{O_e}(\mathbb{H}_e^o,\mathbb{H}_e'^o)|$ is small at $\mathbb{H}_e^o=\mathbb{H}_e'^o$ to ensure that $O_e(\mathbb{H}_e'^o)$ gives the expectation values of $\hat{O}_e$ up to $\mathcal{O}(t)$. Then, the proof of Eq. (72) follows three steps of calculations. In the first step, we will consider the actions of $\hat{O}_e$ on the state $|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle$, which involve the actions of holonomy and flux operator on the matrix element functions of h_e selected by Perelomov type coherent state of $SO(D+1)$. In the second step, we will construct the identity

$$\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{O}_e|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle = E_{O_e}(\mathbb{H}_e'^o)\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle \tag{73}$$

with E_{O_e} being a function of $\mathbb{H}_e'^o$. In the construction of this identity, we take advantage of the property of the Gaussian superposition, and utilize the properties of the crucial factor $\langle N_e, V_e'|\tau^{IJ}|N_e, V_e\rangle$ and the Clebsh-Gordan coefficients related to the Perelomov type coherent states of $SO(D+1)$. Then, in the third step, we will prove that $E_{O_e}(\mathbb{H}_e'^o)$ is well evaluated by $O_e(\mathbb{H}_e'^o)$ up to an error controlled by *t*. In the following parts of this subsection and Appendixes C and D, we will show the details and results of the proof of Eq. (72) for holonomy and flux operators respectively.

1. Matrix elements of the flux operator

We consider the matrix elements of the flux operator in the twisted geometry coherent state basis, which are denoted by $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{IJ}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}$. The numerator can be calculated as follows:

$$\begin{aligned}
&\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{IJ}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}}\\
&=-\mathbf{i}\beta t\sum_{N_e}(\dim(\pi_{N_e}))^2\\
&\times\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)\\
&\cdot e^{\mathbf{i}d_{N_e}(\xi_e^o-\xi_e'^o)}\langle N_e,V_e'|\tau^{IJ}|N_e,V_e\rangle\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle+\beta\sqrt{t}\\
&\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}}).
\end{aligned}\tag{74}$$

It is easy to see that the calculation of Eq. (74) is similar to that of Eq. (59), except the appearance of the factor $\langle N_e, V_e'|\tau^{IJ}|N_e, V_e\rangle$. Thus, this calculation can be proceeded with the property of $\langle N_e, V_e'|\tau^{IJ}|N_e, V_e\rangle$ being clarified; see more details in Appendix C. The result of the calculation of Eq. (74) gives the first main result of this paper.

(i) The matrix elements of the flux operators with respect to the twisted geometry coherent states can be estimated by

$$\begin{aligned}
&\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{IJ}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}-\frac{\eta_e'}{2}V'^{IJ}\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}\right|\\
&\overset{\text{large }\eta_e'}{\lesssim} t|f_X(\mathbb{H}_e^o,\mathbb{H}_e'^o)|\cdot\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}\right|,
\end{aligned}\tag{75}$$

where $f_X(\mathbb{H}_e^o,\mathbb{H}_e'^o)$ is a function of $\mathbb{H}_e^o$, $\mathbb{H}_e'^o$ whose growth is always suppressed by $\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{\|\breve{\Psi}_{\mathbb{H}_e'^o}\|\|\breve{\Psi}_{\mathbb{H}_e^o}\|}\right|$ exponentially as $|\eta_e-\eta_e'|$, $\tilde{\Theta}_e$ and $|\xi_e^o-\xi_e'^o|$ going large for large η_e'.

2. Matrix elements of the holonomy operator

The matrix elements of the holonomy operator in the twisted geometry coherent basis will be considered for the cases that $(D+1)$ is even or odd separately. In the case of $(D+1)$ being even, notice that each one of the matrix element of the classical holonomy in the definition representation space of $SO(D+1)$ corresponds to a holonomy operator which acts on the twisted geometry coherent state by multiplying. In order to give a specific holonomy operator, one needs to consider an orthonormal and complete basis $\{|1,V_{\iota\jmath}\rangle|(\iota,\jmath)\in\{(1,2),(2,1),(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}\}$ of the definition representation space of $SO(D+1)$, where $V_{\iota\jmath}$ are the elements in a set of bivectors $\{V_{\iota\jmath}=2\delta_\iota^{[I}\delta_\jmath^{J]}|(\iota,\jmath)\in\{(1,2),(2,1),(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}\}$ in $\mathbb{R}^{D+1}$, and the interpretation of these notations of this basis are explained in Appendix A explicitly. Then, the matrix elements of the classical holonomy h_e in the basis $\{|1,V_{\iota\jmath}\rangle$ of the definition representation space of $SO(D+1)$ can be promoted as holonomy operators as

$$\langle 1,V_{\iota\jmath}|h_e|1,V_{\iota'\jmath'}\rangle\mapsto(\widehat{h_e})_{\iota\jmath,\iota'\jmath'}. \tag{76}$$

For a given twisted geometry coherent state $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$, the label $\vec{\mathbb{H}}_e^o$ assigns classical labels $u_e=u(V_e)$ and $\tilde{u}_e=\tilde{u}(\tilde{V}_e)$ to each edge e. Then, in order to adapt the holonomy operators to the state $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$, let us consider two orthonormal and complete basis $\{\tilde{u}_e|1,V_{\iota\jmath}\rangle\}$ and $\{u_e|1,V_{\iota\jmath}\rangle\}$ of the definition representation space of $SO(D+1)$. These two orthonormal and complete bases select the matrix elements $(u_e^{-1}h_e\tilde{u}_e)_{\iota\jmath,\iota'\jmath'}:=\langle 1,V_{\iota\jmath}|u_e^{-1}h_e\tilde{u}_e|1,V_{\iota'\jmath'}\rangle$ of the classical holomomy h_e, which can be promoted as the holonomy operator

$$(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,\iota'\jmath'} := \langle 1, V_{\iota\jmath}|\widehat{u_e^{-1}h_e\tilde{u}_e}|1, V_{\iota'\jmath'}\rangle \quad (77)$$

and it acts on the coherent state $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}_e^o}$ by multiplying. In the case of $(D+1)$ being odd, we still have the holonomy operators $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,\iota'\jmath'}$ with $(\iota,\jmath)$ and $(\iota',\jmath')\in\{(3,4),(4,3),\ldots,(D-1,D),(D,D-1)\}$. Besides, there are extra holonomy operators $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,(D+1)}$ and $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{(D+1),\iota\jmath}$ with $(\iota,\jmath)\in\{(1,2),(2,1),(3,4),\ldots,(D-1,D),(D,D-1)\}$ in this case, which are defined by

$$(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,(D+1)} := \langle 1, V_{\iota\jmath}|\widehat{u_e^{-1}h_e\tilde{u}_e}|1, \delta_{D+1}\rangle \quad (78)$$

and

$$(\widehat{u_e^{-1}h_e\tilde{u}_e})_{(D+1),\iota\jmath} := \langle 1, \delta_{D+1}|\widehat{u_e^{-1}h_e\tilde{u}_e}|1, V_{\iota\jmath}\rangle \quad (79)$$

respectively, where $|1,\delta_{D+1}\rangle$ is defined in Appendix A.

We show the details of the action of these holonomy operators on the adapting states in Appendix A. Then, one can proceed the calculation of the matrix elements $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\widehat{u_e'^{-1}h_e\tilde{u}_e'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ of the holonomy operators in the twisted geometry coherent state basis; see more details in Appendix D. The results of this calculation give the second main result of this paper as follows.

(ii) The matrix elements of the holonomy operators with respect to the twisted geometry coherent states can be estimated by

$$\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\widehat{u_e'^{-1}h_e\tilde{u}_e'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} - u_e'^{-1}h_e'^s\tilde{u}_e'\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large }\eta_e'}{\lesssim} t|f_h(\mathbb{H}_e^o,\mathbb{H}_e'^o)|\cdot\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|, \quad (80)$$

where $f_h(\mathbb{H}_e^o,\mathbb{H}_e'^o)$ is a function whose growth is always suppressed by $\frac{|\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle|}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ exponentially as $|\eta_e-\eta_e'|$, $\tilde{\Theta}_e$ and $|\xi_e^o-\xi_e'^o|$ going large for large η_e', and $u_e'^{-1}h_e'^s\tilde{u}_e'$ is defined by $\mathbb{H}_e'^o$. The matrix elements of $u_e'^{-1}h_e'^s\tilde{u}_e'$ in the definition representation space of $SO(D+1)$ can be given for the cases that $(D+1)$ is even or odd separately. In the case of $(D+1)$ being even, one has

$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{12,12} = e^{\mathbf{i}\xi_e'^o}, \qquad (u_e'^{-1}h_e'^s\tilde{u}_e')_{21,21} = e^{-\mathbf{i}\xi_e'^o},$$
$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{12,21} = (u_e'^{-1}h_e'^o\tilde{u}_e')_{21,12} = 0 \quad (81)$$

and

$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{12,\iota\jmath} = (u_e'^{-1}h_e'^s\tilde{u}_e')_{21,\iota\jmath} = 0, \quad \text{for } (\iota,\jmath)\neq(1,2) \text{ or } (2,1),$$
$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{\iota\jmath,12} = (u_e'^{-1}h_e'^s\tilde{u}_e')_{\iota\jmath,21} = 0, \quad \text{for } (\iota,\jmath)\neq(1,2) \text{ or } (2,1),$$
$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{\iota\jmath,\iota'\jmath'} = 0, \quad \text{for } (\iota,\jmath)\neq(1,2) \text{ or } (2,1), \quad (82)$$

with $(u_e'^{-1}h_e'^s\tilde{u}_e')_{\iota\jmath,\iota'\jmath'} := \langle 1, V_{\iota\jmath}|u_e'^{-1}h_e'^s\tilde{u}_e'|1, V_{\iota'\jmath'}\rangle$. In the case of $(D+1)$ being even, one still has the components given by Eqs. (81) and (82). Besides, there are some extra components

$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{\iota\jmath,(D+1)} := \langle 1, V_{\iota\jmath}|u_e'^{-1}h_e'^s\tilde{u}_e'|1, \delta_{D+1}\rangle = 0,$$
$$(u_e'^{-1}h_e'^s\tilde{u}_e')_{(D+1),\iota\jmath} := \langle 1, \delta_{D+1}|u_e'^{-1}h_e'^s\tilde{u}_e'|1, V_{\iota\jmath}\rangle = 0 \quad (83)$$

with $(\iota,\jmath)\in\{(1,2),(2,1),(3,4),\ldots,(D-1,D),(D,D-1)\}$.

Here we would like to emphasize that the matrix elements of the holonomy operators $\hat{h}_e$ with respect to the twisted geometry coherent states are not estimated by the corresponding classical holonomies $h_e' = u_e' e^{\bar{\xi}_e'^\mu\bar{\tau}_\mu} e^{\xi_e'^o\tau_o}\tilde{u}_e'^{-1}$, but by the corresponding simplicity resolved holonomies $h_e'^s$ which are independent with the gauge component $e^{\bar{\xi}_e'^\mu\bar{\tau}_\mu}$.

C. Expectation values of nonpolynomial operators

Let us consider the construction of the Ehrenfest property of the twisted geometry coherent state for nonpolynomial operators in this subsection. Similar to that of the heat-kernel coherent state in $SU(2)$ LQG [11], the expectation values of nonpolynomial operators with respect to the twisted geometry coherent state in all dimensional LQG can be studied by reformulating it as the Hamburger moment problem.

Theorem (Hamburger).—Given a sequence of real numbers $a_n\in\mathbb{R}$, $n=0,1,2,\ldots$ a sufficient and necessary condition for the existence of a positive, finite measure $d\rho(x)$ on $\mathbb{R}$ such that the a_n are its moments, that is,

$$a_n = \int_{\mathbb{R}} d\rho(x)x^n \quad (84)$$

is that for arbitrary natural number $0\le M<\infty$ and arbitrary complex numbers z_i, $i=0,\ldots,M$ it holds that

$$\sum_{i,j=0}^{M}\bar{z}_i z_j a_{i+j}\geq 0. \quad (85)$$

The measure is faithful if equality in (85) occurs only for $z_i=0$. Moreover, the measure ρ is unique if there exist constants $\alpha,\beta>0$ such that $|a_n|\le\alpha\beta^n(n!)$ for all n.

The proof of this theorem can be found in Refs. [11,45]. In this section, we consider the operators whose arbitrary powers are densely defined on a common domain. Then, by using the Hamburger theorem, we can extend the Theorem 3.6, Corollary 3.1, and Theorem 3.7 in the Ref. [11], which consider the heat-kernel coherent state in $SU(2)$ LQG, to the case of the twisted geometry coherent state in all dimensional LQG. The result of this extension leads to the following four corollaries which can be used to evaluate the expectation values of nonpolynomial operators on $\mathcal{H}^s_\gamma$ with respect to the twisted geometry coherent state.

Corollary (i).—Consider a self-adjoint operator $\hat{O} = O((\hat{X}^{IJ}_e, \hat{h}_e)_{e\in\gamma})$ on $\mathcal{H}^s_\gamma$ which is constructed from $\{(\hat{X}^{IJ}_e, \hat{h}_e)|e\in\gamma\}$. Let $O = O((X^{IJ}_e, h_e)_{e\in\gamma})$ be its real valued classical counterpart. Define its real valued and simplicity resolved classical counterpart $O^s(\vec{\mathbb{H}}^o) = O((X^{IJ}_e(\mathbb{H}^o_e), h^s_e(\mathbb{H}^o_e))_{e\in\gamma})$ of $\hat{O}$ by replacing (X^{IJ}_e, h_e) with $(X^{IJ}_e(\mathbb{H}^o_e), h^s_e(\mathbb{H}^o_e))$ in the expression of $O = O((X^{IJ}_e, h_e)_{e\in\gamma})$, where $X^{IJ}_e(\mathbb{H}^o_e) = \frac{1}{2}\eta_e V^{IJ}_e$ and $h^s_e(\mathbb{H}^o_e)$ is given by Eqs. (81), (82), and (83). Suppose that for every $n\in\mathbb{N}$

$$\lim_{t\to 0}\langle\hat{O}^n\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = (O^s(\vec{\mathbb{H}}^o))^n, \tag{86}$$

where $\langle\hat{o}\rangle^t_{\gamma,\vec{\mathbb{H}}^o} \coloneqq \langle\phi^t_{\gamma,\vec{\mathbb{H}}^o}|\hat{o}|\phi^t_{\gamma,\vec{\mathbb{H}}^o}\rangle$ with $\vec{\mathbb{H}}^o \coloneqq \{\ldots,\mathbb{H}^o_e,\ldots\}_{e\in\gamma}$ and $\phi^t_{\gamma,\vec{\mathbb{H}}^o} \coloneqq \breve{\Psi}_{\gamma,\vec{\mathbb{H}}^o_e}/||\breve{\Psi}_{\gamma,\vec{\mathbb{H}}^o_e}||$ being the normalized formulation of $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}^o_e}$. Then for arbitrary Borel measurable function f on $\mathbb{R}$ such that $\langle f(\hat{O})^\dagger f(\hat{O})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} < \infty$ we have

$$\lim_{t\to 0}\langle f(\hat{O})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = f(O^s(\vec{\mathbb{H}}^o)). \tag{87}$$

Proof.—This corollary is a direct generalization of the Theorem 3.6 in Ref. [11] which considers the heat-kernel coherent state in $SU(2)$ LQG. Let us give the main idea of this proof as follows. Denoted by $E(x)$, $x\in\mathbb{R}$ the spectral projection of $\hat{O}$. Then, by assumption and the spectral theorem we have

$$\lim_{t\to 0}\int_{\mathbb{R}} d\langle\phi^t_{\gamma,\vec{\mathbb{H}}^o}|E(x)|\phi^t_{\gamma,\vec{\mathbb{H}}^o}\rangle x^n = (O^s(\vec{\mathbb{H}}^o))^n. \tag{88}$$

Define $a_n \coloneqq (O^s(\vec{\mathbb{H}}^o))^n$; it obviously satisfies all the criteria of the Hamburger theorem and we conclude that there exists a measure $d\rho_{\vec{\mathbb{H}}^o}(x)$ on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} d\rho_{\vec{\mathbb{H}}^o}(x)x^n = (O^s(\vec{\mathbb{H}}^o))^n. \tag{89}$$

It is obvious that the Dirac measure $d\rho_{\vec{\mathbb{H}}^o}(x) = \delta_{\mathbb{R}}(x, O^s(\vec{\mathbb{H}}^o))dx$ satisfies (89) and it satisfies the uniqueness part of the criterion by choosing $\alpha = 1, \beta = |O^s(\vec{\mathbb{H}}^o)|$ in the Hamburger theorem. Hence we can conclude that the Dirac measure is the unique solution of this moment problem and it follows that the spectral measure $d\rho^t_{\vec{\mathbb{H}}^o}(x) \coloneqq d\langle\phi^t_{\gamma,\vec{\mathbb{H}}^o}|E(x)|\phi^t_{\gamma,\vec{\mathbb{H}}^o}\rangle$ approaches the Dirac measure in the limit $t\to 0$. Now, for arbitrary Borel measurable function f on $\mathbb{R}$ such that $\langle f(\hat{O})^\dagger f(\hat{O})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} < \infty$, the spectral theorem applies and one can get

$$\langle f(\hat{O})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = \int_{\mathbb{R}} d\rho^t_{\vec{\mathbb{H}}^o}(x)f(x), \tag{90}$$

and then

$$\begin{aligned}\lim_{t\to 0}\langle f(\hat{O})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} &= \lim_{t\to 0}\int_{\mathbb{R}} d\rho^t_{\vec{\mathbb{H}}^o}(x)f(x)\\ &= \int_{\mathbb{R}} dx\delta_{\mathbb{R}}(x, O^s(\vec{\mathbb{H}}^o))f(x)\\ &= f(O^s(\vec{\mathbb{H}}^o)).\end{aligned} \tag{91}$$

