Wormholes and half wormholes under irrelevant deformation

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Recently it has been shown by Almheiri and Lin [1] that the reconstruction of a black-hole interior is sensitive to knowing the exact coupling of the boundary theory even if the coupling is irrelevant. This motivates us to enlarge the set of the one-time and two-time toy models inspired from the Sachdev-Ye-Kitaev by deforming the same with "irrelevant" coupling. We find that half-wormholes, as well as the wormholes, persist in the presence of the deformation, leading to a similar mechanism for curing the factorization problem. While for the one-time case, the deformed partition function and its moments change by an overall factor, which can completely be absorbed into a renormalization of coupling; for the two-time (or coupled one-time) Sachdev-Ye-Kitaev we find nontrivial dynamics of the saddles as the couplings are varied. Curiously, the irrelevant deformations that we consider can also be thought of as an ensemble average over an overall scaling of the original undeformed Hamiltonian with an appropriate probability distribution; this allows for the possibility that half-wormholes may also be present in suitably defined ensemble of theories.

I. INTRODUCTION AND SUMMARY

In AdS/CFT, the factorization problem arises if we have a bulk manifold X with two (or multiple) disconnected conformal boundaries say L and R. From the boundary perspective, the partition function $Z_{L \cup R}$ on $L \cup R$ should factorize as a product of two partition functions Z_L and Z_R, defined solely on the L and R component respectively. On the other hand, the holographic dictionary instructs us to perform a bulk computation summing up contribution from all possible choices of X. This allows for, in particular, a contribution from wormhole geometries connecting the L and R. Therefore, the bulk computation seems to capture the correlation between two boundary quantities, i.e., $\langle Z_L Z_R \rangle - \langle Z_L \rangle \langle Z_R \rangle$, leading to nonfactorization. Following [2], let us call these kinds of contributions "connected amplitudes with disconnected boundaries" i.e., CADB. They have recently gained lot of interest. In particular, the wormhole saddle contributions are responsible for the late time behaviors of the spectral form factor [3,4] and correlators [5] of holographic theories in the ramp

regime, as well as the late time Hawking radiation entropy [6,7].

The CADB lead to factorization problem only if one insists on having a unitary dual conformal field theory with fixed Hamiltonian. The puzzle can be avoided if one, instead, assumes the Hamiltonian is drawn from a random matrix ensemble. In fact, in 2D Jackiw–Teitelboim (JT) gravity, this is precisely what happens; the dual theory of JT gravity is not a single fixed theory, rather an ensemble of theories [4] and hence provides a plausible explanation why CADB exist from the boundary perspective. Even though the ensemble average saves a potential embarrassment, this raises a puzzle in cases where we expect and know the AdS/CFT duality to work without any averaging, i.e., when a theory with a specific set of couplings in the boundary is dual to a bulk theory with a specific set of parameters (as in the original examples of AdS/CFT based on maximally supersymmetric quantum field theory (QFT) models).

The dichotomy is that on one hand wormhole contributions are important, for instance, to find the ramp in the spectral form factor (SFF), while on the other hand they spoil factorization. Therefore, it is desirable to search for wormholes even in the factorizable observables, and naturally, we are led to find other contributions which restore factorization in those scenarios. Moreover, the SFF is not self-averaging [8]; for a single fixed theory there are erratic fluctuations around the ramp, hence there should be some bulk contribution responsible for these fluctuations. As the problem for the full Sachdev-Ye-Kitaev (SYK) model is a difficult

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one, the authors of [9,10] focused on the SYK model at a single time instance and within this toy problem resolved factorization by discovering "half"-wormholes on top of the usual wormhole contributions. We emphasize that in these toy models, the wormholes survive even in the unaveraged theory, and the half-wormholes added to it restore factorization. See also Refs. [11,12] on the general mechanism of how half-wormholes can possibly cure the factorization puzzle. It is worth mentioning that [13,14] discuss two-dimensional gravity models, where they can successfully address the factorization puzzle. While the former discusses an interpolating model between a random matrix ensemble and a fixed Hamiltonian, the latter utilizes nonlocal interactions in the action.

In this short paper we enlarge the set of such toy examples by considering irrelevant deformations of the one- (and two-) time SYK model. Since the toy models are zero dimensional, any deformation is marginal. Our nomenclatures are made with reference to the actual SYK model. Let us motivate why we care about irrelevant deformations. First of all, one might wonder how robust is the prediction of the one-time SYK model in terms of having wormholes and half-wormholes. More importantly, it has been pointed out recently that the reconstruction of black-hole interior is highly theory sensitive [1], in particular, it depends on the marginal and irrelevant couplings nontrivially. Hence, it is a natural step to deform the original one-time SYK model by irrelevant deformations and ask about the presence of wormholes and half-wormholes, to which we answer affirmatively. On top of the one-time case, we also consider the coupled (two-time instances) case, which is closer to the actual SYK model.

The relevant quantities that have been studied in [9,10] for the undeformed case are given by

$$z_{\text{SYK}} = \int d^N \psi \, \exp(H_0),$$

$$\zeta_{\text{SYK}}(\mu) = \int d^{2N} \psi \, \exp\left(H_{0L} + H_{0R} + \mu \psi_i^L \psi_i^R\right) \quad (1)$$

We note that $\zeta_{\text{SYK}}(0) = z_{\text{SYK}}^2$. Nonzero μ can be thought of as a two-time or coupled one-time SYK model. One of the salient conclusions from [9,10] is

$$z_{\text{SYK}}^2 \simeq \langle z_{\text{SYK}}^2 \rangle + \text{half-wormhole}, \quad (2)$$

and $\langle z_{\text{SYK}}^2 \rangle$ is identified with the wormhole contribution. Later, we will refer to them as moments of partition function. Furthermore, it has been shown in [10] that the wormholes appear as the final term in a perturbation series around the half-wormholes. Finally, the half-wormholes are responsible for curing the factorization problem.

We study the above physical observables for irrelevant deformation of H_0, denoted as $f(H_0)$. In particular, we expound upon

$$z_{\text{deformed}} = \int d^N \psi \, \exp\{f(H_0)\},$$

$$\zeta_{\text{deformed}}(\mu) = \int d^{2N} \psi \, \exp\{f(H_{0L}) + f(H_{0R}) + \mu \psi_i^L \psi_i^R\}.$$
$$(3)$$

Several interesting features come out of the analysis.
(1) *(Half)-wormholes persist after deformation:*
    We find that half-wormholes, as well as the wormhole, persist in the presence of the deformation, leading to a similar mechanism for curing the factorization problem.
(2) *Wormholes as large fluctuations around half-wormholes even after deformation:*
    Just like the original undeformed model, the wormhole can be thought of as a large fluctuation around the half-wormhole saddle. One need not add it separately. The scenario mimics the case of the tensionless string, where one can prove the background independence of the partition function [15]. See also Ref. [16] for stringy realizations which point out several limitations of ensemble QFTs in absence of a UV completion.
(3) Deformation = renormalization:
    For the one-time case, the deformed partition function changes by an overall factor, which can completely be absorbed into a renormalization of coupling J amongst fermions. Quantitatively, consider the following deformation $f(H_0)$ (E_0 is the ground state energy of the undeformed model) and define the function B,

$$f(H_0) = \sum_{\alpha=0}^{\infty} h(E_0, \alpha) H_0^\alpha,$$

$$B(r_1) \equiv \sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_i \alpha_i = p}} \left(\prod_{i=1}^{r_1} h(E_0, \alpha_i) \right), \quad (4)$$

so that for the one-time model, the renormalized coupling $J_{\text{renormalized}}$ is given by

$$J_{\text{renormalized}} = J \left(p! \sum_{r_1=1}^{p} \frac{1}{r_1!} B(r_1) \right)^{1/p}. \quad (5)$$

For more details, see the end of Sec. II, in particular, Eq. (41). For the two-time (or coupled one-time) SYK, we find nontrivial dynamics of the saddles as the couplings are varied. In particular, one can understand the deformation for the two-time case as renormalization of the two-time coupling μ. In this case the wormholes lose their self-averaging characteristics if the renormalized $\mu_{\text{renormalized}}$ becomes very large. On the other hand, in the

regime, $1/N \ll \mu_{\texttt{renormalized}} \ll 1$, the wormholes persist and dominate over half-wormhole just like in the undeformed case.

