Retrieving black hole information from the main Lorentzian saddle point

Cristiano Germani[*]

Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain
and Departament de Física Quàntica i Astrofísica, Facultat de Física, Universitat de Barcelona,
Martí i Franquès 1, 08028 Barcelona, Spain

One of the most striking evidences of the information loss paradox is that, according to the Hawking's calculation, the correlation functions of a test scalar field exponentially decay in time. In this paper, I argue that a judicious use of the steepest descent expansion on the classical saddle point (the black hole background) is enough to change this early time decay into a late time growth in agreement with information retrieval. I will explicitly show this in the Jackiw-Teitelboim gravity. There, the so-called "ramp" in the bulk tow-point function is analytically obtained without the need of any other subdominant configurations of the gravity path integral.

I. INTRODUCTION

Black holes evaporate [1]. According to the standard Hawking calculation, they do by releasing a thermal black body spectrum that is only parametrized by the black hole's mass,[1] thus carrying negligible information about the quantum state of the black hole. Trusting the Hawking result until full evaporation would then lead to an information paradox [2], namely that quantum systems do not conserve information.

One might think that all this information is stored inside the event horizon and for this reason, inaccessible. However, Page [3] has shown that trace of information should start to emerge already when the black hole is roughly evaporated half of its mass (Page time). The intriguing fact is that, at the Page time, a large black hole is still large, and the Hawking calculation, based on semiclassical gravity, should be correct.

It is then clear that the Hawking calculation must contain an approximation that breaks down well before the black hole's curvature becomes large, i.e., that full quantum gravity effects become dominant.

The situation is perhaps clearer by looking at correlation functions on a Black Hole spacetime. Let us consider a

scalar field. The two point function is formally defined with the following path integral[2]

$$\langle \phi(t', x')\phi(t, x)\rangle = N \int \mathcal{D}\phi \mathcal{D}g_{\mu\nu}\phi(t', x')\phi(t, x)e^{iS[\phi, g]},$$

where S is the action of the scalar-gravity system, and N the normalization.[3] In the approximation of a test scalar field and low curvatures, we can consider the saddle point of the sole gravitational action, i.e., the point in which $\delta S_{\text{gravity}} = 0$. This defines the semiclassical approximation. Then, we would naively expect (this is the core of Hawking's calculation)

$$\langle \phi(t', x')\phi(t, x)\rangle \simeq \bar{N} \int \mathcal{D}\phi\phi(t', x')\phi(t, x)e^{iS_{\text{scalar}}[\phi, \bar{g}]}, \quad (1)$$

where S_{scalar} is the scalar field Lagrangian evaluated in the saddle point metric $\bar{g}_{\mu\nu}$, and $\bar{N}$ is its related normalization.

To make the point, I will consider a massless scalar field and calculate the correlation function (1) in the vicinity of a spherically symmetric black hole horizon with metric $ds^2 = -\bar{g}(r)dt^2 + \frac{dr^2}{\bar{g}(r)} + r^2 d\Omega_2^2$. Here, the horizon is the sphere $r = r_0$ in which $\bar{g}(r_0) = 0$. Expanding around that, we get the Rindler metric $ds^2 \simeq -(\frac{2\pi}{\beta})^2 x^2 dt^2 + dx^2 + d\ell_2^2$, where $x^2 = \frac{4}{\bar{g}'(r_0)}(r - r_0)$ and $\beta \equiv \frac{4\pi}{\bar{g}'(r_0)}$ is the inverse Bekenstein-Hawking temperature. Finally $d\ell_2^2 = r_0^2 d\Omega^2$. To simplify the calculation, I follow [4]. We can consider the optical metric $\tilde{g}_{\mu\nu} = \frac{g_{\mu\nu}}{g_{tt}}$, the rescaled scalar $\tilde{\phi} = x^{-2}\phi$,

---

[*]germani@icc.ub.edu
[1]For simplicity, we will only discuss uncharged and not rotating black holes.