This finishes the proof. ■

Corollary (ii).—Consider the self-adjoint, not necessarily commuting, operators

$$\hat{O}_1 = O_1((\hat{X}^{IJ}_e, \hat{h}_e)_{e\in\gamma}), \ldots, \hat{O}_m = O_m((\hat{X}^{IJ}_e, \hat{h}_e)_{e\in\gamma}) \tag{92}$$

on $\mathcal{H}^s_\gamma$ which is constructed from $\{(\hat{X}^{IJ}_e, \hat{h}_e)|e\in\gamma\}$. Let $O^s_1(\vec{\mathbb{H}}^o) = O_1((X^{IJ}_e(\mathbb{H}^o_e), h^s_e(\mathbb{H}^o_e))_{e\in\gamma})$, ..., $O^s_m(\vec{\mathbb{H}}^o) = O_m((X^{IJ}_e(\mathbb{H}^o_e), h^s_e(\mathbb{H}^o_e))_{e\in\gamma})$ be their real valued and simplicity resolved classical counterpart. Suppose that for every $n_k\in\mathbb{N}$

$$\lim_{t\to 0}\left\langle\prod_{k=1}^m \hat{O}^{n_k}_k\right\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = \prod_{k=1}^m O^s_k(\vec{\mathbb{H}}^o)^{n_k}. \tag{93}$$

Then for arbitrary Borel measurable function f on $\mathbb{R}^m$ such that $\langle f(\{\hat{O}_k\}^m_{k=1})^\dagger f(\{\hat{O}_k\}^m_{k=1})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} < \infty$ we have

$$\lim_{t\to 0}\langle f(\{\hat{O}_k\}^m_{k=1})\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = f(\{O^s_k(\vec{\mathbb{H}}^o)\}^m_{k=1}). \tag{94}$$

Proof.—Similar to the proof of Corollary (i), this corollary can be proven directly by using the spectral theorem

$$\begin{aligned}&\lim_{t\to 0}\int_{\mathbb{R}} d^m\langle\phi^t_{\gamma,\vec{\mathbb{H}}^o}|E_1(x_1)\ldots E_m(x_m)|\phi^t_{\gamma,\vec{\mathbb{H}}^o}\rangle\prod_{k=1}^m x^{n_k}_k\\ &\quad= \prod_{k=1}^m O^s_k(\vec{\mathbb{H}}^o)^{n_k}\end{aligned} \tag{95}$$

and the uniqueness part of Hamburger theorem. ■

Corollary (iii).—Consider the self-adjoint, not necessarily commuting, operators $\hat{O}_1 = O_1((\hat{X}^{IJ}_e, \hat{h}_e)_{e\in\gamma})$ and

$\hat{O}_2 = O_2((\hat{X}_e^{IJ}, \hat{h}_e)_{e\in\gamma})$ on $\mathcal{H}_\gamma^s$ which is constructed from $\{(\hat{X}_e^{IJ}, \hat{h}_e)|e\in\gamma\}$. Let $O_1 = O_1((X_e^{IJ}, h_e)_{e\in\gamma})$ and $O_2 = O_2((X_e^{IJ}, h_e)_{e\in\gamma})$ be their real valued classical counterpart, and $O_1^s(\vec{\mathbb{H}}^o) = O_1((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma})$ and $O_2^s(\vec{\mathbb{H}}^o) = O_2((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma})$ be their real valued and simplicity resolved classical counterpart. Suppose that $\hat{O}_1$, $\hat{O}_2$ satisfy the assumption (93) of Corollary (ii), $\hat{O}_1$ is positive semidefinite and

$$\lim_{t\to 0}\frac{\langle[\hat{O}_1, \hat{O}_2]\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} = O_{\{1,2\}}((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma}) \tag{96}$$

with $O_{\{1,2\}}((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma})$ being given by replacing (X_e^{IJ}, h_e) with $(X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))$ in the result of the Poisson bracket $O_{\{1,2\}}((X_e^{IJ}, h_e)_{e\in\gamma})$, where $O_{\{1,2\}}((X_e^{IJ}, h_e)_{e\in\gamma}) \coloneqq \{O_1((X_e^{IJ}, h_e)_{e\in\gamma}), O_2((X_e^{IJ}, h_e)_{e\in\gamma})\}$ is the Poisson bracket between $O_1((X_e^{IJ}, h_e)_{e\in\gamma})$ and $O_2((X_e^{IJ}, h_e)_{e\in\gamma})$. Then for arbitrary rational number $r = m/n$ with m, n integers and $n > 0$, we have

$$\lim_{t\to 0}\frac{\langle[(\hat{O}_1)^r, \hat{O}_2]\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} = O_{\{1(r),2\}}((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma}) \tag{97}$$

with $O_{\{1(r),2\}}((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma})$ being given by replacing (X_e^{IJ}, h_e) with $(X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))$ in the result of the Poisson bracket $O_{\{1(r),2\}}((X_e^{IJ}, h_e)_{e\in\gamma})$, where

$$\begin{aligned} &O_{\{1(r),2\}}((X_e^{IJ}, h_e)_{e\in\gamma}) \\ &\quad \coloneqq \{(O_1((X_e^{IJ}, h_e)_{e\in\gamma}))^r, O_2((X_e^{IJ}, h_e)_{e\in\gamma})\} \end{aligned} \tag{98}$$

is the Poisson bracket between $(O_1((X_e^{IJ}, h_e)_{e\in\gamma}))^r$ and $O_2((X_e^{IJ}, h_e)_{e\in\gamma})$.

Proof.—Following the proof of Theorem 3.7 in Ref. [11], this corollary can be proven similarly by using the completeness relation of the twisted geometry coherent states and applying the Corollary (ii). ■

Corollary (iv).—Consider the self-adjoint, not necessarily commuting, operator $\hat{O}_1 = O_1((\hat{X}_e^{IJ}, \hat{h}_e)_{e\in\gamma})$ on $\mathcal{H}_\gamma^s$ which is constructed from $\{(\hat{X}_e^{IJ}, \hat{h}_e)|e\in\gamma\}$. Let $O_1 = O_1((X_e^{IJ}, h_e)_{e\in\gamma})$ be its real valued classical counterpart, and $O_1^s(\vec{\mathbb{H}}^o) = O_1((X_e^{IJ}(\mathbb{H}_e^o), h_e^s(\mathbb{H}_e^o))_{e\in\gamma})$ be its real valued and simplicity resolved classical counterpart. Suppose that $\hat{O}_1$ is positive semidefinite, and it satisfies the assumption (93) of Corollary (ii) and

$$\lim_{t\to 0}\frac{\langle[\hat{O}_1, \hat{h}_e]\hat{h}_e^{-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} = (\{O_1, h_e\}h_e^{-1})|_{(X_e^{IJ},h_e)=(X_e^{IJ}(\mathbb{H}_e^o),h_e^s(\mathbb{H}_e^o))}. \tag{99}$$

Then for arbitrary rational number $r = m/n$ with m, n integers and $n > 0$, we have

$$\begin{aligned} &\lim_{t\to 0}\frac{\langle[(\hat{O}_1)^r, \hat{h}_e]\hat{h}_e^{-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} \\ &\quad = (\{(O_1)^r, h_e\}h_e^{-1})|_{(X_e^{IJ},h_e)=(X_e^{IJ}(\mathbb{H}_e^o),h_e^s(\mathbb{H}_e^o))}. \end{aligned} \tag{100}$$

Proof.—We have the identity

$$\begin{aligned} \frac{[\hat{O}_1^m, \hat{h}_e]\hat{h}_e^{-1}}{\mathbf{i}t} &= \sum_{k=1}^m \frac{\hat{O}_1^{k-1}[\hat{O}_1, \hat{h}_e]\hat{O}_1^{m-k}\hat{h}_e^{-1}}{\mathbf{i}t} \\ &= \sum_{k=1}^n \frac{\hat{O}_1^{r(k-1)}[\hat{O}_1^r, \hat{h}_e]\hat{O}_1^{r(n-k)}\hat{h}_e^{-1}}{\mathbf{i}t}. \end{aligned} \tag{101}$$

Notice that $\hat{O}_1^{m-k}\hat{h}_e^{-1} = \hat{h}_e^{-1}\hat{O}_1^{m-k} + \mathcal{O}(t)$ and $\hat{O}_1^{r(n-k)}\hat{h}_e^{-1} = \hat{h}_e^{-1}\hat{O}_1^{r(n-k)} + \mathcal{O}(t)$ hold, thus we have

$$\begin{aligned} \lim_{t\to 0}\frac{[\hat{O}_1^m, \hat{h}_e]\hat{h}_e^{-1}}{\mathbf{i}t} &= \lim_{t\to 0}\sum_{k=1}^m \frac{\hat{O}_1^{k-1}[\hat{O}_1, \hat{h}_e]\hat{h}_e^{-1}\hat{O}_1^{m-k}}{\mathbf{i}t} \\ &= \lim_{t\to 0}\sum_{k=1}^n \frac{\hat{O}_1^{r(k-1)}[\hat{O}_1^r, \hat{h}_e]\hat{h}_e^{-1}\hat{O}_1^{r(n-k)}}{\mathbf{i}t}. \end{aligned} \tag{102}$$

Now for any measurable function f, by assumption and Corollary (ii), we know that

$$\langle f(\hat{O}_1)\rangle^t_{\gamma,\vec{\mathbb{H}}^o,\vec{\mathbb{H}}'^o} \coloneqq \langle \phi^t_{\gamma,\vec{\mathbb{H}}^o}|f(\hat{O}_1)|\phi^t_{\gamma,\vec{\mathbb{H}}'^o}\rangle \tag{103}$$

is concentrated at $\vec{\mathbb{H}}^o, \vec{\mathbb{H}}'^o$ as

$$\lim_{t\to 0}\langle f(\hat{O}_1)\rangle^t_{\gamma,\vec{\mathbb{H}}^o,\vec{\mathbb{H}}'^o} = \lim_{t\to 0}\langle f(\hat{O}_1)\rangle^t_{\gamma,\vec{\mathbb{H}}^o}\langle \phi^t_{\gamma,\vec{\mathbb{H}}^o}|\phi^t_{\gamma,\vec{\mathbb{H}}'^o}\rangle. \tag{104}$$

We therefore find

$$\begin{aligned} &m\lim_{t\to 0}\langle \hat{O}_1^{m-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}\frac{\langle[\hat{O}_1, \hat{h}_e]\hat{h}_e^{-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} \\ &\quad = n\lim_{t\to 0}\langle \hat{O}_1^{\frac{m}{n}(n-1)}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}\frac{\langle[\hat{O}_1^r, \hat{h}_e]\hat{h}_e^{-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} \end{aligned} \tag{105}$$

for the expectation value of Eq. (102) by using the completeness relation. Using the assumptions of this corollary we thus find

$$\begin{aligned} &\lim_{t\to 0}\frac{\langle[\hat{O}_1^r, \hat{h}_e]\hat{h}_e^{-1}\rangle^t_{\gamma,\vec{\mathbb{H}}^o}}{\mathbf{i}t} \\ &\quad = \frac{m}{n}((O_1)^{\frac{m}{n}-1}\{O_1, h_e\}h_e^{-1})|_{(X_e^{IJ},h_e)=(X_e^{IJ}(\mathbb{H}_e^o),h_e^s(\mathbb{H}_e^o))} \\ &\quad = (\{(O_1)^r, h_e\}h_e^{-1})|_{(X_e^{IJ},h_e)=(X_e^{IJ}(\mathbb{H}_e^o),h_e^s(\mathbb{H}_e^o))} \end{aligned} \tag{106}$$

as claimed. This finish the proof. ■

By combining these corollaries we reach the main result of this section, that is, at zero order of t, the expectation values of a nonpolynomial operator on $\mathcal{H}^s_\gamma$ with respect to twisted geometry coherent state in all dimensional LQG, is given by its classical correspondence on the reduced phase space with respect to the edge-simplicity constraint. To explain this result, let us consider two examples of the nonpolynomial operator. The first example is the elementary D-volume operator $\hat{V}_v$ at v which takes the formulation

$$\hat{V}_v = \sqrt[D-1]{|\hat{Q}_v|}, \tag{107}$$

wherein $\hat{Q}_v$ is a polynomial of the flux operator $\hat{X}_e$. Following the results of Sec. III B 1, we have

$$\lim_{t\to 0}\langle(\hat{Q}_v)^n\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = (Q_v)^n, \tag{108}$$

where Q_v is given by replacing $\hat{X}^{IJ}_e$ with $X^{IJ}_e(\mathbb{H}^o_e)$ in $\hat{Q}_v$. Then, by using the Corollary (i), we can immediately give the expectation value of $\hat{V}_v$ as

$$\lim_{t\to 0}\langle\hat{V}_v\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = \sqrt[D-1]{|Q_v|}. \tag{109}$$

The second example of the nonpolynomial operator is the dedensitized dual momentum operator, which is given as [39]

$$\underline{\hat{\pi}}_e = -\frac{(D-1)}{\mathbf{i}\beta a^2 t}[\hat{V}_v, \hat{h}_e]\hat{h}_e^{-1}, \qquad b(e) = v. \tag{110}$$

This operator contains the quantum commutator and it appears in the length operator and scalar constraint operator in all dimensional LQG. The expectation value of the dedensitized dual momentum operator with respect to the twisted geometry coherent state can be evaluated by using the above Corollary (iv), which reads

$$\lim_{t\to 0}\langle\underline{\hat{\pi}}_e\rangle^t_{\gamma,\vec{\mathbb{H}}^o} = -\frac{(D-1)}{\beta a^2}(\{\sqrt[D-1]{|Q_v|}, h_e\}h_e^{-1})\Big|_{(X^{IJ}_e,h_e)=(X^{IJ}_e(\mathbb{H}^o_e),h^s_e(\mathbb{H}^o_e))}. \tag{111}$$

IV. CONCLUSION AND DISCUSSION

The coherent state in all dimensional LQG is a necessary tool in the study of the semiclassical limit of this theory. Since the heat-kernel coherent state for $SO(D+1)$ gauge theory is too complicated to proceed the explicit calculations, we construct a new type of coherent state based on the twisted geometry parametrization of the $SO(D+1)$ holonomy-flux phase in all dimensional LQG. The twisted geometry coherent state is given by selecting the terms corresponding to the highest and lowest weight vectors of representation of $SO(D+1)$ in the simple heat-kernel coherent state, and these terms give the superpositions over quantum numbers and holonomy matrix element selected by the Perelomov type coherent state of $SO(D+1)$. With the "Peakedness property" of the twisted geometry coherent state having been studied in our companion paper [1], we show that the Ehrenfest property holds for the twisted geometry coherent state in this paper. In other words, the expectation values of polynomials of the elementary operators as well as the operators which are not polynomial functions of the elementary operators, reproduce, to zeroth order in $t := \frac{\hbar\kappa}{a^2}$, the values of the corresponding classical functions at the twisted geometry space point where the coherent state is peaked. More explicitly, based on the completeness relation and the peakedness property of the twisted geometry coherent state, it is shown that in order to establish Ehrenfest property for polynomials of elementary operators, it is completely sufficient to prove that the matrix elements of holonomy and flux operators in the twisted geometry coherent state basis are estimated by their corresponding classical values up to first order of t. Then, with the Clebsh-Gordan coefficients related to the states in the simple representation space of $SO(D+1)$ being given, we complete this proof by using the properties of the Perelomov type coherent states of $SO(D+1)$ and the Gaussian functions. Besides, it is shown that the expectation values of nonpolynomial operators with respect to twisted geometry coherent state in all dimensional LQG can be reformulated as the Hamburger moment problem. By extending the similar researches for the heat-kernel coherent state in $SU(2)$ LQG, we show that the Ehrenfest property for nonpolynomial operators can be established at zeroth order of t.

It is necessary to have a discussion on the quantum simplicity constraint. The twisted geometry coherent states vanish the edge-simplicity constraint operator and provide an overcomplete basis of the solution space $\bigoplus_\gamma \mathcal{H}^s_\gamma$ of the edge-simplicity constraint. Besides, following the results of the twisted geometry parametrization of $SO(D+1)$ holonomy-flux space, we still need to solve the vertex simplicity constraint weakly. Notice that the vertex simplicity constraint operator $\hat{X}^{[IJ}_e\hat{X}^{KL]}_{e'}$ with $b(e)=b(e')=v$ is a monomial of flux operators; its matrix elements in the twisted geometry coherent state basis, at zeroth order of t, is evaluated by its classical counterpart $X^{[IJ}_e X^{KL]}_{e'}$ which is proportion to $V^{[IJ}_e V^{KL]}_{e'}$. Thus, we claim that the weak solution space of the vertex simplicity constraint can be composed by the twisted geometry coherent states $\breve{\Psi}_{\gamma,\vec{\mathbb{H}}^o_e}$ whose labels $\vec{\mathbb{H}}^o_e$ satisfy $V^{[IJ}_e V^{KL]}_{e'}=0$ with $b(e)=b(e')=v$. With the Ehrenfest property being constructed, we can have an outlook on the application of twisted geometry coherent state in the study of the dynamics of all dimensional LQG. First, one may consider the expectation value of the Hamiltonian constraint operator based on the cosmology model in higher dimensional LQG [41] to study higher dimensional loop quantum cosmology from the perspective of the effective dynamics of full theory. Moreover, one can also explore the effective dynamics of all dimensional LQG based on the coherent state path integral [12,13] to give the effective action and equation of motion with the twisted geometry variables. Therefore, the twisted geometry coherent state provides us a reliable candidate for the study of the effective dynamics of all dimensional LQG.

