

Semiclassical spectrum of a Jordanian deformation of $\text{AdS}_5 \times S^5$

Riccardo Borsato^{1,*}, Sibylle Driezen^{1,†}, Juan Miguel Nieto García^{2,‡} and Leander Wyss^{2,§}

¹*Instituto Gallego de Física de Altas Energías (IGFAE) and Departamento de Física de Partículas, Universidade de Santiago de Compostela, Campus Vida, Santiago de Compostela 15782, Galicia, Spain*

²*Department of Mathematics, University of Surrey, Guildford, GU2 7XH, United Kingdom*



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We study a Jordanian deformation of the $\text{AdS}_5 \times S^5$ superstring that preserves 12 superisometries. It is an example of homogeneous Yang-Baxter deformations, a class that generalizes TsT deformations to the non-Abelian case. Many of the attractive features of TsT carry over to this more general class, from the possibility of generating new supergravity solutions to the preservation of world sheet integrability. In this paper, we exploit the fact that the deformed σ -model with periodic boundary conditions can be reformulated as an undeformed one with twisted boundary conditions and discuss the construction of the classical spectral curve and its semiclassical quantization. First, we find global coordinates for the deformed background and identify the global time corresponding to the energy that should be computed in the spectral problem. Using the curve of the twisted model, we obtain the one-loop correction to the energy of a particular solution, and we find that the charge encoding the twisted boundary conditions does not receive an anomalous correction. Finally, we give evidence suggesting that the unimodular version of the deformation (giving rise to a supergravity background) and the nonunimodular one (whose background does not solve the supergravity equations) have the same spectrum at least to one-loop.

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I. INTRODUCTION

We currently know various examples of deformations of σ -models that preserve integrability; see [1] for a review. Important representatives are the λ -deformation [2–4], the inhomogeneous deformation (also called η -deformation) [5–8], and the class of homogeneous Yang-Baxter deformations [9,10]. These are being studied with various motivations, for example, to clarify a possible link between integrability and renormalizability of the σ -models [11–13]. When these deformations are applied to string σ -models, they lead to deformations of the corresponding target-space backgrounds. By now, the conditions when they give rise to backgrounds that solve the supergravity equations are well understood [14–19]. Interestingly, they were also reformulated in the language of double field theory (see e.g., [19–22]), which in turn can be used to identify their α' -corrections, at least to the first order (i.e., two loops in the

σ -model) in the α' -expansion [23]; see also [24–26] for the generalization to other solution-generating techniques. One reason why they have attracted so much attention is that they can be used to deform a very large class of string backgrounds, including the maximally symmetric ones with anti-de Sitter (AdS) factors, which were paramount in the development of the AdS/CFT correspondence. The possibility of deforming such backgrounds (e.g., $\text{AdS}_5 \times S^5$ or $\text{AdS}_3 \times S^3 \times T^4$) while preserving the underlying integrability is an exciting direction to identify possible generalizations of the AdS/CFT correspondence, with new gauge theories that are potentially exactly solvable (in the same sense as $\mathcal{N} = 4$ is exactly solvable in the large- N limit).

In this article, we will focus on homogeneous Yang-Baxter deformations (and for simplicity, we will always omit the prefix “homogeneous” from now on). This is a class of integrable deformations that can be understood as a generalization of TsT deformations (which in the double field theory language would be called β -shifts) [27]. While TsT deformations [28–30] can be applied whenever we have two commuting isometries of the “seed” undeformed background, Yang-Baxter deformations generalize this possibility to the non-Abelian case. The central ingredient in the construction is a constant and antisymmetric R -matrix that must solve the classical Yang-Baxter equation on the Lie (super)algebra of isometries. When restricting to supercosets (for example, the one of $\text{AdS}_5 \times S^5$), this family of deformations has a rich list of possibilities, and

*riccardo.borsato@usc.es

†sib.driezen@gmail.com

‡j.nietogarcia@surrey.ac.uk

§l.wyss@surrey.ac.uk

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everything is known about their supergravity embedding. In particular, the deformed background (which in general can have all NSNS and RR fields nontrivially turned on) is a solution of the type IIB supergravity equations if and only if the R -matrix solves a very simple and algebraic “unimodularity” condition [14,31].¹

On the other hand, not much has been done to extend the methods of integrability, that were developed for the undeformed $\text{AdS}_5 \times S^5$ case, to the Yang-Baxter deformed models.² For the very restricted subclass of “diagonal” TsT deformations (i.e., combinations of TsT deformations that involve only the Cartan isometries of $\text{AdS}_5 \times S^5$), the solution to the spectral problem is understood in terms of a simple deformation of the Bethe equations and thermodynamic Bethe ansatz [36,37] or quantum spectral curve [38]. But as soon as one goes beyond this special subclass, various obstructions do not yet allow for the application of the integrability techniques. This happens already for the rather simple class of “nondiagonal” TsT deformations (i.e., involving at least one non-Cartan isometry) [39].

A natural strategy to try to extend the methods of integrability to the full class of Yang-Baxter deformations is to do what worked in the case of diagonal TsT deformations, where one replaces the study of the spectral problem of the *deformed* σ -model with that of an *undeformed* model which, however, has *twisted* boundary conditions on the world sheet. The two σ -models (the deformed one and the undeformed yet twisted one) are equivalent on shell; i.e., there is a map that relates solutions of the σ -model equations of motion on the two sides. In the case of TsT deformations, the twisted boundary conditions for the alternative picture are very easy to write down, essentially because they are linear in terms of the fields [29,30,40]. For more generic Yang-Baxter deformations, writing down the equivalent twisted boundary conditions is more complicated. Previous expressions were written in terms of a path-ordered exponential, and the nonlocalities that this introduces were making it impossible to make any progress [41–43]. This problem was solved in [44], where the twisted boundary conditions were rewritten in terms of the convenient degrees of freedom, namely those of the twisted model itself. While the twisted boundary conditions can lead to complicated nonlinear relations among the fields—a fact which is related to the non-Abelian nature of the deformations—the expressions are *local*.

¹See [32] for the generic treatment of a Green-Schwarz string with a semisimple Lie algebra of isometries. Recently, Yang-Baxter deformations have also been constructed for symmetric space sigma models with nonsemisimple symmetry algebra [33].

²We stress that in this paper we focus on *homogeneous* Yang-Baxter deformations. For the case of the inhomogeneous deformation, the integrability methods have been pushed to the level of the thermodynamic Bethe ansatz [34] and quantum spectral curve [35].

Among the methods of integrability that we can apply, the classical spectral curve has the right balance of complexity and computational power. The curve is defined as the N -sheeted Riemann surface obtained from the eigenvalue-problem of a monodromy matrix of size $N \times N$ [45]. Its power was originally applied in the context of string theory in $\text{AdS}_5 \times S^5$ by reformulating the construction of classical string solutions as a Riemann-Hilbert problem [46–50], as all their information is encoded in terms of cuts and poles on the Riemann surface. It gained even more traction after it was realized that the classical curve can also be used to obtain 1-loop information related to these classical solutions [51–55]. This information is retrieved by adding microscopic cuts and poles to the Riemann surface, which will behave as quantum fluctuations around the classical solution. The classical spectral curve was soon generalized from $\text{AdS}_5 \times S^5$ to other AdS_d backgrounds; see [56] and references therein, as well as deformed backgrounds, such as the flux-deformed $\text{AdS}_3 \times S^3$ [57–59] or the Schrödinger background obtained from TsT [60].

In this article, we will use the equivalence between deformed models and undeformed yet twisted model to apply the method of the classical spectral curve and its semiclassical quantization to a particular example of a Yang-Baxter deformation of $\text{AdS}_5 \times S^5$ of non-Abelian type. It belongs to the so-called class of Jordanian deformations, which make use of an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of the Lie algebra of superisometries [9,10,61,62]. Importantly, the simplest version of Jordanian deformations [i.e., the one needing just the $\mathfrak{sl}(2, \mathbb{R})$] is nonunimodular. Therefore, the deformed background does not lead to a solution of the type IIB supergravity equations but rather to one of the modified supergravity equations of [63,64]. It is however possible to cure this problem by constructing “extended” Jordanian deformations [65] that exploit a *superalgebra* of isometries containing $\mathfrak{sl}(2, \mathbb{R})$. They lead to unimodular R -matrices and to backgrounds that are solutions of the standard type IIB supergravity equations. Notice that the unimodular and nonunimodular versions share the same background metric and Kalb-Ramond field.³ Interestingly, both cases are closely related to non-Abelian T-duality [62,66,67]: a nonvanishing deformation parameter can be rescaled away in the background by a simple coordinate transformation, and the solution is equivalent to the background resulting from doing non-Abelian T-duality on the two-dimensional Borel subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ (or of its supersymmetric extension if considering the unimodular case).

³In the unimodular case, these are supplemented by a dilaton and RR fluxes that solve the type IIB equations. In the nonunimodular case, the RR fields are replaced by fields that do not satisfy the standard Bianchi identities, and the background is supplemented by a nondynamical Killing vector that enters the modified supergravity equations.

The article is organised as follows. In Sec. II, we review the construction of Yang-Baxter deformations of semisymmetric space σ -models, and we specify the particular Jordanian deformation, which we will study. After displaying its deformed metric and Kalb-Ramond field, we identify in Sec. II B the residual bosonic isometries that survive. In Sec. II C, we then propose a coordinate system for which we explicitly identify the time coordinate. We prove that this system provides global coordinates of the deformed spacetime for every value of the deformation parameter by analyzing its geodesic completeness. In Sec. III, we derive a particular pointlike string solution of the deformed σ -model, which can be interpreted as the analog of the Berenstein-Maldacena-Nastase (BMN) solution of $\text{AdS}_5 \times S^5$.

In Sec. IV, we transfer from the deformed Yang-Baxter picture to the undeformed model with twisted boundary conditions. We review the results of [44] regarding the on shell equivalence in Sec. IV A and identify the symmetries of the twisted model in IV B. In Sec. IV C, we show how the twist of the Jordanian model can be simplified. The transformation of our BMN-like solution to the twisted model is then performed in Sec. IV D, where we pay particular attention to the gauge ambiguities of the supercoset. Additionally, in this section, we analyze also solutions of the twisted model in more generality and show, under certain assumptions, that they lead to extended string solutions with profiles involving Airy functions.

We start the program of the classical spectral curve and its quantization in Sec. V with a review of the construction of the classical curve in terms of the quasimomenta of the (general) twisted supercoset model on $PSU(2, 2|4)$. We show in Sec. V A how the asymptotics of the quasimomenta give rise to local Cartan charges, involving the energy, of the symmetry algebra of our Jordanian twisted model. We then construct the algebraic curve for our BMN-like solution in Sec. V B. In Sec. VI, we study the semiclassical quantum corrections to the corresponding BMN-like quasimomenta by applying the recipe of [51, 55] to our twisted case. We identify the frequencies of all possible excitations whose resummation, which we perform in Sec. VI C, gives the one-loop correction to the energy of our string solution. In Sec. VI B, we furthermore show that the charge identifying the twist of the boundary conditions is protected—it does not receive any anomalous correction—at one-loop. Finally, in Sec. VII, we discuss how our results regarding the classical spectral curve and its semiclassical quantization are insensitive to whether we consider the unimodular or nonmodular version of our Jordanian deformation.

We end in Sec. VIII with a conclusion and outlook. In Appendix A, we show how the global coordinates defined in Sec. II C are related to the embedding coordinates of (undeformed) AdS . For completeness, we discuss in Appendix B why the geodesic incompleteness of the Poincaré coordinate system persists along the deformation.

In Appendix C, we identify the possible Cartan subalgebras of the residual isometry algebra of the deformed model. In Appendix D, we present the details regarding the calculation of the frequencies of excitations presented in Sec. VI. We cross-check these results in Appendix E by computing explicitly, in the picture of the deformed model, the frequencies of the quadratic bosonic fluctuations around the classical solution.

II. A JORDANIAN-DEFORMED BACKGROUND

A. The deformed σ -model and the background fields

In order to set up the notation, let us review the construction of the homogeneous Yang-Baxter deformation of a string σ -model on a semisymmetric space. The starting point is a Lie supergroup G with a corresponding Lie superalgebra $\mathfrak{g}$ that admits a $\mathbb{Z}_4$ -graded decomposition; i.e., $\mathfrak{g} = \bigoplus_{i=0}^3 \mathfrak{g}^{(i)}$ such that $[[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}]] \subset \mathfrak{g}^{(i+j \bmod 4)}$. Here, we are using $[[\cdot, \cdot]]$ to denote the graded bracket on $\mathfrak{g}$. The action of the string σ -model on the supercoset $G \backslash G^{(0)}$ can be written as

$$S_0 = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Pi_{(-)}^{\alpha\beta} \text{STr}(J_\alpha \hat{d}J_\beta), \quad (2.1)$$

where $J = g^{-1}dg \in \mathfrak{g}$ is the Maurer-Cartan form constructed from the group element $g(\tau, \sigma) \in G$ that depends on the world sheet coordinates τ, σ . The linear operator $\hat{d}: \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear combination $\hat{d} = \frac{1}{2}P^{(1)} + P^{(2)} - \frac{1}{2}P^{(3)}$ of projectors $P^{(i)}$, which by definition project onto the subspace $\mathfrak{g}^{(i)}$.⁴ Moreover, STr denotes the supertrace on $\mathfrak{g}$, which we use to obtain an ad-invariant graded-symmetric nondegenerate bilinear form on $\mathfrak{g}$. The world sheet indices α, β in the action are contracted with the projector $\Pi_{(\pm)}^{\alpha\beta} = \frac{1}{2}(\sqrt{|h|}h^{\alpha\beta} \pm \epsilon^{\alpha\beta})$, where $h_{\alpha\beta}$ is the world sheet metric and $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = -1$. Finally, we use $\frac{\sqrt{\lambda}}{4\pi}$ to denote the string tension.

To construct the homogeneous Yang-Baxter deformation of the above action, one needs a linear operator $R: \mathfrak{g} \rightarrow \mathfrak{g}$ that is antisymmetric with respect to the supertrace,

$$\text{STr}(Rx y) = -\text{STr}(x Ry), \quad \forall x, y \in \mathfrak{g}, \quad (2.2)$$

and that solves the classical Yang-Baxter equation (CYBE) on $\mathfrak{g}$,

$$[[Rx, Ry]] - R([[Rx, y]] + [[x, Ry]]) = 0, \quad \forall x, y \in \mathfrak{g}. \quad (2.3)$$

⁴For algebra elements $x \in \mathfrak{g}$, we will use the notation $x^{(i)} = P^{(i)}x$. When there is no risk of ambiguity, we will always use a notation such that a linear operator on the Lie algebra acts on what sits to its right, e.g., $\hat{d}J = d(J)$.

If we choose a basis T_A for $\mathfrak{g}$, such that $[[T_A, T_B]] = f_{AB}^C T_C$, then we can think of R as a matrix identified by $RT_A = R_A^B T_B$. Moreover, if $K_{AB} = \text{STr}(T_A T_B)$ is the metric on $\mathfrak{g}$ and K^{AB} its inverse, we can define the dual generators $T^A = T_B K^{BA}$ that satisfy⁵ $\text{STr}(T_A T^B) = \delta_A^B$. We can then equivalently think of R as $r = -\frac{1}{2} R^{AB} T_A \wedge T_B \in \mathfrak{g} \wedge \mathfrak{g}$, where we use the graded wedge product $x \wedge y = x \otimes y - (-1)^{\deg(x)\deg(y)} y \otimes x$, and $R_A^B = K_{AC} R^{CB}$, so that $Rx = \text{STr}_2(r(1 \otimes x))$ in which STr_2 denotes the supertrace on the second factor of the tensor product. Naturally, we only consider R -matrices that do not mix even and odd gradings, meaning that $r = -\frac{1}{2} R^{ab} T_a \wedge T_b - \frac{1}{2} R^{\hat{a}\hat{b}} T_{\hat{a}} \wedge T_{\hat{b}}$, where a, b are indices of even (bosonic) generators while $\hat{a}, \hat{b}$ are indices of odd (fermionic) generators.

Notice that (2.2) implies that R^{ab} is antisymmetric while $R^{\hat{a}\hat{b}}$ is symmetric. With these ingredients the deformed action is now given by [8]

$$S_\eta = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Pi_{(-)}^{\alpha\beta} \text{STr} \left(J_\alpha \hat{d} \frac{1}{1 - \eta R_{\hat{g}} \hat{d}} J_\beta \right), \quad (2.4)$$

where $R_g = Ad_g^{-1} R Ad_g$ (with $Ad_g x = gxg^{-1}$) and η is a deformation parameter. To have a real action, we take $\eta \in \mathbb{R}$ and R^{AB} anti-Hermitian, so that R^{ab} is real while $R^{\hat{a}\hat{b}}$ has imaginary components.

The space defined by the image of the R -matrix, $\mathfrak{f} = \text{Im}(R)$, will play a central role in our computations. We can check that it is a subalgebra of $\mathfrak{g}$ as a consequence of the CYBE. We will denote its corresponding Lie group by F . Later, we will use indices I, J for generators $T_I \in \mathfrak{f}$, and indices i, j when restricting to the bosonic subalgebra. Using the metric on $\mathfrak{g}$ induced by the supertrace, we can also define the dual to $\mathfrak{f}$, which we denote by $\mathfrak{f}^*$. It is a subspace spanned by $T^I = T_A K^{AI}$ such that $\text{STr}(T_I T^J) = \delta_I^J$.

So far we have reviewed the main ingredients—in particular, Eqs. (2.2) and (2.3)—that make the action (2.4) an integrable deformation of the seed supercoset action (2.1) [6–8]. See later for more details. In general, extra conditions are necessary in order to make sure that the deformed model can be interpreted also as a consistent string σ -model. Since we are deforming a σ -model on a semisymmetric space, the necessary and sufficient condition for the background fields of (2.4) to satisfy the type IIB supergravity equations, is that R also solves the unimodularity condition [14],

$$0 = K^{AB} [[T_A, RT_B]] = R^{AB} f_{AB}^D T_D. \quad (2.5)$$

⁵Extra care must be taken when the elements are of odd grading, because in that case $\text{STr}(xy) = -\text{STr}(yx)$.

When this is not satisfied, the background fields solve the more general equations of “modified supergravity” [63,64].

In this article, we are interested in deformations of the superstring on $\text{AdS}_5 \times S^5$ and thus, we will take $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$. The original supercoset is then $PSU(2, 2|4) \backslash (SO(1, 4) \times SO(5))$.⁶ Furthermore, we want to study deformations that are not interpretable as simple TsT transformations. This implies that we must deform the AdS factor of the background. Therefore, it will be sufficient to state only the commutation relations for the $\mathfrak{so}(2, 4) \cong \mathfrak{su}(2, 2)$ conformal algebra, which is a subalgebra of $\mathfrak{g}$ and corresponds to the isometries of AdS_5 . Given the Lorentz indices $\mu, \nu = 0, \dots, 3$ in $3+1$ dimensions, we will use the Lorentz generators $M^{\mu\nu}$, the translations p^μ , the special conformal transformations k^μ , and the dilatation d that close into the following commutation relations:

$$\begin{aligned} [M^{\mu\nu}, p^\rho] &= \eta^{\nu\rho} p^\mu - \eta^{\mu\rho} p^\nu, & [d, p^\mu] &= +p^\mu, \\ [M^{\mu\nu}, k^\rho] &= \eta^{\nu\rho} k^\mu - \eta^{\mu\rho} k^\nu, & [d, k^\mu] &= -k^\mu, \\ [M^{\mu\nu}, d] &= 0, & [p^\mu, k^\nu] &= 2M^{\mu\nu} + 2\eta^{\mu\nu} d, \\ [M^{\mu\nu}, M^{\rho\sigma}] &= -\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho}, \end{aligned} \quad (2.6)$$

with the Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

An interesting and rich class of deformations that deform AdS are given by Jordanian ones [9,10,61,62]. Here, we choose to work with⁷ [65]

$$r = \mathbf{e} \wedge \mathbf{h} + \frac{i}{2} \zeta (\mathbf{Q}_1 \wedge \mathbf{Q}_1 + \mathbf{Q}_2 \wedge \mathbf{Q}_2), \quad (2.7)$$

where⁸

$$\mathbf{h} = \frac{d + M^{01}}{2}, \quad \mathbf{e} = \frac{p^0 + p^1}{\sqrt{2}}, \quad (2.8)$$

and they satisfy the commutation relation,

$$[\mathbf{h}, \mathbf{e}] = \mathbf{e}. \quad (2.9)$$

The supercharges $\mathbf{Q}_1, \mathbf{Q}_2$ complete the (anti)commutation relations to an $\mathcal{N} = 1$ super Weyl algebra in one

⁶We will not need to give the full set of (anti)commutation relations of this superalgebra, nor provide an explicit matrix realization of its elements. Conventions for these can be found in various places in the literature; see, e.g., [68].

⁷Comparing to [65], one should rescale the deformation parameter there as $\eta \rightarrow -\eta/2$.

⁸If we denote $T_{\mathbf{h}} = \mathbf{h}, T_{\mathbf{e}} = \mathbf{e}$ as generators of $\mathfrak{f}$, then we can identify the duals $T^{\mathbf{h}} = 2T_{\mathbf{h}}$ and $T^{\mathbf{e}} = \frac{k^1 - k^0}{2\sqrt{2}} = T_{\mathbf{e}}$ as generators of $\mathfrak{f}^*$, so that $\text{STr}(T^a T_b) = \delta_b^a$. Moreover, we can check that they form an $\mathfrak{sl}(2, \mathbb{R})$ algebra, $[T_{\mathbf{h}}, T_{\pm}] = \pm T_{\pm}$ and $[T_{\mathbf{e}}, T_{\pm}] = 2T_{\mathbf{h}}$. According to our conventions, the R -matrix acts as $R(T^{\mathbf{h}}) = T_{\mathbf{e}}$ and $R(T^{\mathbf{e}}) = -T_{\mathbf{h}}$.

dimension. In this article, we will not need to specify them further, and more information can be found in [65]. For our convenience, we have also introduced the parameter ζ . It is worth to stress that r solves the CYBE only for $\zeta = 0$ or $\zeta = 1$. When $\zeta = 0$, the purely bosonic r -matrix is *not* unimodular, and it gives rise to background fields that solve only the modified supergravity equations; when $\zeta = 1$, we have an additional “fermionic tail” such that r is also unimodular, and therefore, one can obtain a standard supergravity background [14,65]. In this article, we will find it interesting to consider both cases and compare them.

Let us identify the bosonic part of (2.4) (i.e., when setting fermions to zero) with $S_\eta = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Pi_{(-)}^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \times (G_{mn} + B_{mn})$. Obviously, the background metric G_{mn} and the Kalb-Ramond field B_{mn} do not depend on ζ . In order to find an explicit expression, we can parametrize the group element as $g = g_a g_s$, where $g_a \in SO(2, 4)$ parametrizes the AdS spacetime in the undeformed limit, and $g_s \in SO(6)$ the sphere factor. We take

$$g_a = \exp(\theta M^{23}) \exp(\rho p^3 + x^0 p^0 + x^1 p^1) \exp(\log(z)d), \quad (2.10)$$

which gives⁹

$$ds^2 = \frac{dz^2 + d\rho^2 + \rho^2 d\theta^2 - 2dx^+ dx^-}{z^2} - \frac{\eta^2(z^2 + \rho^2)dx^{-2}}{4z^6} + ds_{S^5}^2, \\ B = \frac{1}{2} B_{mn} dX^m \wedge dX^n = \frac{\eta}{2} \left(\frac{\rho d\rho \wedge dx^-}{z^4} + \frac{dz \wedge dx^-}{z^3} \right), \quad (2.11)$$

with $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ and $ds_{S^5}^2$ the metric of the undeformed 5-sphere. The background is supported by an F_3 RR-flux, which vanishes for $\eta \rightarrow 0$,¹⁰ and by the undeformed F_5 RR-flux of $\text{AdS}_5 \times S^5$. Furthermore, the dilaton remains constant, $\Phi = \Phi_0$, in contrast to many other known deformations. Let us recall that it is only for $\zeta = 1$ that one can find this set of Ramond-Ramond fields and dilaton that complete the background to a standard supergravity one.

Notice that when $\eta \neq 0$, this deformation parameter can be reabsorbed by a simple coordinate redefinition, which is

⁹In terms of the usual Poincaré coordinates from $g_a = \exp(x^2 p^2 + x^3 p^3 + x^0 p^0 + x^1 p^1) \exp(\log(z)d)$, we would get the metric $z^{-2}(dz^2 + (dx^2)^2 + (dx^3)^2 - 2dx^+ dx^-) - \eta^2 z^{-6}/4(z^2 + (x^2)^2 + (x^3)^2)dx^{-2}$ for the deformed AdS factor. Instead, we choose to use polar coordinates $x^2 = \rho \sin \theta$ and $x^3 = \rho \cos \theta$ in the 2–3 plane, such that a $U(1)$ isometry corresponding to shifts of θ , which survives the deformation, becomes manifest. In the next subsection, we will elaborate on the residual isometries of the deformed model.

¹⁰The explicit expression for F_3 can be found in [65].

however singular in the $\eta \rightarrow 0$ limit. This property is in fact a consequence of the fact that the Jordanian deformation is related to a non-Abelian T-duality transformation of the original background [66,67].