[2]We use $\hbar = 1$ units.
[3]$N = (\int \mathcal{D}\phi \mathcal{D}g_{\mu\nu}e^{iS[\phi, g]})^{-1}$.

and make use of the Euclidean time $t \to i\frac{\beta}{2\pi}\tau$. In this setup, we can Fourier-Bessel expand $\tilde{\phi}$ as $\tilde{\phi}(x,\tau) = x\sum_{n,\omega} c_{n,\omega}\psi_\omega(x)e^{-i2\pi n\tau}$, where

$$\psi_\omega(\vec{x}) \equiv \frac{\sqrt{2\omega \sinh \pi\omega}}{(2\pi)^{3/2}\pi} e^{-ip_i\ell^i} K_{i\omega}(px),$$

and K is the Bessel function of second kind.

Thus, around the horizon, $iS \to -S_E$ with $S_E = \sum_{n,\omega} S_E(n,\omega)$, where

$$S_E(n,\omega) = 4\pi^2(n^2 + \omega^2)c_{n,\omega}^2.$$

The path integral at coincident spacial points is now Gaussian, and it can be easily evaluated [5]. At large Lorentzian times $(t' - t \gg \beta)$ [6], one gets

$$\langle\phi(t',x)\phi(t,x)\rangle \xrightarrow[(t'-t)\to\infty]{} \sim e^{-\frac{2\pi}{\beta}(t'-t)},$$

which means the correlation function decays exponentially to zero. The same can be checked for higher correlators. This clearly shows that, after a long time, correlations

$$e^{-\frac{h(x)}{g^2}} = e^{-\frac{h(x_0)}{g^2}} e^{-\frac{y^2}{2}h''(x_0)}\left(1 - g\frac{y^3}{6}h'''(x_0) - y^4 h''''(x_0) - g^2\frac{3}{72}y^6 h'''(x_0))^2 + \dots\right).$$

A similar expansion can be done for the function $f(x)$

$$f(x) = f(x_0)\left(1 + gy\frac{f'(x_0)}{f(x_0)} + g^2 y^2\frac{f''(x_0)}{2f(x_0)} + \dots\right). \quad (2)$$

In (2), we already see something that will be crucial for the following. In the limit $f(x_0) \to 0$, the first dominant term in the steepest descent approximation is not $f(x_0)$ but rather its derivatives, which we assume not to vanish. Note that, however, in this limit, the saddle point approximation is still valid.

Combining the above expansions, we get, up to $\mathcal{O}(g^2)$,[4]

$$I(g) \simeq gf(x_0)e^{-\frac{h(x_0)}{g^2}} \int_{-\infty}^{\infty} dy\, e^{-y^2\frac{h''(x_0)}{2}}(1 + g^2 P(y)), \quad (3)$$

where P is a polynomial in y.

Let us think about (3) as a path integral in which h plays the role of the gravitational action, x of the metric, g the Newtonian constant, and finally f the matter part. The semiclassical approximation is immediately recognized. Whenever $f(x_0)$ is not small, we can neglect the g^2 term

functions as calculated above do not longer carry information. What went wrong?

In [7], Maldacena proposed that after a long time, other saddle points of the Euclideanized gravitational action should dominate. However, none has been found that would recover the right information so far. Lately instead [8], it has been conjectured that other configurations of the Euclidean gravitational action maximizing the entanglement entropy (the so-called islands) should dominate the very long time behavior of correlators. While this seems to work in two dimensions [8], it's doubtful that it would in higher [9]. Here, I will propose a more conservative way out to the exponential decay. Before, it is useful to remind the reader about some basis of the steepest descent method.