ACKNOWLEDGMENTS

This work is supported by the project funded by China Postdoctoral Science Foundation with Grant No. 2021M691072, and the National Natural Science Foundation of China (NSFC) with Grants No. 12047519, No. 11875006, and No. 11961131013.

APPENDIX A: THE MATRIX ELEMENTS OF HOLONOMY OPERATOR IN SPIN-NETWORK BASIS

As we mentioned in Sec. III B, the action of the holonomy operator on the twisted geometry coherent state is the first step in the calculation of the matrix element of the holonomy operator. Since the twisted geometry coherent states are some kinds of superpositions of the spin-network states, it is necessary to study the matrix elements of the holonomy operator in the spin-network basis in all dimensional LQG. To calculate the matrix elements of the holonomy operator in the spin-network basis, one first needs to consider the Clebsh-Gordan coefficients related to the states in the simple representation space of $SO(D+1)$. Recall that the space $\mathfrak{H}^N_{D+1}$ of the sphere harmonic function on S^D with degree N is the simple representation space of $SO(D+1)$ labeled by N, and it has the orthonormal basis $\{\Xi^{N,\mathbf{M}}(\boldsymbol{x})\}$ (or $\{|N,\mathbf{M}\rangle\}$ in Dirac bracket formulation). Then, the Clebsh-Gordan coefficient can be given by

$$\begin{aligned}&\langle N',\mathbf{M}';N'',\mathbf{M}''|N,\mathbf{M}\rangle\langle N,\mathbf{0}|N',\mathbf{0};N'',\mathbf{0}\rangle\\ &\quad=\dim(\pi_N)\int_{SO(D+1)}dg\overline{D^N_{(\mathbf{M},\mathbf{0})}(g)}D^{N'}_{(\mathbf{M}',\mathbf{0})}(g)D^{N''}_{(\mathbf{M}'',\mathbf{0})}(g),\end{aligned}\tag{A1}$$

where $\dim(\pi_N)=\frac{(D+N-2)!(2N+D-1)}{(D-1)!N!}$, $|N',\mathbf{M}';N'',\mathbf{M}''\rangle\coloneqq|N',\mathbf{M}'\rangle\otimes|N'',\mathbf{M}''\rangle$ and

$$D^N_{(\mathbf{M},\mathbf{0})}(g)\coloneqq\langle N,\mathbf{M}|g|N,\mathbf{0}\rangle\tag{A2}$$

is the matrix element function on $SO(D+1)$ selected by $|N,\mathbf{M}\rangle$ and $|N,\mathbf{0}\rangle$. Based on Eq. (A1), it is easy to see that

$$\begin{aligned}&|\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle|^2\\ &\quad=\dim(\pi_{N+1})\int_{SO(D+1)}dg\overline{D^{N+1}_{(\mathbf{0},\mathbf{0})}(g)}D^N_{(\mathbf{0},\mathbf{0})}(g)D^1_{(\mathbf{0},\mathbf{0})}(g).\end{aligned}\tag{A3}$$

Moreover, let us note that [43]

$$\begin{aligned}&D^N_{(\mathbf{0},\mathbf{0})}(g)=D^N_{(\mathbf{0},\mathbf{0})}(\theta)=\frac{(D-2)!N!}{(D+N-2)!}C^{\frac{D-1}{2}}_N(\cos\theta),\\ &C^{\frac{D-1}{2}}_1(\cos\theta)=(D-1)\cos\theta,\end{aligned}\tag{A4}$$

and

$$\begin{aligned}C^{\frac{D-1}{2}}_{N+1}(\cos\theta)=&\frac{2N+D-1}{N+1}\cos\theta C^{\frac{D-1}{2}}_N(\cos\theta)\\ &-\frac{N+D-2}{N+1}C^{\frac{D-1}{2}}_{N-1}(\cos\theta).\end{aligned}\tag{A5}$$

Then, we can calculate

$$|\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle|^2=\frac{D+N-1}{2N+D-1},\tag{A6}$$

and similarly we have

$$|\langle N-1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle|^2=\frac{N}{2N+D-1},\tag{A7}$$

Furthermore, the relation between the function $D^N_{(\mathbf{M},\mathbf{0})}(g)$ and $\Xi^{N,\mathbf{M}}(\boldsymbol{x})$ can be found in Ref. [43] and it leads to

$$\begin{aligned}&\langle N',\mathbf{M}';N'',\mathbf{M}''|N,\mathbf{M}\rangle\langle N,\mathbf{0}|N',\mathbf{0};N'',\mathbf{0}\rangle\\ &\quad=\sqrt{\frac{\dim(\pi_N)}{\dim(\pi_{N'})\dim(\pi_{N''})}}\int_{SO(D+1)}d\boldsymbol{x}\overline{\Xi^{N,\mathbf{M}}(\boldsymbol{x})}\\ &\qquad\times\Xi^{N',\mathbf{M}''}(\boldsymbol{x})\Xi^{N'',\mathbf{M}''}(\boldsymbol{x}).\end{aligned}\tag{A8}$$

Now let us turn to considering the functions which are involved in the main part of this article. The normalized harmonic function $c_N(x_\imath+\mathbf{i}x_\jmath)$ can be denoted by $|N,V_{\imath\jmath}\rangle$ with $\imath,\jmath=1,\ldots,D+1$, $\boldsymbol{x}=(x_1,\ldots,x_{D+1})\in S^D$, $V_{\imath\jmath}\coloneqq2\delta^{[I}_\imath\delta^{J]}_\jmath$, and c_N being the normalization factor. A harmonic function basis of the definition representation space of $SO(D+1)$ can be given by

$$\begin{aligned}&(x_1+\mathbf{i}x_2),\quad(x_1-\mathbf{i}x_2),\quad(x_3+\mathbf{i}x_4),\ldots,(x_D+\mathbf{i}x_{D+1}),\\ &(x_D-\mathbf{i}x_{D+1})\end{aligned}\tag{A9}$$

for $D+1$ being even, and

$$(x_1+\mathbf{i}x_2),\quad (x_1-\mathbf{i}x_2),\quad (x_3+\mathbf{i}x_4),\ldots,(x_{D-1}+\mathbf{i}x_D),\quad (x_{D-1}-\mathbf{i}x_D),\quad x_{D+1} \tag{A10}$$

for $D+1$ being odd. These functions can be expressed as the following normalized states by using Dirac bracket notation, which reads

$$\{|1,V_{\imath\jmath}\rangle\},\quad (\imath,\jmath)\in\{(1,2),(2,1),(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\} \tag{A11}$$

for $D+1$ being even, and

$$\{|1,V_{\imath\jmath}\rangle,|1,\delta_{D+1}\rangle\},\quad (\imath,\jmath)\in\{(1,2),(2,1),(3,4),(4,3),\ldots,(D-1,D),(D,D-1)\} \tag{A12}$$

for $D+1$ being odd. First, one can check that

$$\begin{aligned}|1,V_{12};N,V_{12}\rangle &\coloneqq |1,V_{12}\rangle\otimes|N,V_{12}\rangle\\ &= |N+1,V_{12}\rangle.\end{aligned} \tag{A13}$$

Then, one can further calculate some of the other Clebsh-Gordan coefficients. Following Eqs. (A8) and (A13), let us consider

$$\begin{aligned}&\langle N,V_{12};1,V_{12}|N+1,V_{12}\rangle\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle\\ &\quad=\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle\\ &\quad=\sqrt{\frac{\dim(\pi_{N+1})}{(D+1)\cdot\dim(\pi_N)}}\int_{SO(D+1)}d\boldsymbol{x}\overline{\Xi^{N+1,V_{12}}(\boldsymbol{x})}\\ &\qquad\times\Xi^{N,V_{12}}(\boldsymbol{x})\Xi^{1,V_{12}}(\boldsymbol{x}),\end{aligned} \tag{A14}$$

and

$$\begin{aligned}&\langle N+1,V_{12};1,V_{21}|N,V_{12}\rangle\langle N,\mathbf{0}|N+1,\mathbf{0};1,\mathbf{0}\rangle\\ &\quad=\sqrt{\frac{\dim(\pi_N)}{(D+1)\cdot\dim(\pi_{N+1})}}\int_{SO(D+1)}d\boldsymbol{x}\overline{\Xi^{N,V_{12}}(\boldsymbol{x})}\\ &\qquad\times\Xi^{N+1,V_{12}}(\boldsymbol{x})\Xi^{1,V_{21}}(\boldsymbol{x}).\end{aligned} \tag{A15}$$

By substituting (A6) into (A14) we get

$$\begin{aligned}\frac{c_Nc_1}{c_{N+1}}&=\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle\sqrt{\frac{(D+1)\cdot\dim(\pi_N)}{\dim(\pi_{N+1})}}\\ &=e^{-\mathbf{i}\alpha}\sqrt{\frac{(N+1)(D+1)}{2N+D+1}}.\end{aligned} \tag{A16}$$

Notice that $\Xi^{1,V_{21}}(\boldsymbol{x})=\overline{\Xi^{1,V_{12}}(\boldsymbol{x})}$, and then Eqs. (A14) and (A15) give

$$\langle N,V_{12}|N+1,V_{12};1,V_{21}\rangle=\frac{\dim(\pi_N)}{\dim(\pi_{N+1})}\frac{\langle N+1,\mathbf{0}|N,\mathbf{0};1,\mathbf{0}\rangle}{\langle N+1,\mathbf{0};1,\mathbf{0}|N,\mathbf{0}\rangle}. \tag{A17}$$

Let us denote

$$\begin{aligned}|1,V_{21};N,V_{12}\rangle &\coloneqq |1,V_{21}\rangle\otimes|N,V_{12}\rangle\\ &=\alpha_1(N)|N-1,V_{12}\rangle+\alpha_2(N)|N+1,\ldots\rangle\\ &\quad+\alpha_3(N)|\text{not simple}\rangle,\end{aligned} \tag{A18}$$

where $|N+1,\ldots\rangle$ is a state in the simple representation space $\mathfrak{H}^{N+1}_{D+1}$ and $|\text{not simple}\rangle$ is a state not belonging to any simple representation. By using Eq. (A17), one can get

$$\begin{aligned}|\alpha_1(N)|^2&=\left|\frac{\dim(\pi_{N-1})}{\dim(\pi_N)}\frac{\langle N,\mathbf{0}|N-1,\mathbf{0};1,\mathbf{0}\rangle}{\langle N,\mathbf{0};1,\mathbf{0}|N-1,\mathbf{0}\rangle}\right|^2\\ &=\frac{N(2N+D-3)}{(D+N-2)(2N+D-1)}\end{aligned} \tag{A19}$$

and $|\alpha_2(N)|^2\leq 1-|\alpha_1(N)|^2$.

We are also interested in the special Clebsh-Gordan coefficient $\langle 1,V_{\imath\jmath};N,V_{12}|N',V'\rangle$ with $(\imath,\jmath)\in\{(3,4),(4,3),\ldots\}$, which can be given by

$$\begin{aligned}&\langle 1,V_{\imath\jmath};N,V_{12}|N',V'\rangle\langle N',\mathbf{0}|1,\mathbf{0};N,\mathbf{0}\rangle\\ &\quad=\sqrt{\frac{\dim(\pi_{N'})}{(D+1)\cdot\dim(\pi_N)}}\int_{SO(D+1)}d\boldsymbol{x}\overline{\Xi^{N',V'}(\boldsymbol{x})}\\ &\qquad\times\Xi^{1,V_{\imath\jmath}}(\boldsymbol{x})\Xi^{N,V_{12}}(\boldsymbol{x}).\end{aligned} \tag{A20}$$

Notice that we have the relation

$$\begin{aligned}\Xi^{1,V_{\imath\jmath}}(\boldsymbol{x})\Xi^{N,V_{12}}(\boldsymbol{x})&=-\frac{c_Nc_1}{c_{N+1}}\frac{1}{N+1}(\tau^{1\imath}+\tau^{2\jmath})\Xi^{N+1,V_{12}}(\boldsymbol{x}),\\ &\qquad\text{for } \imath<\jmath,\ \text{and } \imath,\jmath\neq 1,2,\end{aligned} \tag{A21}$$

$$\begin{aligned}\Xi^{1,V_{\imath\jmath}}(\boldsymbol{x})\Xi^{N,V_{12}}(\boldsymbol{x})&=-\frac{c_Nc_1}{c_{N+1}}\frac{1}{N+1}(\tau^{1\imath}-\tau^{2\jmath})\Xi^{N+1,V_{12}}(\boldsymbol{x}),\\ &\qquad\text{for } \imath>\jmath,\ \text{and } \imath,\jmath\neq 1,2.\end{aligned} \tag{A22}$$

Then, by using the above equations and substituting (A6) and (A16) into Eq. (A20), we can check that

$$\begin{aligned}&\langle 1,V_{\imath\jmath};N,V_{12}|N',V'\rangle\\ &\quad=-\frac{1}{(N+1)}\langle N',V'|(\tau^{1\imath}+\tau^{2\jmath})|N+1,V_{12}\rangle,\\ &\qquad\text{for } \imath<\jmath,\ \text{and } \imath,\jmath\neq 1,2,\end{aligned} \tag{A23}$$

and

$$\begin{aligned}&\langle 1, V_{\imath\jmath}; N, V_{12}|N', V'\rangle\\&\quad= -\frac{1}{(N+1)}\langle N', V'|(\tau^{1\imath} - \tau^{2\jmath})|N+1, V_{12}\rangle,\\&\qquad \text{for } \imath > \jmath, \text{ and } \imath, \jmath \neq 1, 2. \end{aligned} \tag{A24}$$

Similarly, we can also consider

$$\begin{aligned}&\langle 1, \delta_{D+1}; N, V_{12}|N', V'\rangle\langle N', \mathbf{0}|1, \mathbf{0}; N, \mathbf{0}\rangle\\&\quad= \sqrt{\frac{\dim(\pi_{N'})}{(D+1)\cdot \dim(\pi_N)}}\int_{SO(D+1)} d\boldsymbol{x}\overline{\Xi^{N',V'}(\boldsymbol{x})}\\&\qquad \times \Xi^{1,\delta_{D+1}}(\boldsymbol{x})\Xi^{N,V_{12}}(\boldsymbol{x}),\end{aligned} \tag{A25}$$

where $\Xi^{1,\delta_{D+1}}(\boldsymbol{x}) = \sqrt{D+1}x_{D+1}$. Note that

$$\Xi^{1,\delta_{D+1}}(\boldsymbol{x})\Xi^{N,V_{12}}(\boldsymbol{x}) = -\frac{\sqrt{2}c_1 c_N}{c_{N+1}}\frac{1}{N+1}\tau^{1,D+1}\Xi^{N+1,V_{12}}(\boldsymbol{x}), \tag{A26}$$

with $c_1 = \sqrt{\frac{D+1}{2}}$. Then, we have

$$\begin{aligned}&\langle 1, \delta_{D+1}; N, V_{12}|N', V'\rangle\\&\quad= -\frac{\sqrt{2}}{(N+1)}\langle N', V'|\tau^{1,D+1}|N+1, V_{12}\rangle.\end{aligned} \tag{A27}$$

Now, we can consider the holonomy operator $\hat{h}$ acting on the matrix element function $\Xi^N_{u^{-1},\tilde{u}}(h) \coloneqq \langle N, V_{12}|u^{-1}h\tilde{u}|N, V_{12}\rangle$. For the case of $D+1$ being even, the action of the holonomy operator corresponding to the holonomy component $\langle 1, V_{\imath\jmath}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle$ is given by

$$\begin{aligned}&\langle 1, V_{\imath\jmath}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle \circ \Xi^N_{u^{-1},\tilde{u}}(h)\\&\quad \coloneqq \langle 1, V_{\imath\jmath}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot \Xi^N_{u^{-1},\tilde{u}}(h)\\&\quad= \langle 1, V_{\imath\jmath}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot\langle N, V_{12}|u^{-1}h\tilde{u}|N, V_{12}\rangle\\&\quad= \langle N, V_{12}; 1, V_{\imath\jmath}|u^{-1}h\tilde{u}|N, V_{12}; 1, V_{\imath'\jmath'}\rangle\end{aligned} \tag{A28}$$

with $(\imath,\jmath),(\imath',\jmath') \in \{(1,2),(2,1),\ldots\}$. Based on this action, the matrix elements of the operator $\langle 1, V_{\imath\jmath}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle$ in the basis spanned by the states $|N, u^{-1}, \tilde{u}\rangle$ corresponding to $\Xi^N_{u^{-1},\tilde{u}}(h)$ can be given by

$$\begin{aligned}&\langle N', u'^{-1}, \tilde{u}'|\langle 1, V_{\imath\jmath}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle|N, u^{-1}, \tilde{u}\rangle\\&\quad \coloneqq \int_{SO(D+1)} dh\overline{\langle N', V_{12}|u'^{-1}h\tilde{u}'|N', V_{12}\rangle}\cdot\langle 1, V_{\imath\jmath}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot\langle N, V_{12}|u^{-1}h\tilde{u}|N, V_{12}\rangle\\&\quad= \frac{1}{\dim(\pi_{N'})}\langle 1, V_{\imath\jmath}; N, V_{12}|u^{-1}u'|N', V_{12}\rangle\cdot\langle N', V_{12}|\tilde{u}'^{-1}\tilde{u}|1, V_{\imath'\jmath'}; N, V_{12}\rangle,\end{aligned} \tag{A29}$$

where $\dim(\pi_N) = \frac{(D+N-2)!(2N+D-1)}{(D-1)!N!}$. Then, by using Eqs. (A13), (A18), (A23), (A24), (A27), and (A29) can be further calculated and the results are obvious. For instance, we have

$$\text{Eq. (A29)} = \frac{1}{\dim(\pi_{N+1})}\langle N+1, V_{12}|u^{-1}u'|N', V_{12}\rangle\langle N', V_{12}|\tilde{u}'^{-1}\tilde{u}|N+1, V_{12}\rangle, \quad \text{if } (\imath,\jmath) = (1,2), (\imath',\jmath') = (1,2), \tag{A30}$$

$$\begin{aligned}\text{Eq. (A29)} &= \frac{1}{\dim(\pi_{N-1})}|\alpha_1(N)|^2\langle N-1, V_{12}|u^{-1}u'|N', V_{12}\rangle\langle N', V_{12}|\tilde{u}'^{-1}\tilde{u}|N-1, V_{12}\rangle\\&\quad+\frac{1}{\dim(\pi_{N+1})}|\alpha_2(N)|^2\langle N+1, \ldots|u^{-1}u'|N', V_{12}\rangle\langle N', V_{12}|\tilde{u}'^{-1}\tilde{u}|N+1, \ldots\rangle,\\&\quad \text{if } (\imath,\jmath) = (2,1), (\imath',\jmath') = (2,1),\end{aligned} \tag{A31}$$

and

$$\begin{aligned}\text{Eq. (A29)} &= -\frac{1}{\dim(\pi_{N+1})}\frac{1}{(N+1)^2}\langle N', V_{12}|u'^{-1}u(\tau^{1\imath}\pm\tau^{2\jmath})|N+1, V_{12}\rangle\langle N+1, V_{12}|(\tau^{1\imath'}\pm\tau^{2\jmath'})\tilde{u}^{-1}\tilde{u}'|N', V_{12}\rangle,\\&\quad \text{if } \imath,\jmath,\imath',\jmath', \neq 1, 2,\end{aligned} \tag{A32}$$

where $\tau^{1\imath}\pm\tau^{2\jmath}$ takes $\tau^{1\imath}+\tau^{2\jmath}$ if $\imath < \jmath$, and $\tau^{1\imath}-\tau^{2\jmath}$ if $\imath > \jmath$.