(4) Deformation = ensemble *average over overall scale*:

The irrelevant deformations that we consider can also be thought of as an ensemble average over an overall scaling of the original undeformed Hamiltonian (along with a possible shift in ground state energy) with an appropriate probability distribution. This allows for the possibility that the half-wormhole may persist in a suitably defined ensemble theory as well. In particular, we can express Eq. (3) for appropriate $p(\ell)$, which depends on the deformation coupled with a possible shift in the ground state energy E_0:

$$z_{\texttt{deformed}} = \int d\ell\, p(\ell) \int d^N \psi\, e^{\ell(H_0 - E_0)},$$

$$\zeta_{\texttt{deformed}}(\mu) = \int d\ell_L\, d\ell_R\, p(\ell_L) p(\ell_R) \int d^{2N} \psi\, e^{\ell_L(H_{0L} - E_0) + \ell_R(H_{0R} - E_0) + \mu \psi_i^L \psi_i^R}. \tag{6}$$

Specifically, for $T\bar{T}$ deformation and Gaussian deformation, we have

$$p_{T\bar{T}\texttt{deformed}} = \frac{1}{\sqrt{8\pi\lambda\ell^3}} \exp\left\{ -\frac{(\ell - 1)^2}{8\lambda\ell} \right\} \qquad \ell \in [0, \infty),$$

$$p_{\texttt{Gaussiandeformed}} = \frac{1}{\sqrt{2\pi}s} \exp\left\{ -\frac{(\ell - \ell_*)^2}{2s^2} \right\} \qquad \ell \in (-\infty, \infty). \tag{7}$$

Here λ, s, ℓ_* parametrize the deformation. For more details, see Sec. II and Sec. IV.

The einbein averaging mentioned above is, in fact, an annealed one in contrast to the usual disordered averaging in the q fermion coupling in $z_{\texttt{SYK}}$, which is a quenched average. This curious feature of deformation being captured by annealed average is, in fact, more generic than being a model dependent feature. For example, in the case of Lagrangian field theories the annealed ensemble averaging over couplings g_i, corresponding to operator $\mathcal{O}_i$ is equivalent to deforming the original theory by local composite operators[1] built out of $\mathcal{O}_i$. In Eq. (6), the role of coupling g is played by an overall scaling of the original Hamiltonian, which plays the role of $\mathcal{O}$.

It is worth mentioning that in [17] it has been shown, using harmonic analysis on $\text{SL}(2, \mathbb{Z})$, that the fixed N ensemble average of $\text{SL}(2, \mathbb{Z})$ invariant physical observable over the $\mathcal{N} = 4$ supersymmetric conformal manifold upon taking the large N limit reproduces the large N, large 't-Hooft coupling limit. This provides another example where the ensemble average is equivalent to a theory with particular coupling, albeit in some particular limit. See also Ref. [18] for the implication of harmonic analysis of $\text{SL}(2, \mathbb{Z})$ in context of 2D CFT. In recent years, the ensemble average over some appropriate moduli space of 2D CFT has been considered in many papers, which includes [19–29], see also Refs. [30–32]. It deserves mentioning that the ensemble average over overall scaling induces a deformation that is briefly discussed in the concluding section of [1].

Deformations of the SYK theory have previously been studied in [33,34]. The 1D analog of the $T\bar{T}$ deformation was shown to be equivalent to coupling an undeformed theory to the worldline quantum gravity. The generic deformation $H_0 \to f(H_0)$ is speculated to couple the undeformed theory to other 1D quantum gravities [33]. For the SYK model, [34] finds that the effect of the deformation, $H(J) \to H(J) + \lambda T\bar{T}$, can essentially be captured in the renormalization of the coupling of the theory: $J \to J_{\texttt{renormalized}} = J(\lambda)$. It is also quite important to normalize the vacuum energy by adding a shift: $H_0 \to H_0 - E_0$. This constant shift, though trivial for the undeformed theory, gives inequivalent deformed theories. In particular, the renormalization becomes trivial when $E_0 = 0$. We show that similar features are also present in deformation of one- and two-time SYK models. It deserves mentioning that the $T\bar{T}$ deformation in context of JT gravity is studied in [35] (see also Refs. [36–38] and [39,40] for $T\bar{T}$ deformation of JT gravity and various quantum mechanical models).

The organization of the rest of the paper, along with glimpses of some results, are as follows. In Sec. II we review the one-time undeformed SYK model, then go on studying its $T\bar{T}$ deformation. The deformed version is investigated in two different ways. The first method

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[1]Unlike the one-point theories described in the paper, the quantum field theory in the higher dimension has coincidence singularities and usually one needs to be careful about handling the composite operators.

involves using einbein to recast the model as an original SYK model with an integral over the einbein variable. The integral over einbein for the $T\bar{T}$ deformation can also be interpreted as ensemble averaging. From the exact answer we conclude that wormholes, as well as half-wormholes, are present in the deformed theory. The other method involves explicitly evaluating the z_{deformed}^2 as a finite perturbation series around the half-wormhole saddle, where the wormhole appears as the final term in the series. We further generalize this second method to any arbitrary deformation $f(H_0)$. In Sec. III, we look at two-time SYK and its deformed version, i.e., we study ζ_{deformed} and dynamics of saddle points. As our work reveals the interpretation of deformation in terms of ensemble averaging, in Sec. IV we study the Gaussian deformation. In particular, one chooses the overall scaling of the Hamiltonian from a Gaussian ensemble; this model is exactly equivalent to a quadratic deformation of the original Hamiltonian. Since this falls under the class of arbitrary deformation, the conclusion regarding the presence of half-wormholes remains true.

II. THE ONE-TIME SYK UNDER $T\bar{T}$

In this section after a short review of the undeformed single-time SYK model, we analyze the deformed case. We write down the exact deformed partition function and its moments using both the einbein approach, as well as through a Taylor expansion in the deformation parameter. We find that the deformed partition function is proportional to the undeformed single-time z_{SYK}. Recalling that z_{SYK} has half-wormholes and wormholes, we establish the presence of half-wormholes and wormholes in the deformed theory. The method involving the einbein further reveals that one can view the deformation as an ensemble average. Using the Taylor expansion method, one can conclude the presence of half-wormholes for $H \to f(H)$ for any nice enough function f.

A. Undeformed theory

We start with the toy version of the undeformed SYK model wherein we focus on a single time instance. The analog of the partition function is built out of the following multi-Grassmann valued number:

$$H_0 = i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1 \cdots i_q} \psi_{i_1 \cdots i_q}, \quad \psi_{i_1 \cdots i_q} \equiv \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}. \tag{8}$$

The couplings $J_{i_1 \cdots i_q}$ are drawn randomly from a Gaussian distribution:

$$\langle J_{i_1 \cdots i_q} \rangle = 0, \quad \langle J_{i_1 \cdots i_q} J_{j_1 \cdots j_q} \rangle = \underbrace{\frac{(q-1)!}{N^{q-1}}}_{\bar{J}^2} \delta_{i_1 j_1} \cdots \delta_{i_q j_q}. \tag{9}$$

We also assumed that both N, q are even integers, and the ratio $N/q = p$ is a positive integer. The partition function can be expressed as a Grassmannian integral,

$$z_{\text{SYK}} = \int \mathrm{d}^N \psi \exp(H_0) = \sum_{A_1 < \cdots < A_p} \mathrm{sgn}(A) J_{A_1} \cdots J_{A_p}, \tag{10}$$

where in the last line the fermions were integrated out. The labels A_i denote ordered nonintersecting subsets of $\{1, \ldots, N\}$ with cardinality q. We remark that on nonintersecting subsets $A_i = \{a_{i1}, a_{i2}, \cdots a_{iq}\}$ of $\{1, \ldots, N\}$ with cardinality q and $a_{im} < a_{in}$ if $m < n$; "ordering" is defined as follows: $a_{i1} < a_{j1} \Leftrightarrow A_i < A_j$ for $i \ne j$. In the undeformed model with the explicit form of z_{SYK}, we may evaluate the averaged (over J) moments: $\langle z_{\text{SYK}}^k \rangle$. Clearly, $\langle z_{\text{SYK}} \rangle = 0$ since there are no possible Wick contractions of J indices. The two lowest nontrivial moments come from

$$\langle z_{\text{SYK}}^2 \rangle = \frac{N!}{p!(q!)^p} (\bar{J}^2)^p,$$

$$\langle z_{\text{SYK}}^4 \rangle = \left(\frac{\bar{J}^2}{q!}\right)^{2p} \sum_{n_1 + n_2 + n_3 = p} \frac{(qn_1)!(qn_2)!(qn_3)!}{(n_1! n_2! n_3!)^2}. \tag{11}$$

In the large N limit, it follows that

$$\langle z_{\text{SYK}}^4 \rangle \approx 3 \langle z_{\text{SYK}}^2 \rangle^2. \tag{12}$$

The correlations in the $\langle z_{\text{SYK}}^2 \rangle = \langle z_L z_R \rangle_{\text{SYK}}$ contraction are the "wormhole" contribution. We have added replica indices L and R to distinguish the replicas. For $\langle z_{\text{SYK}}^4 \rangle$ we therefore have $\langle z_L z_R z_{L'} z_{R'} \rangle_{\text{SYK}}$. Among all the possible contractions in the large N limit, the ones that dominate turn out to be $\langle z_{\text{SYK}}^4 \rangle \approx \langle z_L z_R \rangle_{\text{SYK}} \langle z_{L'} z_{R'} \rangle_{\text{SYK}} + \langle z_L z_{L'} \rangle_{\text{SYK}} \langle z_R z_{R'} \rangle_{\text{SYK}} + \langle z_L z_{R'} \rangle_{\text{SYK}} \langle z_R z_{L'} \rangle_{\text{SYK}}$. It also turns out that each of the averages are equal to each other and equal to the wormhole correlation, from whence Eq. (12) follows.