B. Residual (super)isometries

As is well known, the (super)isometries are realized in the generic σ -model action (2.1) as global transformations which act as $g \rightarrow g_L g$ with g_L constant, in this case, an element of $G = PSU(2, 2|4)$. The Lie algebra of bosonic isometries of the undeformed $\text{AdS}_5 \times S^5$ background is $\mathfrak{so}(2, 4) \oplus \mathfrak{so}(6) \cong \mathfrak{su}(2, 2) \oplus \mathfrak{su}(4)$, corresponding, respectively, to the AdS_5 and S^5 factor of the spacetime. The deformed action (2.4) is invariant only under a subset of these global transformations, in particular, those which are preserved by the operator R ,

$$Ad_{g_L}^{-1} R Ad_{g_L} = R. \quad (2.12)$$

At the level of the algebra, this condition translates to

$$[[T_{\bar{A}}, R(x)]] = R[[T_{\bar{A}}, x]], \quad \forall x \in \mathfrak{g}, \quad (2.13)$$

where $T_{\bar{A}}$ denotes a generator of a (super)isometry. The subset $\text{span}(T_{\bar{A}}) \subset \mathfrak{g}$ forms a subalgebra by means of the Jacobi identity. Using the above condition it is easy to check that the nonunimodular deformation ($\zeta = 0$) preserves 16 superisometries, while the unimodular deformation ($\zeta = 1$) preserves 12 superisometries. In the non-unimodular case, since the R -matrix acts trivially on the sphere, all of the $\mathfrak{so}(6)$ isometries will be preserved by the deformation, and its Noether currents correspond as usual to the Maurer-Cartan forms $g_s^{-1} dg_s$; see, e.g., [68]. In the unimodular case, instead, some of the $\mathfrak{so}(6)$ isometries are broken by the additional fermionic tail of the R -matrix. This is reflected, for example, at the level of the RR fields, which are not invariant under $\mathfrak{so}(6)$. Nevertheless, classical string configurations are insensitive to this fermionic tale, and at that level, one can work as if there is $\mathfrak{so}(6)$ invariance. More generally, in the deformed model, the Noether currents corresponding to the residual (super) isometries are given by¹¹

$$\mathcal{J}_{\bar{A}\pm} = \text{STr}(\mathcal{J}_\pm T_{\bar{A}}), \quad \mathcal{J}_\pm = Ad_g \left(A_\pm^{(2)} \mp \frac{1}{2}(A_\pm^{(1)} - A_\pm^{(3)}) \right), \quad (2.14)$$

where the world sheet one-form $A = A_+ d\sigma^+ + A_- d\sigma^-$, here written in terms of the world sheet light cone coordinates $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$, is defined as

¹¹Because of the factorization of the AdS and sphere algebra, if we are interested only in the residual AdS isometries, we may simply substitute g with g_a in the formulas above.

$$A_{\pm} = \frac{1}{1 \pm \eta R_g \hat{d}} g^{-1} \partial_{\pm} g. \quad (2.15)$$

From these Noether currents, one may also define the Noether charges,

$$Q_{\bar{A}} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \mathcal{J}_{\bar{A}\tau}. \quad (2.16)$$

For a generic σ -model with background metric and Kalb-Ramond field, the Killing vectors $k_{\bar{a}}$ associated to *bosonic* isometries can then be found from solving (see, e.g., [69])

$$\mathcal{J}_{\bar{a}\pm} = k_{\bar{a}}^m (G_{mn} \mp B_{mn}) \partial_{\pm} X^n \pm \omega_{\bar{a}n} \partial_{\pm} X^n, \quad (2.17)$$

where $\mathcal{L}_{k_{\bar{a}}} G = 0$ and $\mathcal{L}_{k_{\bar{a}}} B = d\omega_{\bar{a}}$, with $\mathcal{L}$ the Lie derivative, and $\omega_{\bar{a}}$ an arbitrary one form. Since the deformation only acts nontrivially on AdS, we will focus on this subsector for most of the remainder of this paper.

Solving (2.13) for the R matrix (2.7) in question then gives five residual isometries of $\mathfrak{so}(2,4)$, forming the subalgebra,¹²

$$\begin{aligned} \mathfrak{k} &= \text{span}(\mathbf{T}_{\bar{a}}, \bar{a} = 1, \dots, 5) \\ &= \text{span}(d - M^{01}, p^0, p^1, k^0 + k^1, M^{23}). \end{aligned} \quad (2.18)$$

The corresponding Killing vectors in the (polar) Poincaré coordinates read

$$\begin{aligned} k_1^m \partial_m &= 2x^- \partial_{x^-} + \rho \partial_{\rho} + z \partial_z, & k_2^m \partial_m &= \frac{\partial_{x^+} + \partial_{x^-}}{\sqrt{2}}, & k_3^m \partial_m &= \frac{\partial_{x^+} - \partial_{x^-}}{\sqrt{2}}, \\ k_4^m \partial_m &= \sqrt{2}(z^2 + \rho^2) \partial_{x^+} + \sqrt{2}2(x^-)^2 \partial_{x^-} + \sqrt{2}2x^- \rho \partial_{\rho} + \sqrt{2}2x^- z \partial_z, & k_5^m \partial_m &= \partial_{\theta}, \end{aligned} \quad (2.19)$$

which shows that we have in particular three translation isometries in x^+ , x^- and θ .¹³

C. Global coordinates

In view of the applications of the Jordanian deformation of $\text{AdS}_5 \times S^5$ in holography, it is important to identify global coordinates for the background for generic values of η . As it is well known, the Poincaré coordinates are globally incomplete already for the undeformed AdS spacetime. Thus, our first task is to find an appropriate coordinate transformation. A first obvious attempt is to transform to the usual global coordinates of AdS and then check if they remain global coordinates also when $\eta \neq 0$. However, we find that the deformed metric in these coordinates is complicated and does not exhibit any of the manifest residual isometries. We have therefore not pursued this coordinate system further, as it would not be useful for practical purposes.

In fact, similar issues were found in [70], which analyzed the geometry of Schrödinger spacetimes Sch_z . Such geometries possess an anisotropic scale invariance $(t, x^i) \rightarrow (\lambda^z t, \lambda x^i)$ characterized by a critical exponent $z > 1$, where the case $z = 1$ corresponds to usual AdS. A global coordinate system different from the usual global AdS

coordinates was then obtained for $z = 2$. Conveniently, the Jordanian deformed background (2.11) is extremely reminiscent of the Sch_2 case and, in fact, exhibits the same anisotropic scale symmetry.¹⁴ It is therefore conceivable that a good candidate for a global coordinate system of our Jordanian background coincides with that found in [70]. The transformation from polar Poincaré coordinates to the global ones $X^m = (T, V, P, \Theta, Z)$ is obtained from¹⁵

$$\begin{aligned} x^+ &= V + \frac{1}{2}(Z^2 + P^2) \tan T, & z &= \frac{Z}{\cos T}, & \rho &= \frac{P}{\cos T}, \\ \theta &= \Theta, & x^- &= \tan T. \end{aligned} \quad (2.20)$$

We can cover the full spacetime if we take the coordinates in the ranges $T, V \in]-\infty, +\infty[$, $\Theta \in [0, 2\pi[$ and $P, Z \in]0, +\infty[$. The only coordinate that we allow to be periodically identified is Θ , with period 2π . These ranges have been derived from the relation between these coordinates and the embedding coordinates of AdS, which we discuss in Appendix A. In the new coordinates, the background fields (2.11) then read

¹⁴In particular, when defining $(x^2, x^3) = (\rho \sin \theta, \rho \cos \theta)$, the Jordanian background is invariant under $z \rightarrow \lambda z$, $x^2 \rightarrow \lambda x^2$, $x^3 \rightarrow \lambda x^3$, $x^- \rightarrow \lambda^2 x^-$ and $x^+ \rightarrow x^+$, which translates to Eq. (2.1) of [70] by identifying (z, x^2, x^3, x^-, x^+) with (r, x^2, x^3, t, ξ) for the critical exponent $z = 2$. Notice that the only difference between the Jordanian metric and the Schrödinger Sch_2 metric, given in Eq. (1.1) of [70], is the term $\frac{\rho^2}{4z^6} dx^{-2} = \frac{(x^2)^2 + (x^3)^2}{4z^6} dx^{-2}$.

¹⁵Strictly speaking, the relation to the global Schrödinger coordinates of [70] is obtained by transforming (P, Z) to $(X^2, X^3) = (P \sin \Theta, P \cos \Theta)$.

¹²From an algebraic perspective, we thus see that indeed M^{23} is a residual $U(1)$ isometry that corresponds to shifts in an angular coordinate.

¹³One can also derive the one-forms $\omega_{\bar{a}}$ needed to relate the Noether currents $\mathcal{J}_{\bar{a}}$ to the Killing vectors $k_{\bar{a}}$. In this case, they are found to be nonvanishing but closed for generic values of η . The B -field itself is therefore isometric with respect to the residual isometries.

$$ds^2 = \frac{dZ^2 + dP^2 + P^2 d\Theta^2 - 2dTdV}{Z^2} - \frac{4Z^4(Z^2 + P^2) + \eta^2(Z^2 + P^2)}{4Z^6} dT^2 + ds_{S^2}^2,$$

$$B = \frac{\eta}{2} \left(\frac{PdP \wedge dT}{Z^4} + \frac{dZ \wedge dT}{Z^3} \right). \quad (2.21)$$

Clearly, this coordinate system will be useful for practical purposes, since it manifestly displays three translational isometries, namely in T , V , and Θ . Because the Cartan subalgebra of the algebra of residual isometries is three-dimensional (see Appendix C), this is in fact the maximal number of manifest translational isometries that a coordinate system could display. Explicitly, the Killing vectors introduced earlier read

$$\begin{aligned} k_1^m \partial_m &= \sin(2T) \partial_T - (P^2 + Z^2) \sin(2T) \partial_V + \cos(2T) (P \partial_P + Z \partial_Z), \\ k_2^m \partial_m &= \frac{\cos^2(T)}{\sqrt{2}} \partial_T - \frac{(P^2 + Z^2) \cos(2T) - 2}{2\sqrt{2}} \partial_V - \frac{\sin(2T)}{2\sqrt{2}} (P \partial_P + Z \partial_Z), \\ k_3^m \partial_m &= -\frac{\cos^2(T)}{\sqrt{2}} \partial_T + \frac{(P^2 + Z^2) \cos(2T) + 2}{2\sqrt{2}} \partial_V + \frac{\sin(2T)}{2\sqrt{2}} (P \partial_P + Z \partial_Z), \\ k_4^m \partial_m &= 2\sqrt{2} \sin^2(T) \partial_T + \sqrt{2} (P^2 + Z^2) \cos(2T) \partial_V + \sqrt{2} \sin(2T) (P \partial_P + Z \partial_Z), \\ k_5^m \partial_m &= \partial_\Theta. \end{aligned} \quad (2.22)$$

Note that $(k_2^m + k_3^m) \partial_m = \sqrt{2} \partial_V$ and $(2k_2^m - 2k_3^m + k_4^m) \partial_m = 2\sqrt{2} \partial_T$.

Following the above proposal, in the rest of this section, we will prove that the coordinate system $X^m = (T, V, X^2, X^3, Z)$ with $(X^2, X^3) = (P \sin \Theta, P \cos \Theta)$ gives rise to a geodesically complete spacetime.

1. On the generators of time translations

Before performing the full analysis of geodesic completeness of the deformed spacetime, let us first identify the possible Hamiltonians (generators of time translations) within the isometry algebra $\mathfrak{k}$ in (2.18). We will look for generators of this kind that are part of the Cartan subalgebra of $\mathfrak{k}$ because in that case, one can find an adapted coordinate system where the generator of time translations simply shifts one of the coordinates, which can be identified as time.

The first natural question is thus to identify all the possible Cartan subalgebras of $\mathfrak{k}$, as explained in Appendix C. Up to automorphisms, there are two possible inequivalent choices of Cartan subalgebras, namely,

$$\begin{aligned} \text{(I): } & \text{span}\{d - M^{01}, p^0 + p^1, M^{23}\}, \\ \text{(II): } & \text{span}\{p^0 - p^1 + \alpha(k^0 + k^1), p^0 + p^1, M^{23}\}, \end{aligned} \quad (2.23)$$

where we leave a possible $\alpha > 0$ coefficient for later convenience.

Identifying the possible generators of time translations is now a very simple task. One can construct a generic linear combination of the Cartan generators and demand that the corresponding Killing vector $\bar{k}^m$ has a strictly negative norm in every region of spacetime. Notice, however, that $p^0 + p^1$ and M^{23} are not timelike and, being central elements, it is easy to remove their contribution by a spacetime-coordinate redefinition. We can therefore simply compute the norm of

the Killing vectors associated to the two possible Cartans of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of $\mathfrak{k}$. We find

$$\begin{aligned} \text{(I): } \bar{k}^m &= k_1^m, \quad \bar{k}^m \bar{k}_m = \frac{(P^2 + Z^2)(4Z^4 - \eta^2 \sin(2T)^2)}{4Z^6} \\ \text{(II): } \bar{k}^m &= k_{(2)}^m - k_{(3)}^m + \frac{1}{2} k_4^m, \quad \bar{k}^m \bar{k}_m = -\frac{(Z^2 + P^2)(\eta^2 + 4Z^4)}{2Z^6}, \end{aligned} \quad (2.24)$$

where in the latter case, we took $\alpha = 1/2$. While other choices of $\alpha > 0$ are possible, we make this choice here to match with the expression $(2k_{(2)}^m - 2k_{(3)}^m + k_4^m) \partial_m = 2\sqrt{2} \partial_T$ found above for the generator of T translations. Whilst in case (I), the norm of the Killing vector is not everywhere negative (set, e.g., $T = 0$); in case (II), we do have a strictly timelike Killing vector for generic values of the coordinates and of the deformation parameter η . The relevant Cartan subalgebra is thus that of class (II) with $\alpha = 1/2$. Concluding, the coordinate T will be our preferred global time coordinate and the Hamiltonian is¹⁶

$$H = \frac{1}{\sqrt{2}} \left(p^0 - p^1 + \frac{1}{2} (k^0 + k^1) \right). \quad (2.25)$$

¹⁶In the literature, in similar setups, sometimes the Hamiltonian is identified with $d - M^{01} \sim h$. While this is of course possible when considering the complexified algebra [where any choice of Cartan can be brought to $d - M^{01}$; see, e.g., (C1)], here we insist on respecting reality conditions. Notice that the square of $d - M^{01}$ has a positive trace and thus, cannot correspond to a timelike Killing vector. In fact, if we take for the sake of the argument the case of group manifolds, the Killing vectors can be identified by $k_a^m = \text{Tr}(T^b A d_g^{-1} T_a) \ell_b^m$ where ℓ_b^m is the inverse of ℓ_m^b and $dX^m \ell_\mu^a T_a = g^{-1} dg$. When moving to the identity of the group (i.e., at $g = 1$), the norm $k_a^m k_{am}$ of the Killing vector reduces to $\text{Tr}(T_a T_a)$. Therefore, a necessary condition for k_a to be timelike is that $\text{Tr}(T_a T_a) < 0$.

Interestingly, when taking the undeformed limit $\eta \rightarrow 0$, the above Hamiltonian does not reduce to something equivalent to the BMN generator of time translations [71], because it is possible to check that there is no inner automorphism of $SU(2, 2)$ that relates H to $p_0 + k_0$.

2. Analysis of geodesic completeness

In this section, we will analyse whether the coordinate system $X^m = (T, V, P, \Theta, Z)$, which describes Sch_2 spacetimes globally, are also well-defined global coordinates for the Jordanian deformation. This analysis is necessary because when comparing the Sch_2 metric, given in Eq. (3.18) of [70], and the Jordanian metric, given in (2.21), we see that the Jordanian case has an additional term $\frac{P^2}{4Z^6} dT^2$, which may lead to pathologies.

There are typically two common ways to analyze whether coordinates are global or not. First, one can try to find the embedding in a higher dimensional spacetime¹⁷ and argue whether or not the embedding coordinates cover the full hypersurface. Another possibility is to consider geodesics $\gamma(t)$, parametrized by a geodesic parameter $t \in \mathbb{R}$, and investigate geodesic completeness of the coordinate system. We will follow the latter method. Before doing so explicitly, let us recall the criteria concerning geodesic (in)completeness and singular spacetimes.

A coordinate system is called geodesically complete if all geodesics $\gamma(t)$ are defined for all values of the geodesic parameter $t \in \mathbb{R}$, within the ranges of coordinates that parametrize the manifold. A spacetime is said to be geodesically incomplete if there exists a *finite* value t_0 of the geodesic parameter such that the geodesic $\gamma(t_0)$ hits an extremal value allowed by the range of spacetime coordinates, and the geodesic $\gamma(t)$ cannot be extended to $t > t_0$. Hitting an (apparent) metric singularity would not be

considered pathological—and therefore would not classify as geodesic incompleteness—if the geodesic “bounces back” (i.e., if it can be continued to allowed ranges of the spacetime coordinates at later t) or if the singularity is reached only in the limit $t \rightarrow \infty$.¹⁸

Rather than solving the five second-order geodesic equations for $X^m(t)$, we can make use of the five residual Killing isometries to perform a *single* integration and simplify the calculations considerably. In particular, recall that for each Killing vector field k_a^m , there is an independent charge,

$$Q_a = k_a^m G_{mn} \dot{X}^n, \quad (2.26)$$

which is conserved along the geodesics.¹⁹ In other words, since the full system of geodesic equations can be rewritten as $\dot{Q}_a = 0$ and we have as many Killing vectors as independent coordinates, the geodesics can be analysed by studying the system (2.26) rather than the usual geodesic equations. To do so, it will be useful to introduce the conserved quantities related to the manifest translational isometries in T , V and Θ ,²⁰

$$\begin{aligned} k_T = \partial_T: \quad Q_T &= -\frac{4Z^4(Z^2 + P^2) + \eta^2(Z^2 + P^2)}{4Z^6} \dot{T} - \frac{1}{Z^2} \dot{V}, \\ k_V = \partial_V: \quad Q_V &= -\frac{1}{Z^2} \dot{T}, \\ k_\Theta = \partial_\Theta: \quad Q_\Theta &= \frac{P^2}{Z^2} \dot{\Theta}. \end{aligned} \quad (2.27)$$

An interesting feature now emerges when analyzing the system (2.26): instead of five first order differential equations for $X^m(t)$, one finds that (2.26) decouples into four first order differential equations and one algebraic relation among the coordinates, which restricts the geodesics to a hypersurface. We choose to decouple (2.26) as

$$\begin{aligned} \dot{T}(t) &= -Q_V Z(t)^2, \\ \dot{V}(t) &= \frac{Q_V}{4} (\eta^2 + 4Z(t)^4) \left(1 + \frac{P(t)^2}{Z(t)^2} \right) - Q_T Z(t)^2, \\ \dot{\Theta}(t) &= \frac{Z(t)^2}{P(t)^2} Q_\Theta, \\ \dot{P}(t) &= \frac{Z(t)}{P(t)} \left(\frac{Z(t)(Q_\Theta + \sin 2T(t)(Q_V Z(t)^2 + P(t)^2) - Q_T)}{\cos 2T(t)} - \dot{Z}(t) \right), \end{aligned} \quad (2.28)$$

¹⁷In the case of undeformed AdS_5 , this is of course $\mathbb{R}^{2,4}$, which is invariant under the conformal algebra $SO(2, 4)$. We show the relation between the coordinate system $X^m = (T, V, P, \Theta, Z)$ and the AdS embedding coordinates in Appendix A.

¹⁸When at least one geodesic ends in a point p of the manifold, then p is a singularity. In this case, one should further study whether p is a physical singularity or just a coordinate singularity. It is only in the latter case that one speaks of geodesic incompleteness. To distinguish the nature of the singularity, one typically computes curvature scalars or tidal forces. The point p is an honest curvature singularity when at least one of these become infinite at p . If all of them remain finite, and the geodesics appear to end at p , then the coordinate system is considered to be incomplete. A fully general criterion about whether or not an apparent singularity is honest, however, does not exist. Fortunately, we will not find such singularities, and therefore, we will not need to perform this analysis.

¹⁹Here, the dot refers to derivation with respect to the geodesic parameter t .

²⁰The relation to the previously defined Noether charges is $Q_2 = \frac{1}{4}(2\sqrt{2}Q_T + 2\sqrt{2}Q_V - Q_4)$, $Q_3 = \frac{1}{4}(-2\sqrt{2}Q_T + 2\sqrt{2}Q_V + Q_4)$ and $Q_5 = Q_\Theta$.

subjected to the algebraic relation, which we will call the “hypersurface equation”,

$$2Q_V P^2(t) = 2Q_T - (2Q_T - \sqrt{2}Q_4) \cos 2T(t) - 2Q_1 \sin 2T(t) - 2Q_V Z(t)^2. \quad (2.29)$$

In fact, this equation defines a circle in the (P, Z) plane with a varying (but bounded) radius depending on $T(t)$, i.e., $P(t)^2 + Z(t)^2 = R^2(T(t))$.²¹ In addition, there is another useful constant of motion along the geodesics, namely,

$$\epsilon = G_{mn} \dot{X}^m \dot{X}^n, \quad (2.30)$$

with $\epsilon = 0, -1, 1$ for null, timelike, or spacelike paths. Substituting the system (2.28) into (2.30) leads, in general, to a complicated differential equation for $\dot{Z}$ subjected to the hypersurface equation.

Let us now analyze potential pathological regions of the deformed background. Notice that we can completely exclude the coordinates T, V, Θ from our analysis, as translations in these directions are manifest isometries and, thus, will not give rise to pathologies. From the background fields (2.21), we see the only regions that can potentially be pathological are $Z, P = \{0, +\infty\}$, as the (inverse) spacetime metric appears to blow up there.

First, let us consider the regions $Z \rightarrow \infty$ or $P \rightarrow \infty$, in which case the hypersurface equation plays an important role in the analysis. In particular, the hypersurface bounds the geodesic motion in Z and P , and forces both to a maximal absolute value, $|Z|, |P| \leq |R|$, which is always finite for a finite value of the conserved charges $Q_{\bar{a}}$ and for any finite value of $t \in \mathbb{R}$. The only exception to this is when the charge $Q_V = 0$, for which the dependence of P and Z drops out of (2.29) (equivalently, the radius R becomes infinite). However, we can argue that also in this case there is no pathology. First, for $Q_V = 0$ the hypersurface equation forces $T(t)$ to be a constant in t , which is consistent with its geodesic equation $\dot{T} = -Q_V Z^2$. Hence, the coordinate time does not evolve in the geodesic parameter. This should therefore be an extreme spacelike path, as we also see by analyzing the equation for $\dot{Z}$ obtained from (2.30). Remarkably, for $Q_V = 0$, this equation in fact greatly simplifies to $\dot{Z}^2(t) = \epsilon Z(t)^2$ which is immediately solved by

$$Z(t) = A e^{\pm \sqrt{\epsilon} t}, \quad (2.31)$$

where A is an integration constant. Reality conditions thus tell us that paths with $Q_V = 0$ are only possible when they are spacelike or null (the latter being trivial in that case).

²¹There are a number of conditions that we can derive for the values of the charges such that $R^2 \geq 0$, which should be interpreted as reality conditions for Z and P . They should be further subjected to a condition ensuring also that T is real when T remains constant along the geodesic.

The spacelike paths can in fact reach $Z = \infty$ (as well as $Z = 0$, see also later), but they do so only at an infinite value of the geodesic parameter t . We can thus conclude that $Z \rightarrow \infty$ is not a pathological region. Concerning the behavior at $P \rightarrow \infty$, one finds that taking $Q_V = 0$ would lead to paths that are not defined due to reality conditions. Given that for generic Q_V , the motion in P is bounded, we can again conclude that also the region $P \rightarrow \infty$ is not pathological. In fact, even though P can become very large for small Q_V , the region $P \rightarrow \infty$ will never be reached by any geodesic in finite t .

Next, let us consider the $Z \rightarrow 0$ region and analyze the small Z behavior. Note that we can just focus on the system T, P, Z (the solutions for Θ and V will be given once solutions for T, P, Z are found). We will write $Z(t) = \lambda \hat{Z}(t)$ and expand the equations around $\lambda = 0$ up to $\mathcal{O}(\lambda)$, unless stated otherwise. From the first equation in (2.28) and (2.29), we will find that T and P are constant in t up to linear order in λ . In particular,²²

$$\begin{aligned} T(t) &= T_0 + \mathcal{O}(\lambda^2), \\ P(t) &= \sqrt{R^2(T_0) + \mathcal{O}(\lambda^2)} \\ &= \sqrt{\frac{2Q_T - (2Q_T - \sqrt{2}Q_4) \cos 2T_0 - 2Q_1 \sin 2T_0}{2Q_V}} \\ &\quad + \mathcal{O}(\lambda^2), \end{aligned} \quad (2.32)$$

with T_0 the constant value of time.²³ Now let us analyze the equation for $\dot{Z}$ obtained from (2.30) and (2.28) for small Z . Up to subleading order in λ , where $T(t) = T_0 + \lambda^2 \mathcal{T}(t) + \mathcal{O}(\lambda^3)$ and $P^2(t) = R^2(T(t)) - \lambda^2 \hat{Z}^2(t) + \mathcal{O}(\lambda^3) = R^2(T_0) + \lambda^2 (\partial_{T_0} R^2(T_0) \mathcal{T}(t) - \hat{Z}^2(t)) + \mathcal{O}(\lambda^3)$, where $\mathcal{T}(t)$ is such that $\dot{\mathcal{T}}(t) = -Q_V Z(t)^2$ and $\mathcal{T}(t=0) = 0$, we find

$$\begin{aligned} \epsilon &= \frac{1}{\lambda^2} \left(\frac{\eta^2 Q_V^2 R^2(T_0) + \dot{Z}(t)^2}{\hat{Z}(t)^2} \right) \\ &\quad + \left(\frac{\dot{Z}(t)^2}{R^2(T_0)} + \frac{\eta^2 Q_V^2 \partial_{T_0} R^2(T_0) \mathcal{T}(t)}{\hat{Z}(t)^2} \right) + \mathcal{O}(\lambda), \end{aligned} \quad (2.33)$$

²²Recall that $P \in]0, \infty[$.

²³Note that if T_0 is such that $R^2(T_0) = 0$, then (2.29) implies that $P(t)$ will have a linear in λ contribution, namely, $P(t) = \sqrt{-\hat{Z}(t)^2} \lambda + \mathcal{O}(\lambda^2)$. However, due to reality conditions, the latter is only possible when $\hat{Z} = 0$, and thus, $P = \mathcal{O}(\lambda^2)$. Hence, this is consistent with (2.32) at $R^2(T_0) = 0$.

which one can analyze order by order in λ . The leading term thus gives

$$\dot{Z}(t)^2 = -\eta^2 Q_V^2 R(T_0)^2 + \mathcal{O}(\lambda^2), \quad (2.34)$$

which in general would give an imaginary solution for Z , i.e., $Z(t) = Z_0 \pm t\sqrt{-\eta^2 Q_V^2 R(T_0)^2 + \mathcal{O}(\lambda^2)}$. This means that we cannot get arbitrarily close to $Z = 0$ except if (i) $\eta = 0$ (the undeformed limit, in which case the spacetime is geodesically complete, see also Appendix A), (ii) $Q_V = 0$ (a case which we already analyzed in generality above and where it was found that geodesics can reach $Z = 0$ only in an infinite amount of the geodesic time), and (iii) $R^2(T_0) = 0$ [in which case $P = \mathcal{O}(\lambda^2)$ and $\dot{Z} = 0$ (see also footnote 23) and thus, all the coordinates are constant in t]. Finally, we must analyze consistency with the equation for $\dot{P}$ given in (2.28), which can be interpreted as $R\dot{R} - Z\dot{Z} = P\dot{P}$. From the leading solution for P in (2.32), we know that $\dot{P} = \mathcal{O}(\lambda^2)$. Therefore, also $P\dot{P} = \mathcal{O}(\lambda^2)$, and we can rewrite $R\dot{R} - Z\dot{Z} = P\dot{P}$ as

$$\lambda^2 \frac{\dot{Z}(t)^2 (Q_1 + \sin 2T_0 (Q_V R^2(T_0) - Q_T))}{\cos 2T_0} - \lambda \dot{Z}(t) \dot{Z}(t) = \mathcal{O}(\lambda^2); \quad (2.35)$$

i.e., $\dot{Z}(t) = \mathcal{O}(\lambda)$. This is consistent with what we have found previously, i.e., Eq. (2.34), requiring either (i) $\eta = 0$, (ii) $Q_V = 0$, or (iii) $R(T_0)^2 = 0$. We can thus conclude that the geodesics will never reach $Z = 0$ by an evolution in t and that this region is not pathological. Finally, analyzing the region $P \rightarrow 0$ using a similar strategy does not give us a conclusive answer. However, this is of course only the apparent pathology of polar coordinates: neither the metric nor its inverse is singular for $X^2 = X^3 = 0$.