For the case of $D+1$ being odd, the previous discussion of the operator $\langle 1, V_{\imath\jmath}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle$ with $(\imath,\jmath),(\imath',\jmath')\in\{(1,2),(2,1),\ldots\}$ still holds. Besides, we have the extra holonomy operator $\langle 1,\delta_{D+1}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle$ and $\langle 1, V_{\imath\jmath}|\widehat{u^{-1}h\tilde{u}}|1,\delta_{D+1}\rangle$. Let us consider $\langle 1,\delta_{D+1}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle$ as an example, whose action on $\Xi^N_{u^{-1},\tilde{u}}(h)$ is given by

$$\begin{aligned}
&\langle 1,\delta_{D+1}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle\circ\Xi^N_{u^{-1},\tilde{u}}(h)\\
&\quad:=\langle 1,\delta_{D+1}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot\Xi^N_{u^{-1},\tilde{u}}(h)\\
&\quad=\langle 1,\delta_{D+1}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot\langle N, V_{12}|u^{-1}h\tilde{u}|N, V_{12}\rangle\\
&\quad=\langle N, V_{12};1,\delta_{D+1}|u^{-1}h\tilde{u}|N, V_{12};1, V_{\imath'\jmath'}\rangle
\end{aligned}\tag{A33}$$

with $(\imath,\jmath),(\imath',\jmath')\in\{(1,2),(2,1),\ldots\}$. Then, similar to Eq. (A29), we have

$$\begin{aligned}
&\langle N', u'^{-1},\tilde{u}'|\langle 1,\delta_{D+1}|\widehat{u^{-1}h\tilde{u}}|1, V_{\imath'\jmath'}\rangle|N, u^{-1},\tilde{u}\rangle\\
&:=\int_{SO(D+1)}dh\overline{\langle N', V_{12}|u'^{-1}h\tilde{u}'|N', V_{12}\rangle}\cdot\langle 1,\delta_{D+1}|u^{-1}h\tilde{u}|1, V_{\imath'\jmath'}\rangle\cdot\langle N, V_{12}|u^{-1}h\tilde{u}|N, V_{12}\rangle\\
&=\frac{1}{\dim(\pi_{N'})}\langle 1,\delta_{D+1};N, V_{12}|u^{-1}u'|N', V_{12}\rangle\cdot\langle N', V_{12}|\tilde{u}'^{-1}\tilde{u}|1, V_{\imath'\jmath'};N, V_{12}\rangle\\
&=-\frac{1}{\dim(\pi_{N+1})}\frac{\sqrt{2}}{(N+1)}\delta_{N',N+1}\langle N+1, V_{12}|u'^{-1}u\tau^{1,D+1}|N+1, V_{12}\rangle\cdot\langle N+1, V_{12}|\tilde{u}'^{-1}\tilde{u}|1, V_{\imath'\jmath'};N, V_{12}\rangle.
\end{aligned}\tag{A34}$$

By using Eqs. (A13), (A18), (A23), (A24), and (A34) can be further calculated and it will be used in the calculation of the matrix elements of holonomy operator in the twisted geometry coherent basis.

APPENDIX B: POISSON SUMMATION FORMULA AND RELEVANT CALCULATIONS

As we mentioned in Sec. III B, the second step of the calculation of the matrix elements of holonomy and flux operators involves the property of the Gaussian superpositions. Usually, the property of Gaussian superpositions can be analyzed by using the Poisson summation formula. Let us introduce it and show three cases of its application scenarios in this section. Let f be a function in $L_1(\mathbb{R}, dx)$ such that the series

$$\phi(y)=\sum_{n=-\infty}^{\infty}f(y+ns)\tag{B1}$$

is absolutely and uniformly convergent for $y\in[0,s]$, $s>0$. Then

$$\sum_{n=-\infty}^{\infty}f(ns)=\frac{2\pi}{s}\sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{s}\right),\tag{B2}$$

where $\tilde{f}(k):=\int_{\mathbf{R}}\frac{dx}{2\pi}e^{-\mathbf{i}kx}f(x)$ is the Fourier transform of f. The proof of this theorem can be found in [46]. In this paper, the application of the Poisson summation formula is involved in the calculation of the expression

$$\sum_{N=0}^{\infty}F(N)e^{\mathbf{i}N\xi}\exp(-2t(\alpha_\eta-N)^2),\tag{B3}$$

where we defined $\alpha_\eta:=c_1\frac{\eta}{t}+c_2>0$, where c_1 and c_2 are constants satisfying $c_1>0$ and $|c_2|\ll c_1\frac{\eta}{t}$. Now let us consider three cases of $F(x)$ separately.

Case I: $F(x)=x^\ell$ is a polynomial with $\ell\in\mathbb{N}$. Consider the following calculations:

$$\begin{aligned}
&\sum_{N=0}^{\infty}F(N)e^{\mathbf{i}N\xi}\exp(-2t(N-\alpha_\eta)^2)\\
&\quad=e^{\mathbf{i}\alpha_\eta\xi}\sum_{k=-\alpha_\eta}^{\infty}(\alpha_\eta+k)^\ell\exp(-2tk^2)e^{\mathbf{i}k\xi}\\
&\quad=(\alpha_\eta)^\ell e^{\mathbf{i}\alpha_\eta\xi}\sum_{[k]=[-\alpha_\eta]}^{\infty}\left(1+\frac{([k]+r)}{\alpha_\eta}\right)^\ell\\
&\qquad\times\exp(-2t([k]+r)^2)e^{\mathbf{i}([k]+r)\xi},
\end{aligned}\tag{B4}$$

where $k:=N-\alpha_\eta$, $r=-\alpha_\eta-[-\alpha_\eta]$ and $[k]$ represents the maximal integer no greater than k. Note that we have

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty}(m+r)^{\ell'}e^{(-2t(m+r)^2)}e^{\mathbf{i}(m+r)\xi}\\
&\quad=\sum_{m=-\infty}^{\infty}e^{2\pi\mathbf{i}mr}\sqrt{\frac{\pi}{2t}}\check{P}(2\pi m-\xi)e^{-\frac{(2\pi m-\xi)^2}{8t}},\quad \ell'\in\mathbb{N};
\end{aligned}\tag{B5}$$

here we defined $\check{P}(x):=(\mathbf{i})^{\ell'}(\frac{d^{\ell'}}{dx^{\ell'}}e^{-\frac{(x)^2}{8t}})e^{\frac{(x)^2}{8t}}$. Then by setting $[k]=m$ in Eq. (B4), we have

$$\sum_{N=0}^{\infty} F(N)\exp(-2t(\alpha_\eta - N)^2)e^{\mathbf{i}N\xi} \overset{\text{large}\,\eta}{=} \sqrt{\frac{\pi}{2t}}(\alpha_\eta)^{\ell} e^{\mathbf{i}\alpha_\eta\xi} e^{-\frac{(\xi)^2}{8t}}\left(1+\mathcal{O}\left(\frac{t}{\eta}\right)+\mathcal{O}(e^{-\frac{\pi^2}{2t}})\right). \tag{B6}$$

Case II: $F(x) = (P(x))^{1/4}$, where $P(x)$ is a polynomial of x with $P(x) > 0$ if $x > 0$. Let us focus on the case of $\alpha_\eta > 0, x > 0$ involved in this paper. We can reformulate $F(x)$ as $F(x) = (P(\alpha_\eta))^{1/4} f(z)$ with $f(z) \coloneqq (1+z)^{1/4}$ and $z \coloneqq \frac{P(x)}{P(\alpha_\eta)} - 1 > -1$. Then by Taylor's theorem, we have

$$f(z) = 1 + \sum_{n=1}^{\infty}\binom{q}{n} z^n, \qquad \binom{q}{n} = (-1)^{n+1}\frac{q(1-q)\ldots(n-1+q)}{n!} \tag{B7}$$

with $q = 1/4$ here. To proceed the next step of the calculation, we introduce a lemma as follows.

Lemma.—For each $l \geq 0$ there exist $0 < \beta_l < \infty$ such that

$$f_{2l+1}(z) - \beta_l z^{2l+2} \leq f(z) \leq f_{2l+1}(z), \tag{B8}$$

where $f_l(z) = 1 + \sum_{n=1}^{l}\binom{q}{n} z^n$, $\binom{q}{n}$ denotes the partial Taylor series of $f(z) = (1+z)^q$, $0 < q \leq 1/4$, up to order z^k.

The proof of this lemma can be find in [47].

Now let us set $x = N$ in $P(x)$ so that $z = \frac{P(N)}{P(\alpha_\eta)} - 1$. Then by using the results of case I, we can give

$$\sqrt{\frac{2t}{\pi}}\sum_{N=0}^{\infty} z^{\ell'}\exp(-2t(\alpha_\eta - N)^2)e^{\mathbf{i}N\xi} e^{\frac{(\xi)^2}{8t}} e^{-\mathbf{i}\alpha_\eta\xi} \overset{\text{large}\,\eta}{=} \mathcal{O}\left(\frac{t}{\eta}\right) + \mathcal{O}(e^{-\frac{\pi^2}{2t}}), \quad \text{for } \ell' \in \mathbb{N}_+. \tag{B9}$$

Further by using the above lemma, we get

$$\sqrt{\frac{2t}{\pi}}\sum_{N=0}^{\infty} F(N)\exp(-2t(\alpha_\eta - N)^2)e^{\mathbf{i}N\xi} \overset{\text{large}\,\eta}{=} (P(\alpha_\eta))^{1/4} e^{\mathbf{i}\alpha_\eta\xi} e^{-\frac{(\xi)^2}{8t}}\left(1+\mathcal{O}\left(\frac{t}{\eta}\right)+\mathcal{O}(e^{-\frac{\pi^2}{2t}})\right). \tag{B10}$$

Case III: $F(x) = (P_1(x))^{1/4}(\frac{P_2(x)}{P_3(x)})^{\frac{1}{m}}$, $m \in \mathbb{N}_+$, where $P_1(x)$ is a polynomial of x with degree larger than 4 and it satisfies $P_1(x) \geq 0$, $\frac{d}{dx}(P_1(x)) > 0$ for $x \geq 0$, $P_2(x)$ and $P_3(x)$ are both polynomials of x which satisfy $P_3(x) > P_2(x) \geq 0$ for $x \geq 0$, $\frac{d}{dx}(\frac{P_2(x)}{P_3(x)}) < 0$ for $x \geq 1$ and the degree of $P_3(x)$ larger than that of $P_2(x)$. We can evaluate that

$$\begin{aligned}
&\sqrt{\frac{2t}{\pi}}\left|\sum_{N=0}^{\infty} F(N)\exp(-2t(\alpha_\eta - N)^2)e^{\mathbf{i}N\xi}\right| \\
&< \sqrt{\frac{2t}{\pi}}\sum_{N=0}^{\infty} F(N)\exp(-2t(\alpha_\eta - N)^2) \\
&< \sqrt{\frac{2t}{\pi}}\left(\left(P_1\left(\frac{\alpha_\eta}{2}\right)\right)^{1/4}\sum_{N=0}^{[\frac{\alpha_\eta}{2}]}\exp(-2t(\alpha_\eta - N)^2) + \sum_{N=[\frac{\alpha_\eta}{2}]+1}^{\infty}(P_1(N))^{1/4}\frac{P_2(\frac{\alpha_\eta}{2}+1)}{P_3(\frac{\alpha_\eta}{2}+1)}\exp(-2t(\alpha_\eta - N)^2)\right) \\
&\lesssim \sqrt{\frac{2t}{\pi}}\left(\left[\frac{\alpha_\eta}{2}\right]+1\right)(P_1(\alpha_\eta))^{1/4}\exp\left(-\frac{1}{2}t(\alpha_\eta)^2\right) + P_1(\alpha_\eta)^{1/4}\left(\frac{P_2(\frac{\alpha_\eta}{2}+1)}{P_3(\frac{\alpha_\eta}{2}+1)}\right)^{\frac{1}{m}}\left(1+\mathcal{O}\left(\frac{t}{\eta}\right)+\mathcal{O}(e^{-\frac{\pi^2}{2t}})\right) \\
&\simeq (P_1(\alpha_\eta))^{1/4}\left(\alpha_\eta\sqrt{t}\,\mathcal{O}(e^{-\frac{\eta^2}{8t}}) + \mathcal{O}\left(\left(\frac{t}{\eta}\right)^{\frac{1}{m}}\right)\right)
\end{aligned} \tag{B11}$$

for large η.

As we will see in the next two subsections, the three cases discussed above will be used in the calculations of the matrix elements of the flux and holonomy operators in the twisted geometry coherent state basis.

APPENDIX C: MATRIX ELEMENTS OF THE FLUX OPERATOR

We consider the matrix elements of the flux operator in the twisted geometry coherent state basis, which are denoted by $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{IJ}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$. The numerator can be calculated as follows:

$$\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{IJ}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} = -\mathbf{i}\beta t\sum_{N_e}(\dim(\pi_{N_e}))^2\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot e^{\mathbf{i}d_{N_e}(\xi_e^o-\xi'^o_e)}\langle N_e,V_e'|\tau^{IJ}|N_e,V_e\rangle\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}}). \quad \text{(C1)}$$

Without loss of generality, we set $V_e'=2\delta_1^{[I}\delta_2^{J]}$ to simplify the expressions. Then, the components of $\hat{X}_e^{IJ}$ can be decomposed as three sets and we can discuss them separately.