II. STEEPEST DESCENT

Suppose we would like to evaluate the following integral

$$I(g) = \int_{x_1}^{x_2} dx f(x)e^{-\frac{1}{g^2}h(x)}$$

in the limit of small g and in the case in which $h(x)$ has a minimum in a point $x = x_0$ (the saddle point). By shifting $x = x_0 + gy$, we can expand the exponential as

and $I(g) \propto f(x_0)$. While, whenever $f(x_0)$ is exponentially small, the dominant part of the integral is

$$I(g) \propto g^2\left(\frac{f''(x_0)}{4h''(x_0)} - \frac{f'(x_0)h'''(x_0)}{8h''(x_0)^2}\right). \quad (4)$$

I'd like to pause here by stressing once more that the result (4) has been found by using the saddle point of $h(x)$. Note, however, that whenever $f \to 0$, the whole function $f(x)e^{-h/g}$ does not have a maximum in $x = x_0$ but rather, for a sufficiently steep exponential, two new maximums surrounding a valley in $x = x_0$.[5] One might be tempted to solve the integral by considering these new peaks as new saddle points. This would be indeed correct if the curve in between the two maximums is exponentially steep.

My conjecture, which I shall prove in the two dimensional Jackiw-Teitelboim gravity (JT) [10], is that, at least in the perturbative regime of the evaporation:

The black hole information encoded in the correlations functions starts to emerge when the leading order in the steepest descent expansion of the gravitational path integral becomes of order of the next to leading order. Moreover,

---

[4]Odd polynomials of y integrate to 0.

[5]I thank Roberto Emparan for pointing this out.

this information can be largely retrieved by the sole use of the main Lorentzian saddle point (the black hole background).

III. JT GRAVITY

Two-dimensional gravity has been always the playground to capture certain properties of the significantly more difficult higher dimensional gravity. For example, one might define a black hole generated by a conical singularity where the interior is characterized by a timelike Liouville theory while the exterior by a spacelike one [11]. There, the semiclassical correlators in the interior are dominated by two equally important saddle points; this can be seen either by direct calculation or via holography [11,12].

In this paper, I will consider instead a different theory, the JT gravity.[6] The reason is that, in this theory, it has been lately argued that island configurations should be the key ingredients to recover the information in correlations functions, although, in this case, one has to introduce an external topological term to the path integral "weighting" the different topologies (islands).

Without this extra term, the JT gravity has the following action (for a full review see Ref. [14])

$$S_{\mathrm{JT}} = \frac{1}{16\pi G}\left[\int d^2x\,\Phi\sqrt{-g}(R+2) + 2\int_{\mathrm{bdy}}\Phi_b K\right],$$

where Φ is the dilaton and "bdy" stands for boundary. The last integral is the Gibbons-Hawking term multiplied by the dilaton boundary value $\Phi = \Phi_b$. Units are such that the AdS$_2$ length is dimensionless.

The dilaton acts like a Lagrange multiplier fixing (at the full quantum level) the metric to be locally AdS$_2$. The connection to the black hole is done by taking the Rindler patch of AdS$_2$ and fixing appropriate boundary conditions that make the choice of coordinates physical.[7] Explicitly, the metric is $ds^2 = -4(\rho^2 - \frac{\pi^2}{\beta^2})d\tilde{t}^2 + \frac{d\rho^2}{\rho^2 - \frac{\pi^2}{\beta^2}}$ with $\Phi^2 = 1 + \rho$. The horizons of the two copies of the black hole are in $\rho = \pm\frac{\pi}{\beta}$. The same metric might also be written in a conformal form: After rotating to Euclidean time $t_E = i\frac{2\pi}{\beta}t$, one has [15]

$$ds_E^2 = \frac{4\pi^2}{\beta^2}\frac{dt_E^2 + dz^2}{\sinh^2\frac{2\pi}{\beta}z}. \tag{5}$$

Note that the dilaton diverges at spacial infinity. We will then fix a boundary at finite distance from the horizon at $\rho = \rho_b$.

---

[6]Interestingly, JT gravity and the timelike Liouville theory behave similarly in a certain corner of their parameter's space [13].
[7]Adding a boundary breaks the gauge invariance.