The arguments above prove that the coordinate system $X^m = (T, V, X^2, X^3, Z)$ is globally well-defined for any value of the deformation parameter. We will however prefer to work in terms of $X^m = (T, V, P, \Theta, Z)$, in which the shift isometry in Θ is manifest. This concludes our discussion and, combined with the previous section, shows that the coordinate T is our *global* time coordinate.

III. BMN-LIKE SOLUTION OF THE σ -MODEL

In this section, we give the derivation of a particular pointlike string solution to the equations of motion of our Jordanian deformation. As we will show in Sec. V, it will lead to relatively simple quasimomenta. For us, it will be the analog of the BMN solution valid in $\text{AdS}_5 \times S^5$, and we will call it a BMN-like solution.

For a general bosonic string configuration $X^m(\tau, \sigma)$, recall that the equations of motion of the action $S_\eta = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Pi_{(-)}^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n (G_{mn} + B_{mn})$ can be written in conformal gauge as

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta X^m + \left(\eta^{\alpha\beta} \Gamma_{nl}^m + \frac{1}{2} \epsilon^{\alpha\beta} H_{nl}^m \right) \partial_\alpha X^n \partial_\beta X^l = 0, \quad (3.1)$$

which is a generalization of the geodesic equation of a point particle. Here, Γ_{nl}^m are the usual Christoffel symbols, and H_{mnl} are the components of the torsion 3-form $H = dB$. Interestingly, it can be checked that these equations evaluated for the Jordanian deformation admit a consistent truncation on $x_2(\tau, \sigma) = x_3(\tau, \sigma) = 0$, or equivalently, $P(\tau, \sigma) = \Theta(\tau, \sigma) = 0$,²⁴ for any value of the deformation parameter.

Let us now consider the following σ -independent ansatz on the coordinates:

$$X^m = a_m \tau + b_m, \quad (3.2)$$

with a_m and b_m constant variables which will be determined upon imposing (3.1). Furthermore, we will work on the consistent truncation $P(\tau, \sigma) = \Theta(\tau, \sigma) = 0$, equivalently we set $a_P = a_\Theta = b_P = b_\Theta = 0$. Since there is no σ -dependence, this pointlike string will not couple to the Kalb-Ramond two-form, and its equations of motion are therefore simply the geodesic equations. Solving them for the variables a_m and b_m while imposing that fields are real singles out the solution,²⁵

$$a_Z = 0, \quad \text{and} \quad a_V = -\frac{\eta^2 a_T}{4b_Z^2}, \quad (3.3)$$

while the other variables remain free. Because of the manifest isometries in T , V , and Θ , we can however set $b_T = b_V = b_\Theta = 0$ without loss of generality. In summary, our pointlike string solution thus evolves as

$$T = a_T \tau, \quad V = -\frac{\eta^2 a_T}{4b_Z^2} \tau, \quad Z = b_Z, \quad P = 0, \quad \Theta = 0. \quad (3.4)$$

This solution is in fact exactly the same as the one studied for the Schrödinger $Sch_5 \times S^5$ background (at least, in the deformed AdS subsector) in [39]. The reason is that on the

²⁴In the latter case, the limit $P(\tau, \sigma) \rightarrow 0$ on the equations of motion (in particular, the equation of motion for Θ) should be taken with care. First, one must set $P(\tau, \sigma)$ to a constant P to then find that in the resulting equations of motion the limit $P \rightarrow 0$ can indeed be taken smoothly.

²⁵When one does not impose the consistent truncation $P(\tau, \sigma) = \Theta(\tau, \sigma) = 0$, but still assumes the pointlike ansatz (3.2), there is one other real solution namely $a_Z = a_P = 0$, $a_V = -\frac{\eta^2 a_T (b_Z^2 + b_P^2)}{4b_Z^2}$ and $a_\Theta = a_T \sqrt{\eta^2 + 4b_Z^4 2b_Z^2}$ while the other variables remain free. Carrying the analysis of this solution through (again using the isometries to set $b_T = b_V = b_\Theta = 0$), one will find the associated quasimomenta to be quite complicated, and we will therefore not consider this possibility further.

consistent truncation $P = 0$, the Jordanian metric given in (2.11) coincides precisely with the Schrödinger metric. Given that this solution is pointlike, the isometry charges defined in (2.26) will coincide with the Noether charges defined in (2.16). In terms of (2.26) and (2.27), we get

$$\begin{aligned} Q_T &= -a_T, & Q_V &= -a_T b_Z^{-2}, & Q_\Theta &= 0, \\ Q_{\bar{1}} &= 0, & Q_{\bar{4}} &= -\sqrt{2}a_T. \end{aligned} \quad (3.5)$$

Note that these charges are independent of the deformation parameter. As a consistency check, one can furthermore verify that the hypersurface equation (2.29) is satisfied, as it should.

The solution we described here is further subjected to the Virasoro constraints, which will couple the deformed AdS with the undeformed sphere subsector. For Yang-Baxter deformations, the Virasoro constraints, obtained by varying the action (2.4) with respect to the world sheet metric, can be written as [8]

$$\text{STr}(A_{(\pm)}^{\alpha(2)} A_{(\pm)}^{\beta(2)}) = 0, \quad (3.6)$$

where we defined the world sheet projections $A_{(\pm)}^\alpha = \Pi_{(\pm)}^{\alpha\beta} A_\beta$ of the one-form A defined in (2.15). In conformal gauge, the Virasoro constraints would read $\text{STr}(A_{\pm}^{(2)} A_{\pm}^{(2)}) = 0$, and parametrizing the group element as before as $g = g_a g_{\mathfrak{s}}$, we can identify an AdS and sphere contribution as

$$\begin{aligned} \text{STr}(A_{a\pm}^{(2)} A_{a\pm}^{(2)}) + \text{STr}(A_{\mathfrak{s}\pm}^{(2)} A_{\mathfrak{s}\pm}^{(2)}) &= 0 \Leftrightarrow \text{Tr}(A_{a\pm}^{(2)} A_{a\pm}^{(2)}) \\ &= \text{Tr}(A_{\mathfrak{s}\pm}^{(2)} A_{\mathfrak{s}\pm}^{(2)}), \end{aligned} \quad (3.7)$$

where we used that the supertrace is defined as the trace in the AdS algebra while it is *minus* the trace in the sphere algebra, and where $A_{a\pm}$ and $A_{\mathfrak{s}\pm}$ are the projections of A on the AdS and sphere subalgebra, respectively. On our solution, (3.4) the Virasoro constraints then require the following relation between sphere and AdS variables:

$$\left(1 - \frac{\eta^2}{4b_Z^4}\right) a_T^2 = -\text{Tr}(A_{\mathfrak{s}\pm}^{(2)} A_{\mathfrak{s}\pm}^{(2)}). \quad (3.8)$$

IV. MAP TO AN UNDEFORMED YET TWISTED MODEL

Starting from this section, for simplicity, we specify our discussion to the nonunimodular case. We therefore consider a deformation generated by an R -matrix as in (2.7) with $\zeta = 0$. We remind that this case gives rise to background fields that do not satisfy the type IIB supergravity equations but only the “modified” supergravity equations. This study will be however preparatory to the unimodular case, to which we will go back in Sec. VII.

A. Review of the twist

Homogeneous Yang-Baxter deformed σ -models are known to be on shell equivalent to undeformed yet twisted models; see, e.g., [42,43]. This is due to the fact that it is possible to identify the Lax connections of the two models, while achieving a map of the equations of motion of the two sides. In other words, there is an on shell relation between the σ -model on the η -dependent deformed background and the σ -model on the undeformed one, and there is a one-to-one map of the solutions to the equations of motion of the two sides. This map, however, implies that if we choose to have *periodic* boundary conditions on the world sheet for the *deformed* model (which may be motivated by the choice of studying closed strings on the deformed background), then we must generically have *twisted* boundary conditions for the *undeformed* one. This situation is a generalization of what happens for TsT transformations, where the twisted boundary conditions can be derived and written explicitly in a local and linear way [29,30]. From now on, we will use tildes on all objects of the undeformed yet twisted model. For example, if $g \in G$ denotes the group element used to construct the periodic Yang-Baxter model, we will use $\tilde{g} \in G$ to denote the group element of the η -independent twisted model. The map between the two sets of degrees of freedom can be written as

$$g = \mathcal{F} \tilde{g} h, \quad (4.1)$$

where $\mathcal{F} \in F$ is called the “twist field” and is responsible for translating between the two formulations. Recall that F is the Lie group of $\mathfrak{f} = \text{Im}(R)$. In these expressions, we are also allowing for a possible gauge transformation, implemented as a right-multiplication by $h \in G^{(0)}$. This possibility is always present in the (super)coset case, given the local $G^{(0)}$ -invariance of the σ -model action. The twist field $\mathcal{F}$ is fixed by demanding that the Lax connections of the deformed and undeformed models can be identified [see also (5.1)]. This amounts to require that the “modified current” $A_\pm$ of the deformed model defined in (2.15) can be identified with the Maurer-Cartan current $\tilde{J}_\pm = \tilde{g}^{-1} \partial_\pm \tilde{g}$ of the undeformed model, *up to* a possible gauge transformation h , so that

$$A_\pm = h^{-1} \tilde{J}_\pm h + h^{-1} \partial_\pm h. \quad (4.2)$$

It is important to remark that the Virasoro constraints are compatible with the on shell identification. In fact, it is easy to see that using (4.2) the Virasoro conditions for the deformed model written in (3.6) give $\text{STr}(\tilde{J}_{(\pm)}^{\alpha(2)} \tilde{J}_{(\pm)}^{\beta(2)}) = 0$, which are precisely the Virasoro conditions for the undeformed model. This is actually a crucial point to claim on shell equivalence in the case of a string σ -model.

The fact that g has periodic world sheet boundary conditions now indeed implies that $\tilde{g}$ is not periodic in

$\sigma \in [0, 2\pi]$ but rather satisfies the following twisted boundary conditions:

$$\tilde{g}(2\pi) = W\tilde{g}(0)h(0)h^{-1}(2\pi), \quad (4.3)$$

where $W = \mathcal{F}^{-1}(2\pi)\mathcal{F}(0)$ is called the twist.²⁶ Both the twist W and the twist field $\mathcal{F}$ are gauge-invariant under the right-multiplication by h . Moreover, it is clear that it is always possible to use a compensating gauge transformation to fix $h = 1$ in (4.1). In this sense, only W controls the physically relevant twisted boundary conditions, and we can talk about a *left* twist only. At the same time, we want to include the possibility of having a nontrivial $h \neq 1$ because it may be important in certain calculations. It becomes crucial, for example, if we want to insist on using the same parametrization for g and $\tilde{g}$ in terms of coordinates.²⁷ The reader may worry that the need of determining h will introduce unnecessary complications in the calculations, but in Sec. IV D 1, we will explain how to efficiently bypass it and, if needed, easily derive the relevant h .

By identifying the Lax connections, it is in principle straightforward to write down an expression for $\mathcal{F}$ and W in terms of the degrees of freedom g of the deformed model; see [42,43]. In addition to being an unnatural expression for the twisting of the undeformed model, that solution is however written in terms of a path-order exponential. It is therefore plagued by nonlocalities that make it unusable for practical purposes. In [44], this problem was solved by rewriting the solution for W in terms of the degrees of freedom $\tilde{g}$ instead. Depending on the choice of the R -matrix, the twisted boundary conditions may correspond to complicated nonlinear relations among the coordinates evaluated at $\sigma = 2\pi$ and $\sigma = 0$ (a fact which is related to the non-Abelian nature of the deformation) but W itself takes a simple expression in terms of $\tilde{g}$ that only involves single (as opposed to nested) integrals. Importantly, W is constant and, therefore, it is written in terms of conserved charges of the twisted σ -models. These do not need to be Noether charges, as they may correspond to some hidden symmetries. We refer to [44] for a more detailed explanation on how to construct W from a given R -matrix, where the derivation was done for the case of the PCM as well as that of (super)cosets on (semi)symmetric

spaces. In fact, the generalization from the former to the latter only involves some decorations of the formulas with certain linear combinations of projectors on the $\mathbb{Z}_n$ -graded subspaces of $\mathfrak{g}$ (with $n = 2, 4$). In particular, the derivation of the Jordanian twist of Sec. 4.3 of [44] is still valid also in the supercoset case, and it is therefore given by²⁸

$$W = \exp(\mathbf{Q}(h - \mathbf{q}e)), \quad (4.4)$$

which takes values in the Lie subgroup F , and where we have the following expressions for the conserved charges:

$$\mathbf{Q} \equiv \log \frac{1 - \eta Y_+(\tau, 0)}{1 - \eta Y_+(\tau, 2\pi)}, \quad \mathbf{q} \equiv \frac{Y_h(\tau, 2\pi) - Y_h(\tau, 0)}{Y_+(\tau, 2\pi) - Y_+(\tau, 0)}, \quad (4.5)$$

with Y_h, Y_+ projections of the Lie-algebra valued field,

$$\begin{aligned} Y(\tau, \sigma) &= Y_h(\tau, \sigma)\mathbf{T}^h + Y_+(\tau, \sigma)\mathbf{T}^+ \\ &= P^T \left(\int_0^\sigma d\sigma' \hat{d}_{\tilde{g}^{-1}}(\partial_\tau \tilde{g} \tilde{g}^{-1})(\tau, \sigma') \right. \\ &\quad \left. + \int_0^\tau d\tau' \hat{d}_{\tilde{g}^{-1}}(\partial_\sigma \tilde{g} \tilde{g}^{-1})(\tau', 0) \right) + Y(0, 0). \end{aligned} \quad (4.6)$$

As anticipated, the above expression differs from the one valid in the PCM case (which can be found in [44]) just by the insertion of the linear operator $\hat{d}_{\tilde{g}^{-1}} = \text{Ad}_{\tilde{g}} \hat{d} \text{Ad}_{\tilde{g}}^{-1}$, where $\hat{d}$ was defined in Sec. II A. Note that also Y is gauge invariant under the local right-multiplication of $\tilde{g}$ by h . The above formula for the Jordanian twist W will be our starting point for the construction of the classical spectral curve of our model, as well as for the computation of the one-loop quantum corrections to its spectrum.

B. Manifest symmetries of the twisted model

Before continuing with the study of the Jordanian deformation and the corresponding twisted model, it is useful to analyze the manifest symmetries of the latter and understand how they are related to those of the former, which were discussed in Sec. II B. The following discussion is valid for the whole family of homogeneous Yang-Baxter deformations, so we do not need to specify to the Jordanian case. We give the presentation for the supercoset, but the same considerations can be made already in the simpler PCM setup.

Let us define

$$\hat{Q} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \hat{J}_\tau, \quad (4.7)$$

²⁶Notice that although $g, \tilde{g}, h, \mathcal{F}$ depend on both world sheet coordinates τ and σ , when writing the boundary conditions, we will omit the explicit τ dependence and write down only the values that σ takes, to have a lighter notation.

²⁷For example, as we will do later, this happens if we parametrize g as in (2.10) in terms of coordinates $\{x^0, x^1, \rho, \theta, z\}$ and $\tilde{g}$ in the same way but with coordinates $\{\tilde{x}^0, \tilde{x}^1, \tilde{\rho}, \tilde{\theta}, \tilde{z}\}$ [or equivalently in terms of the corresponding global coordinates under the transformation (2.20)]. In this case, the on shell identification condition requires $h \neq 1$. A gauge transformation can be used to reabsorb h in g or $\tilde{g}$, but that would imply that the parametrization of the coset representative would not be given anymore by (2.10).

²⁸Compared to [44], here we use the notation $\mathbf{Q} = \log Q_A$ and $\mathbf{q} = Q_B$.

where²⁹

$$\hat{\mathcal{J}}_\alpha = \text{Ad}_{\tilde{g}} \left(\tilde{\mathcal{J}}_\alpha^{(2)} - \frac{1}{2} \gamma_{\alpha\beta} \epsilon^{\beta\gamma} (\tilde{\mathcal{J}}_\gamma^{(1)} - \tilde{\mathcal{J}}_\gamma^{(3)}) \right). \quad (4.8)$$

In the undeformed and periodic (i.e., $W = 1$) supercoset σ -model, (4.7) are the conserved Noether charges and (4.8) the Noether currents corresponding to the (left) G invariance of the action [68]. In fact, the Noether charges Q of the deformed model defined in (2.16) agree with $\hat{Q}$ in the limit $\eta \rightarrow 0$. In the twisted case ($W \neq 1$), $\hat{Q}$ in (4.7) is not necessarily a conserved charge. In fact, although the Noether current is still conserved ($\partial_\alpha \hat{\mathcal{J}}^\alpha = 0$) simply as a consequence of the equations of motion, the twisted boundary conditions may in general break the constancy in time of $\hat{Q}$, since

$$\begin{aligned} \partial_\tau \hat{Q} &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau \hat{\mathcal{J}}_\tau = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \partial_\sigma \hat{\mathcal{J}}_\sigma \\ &= (Ad_W - 1)(\hat{\mathcal{J}}_\sigma)|_{\sigma=0}, \end{aligned} \quad (4.9)$$

where (4.3) was used. In particular, if we consider the projection of $\hat{Q}$ along a given generator $T_A \in \mathfrak{g}$ as $\hat{Q}_A = \text{STr}(\hat{Q}T_A)$, we have $\partial_\tau \hat{Q}_A = \text{STr}[(\hat{\mathcal{J}}_\sigma)|_{\sigma=0}(W^{-1}T_A W - T_A)]$. It is now clear that if some $T_{\tilde{A}} \subset T_A$ commutes with the whole subalgebra $\mathfrak{f}$, then $\hat{Q}_{\tilde{A}}$ is a conserved charge for the twisted model. Note that $\text{span}(T_{\tilde{A}})$ forms a subalgebra of $\mathfrak{g}$. For the specific Jordanian deformation, we consider in this paper, we have that the manifest bosonic symmetries are

$$T_{\tilde{a}} = \{d - M^{01}, p^0 - p^1, k^0 + k^1, M^{23}\}. \quad (4.10)$$

Importantly, if $T_{\tilde{A}}$ commutes with $\mathfrak{f}$, then it is also a symmetry of the R -matrix and of the deformed model because it automatically solves (2.13).³⁰ Therefore, manifest symmetries of the twisted model correspond to isometries of the YB model.

It is worth stressing that the opposite is not always true: one may have an isometry of the YB model satisfying (2.13), which is not a manifest symmetry of the twisted model because T_A does not commute with $\mathfrak{f}$. In our case of a Jordanian R -matrix, an example of this kind is given by $\mathbf{e} = (p^0 + p^1)/\sqrt{2}$. In fact, $\mathbf{e}$ is a symmetry of the R -matrix and therefore an isometry of the YB model, but it does not commute with $\mathfrak{f}$, so $\hat{Q}_{\mathbf{e}} = \text{STr}(\hat{Q}\mathbf{e})$ is not a conserved charge (in particular $W^{-1}\mathbf{e}W \neq \mathbf{e}$).

Notice that when T_A commutes with $\mathfrak{f}$, the expression for the charges of the twisted model agree with those of the Noether charges of the deformed one, and they are related by the on shell map (4.2). In fact, let us start from

$$Q = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \mathcal{J}_\tau, \quad (4.11)$$

where $\mathcal{J}$ was defined in (2.14), so that $Q_A = \text{STr}(Q T_A)$ is a conserved Noether charge of the deformed model already defined in (2.16) if T_A satisfies (2.13). Then it is straightforward to see that

$$\begin{aligned} [[T_A, \mathfrak{f}] = 0 \Rightarrow \hat{Q}_A &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \text{STr} \left[\tilde{g} \left[\tilde{\mathcal{J}}_\tau^{(2)} - \frac{1}{2} (\tilde{\mathcal{J}}_\sigma^{(1)} - \tilde{\mathcal{J}}_\sigma^{(3)}) \right] \tilde{g}^{-1} T_A \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \text{STr} \left[\mathcal{F}^{-1} g \left[A_\tau^{(2)} - \frac{1}{2} (A_\sigma^{(1)} - A_\sigma^{(3)}) \right] g^{-1} \mathcal{F} T_A \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \text{STr} \left[g \left[A_\tau^{(2)} - \frac{1}{2} (A_\sigma^{(1)} - A_\sigma^{(3)}) \right] g^{-1} \mathcal{F} T_A \mathcal{F}^{-1} \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \text{STr} \left[g \left[A_\tau^{(2)} - \frac{1}{2} (A_\sigma^{(1)} - A_\sigma^{(3)}) \right] g^{-1} T_A \right] \\ &= Q_A, \end{aligned} \quad (4.12)$$

where we used $P^{(i)}(\tilde{J}) = hP^{(i)}(A)h^{-1}$ for $i \neq 0$. According to the discussion above, all manifest symmetries $\hat{Q}_{\tilde{A}}$ can be written as isometries $Q_{\tilde{A}}$ of the deformed model, but not all isometries $Q_{\tilde{A}}$ can be written as a manifest symmetry $\hat{Q}_{\tilde{A}}$, so

that schematically $\{\hat{Q}_{\tilde{A}}\} \subset \{Q_{\tilde{A}}\}$. Because of the classical equivalence between the two models, we expect that the isometries of the deformed model that are not manifest symmetries of the twisted model should correspond to a more general family of hidden symmetries, which is always there for integrable models.

C. A simpler twist

In general, there is not a unique way to write the twist W that controls the boundary conditions. We should rather

²⁹In conformal gauge, it reads $\hat{\mathcal{J}}_\tau = \text{Ad}_{\tilde{g}}(\tilde{\mathcal{J}}_\tau^{(2)} - \frac{1}{2}(\tilde{\mathcal{J}}_\sigma^{(1)} - \tilde{\mathcal{J}}_\sigma^{(3)}))$.

³⁰In fact, $[[T_A, RX]] = R[[T_A, X]]$ because the left-hand-side is obviously 0, and because using $Rx = -\frac{1}{2}R^{IJ}\text{STr}(xT_J)T_I$ with $T_I \in \mathfrak{f}$, the right-hand-side is $-\frac{1}{2}R^{IJ}\text{STr}([T_A, X]T_J)T_I = -\frac{1}{2}R^{IJ}\text{STr}([T_J, T_A]X)T_I = 0$.

think in terms of *equivalent classes* of twists. For example, if $\tilde{g}$ satisfies the twisted boundary conditions (4.3), and we define a new field $\tilde{g}'$ related to $\tilde{g}$ by the simple field redefinition $\tilde{g}' = u\tilde{g}$ with $u \in G$ constant, then it follows that the new field satisfies the boundary conditions,

$$\tilde{g}'(2\pi) = W'\tilde{g}'(0)h(0)h^{-1}(2\pi), \quad W' = uWu^{-1}. \quad (4.13)$$

The fields $\tilde{g}$ and $\tilde{g}'$ are equivalent representations of the same model in terms of different variables, so that we are free to choose whether we want to work with $\tilde{g}$ or $\tilde{g}'$. Therefore, we should not think of the twisted boundary conditions as being identified by a unique expression W , since $W' = uWu^{-1}$ belonging to the same equivalence class actually describes the same physics.

Notice that if $W \in F$ then in general W' belongs to $F' = uFu^{-1}$; in other words, it belongs to an adjoint orbit of G on F . The relation to the degrees of freedom of the YB model now is $g = \mathcal{F}u^{-1}\tilde{g}'h$. In [44], the solution for the twist field was constructed by fixing the initial condition $\mathcal{F}(0,0)$, and while the above field redefinition breaks this choice, it can be restored by the compensating field redefinition $g' = ug$, so that we can write

$$g' = \mathcal{F}'\tilde{g}'h, \quad \mathcal{F}' = u\mathcal{F}u^{-1}. \quad (4.14)$$

Notice that in general the YB model written in terms of these new degrees of freedom g' will be constructed not in terms of R but rather by the R -matrix $R' = \text{Ad}_u R \text{Ad}_u^{-1}$. This is obviously a physically equivalent antisymmetric solution of the CYBE.³¹ It is worth remarking that the Lax connections constructed out of $\tilde{g}$ and $\tilde{g}'$ are equal because u is assumed to be constant, and therefore, also the two corresponding monodromy matrices (and not just their eigenvalues) are equal to each other.

We now want to exploit the above possibility to define a new twist W' for the Jordanian deformation, that has the advantage of having a simpler expression compared to W . We start from (4.4), and we rewrite it in a “factorized” form by using the following identities coming from the Baker-Campbell-Hausdorff (BCH) formula:

$$\begin{aligned} \exp(A)\exp(B) &= \exp\left(A + \frac{s}{1-e^{-s}}B\right), \quad \text{and} \\ \exp(A)\exp(B)\exp(-A) &= \exp(e^s B), \end{aligned} \quad (4.15)$$

which hold when $[A, B] = sB$ for $s \neq 2\pi i n$. First, we parametrize W in (4.4) as

³¹While the above considerations are valid for a generic $u \in G$, if u is also a symmetry of R then $R' = R$, and both $\mathcal{F}'$ and W' are still elements of the unprimed F subgroup; see also [44]. It turns out that the u that we are about to consider is of this type.

$$W = \exp\left(A + \frac{s'}{1-e^{-s'}}B'\right), \quad (4.16)$$

where we have to identify

$$A = \mathbf{Q}h, \quad B' = \mathbf{q}(e^{-Q} - 1)\mathbf{e}, \quad s' = \mathbf{Q}, \quad (4.17)$$

which is consistent with $[h, \mathbf{e}] = \mathbf{e}$. Now, using the above identities, we can write

$$\begin{aligned} \exp\left(A + \frac{s'}{1-e^{-s'}}B'\right) &= \exp(s'B')\exp(A) \\ &= \exp((e^s - 1)B)\exp(A) \\ &= \exp(-B)\exp(e^s B)\exp(A) \\ &= \exp(-B)\exp(A)\exp(B), \end{aligned} \quad (4.18)$$

where we defined $B \equiv e^{s'}B'/(e^s - 1)$. For consistency with $[A, B] = sB$, we must have $s = s'$.³² Notice that in our case we simply have $B = -\mathbf{q}\mathbf{e}$. To conclude, the twist can be written as

$$W = u^{-1}W'u, \quad \text{where } W' = \exp(\mathbf{Q}h), \quad \text{and } u = \exp(-\mathbf{q}\mathbf{e}). \quad (4.19)$$

This is remarkable, because the new twist is identified by the Lie algebra element $\mathbf{h}$ only, and all information regarding $\mathbf{e}$ is lost in the $\tilde{g}'$ -model. The above arguments also imply that the spectrum of both the $\tilde{g}$ - and the $\tilde{g}'$ -models do not depend on the charge $\mathbf{q}$, at least at the classical and one-loop level. Naively, the charge $\mathbf{q}$ controls the boundary conditions of the twisted model, but as we just showed, this dependence can be removed by a simple field redefinition. In the next section, we will see this independence of the spectrum on $\mathbf{q}$ more explicitly. Notice that in the derivation of factorizing the twist, there is a subtlety when $\mathbf{Q} \rightarrow 0$, because then some formulas diverge. We will come back to this point in Sec. VII, where we will discuss the factorization of the twist in a more general setup that encompasses also the case of unimodular deformations.