1. Half-wormholes of the undeformed theory

The half-wormholes are the contributions to the unaveraged z_{SYK}^2, in addition to $\langle z_{\text{SYK}}^2 \rangle$ which restore factorization. The regime where the half-wormholes dominate is best seen by introducing collective fields $g \propto \sum_i \psi_i^L \psi_i^R$ and its conjugate σ. It turns out that the factorized answer can be expressed as follows:

$$z_{\text{SYK}}^2 = \int \mathrm{d}\sigma \, \Phi_0(\sigma) \Psi_0(\sigma), \tag{13}$$

where

$$\Phi_0(\sigma) = \int d^{2N}\psi \ \exp[ie^{-i\pi/q}\sigma\psi_i^L\psi_i^R - \bar{J}^2(\psi_A^L\psi_A^R) + i^{q/2}J_A(\psi_A^L + \psi_A^R)]$$

$$= \int d^{2N}\psi \ \exp\left[ie^{-i\pi/q}\sigma\psi_i^L\psi_i^R - \frac{N}{q}\left(\frac{1}{N}\psi_i^L\psi_i^R\right)^q + i^{q/2}J_A(\psi_A^L + \psi_A^R)\right]. \quad (14)$$

Here we have used $\bar{J}^2 = \frac{(q-1)!}{N^{q-1}}$ and $(\psi_A^L\psi_A^R) = \frac{1}{q!}(\psi_i^L\psi_i^R)^q$ to go from the second line to the third line. The $\Psi_0(\sigma)$ is given by

$$\Psi_0(\sigma) = \int_{-\infty}^{\infty}\frac{dg}{2\pi/N}\exp\left[N\left(-i\sigma g - \frac{1}{q}g^q\right)\right] \quad (15)$$

and is independent of the disorder. Therefore, disorder average $\langle\Phi_0(\sigma)\rangle$ is well approximated by the contributions of the wormhole saddles. On the complex σ plane these lie symmetrically on the unit circle separated out by angle $2\pi/q$. In order to compute the RMS, we need

$$\langle\Phi_0^2(\sigma)\rangle = \int d^4g_{ab}d^4\sigma_{ab}I(\sigma, \sigma_{ab}, g_{ab}), \quad (16)$$

where we have

$$I(\sigma, \sigma_{ab}, g_{ab}) \equiv \frac{1}{(2\pi/N)^4}\exp\left[N\log(\sigma^2 + \sigma_{LR'}\sigma_{RL'}\right.$$
$$\left. - \sigma_{LL'}\sigma_{RR'}) - N\left(i\sigma_{ab}g_{ab} + \frac{g_{ab}^q}{q}\right)\right], \quad (17)$$

where $(a,b) \in \{(L,L'),(L,R'),(R,R'),(R,L')\}$ and $\sigma_{LR} = \sigma_{L'R'}$ fixed to σ and not integrated over. Note that $\sigma_{ab} = 0$ is a trivial saddle for any value of σ and responsible for $\langle\Phi_0(\sigma)\rangle^2$ contribution to $\langle\Phi_0^2(\sigma)\rangle$. When $\sigma = 0$, the nontrivial saddles lie on the unit circle, and there are $q \times q \times 2$ of them. One of the two kinds corresponds to $(L,R'),(L',R)$ contraction while the other represents $(L,L'),(R,R')$ contraction. As we make $\sigma \neq 0$, these nontrivial saddles take the form $\sigma_{LR'}\sigma_{RL'} = s^2e^{2\pi im/q}$, $\sigma_{LL'}\sigma_{RR'} = s^2e^{2\pi im/q}$. Without loss of generality, let us focus on the case $\sigma_{LR'}\sigma_{RL'} = s^2e^{2\pi im/q}$ and $\sigma_{LL'}\sigma_{RR'} = 0$. Let us name the saddles as σ_{ab*}, g_{ab*}. So we have two contributions:

$$\langle\Phi_0^2(\sigma)\rangle = I(\sigma, 0, 0) + \sum_{\text{nontrivial saddles}}I(\sigma, \sigma_{ab*}, g_{ab*}). \quad (18)$$

Now as we vary σ on the complex plane, $\langle\Phi_0^2(\sigma)\rangle$ gets most of the contribution either from the trivial saddle or from the nontrivial ones. The region dominated by the trivial saddle is a self-averaging one, while the other region is non-self-averaging. Thus, we can write

$$\langle\Phi_0^2(\sigma)\rangle = \langle\Phi_0(\sigma)\rangle^2 + \sum_{\text{nontrivial saddles}}I(\sigma, \sigma_{ab*}, g_{ab*}). \quad (19)$$

It turns out that for the one-time SYK model, the wormhole saddles (which correspond to some specific value of σ) always lie in the self-averaging region. The non-self-averaging region takes the shape of a scallop (see Fig. 1 in [9] for $q = 4$). For future reference, we define

$$\Theta(\sigma) = \frac{\sum_{\text{nontrivial saddles}}I(\sigma, \sigma_{ab*}, g_{ab*})}{\langle\Phi_0(\sigma)\rangle^2}, \quad (20)$$

and use it as diagnostic of the self-averaging (or non-self-averaging) region. The above exercise shows that the wormholes persist for the one-point SYK model before averaging since they live in the self-averaging region.

Now we turn our attention to the half-wormholes. While the wormhole saddles are in the self-averaging regime, we know that z^2 is factorizable and cannot be self-averaging exactly. This can be seen by setting $\sigma = 0$ and noticing that the dominant contribution to $\langle\Phi_0^2\rangle$ comes from nontrivial saddles, hence $\sigma = 0$ is in the non-self-averaging region. It turns out that the following is a good approximation:

$$z_{\text{SYK}}^2 \simeq \langle z_{\text{SYK}}^2\rangle + \Phi_0(0), \quad (21)$$

where $\Phi_0(0)$ is dubbed as half-wormhole [9,10] and responsible for factorization. The RMS value of the fluctuation is $\langle\Phi_0^2(0)\rangle$, which is dominated by nontrivial saddles is of the same order as the trivial contribution:

$$\sqrt{\langle\Phi_0^2(0)\rangle} \simeq \langle z_{\text{SYK}}^2\rangle. \quad (22)$$

Furthermore, one can show that other contributions to z_{SYK}^2 are typically suppressed [9,10].

B. $T\bar{T}$ deformed theory

In this subsection we focus on the $\lambda T\bar{T}$ deformation. The deformation of the spectrum indexed by energy levels E can fully be solved from the nonperturbative flow equation:

$$\frac{\partial E}{\partial \lambda} = \frac{E^2}{1/2 - 2\lambda E}. \quad (23)$$

The above can be integrated to find the full deformed Hamiltonian in terms of the original one as follows:

$$H(\lambda) = \frac{1}{4\lambda}(1 \pm \sqrt{1 - 8\lambda(H_0 - E_0)}), \qquad (24)$$

where H_0 is the undeformed SYK Hamiltonian. We have also included a constant E_0 shift to the ground state energy of the undeformed theory. The solution with negative sign is perturbatively connected from the undeformed spectrum, hence, we consider $H(\lambda)$ with only the negative sign in front of the square root.

In what follows we will compute the moments of this model using two methods. The first one involves introducing an einbein and then doing an ensemble average over the einbein to implement the deformation. The second method proceeds without the einbein by doing simple brute force Taylor expansion of deformed Hamiltonian around the original undeformed one.

1. Moments from exact computation-I via einbeins

A convenient way to implement the $T\bar{T}$ deformation given by Eq. (24) is to use an integral over einbeins. The zeroth moment, i.e., the partition function, now involves also an integral over the einbein ℓ as follows[2]:

$$z_{\text{deformed}} = \int_0^\infty d\ell \, \frac{1}{\sqrt{8\pi\lambda\ell^3}} \exp\left\{-\frac{(\ell-1)^2}{8\lambda\ell}\right\}$$
$$\times \int d^N \psi \, \exp\left\{\ell(H_0 - E_0)\right\}. \qquad (25)$$

We note that $\frac{1}{\sqrt{2\pi\lambda\ell^3}} \exp\left\{-\frac{(\ell-1)^2}{8\lambda\ell}\right\}$ can be thought of as a probability distribution of ℓ, the overall scaling of shifted Hamiltonian $H_0 - E_0$. The distribution has its support on $[0,\infty)$ with unit mean and 4λ variance. This is only when the deformation coupling λ is positive.[3] As promised in the Introduction, we see the ensemble average over overall scaling implements a deformation of the original Hamiltonian.