1. Set I: Matrix elements of $\hat{X}_e^{12}$

Similar to the calculation of the overlap function of the twisted geometry coherent state, we defined $\tilde{\Theta}_e\coloneqq\Theta(u_e,u_e')+\Theta(\tilde{u}_e,\tilde{u}_e')$ and used the invention

$$\langle N_e,V_e'|N_e,V_e\rangle=\exp\left(-N_e\Theta(u_e,u_e')\right)e^{\mathbf{i}N_e\varphi(u_e,u_e')}, \quad \text{(C2)}$$

$$\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle=\exp\left(-N_e\Theta(\tilde{u}_e,\tilde{u}_e')\right)e^{\mathbf{i}N_e\varphi(\tilde{u}_e,\tilde{u}_e')}, \quad \text{(C3)}$$

with $\Theta(u_e,u'_e)\coloneqq-\frac{\ln|\langle N_e,V_e'|N_e,V_e\rangle|}{N_e}\geq0$, $\Theta(\tilde{u}_e,\tilde{u}_e')\coloneqq-\frac{\ln|\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle|}{N_e}\geq0$, $e^{\mathbf{i}N_e\varphi(u_e,u_e')}\coloneqq\frac{\langle N_e,V_e'|N_e,V_e\rangle}{|\langle N_e,V_e'|N_e,V_e\rangle|}$, and $e^{\mathbf{i}N_e\varphi(\tilde{u}_e,\tilde{u}_e')}\coloneqq\frac{\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle}{|\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle|}$ are variables independent with N_e. Then, we have

$$\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} = \beta t\sum_{N_e}N_e(\dim(\pi_{N_e}))^2\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot e^{\mathbf{i}d_{N_e}(\xi_e^o-\xi'^o_e)}\langle N_e,V_e'|N_e,V_e\rangle\langle N_e,-\tilde{V}_e|N_e,-\tilde{V}_e'\rangle+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}})$$
$$=\beta t e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o-\xi'^o_e)}\sum_{N_e}N_e(\dim(\pi_{N_e}))^2\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot e^{\mathbf{i}N_e(\xi_e^o-\xi'^o_e+\varphi(u_e,u_e')+\varphi(\tilde{u}_e,\tilde{u}_e'))}\exp(-N_e\tilde{\Theta}_e)+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}}) \quad \text{(C4)}$$

for large η_e', where $\tilde{\varphi}_e\coloneqq\varphi(u_e,u_e')+\varphi(\tilde{u}_e,\tilde{u}_e')$ and $\tilde{\Theta}_e\coloneqq\Theta(u_e,u_e')+\Theta(\tilde{u}_e,\tilde{u}_e')$. The calculation of Eq. (C4) follows the similar procedures of the calculation of $\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle$ in [1]. Let us consider the cases of $\tilde{\Theta}_e\ll\eta_e+\eta_e'$ and $\tilde{\Theta}_e\simeq\eta_e+\eta_e'$ or $\tilde{\Theta}_e\gg\eta_e+\eta_e'$ separately.

a. Case $\tilde{\Theta}_e\ll\eta_e+\eta_e'$

For the case $\tilde{\Theta}_e\ll\eta_e+\eta_e'$, Eq. (C4) reads

$$\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} = \beta t e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o-\xi'^o_e)}\sum_{N_e}N_e(\dim(\pi_{N_e}))^2\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot e^{\mathbf{i}N_e(\xi_e^o-\xi'^o_e+\varphi(u_e,u_e')+\varphi(\tilde{u}_e,\tilde{u}_e'))}\exp(-N_e\tilde{\Theta}_e)+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}})$$
$$=\beta t e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o-\xi'^o_e)}e^{-t(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t})^2+2t(\frac{\eta_e}{4t}-\frac{\eta_e'}{4t}-\frac{\tilde{\Theta}_e}{4t})^2}e^{\mathbf{i}(\frac{\eta_e}{4t}-\frac{D-1}{2}+\frac{\eta_e'}{4t}-\frac{\tilde{\Theta}_e}{4t})(\xi_e^o-\xi'^o_e+\tilde{\varphi}_e)}$$
$$\cdot\exp\left(-\left(\frac{\eta_e'}{2t}-\frac{D-1}{2}\right)\tilde{\Theta}_e\right)\sum_{[\tilde{k}_e]}\tilde{\mathrm{P}}(\tilde{k}_e)(\exp(-2t\tilde{k}_e^2)e^{\mathbf{i}\tilde{k}_e(\xi_e^o-\xi'^o_e+\tilde{\varphi}_e)})+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}})$$
$$=\beta t\frac{\sqrt{\pi}}{\sqrt{2t}}e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o-\xi'^o_e)}e^{-t(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t})^2+2t(\frac{\eta_e}{4t}-\frac{\eta_e'}{4t}-\frac{\tilde{\Theta}_e}{4t})^2}e^{\mathbf{i}(\frac{\eta_e}{4t}-\frac{D-1}{2}+\frac{\eta_e'}{4t}-\frac{\tilde{\Theta}_e}{4t})(\xi_e^o-\xi'^o_e+\tilde{\varphi}_e)}e^{-(\frac{\eta_e'}{2t}-\frac{D-1}{2})\tilde{\Theta}_e}$$
$$\cdot\sum_{n=-\infty}^{\infty}\check{\mathrm{P}}(2\pi n-(\xi_e^o-\xi'^o_e+\tilde{\varphi}_e))\exp\left(-\frac{(2\pi n-(\xi_e^o-\xi'^o_e+\tilde{\varphi}_e))^2}{8t}\right)$$
$$\times e^{\mathbf{i}2\pi n\ \mathrm{mod}\ (\tilde{k}_e,1)}\left(1+\mathcal{O}\left(\frac{t}{\eta_e'}\right)\right)+\beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}}) \quad \text{(C5)}$$

for large η'_e, where $\tilde{k}_e := d_{N_e} - \frac{\eta'_e}{4t} - \frac{\eta_e}{4t} + \frac{\tilde{\Theta}_e}{4t} = [\tilde{k}_e] + \text{mod}\ (\tilde{k}_e, 1)$ with $[\tilde{k}_e]$ being the maximum integer less than or equal to $\tilde{k}_e$ and mod $(\tilde{k}_e, 1)$ being the corresponding remainder, and $\tilde{\mathrm{P}}(\tilde{k}_e)$ is a polynomial of $\tilde{k}_e$ defined by $\tilde{\mathrm{P}}(\tilde{k}_e) = N_e(\dim(\pi_{N_e}))^2$. Besides, $\check{\mathrm{P}}(x)$ is also a polynomial which is given by

$$\check{\mathrm{P}}(x) = \left((\mathbf{i})^n a_n \frac{d^n}{dx^n}\exp\left(-\frac{x^2}{8t}\right) + (\mathbf{i})^{n-1} a_{n-1}\frac{d^{n-1}}{dx^{n-1}}\exp\left(-\frac{x^2}{8t}\right) + \cdots + a_0 \exp\left(-\frac{x^2}{8t}\right)\right)\exp\left(\frac{x^2}{8t}\right) \tag{C6}$$

with $a_n, a_{n-1}, \ldots, a_0$ given by the expanding $\tilde{\mathrm{P}}(\tilde{k}_e) = a_n \tilde{k}_e^n + a_{n-1}\tilde{k}_e^{n-1} + \cdots + a_0$, wherein $a_0 = (\frac{\eta'_e}{4t} + \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t})(\check{\mathrm{P}}(\frac{\eta'_e}{4t} + \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t}))^2$, $\frac{a_0}{a_m} \simeq (\frac{\eta_e}{4t})^m$, $0 \leq m \leq n$ for large η'_e by the definition of $\tilde{\mathrm{P}}(\tilde{k}_e)$. In addition, we used the Poisson summation formula in the third step of Eq. (C7), which converts a slowly converging series into a rapidly converging series which in our case almost only the term with $n = 0$ need be relevant. Following this analysis, we can further give

$$\begin{aligned}
\langle \check{\Psi}_{\mathbb{H}^o_e} | \hat{X}^{12}_e | \check{\Psi}_{\mathbb{H}'^o_e}\rangle &= \beta t \frac{\sqrt{\pi}}{\sqrt{2t}} e^{\mathbf{i}\frac{D-1}{2}(\xi^o_e - \xi'^o_e)} e^{\frac{(\eta_e)^2 + (\eta'_e)^2 + 2t^2(D-1)^2}{4t}} e^{-t(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2 + 2t(\frac{\eta_e}{4t} - \frac{\eta'_e}{4t} - \frac{\tilde{\Theta}_e}{4t})^2} e^{\mathbf{i}(\frac{\eta_e}{4t} - \frac{D-1}{2} + \frac{\eta'_e}{4t} - \frac{\tilde{\Theta}_e}{4t})(\xi^o_e - \xi'^o_e + \tilde{\varphi}_e)} \\
&\quad \cdot e^{-(\frac{\eta'_e}{2t} - \frac{D-1}{2})\tilde{\Theta}_e}\left(\frac{\eta'_e}{4t} + \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t}\right)\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t} + \frac{\eta_e}{4t} - \frac{\tilde{\Theta}_e}{4t}\right)\right)^2 \exp\left(-\frac{(\xi^o_e - \xi'^o_e + \tilde{\varphi}_e)^2}{8t}\right)\left(1 + \mathcal{O}\left(\frac{t}{\eta'_e}\right) + \mathcal{O}(e^{-\frac{1}{t}})\right) \\
&= \beta\left(\frac{\eta'_e}{4} + \frac{\eta_e}{4} - \frac{\tilde{\Theta}_e}{4}\right)\langle \check{\Psi}_{\mathbb{H}^o_e} | \check{\Psi}_{\mathbb{H}'^o_e}\rangle \left(1 + \mathcal{O}\left(\frac{t}{\eta'_e}\right) + \mathcal{O}(e^{-\frac{1}{t}})\right)
\end{aligned} \tag{C7}$$

for large η'_e. Notice that the classical evaluation of operator $\hat{X}^{12}$ is given by $X^{12}(\mathbb{H}'^o_e) = \frac{\eta'_e}{2}$. Hence one can estimate the matrix elements of $\hat{X}_{12}$ with respect to the twisted geometry coherent states by

$$\begin{aligned}
&\frac{\langle \check{\Psi}_{\mathbb{H}^o_e} | \hat{X}^{12}_e | \check{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\check{\Psi}_{\mathbb{H}'^o_e}|| \, ||\check{\Psi}_{\mathbb{H}^o_e}||} - \beta\frac{\eta'_e}{2}\frac{\langle \check{\Psi}_{\mathbb{H}^o_e} | \check{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\check{\Psi}_{\mathbb{H}'^o_e}|| \, ||\check{\Psi}_{\mathbb{H}^o_e}||} \\
&\overset{\text{large } \eta'_e}{=} \beta\left(\left(\frac{\eta_e}{4} - \frac{\eta'_e}{4} - \frac{\tilde{\Theta}_e}{4}\right) + \left(\frac{\eta_e}{4} + \frac{\eta'_e}{4} - \frac{\tilde{\Theta}_e}{4}\right)\left(\mathcal{O}\left(\frac{t}{\eta'_e}\right) + \mathcal{O}(e^{-\frac{1}{t}})\right)\right)\frac{\langle \check{\Psi}_{\mathbb{H}^o_e} | \check{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\check{\Psi}_{\mathbb{H}'^o_e}|| \, ||\check{\Psi}_{\mathbb{H}^o_e}||}.
\end{aligned} \tag{C8}$$

Recall the overlap function $i^t(\mathbb{H}^o_e, \mathbb{H}'^o_e)$ is sharply peaked at $\eta_e = \eta'_e$ and $\tilde{\Theta}_e = 0$, and it decays exponentially for $\eta_e \neq \eta'_e$ or $\tilde{\Theta}_e \neq 0$. Hence we can conclude that the right-hand side of Eq. (C8) is bounded by a correction term which tends to zero in the limit $t \to 0$ for large η'_e.

b. Case $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$

For the case $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$, similar to the analysis of the overlap function in [1], we have

$$\begin{aligned}
&\langle \check{\Psi}_{\mathbb{H}^o_e} | \hat{X}^{12}_e | \check{\Psi}_{\mathbb{H}'^o_e}\rangle e^{-\frac{(\eta_e)^2 + (\eta'_e)^2 + 2t^2(D-1)^2}{4t}} \\
&= \beta t e^{\mathbf{i}\frac{D-1}{2}(\xi^o_e - \xi'^o_e)} \sum_{N_e} N_e(\dim(\pi_{N_e}))^2 \exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\quad \cdot e^{\mathbf{i}N_e(\xi^o_e - \xi'^o_e + \varphi(u_e, u'_e) + \varphi(\tilde{u}_e, \tilde{u}'_e))} \exp(-N_e\tilde{\Theta}_e) + \beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}}) \\
&< \beta t[\eta'_e/4t]\exp\left(-t\left(\frac{\eta'_e}{4t} - \frac{D+1}{2}\right)^2 - t\left(\frac{\eta_e}{2t} - \frac{\eta'_e}{4t} - \frac{D+1}{2}\right)^2\right)\left(\frac{\eta'_e}{4t}\right)\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}\right)\right)^2 \exp(-\tilde{\Theta}_e) \\
&\quad + \beta t \sum_{N_e = [\frac{\eta'_e}{4t}]+1}^{+\infty} N_e(\dim(\pi_{N_e}))^2\left(\exp\left(-t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2 - t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2\right)\exp\left(-\left[\frac{\eta'_e}{4t}\right]\tilde{\Theta}_e\right)\right) + \beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}}) \\
&\simeq \beta t[\eta'_e/4t]\exp\left(-t\left(\frac{\eta'_e}{4t} - \frac{D+1}{2}\right)^2 - t\left(\frac{\eta_e}{2t} - \frac{\eta'_e}{4t} - \frac{D+1}{2}\right)^2\right)\left(\frac{\eta'_e}{4t}\right)\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}\right)\right)^2 \exp(-\tilde{\Theta}_e) \\
&\quad + \beta t\sqrt{\frac{\pi}{2t}}\left(\frac{\eta_e}{4t} + \frac{\eta'_e}{4t}\right)\left(\check{\mathrm{P}}\left(\frac{\eta_e}{4t} + \frac{\eta'_e}{4t}\right)\right)^2 e^{-\frac{t}{2}(\frac{\eta_e}{2t} - \frac{\eta'_e}{2t})^2}\exp\left(-\left[\frac{\eta'_e}{4t}\right]\tilde{\Theta}_e\right) + \beta\sqrt{t}\cdot\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})
\end{aligned} \tag{C9}$$

for η'_e being large, wherein the sign "<" means that the module of its left side is less than that of its right side. Then, in this case the matrix element of $\hat{X}_{12}$ is estimated by

$$0 < \left| \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{12} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}|| \, ||\breve{\Psi}_{\mathbb{H}_e^o}||} \right| \lesssim \beta t \frac{\sqrt{\frac{2t}{\pi}} f_1(\eta_e, \eta'_e) e^{-t(\frac{\eta'_e}{4t})^2} e^{-\tilde{\Theta}_e} + (\frac{\eta_e}{4t} + \frac{\eta'_e}{4t})(\check{\mathrm{P}}(\frac{\eta_e}{4t} + \frac{\eta'_e}{4t}))^2 e^{-\frac{t}{2}(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2} e^{-[\frac{\eta'_e}{4t}]\tilde{\Theta}_e}}{\check{\mathrm{P}}(\frac{\eta_e}{2t})\check{\mathrm{P}}(\frac{\eta'_e}{2t})} \tag{C10}$$

for large η'_e, where $f_1(\frac{\eta_e}{t}, \frac{\eta'_e}{t}) \coloneqq [\eta'_e/4t] \exp(-t(\frac{\eta_e}{2t} - \frac{\eta'_e}{4t} - \frac{D+1}{2})^2)(\frac{\eta'_e}{4t})(\check{\mathrm{P}}(\frac{\eta'_e}{4t}))^2$. Notice that $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$ in this case, hence we can conclude that $|\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{12} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}|| \, ||\breve{\Psi}_{\mathbb{H}_e^o}||}|$ is always suppressed exponentially by the factors $e^{-t(\frac{\eta'_e}{4t})^2}$ and $e^{-\frac{t}{2}(\frac{\eta'_e}{2t} - \frac{\eta_e}{2t})^2} e^{-[\frac{\eta'_e}{4t}]\tilde{\Theta}_e}$ in Eq. (C10).

Now we are ready to combine the results in the cases for $\tilde{\Theta}_e \ll \eta_e + \eta'_e$ and $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$ or $\tilde{\Theta}_e \gg \eta_e + \eta'_e$. In general, the matrix elements of $\hat{X}_{12}$ with respect to the twisted geometry coherent state is estimated by

$$\left| \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{12} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}|| \, ||\breve{\Psi}_{\mathbb{H}_e^o}||} - \frac{\eta'_e}{2} \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}|| \, ||\breve{\Psi}_{\mathbb{H}_e^o}||} \right| \overset{\text{large } \eta'_e}{\lesssim} t |f_{12}(\mathbb{H}_e^o, \mathbb{H}_e'^o)| \cdot \left| \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}|| \, ||\breve{\Psi}_{\mathbb{H}_e^o}||} \right|, \tag{C11}$$

where $f_{12}(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ is a function that grows no faster than the exponentials as $|\eta_e - \eta'_e|$, $\tilde{\Theta}_e$ and $|\xi_e^o - \xi_e'^o|$ going large for large η'_e.