D. Classical solution in the twisted model

1. Map from the classical solution of the deformed model

In this section, we return to our discussion of the particular pointlike solution (3.4) of the Jordanian deformed model and show how it can be mapped to a twisted solution of undeformed AdS_5 . This means that we need to apply the reverse logic of [44] and derive the twist

³²In fact, as soon as we have $[A, B] = sB$, and we define $B' = \alpha B$ (for any nonvanishing α) while requiring that $[A, B'] = s'B'$, it is then obvious that we must have $s' = s$.

field $\mathcal{F}$ starting from the deformed variables. As we mentioned in IV A, its expression is then in general a complicated path-ordered exponential, but on our specific simple solution (3.4), it can however be evaluated explicitly. In this case, the easiest way to obtain $\mathcal{F}$ is actually to work with the associated differential system, namely,

$$\partial_{\pm}\mathcal{F} = V_{\pm}\mathcal{F}, \quad V_{\pm} = \pm\eta R \text{Ad}_g \hat{d}A_{\pm}, \quad (4.20)$$

which is obtained from identifying the Lax connections of the two models. When using the initial condition $\mathcal{F}(0,0) = 1$, we have $\mathcal{F} \in F$, and thus in full generality, we can write

$$\mathcal{F} = \exp(f_h(\tau, \sigma)\mathbf{h} + f_e(\tau, \sigma)\mathbf{e}). \quad (4.21)$$

Solving the differential Eqs. (4.20) on the solution (3.4) is then rather straightforward, and after imposing $\mathcal{F}(0,0) = 1$, we obtain

$$f_h(\tau, \sigma) = \frac{\eta a_T \sigma}{b_Z^2}, \quad f_e(\tau, \sigma) = \frac{\eta^3 a_T^2 \tau \sigma}{4 \left(1 - e^{-\frac{\eta a_T \sigma}{b_Z^2}}\right) b_Z^4}. \quad (4.22)$$

Note that the twist field $\mathcal{F}$ reduces to the identity element in the undeformed $\eta \rightarrow 0$ limit, as it should.

Given the solution for $\mathcal{F}$, the transformation to the undeformed variables is done in principle by employing the map $g = \mathcal{F}\tilde{g}h$, although this may be complicated by possible gauge transformations. In fact, in our case, if we insist on parametrizing both g and $\tilde{g}$ in the same way, then we cannot relate the two models if $h = 1$. Guessing the appropriate gauge transformation is not straightforward, if not unsatisfactory, and thus the translation between deformed and twisted variables begs a gauge-invariant procedure in the case of (super)coset models. To do this, we introduce the gauge-invariant objects,

$$G = gKg^t, \quad \tilde{G} = \tilde{g}K\tilde{g}^t, \quad (4.23)$$

where t denotes usual matrix transposition and K satisfies $hKh^t = K$ for $h \in SO(1,4)$ (see, e.g., Sec. 1.5.2 of [68]).³³ Then indeed G and $\tilde{G}$ are invariant under the local right-multiplications $g \rightarrow gh$ and $\tilde{g} \rightarrow \tilde{g}h$, respectively. Using these objects, the map $g = \mathcal{F}\tilde{g}h$ can then be rewritten in the following gauge-invariant way:

$$G = \mathcal{F}\tilde{G}\mathcal{F}^t, \quad (4.25)$$

³³In the matrix realization of $\mathfrak{su}(2,2|4)$ that is typically used (see, e.g., [68]), one takes

$$K = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.24)$$

and also the twisted boundary conditions (4.3) can be rewritten as

$$\tilde{G}(2\pi) = W\tilde{G}(0)W^t. \quad (4.26)$$

None of these formulae now suffer from any possible complications arising from gauge ambiguities.

Using (4.25), it is now very easy to translate to the undeformed variables. We find, for $\tilde{g}$ parametrized in terms of the global coordinates, that (3.4) is mapped to

$$\tilde{T} = a_T \tau, \quad \tilde{V} = 0, \quad \tilde{Z} = \exp\left(-\frac{\eta a_T \sigma}{2b_Z^2}\right) b_Z, \quad \tilde{P} = 0, \quad (4.27)$$

while $\tilde{\Theta}$ remains free. The latter is however only a redundancy of the chosen parametrization, and we may set $\tilde{\Theta} = 0$ without loss of generality. Having already determined $g, \tilde{g}, \mathcal{F}$, the unknown gauge transformation h can now be simply derived as $h = \tilde{g}^{-1}\mathcal{F}^{-1}g$. Furthermore, knowing $\mathcal{F}$, we can also obtain the twist W of the solution, i.e.,

$$W = \mathcal{F}^{-1}(2\pi)\mathcal{F}(0) = \exp\left(-\frac{2\pi\eta a_T}{b_Z^2}\mathbf{h}\right). \quad (4.28)$$

Consequently, we can identify the Jordanian charges $\mathbf{Q}$ and $\mathbf{q}$ from (4.4) giving precisely

$$\mathbf{Q} = -\frac{2\pi\eta a_T}{b_Z^2}, \quad \mathbf{q} = 0. \quad (4.29)$$

Thus, for this particular classical solution, there is no need to simplify the twist following Sec. IV C, because $\mathbf{q}$ is already zero. Translating the twisted boundary conditions for $\tilde{g}$ in terms of explicit expressions for the associated coordinates, we see that all of them are periodic except $\tilde{Z}$, which satisfies $\tilde{Z}(2\pi) = \exp(\mathbf{Q}/2)\tilde{Z}(0)$.

There are now several consistency checks of the calculations just performed. First, one can verify that (4.26) holds, while the twisted boundary conditions for $\tilde{g}$ have a nontrivial gauge transformation h in their expression. Secondly, knowing the solution $\tilde{g}$, one can calculate the gauge-invariant object Y defined in (4.6). We find

$$Y_h(\tau, \sigma) = -\frac{\eta a_T \tau}{4b_Z^2}, \quad Y_+(\tau, \sigma) = \eta^{-1} \left(1 - \exp\left(\frac{\eta a_T \sigma}{b_Z^2}\right)\right), \quad (4.30)$$

which satisfies $Y(0,0) = 0$, consistent with $\mathcal{F}(0,0) = 1$. Comparing now the Jordanian charges $\mathbf{Q}$ and $\mathbf{q}$ obtained from (4.5) with the expressions (4.29) obtained directly from the twist field, we can check that the two results agree.

At last, one can verify that (4.27) is a solution to the equations of motion of the undeformed σ -model.

2. More general classical solutions of the twisted model

We now want to explore the question of finding more general solutions of the undeformed twisted model. Although the boundary conditions will depend on the deformation parameter η , the equations of motion will not, and thus, one may hope to find more solutions than (4.27). To maintain some level of simplicity, however, we will continue to work on the consistent truncation $\tilde{P}(\tau, \sigma) = \tilde{\Theta}(\tau, \sigma) = 0$ and assume that the global coordinate time $\tilde{T}$ evolves linearly with the world sheet time τ , i.e., as before $\tilde{T}(\tau, \sigma) = a_T \tau$.

On these assumptions, let us first determine the most general solution to the twisted boundary conditions (4.26) for the Jordanian twist W (4.4) in terms of the coordinates $\tilde{Z}(\tau, \sigma)$ and $\tilde{V}(\tau, \sigma)$. It is not difficult to show that in general they must satisfy

$$\begin{aligned}\tilde{Z}(\tau, 2\pi) &= \exp(\mathbf{Q}/2)\tilde{Z}(\tau, 0), \\ \tilde{V}(\tau, 2\pi) &= \exp(\mathbf{Q})\tilde{V}(\tau, 0) + (1 - \exp(\mathbf{Q}))\mathbf{q}.\end{aligned}\quad (4.31)$$

As before, one will have, however, $\tilde{g}(2\pi) \neq W\tilde{g}(0)$ implying that the group element used to construct the Lax connection of the twisted model must be dressed with a gauge transformation.

Next, we compute the equations of motion of the undeformed model and find that they reduce to the following three differential equations:

$$0 = \partial_\tau \tilde{Z}, \quad (4.32)$$

$$0 = 2a_T \partial_\tau \tilde{V} - (\partial_\sigma \tilde{Z})^2 + \tilde{Z} \partial_\sigma^2 \tilde{Z}, \quad (4.33)$$

$$0 = 2\partial_\sigma \tilde{V} \partial_\sigma \tilde{Z} - \tilde{Z} \partial_\sigma^2 \tilde{V}. \quad (4.34)$$

From (4.32) and (4.33), we find that $\tilde{V}$ must be linear in τ as $\tilde{V} = \frac{1}{2a_T} \tilde{V}_1 \tau + \tilde{V}_0$ with

$$\tilde{V}_1 \equiv (\partial_\sigma \tilde{Z})^2 - \tilde{Z} \partial_\sigma^2 \tilde{Z}, \quad (4.35)$$

and $\tilde{V}_0 \equiv \tilde{V}_0(\sigma)$ an unknown function of σ . Now (4.34) holds two equations, one at $\mathcal{O}(\tau^0)$ and one at $\mathcal{O}(\tau)$, namely,

$$\begin{aligned}-2\partial_\sigma \tilde{V}_0 \partial_\sigma \tilde{Z} + \tilde{Z} \partial_\sigma^2 \tilde{V}_0 &= 0, \quad \text{and} \\ -2\partial_\sigma \tilde{V}_1 \partial_\sigma \tilde{Z} + \tilde{Z} \partial_\sigma^2 \tilde{V}_1 &= 0,\end{aligned}\quad (4.36)$$

respectively. They can be rewritten as

$$2\partial_\sigma \log \tilde{Z} = \partial_\sigma \log \tilde{V}'_0 = \partial_\sigma \log \tilde{V}'_1, \quad (4.37)$$

where the prime denotes derivation to σ , $\tilde{V}'_0 \equiv \partial_\sigma \tilde{V}_0$ and $\tilde{V}'_1 \equiv \partial_\sigma \tilde{V}_1$. Since all the objects involved only depend on σ , we can perform one integration to find

$$\tilde{V}'_0 = c_0 \tilde{Z}^2, \quad \tilde{V}'_1 = c_1 \tilde{Z}^2, \quad (4.38)$$

where c_0 and c_1 are integration constants. Only the latter equation of (4.38) is now a differential equation for $\tilde{Z}$, and once we obtain its solution, the solution for $\tilde{V}_0$ (and consequently of $\tilde{V}$) is obtained immediately by performing a straightforward integration of the first equation of (4.38). First, note that we can rewrite the expression for $\tilde{V}'_1$ as

$$\tilde{V}'_1 = \tilde{Z}' \tilde{Z}'' - \tilde{Z} \tilde{Z}''' = -\tilde{Z}^2 \left(\frac{\tilde{Z}''}{\tilde{Z}} \right)'. \quad (4.39)$$

Then the second equation of (4.38) becomes $(\tilde{Z}'' \tilde{Z}^{-1})' = -c_1$, which means we can do another simple integration to find

$$\tilde{Z}'' + (c_1 \sigma + c_2) \tilde{Z} = 0, \quad (4.40)$$

with c_2 another integration constant. Here the experienced reader may recognize the Airy (or Stokes) differential equation that has as linearly independent solutions the Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$. Concluding, within our assumptions given in the beginning of this section, the most general solution for the variable $\tilde{Z}$ of the undeformed model is

$$\tilde{Z}(\tau, \sigma) = \alpha \text{Ai} \left(-\frac{c_1 \sigma + c_2}{(-c_1)^{2/3}} \right) + \beta \text{Bi} \left(-\frac{c_1 \sigma + c_2}{(-c_1)^{2/3}} \right), \quad (4.41)$$

before imposing any boundary condition. These are oscillatory functions that at a certain turning point become exponential. In principle, we can now also write down an explicit solution for $\tilde{V}$, as mentioned before; however we will refrain from doing so because it is not particularly enlightening. Nevertheless, note that, because of the additional integration performed to obtain $\tilde{V}_0$, the system has one additional integration constant (say c_3). Before imposing boundary conditions, the full solution thus has in total 7 free parameters, namely a_T , c_0 , c_1 , c_2 , c_3 , α , and β .

Imposing the twisted boundary conditions (4.31) gives three equations for $\tilde{Z}$, $\tilde{V}_1$, and $\tilde{V}_0$, obtained order by order in τ . The equation for $\tilde{Z}$ fixes an expression for $\mathbf{Q}$ while the one for $\tilde{V}_0$ fixes $\mathbf{q}$. The remaining condition on $\tilde{V}_1$ becomes a constraint between the parameters,³⁴ reducing the number of free parameters to 6. Two further constraints between the parameters may arise when comparing the expressions of $\mathbf{Q}$ and $\mathbf{q}$ with those obtained from (4.5) and (4.6), as these

³⁴On the solution for $\tilde{Z}$ and $\tilde{V}_1$, the constraint simply reads $\tilde{V}_1(2\pi)\tilde{Z}(0)^2 = \tilde{V}_1(0)\tilde{Z}(2\pi)^2$.

must of course coincide for consistency. We have not looked into this further.

We close this section by discussing the special $c_1 = 0$ case. From (4.40), it is obvious that the most generic solution for $\tilde{Z}$ then is simply,

$$\tilde{Z}(\sigma) = \alpha \exp(\sqrt{-c_2}\sigma) + \beta \exp(-\sqrt{-c_2}\sigma). \quad (4.42)$$

Depending on the sign of c_2 these are exponential or oscillatory functions. The simple solution given in (4.27), which is the one we study in the remainder of this paper, falls into the former class. It has $\alpha = 0$, $\beta = b_Z$, and $\sqrt{-c_2} = \frac{\eta a_T}{2b_Z}$. In general, for $\tilde{V}$, we then get

$\tilde{V} = -c_0 \frac{b_Z^2 \exp(-\frac{\eta a_T \sigma}{b_Z^2})}{\eta a_T} + c_3$, which seems more general than (4.27). Consistency with the boundary conditions (4.31) however sets $c_0 = 0$ and $c_3 = \mathbf{q}$. Here, we see explicitly that a possible contribution from the $\mathbf{q}$ charge can be eliminated by a field redefinition, which is in fact a translational isometry in the V coordinate.

V. CLASSICAL SPECTRAL CURVE

Knowing a Lax connection that is flat on the σ -model equations of motion allows one to apply the methods of classical integrability. While in principle it is possible to work from the point of view of the Yang-Baxter deformed σ -model, we find it more convenient to work from the point of view of the undeformed yet twisted σ -model, which is on shell equivalent to the former. In fact, this is in analogy to how the classical and quantum spectrum of TsT deformations is obtained; see, e.g., [30,36]. Here, we want to give a brief summary of facts that are valid in the undeformed and periodic case and that carry over also to the twisted one. For more details, we refer to the original literature developed in the periodic case; see, e.g., [47].

The Lax connection of the twisted σ -model on the $\text{AdS}_5 \times S^5$ supercoset is³⁵

$$\begin{aligned} \mathcal{L}_\alpha = & \tilde{J}_\alpha^{(0)} + \frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) \tilde{J}_\alpha^{(2)} - \frac{1}{2} \left(z^2 - \frac{1}{z^2} \right) \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \tilde{J}_\gamma^{(2)} \\ & + z \tilde{J}_\alpha^{(1)} + \frac{1}{z} \tilde{J}_\alpha^{(3)}, \end{aligned} \quad (5.1)$$

which agrees with the Lax connection of the undeformed and periodic case [72,73] because boundary conditions play no role in its construction. Here, $\tilde{J}^{(i)} = P^{(i)} \tilde{J}$ are projections of the currents on the $\mathbb{Z}_4$ -graded subspaces of $\mathfrak{psu}(2, 2|4)$, and $\gamma^{\alpha\beta} = \sqrt{|h|} h^{\alpha\beta}$. From the Lax connection, one can construct the monodromy matrix,

³⁵In conformal gauge, this is just $\mathcal{L}_\pm(z) = \tilde{J}_\pm^{(0)} + z \tilde{J}_\pm^{(1)} + z^\mp \tilde{J}_\pm^{(2)} + z^{-1} \tilde{J}_\pm^{(3)}$. The Lax connection of the Yang-Baxter deformed model is obtained simply by replacing $\tilde{J}$ by A .

$$\Omega(z, \tau) = \mathcal{P} \exp \left(- \int_0^{2\pi} d\sigma' \mathcal{L}_\sigma(z, \tau, \sigma') \right), \quad (5.2)$$

with $z \in \mathbb{C}$ the spectral parameter. In the undeformed and periodic case, the eigenvalues $\lambda(z)$ of $\Omega(z, \tau)$ are conserved because $\mathcal{L}_\tau(2\pi) = \mathcal{L}_\tau(0)$ as a consequence of the periodic boundary conditions. When considering twisted boundary conditions as in (4.3), $\mathcal{L}_\tau$ remains periodic as long as we take into account the compensating gauge transformation. Therefore, the eigenvalues $\lambda(z)$ of $\Omega(z, \tau)$ are again time independent and encode the infinite number of conserved quantities (when expanding in powers of the spectral parameter z), rendering the model classically integrable. The eigenvalues depend analytically on z except at $z = 0, \infty$ [where the poles of the Lax connection imply essential singularities for $\lambda(z)$] and at the points where two (or more) eigenvalues degenerate. Instead of working with the eigenvalues, it is simpler to work with the quasimomenta $p(z)$, defined as $\lambda = e^{ip}$, as the essential singularities of λ at $z = 0, \infty$ become poles. For the quasimomenta, we use the same notation as for the undeformed and periodic case

$$\{\hat{p}_1(z), \hat{p}_2(z), \hat{p}_3(z), \hat{p}_4(z) | \tilde{p}_1(z), \tilde{p}_2(z), \tilde{p}_3(z), \tilde{p}_4(z)\}, \quad (5.3)$$

where $\hat{p}_i(z)$ with $i = 1, \dots, 4$ are the quasimomenta of the (deformed/twisted) AdS factor corresponding to $SU(2, 2) \subset PSU(2, 2|4)$ and $\tilde{p}_i(z)$ with $i = 1, \dots, 4$ are those of the (in this case, undeformed/untwisted) sphere factor corresponding to $SU(4) \subset PSU(2, 2|4)$. When two eigenvalues of the same type (i.e., both $\hat{p}$ or both $\tilde{p}$) degenerate, they give rise to branch points of square-root cuts, that correspond to collective *bosonic* excitations. When two eigenvalues of different type (i.e., one $\hat{p}$ and the other $\tilde{p}$) degenerate, they give rise to poles, that correspond to *fermionic* excitations. The quasimomenta are everywhere analytic except at these bosonic branch points and fermionic poles, and at $z = 0, \infty$. Because of its definition, the quasimomentum has a $2\pi\mathbb{Z}$ ambiguity, but this ambiguity is lost when considering dp , which can be thought of as an Abelian differential. The spectral problem can therefore be reformulated in terms of the classification of the admissible algebraic curves with Abelian differential dp . These curves can be understood as a collection of $4 + 4$ sheets (one for each of the quasimomenta) connected by the above cuts and poles. We follow the usual conventions and employ an alternative useful parametrization of the spectral parameter, taking $z = \sqrt{\frac{1+x}{1-x}}$ so that the points $z = 0, \infty$ correspond to $x = -1, +1$.

Let us continue with the list of the facts that are valid in the undeformed and periodic $\text{AdS}_5 \times S^5$ case and that continue to be valid also in the twisted one. The Lax connection is supertraceless, which implies that the

superdeterminant of the monodromy matrix is 1. In turn, this implies the “supertraceless condition,”³⁶

$$\sum_i \hat{p}_i = \sum_i \tilde{p}_i. \quad (5.4)$$

Moreover, the $\mathbb{Z}_4$ automorphism of $\mathfrak{psu}(2, 2|4)$ is still implemented by sending $z \rightarrow iz$ or equivalently, $x \rightarrow 1/x$ at the level of the Lax connection. We will call this the “inversion symmetry”. At the level of the quasimomenta, it implies [47]

$$\hat{p}_k(1/x) = -\hat{p}_{k'}(x), \quad \tilde{p}_k(1/x) = -\tilde{p}_{k'}(x) + 2\pi m \epsilon_k, \quad (5.5)$$

where $k = (1, 2, 3, 4)$ and $k' = (2, 1, 4, 3)$. Notice that this permutation is related to the choice of charge conjugation matrix in $\mathfrak{psu}(2, 2|4)$. For the sphere quasimomenta, the inversion symmetry allows for a possible shift with $\epsilon_k = (1, 1, -1, -1)$ and $m \in \mathbb{Z}$, which is related to winding around the S^5 space. For the AdS quasimomenta, this possibility is not allowed by the requirement of absence of winding for the time coordinate.

Combining the supertraceless condition, the inversion symmetry, and the Virasoro constraints, one has a synchronization³⁷ of the poles of the quasimomenta at $x = \pm 1$ also in the twisted case,

$$\begin{aligned} p(x) &= \frac{\text{diag}(\alpha^{(\pm)}, \alpha^{(\pm)}, -\alpha^{(\pm)}, -\alpha^{(\pm)} | \alpha^{(\pm)}, \alpha^{(\pm)}, -\alpha^{(\pm)}, -\alpha^{(\pm)})}{x \pm 1} \\ &+ \mathcal{O}(x \pm 1)^0. \end{aligned} \quad (5.6)$$

The bosonic cuts and the fermionic poles must respect the inversion symmetry, which means that in both cases, they must come in even number. For example, we can write the gluing condition of two quasimomenta on a cut (or more generally on a collection of cuts) $\hat{\mathcal{C}}_{(k,l)}$ or $\tilde{\mathcal{C}}_{(k,l)}$ as

³⁶In principle, one can modify the above equation by adding a shift by $2\pi n$ with $n \in \mathbb{Z}$. This can be removed by the freedom of shifting the quasimomenta by any integer multiple of 2π . In the periodic case, one can use this freedom so that at large x the quasimomenta go like $p(x) \sim \mathcal{O}(1/x)$. In the twisted case, the twist can introduce additional finite terms in the strict $x \rightarrow \infty$ limit. However, Eq. (5.4) remains valid because these finite shifts cancel each other in the above equation, as a consequence of $\det W = 1$.

³⁷When we later study one-loop corrections, we will slightly relax this synchronization of the poles to $p(x) \sim \frac{\text{diag}(\alpha^{(\pm)}, \alpha^{(\pm)}, \beta^{(\pm)}, \beta^{(\pm)} | \alpha^{(\pm)}, \alpha^{(\pm)}, \beta^{(\pm)}, \beta^{(\pm)})}{x \pm 1} + \mathcal{O}(x \pm 1)^0$, because in that case, one should consider also fermionic contributions and the tracelessness condition of $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(4)$ of the purely bosonic classical case is no longer enforced.

$$\begin{aligned} \hat{p}_k(x + i\epsilon) - \hat{p}_l(x - i\epsilon) &= 2\pi \hat{n}_{(k,l)}, \quad \text{for } x \in \hat{\mathcal{C}}_{(k,l)}, \\ \tilde{p}_k(x + i\epsilon) - \tilde{p}_l(x - i\epsilon) &= 2\pi \tilde{n}_{(k,l)}, \quad \text{for } x \in \tilde{\mathcal{C}}_{(k,l)}, \end{aligned} \quad (5.7)$$

and similar equations must hold for the cuts, which are the images of $\hat{\mathcal{C}}_{(k,l)}$ and $\tilde{\mathcal{C}}_{(k,l)}$ under $x \rightarrow 1/x$. Similarly, if there is a fermionic pole at $x_{(k,l)}^*$, by inversion symmetry there will be another pole for quasimomenta at $1/x_{(k,l)}^*$. Because of this property, we can declare a “physical” region as the one for which $|x| > 1$. The cuts in the physical region (i.e., half of the cuts in the pair of cuts connected by inversion symmetry) will be considered the “fundamental” ones.

To conclude, let us also say that we can still define the filling fractions for sphere cuts $\tilde{\mathcal{C}}_{(k,l)}$ connecting the sheets $\tilde{k}$ and $\tilde{l}$ as

$$\begin{aligned} \tilde{K}_{(k,l)} &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint dx \left(1 - \frac{1}{x^2}\right) \tilde{p}_k = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint \left(x + \frac{1}{x}\right) d\tilde{p}_k \\ &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint \left(x + \frac{1}{x}\right) d\tilde{p}_l, \end{aligned} \quad (5.8)$$

where the integration path is defined by a closed loop that encircles the *fundamental* cut $\tilde{\mathcal{C}}_{(k,l)}$. Similarly, we define the filling fractions for AdS cuts $\hat{\mathcal{C}}_{(k,l)}$ connecting the sheets $\hat{k}$ and $\hat{l}$ as

$$\begin{aligned} \hat{K}_{(k,l)} &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint dx \left(1 - \frac{1}{x^2}\right) \hat{p}_l = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint \left(x + \frac{1}{x}\right) d\hat{p}_l \\ &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint \left(x + \frac{1}{x}\right) d\hat{p}_k, \end{aligned} \quad (5.9)$$

where the integration path is defined by a closed loop that encircles the *fundamental* cut $\hat{\mathcal{C}}_{(k,l)}$. In the periodic case, these filling fractions are identified as the action variables of the integrable model [48], and they are taken to be quantized as integers in a semiclassical treatment [47]. In the twisted case, we can do the same, because the identification of the definition of the filling fractions comes from the position of the poles of the Lax connection, and these are still at $x = \pm 1$. The filling fractions can also be written in terms of the so-called twist function [74], and this is invariant under an homogeneous Yang-Baxter deformation [42].

In the periodic case, the expansion of the monodromy matrix around $z = 1$ (or equivalently, $x = \infty$) yields (at the lowest nontrivial order) the conserved Noether charges coming from the $\mathfrak{psu}(2, 2|4)$ superisometry. In the twisted case, this picture is modified by the presence of W . In fact,

after considering a gauge transformed Lax connection $\mathcal{L}'_\sigma = \tilde{g}\mathcal{L}_\sigma\tilde{g}^{-1} - \partial_\sigma\tilde{g}\tilde{g}^{-1}$, one finds³⁸

$$\tilde{g}(0)\Omega(z)\tilde{g}(0)^{-1} = W^{-1}\mathcal{P}\exp\left(-\int_0^{2\pi} d\sigma\mathcal{L}'_\sigma\right), \quad (5.10)$$

which, when writing $z = 1 + \epsilon$, leads to

$$\tilde{g}(0)\Omega(z = 1 + \epsilon)\tilde{g}(0)^{-1} = W^{-1} + 4\pi\epsilon W^{-1}\hat{Q} + \mathcal{O}(\epsilon^2), \quad (5.11)$$

where $\hat{Q}$ was defined in (4.7). In the periodic case ($W = 1$), $\hat{Q}$ is the Lie-algebra valued conserved Noether charge, and one can use a $PSU(2, 2|4)$ automorphism to put it in the Cartan subalgebra. After doing that, the monodromy matrix is diagonal at least to order ϵ , and the eigenvalues are therefore written in terms of the Cartan charges. In the twisted case, the monodromy matrix is nontrivial already at order zero in the ϵ -expansion, and the quasimomenta will depend also on the eigenvalues of W . Generically, $\hat{Q}$ on its own will not be conserved in the twisted case. As we will see in the next section, when computing the eigenvalues of the above expansion they are not necessarily written in terms of the standard Cartan charges of $\mathfrak{psu}(2, 2|4)$: there is a “polarization” induced by the presence of W , so that instead the quasimomenta are written in terms of the conserved Cartan charges of the algebra of symmetries of the twisted model (which we discussed in Sec. IV B). The symmetries of the twisted model form a subalgebra of $\mathfrak{psu}(2, 2|4)$, and the corresponding choice of Cartan subalgebra may not be equivalent to the one of the periodic case. We will now be more explicit about the computation of the eigenvalues of the monodromy matrix around $z = 1$.