The scalar Euclidean correlators calculated in the Poincarré patch of AdS$_2$, i.e., $ds^2 = Z^{-2}(dT_E^2 + dZ^2)$, correspond to the Hartle-Hawking vacuum of the black hole (5) whenever the periodicity of T_E is $2\pi/\beta$ [15]. Thus, we define, in Poincarré coordinates, the boundary as the closed curve $(f(\tau),\zeta(\tau))$ parametrized by the Euclidean boundary time τ [16]. This curve has a fixed proper length square (ϵ^{-2}) defined by the equation $f'(\tau)^2 + \zeta'(\tau)^2 = \frac{\zeta(\tau)^2}{\epsilon^2}$.[8] The boundary value for the dilaton is $\Phi_b = \frac{\Phi_r(\rho_b)}{\epsilon}$, where Φ_r (finite for $\epsilon \to 0$) is the "renormalized" dilaton. For $\epsilon \to 0$, we recover full AdS$_2$. In the following, we will only consider the leading order in small ϵ; thus, $\zeta(\tau) \simeq \epsilon f'(\tau)$.

At this order, we get [16]

$$S_{\mathrm{JT}} \xrightarrow[\epsilon\to 0]{} \frac{i}{2g^2}\int d\tau\,\mathrm{Sch}(f,\tau), \tag{6}$$

where

$$\mathrm{Sch}(f,u) = -\frac{1}{2}\frac{f''^2}{f'^2} + \left(\frac{f''}{f'}\right)', \tag{7}$$

is the Schwarzian derivative of $f = f(\tau)$. In (6), I have redefined $g^2 \equiv \frac{4\pi G}{\Phi_r} \ll 1$.

Varying this action with respect to f,[9] we find the periodic solution $f = \tan(\frac{\tau}{2})$ that corresponds to the black hole configuration. Indeed, the bulk metric might be equivalently written as [17] $ds^2 = \frac{\dot{f}(u)\dot{f}(v)}{(f(u)-f(v))^2}dudv$, where $u = i\frac{2\pi}{\beta}(t+z)$ and $v = i\frac{2\pi}{\beta}(t-z)$ are now bulk "Euclidean" coordinates. With the Euclidean saddle solution $f = \tan\frac{\tau}{2}$, we recover the black hole metric (5). It is then clear that the gravitational degree of freedom is completely mapped into a function f of the boundary time.

A. Adding matter

In this section, for simplicity but without loss of generality, I will consider a massless scalar field and calculate its two-point function. Alternatively, as it is done repeatedly in the literature, one may calculate its dual correlator living at the boundary of the theory (from now on, dual correlator). The full numerical calculation of this exists in the related one-dimensional Sachdev-Ye-Kitaev theory (SYK) [18] (see, e.g., Ref. [19]). There, one finds an exponential decay of the dual correlation function superseded by a linear growth (the "ramp"), in compatibility with the Page curve. However, the literature interpretation of this result in the path integral language (see, e.g., Refs. [17,20]) differs greatly from what I shall present here. In [17,20], by using

---

[8]$' \equiv d/d\tau$.
[9]We only consider disk topologies.

analytical methods (see also Ref. [21] for alternative techniques), the information recovery is attributed to new nonsaddle gravitational configurations of the Euclidean action weighted by an external topological term related to the Euler characteristic of the configuration chosen. Here instead, I claim that the information retrieving behavior of the bulk correlators is just the result of a correct implementation of the steepest descent method around the Lorentzian saddle point: the classical black hole geometry.