2. Set II: Matrix elements of $\hat{X}_e^{IJ}$ with $I, J \in \{3, \ldots, D+1\}$

Notice that $\langle N_e, V'_e | \tau^{IJ} | N_e, V_e \rangle = 0$ with $V'_e = 2\delta_1^{[I}\delta_2^{J]}$ and $I, J \in \{3, \ldots, D+1\}$; it is straightforward to get that

$$\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{IJ} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle = 0, \quad \text{for } I, J \in \{3, \ldots, D+1\}. \tag{C12}$$

3. Set III: Matrix elements of $\hat{X}_e^{1J}$ and $\hat{X}_e^{2J}$ with $J \in \{3, \ldots, D+1\}$

Let us consider the rest of the components $\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{1J} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle$ and $\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{2J} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle$ with $J \in \{3, \ldots, D+1\}$. We first have

$$\langle N_e, V'_e | \tau^{1J} | N_e, V_e \rangle = N_e \langle 1, V'_e | \tau_{1J} | 1, V_e \rangle \langle (N_e - 1), V'_e | (N_e - 1), V_e \rangle, \quad \text{for } J \in \{3, \ldots, D+1\}. \tag{C13}$$

Here we note that $\langle 1, V'_e | \tau_{1J} | 1, V_e \rangle$ with $J \neq 1, 2$ as functions of V_e vanish at $V_e = V'_e$ and the growth of their modules are restricted by their derivatives evaluated by Eq. (38) as V_e being transformed by $e^{t\tau_{KL}} \in SO(D+1)$. Then, by checking how $\Theta(u_e, u'_e) = -\ln |\langle 1, V'_e | 1, V_e \rangle|$ grows as V_e being transformed by $e^{t\tau_{KL}} \in SO(D+1)$, we can conclude that $\langle 1, V'_e | \tau_{1J} | 1, V_e \rangle$ with $J \neq 1, 2$ grows no faster than the exponentials as $\Theta(u_e, u'_e) = -\ln |\langle 1, V'_e | 1, V_e \rangle|$ going large. Then, for large η'_e we can give

$$\begin{aligned}
&\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{1J} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle e^{-\frac{(\eta_e)^2 + (\eta'_e)^2 + 2t^2(D-1)^2}{4t}} \\
&\quad = -\mathbf{i}\beta t \langle 1, V'_e | \tau_{1J} | 1, V_e \rangle \sum_{N_e} N_e (\dim(\pi_{N_e}))^2 \exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\qquad \cdot e^{\mathbf{i} d_{N_e}(\xi_e^o - \xi_e'^o)} \langle (N_e - 1), V'_e | (N_e - 1), V_e \rangle \langle N_e, -\tilde{V}_e | N_e, -\tilde{V}'_e \rangle + \beta \sqrt{t} \mathcal{O}(e^{-(\eta'_e)^2/(8t)}) \\
&\quad = -\mathbf{i}\beta t e^{\mathbf{i}\frac{D-1}{2}(\xi_e^o - \xi_e'^o)} \frac{\langle 1, V'_e | \tau_{1J} | 1, V_e \rangle}{\langle 1, V'_e | 1, V_e \rangle} \sum_{N_e} N_e (\dim(\pi_{N_e}))^2 \exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\qquad \cdot e^{\mathbf{i} N_e(\xi_e^o - \xi_e'^o + \varphi(u_e, u'_e) + \varphi(\tilde{u}_e, \tilde{u}'_e))} \exp(-N_e \tilde{\Theta}_e) + \beta \sqrt{t} \mathcal{O}(e^{-(\eta'_e)^2/(8t)}),
\end{aligned} \tag{C14}$$

where we only consider the case of $\langle 1, V'_e | 1, V_e \rangle \neq 0$ since one have $\langle 1, V'_e | \tau_{1J} | 1, V_e \rangle = 0$ and $\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{1J} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle = 0$ if $\langle 1, V'_e | 1, V_e \rangle = 0$. By comparing Eq. (C17) with Eq. (C4), one has

$$\langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{1J} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle \overset{\text{large } \eta'_e}{=} -\mathbf{i} \frac{\langle 1, V'_e | \tau_{1J} | 1, V_e \rangle}{\langle 1, V'_e | 1, V_e \rangle} \langle \breve{\Psi}_{\mathbb{H}_e^o} | \hat{X}_e^{12} | \breve{\Psi}_{\mathbb{H}_e'^o} \rangle + e^{\frac{(\eta_e)^2 + (\eta'_e)^2 + 2t^2(D-1)^2}{4t}} \beta \sqrt{t} \mathcal{O}(e^{-(\eta'_e)^2/(8t)}) \tag{C15}$$

and further

$$\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{1J}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large } \eta'_e}{=} \left|\frac{\langle 1, V'_e|\tau_{1J}|1, V_e\rangle}{\langle 1, V'_e|1, V_e\rangle}\right| \frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} + \beta t\mathcal{O}(e^{-(\eta'_e)^2/(8t)}), \tag{C16}$$

wherein one should note that $|\langle 1, V'_e|1, V_e\rangle| = e^{-\Theta(u_e,u'_e)}$, $0 \leq |\langle 1, V'_e|\tau_{1J}|1, V_e\rangle| \leq 1$ with $\langle 1, V'_e|\tau_{1J}|1, V_e\rangle = 0$ at $V'_e = V_e$, and $\langle 1, V'_e|\tau_{1J}|1, V_e\rangle$ with $J \neq 1$, 2 grows no faster than the exponentials as $\Theta(u_e, u'_e) = -\ln|\langle 1, V'_e|1, V_e\rangle|$ going large. Then let us recall Eq. (C4) and notice that $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ is unity at $\mathbb{H}_e'^o = \mathbb{H}_e^o$ and decays exponentially fast to 0 as $|\eta_e - \eta'_e|$, $\tilde{\Theta}_e$ and $|\xi_e^o - \xi_e'^o|$ going large for large η'_e, and we can conclude that

$$\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{1J}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large } \eta'_e}{\lesssim} t|f_{1J}(\mathbb{H}_e^o, \mathbb{H}_e'^o)|\cdot\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|, \quad \text{for } J \in \{3, \ldots, D+1\}, \tag{C17}$$

where $f_{1J}(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ is a function of $\mathbb{H}_e^o, \mathbb{H}_e'^o$ which vanishes for $\Theta(u_e, u'_e) = 0$ and whose growth is always suppressed by $|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}|$ exponentially as $|\eta_e - \eta'_e|$, $\tilde{\Theta}_e$ and $|\xi_e^o - \xi_e'^o|$ going large for large η'_e. A similar discussion can be given for $\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{2J}|\breve{\Psi}_{\mathbb{H}_e^o}\rangle$ and we reach that

$$\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\hat{X}_e^{2J}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} \overset{\text{large } \eta'_e}{\lesssim} t|f_{2J}(\mathbb{H}_e^o, \mathbb{H}_e'^o)|\cdot\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|, \quad \text{for } J \in \{3, \ldots, D+1\}, \tag{C18}$$

where $f_{2J}(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ is a function of $\mathbb{H}_e^o, \mathbb{H}_e'^o$ which vanishes for $\Theta(u_e, u'_e) = 0$ and whose growth is always suppressed by $\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|$ exponentially as $|\eta_e - \eta'_e|$, $\tilde{\Theta}_e$ and $|\xi_e^o - \xi_e'^o|$ going large for large η'_e.

Finally, collecting Eqs. (C11), (C12), (C17), and (C18), we arrive at the main result of Sec. III B 1.

APPENDIX D: MATRIX ELEMENTS OF THE HOLONOMY OPERATOR

Let us consider the matrix elements of the holonomy operator in the twisted geometry coherent state basis. Also, the cases of $(D+1)$ being even and odd are considered separately.

1. $D+1$ even

In the case of $(D+1)$ being even, we need to calculate all of the components of $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{IJ,I'J'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$. We first consider the simplest component $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{12,12}$ of the holonomy operator, whose matrix elements in twisted geometry coherent basis is given by

$$\begin{aligned}
&\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{12,12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}} \\
&\overset{\text{large } \eta'_e}{=} e^{\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e+1}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\quad \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o - \xi_e'^o)}\langle N_e+1, V'_e|N_e+1, V_e\rangle\langle N_e+1, -\tilde{V}_e|N_e+1, -\tilde{V}'_e\rangle + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}}),
\end{aligned} \tag{D1}$$

wherein we used that $|1, V_{12}\rangle \otimes |N_e, V_{12}\rangle = |N_e+1, V_{12}\rangle$. To calculate Eq. (D1), let us consider two cases separately. In the first case, we consider $\tilde{\Theta}_e \gg \eta_e + \eta'_e$ or $\tilde{\Theta}_e \simeq \eta_e + \eta'_e$. Similar to the analysis of Eqs. (C9) and (C10), we have

$$\begin{aligned}
&\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{12,12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}} \\
&= e^{\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t} - d_{N_e+1}\right)^2 - t\left(\frac{\eta'_e}{2t} - d_{N_e}\right)^2\right) \\
&\quad \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o - \xi_e'^o)}\langle N_e+1, V'_e|N_e+1, V_e\rangle\langle N_e+1, -\tilde{V}_e|N_e+1, -\tilde{V}'_e\rangle + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}}) \\
&< ([\eta'_e/4t]+1)\exp\left(-t\left(\frac{\eta'_e}{4t} - \frac{D+1}{2}\right)^2 - t\left(\frac{\eta_e}{2t} - \frac{\eta'_e}{4t} - \frac{D-1}{2}\right)^2\right)\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}+1\right)\right)^{1/2}\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}\right)\right)^{3/2} \\
&\quad + \sum_{N_e=[\frac{\eta'_e}{4t}]+1}^{+\infty}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\left(\exp\left(-t\left(\frac{\eta'_e}{2t} - d_{N_e+1}\right)^2 - t\left(\frac{\eta_e}{2t} - d_{N_e}\right)^2\right)e^{-[\frac{\eta'_e}{4t}]\tilde{\Theta}_e}\right) + \frac{1}{\sqrt{t}}\cdot\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})
\end{aligned}$$

$$\simeq ([\eta'_e/4t]+1)\exp\left(-t\left(\frac{\eta'_e}{4t}-\frac{D+1}{2}\right)^2-t\left(\frac{\eta_e}{2t}-\frac{\eta'_e}{4t}-\frac{D-1}{2}\right)^2\right)\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}+1\right)\right)^{1/2}\left(\check{\mathrm{P}}\left(\frac{\eta'_e}{4t}\right)\right)^{3/2}$$
$$+\sqrt{\frac{\pi}{2t}}\left(\check{\mathrm{P}}\left(\frac{\eta_e}{4t}+\frac{\eta'_e}{4t}\right)\right)^{3/2}\left(\check{\mathrm{P}}\left(\frac{\eta_e}{4t}+\frac{\eta'_e}{4t}+1\right)\right)^{1/2}e^{-\frac{t}{2}(\frac{\eta_e}{2t}-\frac{\eta'_e}{2t})^2}\exp\left(-\left[\frac{\eta'_e}{4t}\right]\tilde{\Theta}_e\right)+\frac{1}{\sqrt{t}}\cdot\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}}) \tag{D2}$$

for η'_e being large, where "<" represents that the module of its left-hand side is less than the module of its right-hand side. Then, in this case the matrix elements of $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{12,12}$ are estimated by

$$0<\left|\frac{\langle\check{\Psi}_{\mathbb{H}'^o_e}|(\widehat{u'^{-1}_e h_e\tilde{u}'_e})_{12,12}|\check{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\check{\Psi}_{\mathbb{H}'^o_e}||||\check{\Psi}_{\mathbb{H}^o_e}||}\right|\lesssim\frac{\sqrt{\frac{2t}{\pi}}\tilde{f}_1(\frac{\eta_e}{t},\frac{\eta'_e}{t})e^{-t(\frac{\eta'_e}{4t}-\frac{D+1}{2})^2}+(\check{\mathrm{P}}(\frac{\eta_e}{4t}+\frac{\eta'_e}{4t}))^{3/2}(\check{\mathrm{P}}(\frac{\eta_e}{4t}+\frac{\eta'_e}{4t}+1))^{1/2}e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t})^2}e^{-[\frac{\eta'_e}{4t}]\tilde{\Theta}_e}}{\check{\mathrm{P}}(\frac{\eta_e}{2t})\check{\mathrm{P}}(\frac{\eta'_e}{2t})} \tag{D3}$$

for large η'_e, where $\tilde{f}_1(\frac{\eta_e}{t},\frac{\eta'_e}{t}) := ([\eta'_e/4t]+1)\exp(-t(\frac{\eta_e}{2t}-\frac{\eta'_e}{4t}-\frac{D-1}{2})^2)(\check{\mathrm{P}}(\frac{\eta'_e}{4t}+1))^{1/2}(\check{\mathrm{P}}(\frac{\eta'_e}{4t}))^{3/2}$. Notice $\tilde{\Theta}_e\simeq\eta_e+\eta'_e$ or $\tilde{\Theta}_e\gg\eta_e+\eta'_e$ in this case, hence we can conclude that $\left|\frac{\langle\check{\Psi}_{\mathbb{H}'^o_e}|(\widehat{u'^{-1}_e h_e\tilde{u}'_e})_{12,12}|\check{\Psi}_{\mathbb{H}^o_e}\rangle}{||\check{\Psi}_{\mathbb{H}'^o_e}||||\check{\Psi}_{\mathbb{H}^o_e}||}\right|$ is always suppressed exponentially by the factors $e^{-t(\frac{\eta'_e}{4t}-\frac{D+1}{2})^2}$ and $e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t})^2}e^{-[\frac{\eta'_e}{4t}]\tilde{\Theta}_e}$ in Eq. (D3). In the second case, we consider $\tilde{\Theta}_e\ll\eta_e+\eta'_e$,

$$\begin{aligned}
&\langle\check{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e\tilde{u}'_e})_{12,12}|\check{\Psi}_{\mathbb{H}'^o_e}\rangle e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}}\\
&\overset{\text{large }\eta'_e}{=} e^{\mathbf{i}\xi'^o_e}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2-t\left(\frac{\eta'_e}{2t}-d_{N_e}\right)^2\right)\\
&\quad\cdot e^{\mathbf{i}d_{N_e+1}(\xi^o_e-\xi'^o_e)}\langle N_e+1,V'_e|N_e+1,V_e\rangle\langle N_e+1,-\tilde{V}_e|N_e+1,-\tilde{V}'_e\rangle+\frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}})\\
&= e^{\mathbf{i}\xi'^o_e}e^{\mathbf{i}\frac{D+1}{2}(\xi^o_e-\xi'^o_e)}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-1-d_{N_e}\right)^2-t\left(\frac{\eta'_e}{2t}-d_{N_e}\right)^2\right)\\
&\quad\cdot e^{\mathbf{i}(N_e+1)(\xi^o_e-\xi'^o_e+\varphi(u_e,u'_e)+\varphi(\tilde{u}_e,\tilde{u}'_e))}e^{-\tilde{\Theta}_e}\exp(-N_e\tilde{\Theta}_e)+\frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}})\\
&= e^{\mathbf{i}\xi'^o_e}e^{\mathbf{i}\frac{D+1}{2}(\xi^o_e-\xi'^o_e)}e^{\mathbf{i}(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)}e^{-t(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2+2t(\frac{\eta_e}{4t}-\frac{1}{2}-\frac{\eta'_e}{4t}-\frac{\tilde{\Theta}_e}{4t})^2}e^{\mathbf{i}(\frac{\eta_e}{4t}-\frac{1}{2}-\frac{D-1}{2}+\frac{\eta'_e}{4t}-\frac{\tilde{\Theta}_e}{4t})(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)}\\
&\quad\cdot e^{-\tilde{\Theta}_e}\exp\left(-\left(\frac{\eta'_e}{2t}-\frac{D-1}{2}\right)\tilde{\Theta}_e\right)\sum_{[\tilde{k}_e]}(\tilde{\mathrm{P}}(\tilde{k}_e))^{1/4}(\exp(-2t\tilde{k}_e^2)e^{\mathbf{i}\tilde{k}_e(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)})+\frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}}).
\end{aligned} \tag{D4}$$

Here we defined $\tilde{k}_e := d_{N_e}-\frac{\eta'_e}{4t}-\frac{\eta_e}{4t}+\frac{1}{2}+\frac{\tilde{\Theta}_e}{4t}=[\tilde{k}_e]+\text{mod}\,(\tilde{k}_e,1)$ with $[\tilde{k}_e]$ being the maximum integer less than or equal to $\tilde{k}_e$ and mod $(\tilde{k}_e,1)$ being the corresponding remainder, and $\tilde{\mathrm{P}}(\tilde{k}_e)$ is a polynomial of $\tilde{k}_e$ defined by $(\tilde{\mathrm{P}}(\tilde{k}_e))^{1/4}=(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}$. By applying the result of case II discussed in Appendix B, we can immediately get

$$\begin{aligned}
&\langle\check{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e\tilde{u}'_e})_{12,12}|\check{\Psi}_{\mathbb{H}'^o_e}\rangle e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}}\\
&\overset{\text{large }\eta'_e}{=} e^{\mathbf{i}\xi'^o_e}e^{\mathbf{i}\frac{D+1}{2}(\xi^o_e-\xi'^o_e)}e^{\mathbf{i}(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)}e^{-t(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2+2t(\frac{\eta_e}{4t}-\frac{1}{2}-\frac{\eta'_e}{4t}-\frac{\tilde{\Theta}_e}{4t})^2}e^{\mathbf{i}(\frac{\eta_e}{4t}-\frac{1}{2}-\frac{D-1}{2}+\frac{\eta'_e}{4t}-\frac{\tilde{\Theta}_e}{4t})(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)}\\
&\quad\cdot\exp(-\tilde{\Theta}_e)\exp\left(-\left(\frac{\eta'_e}{2t}-\frac{D-1}{2}\right)\tilde{\Theta}_e\right)\sqrt{\frac{\pi}{2t}}(\tilde{\mathrm{P}}(0))^{1/4}e^{-\frac{(\xi^o_e-\xi'^o_e+\tilde{\varphi}_e)^2}{8t}}\left(1+\mathcal{O}\left(\frac{t}{\eta'_e}\right)+\mathcal{O}(e^{-1/t})\right)\\
&\quad+\frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}})
\end{aligned} \tag{D5}$$

and then

$$\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{12,12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} \overset{\text{large } \eta_e'}{=} e^{\mathbf{i}\xi_e'^o} e^{-\tilde{\Theta}_e/2} e^{-t/2} e^{\mathbf{i}\frac{D+1}{2}(\xi_e^o-\xi_e'^o)} e^{\frac{\mathbf{i}}{2}(\xi_e^o-\xi_e'^o+\tilde{\varphi}_e)} e^{(\eta_e/2-\eta_e'/2)} \frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\left(1+\mathcal{O}\left(\frac{t}{\eta_e'}\right)+\mathcal{O}(e^{-1/t})\right). \tag{D6}$$

Combining the results (D3) and (D6) of these two cases, we reach

$$\left|\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{12,12}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} - e^{\mathbf{i}\xi_e'^o}\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large } \eta_e'}{\lesssim} t|\tilde{f}_{12}(\mathbb{H}_e^o,\mathbb{H}_e'^o)|\cdot\left|\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|, \tag{D7}$$

where $\tilde{f}_{12}(\mathbb{H}_e^o,\mathbb{H}_e'^o)$ is a function whose growth is always suppressed by $\left|\frac{\langle \breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|$ exponentially as $|\eta_e-\eta_e'|$, $\tilde{\Theta}_e$ and $|\xi_e^o-\xi_e'^o|$ going large for large η_e'.