Let us emphasize the great advantage of making the above discussion from the point of view of the *twisted* rather than *deformed* model. In the latter case, the expansion of the monodromy matrix around $z = 1$ would give rise to complicated nonlocal expressions that would be very hard to work with.

A. Asymptotics of the quasimomenta

To obtain the expression for the quasimomenta around $z = 1$, we must compute the eigenvalues of the expression (5.11). Since the Jordanian twist W given in (4.4) is diagonalizable, we can restrict our discussion to diagonalizable matrices. The explanation that we are about to give is actually valid for a generic diagonalizable twist.

The diagonalization of (5.11) to order ϵ follows the same route as the diagonalization of a Hamiltonian in perturbation theory of quantum mechanics. Let us rephrase that

procedure in a different language. First, consider the diagonalization of W^{-1} as $D = S^{-1}W^{-1}S$. To be as general as possible, we allow eigenvalues to be degenerate

$$D = \text{diag}(d_1, \dots, d_1, d_2, \dots, d_2, \dots), \quad (5.12)$$

where the eigenvalue d_i comes with multiplicity m_i . We will introduce projectors $P^{[i]} = \text{diag}(0, 0, \dots, 0, 1, 1, \dots, 1, 0, 0, \dots, 0)$ that have only m_i entries with value 1 in correspondence with the eigenvalues d_i , and 0 otherwise. Note that $P^{[i]}D = DP^{[i]} = d_iP^{[i]}$. We can then write for $\Omega'(z) = \tilde{g}(0)\Omega(z)\tilde{g}(0)^{-1}$,

$$\begin{aligned} \Omega'(z = 1 + \epsilon) &= S(D + \epsilon X)S^{-1} + \mathcal{O}(\epsilon^2), \\ X &\equiv 4\pi DS^{-1}\hat{Q}S. \end{aligned} \quad (5.13)$$

From simple reasoning in linear algebra, we know that to compute the ϵ correction to the eigenvalues of D , we must first project

$$X^{[i]} \equiv P^{[i]}XP^{[i]}, \quad (5.14)$$

and then compute the eigenvalues of $X^{[i]}$. If there is a block corresponding to eigenvalues all of which have multiplicity 1, then the above is equivalent to projecting on this nondegenerate block, and the diagonal part of this block will give the correction to the eigenvalues.

With this procedure, it is easy to compute the expansion of the eigenvalues. In our case, we find in the AdS sector,

$$\begin{aligned} d_1 &= 1 + 2\pi\epsilon(-C_1 - C_3) + \mathcal{O}(\epsilon^2), \\ d_2 &= e^{+Q/2}(1 + 2\pi\epsilon(C_1 - C_2)) + \mathcal{O}(\epsilon^2), \\ d_3 &= e^{-Q/2}(1 + 2\pi\epsilon(C_1 + C_2)) + \mathcal{O}(\epsilon^2), \\ d_4 &= 1 + 2\pi\epsilon(-C_1 + C_3) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} C_1 &= i\hat{Q}_{[M^{23}]}, \\ C_2 &= \hat{Q}_{[d+M^{01}]} - \sqrt{2}\mathbf{q}\hat{Q}_{[p^0+p^1]}, \\ C_3 &= \sqrt{(\hat{Q}_{[d-M^{01}]}^2 + \hat{Q}_{[-p^0+p^1]}\hat{Q}_{[k^0+k^1]}}, \end{aligned} \quad (5.16)$$

and $\hat{Q}_i = \text{STr}(\hat{Q}\mathbf{T}_i)$. The explicit expression for $\mathbf{q}$ given in (4.5) can be written precisely as $\mathbf{q} = \hat{Q}_h/\hat{Q}_e = \hat{Q}_{[d+M^{01}]} / (\sqrt{2}\hat{Q}_{[p^0+p^1]})$, which implies $C_2 = 0$. This fact is very important. It means that the only components of $\hat{Q}$ that contribute to the eigenvalues of the monodromy matrix are those which correspond to the *symmetries of the twisted model*; see (4.10). In fact, we managed to get rid of both $d + M^{01}$ and $p^0 + p^1$, which do not have this property. We remind that all of the symmetries of the twisted model are also isometries of the YB model, and in terms of the isometric charges $Q_{\bar{a}} = \text{STr}(\mathbf{T}_{\bar{a}}Q)$ with Q defined in (4.11) and the isometric generators $\mathbf{T}_{\bar{a}}$ in (2.18), we can write

³⁸For simplicity, we are writing these formulas in the case that the twisted boundary conditions are just $\tilde{g}(2\pi) = W\tilde{g}(0)$; i.e., $h = 1$ in (4.3). If $h \neq 1$, one simply has to use a gauge transformation by $\tilde{g}h$ rather than just $\tilde{g}$, and the right-hand-side of (5.11) is still given by the same expression.

$$C_1 = iQ_5, \quad C_3 = \sqrt{(Q_1)^2 + (-Q_2 + Q_3)Q_4}. \quad (5.17)$$

Again, the combination $Q_2 + Q_3$ does not appear: it would be an isometry of the deformed model, but it is not a symmetry of the twisted model.

The appearance of a square root is archetypal in the computations of eigenvalues, and in fact, it appears also in the undeformed periodic case. It is, however, an awkward feature for practical computations, especially when computing semiclassical corrections to the conserved charges of the model. Nevertheless, as in the periodic case, see, e.g., [46], one can use an automorphism to get rid of the square root and in fact, transform the expressions to ones which only involve Cartans of the symmetry algebra. First, as done in Appendix C, we identify $\mathfrak{h} = \frac{1}{2}(d - M^{01})$, $\mathbf{e}_+ = \frac{1}{2}(p^0 - p^1)$, $\mathbf{e}_- = -\frac{1}{2}(k^0 + k^1)$, which close into a $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of the algebra of isometries $\mathfrak{k}$. We can construct dual generators $\mathbf{h}^* = \frac{1}{2}\mathfrak{h}$, $\mathbf{e}^\pm = \mathbf{e}_\mp$ spanning a dual $\mathfrak{sl}(2, \mathbb{R})$ algebra. We can then construct $Q_{\mathfrak{sl}(2, \mathbb{R})} = Q_{\mathfrak{h}}\mathbf{h}^* + Q_+\mathbf{e}_+ + Q_-\mathbf{e}_- = (Q_1\mathbf{h}^* + (Q_2 - Q_3)\mathbf{e}_+ - Q_4\mathbf{e}_-)/2$. It is easy to check that $2\text{STr}[(Q_{\mathfrak{sl}(2, \mathbb{R})})^2] = (Q_1)^2 + (-Q_2 + Q_3)Q_4$, and thus,

$$C_3 = \sqrt{2\text{STr}[(Q_{\mathfrak{sl}(2, \mathbb{R})})^2]}. \quad (5.18)$$

Now recall that the generator of global time translations is (2.25), which we can write as $\mathbf{H} = \frac{1}{\sqrt{2}}(2\mathbf{e}_+ - \mathbf{e}_-)$. The global energy is thus $E = \text{STr}(Q_{\mathfrak{sl}(2, \mathbb{R})}\mathbf{H}) = \frac{1}{\sqrt{2}}(Q_2 - Q_3 + \frac{1}{2}Q_4)$. The other generators in $\mathfrak{sl}(2, \mathbb{R})$ can then be rotated such that they close into an $\mathfrak{so}(2, 1) \sim \mathfrak{sl}(2, \mathbb{R})$ algebra. Indeed, define $\mathbf{T}_E \equiv \mathbf{H}/2$, $\mathbf{T}_P \equiv (\mathbf{e}_+ + \mathbf{e}_-)/\sqrt{2}$ then, $\mathbf{T}_J \equiv \mathfrak{h}$ [$\mathbf{T}_J, \mathbf{T}_E] = \mathbf{T}_P$, $[\mathbf{T}_J, \mathbf{T}_P] = \mathbf{T}_E$ and $[\mathbf{T}_E, \mathbf{T}_P] = \mathbf{T}_J$ hold.³⁹ We can now define the dual generator $\mathbf{H}^*$ of global time translations as $\text{STr}(\mathbf{H}^*\mathbf{T}_P) = \text{STr}(\mathbf{H}^*\mathbf{T}_J) = 0$ and $\text{STr}(\mathbf{H}^*\mathbf{H}) = 1$, which fixes

$$\mathbf{H}^* = -\frac{\mathbf{e}_+}{\sqrt{2}} + \frac{\mathbf{e}_-}{2\sqrt{2}} = -\frac{\mathbf{T}_2 - \mathbf{T}_3}{2\sqrt{2}} - \frac{\mathbf{T}_4}{4\sqrt{2}}. \quad (5.19)$$

At this point, we use the classification of adjoint orbits in $\mathfrak{sl}(2, \mathbb{R})$ (see, e.g., Appendix C). Depending on the algebra element, the adjoint action may bring it to something proportional to $\mathfrak{h}$, $\mathbf{e}_+$ or $\mathbf{e}_+ - \mathbf{e}_-$. When discussing the classical spectral curve, we always assume that we are considering classical solutions that correspond to highest weight states belonging to the same orbit as the generator of time translations $\mathbf{H}$. In other words, we assume that by an inner automorphism, we can transform $Q_{\mathfrak{sl}(2, \mathbb{R})}$ to the form

$Q_{\mathfrak{sl}(2, \mathbb{R})} = E\mathbf{H}^*$. In that case, one has $\text{STr}[(Q_{\mathfrak{sl}(2, \mathbb{R})})^2] = -E^2/2$ such that we simply get

$$C_3 = iE. \quad (5.20)$$

The AdS quasimomenta in the x variable now simply expand as

$$\begin{aligned} \hat{p}_1 &\sim -\frac{2\pi(Q_\Theta + E)}{x} + \mathcal{O}(x^{-2}), \\ \hat{p}_2 &\sim -\frac{i}{2}\mathbf{Q} + \frac{2\pi Q_\Theta}{x} + \mathcal{O}(x^{-2}), \\ \hat{p}_3 &\sim +\frac{i}{2}\mathbf{Q} + \frac{2\pi Q_\Theta}{x} + \mathcal{O}(x^{-2}), \\ \hat{p}_4 &\sim \frac{2\pi(-Q_\Theta + E)}{x} + \mathcal{O}(x^{-2}), \end{aligned} \quad (5.21)$$

where we defined $Q_\Theta \equiv Q_5$, since it is the angular momentum charge related to rotations by the angle Θ . Notice that the asymptotics of the AdS quasimomenta depend only on three conserved charges, namely the energy E and the angular momentum Q_Θ at $\mathcal{O}(x^{-1})$, and the charge $\mathbf{Q}$ at order $\mathcal{O}(x^0)$, which controls the twisted boundary conditions. This situation should be compared to the three charges (energy and two spins) that appear in the asymptotics of the quasimomenta in the periodic case at $\mathcal{O}(x^{-1})$.

B. Classical algebraic curve for the BMN-like solution

Let us now consider a specific classical solution and derive the corresponding quasimomenta. We will do this for the BMN-like solution that we have constructed. We can compute the quasimomenta using either the deformed or undeformed description. On the one hand, in the deformed picture [when the solution is given by (3.4)], the Lax connection is clearly σ -independent, so that the path-ordered exponential in the definition of the monodromy matrix reduces to a simple matrix exponential of $\mathcal{L}_\sigma$. On the other hand, in the twisted picture [when the solution is given by (4.27)], the solution depends on σ , making the computation potentially more involved. However, this dependence can be eliminated via a gauge transformation and leads to the same result. This is in fact not surprising, since the two models share the same Lax connection, so that the computation is bound to be the same. Given that we can ignore the path-ordering in the exponential, to obtain the quasimomenta, we can then simply compute the eigenvalues of $\mathcal{L}_\sigma$ itself. In terms of the spectral parameter x , we find

$$\begin{aligned} \hat{p}_1(x) &= -\hat{p}_4(x) = \frac{2\pi a_T \sqrt{\frac{x^2 - \beta^2}{(x-1)^2}}}{(x+1)}, \\ \hat{p}_2(x) &= -\hat{p}_3(x) = \frac{2\pi a_T x \sqrt{\frac{1 - \beta^2 x^2}{(x-1)^2}}}{(x+1)}, \end{aligned} \quad (5.22)$$

³⁹This is the algebra of AdS_2 , or the Lorentz algebra in three dimensions. Hence, $\mathbf{T}_P$ can be interpreted as a momentum generator, and $\mathbf{T}_J$ as a boost.

where we defined $\beta \equiv \frac{\eta}{2b_Z^2}$, and we remind that a_T and b_Z are parameters that enter the classical solution; see, e.g., (3.4). In the vanishing deformation limit, which now corresponds to the $\beta \rightarrow 0$ limit, they reduce to the meromorphic quasimomenta associated to the BMN solution of undeformed $\text{AdS}_5 \times S^5$ [51], and in that sense, the solution (3.4) [equivalently (4.27)] is a “BMN-like” solution. The deformation, however, introduces a cut between sheet $\hat{1}$ and $\hat{4}$ and a cut between sheet $\hat{2}$ and $\hat{3}$. Note that the inversion symmetry (5.5) is satisfied. For later convenience, let us write the quasimomenta in the physical region $|x| > 1$,

$$\begin{aligned}\hat{p}_1(x) &= -\hat{p}_4(x) = \frac{2\pi a_T \sqrt{x^2 - \beta^2}}{x^2 - 1}, \\ \hat{p}_2(x) &= -\hat{p}_3(x) = \frac{2\pi a_T x \sqrt{1 - x^2 \beta^2}}{x^2 - 1}.\end{aligned}\quad (5.23)$$

Matching these explicit expressions with the general asymptotics of the quasimomenta (5.21) gives us the following expressions of the charges involved:

$$Q_\Theta = 0, \quad E = Q_T = -a_T, \quad \mathbf{Q} = -4\pi a_T \beta, \quad (5.24)$$

with matches with (3.5) and (4.29).

In addition to the kinematics in the deformed AdS_5 space, we will take our solution to have nontrivial kinematics in the S^5 space as well. Specifically, we consider a pointlike string traveling with constant velocity around a big circle of S^5 , $\phi = \omega\tau$. Here, ω is proportional to the conserved charge associated to an angular momentum on S^5 . As the S^5 part of our space is not deformed, we can borrow the quasimomenta from the usual BMN solution [51],

$$\tilde{p}_1 = \tilde{p}_2 = -\tilde{p}_3 = -\tilde{p}_4 = \frac{2\pi\omega x}{x^2 - 1}. \quad (5.25)$$

In the solution that we are considering, the Virasoro condition (3.8) reads

$$\omega = a_T \sqrt{1 - \beta^2}. \quad (5.26)$$

This implies that if want both ω and a_T to be real, we must have $|\beta| \leq 1$. Notice that, as we commented above, the Virasoro constraint can be reinterpreted as a synchronization condition on the poles of the quasimomenta at $x = \pm 1$.

VI. QUANTUM CORRECTIONS TO THE BMN-LIKE SOLUTION

In this section, we study the quantum corrections to the quasimomenta related to our specific BMN-like solution. Specifically, we apply the recipe presented in [51,55] to compute the one-loop correction to its energy. Despite the fact that those articles originally present this technique for

undeformed $\text{AdS}_5 \times S^5$, it has been successfully applied also to deformed backgrounds, such as the flux-deformed $\text{AdS}_3 \times S^3$ [57–59] or the Schrödinger background [60].

A. Corrections to the classical quasimomenta

Although we are not working on undeformed $\text{AdS}_5 \times S^5$, most of the road map to construct quantum fluctuations described in [51,55] applies to our deformed background. The main idea behind this computation is to introduce (quantum) excitations in the form of cuts so small that we can treat them as poles. As in the macroscopic case, if the microscopic excitation we are adding connects two sheets associated to hatted quasimomenta or two sheets associated to a tilded quasimomenta, the excitation is bosonic. If the excitation we are adding connects a tilded quasimomenta with a hatted quasimomenta, the excitation is fermionic.

These new cuts will modify the quasimomenta we computed in the previous section, but the properties that they must fulfil are restrictive enough to fully constrain how they are altered. The starting point is the fact that the corrections to the quasimomenta cannot alter the gluing condition on a cut (5.7), that is

$$\begin{aligned}(p_k + \delta p_k)(x + i\epsilon) - (p_l + \delta p_l)(x - i\epsilon) &= 2\pi n_{(k,l)}, \\ \text{for } x \in \hat{\mathcal{C}}_{(k,l)} \quad \text{or} \quad x \in \tilde{\mathcal{C}}_{(k,l)}.\end{aligned}\quad (6.1)$$

This condition is true for the macroscopic cuts associated to the classical solution, which imposes the following condition on the perturbations:

$$\delta p_k(x + i\epsilon) - \delta p_l(x - i\epsilon) = 0, \quad \text{for } x \in \hat{\mathcal{C}}_{(k,l)} \quad \text{or} \quad x \in \tilde{\mathcal{C}}_{(k,l)}. \quad (6.2)$$

A similar condition is imposed for the *microscopic cuts* that we are introducing to compute the quantum corrections, and this fixes the positions, x_n^{kl} , where these cuts can be placed by

$$p_k(x_n^{kl}) - p_l(x_n^{kl}) = 2\pi n, \quad (6.3)$$

where the mode number n takes integer values. Notice that, due to inversion symmetry (5.5), we only focus on solutions with $|x_n^{kl}| > 1$. We say that an excitation is in the *physical region* if the position of the pole fulfils that property.

Now that we know where to place the new cuts/poles, we also need information regarding their residues at x_n^{kl} . We will not discuss the explicit expression of the residues at this point but instead will only talk about the relative sign between the residues in the two sheets. We will provide more details on the residues in Appendix D. For bosonic excitations, the poles are actually infinitesimally small

square-root cuts connecting two sheets. This means that the residues on the two sheets have to have opposite signs. For fermionic excitations, the poles are actual poles, and they have to have the same residue. This can be summarized as

$$\begin{aligned}\text{Res}_{x=x_n^{ij}} \hat{p}_k &= (\delta_{i\hat{k}} - \delta_{j\hat{k}}) \alpha(x_n^{ij}) N_n^{ij}, \\ \text{Res}_{x=x_n^{ij}} \tilde{p}_k &= (\delta_{j\tilde{k}} - \delta_{i\tilde{k}}) \alpha(x_n^{ij}) N_n^{ij},\end{aligned}\quad (6.4)$$

with $i < j$ taking values $\hat{1}, \hat{2}, \hat{3}, \hat{4}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}$. Here, N_n^{ij} indicates the number of quantum excitations in the physical region that connect p_i with p_j and have mode number n , and $\alpha(x_n^{ij})$ is the residue of the pole associated to that excitation. A more detailed explanation on the origin of these signs can be found in [47].

The number of excitations N_n^{ij} are not generically free: they have to satisfy the level-matching condition,

$$\sum_n n \sum_{\text{all } ij} N_n^{ij} = 0. \quad (6.5)$$

This condition can be understood from a mathematical perspective as a consequence of the Riemann bilinear identity [47]. Physically, because the new poles modify the value of the filling fractions (see also footnote 40), the N_n^{ij} can be interpreted as the amplitudes of the physical modes, which are related to mode numbers through conventional string level matching.

In addition to these pieces of information, these corrections have some important features that they inherit from properties of the classical quasimomenta. Among those, the most relevant for us are the inversion symmetry of (5.5)

$$\delta p_k(x) = -\delta p_k(1/x), \quad (6.6)$$

and the synchronization of the poles at ± 1 ,

$$\delta p(x) \sim \frac{\text{diag}(\delta\alpha^{(\pm)}, \delta\alpha^{(\pm)}, \delta\beta^{(\pm)}, \delta\beta^{(\pm)} | \delta\alpha^{(\pm)}, \delta\alpha^{(\pm)}, \delta\beta^{(\pm)}, \delta\beta^{(\pm)})}{x \pm 1} + \dots \quad (6.7)$$

Notice that we are using the relaxed version of the synchronization of poles as commented on in footnote 37, because we will have to consider also fermionic excitations when computing quantum corrections.

The final piece of information that we need to compute the perturbations is their asymptotic behavior for large values of x . This can be obtained by analyzing the asymptotic behavior of the quasimomenta (5.21). We require⁴⁰

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_4 \\ \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} = \frac{4\pi}{x\sqrt{\lambda}} \begin{pmatrix} +\frac{\delta\Delta}{2} + N_{\hat{1}\hat{4}} + N_{\hat{1}\hat{3}} + N_{\hat{1}\tilde{3}} + N_{\hat{1}\tilde{4}} \\ -\frac{\delta\Delta}{2} - N_{\hat{1}\hat{4}} - N_{\hat{2}\hat{4}} - N_{\tilde{2}\hat{4}} - N_{\tilde{1}\hat{4}} \\ -N_{\tilde{1}\tilde{3}} - N_{\tilde{1}\tilde{4}} - N_{\tilde{1}\tilde{3}} - N_{\tilde{1}\tilde{4}} \\ -N_{\tilde{2}\tilde{3}} - N_{\tilde{2}\tilde{4}} - N_{\tilde{2}\tilde{3}} - N_{\tilde{2}\tilde{4}} \\ +N_{\tilde{1}\tilde{3}} + N_{\tilde{2}\tilde{3}} + N_{\tilde{1}\tilde{3}} + N_{\tilde{2}\tilde{3}} \\ +N_{\tilde{1}\tilde{4}} + N_{\tilde{2}\tilde{4}} + N_{\tilde{1}\tilde{4}} + N_{\tilde{2}\tilde{4}} \end{pmatrix} + \mathcal{O}(x^{-2}), \quad (6.8)$$

⁴⁰In all the following expressions, we rescale our conserved charges with a factor of $\sqrt{\lambda}$, so that the reader can compare them with the previous literature more easily. We want to stress that the normalization factor $\frac{4\pi}{\sqrt{\lambda}}$ can be modified in the sense that it does not affect the computation of the anomalous contribution to the energy $\delta\Delta$, at the condition that we also modify the normalization of the residues $\alpha(x_n^{ij})$ consistently. However, the natural way to fix this normalization factor is by demanding that the correction to the filling fractions $K_{(i,j)}$ is given by N_{ij} , i.e., $-\frac{\sqrt{\lambda}}{8\pi^2} \oint dx (1 - \frac{1}{x})(p_i + \delta p_i) = K_{(i,j)} + N_{ij}$. In this sense, the normalization chosen for the filling fractions fixes also the normalization of the asymptotics of δp .

where $\delta\Delta$ is the anomalous correction of the energy and where $N_{ij} = \sum_n N_n^{ij}$ is the total number of poles in the physical region connecting sheet i and j . The above formula can be justified in the same way as in the undeformed periodic case. In particular, one can assume that the addition of quantum excitations produces integer shifts in the charges that appear at order $1/x$ in the large- x asymptotics of the quasimomenta (5.21), which justifies the presence of N_{ij} . Here, we are also assuming that only the energy is allowed to receive anomalous corrections, and we will see that the results will be consistent with this. Notice that above we have not included the asymptotics of $\hat{p}_2$ and $\hat{p}_3$. The reason is that the classical asymptotic behavior (5.21) suggests that their form is nonstandard (i.e., they might be finite in the $x \rightarrow \infty$ limit) as a consequence of the twist. We will actually not need them to compute $\delta\Delta$, and we refer to [60] for a similar strategy applied in the case of the Schrödinger background. Nevertheless, we discuss the asymptotic behavior of $\hat{p}_2$ in detail in Sec. VIB and, importantly, we will show that the charge $\mathbf{Q}$ does not get any anomalous correction.

Combining all the properties we have enumerated, we possess enough information about the analytic structure of these corrections to be able to reconstruct them. As this is a tedious and sometimes repetitive process, we have relegated the details regarding this reconstruction to Appendix D, while here we collect the final results. If we denote each contribution to $\delta\Delta$ as

$$\delta\Delta = \sum_n \Omega^{ij}(x_n) N_n^{ij}, \quad (6.9)$$

we find the following expressions:

$$\Omega^{\hat{1}\hat{3}}(x_n) = \Omega^{\hat{1}\hat{4}}(x_n) = \Omega^{\hat{2}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) = 2 \frac{K(1)}{x_n^2 - 1}, \quad (6.10)$$

$$\Omega^{\hat{1}\hat{4}}(x_n) = 2 \frac{x_n^2 K(1/x_n)}{x_n^2 - 1} - 2, \quad (6.11)$$

$$\Omega^{\hat{2}\hat{3}}(x_n) = \frac{2K(x_n)}{x_n^2 - 1}, \quad (6.12)$$

$$\Omega^{\hat{1}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) = \frac{x_n^2 K(1/x_n) + K(x_n)}{x_n^2 - 1} - 2, \quad (6.13)$$

$$\begin{aligned} \Omega^{\hat{1}\hat{3}}(x_n) &= \Omega^{\hat{1}\hat{4}}(x_n) = \Omega^{\hat{2}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) \\ &= \frac{x_n^2 K(1/x_n) + K(1)}{x_n^2 - 1} - 1, \end{aligned} \quad (6.14)$$

$$\begin{aligned} \Omega^{\hat{2}\hat{3}}(x_n) &= \Omega^{\hat{2}\hat{4}}(x_n) = \Omega^{\hat{1}\hat{3}}(x_n) = \Omega^{\hat{1}\hat{4}}(x_n) \\ &= \frac{K(x_n) + K(1)}{x_n^2 - 1}, \end{aligned} \quad (6.15)$$

where $K(x) = \sqrt{1 - x^2 \beta^2}$.