We can rescale ψ in H_0 to separate out the ℓ integral so that we obtain

$$z_{\text{deformed}} = \int d\ell \, e^{-S'(\ell)} \int d^N \psi \exp\left\{i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1 \cdots i_q} \psi^L_{i_1 \cdots i_q}\right\}$$

$$\text{with,} \qquad S'(\ell) = \frac{(\ell-1)^2}{8\lambda\ell} + \ell E_0 + \frac{1}{2}\log(8\pi\lambda) + \left(\frac{3}{2} - \frac{N}{q}\right)\log\ell. \qquad (26)$$

Note that we have used the Grassmannian identity $d(a\psi) = d\psi/a$ above. The ℓ integral can now be done explicitly to obtain

$$z_{\text{deformed}} = \left(\frac{e^{\frac{1}{4\lambda}}(1 + 8\lambda E_0)^{\frac{1}{4} - \frac{N}{2q}}}{\sqrt{2\pi\lambda}} K_{\frac{1}{2} - \frac{N}{q}}\left[\frac{\sqrt{1 + 8\lambda E_0}}{4\lambda}\right]\right) z_{\text{SYK}}. \qquad (27)$$

Since the effect of the deformation appears as an overall factor to z_{SYK}, the existence of wormholes and half-wormholes follow naturally. One can also check this by looking at the variance and moments, e.g., the kth moment of the partition function upon performing the SYK-like average over the coupling J evaluates to

$$\langle z^k_{\text{deformed}} \rangle \equiv \int \left(\prod_A \frac{dJ_A}{\sqrt{2\pi}\bar{J}}\right) e^{-\frac{1}{2\bar{J}^2}J_A^2} z^k_{\text{deformed}}(J) \qquad (28)$$

and noting the integral only affects the z^k_{SYK} factor of z^k_{deformed}. Thus,

$$\langle z^k_{\text{deformed}} \rangle = \left(\frac{e^{\frac{1}{4\lambda}}(1 + 8\lambda E_0)^{\frac{1}{4} - \frac{N}{2q}}}{\sqrt{2\pi\lambda}} K_{\frac{1}{2} - \frac{N}{q}}\left[\frac{\sqrt{1 + 8\lambda E_0}}{4\lambda}\right]\right)^k \langle z^k_{\text{SYK}} \rangle. \qquad (29)$$

Note that in the limit $\lambda \to 0$ with λE_0 held fixed, we can use the asymptotics of Bessel K to obtain

$$\langle z^2_{\text{deformed}} \rangle \approx \frac{e^{2E_0'}}{(1 + 8\lambda E_0)^p} \langle z^2_{\text{SYK}} \rangle, \qquad E_0' = \frac{1 - \sqrt{1 + 8E_0\lambda}}{4\lambda}. \qquad (30)$$

Similarly, we have

$$\langle z^k_{\text{deformed}} \rangle \approx \frac{e^{kE_0'}}{(1 + 8\lambda E_0)^{pk/2}} \langle z^k_{\text{SYK}} \rangle. \qquad (31)$$

The same conclusion will be reached using the exact computation without using einbein.

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[3]When $\lambda < 0$, the analog process with the probability distribution $\frac{1}{\sqrt{2\pi\lambda\ell^3}} \exp\left\{-\frac{(\ell+1)^2}{8\lambda\ell}\right\}$ having support on $\ell \in (-\infty, 0]$ with mean -1 and variance -4λ, yields, $e^{-H(\lambda)}$.

2. Moments from exact computation-II

In this subsection we evaluate the Grassmannian integrals exactly to compute one-time z and its higher moments. We find that in the leading order in N, and more specifically when $N\lambda \ll q$, the answers from the einbeins are reproduced. Our methods allow for an easy generalization to arbitrary deformations of the type $H_0 \to f(H_0)$ of which $T\bar{T}$ is only a special case. For the latter we can write,

$$z_{\text{deformed}} = \int d^N \psi \, e^{H(\lambda)}$$

$$= \int d^N \psi \, \exp\left\{ \frac{1}{4\lambda} \left(1 - \sqrt{1 - 8\lambda(H_0 - E_0)}\right) \right\}$$

$$= \int d^N \psi \sum_{r=0}^{\infty} \frac{1}{r!} \left(\sum_{\alpha=0}^{\infty} h(E_0, \alpha) H_0^{\alpha} \right)^r, \qquad (32)$$

where $h(E_0, \alpha)$ is given by

$$h(E_0, \alpha) = \left(\sum_{k \geq \alpha k > 0}^{\infty} \frac{1}{4\lambda E_0} \binom{1/2}{k} \binom{k}{\alpha} (8\lambda E_0)^k (-E_0)^{1-\alpha} \right)$$

$$= \begin{cases} 2^{3\alpha-2} \binom{1/2}{\alpha} \left(\frac{-\lambda}{1+8\lambda E_0} \right)^{\alpha} \frac{\sqrt{1+8\lambda E_0}}{-\lambda} & \text{for } \alpha \geq 1, \\ \frac{1}{4\lambda} \left(1 - \sqrt{1 + 8\lambda E_0}\right) & \text{for } \alpha = 0. \end{cases}$$

$$(33)$$

Now let us consider $\left(\sum_{\alpha=0}^{\infty} h(E_0, \alpha) H_0^{\alpha} \right)^r$ and separate out the $\alpha = 0$ term from the rest, so that we have

$$\left(h(E_0, 0) + \sum_{\alpha=1}^{\infty} h(E_0, \alpha) H_0^{\alpha} \right)^r$$

$$= \sum_{r_1=0}^{r} \binom{r}{r_1} h(E_0, 0)^{r-r_1} \left(\sum_{\alpha=1}^{\infty} h(E_0, \alpha) H_0^{\alpha} \right)^{r_1}. \quad (34)$$

Now the idea is that we need exactly N Grassmannian variables to saturate the integral, thus we need to figure out the coefficient of H^p, where $N/q = p$. From Eq. (34) we find that

$$B(r_1) \equiv \sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_i \alpha_i = p}} \left(\prod_{i=1}^{r_1} h(E_0, \alpha_i) \right). \qquad (35)$$

The sum runs over all possible sets $\{\alpha_i : i = 1, 2, \dots r_1\}$ such that $\sum_i \alpha_i = p$ and $\alpha_i > 0$. The partitions are unordered, e.g., in the length $r_1 = 3$ partition of $p = 4 = 1 + 1 + 2$, the combination contributes with degeneracy 3 as $(1,1,2),(1,2,1)$, and $(2,1,1)$. Also note that

$$B(r_1) = 0, \quad \text{if } r_1 > p.$$

This happens because $\alpha_i \geqslant 1$, and hence $p = \sum_i \alpha_i \geqslant r_1$. Finally, performing the Grassmannian integral, we find that

$$z_{\text{deformed}} = e^{h(E_0, 0)} \left(\sum_A \text{sgn}(A) J_{A_1} J_{A_2} \cdots J_{A_p} \right)$$

$$\times \left(p! \sum_{r_1=1}^{p} \frac{1}{r_1!} B(r_1) \right). \qquad (36)$$

One can check numerically that the above equation is identical to Eq. (27), for more details see Appendix B. Note that $B(r_1)$ is maximized when $r_1 = p$, and the most contributing term for small λ is coming from the set $\alpha_i = 1$. The next leading term comes from $B(p - 1)$, where the most contributing term is when one of the $\alpha_i = 2$ and rest are set to 1:

$$B(p) \simeq (1 + 8\lambda E_0)^{-p/2},$$

$$B(p - 1) \simeq 2(p - 1)\lambda(1 + 8\lambda E_0)^{-p/2-1}. \qquad (37)$$

Hence, in the approximation $p\lambda \ll 1$, we have

$$z_{\text{deformed}} = \frac{1}{(1 + 8\lambda E_0)^{p/2}} \exp\left[\frac{1 - \sqrt{1 + 8E_0\lambda}}{4\lambda} \right]$$

$$\times \left(\sum_A \text{sgn}(A) J_{A_1} J_{A_2} \cdots J_{A_p} \right) + O\left(\frac{p\lambda}{1 + 8\lambda E_0} \right). \qquad (38)$$

If we now consider $\langle z_{\text{deformed}}^2 \rangle$ at the leading order, then it will precisely give us Eq. (30). Going back to Eq. (38), we can use the replacements

$$J \to J(\lambda) = \frac{1}{\sqrt{1 + 8\lambda E_0}} J, \qquad (39)$$

and E_0' from Eq. (30) to rewrite

$$z = e^{E_0'} \left(\sum_A \text{sgn}(A) J_{A_1}(\lambda) \cdots J_{A_p}(\lambda) \right) + O\left(\frac{p\lambda}{1 + (4\lambda E_0')^2} \right). \qquad (40)$$

In leading order, this is therefore the one-time SYK model with renormalized coupling $J(\lambda)$ and ground state energy shifted to E_0'. This is consistent with the results obtained in the full SYK model using diagrammatics and einbeins [34].