The rest of the paper is about calculating the following two-point correlation function

$$\langle \phi(t', z)\phi(t, z)\rangle = N \int \mathcal{D}\phi \mathcal{D}f \phi(t', z)\phi(t, z)e^{iS_\phi(\phi, f)}e^{iS_{JT}},$$

where S_ϕ is the action of the massless scalar ϕ and N the usual normalization. I will follow the procedure outlined in [17]. Firstly, we will do the path integral of ϕ for a given configuration of f. By noticing that different f are related to different time reparameterizations of AdS$_2$, one can use the well-known scalar correlators in AdS$_2$ and insert them into the path integral for f. One finds (see Ref. [17] for details)

$$\langle \phi(t', z)\phi(t, z)\rangle = -\int_v^u d\tau_1 \int_{v'}^{u'} d\tau_2 N \int \mathcal{D}f \frac{\dot{f}(\tau_1)\dot{f}(\tau_2)}{(f(\tau_1) - f(\tau_2))^2} e^{-\frac{1}{2g^2}\int du \, \text{Sch}(f, u)}. \tag{8}$$

The measure $\mathcal{D}f$ of this path integral requires some discussion [22]. First of all, the path integral has redundant configurations due to the invariance of the action with respect to the $SL(2, \mathcal{R})$ group. Thus, we need to factor out those configurations to get a finite result. Defining $f = \tan(\psi(u)/2)$, the natural measure on the disk is the Pfaffian one $\mathcal{D}f \to \prod_u \frac{d\psi(u)}{\psi'(u)}$. By using a new (Majorana) fermionic variable $\eta = d\psi/\psi'$, one finds that the path integral (8) may be recast into [22]

$$\langle \phi(t', z)\phi(t, z)\rangle = -\int_v^u d\tau_1 \int_{v'}^{u'} d\tau_2 N \int \frac{\mathcal{D}\psi \mathcal{D}\eta}{SL(2, \mathcal{R})} \frac{\dot{f}(\tau_1)\dot{f}(\tau_2)}{(f(\tau_1) - f(\tau_2))^2} e^{-\frac{1}{2}\int du(\frac{\psi''^2}{g^2\psi'^2} - \frac{\psi'^2}{g^2} + \frac{\eta''\eta'}{\psi'^2} - \eta'\eta)}, \tag{9}$$

where $\mathcal{D}\psi$ and $\mathcal{D}\eta$ are now flat measures [23]. Finally, I define $G(\Delta t, z) \equiv \text{Re}(\langle \phi(t', z)\phi(t, z)\rangle)$ with $\Delta t = t' - t$.

IV. RECOVERING THE "RAMP" IN THE STEEPEST DESCENT APPROXIMATION

I will now show that the equivalent to the ramp found numerically in [19] for the boundary correlator precisely comes from the order g^2 in the steepest descent approximation of the path integral in the bulk correlator.

We can expand $\psi = \tau + g\gamma(\tau)$ in (9) and follow the steps of Sec. II. At zeroth order in g, it is very easy to get

$$\langle \phi(t', z)\phi(t, z)\rangle \xrightarrow[\frac{2\pi}{\beta}\Delta t \to \infty]{} \left(1 - \cosh\frac{4\pi z}{\beta}\right)e^{-\frac{2\pi}{\beta}\Delta t},$$

where $\Delta t \equiv (t' - t) > 0$.

The $\mathcal{O}(g^2)$ is instead more complicated. Expanding (9), we get

$$G(\Delta t, z) = \langle \phi(t', z)\phi(t, z)\rangle|_{g=0} - \frac{g^2}{8}\int_u^v d\tau_1 \int_{v'}^{u'} d\tau_2 \csc\left(\frac{\tau_1 - \tau_2}{2}\right)^4 \times (\langle \gamma'(\tau_1)\gamma'(\tau_2)\rangle(1 - \cos(\tau_1 - \tau_2)))$$

$$+ (\langle \gamma(\tau_1)^2\rangle + \langle \gamma(\tau_2)^2\rangle)\left(1 - \frac{1}{2}\cos(\tau_1 - \tau_2)\right) - \langle \gamma(\tau_1)\gamma(\tau_2)\rangle(\cos(\tau_1 - \tau_2) + 2)$$

$$- \sin(\tau_2 - \tau_1)(\langle \gamma(\tau_1)\gamma'(\tau_1)\rangle + \langle \gamma(\tau_2)\gamma'(\tau_2)\rangle + \langle \gamma(\tau_1)\gamma'(\tau_2)\rangle - \langle \gamma(\tau_2)\gamma'(\tau_1)\rangle) + \mathcal{O}(\gamma^4)). \tag{10}$$

The correlators of γ can be calculated by expanding $\gamma = \sum_n e^{-in\tau} k_n$ and imposing the reality conditions. They will result on Gaussian integrals. See, for example, [24]. Similarly, this happens for the fermionic variable ψ (see, e.g., Ref. [22]).