The final step is to use Eq. (6.3) to find the value of the spectral parameter at which we have to place the microscopic cuts, x_n , and substitute it in the above equations. As the process is the same for all the cases *mutatis mutandis*, we will illustrate it in the case $\hat{1}\hat{3}$. The algebraic equation $\hat{p}_1(x_n) - \hat{p}_3(x_n) = 2\pi n$ has two solutions,

$$x_n = \frac{a_T \sqrt{1 - \beta^2} \pm \sqrt{a_T^2 + n^2}}{n}. \quad (6.16)$$

Because $\sqrt{a_T^2 + n^2} \geq a_T \geq a_T \sqrt{1 - \beta^2}$ for real values of n and $|\beta| \leq 1$ [as required from reality conditions, see below (5.26)] the solution with minus sign has modulus smaller than one, and we will disregard it. Substituting this result into $\Omega^{\hat{1}\hat{3}}(x_n)$, we find

$$\Omega^{\hat{1}\hat{3}}(x_n) = \frac{x_n^2 K(1/x_n) + K(1)}{x_n^2 - 1} - 1 = \sqrt{\frac{a_T^2 + n^2}{a_T^2}} - 1. \quad (6.17)$$

The remaining contributions take the following form:

$$\begin{aligned} \Omega^{\hat{1}\hat{3}}(x_n) &= \Omega^{\hat{1}\hat{4}}(x_n) = \Omega^{\hat{2}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) \\ &= -\sqrt{1 - \beta^2} + \sqrt{1 - \beta^2 + \frac{n^2}{a_T^2}}, \end{aligned} \quad (6.18)$$

$$\Omega^{\hat{1}\hat{4}}(x_n) = -2 + \sqrt{2 + \frac{n^2}{a_T^2} + 2\sqrt{1 + \frac{n^2}{a_T^2}(1 - \beta^2)}}, \quad (6.19)$$

$$\Omega^{\hat{2}\hat{3}}(x_n) = \sqrt{2 + \frac{n^2}{a_T^2} - 2\sqrt{1 + \frac{n^2}{a_T^2}(1 - \beta^2)}}, \quad (6.20)$$

$$\Omega^{\hat{1}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) = -1 + \sqrt{1 + \beta^2 + \frac{n^2}{a_T^2}}, \quad (6.21)$$

$$\begin{aligned} \Omega^{\hat{1}\hat{3}}(x_n) &= \Omega^{\hat{1}\hat{4}}(x_n) = \Omega^{\hat{2}\hat{3}}(x_n) = \Omega^{\hat{2}\hat{4}}(x_n) \\ &= -1 + \sqrt{1 + \frac{n^2}{a_T^2}}, \end{aligned} \quad (6.22)$$

$$\begin{aligned} \Omega^{\hat{2}\hat{3}}(x_n) &= \Omega^{\hat{2}\hat{4}}(x_n) = \Omega^{\hat{1}\hat{3}}(x_n) = \Omega^{\hat{1}\hat{4}}(x_n) \\ &= -\sqrt{1 - \beta^2} + \sqrt{1 + \frac{n^2}{a_T^2}}. \end{aligned} \quad (6.23)$$

In order to have an independent check of our results, we have computed the bosonic AdS contributions using the method of quadratic fluctuations in the picture of the deformed model. The details of that computation are collected in Appendix E, and the final results are consistent with the ones obtained using the classical spectral curve of the twisted model.

Another nontrivial check that our results pass is the fact that we recover the usual BMN contributions in the undeformed limit (see, for example, [75]), i.e., $\Omega = -1 + \sqrt{1 + \frac{n^2}{a_T^2}}$, upon using the Virasoro constraint (5.26).

B. Anomalous correction to the twist

Before computing the one-loop correction to the dispersion relation of the BMN-like solution, we want to make some comments on the asymptotic behavior of $\delta\hat{p}_2$ and its effect on the charge $\mathbf{Q}$ of the twist. In particular, we will show that there is no contribution to the $\mathcal{O}(x^0)$ term of the expansion. To that end, we will use the inversion symmetry (6.6) to compute $\delta\hat{p}_2$ from the different $\delta\hat{p}_1$ excitations that we constructed in Appendix D.

Let us begin with the excitation $\hat{2}\hat{3}$. Using inversion symmetry on

$$\delta\hat{p}_1(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{K(1)}{K(1/x)} \frac{x_n}{x_n^2 - 1} \frac{2x}{x^2 - 1}, \quad (6.24)$$

we can check that

$$\delta\hat{p}_2(x) = -\delta\hat{p}_1\left(\frac{1}{x}\right) \approx \frac{8\pi}{x\sqrt{\lambda}} \frac{K(1)}{\sqrt{-\beta^2}} \frac{x_n}{x_n^2 - 1} + \mathcal{O}(x^{-2}). \quad (6.25)$$

Despite getting a nontrivial $\mathcal{O}(x^{-1})$ term, we can check that it actually vanishes when we impose the level-matching condition. The other three excitations associated to the S^5 have the same behavior, as $\delta\hat{p}_1$ has a similar form for all four.

The situation is very similar for both the excitations associated to $\hat{1}\hat{4}$ and $\hat{1}\hat{3}$. In particular, we have

$$\delta\hat{p}_2 \approx \sum_n \frac{K(1/x_n)}{\sqrt{-\beta^2}} \frac{\alpha(x_n)N_n}{x_n x} + \mathcal{O}(x^{-2}), \quad (6.26)$$

$$\delta\hat{p}_2 \approx \sum_n \frac{\alpha(x_n)}{x_n \sqrt{-\beta^2}} \frac{K(1/x_n) + K(1)}{2x} N_n + \mathcal{O}(x^{-2}), \quad (6.27)$$

respectively. In both cases, the $1/x$ term is proportional to the pole-fixing condition (6.3) and, thus, proportional to n . This means that both the x^0 and x^{-1} term vanish once the level-matching condition is imposed.

The situation for the excitations associated to $\hat{1}\hat{3}$ and $\hat{2}\hat{3}$ is slightly different. For these kinds of excitations, we find that

$$\delta\hat{p}_2 \approx -\sum_n \left(\frac{4\pi}{\sqrt{\lambda}} + \frac{\alpha(x_n)K(x_n) + K(1/x_n)}{x_n \sqrt{-\beta^2}} \right) \frac{N_n}{2x} + \mathcal{O}(x^{-2}), \quad (6.28)$$

$$\delta\hat{p}_2 \approx \sum_n \left(\frac{4\pi}{\sqrt{\lambda}} + \frac{\alpha(x_n)K(x_n) + K(1)}{x_n \sqrt{-\beta^2}} \right) \frac{N_n}{2x} + \mathcal{O}(x^{-2}), \quad (6.29)$$

respectively. The analysis in both cases is the same: the x^0 contribution vanishes automatically, while the second term in the x^{-1} contribution vanishes due to the level-matching condition. However, in both cases, we have a nonvanishing $\mp \sum_n \frac{2\pi N_n}{\sqrt{\lambda}}$ at order x^{-1} .

The behavior of the remaining excitations can be inferred using the different tricks described in Appendix D, e.g., the composition of excitations described in D 4 or the fact that the classical solution has pairwise symmetric quasimomenta (D26).

Let us summarize and compare our results with what we would expect from the asymptotic behavior of the quasimomenta (5.21). First, we find that the x^0 terms of the expansion of $\delta\hat{p}_2$ for large values of x vanish in all the cases. In (5.21), these terms corresponds to the conserved charge $\mathbf{Q}$, associated to the twist. Hence, we find that $\mathbf{Q}$ does not receive any anomalous correction, $\delta\mathbf{Q} = 0$. On the other hand, depending on the excitation we are considering,

the x^{-1} terms either vanish (after imposing the level-matching condition) or give a constant contribution. In particular, $\hat{1}\hat{3}$, $\hat{2}\hat{4}$, $\hat{2}\hat{3}$, $\hat{2}\hat{4}$, $\hat{1}\hat{3}$, and $\hat{2}\hat{3}$ fall in this second category, where the two bosonic excitations contribute with $-N_n/2$ and the four fermionic excitations with $+N_n/2$.⁴¹ If we try to interpret this from the lens of undeformed $\text{AdS}_5 \times S^5$, the appearance of, e.g., a contribution proportional to $N_n^{\hat{1}\hat{3}}/2$ in $\delta\hat{p}_2$ is surprising. Due to the presence of the twist, however, the contribution of certain excitations to the asymptotics of $\delta\hat{p}_2$ and $\delta\hat{p}_3$ do in fact mix. Note in particular that the only nonvanishing excitations are those which connect one sheet affected by the twist with a sheet that is not affected by the twist. Comparing with (5.21), the new asymptotics suggest a shift of the spin charge Q_Θ as

$$Q_\Theta \rightarrow Q_\Theta - N_{\hat{1}\hat{3}} - N_{\hat{2}\hat{4}} + N_{\hat{2}\hat{3}} + N_{\hat{2}\hat{4}} + N_{\hat{1}\hat{3}} + N_{\hat{2}\hat{3}}. \quad (6.30)$$

Let us emphasize that this contribution is nondynamical, as it only depends on the number of excitations N_n^{ij} but not on the position of the poles x_n^{ij} . Similar shifts of spin charges are found also in undeformed $\text{AdS}_5 \times S^5$, see, e.g., [51], where the spins get corrections in which the contributions of fermionic excitations are weighted with a factor of $1/2$ relative to those of bosonic excitations. It would be interesting to understand this behavior for twisted models in more generality.

C. One-loop correction to the energy

Now we have enough information to compute the one-loop correction to the energy of our classical string. This is done by considering that the excitations we are introducing behave as harmonic oscillators and computing their ground state energy. Mathematically, this means that

$$E \approx E_{\text{class}} + E_{1\text{-loop}} = E_{\text{class}} + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{ij} (-1)^{F_{ij}} \Omega^{ij}(x_n), \quad (6.31)$$

where we have $F_{ij} = 0$ for bosonic and $F_{ij} = 1$ for fermionic contributions, respectively, and where $E_{\text{class}} = -E = \sqrt{\lambda} a_T$ (recall that we rescale charges with $\sqrt{\lambda}$).

Before attempting to perform the sum over n , we should check if the series of the addends converges. We can check that it is the case, as

$$\sum_{ij} (-1)^{F_{ij}} \Omega^{ij}(x_n) = -\frac{2\beta^2}{n^3} + \mathcal{O}(n^{-4}), \quad (6.32)$$

⁴¹Despite not being mentioned in the analysis of [60], a similar behavior can be found in the case of the Schrödinger background.

meaning that the partial sums over the mode number have to converge, and we are allowed to compute the sum over integer numbers.

First of all, we should notice that the mode number n always appears divided by a_T . We can safely assume that $a_T \gg 1$ because it is related to the energy of our classical solution. This allows us to approximate our sum over $n \in \mathbb{Z}$ by an integral over $n \in \mathbb{R}$.⁴² Then,

$$\begin{aligned} \frac{1}{2} \sum_{n \in \mathbb{Z}, ij} (-1)^{F_{ij}} \Omega^{ij}(x_n) &\approx \frac{a_T}{2} \int_{-\infty}^{\infty} dn \sum_{ij} (-1)^{F_{ij}} \Omega^{ij}(x_{a_T n}) \\ &= a_T \int_0^{\infty} dn \sum_{ij} (-1)^{F_{ij}} \Omega^{ij}(x_{a_T n}), \end{aligned} \quad (6.33)$$

where by $x_{a_T n}$ we mean that we are rescaling n by a factor a_T . Notice that, to get the second equality, we have used that the frequencies only depend on n quadratically, and we are therefore dealing with an even integrand.

At this point, it is useful to divide our contributions into two different types,

$$\begin{aligned} \Omega_{\text{nested}} &= \Omega^{\hat{1}\hat{4}}(x_n) + \Omega^{\hat{2}\hat{3}}(x_n) \\ &= \sqrt{2} \sqrt{2 + n^2 + \sqrt{n^2(n^2 + 4\beta^2)}}, \end{aligned} \quad (6.34)$$

$$\Omega_{\text{other}} = -8\sqrt{1 + n^2} + 4\sqrt{1 + n^2 - \beta^2} + 2\sqrt{1 + n^2 + \beta^2}. \quad (6.35)$$

We do so because the nested square root structure makes the integration of Ω_{nested} a bit involved. Let us first address the integral of Ω_{other} . Instead of integrating from 0 to ∞ , we shall integrate only up to a positive cutoff, Λ . We do so because, while the total series is convergent, the two separate contributions Ω_{nested} and Ω_{other} are not. Obviously, they have divergences that later will cancel each other. After some algebra, we find

$$\begin{aligned} \frac{I_{\text{other}}}{a_T} &= 2 \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda} dn \Omega_{\text{other}} = -2\Lambda^2 - 2(1 + \beta^2) \log(2\Lambda) \\ &\quad - [(1 + \beta^2) + (1 + \beta^2) \log(1 + \beta^2) \\ &\quad + 2(1 - \beta^2) \log(1 - \beta^2)]. \end{aligned} \quad (6.36)$$

The integral over Ω_{nested} is more intricate, but it can be simplified using a change of variables inspired by the one proposed in Appendix C of [77]. In particular, after we change into a variable $y(n)$ that eliminates the nested

integral, i.e., $\Omega_{\text{nested}} = \sqrt{4 + y(n)^2}$, we can perform the integral as

$$\lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda} dn \Omega_{\text{nested}} = \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda'} dy \sqrt{4 + y^2} \frac{y^3 + 8y\beta^2}{2\sqrt{y^2 + 4\beta^2}}, \quad (6.37)$$

where $\Lambda' = 2\Lambda + \frac{\beta^2}{\Lambda} + \mathcal{O}(\Lambda^{-3})$. Notice that, although we are interested in the limit of very large Λ , we have kept the next-to-leading order of the map between the two cutoffs. We have done that because the integral will diverge as Λ'^2 , which means that this subleading order of the transformation will give rise to a nontrivial contribution to the integral. After evaluating the integral, and with a healthy dose of algebra, we arrive at

$$\begin{aligned} \frac{I_{\text{nested}}}{a_T} &= 2 \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda} dn \Omega_{\text{nested}} = 2\Lambda^2 + 2(1 + \beta^2) \log(2\Lambda) \\ &\quad + 1 + 2\beta - \beta^2 - (1 + \beta^2) \log(1 - \beta^2) \\ &\quad + (1 + \beta^2) \log\left(\frac{1 - \beta}{1 + \beta}\right). \end{aligned} \quad (6.38)$$

Putting both integrals together, it is immediate to check that the divergent parts perfectly cancel, giving us

$$\begin{aligned} \frac{2E_{1\text{-loop}}}{a_T} &= \frac{I_{\text{nested}} + I_{\text{other}}}{a_T} = 2\beta - 2\beta^2 - (3 - \beta^2) \log(1 - \beta^2) \\ &\quad - (1 + \beta^2) \log(1 + \beta^2) + (1 + \beta^2) \log\left(\frac{1 - \beta}{1 + \beta}\right). \end{aligned} \quad (6.39)$$

As a check of our expression, we can examine two particularly interesting values of β . On the one hand, we can check that $E_{1\text{-loop}} = 0$ for $\beta = 0$. This is consistent with the fact that we recover the BMN solution in the undeformed limit, which does not receive corrections. On the other hand, in the $\beta = 1$ limit, the form of Ω_{nested} simplifies, as we can get rid of the nested square root,

$$\begin{aligned} \lim_{\beta \rightarrow 1} \Omega_{\text{nested}} &= \sqrt{4 + 2n^2 + 2\sqrt{n^2(n^2 + 4)}} \\ &= \sqrt{n^2} + \sqrt{n^2 + 4}. \end{aligned} \quad (6.40)$$

Thus, in this limit all the contributions have the form of a square root, and there is no need to separate the integral into two divergent contributions, avoiding the possible issues that a cutoff may introduce. When taking $\beta \rightarrow 1$, the limit of the integral and the integral of the limit coincide and give $E_{1\text{-loop}} = -a_T \log(8)$.

To end this section, we want to elaborate on our choice of cutoff. As the deformation changes the masses associated to

⁴²Here, we can use the integration method proposed in [76] to argue that the error in our approximation has to be exponentially small in a_T , as none of our contributions have a branch cut for real values of n .

the different quantum fluctuations differently, we may worry that an issue with the choice of regularization similar to the one present in $\text{AdS}_4 \times \mathbb{CP}^3$ may arise, see, e.g., [78,79], where a unique summation prescription does not exist, and one needs further information to choose the correct one. While for AdS_5 all the quantum excitations have the same mass, for the less-symmetrical AdS_4 , they can take two possible values, which allows one to separate them into “heavy” or “light” modes. As the number of bosonic and fermionic heavy modes is the same, and idem for light modes, one is allowed to choose a different cutoff for each type leading to a different result. Although our situation may look similar at first glance, the mass of the excitations in our background are associated to their origin, i.e., if they are bosonic modes associated to the sphere, or to the deformed AdS space, or to fermions. Consequently, although a different regularization scheme may exist, it would be highly nontrivial.

VII. THE UNIMODULAR CASE AND THE SPECTRAL EQUIVALENCE

So far we have considered the nonunimodular deformation; i.e., we have set $\zeta = 0$ in (2.7). Here, we wish to show that these results can be extended to unimodular deformations ($\zeta = 1$) thanks to the observation that even in this case the twist can be factorized as explained in Sec. IV C for the nonunimodular case. First, an extended Jordanian R -matrix of the type (2.7) with $\zeta = 1$ solves the classical Yang-Baxter equation if the odd elements satisfy the (anti) commutation relations [14,65,80],

$$\begin{aligned} [\mathbf{Q}_i, \mathbf{e}] &= 0, & [\mathbf{h}, \mathbf{Q}_1] &= \frac{1}{2} \mathbf{Q}_1 - \xi \mathbf{Q}_2, \\ [\mathbf{h}, \mathbf{Q}_2] &= \frac{1}{2} \mathbf{Q}_2 + \xi \mathbf{Q}_1, & \{\mathbf{Q}_j, \mathbf{Q}_k\} &= -i\delta_{jk} \mathbf{e}, \end{aligned} \quad (7.1)$$

where ξ is a free real parameter. From [44], we know that for a generic homogeneous Yang-Baxter deformation, the twist can be taken to be in the subgroup F , and therefore, it can be written as $W^{-1} = \exp(\eta RQ)$, where Q is a conserved charge that takes values in the dual (with respect to the bilinear form of $\mathfrak{g}$) of $\mathfrak{f}$. That means that we can write it as

$$W^{-1} = \exp(\eta RQ) = \exp[\eta(-\mathcal{Q}_e \mathbf{h} + \mathcal{Q}_h \mathbf{e} - i(\mathcal{Q}_1 \mathbf{Q}_1 + \mathcal{Q}_2 \mathbf{Q}_2))], \quad (7.2)$$

where $\mathcal{Q}_A$ are projections of Q . We will not calculate them explicitly because this will not be needed for the following argument, but notice that, when setting $\zeta = 1$, in principle all of them can receive contributions from the fermionic degrees of freedom. In other words, when going from the nonunimodular to the unimodular case, the difference is not

only the presence of the new (fermionic) charges $\mathcal{Q}_1, \mathcal{Q}_2$, because we can also have different expressions for $\mathcal{Q}_h, \mathcal{Q}_e$.

First, notice that $\mathcal{Q}_1, \mathcal{Q}_2$ are Grassmann variables. We are therefore working with the Grassmann enveloping algebra, in which case anticommutators become standard commutators. In particular, notice that if we define $Z = -i\eta(\mathcal{Q}_1 \mathbf{Q}_1 + \mathcal{Q}_2 \mathbf{Q}_2)$, then $[Z, Z] = 0$, because the only nonvanishing anticommutator in (7.1) is when we take the same odd generator, but then $\mathcal{Q}_i^2 = 0$ because it is Grassmann. The nonunimodular twist is recovered by formally setting $Z = 0$.

When $\xi = 0$ in the commutation relations, the generators $\mathbf{Q}_1, \mathbf{Q}_2$ do not mix, and we can repeat the argument of Sec. IV C to show that $W = v^{-1} W' v$, where $W' = \exp(\eta \mathcal{Q}_e \mathbf{h})$. In fact, first we can write the twist in the form $W^{-1} = \exp(A + \frac{s}{1-e^{-s}} B')$ and repeat the steps in (4.18) if we now identify

$$A = \eta(-\mathcal{Q}_e \mathbf{h} + \mathcal{Q}_h \mathbf{e}), \quad \frac{s}{1-e^{-s}} B' = Z. \quad (7.3)$$

Using that $[\mathbf{h}, Z] = \frac{1}{2} Z$ and imposing $[A, B'] = sB'$, one finds $s = -\eta \mathcal{Q}_e / 2$. This shows that the unimodular twist is related by a Z -dependent similarity transformation to $\exp[\eta(-\mathcal{Q}_e \mathbf{h} + \mathcal{Q}_h \mathbf{e})]$. This has the same form as the nonunimodular twist and, as explained in Sec. IV C, it then follows that it is also related by a similarity transformation to W' .

To prove the factorization of the twist for generic values of ξ , we will now consider a different and even simpler method, that works both for $\zeta = 0$ and $\zeta = 1$ and for any ξ . For definiteness, we will provide the discussion in the $\zeta = 1$ case. First, let us construct

$$v = \exp(\alpha_e \mathbf{e}) \exp(\alpha_1 \mathbf{Q}_1) \exp(\alpha_2 \mathbf{Q}_2), \quad (7.4)$$

where α_1, α_2 are Grassmann, and compute

$$\begin{aligned} v^{-1} \mathbf{h} v &= \mathbf{h} + \alpha_e \mathbf{e} + \left(\frac{1}{2} \alpha_1 + \xi \alpha_2 \right) \mathbf{Q}_1 \\ &\quad + \left(\frac{1}{2} \alpha_2 - \xi \alpha_1 \right) \mathbf{Q}_2 - i \xi \alpha_1 \alpha_2 \mathbf{e}. \end{aligned} \quad (7.5)$$

The above formula implies that

$$\begin{aligned} v^{-1} \exp(c \mathbf{h}) v &= \exp \left[c \left(\mathbf{h} + (\alpha_e - i \xi \alpha_1 \alpha_2) \mathbf{e} \right. \right. \\ &\quad \left. \left. + \left(\frac{1}{2} \alpha_1 + \xi \alpha_2 \right) \mathbf{Q}_1 + \left(\frac{1}{2} \alpha_2 - \xi \alpha_1 \right) \mathbf{Q}_2 \right) \right]. \end{aligned} \quad (7.6)$$

Now we want to match the right-hand-side of the above equation with the right-hand-side of (7.2). We can do this only if we assume $c \neq 0$, and then

$$c = -\eta Q_e, \quad \alpha_e = \eta Q_h/c + i\xi\alpha_1\alpha_2, \quad (7.7)$$

and if we also require

$$\frac{1}{2}\alpha_1 + \xi\alpha_2 = -i\eta Q_1/c, \quad \frac{1}{2}\alpha_2 - \xi\alpha_1 = -i\eta Q_2/c. \quad (7.8)$$

This system has always a solution except if $\xi^2 = -1/4$, but this possibility is excluded because it must be $\xi \in \mathbb{R}$ to respect the real form of $\mathfrak{psu}(2, 2|4)$. To conclude, we can always solve also for $\alpha_1, \alpha_2, \alpha_e$, and then the twist can be put in the wanted form,

$$W = v^{-1}W'v, \quad W' = \exp(\eta Q_e h). \quad (7.9)$$

Let us make a comment on the assumption $c \neq 0$ made above. Given the solution for c in (7.7), there are only two possibilities in which c can vanish. The undeformed limit $\eta \rightarrow 0$ is obviously not problematic: when $\eta \rightarrow 0$ then $W \rightarrow 1$ so the factorization is trivially true. The only real worry should be the case $Q_e = 0$ with $\eta \neq 0$. Let us analyze this situation in the nonunimodular case, when $Q_e = -\mathbf{Q}/\eta$ and $\mathbf{Q}$ was given in (4.5). Then, Q_e can vanish (at finite η) only if $\mathbf{Q} = 0$. Given (4.4), this leads to a trivial twist $W = 1$, unless $\mathbf{q}$ compensates the zero of $\mathbf{Q}$ with a divergence. This is possible only if the field configuration is such that

$$Y_h(\tau, 2\pi) - Y_h(\tau, 0) \neq 0, \quad Y_+(\tau, 2\pi) - Y_+(\tau, 0) = 0. \quad (7.10)$$

In other words, the degrees of freedom in Y_e are periodic while those in Y_h are not. Therefore, in this sector of the theory the nonunimodular twist cannot be put into the form $\exp(ch)$, and it is instead equal to $W = \exp(-w\mathbf{e})$ with $w = \mathbf{Q}\mathbf{q}$ finite and nonzero. This is in fact a nondiagonalizable twist, with all eigenvalues equal to 1. While the leading-order asymptotics of the quasimomenta around $z = 1$ (or $x = \infty$) would match those of the untwisted case, the next-to-leading order will change. A similar analysis is obviously valid also for the unimodular twist.

We conclude this section by arguing the spectral equivalence of the unimodular and the nonunimodular models, up to certain caveats that we are about to point out. In the sector where the twists of the unimodular and nonunimodular models are both equivalent to $W' = \exp(\eta Q_e h)$, the fact that the classical spectra of the two models are the same is rather straightforward. First, in both cases, one can implement field redefinitions as explained in IV C to obtain boundary conditions controlled just by $W' = \exp(\eta Q_e h)$. As remarked above, the explicit expressions for Q_e in the two cases differ, because in the unimodular case, there are additional contributions from the fermionic degrees of freedom.

However, these vanish when considering classical solutions, and therefore, the classical spectral curves of the two models are indistinguishable. One may worry that the nondiagonalizable sector [where the twist is instead $W = \exp(-w\mathbf{e})$] may be problematic from the point of view of the spectral equivalence. But also in this case, the twists of the two models would differ only by contributions of the fermionic degrees of freedom, that do not contribute at the classical level.

We can push this argument even to the one-loop level. In fact, one may use the method of [51,55] to compute quantum corrections to the classical spectrum, as we have done in the previous section for the nonunimodular case. The advantage of this method is that the only data that is needed to compute the one-loop shift is the classical spectral curve itself. Because the two classical spectral curves agree, the unimodular and nonunimodular models will also have the same 1-loop corrections to the spectrum. It would be interesting to understand if it is possible to make any statement beyond one-loop, since it is always possible that the spectral equivalence breaks down at higher loops. However, we want to point out that even at one-loop level, there may be extra subtleties not discussed so far. In fact, the whole argument is done under the assumption that it is possible to reformulate the Yang-Baxter deformed model as the undeformed yet twisted one. While this is certainly true at the classical level because the two σ -models are equivalent on shell, it is possible that the equivalence breaks down at the quantum level. An anomaly may occur especially in the nonunimodular case, when the background fields do not satisfy the standard supergravity equations and the σ -model is not Weyl invariant. In the presence of such an anomaly, even if the (quantum) spectra of the two *twisted* models were the same, the spectra of the corresponding Yang-Baxter *deformations* (with $\zeta = 0$ and with $\zeta = 1$) may not be related in an obvious way. It would be very interesting to investigate these possible scenarios in more details.

VIII. CONCLUSIONS

In this paper, we have considered a Jordanian deformation of the $\text{AdS}_5 \times S^5$ superstring background, that preserves the (classical) integrability and at least 12 superisometries.⁴³ We have identified global coordinates for the deformed background, and in particular, a global time coordinate. This fact is crucial for the correct identification of the energy when considering the spectral problem from first principles, and we believe that it will

⁴³Interestingly, as already noted in [65] using findings of [81], this Jordanian deformation can also be obtained by a sequence of TsT and S-duality transformations. It therefore provides an example of a transformation involving S-duality that preserves integrability. See [82] for more recent examples.

be important in order to have insights into an AdS/CFT interpretation of the deformation.