3. Arbitrary deformations

For arbitrary Hamiltonian deformations $H_0 \to f(H_0)$, we therefore see that once we know the Taylor coefficients $f(H_0) = \sum_\alpha h(E_0, \alpha) H_0^\alpha$, the partition function reduces to z_{SYK} with renormalized couplings,

$$J_{A\text{renormalized}} = J_A \left(p! \sum_{r_1=1}^{p} \frac{1}{r_1!} B(r_1) \right)^{1/p}, \qquad (41)$$

and ground state shifted $E_0 \to h(E_0, 0)$, where $B(r_1)$ is defined in Eq. (35).

III. TWO-TIME SYK AND DEFORMATION

In this section we consider the two-time version of the SYK model and its deformation. We begin with the undeformed theory, which can be thought of as coupling two one-time models with coupling denoted as μ,

$$\zeta(\mu) = \int d^{2N}\psi \, \exp\left\{ i^{q/2} J_A (\psi_A^L + \psi_A^R) + \mu \psi_i^L \psi_i^R \right\}. \quad (42)$$

Note, $\zeta(0) = z_{\text{SYK}}^2$, and we have

$$\zeta(\mu) = \int d\sigma \, \Phi_0(\sigma - i e^{i\pi/q}\mu)\Psi_0(\sigma), \quad (43)$$

where we repeat

$$\Psi_0(\sigma) = \int_{-\infty}^{\infty} \frac{dg}{2\pi/N} \exp\left[N\left(-i\sigma g - \frac{1}{q}g^q \right) \right]. \quad (44)$$

Now averaging leads to

$$\langle \zeta(\mu) \rangle = \int d\sigma \, (i e^{-i\pi/q}\sigma + \mu)^N \int_{-\infty}^{\infty} \frac{dg}{2\pi/N}$$
$$\times \exp\left[N\left(-i\sigma g - \frac{1}{q}g^q \right) \right]. \quad (45)$$

At large N the above integrals over σ and g may be computed by saddle point. If we solve the saddle point equations simultaneously, then we can get the saddle points as specific values of σ. These are the analogs of the wormhole contributions since a nonzero saddle point value of σ implies a nonvanishing correlation between the two replicas. Unlike the one-time case, for a nonzero μ the wormhole saddles are no longer present on the unit circle in the complex σ plane, though there are still q of them. We plot in Fig. 1 for the $q = 4$ case how the saddles behave as a function of μ. When $\mu = 0$ the saddles appear symmetrically on the unit circle as it should since $\zeta(0) = z_{\text{SYK}}^2$. As μ gets turned on, the saddles start to move asymmetrically. One of the saddles always approaches the origin $\sigma = 0$. As μ increases, the rest $q - 1$ saddles collapse and start moving at an angle $\frac{\pi}{2} + \frac{\pi}{q}$.

A. (Half) wormholes of the undeformed (two-time) theory

We repeat the exercise done in the end of II A, but now with $\Phi_0(\sigma - i e^{i\pi/q}\mu)$. When $\mu = 0$ the self-averaging region takes the form of a scallop centered at origin. The wormhole saddles live outside the scallop and hence also exist in the nonaveraged model (see Fig. 2). We contrast this with the large μ, where the scallop is now centered at $\sigma = i e^{i\pi/q}\mu$, i.e., the center moves along a line with an angle $\pi/2 + \pi/q$. For large enough μ, the $\sigma \neq 0$ saddles comes within the scallop. Thus, for large enough μ,
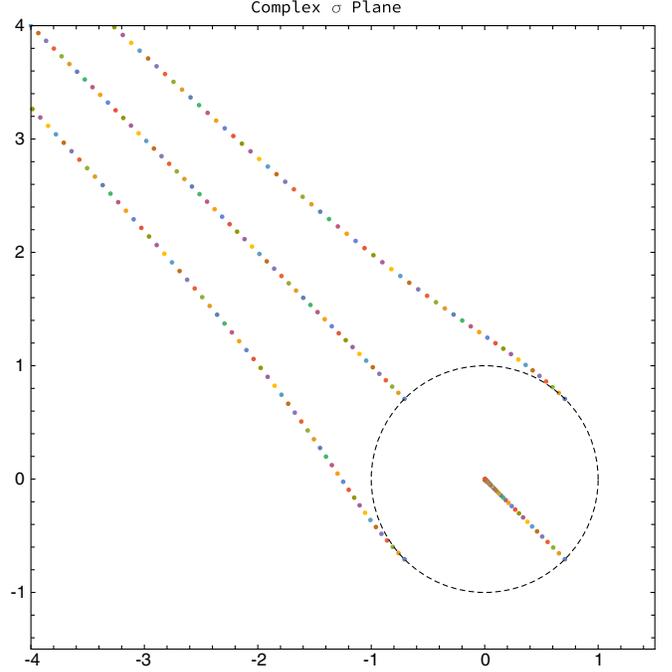

FIG. 1. The saddles contributing to $\langle \zeta(\mu) \rangle$ are plotted for $q = 4$. The saddles start out from the four rotated roots of unity. Different colors indicate different values of μ. As μ increases, while one saddle goes to zero, the other three merge at $\mu e^{i3\pi/4}$.
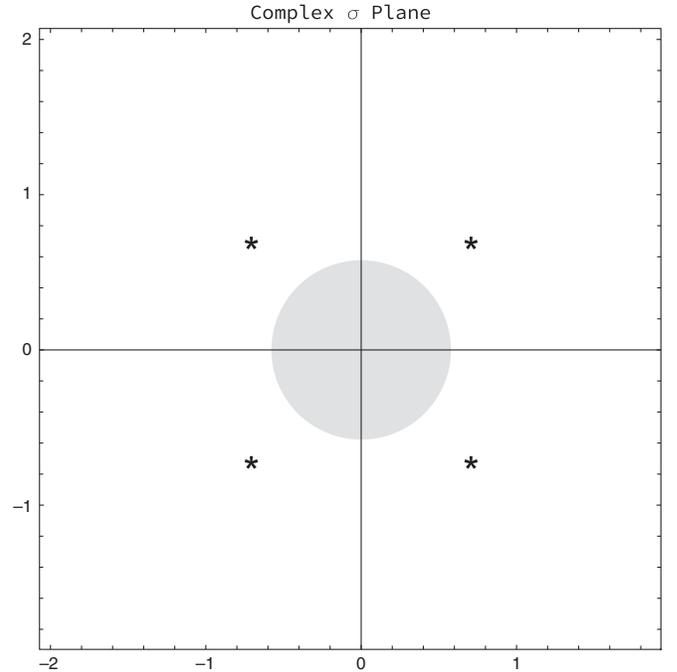

FIG. 2. The plots are for the $q = 4$ case. The gray regions indicate approximately the scallop which is *not* self-averaging in the complex σ plane. The black stars indicate the wormhole saddles. For $\mu = 0$, all the saddles are in the non-self-averaging region.

the nonaveraged model does not have the wormhole saddles. This is expected since increasing μ effectively makes the model a $q = 2$ nonaveraged model, which is a Gaussian integrable theory. Effects of disorder in the spectral form factor for the $q = 2$ SYK model have been analyzed in [41,42], wherein the ramp is exponential. On the other hand, we know that wormholes give rise to a linear ramp.

In Fig. 3 we illustrate the dissolution process of the wormhole saddles with increasing $|\mu|$. The plots take into account both the movement of the wormhole saddles, as well as the motion of the scallop. A simple dimension analysis shows that the threshold value μ_* scales with variance in following manner:

$$\mu_* \propto (\bar{J}^2)^{1/q}. \tag{46}$$

Also, as we see in Appendix A, there is a q dependence on the size of the scallop. With increasing q the scallop gets smaller, and therefore, the saddles survive further out in the complex σ plane as compared with a lower q result. Intuitively, as increasing q increases chaoticity, this is also what one would expect.

B. (Half) wormholes of the deformed (two-time) theory

At this point, it is easy to figure out the effect of deformation. We will be brief and mention the salient features only,

$$\zeta_{\text{deformed}}(\mu) = \int d^2\ell\, e^{-S(\ell_L,\ell_R)} \int d^{2N}\psi\, \exp\left\{i^{q/2}J_A(\ell_L\psi_A^L + \ell_R\psi_A^R) + \mu\psi_i^L\psi_i^R\right\}$$

$$= \int d\sigma\, \Phi_{\text{deformed}}(\sigma - ie^{i\pi/q}\mu)\Psi_0(\sigma), \tag{47}$$

where $\Phi_{\text{deformed}}(\sigma - ie^{i\pi/q}\mu) = \Phi_0(\sigma - ie^{i\pi/q}\mu_{\text{renormalized}})$, which can be obtained by carrying out the einbein integrals via saddle. Thus, we can also express $\zeta_{\text{deformed}}(\mu) = \zeta(\mu_{\text{renormalized}})$. Of course, note $\zeta(0) = z^2$. For the $T\bar{T}$ at leading order in $p\lambda \ll 1$, we find that the dissolution of the wormhole saddles now happen for a different value of μ:

$$\mu_{*\text{renormalized}} \propto (\bar{J}^2_{\text{renormalized}})^{1/q} = \frac{(\bar{J}^2)^{1/q}}{(1 + 8E_0\lambda)^{1/q}}$$

$$= \mu_*(1 + 8E_0\lambda)^{-1/q}. \tag{48}$$

We therefore see for $\lambda < 0$ (recall E_0 is negative), if we increase $|\lambda|$, the renormalized variance $\bar{J}_{\text{renormalized}}$ decreases, and therefore, $\mu_{*\text{renormalized}} < \mu_*$, hence the wormholes dissolve faster.