Similarly, the matrix elements of $(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{21,21}$ in the twisted geometry coherent basis is given by

$$\begin{aligned}\langle \breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}_e'})_{21,21}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle \overset{\text{large } \eta_e'}{=} & \text{FRHS of Eq. (D8)} + \text{SRHS of Eq. (D8)} + \text{TRHS of Eq. (D8)} \\ & + \frac{1}{\sqrt{t}} e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}}\mathcal{O}(e^{-\frac{(\eta_e')^2}{8t}}),\end{aligned} \tag{D8}$$

where we defined

$$\begin{aligned}\text{FRHS of Eq. (D8)} \coloneqq{}& e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} e^{-\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e-1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right) \\ & \cdot e^{\mathbf{i}d_{N_e-1}(\xi_e^o-\xi_e'^o)}\langle N_e-1, V_e'|N_e-1,V_e\rangle\langle N_e-1,-\tilde{V}_e|N_e-1,-\tilde{V}_e'\rangle\end{aligned} \tag{D9}$$

$$\begin{aligned}\text{SRHS of Eq. (D8)} \coloneqq{}& -e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} e^{-\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e-1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right) \\ & \cdot e^{\mathbf{i}d_{N_e-1}(\xi_e^o-\xi_e'^o)}(1-|\alpha_1(N_e)|^2)\langle N_e-1, V_e'|N_e-1,V_e\rangle\langle N_e-1,-\tilde{V}_e|N_e-1,-\tilde{V}_e'\rangle\end{aligned} \tag{D10}$$

and

$$\begin{aligned}\text{TRHS of Eq. (D8)} \coloneqq{}& e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} e^{\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right) \\ & \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}|\alpha_2(N_e)|^2\langle N_e+1,\ldots|u_e'^{-1}u_e|N_e+1,V_{12}\rangle\langle N_e+1,V_{12}|\tilde{u}_e^{-1}\tilde{u}_e'|N_e+1,\ldots\rangle,\end{aligned} \tag{D11}$$

wherein

$$|1,V_{21}\rangle\otimes|N_e,V_{12}\rangle = \alpha_1(N_e)|N_e-1,V_{12}\rangle + \alpha_2(N_e)|N_e+1,\ldots\rangle + \alpha_3(N_e)|\text{not simple}\rangle, \tag{D12}$$

with $|\alpha_1(N)|^2=\frac{N(2N+D-3)}{(D+N-2)(2N+D-1)}$, $|\alpha_2(N)|^2\leq 1-|\alpha_1(N)|^2$; see more details in Appendix A. The three terms in the right-hand side (rhs) of "=" in Eq. (D8) can be calculated separately. (i). The first term in the rhs (Frhs) of "=" in Eq. (D8) is given as

$$\begin{aligned}\text{FRHS of Eq. (D8)} ={}& e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} e^{-\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e-1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right) \\ & \cdot e^{\mathbf{i}d_{N_e-1}(\xi_e^o-\xi_e'^o)}\langle N_e-1, V_e'|N_e-1,V_e\rangle\langle N_e-1,-\tilde{V}_e|N_e-1,-\tilde{V}_e'\rangle.\end{aligned} \tag{D13}$$

By following a similar analysis of Eqs. (D2) and (D4), we can immediately give

$$\frac{\text{FRHS of Eq. (D8)}}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||} \overset{\text{large } \eta_e'}{=} e^{-\mathbf{i}\xi_e'^o} e^{\tilde{\Theta}_e/2} e^{-t/2} e^{\mathbf{i}\frac{D+1}{2}(\xi_e^o-\xi_e'^o)} e^{-\frac{\mathbf{i}}{2}(\xi_e^o-\xi_e'^o+\tilde{\varphi}_e)} e^{-(\eta_e/2-\eta_e'/2)} \frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\left(1+\mathcal{O}\left(\frac{t}{\eta_e'}\right)+\mathcal{O}(e^{-1/t})\right) \quad \text{(D14)}$$

for large $\tilde{\Theta}_e \ll \eta_e + \eta_e'$, and

$$0 < \left|\frac{\text{FRHS of Eq. (D8)}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large } \eta_e'}{\lesssim} \frac{\sqrt{\frac{2t}{\pi}}\tilde{f}_1'(\frac{\eta_e}{t},\frac{\eta_e'}{t})e^{-t(\frac{\eta_e'}{4t}-\frac{D+1}{2})^2} + (\check{\mathrm{P}}(\frac{\eta_e}{4t}+\frac{\eta_e'}{4t}))^{3/2}(\check{\mathrm{P}}(\frac{\eta_e}{4t}+\frac{\eta_e'}{4t}-1))^{1/2}e^{-\frac{t}{2}(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t})^2}e^{-[\frac{\eta_e'}{4t}]\tilde{\Theta}_e}}{\check{\mathrm{P}}(\frac{\eta_e}{2t})\check{\mathrm{P}}(\frac{\eta_e'}{2t})} \quad \text{(D15)}$$

for $\tilde{\Theta}_e \simeq \eta_e + \eta_e'$ or $\tilde{\Theta}_e \gg \eta_e + \eta_e'$, where $\tilde{f}_1'(\frac{\eta_e}{t},\frac{\eta_e'}{t}) \coloneqq ([\eta_e'/4t]+1)\exp(-t(\frac{\eta_e}{2t}-\frac{\eta_e'}{4t}-\frac{D+3}{2})^2)(\check{\mathrm{P}}(\frac{\eta_e'}{4t}-1))^{1/2}(\check{\mathrm{P}}(\frac{\eta_e'}{4t}))^{3/2}$. Also, for the case $\tilde{\Theta}_e \simeq \eta_e + \eta_e'$ or $\tilde{\Theta}_e \gg \eta_e + \eta_e'$, we can conclude that $\left|\frac{\text{FRHS of Eq. (D8)}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|$ is always suppressed exponentially by the factors $e^{-t(\frac{\eta_e'}{4t}-\frac{D+1}{2})^2}$ and $e^{-\frac{t}{2}(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t})^2}e^{-[\frac{\eta_e'}{4t}]\tilde{\Theta}_e}$ based on Eq. (D15). (ii). The second term in the right-hand side (SRHS) of "=" in Eq. (D8) reads

$$\text{SRHS of Eq. (D8)} = -e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} e^{-\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e-1}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot e^{\mathbf{i}d_{N_e-1}(\xi_e^o-\xi_e'^o)}(1-|\alpha_1(N_e)|^2)\langle N_e-1,V_{12}|u_e'^{-1}u_e|N_e-1,V_{12}\rangle\langle N_e-1,V_{12}|\tilde{u}_e^{-1}\tilde{u}_e'|N_e-1,V_{12}\rangle. \quad \text{(D16)}$$

It is easy to see

$$|\text{SRHS of Eq. (D8)}| \le e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}$$
$$\cdot\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e-1}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)\left(1-\frac{N_e(2N_e+D-3)}{(D+N_e-2)(2N_e+D-1)}\right)$$
$$= e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}}e^{-\frac{t}{2}(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t}-1)^2}\sum_{[\tilde{k}_e]}(\tilde{\mathrm{P}}_1(\tilde{k}_e))^{1/4}\frac{\tilde{\mathrm{P}}_2(\tilde{k}_e)}{\tilde{P}_3(\tilde{k}_e)}(\exp(-2t\tilde{k}_e^2)) \quad \text{(D17)}$$

for large η_e'; here $\tilde{k}_e$ is defined by $\tilde{k}_e \coloneqq d_{N_e}-\frac{\eta_e'}{4t}-\frac{\eta_e}{4t}-\frac{1}{2}=[\tilde{k}_e]+\bmod(\tilde{k}_e,1)$, $(\tilde{\mathrm{P}}_1(\tilde{k}_e))^{1/4}$ is defined by $(\tilde{\mathrm{P}}_1(\tilde{k}_e))^{1/4}=(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e-1}))^{1/2}$, and $\frac{\tilde{\mathrm{P}}_2(\tilde{k}_e)}{\tilde{P}_3(\tilde{k}_e)}$ is defined by $\frac{\tilde{\mathrm{P}}_2(\tilde{k}_e)}{\tilde{P}_3(\tilde{k}_e)}=1-\frac{N_e(2N_e+D-3)}{(D+N_e-2)(2N_e+D-1)}$. Then by using the result of case III discussed in Appendix B, we have

$$0 < \left|\frac{\text{SRHS of Eq. (D8)}}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large } \eta_e'}{\lesssim} \frac{e^{-\frac{t}{2}(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t}-1)^2}(\tilde{\mathrm{P}}_1(0))^{1/4}(\frac{\eta_e'+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}})+\mathcal{O}(\frac{t}{\eta_e'}))}{\check{\mathrm{P}}(\frac{\eta_e}{2t})\check{\mathrm{P}}(\frac{\eta_e'}{2t})}. \quad \text{(D18)}$$

Thus we can conclude that $\left|\frac{\text{SRHS of Eq. (D8)}}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right|$ is always suppressed by the factor $(\frac{\eta_e'+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta_e'^2}{8t}})+\mathcal{O}(\frac{t}{\eta_e'}))$ for $\eta_e \simeq \eta_e'$ and by the factor $e^{-\frac{t}{2}(\frac{\eta_e'}{2t}-\frac{\eta_e}{2t}-1)^2}$ for $|\eta_e-\eta_e'|$ being large in the case of large η_e'. (iii). The third term in the right-hand side (TRHS) of "=" in Eq. (D8) is given as

$$\text{TRHS of Eq. (D8)} = e^{\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}}e^{\mathbf{i}\xi_e'^o}\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2-t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\right)$$
$$\cdot|\alpha_2(N_e)|^2e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}\langle N_e+1,\ldots|u_e'^{-1}u_e|N_e+1,V_{12}\rangle\langle N_e+1,V_{12}|\tilde{u}_e^{-1}\tilde{u}_e'|N_e+1,\ldots\rangle, \quad \text{(D19)}$$

wherein $0 \leq |\alpha_2(N_e)|^2 \leq 1 - \frac{N_e(2N_e+D-3)}{(D+N_e-2)(2N_e+D-1)}$. By using the result of the case III discussed in Appendix B, Eq. (D19) can be estimated following a similar procedure of Eq. (D16), which gives

$$0 < \left|\frac{\text{TRHS of Eq. (D8)}}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right| \overset{\text{large } \eta'_e}{\lesssim} \frac{e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2}(\tilde{\mathrm{P}}(0))^{1/4}(\frac{\eta'_e+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})+\mathcal{O}(\frac{t}{\eta'_e}))}{\breve{\mathrm{P}}(\frac{\eta_e}{2t})\breve{\mathrm{P}}(\frac{\eta'_e}{2t})}. \tag{D20}$$

Here we defined $\tilde{k}_e \coloneqq d_{N_e} - \frac{\eta'_e}{4t} - \frac{\eta_e}{4t} + \frac{1}{2}$ and $(\tilde{\mathrm{P}}(\tilde{k}_e))^{1/4} \coloneqq (\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}$. Then we can conclude that $\left|\frac{\text{TRHS of Eq. (D8)}}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right|$ is always suppressed by the factor $(\frac{\eta'_e+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})+\mathcal{O}(\frac{t}{\eta'_e}))$ for $\eta_e \simeq \eta'_e$ and by the factor $e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2}$ for $|\eta_e - \eta'_e|$ being large in the case of large η'_e. Finally, by combining the results of Eqs. (D14), (D15), (D18), and (D20) we get

$$\left|\frac{\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{21,21}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||} - e^{-\mathbf{i}\xi'^o_e}\frac{\langle\breve{\Psi}_{\mathbb{H}^o_e}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right| \overset{\text{large } \eta'_e}{\lesssim} t|\tilde{f}_{21}(\mathbb{H}^o_e,\mathbb{H}'^o_e)|\cdot\left|\frac{\langle\breve{\Psi}_{\mathbb{H}^o_e}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right|, \tag{D21}$$

where $\tilde{f}_{21}(\mathbb{H}^o_e,\mathbb{H}'^o_e)$ is a function whose growth is always suppressed by $\left|\frac{\langle\breve{\Psi}_{\mathbb{H}'^o_e}|\breve{\Psi}_{\mathbb{H}^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right|$ exponentially as $|\eta_e - \eta'_e|$, $\tilde{\Theta}_e$ and $|\xi^o_e - \xi'^o_e|$ going large for large η'_e.

For the off-diagonal component $(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{12,21}$ of holonomy operators, we have

$$\begin{aligned}
&\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{12,21}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle \\
&\overset{\text{large } \eta'_e}{=} e^{\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}}e^{\mathbf{i}\xi'^o_e}\sum_{N_e}(\dim(\pi_{N_e}))^{\frac{3}{2}}(\dim(\pi_{N_e+1}))^{\frac{1}{2}}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta'_e}{2t}-d_{N_e}\right)^2\right) \\
&\quad\cdot e^{\mathbf{i}d_{N_e+1}(\xi^o_e-\xi'^o_e)}\alpha_2(N_e)\langle N_e+1, V'_e|N_e+1, V_e\rangle\langle N_e+1, -\tilde{V}_e|N_e+1, \ldots\rangle + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-(\eta'_e)^2/(8t)}) \\
&\overset{\text{large } \eta'_e}{\lesssim} e^{\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}} \\
&\quad\cdot\sum_{N_e}(\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta'_e}{2t}-d_{N_e}\right)^2\right)\sqrt{1-|\alpha_1(N_e)|^2},
\end{aligned} \tag{D22}$$

where $\alpha_2(N_e)$ satisfies $0 \leq |\alpha_2(N_e)| \leq \sqrt{1-|\alpha_1(N_e)|^2}$. Similar to the calculation of Eq. (D17), the result of the case III discussed in Appendix B is applicable for (D22) and we get

$$0 < \left|\frac{\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{12,21}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right| \overset{\text{large } \eta'_e}{\lesssim} \frac{e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2}(\tilde{\mathrm{P}}(0))^{1/4}\left(\frac{\eta'_e+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})+\mathcal{O}\left(\sqrt{\frac{t}{\eta'_e}}\right)\right)}{\breve{\mathrm{P}}(\frac{\eta_e}{2t})\breve{\mathrm{P}}(\frac{\eta'_e}{2t})}. \tag{D23}$$

Here $\tilde{k}_e$ is defined by $\tilde{k}_e \coloneqq d_{N_e} - \frac{\eta'_e}{4t} - \frac{\eta_e}{4t} + \frac{1}{2}$ and $(\tilde{\mathrm{P}}(\tilde{k}_e))^{1/4} \coloneqq (\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}$. Then, we can conclude that $\left|\frac{\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{12,21}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}\right|$ is always suppressed by the factor $(\frac{\eta'_e+\eta_e}{\sqrt{t}}\mathcal{O}(e^{-\frac{\eta'^2_e}{8t}})+\mathcal{O}(\sqrt{\frac{t}{\eta'_e}}))$ for $\eta_e \simeq \eta'_e$ and by the factor $e^{-\frac{t}{2}(\frac{\eta'_e}{2t}-\frac{\eta_e}{2t}+1)^2}$ for $|\eta_e - \eta'_e|$ being large in the case of large η'_e. Based on Eq. (D23), we can evaluate Eq. (D22) by

$$\frac{|\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{12,21}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle|}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||} \overset{\text{large } \eta'_e}{\lesssim} t|\tilde{f}'(\mathbb{H}^o_e,\mathbb{H}'^o_e)|\frac{|\langle\breve{\Psi}_{\mathbb{H}^o_e}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle|}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}, \tag{D24}$$

where $\tilde{f}'(\mathbb{H}^o_e,\mathbb{H}'^o_e)$ is a function whose growth is always suppressed by $\frac{|\langle\breve{\Psi}_{\mathbb{H}^o_e}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle|}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}$ exponentially as $|\eta_e - \eta'_e|$, $\Theta(u_e, u'_e)$, and $|\xi^o_e - \xi'^o_e|$ going large for large η'_e. Following similar calculations we can also give

$$\frac{|\langle\breve{\Psi}_{\mathbb{H}^o_e}|(\widehat{u'^{-1}_e h_e \tilde{u}'_e})_{21,12}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle|}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||} \overset{\text{large } \eta'_e}{\lesssim} t|\tilde{f}''(\mathbb{H}^o_e,\mathbb{H}'^o_e)|\frac{|\langle\breve{\Psi}_{\mathbb{H}^o_e}|\breve{\Psi}_{\mathbb{H}'^o_e}\rangle|}{||\breve{\Psi}_{\mathbb{H}'^o_e}||||\breve{\Psi}_{\mathbb{H}^o_e}||}, \tag{D25}$$

where $\tilde{f}''(\mathbb{H}_e^o, \mathbb{H}_e'^o)$ is a function whose growth is always suppressed by $\frac{|\langle \breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle|}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ exponentially as $|\eta_e - \eta_e'|, \Theta(u_e, u_e')$ and $|\xi_e^o - \xi_e'^o|$ going large for large η_e'.