Interestingly enough, the higher correlators (which are still at order g^2) sum to zero. The analytical form of G is rather long and not very illuminating. In Fig. 1, I have plotted the correlation function G for different values of

g and z. There, it is clear that the two-point correlation function smoothly joins a linear growth after an exponential decay. The asymptotic linear growth is

$$|G(\Delta t, z)| \sim 32\pi^2 g^2 z^2 \Delta t. \tag{11}$$

It is important to stress that this correlator *is not* the one-loop correction to the dual operator calculated, for example, in [16]. Here, I have worked out the bulk two-point function
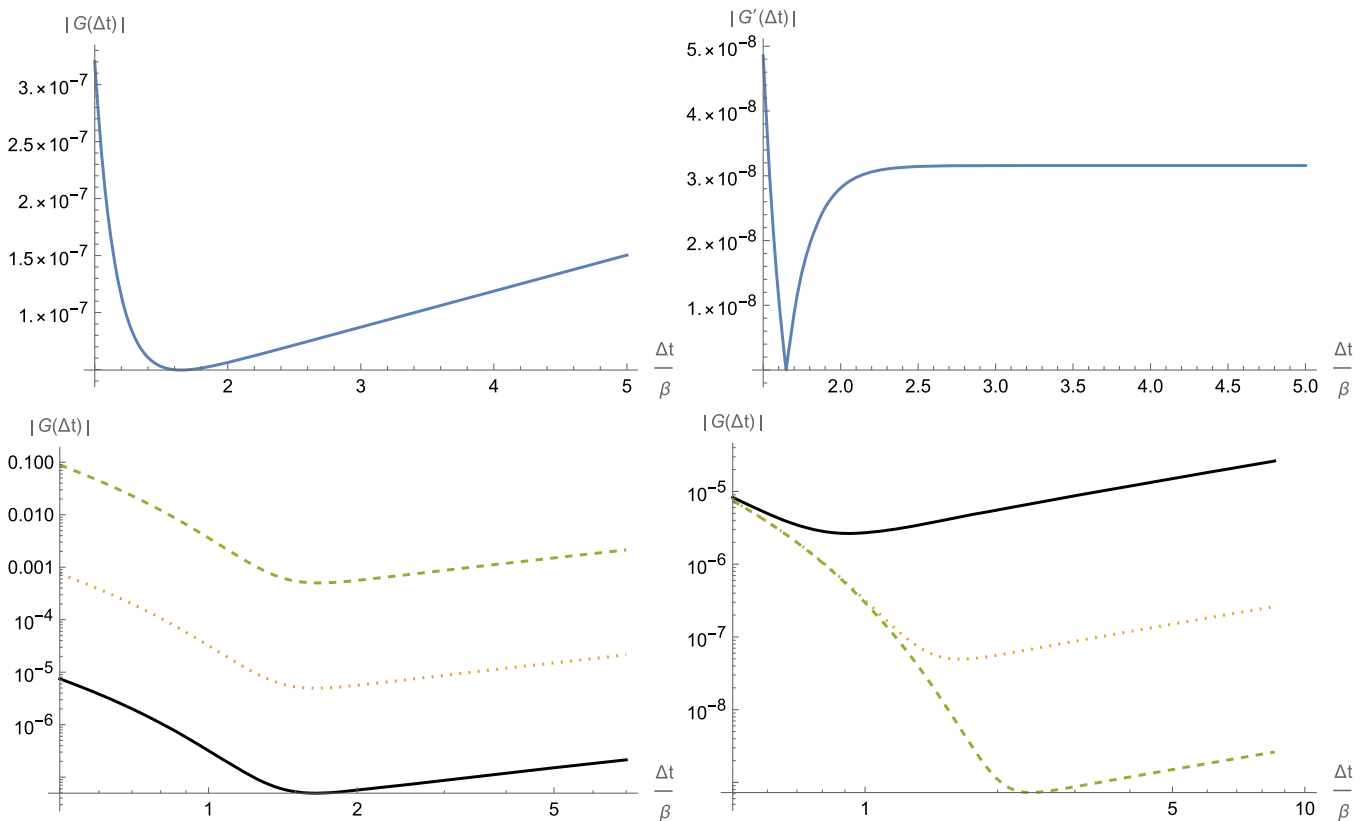
FIG. 1.   First line: correlation function and its time derivative for $z = 0.1$ and $g = 0.01$. We clearly see the emergence of the ramp after the exponential decay. Second line, first plot: correlation functions for different values of z, namely: (solid) $z = 0.1$, (dotted) $z = 0.01$, (dashed) $z = 0.001$ and $g = 0.01$. Second plot second line, $z = 0.1$ and: (solid) $g = 0.1$, (dotted) $g = 0.01$, (dashed) $g = 0.001$. We note that the dependence on z of the dip time is very mild.