Since the $1/N$ expansion in this toy model is finite, it is possible to expand ζ_{deformed} also around the half-wormhole, i.e., $\sigma = 0$ saddle [10]:

$$\zeta_{\text{deformed}} = \sum_{k=0}^{p} \zeta_k(\mu_{\text{renormalized}})\tilde{\Phi}_k, \tag{49}$$

where,

$$\tilde{\Phi}_k = \frac{1}{k!}(\bar{J}^2)^k\left(-\frac{\partial}{\partial\bar{J}^2}\right)^k \Phi_{\text{deformed}}(0), \tag{50}$$

$$\zeta_k(\mu_{\text{renormalized}})$$
$$= \sum_{n=0}^{k} \frac{k!}{(k-n)!(nq)!}\left(\frac{q}{N}\right)^n (N\mu_{\text{renormalized}})^{nq}. \tag{51}$$

In terms of this expansion the wormhole saddles appear as the final terms in the series. Also note that in this expansion the effect of the deformation is encapsulated via $\mu \to \mu_{\text{renormalized}}$. In the next section we shall look into this expansion for the exactly solvable case of Gaussian ensemble of einbeins.

IV. THE GAUSSIAN DEFORMATION

One of the key objects that we have investigated so far is

$$\zeta_{\text{deformed}} = \int d^2\ell\, e^{-S(\ell_L,\ell_R)} \int d^{2N}\psi$$
$$\times \exp\left\{i^{q/2}J_A(\ell_L\psi_A^L + \ell_R\psi_A^R) + \mu\psi_i^L\psi_i^R\right\}. \tag{52}$$

The purpose of the einbein integral is to implement the irrelevant deformation $H \to f(H)$. One can view the above as an ensemble average with a weight factor of $e^{-S(\ell_L,\ell_R)}$. Since $S(\{\ell_i\}) = \prod_i S(\ell_i)$, this averaging does not induce correlations between the "replicas" for $\mu = 0$, rather just renormalizes J_A, which is now a coupling of the irrelevant Hamiltonian $f(H)$. On the other hand for $\mu \neq 0$, things are nontrivial. One cannot simply scale out ℓ and cast the deformation as an overall factor.

The choice of $S(\ell_L, \ell_R)$ provides a class of toy models where one can replace $e^{-S(\ell_L,\ell_R)}$ with a probability measure. An exactly solvable choice is the Gaussian ensemble for the einbeins ℓ_i with means ℓ_i^* and variances s_i. For simplicity, we will assume $\ell_{L*} = \ell_{R*} = \ell_*$. It results in the exact deformation: $H_0 \to \frac{1}{2}(s^2H_0^2 + 2\ell_*H_0)$. In general, for the deformation of the type $H_0 \to \#H_0^\gamma$, the einbein ensemble
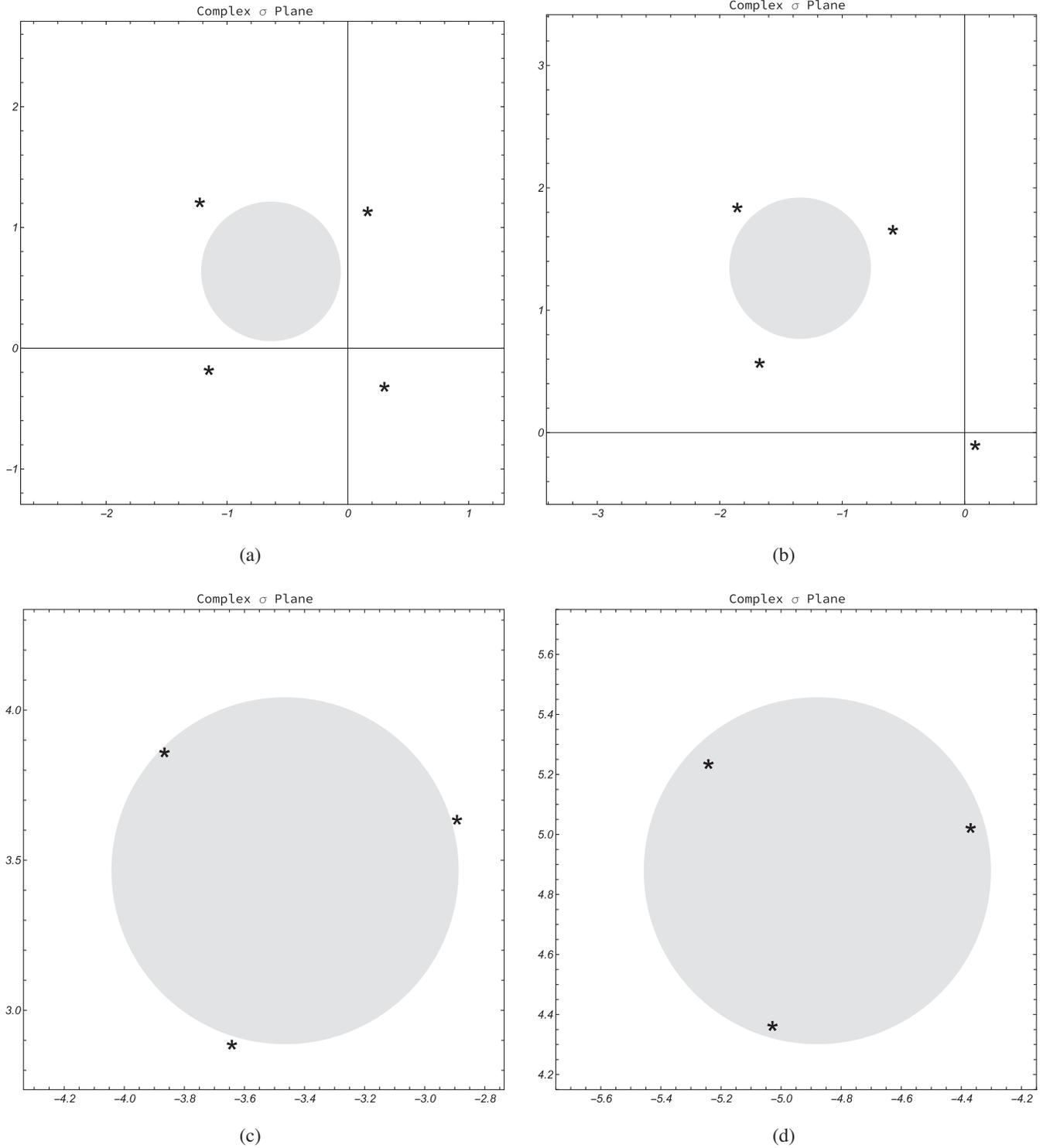
FIG. 3.   The plots are for the $q = 4$ case. The gray regions indicate approximately the scallop which is *not* self-averaging in the complex σ plane. The black stars indicate the wormhole saddles. Progressive plots are for increasing values of μ. The figure focuses on the three saddles which are engulfed, while the other one (not shown in the last two) remains very close to zero.

action that implements this (in the saddle-point approximation) is $S(\ell) = \#\ell^{\gamma/(\gamma-1)}$. Coming back to the Gaussian deformation, the one-time partition function is given by

$$z_{\text{deformed}} = \langle \ell^p \rangle_s z_{\text{SYK}}, \tag{53}$$

where $\langle \ell^k \rangle_s = \int_{-\infty}^{\infty} d\ell \, \ell^k e^{-S(\ell)}$

$$= \frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{\infty} d\ell \, \ell^k \exp \left[-\frac{(\ell - \ell_*)^2}{2s^2} \right]. \tag{54}$$

This integral can be computed exactly and leads to a result in terms of the hypergeometric function. We can arrive at the final answer also by using Eqs. (35) and (36) with appropriate h for the Gaussian case, which now reads for even p,

$$z_{\text{deformed}} = \left(p! \sum_{r=p/2}^{p} \frac{1}{r!} B(r) \right) z_{\text{SYK}}$$

$$= \frac{2^{p/2} s^p}{\sqrt{\pi}} \Gamma \left(\frac{1+p}{2} \right) {}_1F_1 \left(-\frac{p}{2}, \frac{1}{2}, -\frac{\ell_*^2}{2s^2} \right) z_{\text{SYK}}. \tag{55}$$

Note that the sum runs from $p/2$ because $B(r_1) = 0$ for $r_1 < p/2$. This happens because α_i can be 1 or 2 for the Gaussian deformation, leading to $2r \geq \sum_{i=1}^{r} \alpha_i = p$. Thus, $\sum_{i=1}^{r} \alpha_i = p$ cannot have a solution if $r < p/2$. A similar calculation can be done for odd p as well, where the sum runs from $(p+1)/2$, leading to a slightly different final expression. For simplicity, in what follows, we will assume p is even without loosing any physical content.