Let us further consider the components $(\widehat{u_e'^{-1} h_e \tilde{u}_e'})_{\imath\jmath,\imath'\jmath'}$ of the holonomy operator with $(\imath,\jmath),(\imath',\jmath') \in \{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$. Similar to the calculations of Eqs. (D1), (D8), and (D22), the matrix elements of $(\widehat{u_e'^{-1} h_e \tilde{u}_e'})_{\imath\jmath,\imath'\jmath'}$ in the twisted geometry coherent state basis can be evaluated as

$$\begin{aligned}
&\langle \breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1} h_e \tilde{u}_e'})_{\imath\jmath,\imath'\jmath'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta_e')^2+2t^2(D-1)^2}{4t}} \\
&= -e^{\mathbf{i}\xi_e'^o}\sum_{N_e}\Bigg((\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\bigg(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\bigg) \\
&\quad \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}\frac{1}{(N_e+1)^2}\langle N_e+1, V_{12}|u_e^{-1}u_e'(\tau^{1\imath}\pm\tau^{2\jmath})|N_e+1,V_{12}\rangle \\
&\quad \cdot \langle N_e+1, V_{12}|(\tau^{\imath'1}\pm\tau^{\jmath'2})\tilde{u}_e'^{-1}\tilde{u}_e|N_e+1,V_{12}\rangle\Bigg) + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta_e')^2}{8t}}) \\
&= -e^{\mathbf{i}\xi_e'^o}\sum_{N_e}\Bigg((\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\bigg(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\bigg) \\
&\quad \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}\mathcal{T}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)\langle N_e+1,V_{12}|u_e^{-1}u_e'|N_e+1,V_{12}\rangle\langle N_e+1,V_{12}|\tilde{u}_e'^{-1}\tilde{u}_e|N_e+1,V_{12}\rangle\Bigg) \\
&\quad + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta_e')^2}{8t}}) \\
&= \text{FRHS of Eq. (D26)} + \frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta_e')^2}{8t}})
\end{aligned} \tag{D26}$$

for large η_e' and $(\imath,\jmath),(\imath',\jmath') \in \{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$, where we used Eq. (A32) and defined

$$\begin{aligned}
&\text{FRHS of Eq. (D26)} \\
&\coloneqq -e^{\mathbf{i}\xi_e'^o}\sum_{N_e}\Bigg((\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\bigg(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2 - t\left(\frac{\eta_e'}{2t}-d_{N_e}\right)^2\bigg) \\
&\quad \cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}\mathcal{T}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)\langle N_e+1,V_{12}|u_e^{-1}u_e'|N_e+1,V_{12}\rangle\langle N_e+1,V_{12}|\tilde{u}_e'^{-1}\tilde{u}_e|N_e+1,V_{12}\rangle\Bigg)
\end{aligned} \tag{D27}$$

with

$$\begin{aligned}
\mathcal{T}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e) &\coloneqq \frac{\langle 1,V_{12}|\tilde{u}_e'^{-1}\tilde{u}_e(\tau^{1\imath'}\pm\tau^{2\jmath'})|1,V_{12}\rangle}{\langle 1,V_{12}|\tilde{u}_e'^{-1}\tilde{u}_e|1,V_{12}\rangle}\frac{\langle 1,V_{12}|(\tau^{1\imath}\pm\tau^{2\jmath})u_e^{-1}u_e'|1,V_{12}\rangle}{\langle 1,V_{12}|u_e^{-1}u_e'|1,V_{12}\rangle} \\
&= e^{\tilde{\Theta}_e}e^{\mathbf{i}\tilde{\varphi}_e}\tilde{\mathcal{T}}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e),
\end{aligned} \tag{D28}$$

and

$$\tilde{\mathcal{T}}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e) \coloneqq \langle 1,V_{12}|\tilde{u}_e'^{-1}\tilde{u}_e(\tau^{1\imath'}\pm\tau^{2\jmath'})|1,V_{12}\rangle\langle 1,V_{12}|(\tau^{1\imath}\pm\tau^{2\jmath})u_e^{-1}u_e'|1,V_{12}\rangle, \tag{D29}$$

where we used the notation that $\tau^{1\imath}\pm\tau^{2\jmath}$ takes $\tau^{1\imath}+\tau^{2\jmath}$ if $\imath<\jmath$, and $\tau^{1\imath}-\tau^{2\jmath}$ if $\imath>\jmath$. Note that $\mathcal{T}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)$ and $\tilde{\mathcal{T}}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)$ satisfy

$$\mathcal{T}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)=0, \qquad \tilde{\mathcal{T}}_{\pm\imath\jmath,\pm\imath'\jmath'}(u_e^{-1}u_e',\tilde{u}_e'^{-1}\tilde{u}_e)=0, \quad \text{if } \tilde{\Theta}_e=0 \tag{D30}$$

and

$$|\mathcal{T}_{\pm\iota\jmath,\pm\iota'\jmath'}(u_e^{-1}u'_e,\tilde{u}_e'^{-1}\tilde{u}_e)|\leq 4e^{\tilde{\Theta}_e},$$
$$|\tilde{\mathcal{T}}_{\pm\iota\jmath,\pm\iota'\jmath'}(u_e^{-1}u'_e,\tilde{u}_e'^{-1}\tilde{u}_e)|\leq 4, \tag{D31}$$

for $(\iota,\jmath),(\iota',\jmath')\in\{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$. Moreover, by recalling Eq. (38), we also have that $|\tilde{\mathcal{T}}_{\pm\iota\jmath,\pm\iota'\jmath'}(u_e^{-1}u'_e,\tilde{u}_e'^{-1}\tilde{u}_e)|$ with $(\iota,\jmath),(\iota',\jmath')\in\{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$ grows no faster than the exponentials as $\Theta(u_e,u'_e)$ or $\Theta(\tilde{u}_e,\tilde{u}'_e)$ going large. Then, similar to the analysis for Eqs. (D9), (D10), and (D11), we have

$$\begin{aligned}&|\text{FRHS of Eq. (D26)}|\\ &\overset{\text{large }\eta'_e}{=} e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}}|\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle|\\ &\quad\cdot|\mathcal{T}_{\pm\iota\jmath,\pm\iota'\jmath'}(u_e^{-1}u'_e,\tilde{u}_e'^{-1}\tilde{u}_e)|(1+\mathcal{O}(t)+\mathcal{O}(e^{-1/t})).\end{aligned} \tag{D32}$$

Finally, let us combine Eqs. (D30), (D31), and (D32), and notice that $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ is unity at $\mathbb{H}_e'^o=\mathbb{H}_e^o$ and decaying exponentially fast to 0 for $\mathbb{H}_e'^o\neq\mathbb{H}_e^o$, we get

$$\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{\iota\jmath,\iota'\jmath'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \overset{\text{large }\eta'_e}{\lesssim} t|\tilde{f}'''(\mathbb{H}_e^o,\mathbb{H}_e'^o)|\left|\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}\right| \tag{D33}$$

with $(\iota,\jmath),(\iota',\jmath')\in\{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$, where $\tilde{f}'''(\mathbb{H}_e^o,\mathbb{H}_e'^o)$ is a function whose growth is always suppressed by $\frac{|\langle\breve{\Psi}_{\mathbb{H}_e^o}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle|}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ exponentially as $|\eta_e-\eta'_e|$, $\Theta(u_e,u'_e)$ and $|\xi_e^o-\xi_e'^o|$ going large for large η'_e.

The rest of the components of the holonomy operators are $(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{12,\iota\jmath}$ and $(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{21,\iota\jmath}$ and their transpositions $(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{\iota\jmath,12}$ and $(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{\iota\jmath,21}$ with $(\iota,\jmath)\in\{(3,4),(4,3),\ldots,(D,D+1),(D+1,D)\}$. By using the similar techniques utilized in the calculations of Eqs. (D22) and (D26), the matrix elements of these components of the holonomy operators in twisted geometry coherent state basis can be evaluated, and the results combining with Eqs. (D7), (D21), (D24), and (D33) give the main result in Sec. III B 2 in the case of $(D+1)$ being even.

2. $D+1$ odd

Now let us turn to the case of $(D+1)$ being odd. The results of the components $\frac{\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{\iota\jmath,\iota'\jmath'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle}{||\breve{\Psi}_{\mathbb{H}_e'^o}||||\breve{\Psi}_{\mathbb{H}_e^o}||}$ with $(\iota,\jmath)$ and $(\iota',\jmath')\in\{(3,4),(4,3),\ldots,(D-1,D),(D,D-1)\}$ are identical with those in the case of $(D+1)$ being even. Besides, there are extra holonomy operators $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,(D+1)}$ and $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{(D+1),\iota\jmath}$ with $(\iota,\jmath)\in\{(1,2),(2,1),(3,4),\ldots,(D-1,D),(D,D-1)\}$ in the case of $(D+1)$ being odd, which are defined by

$$(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,(D+1)}\coloneqq\langle 1,V_{\iota\jmath}|\widehat{u_e^{-1}h_e\tilde{u}_e}|1,\delta_{D+1}\rangle \tag{D34}$$

and

$$(\widehat{u_e^{-1}h_e\tilde{u}_e})_{(D+1),\iota\jmath}\coloneqq\langle 1,\delta_{D+1}|\widehat{u_e^{-1}h_e\tilde{u}_e}|1,V_{\iota\jmath}\rangle \tag{D35}$$

respectively, where $|1,\delta_{D+1}\rangle$ is defined in Appendix A. Let us analyze $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{(D+1),\iota\jmath}$ as an example. Notice Eq. (A34) in Appendix A, and we have

$$\begin{aligned}&\langle\breve{\Psi}_{\mathbb{H}_e^o}|(\widehat{u_e'^{-1}h_e\tilde{u}'_e})_{D+1,\iota'\jmath'}|\breve{\Psi}_{\mathbb{H}_e'^o}\rangle e^{-\frac{(\eta_e)^2+(\eta'_e)^2+2t^2(D-1)^2}{4t}}\\ &\quad=-e^{\mathbf{i}\xi_e'^o}\sum_{N_e}\Bigg((\dim(\pi_{N_e}))^{3/2}(\dim(\pi_{N_e+1}))^{1/2}\exp\left(-t\left(\frac{\eta_e}{2t}-d_{N_e+1}\right)^2-t\left(\frac{\eta'_e}{2t}-d_{N_e}\right)^2\right)\\ &\quad\cdot e^{\mathbf{i}d_{N_e+1}(\xi_e^o-\xi_e'^o)}\frac{\sqrt{2}}{(N_e+1)}\langle N_e+1,V_{12}|u_e^{-1}u'_e\tau^{1,D+1}|N_e+1,V_{12}\rangle\cdot\langle N_e+1,V_{12}|\tilde{u}_e^{-1}\tilde{u}'_e|1,V_{\iota'\jmath'};N_e,V_{12}\rangle\Bigg)+\frac{1}{\sqrt{t}}\mathcal{O}(e^{-\frac{(\eta'_e)^2}{8t}})\end{aligned} \tag{D36}$$

for large η'_e. Note that the key factor $\langle N_e+1,V_{12}|u_e^{-1}u'_e\tau^{1,D+1}|N_e+1,V_{12}\rangle$ in Eq. (D36) is similar to that in Eq. (D26), thus Eq. (D36) can be calculated following the similar procedures of the calculation of Eq. (D26). Also, the operator $(\widehat{u_e^{-1}h_e\tilde{u}_e})_{\iota\jmath,(D+1)}$ can be evaluated similarly. Finally, by combining all of the results we reach the main result in Sec. III B 2 in the case of $(D+1)$ being odd.

[1] Gaoping Long, Xiangdong Zhang, and Cong Zhang, Twisted geometry coherent states in all dimensional loop quantum gravity: Construction and peakedness properties, Phys. Rev. D **105**, 066021 (2022).

[2] Carlo Rovelli, Graviton Propagator from Background-Independent Quantum Gravity, Phys. Rev. Lett. **97**, 151301 (2006).

[3] Eugenio Bianchi, Elena Magliaro, and Claudio Perini, LQG propagator from the new spin foams, Nucl. Phys. **B822**, 245 (2009).

[4] Eugenio Bianchi, Elena Magliaro, and Claudio Perini, Coherent spin-networks, Phys. Rev. D **82**, 024012 (2010).

[5] Andrea Calcinari, Laurent Freidel, Etera Livine, and Simone Speziale, Twisted geometries coherent states for loop quantum gravity, Classical Quantum Gravity **38**, 025004 (2021).

[6] Laurent Freidel and Simone Speziale, Twisted geometries: A geometric parametrization of SU(2) phase space, Phys. Rev. D **82**, 084040 (2010).

[7] B. Hall, The segal-bargmann "coherent state" transform for compact lie groups, J. Funct. Anal. **122**, 103 (1994).

[8] T. Thiemann, Complexifier coherent states for quantum general relativity, Classical Quantum Gravity **23**, 2063 (2006).

[9] Thomas Thiemann, Gauge field theory coherent states (GCS). I. General properties, Classical Quantum Gravity **18**, 2025 (2001).

[10] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS). II. Peakedness properties, Classical Quantum Gravity **18**, 2561 (2001).

[11] T. Thiemann and O. Winkler, Gauge field theory coherent states (GCS). III. Ehrenfest theorems, Classical Quantum Gravity **18**, 2561 (2001).

[12] Muxin Han and Hongguang Liu, Effective dynamics from coherent state path integral of full loop quantum gravity, Phys. Rev. D **101**, 046003 (2020).

[13] Muxin Han and Hongguang Liu, Semiclassical limit of new path integral formulation from reduced phase space loop quantum gravity, Phys. Rev. D **102**, 024083 (2020).

[14] Cong Zhang, Shicong Song, and Muxin Han, First-order quantum correction in coherent state expectation value of loop-quantum-gravity Hamiltonian, Phys. Rev. D **105**, 064008 (2022).

[15] Gaoping Long and Yongge Ma, Effective dynamics of weak coupling loop quantum gravity, Phys. Rev. D **105**, 044043 (2022).

[16] Gaoping Long, Cong Zhang, and Xiangdong Zhang, Superposition type coherent states in all dimensional loop quantum gravity, Phys. Rev. D **104**, 046014 (2021).

[17] Gaoping Long and Norbert Bodendorfer, Perelomov-type coherent states of $\mathrm{SO}(D+1)$ in all-dimensional loop quantum gravity, Phys. Rev. D **102**, 126004 (2020).

[18] Etera R. Livine and Simone Speziale, New spin-foam vertex for quantum gravity, Phys. Rev. D **76**, 084028 (2007).

[19] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn, New variables for classical and quantum gravity in all dimensions. I. Hamiltonian analysis, Classical Quantum Gravity **30**, 045001 (2013).

[20] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn, New variables for classical and quantum gravity in all dimensions. II. Lagrangian analysis, Classical Quantum Gravity **30**, 045002 (2013).

[21] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn, New variables for classical and quantum gravity in all dimensions. III. Quantum theory, Classical Quantum Gravity **30**, 045003 (2013).

[22] R. Arnowitt, S. Deser, and C. W. Misner, Dynamical structure and definition of energy in general relativity, Phys. Rev. **116**, 1322 (1959).

[23] Abhay Ashtekar and Jerzy Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity **21**, R53 (2004).

[24] Thomas Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007).

[25] Carlo Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2007).

[26] Carlo Rovelli and Francesca Vidotto, *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge University Press, Cambridge, England, 2014).

[27] Muxin Han, M. A. Yongge, and Weiming Huang, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2007).

[28] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn, Towards loop quantum supergravity (LQSG). I. Rarita-Schwinger sector, Classical Quantum Gravity **30**, 045006 (2013).

[29] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn, On the implementation of the canonical quantum simplicity constraint, Classical Quantum Gravity **30**, 045005 (2013).

[30] Maite Dupuis and Etera R. Livine, Revisiting the simplicity constraints and coherent intertwiners, Classical Quantum Gravity **28**, 085001 (2011).

[31] Wojciech Kamiński, Marcin Kisielowski, and Jerzy Lewandowski, Spin-foams for all loop quantum gravity, Classical Quantum Gravity **27**, 095006 (2010).

[32] Jonathan Engle, Roberto Pereira, and Carlo Rovelli, Loop-Quantum-Gravity Vertex Amplitude, Phys. Rev. Lett. **99**, 161301 (2007).

[33] Jonathan Engle, Etera Livine, Roberto Pereira, and Carlo Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. **B799**, 136 (2008).

[34] Laurent Freidel and Kirill Krasnov, A new spin foam model for 4D gravity, Classical Quantum Gravity **25**, 125018 (2008).

[35] L. Freidel, K. Krasnov, and R. Puzio, BF description of higher-dimensional gravity theories, Adv. Theor. Math. Phys. **3**, 1289 (1999).

[36] John W. Barrett and Louis Crane, Relativistic spin networks and quantum gravity, J. Math. Phys. (N.Y.) **39**, 3296 (1998).

[37] Gaoping Long and Chun-Yen Lin, Geometric parametrization of $SO(D+1)$ phase space of all dimensional loop quantum gravity, Phys. Rev. D **103**, 086016 (2021).

[38] Eugenio Bianchi, Pietro Dona, and Simone Speziale, Polyhedra in loop quantum gravity, Phys. Rev. D **83**, 044035 (2011).

[39] Gaoping Long and Yongge Ma, General geometric operators in all dimensional loop quantum gravity, Phys. Rev. D **101**, 084032 (2020).

[40] Gaoping Long and Yongge Ma, Polytopes in all dimensional loop quantum gravity, Eur. Phys. J. C **82**, 41 (2022).
[41] Xiangdong Zhang, Higher dimensional loop quantum cosmology, Eur. Phys. J. C **76**, 395 (2016).
[42] Gaoping Long, Chun-Yen Lin, and Yongge Ma, Coherent intertwiner solution of simplicity constraint in all dimensional loop quantum gravity, Phys. Rev. D **100**, 064065 (2019).
[43] N. Y. Vilenkin and Anatoli Ulianovich Klimyk, *Representation of Lie Groups and Special Functions: Volume 2: Class I Representations, Special Functions, and Integral Transforms* (Springer Science & Business Media, New York, 2013), Vol. 75.
[44] Askold Perelomov, *Generalized Coherent States and Their Applications* (Springer Science & Business Media, New York, 2012).
[45] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics Vol. 2—Fourier Analysis, Self-Adjointness* (Academic, New York, 1975).
[46] S. Bochner, *Vorlesungen über Fouriersche Integrale* (Chelsea Publishing Company, New York, 1948).
[47] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG). III. Semiclassical perturbation theory, Classical Quantum Gravity **24**, 2565 (2007).