of the massless scalar at order g^2, and thus, no input from the AdS/CFT conjecture has been used. Moreover, while in the calculation of the dual operators, there is an ambiguity on defining the Wick rotation back to Lorentzian time, here, as a result of the u and v integration before Wick rotation back to Lorentzian time, this ambiguity disappears.

Finally, one may ask whether the $\mathcal{O}(g^4)$ terms, in the perturbative $g \ll 1$ regime, might also be important whenever the $\mathcal{O}(1)$ term in g becomes subdominant. The latter happens because of the well-known exponential suppression in time due to the presence of a horizon, and thus, it is not parametric in g. On the contrary, because the $\mathcal{O}(g^2)$ is polynomially growing in time, the $\mathcal{O}(g^4)$ is bound to be parametrically smaller than the $\mathcal{O}(g^2)$, at least for the initial part of the ramp. Eventually though, the parametric suppression might be overtaken by the temporal growing of the correlation functions. This is where we expect a plateau behavior [20].

V. CONCLUSIONS AND OUTLOOK

By considering the next to leading order expansion in the steepest descent approximation, I have shown that, at least in the two-dimensional JT gravity, the two-point correlation function in a black hole spacetime changes its exponential decay into a linear growth. This can be immediately compared, at least qualitatively, with the numerical findings in the SYK theory by taking into account that this dual theory lives at the point $z = z_b \simeq \frac{1}{2\rho_b} \ll 1$ of the JT black hole geometry.

It is important to stress that the next to leading order "correction" in steepest descents introduced here is dominant in path integral calculations (gravitational or not) whenever the correlators exponentially decay to zero at the leading order in saddle points. Thus, whether this solves the information paradox or not, it has to be implemented to higher dimensional gravitational theories in the presence of black holes, which I postpone for future work.

Coming back to the JT gravity, I have found that at a finite time Δt, one finds that $G(\Delta t, z_b) \propto z_b^2$. Thus, the dip in the boundary two-point correlation function connecting the exponential decay with the linear growth is suppressed by the large dilaton (note that, conversely from, e.g., [20], here I work with normalized partition functions). Moreover, while this dip happens numerically at the bulk time of order $\Delta t^{\mathrm{dip}} \sim \frac{\beta}{2\pi} \log(\frac{\pi}{g^2})$, as one would expect from Page arguments, from the point of view of the boundary

Lorentzian time (τ_L) on which SYK theory is constructed, we have $\tau_L^{\text{dip}} \sim 4\pi \frac{(4\pi G)^2}{\epsilon g^4 \beta} \Delta t^{\text{dip}} \sim \frac{4\pi G}{g^2} \Phi_b \ln\left(\frac{\pi}{g^2}\right)$.