In the two-time case, when $\mu = 0$, one has $\zeta(0) = z_{\text{SYK}}^2 \prod_{i=L,R} \langle l_i^p \rangle_{s_i}$, which once again may either be computed exactly using Eq. (41) or using einbeins. When $\mu \neq 0$ we may write down the half-wormhole expansion as in Eq. (49):

$$\zeta_{\text{deformed}} = \sum_{k=0}^{p} \tilde{\Phi}_k \sum_{n=0}^{k} \frac{k!}{(k-n)!(nq)!} \left(\frac{q}{N} \right)^n (N\mu)^{nq}$$

$$\times \prod_{i=L,R} \left(\int d\ell_i \, \ell_i^{p-n} e^{-S(\ell_i)} \right). \tag{56}$$

Since $S(\ell_i)$ is Gaussian, the integral in parentheses can be performed exactly:

$$\int_{-\infty}^{\infty} d\ell \, \ell^{p-n} e^{-S(\ell)} = \frac{2^{\frac{p-n}{2}} s^{p-n}}{\sqrt{\pi}} \begin{cases} \Gamma\left(\frac{1}{2}(p-n+1)\right) {}_1F_1\left(\frac{n-p}{2}; \frac{1}{2}; -\frac{\ell_*^2}{2s^2}\right) & n \equiv p \,(\text{mod } 2) \\ \sqrt{2}\frac{\ell_*}{s} \Gamma\left(\frac{p-n}{2}+1\right) {}_1F_1\left(\frac{1}{2}(n-p+1); \frac{3}{2}; -\frac{\ell_*^2}{2s^2}\right) & n+1 \equiv p \,(\text{mod } 2). \end{cases} \tag{57}$$

In spite of the above answer for the integral, the finite sums in Eq. (56) cannot be explicitly performed in closed form. However, in the small variance $s \to 0$ coupled with the large p and $1/N \ll \mu \ll 1$ limit, the final term of the series Eq. (56) (the wormhole saddle $= \langle \zeta_{\text{deformed}} \rangle$) dominates and we find

$$\zeta_{\text{deformed}} \underset{\substack{1/N \ll \mu \ll 1 \\ p \gg 1}}{\approx} \tilde{\Phi}_p \frac{\ell_*^{2p}}{q} \exp \left(\frac{\mu N}{l_*^{2/q}} \right). \tag{58}$$

Here we have used that in the small s limit, the integral given by Eq. (57) gives ℓ_*^{p-n}. This is consistent with the saddle point expectations in the zero variance limit. In this zero "uncertainty" regime of the einbeins, the deformation reduces to the scaling $H_0 \to \ell_* H_0$, which is the same as $\bar{J} \to \ell_* \bar{J}$ and $\mu \to \mu/l_*^{2/q}$. This leads to the ℓ_*^{2p} factor and rescaled μ in Eq. (58).

ACKNOWLEDGMENTS

APPENDIX A: THE q DEPENDENCE OF THE SCALLOP

Here we work with the undeformed theory and look into the q dependence of the size of the scallop, i.e., the half-wormhole dominated region which is, by definition, not self-averaging. To see this, one can calculate the ratio $\Theta(\sigma)$ defined in Eq. (20) on the complex σ plane. If $\Theta(\sigma) > 1$, then the half-wormholes dominate. Here we consider the case $\sigma_{LR'}\sigma_{RL'} = s^2 e^{2\pi im/q}$, $\sigma_{LL'}\sigma_{RR'} = 0$. For the ease of computation, we also focus on $\Theta(\sigma)$ along the real σ line, i.e., for $m = 0$. From Fig. 4, we see that the scallop gets smaller as q increases, thus the non-self-averaging area on the complex σ plane decreases for increasing q.
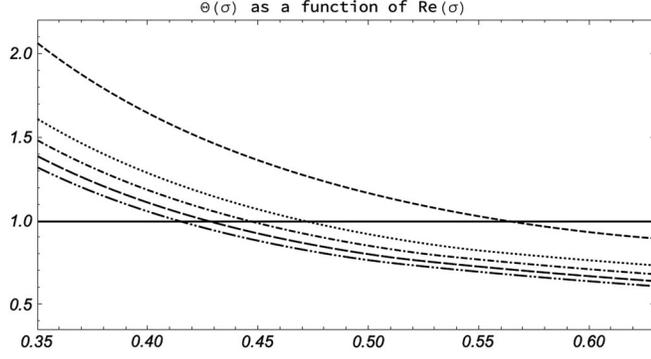
FIG. 4. Here we have plotted $\Theta(\sigma)$ along the real σ line for $q = 4, 8, 12, 20, 40$. As q increases, $\Theta(\sigma)$ becomes less than 1 for smaller values of σ, thus the scallop gets smaller.

APPENDIX B: A MODIFIED BESSEL FUNCTION IDENTITY

We find for various values of p, the one-time prefactors of z_{SYK} from Eq. (36) and Eq. (27) match each other exactly. Below, we show some numerical plots demonstrating this agreement Fig. 5.

If we set $\lambda = 1$ and parametrize $x = \frac{\sqrt{1+8E_0}}{4}$, then the above match follows from the following series representation for modified Bessel K with negative half-integer index:

$$K_{\frac{1}{2}-p}(x) = \sqrt{2\pi}e^{-x}p!(4x)^{p-\frac{1}{2}}\sum_{r=1}^{p}\frac{(-x)^r}{r!}$$

$$\times \sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_{i=1}^{r}\alpha_i=p}}\prod_{i=1}^{r}\binom{\frac{1}{2}}{\alpha_i}(-2x^2)^{-\alpha_i}, \qquad \text{(B1)}$$
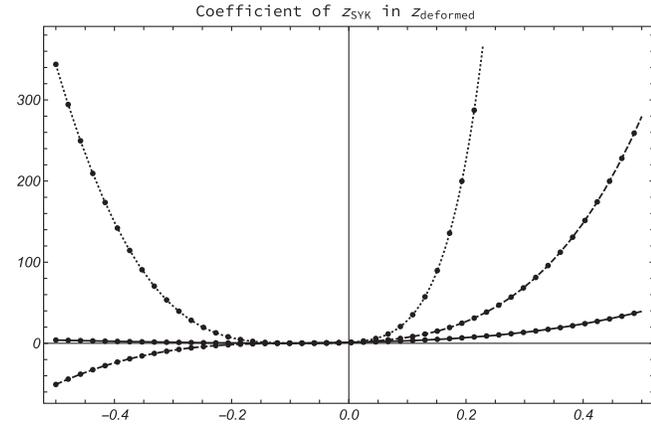
which can be simplified into

FIG. 5. The lines are for different values of p and E_0 for the coefficients in Eq. (36). The black dots are corresponding points obtained using Eq. (27). Solid line is for $p = 3, E_0 = -\frac{1}{16}$, dashed for $p = 4, E_0 = -\frac{1}{40}$, and dotted for $p = 5, E_0 = -\frac{1}{15}$.

$$K_{\frac{1}{2}-p}(x) = \sqrt{\frac{2\pi}{x}}e^{-x}p!2^{p-1}\sum_{r=1}^{p}\frac{(-x)^{r-p}}{r!}\sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_{i=1}^{r}\alpha_i=p}}\prod_{i=1}^{r}\binom{\frac{1}{2}}{\alpha_i}.$$

$$\qquad \text{(B2)}$$

We have checked explicitly the expressions on both sides of the above equality for specific integer values of p and found a match. In what follows, we sketch a proof of the identity.