Importantly, we have reformulated the deformed model in terms of an undeformed yet twisted model. In this twisted picture, we were able to obtain a general class of solutions to the σ -model equations of motion written in terms of Airy functions. Our further study, however, focused on a particular simple solution which we called BMN-like. Given that the difference compared to the standard (periodic) case of $\text{AdS}_5 \times S^5$ is only in a twist appearing in the boundary conditions, we could borrow several methods of integrability that had already been used in the past. We did this not just at the level of the classical spectral curve but also when including its first quantum corrections since, despite the twist, the methods are essentially unaltered.

The only place where the construction of the classical spectral curve is changed with respect to the standard case is in the form that the quasimomenta take at large values of the spectral parameter. In particular, we are forced to reevaluate the ansatz for the asymptotic behavior of the quantum corrections for some of our quasimomenta. For example, in the standard case, we can extract the contribution to the anomalous correction to the energy from any of the corrections to the quasimomenta associated to the AdS directions. However, in the deformed background, we have studied, this contribution does not appear in those quasimomenta which are affected by the twist.

An interesting outcome of our results is that we find no anomalous quantum correction for the charges that control the twisted boundary conditions. The only charge that receives quantum corrections is the energy, and we interpret this as the spectral problem being well-posed. The situation is reminiscent to the β -deformation [28], where the twist appears in the form of phases that enter the Bethe equations [36], for example. Also in that situation, it is the energy of the string (or the dual anomalous dimension of single-trace operators) that receives quantum corrections, while the charges controlling the twist appear as external data, so that the Bethe equations can be solved consistently.

The advantage of working in the language of the twisted model resides also in the fact that the conserved charges that label the full spectrum admit *local* expressions in terms of the variables of the twisted model, while they would be nonlocal in the variables of the deformed one. Moreover, only (the Cartans of) the manifest symmetries of the twisted model appear as labels of the spectrum, and this is crucial because these are only a subset of the isometries of the deformed one.

In the deformed picture, we know that the deformation parameter can be reabsorbed, although this is done in such a way that the undeformed limit $\eta \rightarrow 0$ becomes subtle. One can actually think in terms of two possible cases to be considered, namely $\eta = 0$ and $\eta \neq 0$. This is related to the fact that Jordanian deformation (at $\eta \neq 0$) are essentially

non-Abelian T-duality, without any continuous deformation parameter. However, in the twisted picture, we do not see evidence for a similar interpretation, because the one-loop spectrum seems to depend continuously on the deformation parameter η through the combination $\beta = \frac{\eta}{2b_z^2}$.

It would be interesting to understand this further.

Let us also comment on the fact that we were able to consider both the unimodular Jordanian deformation (giving rise to a type IIB supergravity background) and the nonunimodular one (which does not correspond to a supergravity solution). In this respect, the main message from our results is that the corresponding twisted models share the same spectrum, at least to one-loop. Unless the argument regarding the equivalence of the deformed model to the twisted one at the quantum level fails, this seems to suggest that the unimodular and nonunimodular deformations share the same spectrum at least to that order. It would be interesting to understand these subtleties further and see whether we can make any statement beyond one-loop. A first nontrivial check was obtained already in Appendix E, where we found that the bosonic (AdS) frequencies obtained from the analysis of quadratic fluctuations of the *deformed* σ -model matches with the bosonic (AdS) frequencies obtained from the curve of the twisted σ -model. In addition, let us mention that the equivalence of the quantum spectrum between a unimodular and a non-unimodular Yang-Baxter deformation has been noted also in [83], although in that case, the *inhomogeneous* Yang-Baxter deformation was considered, for which the reinterpretation as a twist of the boundary conditions in the undeformed model does not hold.

Our results offer a first step towards the understanding of how to tackle the spectral problem for Jordanian deformations (and more generally diagonalizable Yang-Baxter deformations) of $\text{AdS}_5 \times S^5$. It would be interesting to start working from the other side of the AdS/CFT duality and construct a corresponding deformation of $\mathcal{N} = 4$ super Yang-Mills. The expectation is that there should be a notion of a deformed/twisted spin-chain in that case, and it is worth exploring the possibility of constructing this spin-chain directly, to obtain insights on the potentially more difficult construction of the deformation of the gauge theory.

From a broader perspective, it would be very interesting to reinterpret the more generic supergravity solution-generating techniques of [18,19] (which include also non-Abelian T-duality, and are integrability-preserving) in terms of twisted models within a first-order formulation. This may open the possibility of tackling the spectral problem in this more generic class of deformed and dual σ -models.

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APPENDIX A: RELATION TO EMBEDDING COORDINATES OF AdS

In this appendix, we want to explain how the global coordinates that we use to parametrize the Jordanian-deformed spacetime are related to the embedding coordinates of AdS. This is both to offer additional geometric intuition on the coordinates used and to give an alternative argument to the fact that they are global coordinates, although here this applies only in the undeformed $\eta = 0$ limit.

AdS_D is defined as the hyperboloid $X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = R^2$ (with R the AdS radius) in $\mathbb{R}^{2,D-1}$ with metric $ds^2 = -dX_0^2 - dX_D^2 + \sum_{i=1}^{D-1} dX_i^2$. Global coordinates in AdS can be obtained by setting $X_0 = R \cosh \rho \cos \tau_G$, $X_D = R \cosh \rho \sin \tau_G$, $X_i = R \sinh \rho \Omega_i$ with the constraint $\sum_{i=1}^{D-1} \Omega_i^2 = 1$. This gives the AdS metric $ds^2 = R^2(-\cosh^2 \rho d\tau_G + d\rho^2 + \sinh^2 \rho d\Omega^2)$, with $d\Omega^2$ the metric of the $(D-2)$ -dimensional sphere. For $D > 2$, the hyperboloid is covered once if we take $\rho \geq 0, 0 \leq \tau_G < 2\pi$, and the universal cover is obtained by decompactifying the global time τ_G . The metric diverges at $\rho \rightarrow \infty$, but that is not a problematic place: geodesics either reach it in an infinite time, or if they do it in a finite time (as it happens for some null geodesics) then they come back to finite ρ at later time. Therefore, the spacetime is geodesically complete in these coordinates.

Alternatively, the identification $X_0 = \frac{1}{2}z(1 + z^{-2}(R^2 + x^i x^i - t^2))$, $X_D = Rt/z$, $X^i = Rx^i/z$ ($i = 1, \dots, D-2$), $X_{D-1} = \frac{1}{2}z(1 - z^{-2}(R^2 - x^i x^i + t^2))$ gives the metric,

$$ds^2 = R^2 \left(\frac{dz^2 + dx^i dx^i - dt^2}{z^2} \right), \quad (\text{A1})$$

in the so-called Poincaré coordinates. These coordinates do not cover the whole hyperboloid and in fact, as we will show in the more generic deformed case in Appendix B, some geodesics are not complete in this patch (e.g., they

reach $z = \infty$ in a finite proper time, and cannot be continued to $z < \infty$). In these coordinates, the boundary of AdS is at $z = 0$, where the metric diverges.

In the coordinates (T, V, Θ, P, Z) of (2.20), we see that the AdS metric (i.e., when setting $\eta = 0$) diverges at $Z = 0$, and because of the relation to the previous coordinates, we can identify that as the boundary of AdS. The composition of relations between the different sets of coordinates leads to the relation of the global coordinates (T, V, Θ, P, Z) to the embedding coordinates. Setting $R = 1$, we find

$$\begin{aligned} X_0 &= \frac{(1 + Z^2 + P^2) \cos T - 2V \sin T}{2Z}, \\ X_1 &= \frac{(P^2 + Z^2 - 2) \sin T + 2V \cos T}{2\sqrt{2}Z}, \\ X_2 &= \frac{P \sin \Theta}{Z}, \\ X_3 &= \frac{P \cos \Theta}{Z}, \\ X_4 &= \frac{(-1 + Z^2 + P^2) \cos T - 2V \sin T}{2Z}, \\ X_5 &= \frac{(P^2 + Z^2 + 2) \sin T + 2V \cos T}{2\sqrt{2}Z}. \end{aligned} \quad (\text{A2})$$

These expressions can be inverted as

$$\begin{aligned} P &= \frac{\sqrt{2} \sqrt{X_2^2 + X_3^2}}{\sqrt{2(X_0 - X_4)^2 + (X_1 - X_5)^2}}, \\ \Theta &= \arctan(X_2/X_3), \\ V &= \frac{X_0(3X_1 + X_5) - X_4(X_1 + 3X_5)}{\sqrt{2}(2(X_0 - X_4)^2 + (X_1 - X_5)^2)}, \\ Z &= \frac{\sqrt{2}}{\sqrt{2(X_0 - X_4)^2 + (X_1 - X_5)^2}}, \\ T &= \arctan\left(\frac{X_1 - X_5}{\sqrt{2}(X_4 - X_0)}\right). \end{aligned} \quad (\text{A3})$$

From these expressions, we see that we can take $P > 0$ and $Z > 0$, and moreover,⁴⁴ $\Theta \in [0, 2\pi]$ and $T \in [0, 2\pi]$, which in both cases are periodically identified with periods 2π . For V , we have instead $V \in]-\infty, +\infty[$. While τ_G is an angle in the (X_0, X_5) plane, T is instead an angle in the plane $(X_4 - X_0, X_1 - X_5)$. Also in this case, we decompactify it and take $T \in]-\infty, +\infty[$.

⁴⁴Given that Θ and T are obtained from the arctan, normally they would take values in $[-\pi/2, \pi/2]$. In particular, $\arctan(y/x)$ is insensitive to a simultaneous change of sign of x and y . To avoid this problem, one can define a function $\arctan(x, y)$ that gives the value of the angle identified by the point in the (x, y) plane taking into account the quadrants in which x and y are placed.

We will now argue that these coordinates cover the whole hyperboloid. First, P and Θ cover all the (X_2, X_3) plane, with a “warping” depending on the other 4 coordinates. If we now define $Y_0^\pm = X_0 \pm X_4$ and $Y_1^\pm = (X_1 \pm X_5)/\sqrt{2}$, then Z and T take care of the whole (Y_0^-, Y_1^-) plane (notice that Z is the inverse of the radial coordinate in that plane). Finally, we need to understand if we can cover the whole (Y_0^+, Y_1^+) plane. Notice that (unlike other coordinates) V depends also on the combinations Y_0^+, Y_1^+ , not just Y_0^-, Y_1^- . One may worry that we have just one degree of freedom (i.e., V) to cover this (Y_0^+, Y_1^+) plane, but in fact this is not an issue, because we should remember that we are constraining the coordinates to satisfy $1 = X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = -X_2^2 - X_3^2 + Y_0^+ Y_0^- - 2Y_1^+ Y_1^-$, so that if for example $Y_0^+, Y_0^-, Y_1^-, X_2, X_3$ have been fixed (because, for example, one solves for Y_0^+ in terms of V), then the remaining Y_1^+ is uniquely identified. Notice that the hyperboloid constraint is quadratic in X_M , but it is linear (for example) in Y_1^+ , so when solving for Y_1^+ the sign is unambiguous. Naively, an issue is present when $Y_1^- = 0$, because then Y_1^+ is undetermined. But when $Y_1^- = 0$ then $V = Y_1^+/Y_0^-$, and we can just swap Y_0^+ and Y_1^+ in the previous reasoning. Then one can solve the hyperboloid constraint for Y_0^+ instead. The issue remains when both $Y_0^- = Y_1^- = 0$, because then V, P and Z diverge. This is not problematic because $Y_0^- = Y_1^- = 0$ is a codimension-2 subspace, which is not an open set in spacetime.

APPENDIX B: GEODESIC INCOMPLETENESS OF POINCARÉ COORDINATES

For completeness, we show in this appendix that the geodesic incompleteness of the parametrization by Poincaré coordinates persists also when turning on the deformation η . As usual, the strategy is to look for solutions to the geodesic equations that are pathological, i.e., the boundary values of the coordinates are reached in a finite amount of geodesic time.

To start, let us construct the conserved quantities $Q_a = k_a^\mu G_{\mu\nu} \dot{X}^\nu$ along the geodesics. They are equivalent to the following equations:

$$\begin{aligned} \dot{\theta} &= \frac{Q_5 z^2}{\rho^2}, \\ \dot{x}^- &= -\frac{(Q_2 + Q_3) z^2}{\sqrt{2}}, \\ \dot{x}^+ &= \frac{\eta^2 (Q_2 + Q_3) \rho^2 - 4(Q_2 - Q_3) z^4 + \eta^2 (Q_2 + Q_3) z^2}{4\sqrt{2} z^2}, \\ \dot{\rho} &= \frac{z(z(\sqrt{2}(Q_3 - Q_2)x^- + Q_1) - z')}{\rho}, \\ (Q_2 + Q_3)(\rho^2 + z^2) &= Q_4 - 2\sqrt{2}Q_1 x^- + 2(Q_2 - Q_3)(x^-)^2. \end{aligned} \quad (\text{B1})$$

At the same time we also have the conserved quantity,

$$\begin{aligned} \epsilon &= \dot{X}^m \dot{X}^n G_{mn} \\ &= -\frac{8z^4 \dot{x}^- \dot{x}^+ + \eta^2 (\dot{x}^-)^2 (\rho^2 + z^2) - 4z^4 (\rho^2 \dot{\theta}^2 + \dot{\rho}^2 + \dot{z}^2)}{4z^6}. \end{aligned} \quad (\text{B2})$$

Let us consider null geodesics, so that $\epsilon = 0$. Now the strategy is to look for solutions that at large z go like $z \propto (\tau - \tau_0)^{-A}$ for some $A > 0$. These are pathological because $z = \infty$ is reached at finite τ . To simplify the analysis, we fix some of the values of the Q_a . At the same time, we must be careful and make sure that the last equation in (B1) does *not* reduce to the form $C_z z^2 + C_\rho \rho^2 + C_-(x^-)^2 = C$ with $C_z, C_\rho, C_- > 0$: if that happened, the geodesic motion would be bounded and no pathological behavior at $z = \infty$ would be possible. To be concrete we take

$$Q_1 = 0, \quad Q_2 \neq 0, \quad Q_3 = 0, \quad Q_4 = 0, \quad Q_5 = 0. \quad (\text{B3})$$

It is of course possible to take more generic situations, but this will already be enough to identify pathological behavior. In this case, (B1) reduce to

$$\begin{aligned} \dot{\theta} &= 0, \quad \dot{x}^- = -\frac{Q_2 z^2}{\sqrt{2}}, \quad \dot{x}^+ = \frac{Q_2 (\eta^2 \rho^2 - 4z^4 + \eta^2 z^2)}{4\sqrt{2} z^2}, \\ \dot{\rho} &= \frac{z(-\sqrt{2}Q_2 z x^- - \dot{z})}{\rho}, \quad 0 = z^2 - 2(x^-)^2 + \rho^2. \end{aligned} \quad (\text{B4})$$

Notice that θ is constant, and that a solution for x^+ can be found once a solution for z is given. Therefore, in the following, we focus on the equations for the remaining coordinates. At this point, we look for geodesic solutions that at large values of z have

$$x^- \sim \alpha_- z, \quad \text{for } z \gg 1, \quad (\text{B5})$$

with α_- a constant. Within this ansatz, the algebraic constraint given in the last equation of (B4) reduces to $(1 - 2\alpha_-^2)z^2 + \rho^2 \sim 0$, which is compatible with large z, ρ only if

$$\alpha_-^2 > 1/2, \quad \Rightarrow \quad \rho \sim z\sqrt{2\alpha_-^2 - 1}. \quad (\text{B6})$$

Demanding compatibility with the equation for $\dot{\rho}$ then gives

$$\dot{z} \sim -\frac{Q_2}{\sqrt{2}\alpha_-} z^2. \quad (\text{B7})$$

This is what we were seeking, since such an equation is solved by⁴⁵ $z \propto (\tau - \tau_0)^{-1}$. The pathological behavior of this

⁴⁵Importantly, this solution is compatible with (B2) in the large- z expansion.

solution is enough to conclude that the Poincaré coordinates are not global coordinates for the deformed spacetime.

APPENDIX C: IDENTIFYING THE CARTAN SUBALGEBRA OF ISOMETRIES

In this appendix, we identify the possible Cartan subalgebras of the algebra of isometries $\mathfrak{k}$ given in (2.18). Since M^{23} and $p^0 + p^1$ are central elements, i.e., they commute with all generators in $\mathfrak{k}$, we can focus on the remaining generators, which span an $\mathfrak{sl}(2, \mathbb{R}) = \text{span}(\mathbf{h}, \mathbf{e}_+, \mathbf{e}_-)$ subalgebra with $[\mathbf{h}, \mathbf{e}_\pm] = \pm \mathbf{e}_\pm$ and $[\mathbf{e}_+, \mathbf{e}_-] = 2\mathbf{h}$ as is found by identifying $\mathbf{h} = \frac{1}{2}(d - M^{01})$, $\mathbf{e}_+ = \frac{1}{2}(p^0 - p^1)$, and $\mathbf{e}_- = -\frac{1}{2}(k^0 + k^1)$. Inequivalent choices of the Cartan subalgebra are then obtained by classifying the inequivalent adjoint orbits of $\mathfrak{sl}(2, \mathbb{R})$ and then requiring that the representative elements have a diagonalizable adjoint action. Although this is a standard exercise, we repeat it explicitly to clarify certain comments.

Let us write a generic element of the group $SL(2, \mathbb{R})$ as $g = \exp(c_h \mathbf{h}) \exp(c_+ \mathbf{e}_+) \exp(c_- \mathbf{e}_-)$ and of the algebra $\mathfrak{sl}(2, \mathbb{R})$ as $x = \alpha_h \mathbf{h} + \alpha_+ \mathbf{e}_+ + \alpha_- \mathbf{e}_-$. For each element x of $\mathfrak{sl}(2, \mathbb{R})$, one can then identify its adjoint orbit $\{gxg^{-1}\}$. One finds three inequivalent possibilities. First, in fact, we can transform the generic element x to the element $\mathbf{h}$ whilst preserving reality conditions as

$$gxg^{-1} = \sqrt{\gamma} \mathbf{h}, \quad \text{with } \gamma = \alpha_h^2 + 4\alpha_- \alpha_+, \quad (\text{C1})$$

by specifying the element g with the coefficients,

$$c_+ = \frac{\alpha_+}{\sqrt{\gamma}}, \quad c_- = \frac{\alpha_h(\sqrt{\gamma} - \alpha_h) - 4\alpha_- \alpha_+}{2\alpha_+ \sqrt{\gamma}}, \quad (\text{C2})$$

if $\gamma > 0$.⁴⁶ The second case is that for $\gamma = 0$, which can be obtained when at least α_+ or α_- are nonzero (otherwise x would also be zero). For definiteness say that $\alpha_+ \neq 0$. One can then transform x into $\mathbf{e}_+$ as

$$gxg^{-1} = e^{c_h} \alpha_+ \mathbf{e}_+, \quad \text{with } c_- = \frac{\alpha_h}{2\alpha_+}. \quad (\text{C3})$$

Alternatively, we can take $\alpha_- \neq 0$, which would transform x into $\mathbf{e}_-$. However, $\mathbf{e}_\pm$ are related by an inner automorphism as $\tilde{g} \mathbf{e}_+ \tilde{g}^{-1} = -\mathbf{e}_-$ with $\tilde{g} = \exp(\pi/2(\mathbf{e}_+ - \mathbf{e}_-))$ so these two possibilities are in fact equivalent. The third and final case is that for $\gamma < 0$, which can be obtained only when both α_+ and α_- are nonvanishing. In this case, one can transform x into $\mathbf{e}_+ - \mathbf{e}_-$ as follows:

$$gxg^{-1} = \frac{1}{2} \alpha_+ \sqrt{-\frac{\gamma}{\alpha_+^2}} (\mathbf{e}_+ - \mathbf{e}_-), \quad (\text{C4})$$

⁴⁶Although the case $\alpha_+ = 0$ seems singular, it can be analyzed on its own (for $\alpha_h \neq 0$) leading to the same conclusion.

by using

$$c_h = \frac{1}{2} \log \left(-\frac{\gamma}{4\alpha_+^2} \right), \quad c_+ = 0, \quad c_- = \frac{\alpha_h}{2\alpha_+}. \quad (\text{C5})$$

Thus, given a generic $x \in \mathfrak{sl}(2, \mathbb{R})$ which we want to simplify by means of inner automorphisms, we have three inequivalent possibilities,⁴⁷

$$\mathbf{h}, \quad \text{or } \mathbf{e}_+, \quad \text{or } \mathbf{e}_+ - \mathbf{e}_-. \quad (\text{C6})$$

To identify the possible Cartan subalgebras, we must now verify whether or not the adjoint actions ad_x for the above inequivalent elements are diagonalizable. A simple computation shows that $ad_{\mathbf{h}}$ is diagonalizable with eigenvalues $(0, 1, -1)$, $ad_{\mathbf{e}_+}$ is not diagonalizable, and $ad_{\mathbf{e}_+ - \mathbf{e}_-}$ is diagonalizable with eigenvalues $(0, 2i, -2i)$.⁴⁸ To conclude, up to automorphisms, there are two possible choices of Cartan subalgebras of $\mathfrak{k}$, namely,

$$\begin{aligned} \text{(I): } & \text{span}\{d - M^{01}, p^0 + p^1, M^{23}\}, \\ \text{(II): } & \text{span}\{p^0 - p^1 + \alpha(k^0 + k^1), p^0 + p^1, M^{23}\}, \end{aligned} \quad (\text{C7})$$

where we leave a possible $\alpha > 0$ coefficient for later convenience. The choice $\mathbf{e}_+ - \mathbf{e}_-$ would correspond to $\alpha = 1$, but in the main text we find it convenient to choose $\alpha = 1/2$.

APPENDIX D: ANALYTIC STRUCTURE OF THE PERTURBATIONS

In this appendix, we detail how to compute the correction to the quasimomenta δp_i that arise from adding small excitations to our classical solution using only their analytic properties, as described in Sec. VI A. Most of our computations mimic those of the $\mathfrak{sl}(2, \mathbb{R})$ circular string in [51] due to the presence of a single cut in the $\hat{p}_i$ quasimomenta, but the asymptotic behavior of our classical quasimomenta and some of their corrections is very different.

1. Excitations associated to the sphere

Excitations that connect two quasimomenta $\tilde{p}_i$ and $\tilde{p}_j$ are related to considering quadratic fluctuations around a classical solution for the modes associated to the sphere. Of those, the only ones that give rise to physical excitations are those that connect either $\tilde{1}$ or $\tilde{2}$ with either $\tilde{3}$ or $\tilde{4}$. Similarly to the case described in [51], the two remaining possible combinations, $\tilde{1}\tilde{2}$ and $\tilde{3}\tilde{4}$, do not give rise to physical excitations.

⁴⁷It is immediate to see that these choices cannot be related by inner automorphisms, given that $\text{Tr}(\mathbf{h}^2) = 1$, $\text{Tr}(\mathbf{e}_+^2) = 0$, $\text{Tr}((\mathbf{e}_+ - \mathbf{e}_-)^2) = -2$.

⁴⁸Notice that $ad_{\mathbf{e}_+ - \mathbf{e}_-}$ is real, only its diagonalized version has imaginary coefficients.

For concreteness, here we will focus on the excitation that connects $\tilde{p}_2$ and $\tilde{p}_3$, which we will denote as $\tilde{2}\tilde{3}$, and turn off any other excitation. The computations for the other three excitations are the same *mutatis mutandis*. Nevertheless, we will make some comments on $\tilde{2}\tilde{4}$, and we will discuss the remaining two in Sec. D 4.

Let us focus first on the correction $\delta\tilde{p}_2$. We know that it has to have poles at ± 1 and at $x_n^{\tilde{2}\tilde{3}}$. Thus, we propose the following ansatz,

$$\delta\tilde{p}_2 = \frac{\delta\alpha_+}{x-1} + \frac{\delta\alpha_-}{x+1} - \sum_n \frac{\alpha(x_n^{\tilde{2}\tilde{3}})N_n^{\tilde{2}\tilde{3}}}{x - x_n^{\tilde{2}\tilde{3}}}. \quad (\text{D1})$$

As there will be no ambiguity, we will drop the superindex of x_n and N_n to alleviate our notation.

We also know that $\delta\tilde{p}_2$ is associated to $\delta\tilde{p}_1$ by inversion symmetry. Thus, we can use the information about their behavior at large values of x to fix the residues of the poles. On the one hand, from Eq. (6.8), we have

$$\delta\tilde{p}_1 \approx \mathcal{O}(x^{-2}), \quad \delta\tilde{p}_2 \approx -\frac{4\pi}{\sqrt{\lambda}} \sum_n \frac{N_n}{x} + \mathcal{O}(x^{-2}). \quad (\text{D2})$$

On the other hand, from our ansatz and inversion symmetry (6.6), we have

$$\begin{aligned} \delta\tilde{p}_1 \approx & \left(\delta\alpha_+ - \delta\alpha_- - \sum_n \frac{\alpha(x_n)N_n}{x_n} \right) \\ & + \frac{\delta\alpha_+ + \delta\alpha_- - \sum_n \frac{\alpha(x_n)N_n}{x_n^2}}{x} + \mathcal{O}(x^{-2}), \end{aligned} \quad (\text{D3})$$

$$\delta\tilde{p}_2 \approx \frac{\delta\alpha_+ + \delta\alpha_- - \sum_n \alpha(x_n)N_n}{x} + \mathcal{O}(x^{-2}). \quad (\text{D4})$$

Consistency between the two expansions gives us enough information to fix the three residues,

$$\alpha(x_n) = \frac{4\pi}{\sqrt{\lambda}} \frac{x_n^2}{x_n^2 - 1}, \quad (\text{D5})$$

$$2\delta\alpha_{\pm} = \sum_n \alpha(x_n)N_n \left(\frac{1}{x_n^2} \pm \frac{1}{x_n} \right). \quad (\text{D6})$$

As we anticipated in Sec. VI A, the function $\alpha(x)$ does not need to be fixed beforehand. The consistency between the positions and signs of the poles with the asymptotic behavior of the corrections is enough to fix its form as a function of the position of the pole. As expected, the function $\alpha(x)$ is unchanged from the undeformed case [51]. This is consistent with the fact that it is related to the weight appearing in the definition of the filling fractions (5.9). In particular, we require that

$$\begin{aligned} -N_n &= -\frac{\sqrt{\lambda}}{8\pi^2 i} \oint_C dx \left(1 - \frac{1}{x^2} \right) \delta\tilde{p}_2 \\ &= -\frac{\sqrt{\lambda}}{4\pi} N_n \alpha(x_n) \left(1 - \frac{1}{x_n^2} \right), \end{aligned} \quad (\text{D7})$$

where C is the integration contour that encircles the small cut we are putting in. As the expressions of the filling fractions are not affected by the deformation, neither are the residues $\alpha(x)$.