The boundary correlation function calculated in this paper entirely comes from the Lorentzian saddle point of the path integral and reproduces, at least qualitatively, the dual one of [17,20] calculated from the so-called "trumpet" configurations. This seemly paradoxical fact might have a simple explanation. As discussed in Sec. II, the saddle point of the gravitational action, as seen from the full path integral of matter + gravity, becomes a valley after the exponential decay of the matter part. This, in turn, might well generate two neighbor peaks that, in terms of gravity configurations, could be related to the trumpets.

Finally, we can expect that the linear growing of the correlation function should change into something else whenever the scalar back-reactions becomes important, i.e., toward the end of the black hole's life [25]. Indeed, in the numerical calculations of [20], the ramp eventually encounters a plateau. This plateau might still be found as a Lorentzian saddle point calculation where backreactions are also included. I leave the proof of this statement for future work.

ACKNOWLEDGMENTS

[1] S. W. Hawking, Nature (London) **248,** 30 (1974).

[2] S. W. Hawking, Phys. Rev. D **14,** 2460 (1976).

[3] D. N. Page, Phys. Rev. Lett. **71,** 3743 (1993).

[4] R. Emparan, Phys. Rev. D **51,** 5716 (1995).

[5] J. L. F. Barbon and R. Emparan, Phys. Rev. D **52,** 4527 (1995).

[6] J. L. F. Barbon and E. Rabinovici, J. High Energy Phys. 11 (2003) 047.

[7] J. M. Maldacena, J. High Energy Phys. 04 (2003) 021.

[8] A. Almheiri, R. Mahajan, J. Maldacena, and Y. Zhao, J. High Energy Phys. 03 (2020) 149; G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, J. High Energy Phys. 03 (2022) 205; A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, J. High Energy Phys. 05 (2020) 013.

[9] R. Li, X. Wang, and J. Wang, Phys. Rev. D **104,** 106015 (2021); H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, J. High Energy Phys. 01 (2022) 182.

[10] C. Teitelboim, Phys. Lett. **126B,** 41 (1983); R. Jackiw, Nucl. Phys. **B252,** 343 (1985).

[11] C. Germani, Phys. Lett. B **733,** 93 (2014).

[12] C. Germani and D. Pereñiguez, Phys. Rev. D **97,** 126011 (2018).

[13] T. G. Mertens and G. J. Turiaci, J. High Energy Phys. 01 (2021) 073.

[14] A. Almheiri and J. Polchinski, J. High Energy Phys. 11 (2015) 014.

[15] M. Spradlin and A. Strominger, J. High Energy Phys. 11 (1999) 021.

[16] J. Maldacena, D. Stanford, and Z. Yang, Prog. Theor. Exp. Phys. **2016,** 12C104 (2016).

[17] A. Blommaert, T. G. Mertens, and H. Verschelde, J. High Energy Phys. 09 (2019) 060.

[18] S. Sachdev and J. Ye, Phys. Rev. Lett. **70,** 3339 (1993); A. Kitaev and S. J. Suh, J. High Energy Phys. 05 (2018) 183.

[19] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, J. High Energy Phys. 05 (2017) 118.

[20] P. Saad, S. H. Shenker, and D. Stanford, arXiv:1806.06840.

[21] T. G. Mertens, G. J. Turiaci, and H. L. Verlinde, J. High Energy Phys. 08 (2017) 136; Z. Yang, J. High Energy Phys. 05 (2019) 205.

[22] D. Stanford and E. Witten, J. High Energy Phys. 10 (2017) 008.

[23] D. Bagrets, A. Altland, and A. Kamenev, Nucl. Phys. **B911,** 191 (2016).

[24] Y. H. Qi, Y. Seo, S. J. Sin, and G. Song, Phys. Rev. D **99,** 066001 (2019).

[25] A. Blommaert, Quantum gravity in two dimensions, Ph.D thesis of University of Ghent, 2020.