Proof.—The identity given by Eq. (B2) can be proven in the following manner. Let us focus on the rhs of (B2), call it $I(x)$. Now we set $s = p - r$ to rewrite

$$I = \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{s=0}^{p-1}\frac{(-x)^{-s}}{(p-s)!}p!2^p\sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_{i=1}^{p-s}\alpha_i=p}}\prod_{i=1}^{p-s}\binom{\frac{1}{2}}{\alpha_i}, \qquad \text{(B3)}$$

which can further be rewritten as

$$I = \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{s=0}^{p-1}\frac{p!s!}{(p-s)!}\frac{(2x)^{-s}}{s!}\left[\sum_{\substack{\{\alpha_i \geq 1\} \\ \sum_{i=1}^{p-s}\alpha_i=p}}\prod_{i=1}^{p-s}\frac{1}{\alpha_i}\binom{2\alpha_i-2}{\alpha_i-1}\right]$$

$$= \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{s=0}^{p-1}\frac{p!s!}{(p-s)!}\frac{(2x)^{-s}}{s!}\left[\sum_{\substack{\{\beta_i \geq 0\} \\ \sum_{i=1}^{p-s}\beta_i=s}}\prod_{i=1}^{p-s}\frac{1}{\beta_i+1}\binom{2\beta_i}{\beta_i}\right]$$

$$= \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{s=0}^{p-1}\frac{p!s!}{(p-s)!}\frac{(2x)^{-s}}{s!}\left[\sum_{\substack{\{\beta_i \geq 0\} \\ \sum_{i=1}^{p-s}\beta_i=s}}\prod_{i=1}^{p-s}C_{\beta_i}\right]. \qquad \text{(B4)}$$

Here C_{β_i} is the β_ith Catalan number. The generating function for the Catalan numbers is

$$\sum_{n=0}^{\infty}C_n x^n = \frac{1-\sqrt{1-4x}}{2x}. \qquad \text{(B5)}$$

Using the generating function, we see that

$$\left[\sum_{\substack{\{\beta_i \geq 0\} \\ \sum_{i=1}^{p-s}\beta_i=s}}\prod_{i=1}^{p-s}C_{\beta_i}\right] = \text{coefficient of } x^s \text{ in } \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{p-s}$$

$$= \frac{p-s}{p+s}\binom{p+s}{s}. \qquad \text{(B6)}$$

A standard way to derive the above coefficient is to do a contour integral.

Now we plug the sum of the product of the Catalan numbers in Eq. (B4) to obtain

$$
\begin{aligned}
I &= \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{s=0}^{p-1} \frac{p!s!}{(p-s)!} \frac{p-s}{p+s} \binom{p+s}{s} \frac{(2x)^{-s}}{s!} \\
&= \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{s=0}^{p-1} \frac{(p+s-1)!}{(p-s-1)!} \frac{(2x)^{-s}}{s!} \\
&= K_{p-1/2}(x) = K_{1/2-p}(x).
\end{aligned}
\tag{B7}
$$

This completes the proof. Note, here we have used $K_\nu(x) = K_{-\nu}(x)$ for $\nu \notin \mathbb{Z}$ and the following identity for the Bessel function with half-integer index [43]:

$$
K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{s=0}^{n} \frac{(n+s)!}{(n-s)!} \frac{(2x)^{-s}}{s!}, \tag{B8}
$$

and set $n = p - 1$.

[1] A. Almheiri and H. W. Lin, The entanglement wedge of unknown couplings, J. High Energy Phys. 08 (2022) 062.

[2] J.-M. Schlenker and E. Witten, No ensemble averaging below the black hole threshold, J. High Energy Phys. 07 (2022) 143.

[3] P. Saad, S. H. Shenker, and D. Stanford, A semiclassical ramp in SYK and in gravity, arXiv:1806.06840.

[4] P. Saad, S. H. Shenker, and D. Stanford, JT gravity as a matrix integral, arXiv:1903.11115.

[5] P. Saad, Late time correlation functions, baby universes, and ETH in JT gravity, arXiv:1910.10311.

[6] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, Replica wormholes and the black hole interior, J. High Energy Phys. 03 (2022) 205.

[7] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Replica wormholes and the entropy of Hawking radiation, J. High Energy Phys. 05 (2020) 013.

[8] R. E. Prange, The Spectral form Factor Is Not Self-Averaging, Phys. Rev. Lett. **78**, 2280 (1997).

[9] P. Saad, S. H. Shenker, D. Stanford, and S. Yao, Wormholes without averaging, arXiv:2103.16754.

[10] B. Mukhametzhanov, Half-wormhole in SYK with one time point, SciPost Phys. **12**, 029 (2022).

[11] P. Saad, S. Shenker, and S. Yao, Comments on wormholes and factorization, arXiv:2107.13130.

[12] B. Mukhametzhanov, Factorization and complex couplings in SYK and in Matrix Models, arXiv:2110.06221.

[13] A. Blommaert and J. Kruthoff, Gravity without averaging, SciPost Phys. **12**, 073 (2022).

[14] A. Blommaert, L. V. Iliesiu, and J. Kruthoff, Gravity factorized, arXiv:2111.07863.

[15] L. Eberhardt, Summing over geometries in string theory, J. High Energy Phys. 05 (2021) 233.

[16] J. J. Heckman, A. P. Turner, and X. Yu, Disorder averaging and its UV discontents, Phys. Rev. D **105**, 086021 (2022).

[17] S. Collier and E. Perlmutter, Harnessing S-duality in $\mathcal{N}=4$ SYM & supergravity as $SL(2,\mathbb{Z})$-averaged strings, arXiv:2201.05093.

[18] N. Benjamin, S. Collier, A. L. Fitzpatrick, A. Maloney, and E. Perlmutter, Harmonic analysis of 2d CFT partition functions, J. High Energy Phys. 09 (2021) 174.

[19] A. Maloney and E. Witten, Averaging over Narain moduli space, J. High Energy Phys. 10 (2020) 187.

[20] N. Afkhami-Jeddi, H. Cohn, T. Hartman, and A. Tajdini, Free partition functions and an averaged holographic duality, J. High Energy Phys. 01 (2021) 130.

[21] S. Datta, S. Duary, P. Kraus, P. Maity, and A. Maloney, Adding flavor to the Narain ensemble, J. High Energy Phys. 05 (2022) 090.

[22] N. Benjamin, C. A. Keller, H. Ooguri, and I. G. Zadeh, Narain to Narnia, Commun. Math. Phys. **390**, 425 (2022).

[23] J. Dong, T. Hartman, and Y. Jiang, Averaging over moduli in deformed WZW models, J. High Energy Phys. 09 (2021) 185.

[24] M. Ashwinkumar, M. Dodelson, A. Kidambi, J. M. Leedom, and M. Yamazaki, Chern-Simons invariants from ensemble averages, J. High Energy Phys. 08 (2021) 044.

[25] A. Pérez and R. Troncoso, Gravitational dual of averaged free CFT's over the Narain lattice, J. High Energy Phys. 11 (2020) 015.

[26] A. Dymarsky and A. Shapere, Quantum stabilizer codes, lattices, and CFTs, J. High Energy Phys. 21 (2020) 160.

[27] A. Dymarsky and A. Shapere, Comments on the holographic description of Narain theories, J. High Energy Phys. 10 (2021) 197.

[28] V. Meruliya and S. Mukhi, AdS$_3$ gravity and RCFT ensembles with multiple invariants, J. High Energy Phys. 08 (2021) 098.

[29] V. Meruliya, S. Mukhi, and P. Singh, Poincaré series, 3d gravity and averages of rational CFT, J. High Energy Phys. 04 (2021) 267.

[30] J. Cotler and K. Jensen, AdS$_3$ wormholes from a modular bootstrap, J. High Energy Phys. 11 (2020) 058.

[31] S. Collier and A. Maloney, Wormholes and spectral statistics in the Narain ensemble, J. High Energy Phys. 03 (2022) 004.

[32] D. Das and S. Datta, Higher spin wormholes from modular bootstrap, J. High Energy Phys. 10 (2021) 010.

[33] D. J. Gross, J. Kruthoff, A. Rolph, and E. Shaghoulian, $T\bar{T}$ in AdS$_2$ and quantum mechanics, Phys. Rev. D **101**, 026011 (2020).

[34] D. J. Gross, J. Kruthoff, A. Rolph, and E. Shaghoulian, Hamiltonian deformations in quantum mechanics, $T\bar{T}$, and the SYK model, Phys. Rev. D **102**, 046019 (2020).

[35] L. V. Iliesiu, J. Kruthoff, G. J. Turiaci, and H. Verlinde, JT gravity at finite cutoff, SciPost Phys. **9**, 023 (2020).

[36] D. Stanford and Z. Yang, Finite-cutoff JT gravity and self-avoiding loops, arXiv:2004.08005.

[37] L. Griguolo, R. Panerai, J. Papalini, and D. Seminara, Nonperturbative effects and resurgence in Jackiw-Teitelboim gravity at finite cutoff, Phys. Rev. D **105**, 046015 (2022).

[38] S. Ebert, C. Ferko, H.-Y. Sun, and Z. Sun, $T\bar{T}$ in JT gravity and BF gauge theory, arXiv:2205.07817.

[39] S. Ebert, C. Ferko, H.-Y. Sun, and Z. Sun, $T\bar{T}$ deformations of supersymmetric quantum mechanics, J. High Energy Phys. 08 (2022) 121.

[40] S. Ebert, H.-Y. Sun, and Z. Sun, $T\bar{T}$-deformed free energy of the Airy model, J. High Energy Phys. 08 (2022) 026.

[41] Y. Liao, A. Vikram, and V. Galitski, Many-Body Level Statistics of Single-Particle Quantum Chaos, Phys. Rev. Lett. **125**, 250601 (2020).

[42] M. Winer, S.-K. Jian, and B. Swingle, An Exponential Ramp in the Quadratic Sachdev-Ye-Kitaev Model, Phys. Rev. Lett. **125**, 250602 (2020).

[43] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1944).