If we now use the condition (6.3) to find the position of the cut, we get

$$\tilde{p}_2(x_n) - \tilde{p}_3(x_n) = 2\pi n \quad \Rightarrow \quad x_n = \frac{\omega \pm \sqrt{\omega^2 + n^2}}{n}. \quad (\text{D8})$$

Here, we are only interested in the solution with the plus sign, as it is the one in the physical region. Substituting the explicit expression for x_n in $\alpha(x_n)$, we find that (D6) gives

$$\begin{aligned} 2\delta\alpha_{\pm} &= \frac{4\pi}{\sqrt{\lambda}} \sum_n N_n \left(\frac{1 \pm x_n}{x_n^2 - 1} \right) \\ &= \frac{4\pi}{\sqrt{\lambda}} \sum_n \frac{nN_n}{2\omega} \left(\frac{n}{\omega - \sqrt{\omega^2 + n^2}} \pm 1 \right). \end{aligned} \quad (\text{D9})$$

The second term in this expression actually vanishes when we impose the level-matching condition $\sum_n nN_n = 0$. This means that $\delta\alpha_+ = \delta\alpha_-$, although we will not make use of this relation in the following equations.

Now that we have computed $\delta\alpha_{\pm}$, we can consider the effect that the excitation has on the quasimomenta associated to the (deformed) AdS part of our space and compute $\delta\hat{p}_i$. The ansätze for these corrections are a bit more involved, as we have to consider the possibility of a shift of the branch points arising from a backreaction of the excitation. Thus, we will assume the following ansätze:

$$\delta\hat{p}_1(x) = f(x) + \frac{g(x)}{K(1/x)}, \quad \delta\hat{p}_4(x) = f(x) - \frac{g(x)}{K(1/x)}, \quad (\text{D10})$$

where $K(x) = \sqrt{1 - x^2\beta^2}$ and the functions $f(x)$ and $g(x)$ are functions to be determined. The second term of our ansatz is inspired by the fact that $\partial_\beta K(x) \propto \frac{1}{K(x)}$ and by Eq. (5.23), as the quasimomentum $\hat{p}_4$ still has to be the analytic continuation of $\hat{p}_1$ through a square-root cut.

Let us start by constraining the function $f(x) = \frac{\delta\hat{p}_1(x) + \delta\hat{p}_4(x)}{2}$. As we are considering a bosonic excitation, the synchronization of the poles at ± 1 forces $\delta\hat{p}_1(x)$ and $\delta\hat{p}_4(x)$ to have residues with opposite signs, meaning that $f(x)$ has no poles at ± 1 . In addition, $\delta\hat{p}_1 \approx -\delta\hat{p}_4 \approx \frac{4\pi}{x\sqrt{\lambda}} \frac{\delta\Delta}{2} + \mathcal{O}(x^{-2})$ for large values of x . As we have not

added any new poles to the AdS directions, this means that $f(x)$ is a holomorphic function that approaches zero at infinity. According to Liouville's theorem, the only function that fulfils these requirements is $f(x) = 0$.

Now, we can fix the function $g(x)$ by demanding that $\delta\hat{p}_1$ has the correct properties. As $\delta\hat{p}_1(x)$ only has poles at ± 1 , we propose the following ansatz for $g(x)$:

$$g(x) = \frac{K(1)\delta\alpha_+}{x-1} + \frac{K(1)\delta\alpha_-}{x+1}, \quad (\text{D11})$$

where we have used the synchronization of the poles at ± 1 as in Eq. (6.7) to fix the residues. Matching the asymptotic behavior of $\delta\hat{p}_1(x)$ with its expected behavior gives us

$$\frac{\delta\Delta}{2} = \frac{\sqrt{\lambda}K(1)}{4\pi K(0)}(\delta\alpha_+ + \delta\alpha_-) = \sqrt{1-\beta^2} \sum_n \frac{N_n}{x_n^2 - 1}. \quad (\text{D12})$$

Notice that $\delta\Delta$ can only be real if $|\beta| \leq 1$, which is exactly what we found from reality conditions implied by the Virasoro constraint.

We should remark that this construction is blind to having the second pole in the sheet $\tilde{3}$ or the sheet $\tilde{4}$, as we did not use that information at any point of the reconstruction. This implies that the contributions to $\delta\Delta$ of $\tilde{2}\tilde{3}$ and $\tilde{2}\tilde{4}$ have the same form as a function of x_n . We will later use the inversion symmetry to show that the other two give us the same result.

2. Excitations associated to the deformed AdS space

Excitations that connect two quasimomenta $\hat{p}_i$ and $\hat{p}_j$ are related to modes associated to the deformed AdS space. Similarly to the case of the sphere, the only ones that give rise to physical excitations are those that connect either $\hat{1}$ or $\hat{2}$ with either $\hat{3}$ or $\hat{4}$. Due to the presence of the twist, the excitations associated to $\hat{1}\hat{4}$ and $\hat{2}\hat{3}$ behave differently to those associated to $\hat{1}\hat{3}$ and $\hat{2}\hat{4}$, and thus, we will treat them separately. We will consider only one case for each respective set because the other one can be obtained through inversion symmetry, as we will discuss in D 4.

Let us start with the case associated to $\hat{1}\hat{4}$ and turn off any other excitation. First, note that we are not modifying the quasimomenta associated to the sphere. This means that all $\delta\hat{p}_i$ decay at infinity as $\mathcal{O}(x^{-2})$. The only possible configuration with this property is the one with $\delta\alpha_{\pm} = \delta\beta_{\pm} = 0$, which allows us to set all the sphere corrections to zero, $\delta\hat{p}_i = 0$. Consequently, none of the corrections $\delta\hat{p}_i$ have poles at ± 1 due to their synchronization.

Similarly to what happened for the excitation $\tilde{2}\tilde{3}$, the residues at x_n of $\delta\hat{p}_1$ and $\delta\hat{p}_4$ have the opposite value. Furthermore, for large values of x , we have that

$\delta\hat{p}_1 \approx -\delta\hat{p}_4 \approx \frac{4\pi}{x\sqrt{\lambda}} \frac{\delta\Delta+2}{2} \sum_n N_n + \mathcal{O}(x^{-2})$. If we consider the ansatz,

$$\delta\hat{p}_1(x) = f(x) + \frac{g(x)}{K(1/x)}, \quad \delta\hat{p}_4(x) = f(x) - \frac{g(x)}{K(1/x)}, \quad (\text{D13})$$

these considerations imply that $f(x) = 0$ and fix $g(x)$ to

$$g(x) = \sum_n \frac{K(1/x_n)\alpha(x_n)N_n}{x - x_n}. \quad (\text{D14})$$

Here, we have used the insight from our sphere computation to argue that the residue at x_n has to be the same as the undeformed one, so we can use the function $\alpha(x)$ we computed above in (D5). Matching the asymptotic behavior of our ansatz with its expected behavior gives us

$$\delta\Delta = \sum_n \left(2 \frac{x_n^2 K(1/x_n)}{x_n^2 - 1} - 2 \right) N_n. \quad (\text{D15})$$

Let us consider now the case associated to $\hat{1}\hat{3}$ and turn off any other excitation. Again, we are not modifying the quasimomenta associated to the sphere, implying that none of the corrections $\delta\hat{p}_i$ have poles at ± 1 .

Similarly to our previous cases, we start by considering the ansatz,

$$\delta\hat{p}_1(x) = f(x) + \frac{g(x)}{K(1/x)}, \quad \delta\hat{p}_4(x) = f(x) - \frac{g(x)}{K(1/x)}. \quad (\text{D16})$$

However, in contrast to previously, $\delta\hat{p}_1(x)$ has a pole at x_n but $\delta\hat{p}_4(x)$ does not, meaning that the function $f(x)$ is not holomorphic and thus, does not vanish in this case. Notice that, instead, $\delta\hat{p}_4(x)$ has a pole at $1/x_n$ inherited from $\delta\hat{p}_3(x)$ due to the inversion symmetry. In fact, assuming that $\delta\hat{p}_3(x)$ has a pole at x_n with residue $-\alpha(x_n)$, inversion symmetry forces $\delta\hat{p}_4(x)$ to have a pole at $1/x_n$ with residue $-\frac{\alpha(x_n)}{x_n^2} = \alpha(\frac{1}{x_n})$.⁴⁹ Substituting this information into an ansatz of the form $f(x) = a + \frac{b}{x-x_n} + \frac{c}{x-1/x_n}$, we arrive to

$$f(x) = \sum_n \frac{4\pi}{\sqrt{\lambda}} \frac{xN_n}{2(x-x_n)(x-1/x_n)}, \quad (\text{D17})$$

which fulfils all our requirements, including consistency with the asymptotic behavior of $\delta\hat{p}_1(x)$ and $\delta\hat{p}_4(x)$, which reads $f(x) \approx \sum_n \frac{4\pi}{x\sqrt{\lambda}} \frac{N_n}{2} + \mathcal{O}(x^{-2})$. Similarly to the function $f(x)$, we can assume that $g(x)$ can be written as $g(x) = a + \frac{b}{x-x_n} + \frac{c}{x-1/x_n}$. These constants can be fixed by demanding that $\delta\hat{p}_1$ has the properties we want; i.e., it has a pole with residue $\alpha(x_n)$ at x_n and no pole at $1/x_n$. After some algebra, we get that

⁴⁹Notice that, although $\delta\hat{p}_2$ and $\delta\hat{p}_4$ have a pole, they are not in the physical region and they do not contribute to N_n^2 .

$$g(x) = \sum_n \frac{4\pi}{\sqrt{\lambda}} \frac{K(x_n)(x - x_n) + x_n^2 K(1/x_n)(x - 1/x_n)}{2(x_n^2 - 1)(x - x_n)(x - 1/x_n)} N_n. \quad (\text{D18})$$

Matching the asymptotic of our ansatz with the expected asymptotic behavior of $\delta\hat{p}_1$ gives us

$$\delta\Delta = \sum_n \left(\frac{x_n^2 K(1/x_n) + K(x_n)}{x_n^2 - 1} - 1 \right) N_n. \quad (\text{D19})$$

3. Fermionic excitations

Let us move finally to fermionic excitations. Excitations that connect a quasimomentum $\hat{p}_i$ and a quasimomentum $\tilde{p}_j$ or vice versa are related to fermionic modes. Here, we will focus only on the excitation associated to $\hat{1}\tilde{3}$ and turn off all other excitations. We will make some comments on the remaining excitations at the end of this section.

We should first focus on the corrections to the quasimomenta associated to the sphere. As we are adding an excitation to the sheet $\tilde{3}$, we will consider the following ansatz:

$$\delta\tilde{p}_3(x) = \frac{\delta\beta_+}{x-1} + \frac{\delta\beta_-}{x+1} + \sum_n \frac{\alpha(x_n)N_n}{x-x_n} = -\delta\tilde{p}_4(1/x), \quad (\text{D20})$$

$$\delta\tilde{p}_1(x) = \delta\tilde{p}_2(x) = 0. \quad (\text{D21})$$

Notice that, as we are considering a fermionic excitation, these corrections fulfil the relaxed synchronization condition (6.7). If we impose the correct value of the residues and the correct asymptotic behavior, we get exactly the same equations as the ones for the $\tilde{2}\tilde{3}$ excitations (up to replacing $\delta\alpha_\pm$ with $\delta\beta_\pm$ and N_n with $-N_n$). Thus, we can borrow the results we found for $\delta\beta_\pm$ and the residue $\alpha(x_n)$ given in (D5) and (D9).

For the correction to the AdS quasimomenta, we will assume the same ansatz as in the previous cases. Similarly to the case $\hat{1}\tilde{3}$, we cannot set the function $f(x)$ to zero because $\delta\hat{p}_1$ has a pole at x_n but $\delta\hat{p}_4$ does not. In addition, $\delta\hat{p}_4$ has poles at ± 1 from the synchronization condition, while $\delta\hat{p}_1$ does not. After imposing these restrictions, as well as the required asymptotic behavior, we get

$$f(x) = \sum_n \frac{4\pi}{\sqrt{\lambda}} \frac{x^2 N_n}{2(x - x_n)(x^2 - 1)}. \quad (\text{D22})$$

With this information, we can now compute the function $g(x)$ by demanding that $\delta\hat{p}_1(x)$ has the correct residues at ± 1 and x_n , which gives us

$$g(x) = \sum_n \frac{4\pi}{\sqrt{\lambda}} \frac{x_n^2(x^2 - 1)K(1/x_n) + (x^2 - x_n^2)K(1)}{2(x - x_n)(x^2 - 1)(x_n^2 - 1)} N_n. \quad (\text{D23})$$

Finally, matching $\delta\hat{p}_1(x)$ with its required asymptotic behavior, we obtain that

$$\delta\Delta = \sum_n \left(\frac{x_n^2 K(1/x_n) + K(1)}{x_n^2 - 1} - 1 \right) N_n. \quad (\text{D24})$$

In the next section, we will discuss how to obtain the contribution of $\hat{1}\tilde{4}$ from this one, but it is easy to see that the computation has to be exactly the same as the one presented above, giving us the same $\delta\Delta$. The same happens for the excitations $\tilde{1}\hat{4}$ and $\tilde{2}\hat{4}$.

The steps to compute the contribution of $\hat{2}\tilde{3}$ are relatively similar. In fact, by analyzing the pole structure, it is easy to reach the conclusion that the expression of $\delta\tilde{p}_2$ is exactly the same one as the expression of $\delta\hat{p}_1$ for the $\hat{1}\tilde{3}$ excitation after substituting $K(1/x)$ by $K(x)$ and adding a constant contribution to $g(x)$. Applying the inversion symmetry to get $\delta\hat{p}_1$ and matching it with its required asymptotic behavior, we obtain that

$$\delta\Delta = \sum_n \left(\frac{K(x_n) + K(1)}{x_n^2 - 1} \right) N_n. \quad (\text{D25})$$

Although at this point, we would have to study also the fermionic excitations that involve either $\hat{3}$ or $\hat{4}$, we can argue that this is not necessary. This happens because our classical solution has pairwise symmetric quasimomenta,

$$\hat{p}_1 = -\hat{p}_4, \quad \hat{p}_2 = -\hat{p}_3, \quad \tilde{p}_1 = -\tilde{p}_4, \quad \tilde{p}_2 = -\tilde{p}_3. \quad (\text{D26})$$

This implies that, after all the appropriate computations, we will find that

$$\Omega^{\hat{1}\hat{4}} = \Omega^{\tilde{1}\tilde{4}}, \quad \Omega^{\hat{2}\hat{4}} = \Omega^{\tilde{1}\tilde{3}}, \quad \Omega^{\hat{1}\tilde{3}} = \Omega^{\tilde{2}\hat{4}}, \quad \Omega^{\hat{2}\tilde{3}} = \Omega^{\tilde{2}\tilde{3}}, \quad (\text{D27})$$

both for Ω understood as a function of the position of the poles and as a function of the mode number n .

4. Composition and inversion

Up to this point, we have computed $\delta\Delta$ for some selected excitations. This is enough for us, as the contribution of the other excitations can be computed using properties such as composition of poles and the inversion property.

The idea is that the procedure we used to compute $\delta\Delta$ does not care about the exact position x_n of the poles/microscopic cuts we are adding. Before substituting the expressions of x_n obtained from (6.3) in the frequency of the corresponding excitation, we can thus think of $\Omega(x)$ as formal, off shell, functions of x . Once we substitute $x = x_n$ the frequency becomes on shell. We can use the off shell expressions of $\Omega(x)$ in our advantage in two different ways. The first one is the inversion symmetry: if we consider an excitation connecting a given pair of sheets, we can move

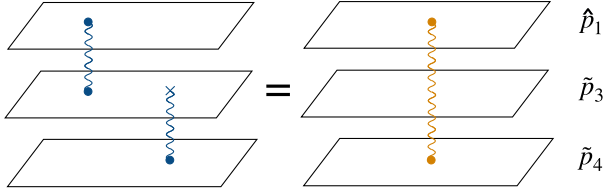


FIG. 1. Illustration of the composition rule (D30). The fermionic frequency $\Omega^{\hat{1}\hat{3}}(x)$ and the bosonic frequency $\Omega^{\hat{3}\hat{4}}(y)$ can be composed, because the two excitations have a pole with opposite residue on sheet $\tilde{p}_3$. In particular, the residue on sheet $\tilde{p}_3$ vanishes in the limit $y \rightarrow x$, giving us the pole structure (and asymptotic behavior) associated with the fermionic excitation $\hat{1}\hat{4}$. Thus, $\Omega^{\hat{1}\hat{3}}(x) + \Omega^{\hat{3}\hat{4}}(x) = \Omega^{\hat{1}\hat{4}}(x)$ off shell.

the pole associated to it to the interior of the unit circle, making it nonphysical, and a new physical pole will emerge in the sheets connected to the original ones by inversion symmetry. The second one is composing two excitations that share a pole with opposite residue on the same sheet, say j , but connect to different sheets, say i and k , respectively. The pole on sheet j will cancel, making the composition of the two excitations ij and jk indistinguishable from having only one excitation ik . We illustrate the latter in Fig. 1. In mathematical terms, these two ideas imply

$$\Omega^{\hat{1}\hat{4}}(x) = -\Omega^{\hat{2}\hat{3}}\left(\frac{1}{x}\right) - 2, \quad (\text{D28})$$

$$\Omega^{\tilde{1}\tilde{4}}(x) = -\Omega^{\tilde{2}\tilde{3}}\left(\frac{1}{x}\right) + \Omega^{\tilde{2}\tilde{3}}(0), \quad (\text{D29})$$

and

$$\Omega^{ij}(x) \pm \Omega^{jk}(x) = \Omega^{ik}(x), \quad (\text{D30})$$

where the $\pm$ is chosen appropriately to cancel the residue in sheet j . We will not reproduce here the derivation of these expressions, which can be found in [55]. Let us stress again

that these relations between the different contributions only hold when they are understood as functions of x , not as functions of n .

Let us now compute some of the remaining frequencies using the ones we obtained above. The process is relatively similar for most of them, so we will consider only some specific cases: the excitations $\hat{2}\hat{3}$, $\tilde{1}\tilde{4}$ and $\hat{1}\tilde{4}$. Both $\hat{2}\hat{3}$ and $\tilde{1}\tilde{4}$ can be obtained by immediate application of the above formulas,

$$\begin{aligned} \Omega^{\hat{2}\hat{3}}(x_n) &= -\Omega^{\hat{1}\hat{4}}\left(\frac{1}{x_n}\right) - 2 = \left(-\frac{2K(x_n)}{1-x_n^2} + 2\right) - 2 \\ &= \frac{2K(x_n)}{x_n^2 - 1}, \end{aligned} \quad (\text{D31})$$

$$\begin{aligned} \Omega^{\tilde{1}\tilde{4}}(x_n) &= -\Omega^{\tilde{2}\tilde{3}}\left(\frac{1}{x_n}\right) + \Omega^{\tilde{2}\tilde{3}}(0) = -2\frac{\sqrt{1-\beta^2}}{x_n^2 - 1} - 2\sqrt{1-\beta^2} \\ &= 2\frac{\sqrt{1-\beta^2}}{x_n^2 - 1}. \end{aligned} \quad (\text{D32})$$

The last one is obtained by considering first the composition of the excitations $\hat{2}\hat{3}$ and $\tilde{2}\tilde{4}$. This gives us that the excitation $\tilde{3}\tilde{4}$ has $\Omega^{\tilde{3}\tilde{4}} = 0$, which is consistent with the fact that it is an unphysical excitation. This result can then be used to show, e.g., that $\Omega^{\hat{1}\tilde{4}} = \Omega^{\hat{1}\hat{3}} + \Omega^{\tilde{3}\tilde{4}} = \Omega^{\hat{1}\hat{3}}$.

APPENDIX E: BOSONIC SECTOR OF THE QUADRATIC FLUCTUATIONS

In order to cross-check the results we obtained using the classical spectral curve method, we can compute the contribution of the bosonic excitations to the one-loop correction to the energy by considering the effective Lagrangian of small fluctuations around the classical solution we are interested in. We will perform this computation in the deformed periodic picture, rather than the undeformed twisted one. The expansion of the deformed AdS_5 sector of the Lagrangian is given by⁵⁰

$$\begin{aligned} \mathcal{L}(x_{\text{clas}} + \epsilon x) &\approx \mathcal{L}(x_{\text{clas}}) + \epsilon(\text{EOM}) + \frac{\epsilon^2}{2b_Z^6} [4b_Z^4(\dot{P}^2 - P'^2 + \dot{Z}^2 - Z'^2) \\ &\quad - (4b_Z^6 + \eta^2 b_Z^2)(\dot{T}^2 - T'^2) - 8b_Z^4(\dot{T}\dot{V} - T'V') + 4a_T b_Z Z(\eta^2 \dot{T} + 4b_Z^2 \dot{V} + 3\eta Z') \\ &\quad - a_T^2(4b_Z^4 + \eta^2)P^2 - 4a_T^2 \eta^2 Z^2 - 4\eta a_T b_Z^2 P P' + 4b_Z \eta(T'\dot{Z} - \dot{T}Z')] + \mathcal{O}(\epsilon^3), \end{aligned} \quad (\text{E1})$$

where, by a slight abuse of notation, we have denoted the fluctuation around the coordinates with the same symbols as the coordinates. Here, the dot and prime represent derivatives with respect to τ and σ , respectively. Notice that neither the coordinate Θ nor the Kalb-Ramond field terms contribute at quadratic order. The fact that Θ does not appear is simply a consequence of the noncanonical kinetic term that it has, which would appear at higher orders in the field expansion.⁵¹

⁵⁰The term $\epsilon(\text{EOM})$ in the expansion vanishes upon the equations of motion (EOM).

⁵¹If we want to do the counting of degrees of freedom correctly, we should change from P and Θ to Cartesian coordinates X^2 , X^3 . However, as we are only interested in independently checking our results in Appendix D, this computation is sufficient.

The next step is to compute the equations of motion of the fluctuations. We will use the fact that we do not want to spoil the periodicity condition of our classical solution by considering the following ansätze for each of the coordinates:

$$X^m = \sum_n A_m \cos(\Omega_n \tau + n\sigma) + B_m \sin(\Omega_n \tau + n\sigma). \quad (\text{E2})$$

After some algebra, the equations of motion become the system of linear equations $Mv = 0$ with v the vector given by $v = (A_Z A_T A_V A_P B_Z B_T B_V B_P)$ and M is the matrix,

$$\begin{pmatrix} \sqrt{\frac{\eta^2}{2\beta^2}}(n^2 + 4a_T^2\beta^2 - \Omega_n^2) & 0 & 0 & 0 & 0 & -a_T\eta^2\Omega_n & -\frac{2a_T\eta\Omega_n}{\beta} & 0 \\ 0 & -\frac{1+\beta^2}{\beta^2}\eta^2(n^2 - \Omega_n^2) & -\frac{2\eta}{\beta}(n^2 - \Omega_n^2) & 0 & \sqrt{2^3\eta^3\beta}a_T\Omega_n & 0 & 0 & 0 \\ 0 & -(n^2 - \Omega_n^2) & 0 & 0 & \sqrt{\frac{2^3\beta}{\eta}}a_T\Omega_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2\eta^3}{\beta^2}}(a_T^2(1+\beta^2) + n^2 - \Omega_n^2) & 0 & 0 & 0 & 0 \\ 0 & a_T\eta^2\Omega_n & \frac{2a_T\eta\Omega_n}{\beta} & 0 & \sqrt{\frac{\eta^2}{2\beta^2}}(n^2 + 4a_T^2\beta^2 - \Omega_n^2) & 0 & 0 & 0 \\ -\sqrt{2^3\eta^3\beta}a_T\Omega_n & 0 & 0 & 0 & 0 & -\frac{1+\beta^2}{\beta^2}\eta^2(n^2 - \Omega_n^2) & -\frac{2\eta}{\beta}(n^2 - \Omega_n^2) & 0 \\ -\sqrt{\frac{2^3\beta}{\eta}}a_T\Omega_n & 0 & 0 & 0 & 0 & -(n^2 - \Omega_n^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2\eta^2}{\beta^2}}(a_T^2(1+\beta^2) + n^2 - \Omega_n^2) \end{pmatrix}. \quad (\text{E3})$$

If we want this system of equations to have a nontrivial solution, we need the matrix of coefficients to have vanishing determinant. This gives us the condition,

$$(n^2 - \Omega_n^2)^2(n^2 + a_T^2(1 + \beta^2) - \Omega_n^2)^2(n^4 - 4a_T^2\beta^2n^2 - 2(2a_T^2 + n^2)\Omega_n^2 + \Omega_n^4)^2 = 0. \quad (\text{E4})$$

We can check that the solutions for Ω_n to this equation, divided by $Q_T = -a_T$, perfectly match (6.19), (6.20), and (6.21) up to a constant term. Additionally, we also get solutions of the form $\Omega_n = \pm n$. The modes associated to these are not physical and are canceled by conformal ghosts [84].

Instead of repeating the same computation for the S^5 sector of the Lagrangian, we can argue that this part of the space is blind to the deformation, and hence, the final result has to be the same as the one for the undeformed background. In fact, we can check that

$$\Omega = \frac{\omega \pm \sqrt{\omega^2 + n^2}}{Q_T} = -\sqrt{1 - \beta^2} \mp \sqrt{1 - \beta^2 + \frac{n^2}{a_T^2}}, \quad (\text{E5})$$

which perfectly matches (6.18).

The mismatch by a constant between the contributions computed using the quadratic fluctuations and the ones computed using the classical spectral curve may be uncomfortable. However, they are also present for undeformed $\text{AdS}_5 \times S^5$ and arise due to each method describing the perturbations around the classical solution using a different frame of reference [51].

Even though these shifts have a nonphysical origin, this does not mean at all that they are harmless, as they can give rise to ambiguities. To discuss those ambiguities, first we

have to distinguish between two kinds of shifts that may appear: a constant shift of Ω (usually proportional to the energy or an angular momentum), and a shift of the mode number n (usually proportional to a winding number). The ones of the first kind are completely harmless, but the ones of the second kind give rise to an ambiguity. To see that, we can compare what would be the contribution to the one-loop energy we obtain with and without performing a shift of the mode number. Although the sum $\sum_{n=-\infty}^{\infty} [2\Omega(x_n) - \Omega(x_{n-m}) - \Omega(x_{n+m})]$ might initially seem to vanish because the terms cancel each other after a relabeling, this is not entirely correct if we consider a partial sum up to a large enough value, Λ . Using that $\Omega(x_n) \approx n$ for large n , we see that, up to subleading orders in Λ ,

$$\begin{aligned} & \sum_{n=-\Lambda}^{\Lambda} [2\Omega(x_n) - \Omega(x_{n-m}) - \Omega(x_{n+m})] \\ &= 2 \sum_{n=\Lambda-m+1}^{\Lambda} [\Omega(x_n) - \Omega(x_{n-m})] \approx 2m. \end{aligned} \quad (\text{E6})$$

Luckily, we do not have shifts of this second kind in our problem. In fact, they are usually associated to winding in the classical solution, which ours does not possess. We therefore conclude that these shifts do not create an ambiguity in our final result for $E_{1\text{-loop}}$.

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Correction: A sign error in Eq. (2.7) has been fixed.